Waves Session 2

Wednesday, November 2, 2022 11:58 AM

Welcome to Martin's Lab Book. The work in the following section was completed on: **Friday, February 17th, 2023** **9am – 12pm** in a synchronous manner becoming of a lab workbook.
Aims
• Finish Part II on Thermal Waves. • Plots all of the provided datasets for Part II and analyse them. • Complete a second *back-of-the-envelope* calculation for the two additional datasets, in order to obtain their respective values of D (with error). • Repeat this analysis for other the rest of the datasets. • Plot D as a function of period discussing our plots. Superimpose expected D. • Complete the Fourier Analysis with n = 3 to create a truncated dataset for τ = 4 min. • Perform the Bessel Analysis of the data. • Perhaps, start Task III on Electrical Waves.

Task 2.4: Physical meaning and key features of all datasets.

a. Plot all datasets on separate axes. Do not forget to label the axes and the period represented on each figure.
b. Explain why none of the measured temperature waves are square waves.
c. Identify the datasets which are affected by strong temperature transients and discard them from the analysis.
d. For the remaining datasets, decide which ones look approximately sinusoidal. In each case, explain your reasoning. Can you identify any pattern for increasing or decreasing period?
e. In which cases could the fundamental mode approximation be sufficient? In which cases would we benefit the most from performing the Fourier analysis of the measured wave?

Task 2.5: More 'back of the envelope' analysis – comparing different values for *D*

a. Now that you have done 4 min, you will look at the datasets for τ = 1 min and τ = 8 min.
   I. Repeat the 'back of the envelope' calculation for these two additional datasets and obtain their respective values of *D* (with error).
b. Time permitting, repeat the analysis for τ = 2 min and τ = 6 min.
c. Plot your values of *D* as a function of period and discuss your plot. Superimpose the expected value of *D* as a constant.
d. Looking at the measured wave + square wave plot for each investigated period, can you identify a pattern in the overall amplitude attenuation? Is this what you expected?

Recall: to obtain the Fourier series of your dataset you need to calculate the constant coefficients a_n and b_n. The numerical integration methods discussed in Task 1.3 might be useful in this process!

Task 2.4b
In order to obtain the measured square waves, we would have to have instant heating and cooling, which is impossible given that instantaneous heat, and therefore energy transfer (nothing can travel faster than the speed of light of course, and thus the vibration of atoms can never be instantaneous). Instead, we have that there is a time lag for which the heat is able to increase and decrease, which can be seen by the approximate sinusoidal shape which this gives rise to. Additionally, the accumulation of heat by the system can be seen by the ever-increasing position in the troughs of the waveforms, e.g. **Figure 5**.

Task 2.4c
To summarise, we have chosen to discard the Data Set B graphs of: τ = 1 min, 2 mins, 4 mins, due to their being strongly affected by temperature transients. Additionally, we find sinusoidal behaviour to exhibited most by the graphs with shorter time periods, such as τ = 1 min, 2 mins, 4 mins, 6 mins, with the best approximation probably being 4 mins, as opposed to any longer time periods, where this begins to approximate a *"shark fin"* shape, like a triangular function.

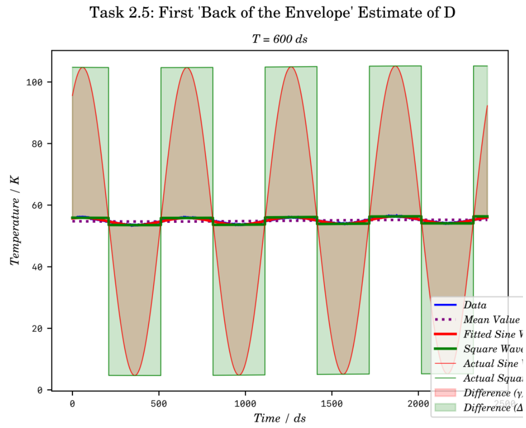
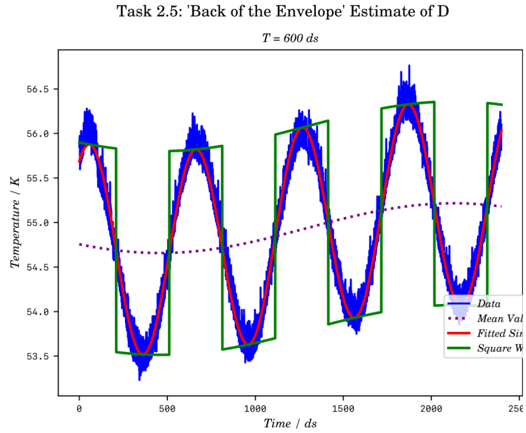
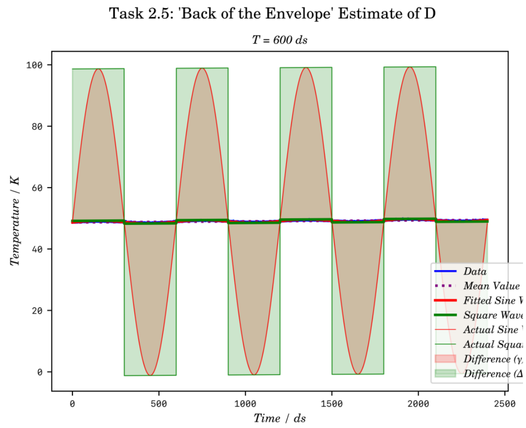
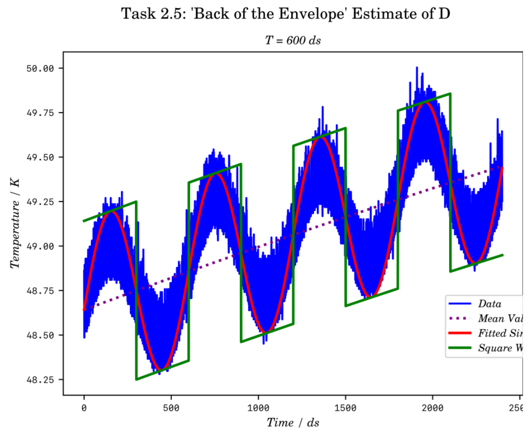
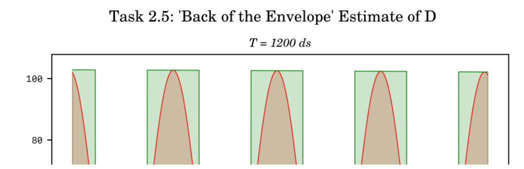
Task 2.4d

Task 2.4e

Note to self: *as wavelengths are being attenuated, the value of γ < 1.*
Note to reader: *The order of terms in Δφ calculations has been swapped where appropriate to ensure that the value obtained is positive, though this doesn't matter as the value is subsequently squared.*

PTFE (Polytetrafluorethylene) at 25 °C	$0.124 mm^2 s^{-1} = \left(1.24 \times 10^{-7}\right) m^2 s^{-1}$

If we make use of the same values of r_{inner} and r_{outer} which we have seen in Task 2.3, which are: r_{outer} = (20.57 ± 0.01) mm, r_{inner} = (2.50 ± 0.05) mm, then we have **Table 1**:

τ / mins	Calculations	Plot(s)
1	***Values: 1min_A*** $\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)mm$ $\gamma = \frac{A_{inner}}{A_{outer}} = \frac{1.1432 \pm 0.0051}{50.0000 \pm 0.0000} = 0.0229 \pm 0.0001$ $\Delta\varphi = Phase_{inner} - Phase_{outer} = (300.000 \pm 0.000) - (209.497 \pm 1.171)$ $= (90.503 \pm 0.117) ds$ $\omega 1 = 2\pi$ $D_{\gamma} = \frac{\omega \Delta r^2}{2ln(\gamma)^2} = \frac{(2\pi)\left(18.07 \times 10^{-3}\right)^2}{2ln(0.0229)^2}$ $= (7.1860 \pm 0.044) \times 10^{-5} m^2 s^{-1}$ $D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{(2\pi)\left(18.07 \times 10^{-3}\right)^2}{2(9.0503)^2}$ $= (1.2524 \pm 0.0331) \times 10^{-5} m^2 s^{-1}$ **Figure 5:** The dataset A for τ = *1 min*. As can be seen, as we increase the time elapsed, the system slowly edges upwards, increasing in temperature. The system gains thermal energy. ***Discarded Values: 1min_B*** $\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)mm$ $\gamma = \frac{A_{inner}}{A_{outer}} = \frac{0.3332 \pm 0.0057}{50.0000 \pm 0.0000} = 0.0066 \pm 0.0001$ $\Delta\varphi = Phase_{inner} - Phase_{outer} = (300.000 \pm 0.000) - (299.835 \pm 0.361)$ $= (0.165 \pm 0.361) ds$ $\omega 1 = 2\pi$ $D_{\gamma} = \frac{\omega \Delta r^2}{2ln(\gamma)^2} = \frac{(2\pi)\left(18.07 \times 10^{-3}\right)^2}{2ln(0.0066)^2} = (4.085 \pm 0.036) \times 10^{-5} m^2 s^{-1}$ $D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{(2\pi)\left(18.07 \times 10^{-3}\right)^2}{2(0.0165)^2} = (3.7689 \pm 16.4417)\ m^2 s^{-1}$ **Figure 6:** The dataset B for τ = *1 min*. As can be seen, as we increase the time elapsed, the system slowly edges upwards, increasing in temperature. The system gains thermal energy. This increase is significant enough for us to class it as strong temperature transients. **DISCARDED**	
2	***Values: 2min_A*** $\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)mm$ $\gamma = \frac{A_{inner}}{A_{outer}} = \frac{2.4595 \pm 0.0042}{(50.0000 \pm 0.0000)} = 0.0492 \pm 0.0001$	

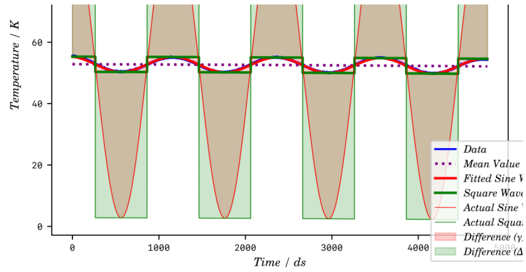
$$\Delta\varphi = Phase_{inner} - Phase_{outer} = (599.655 \pm 0.000) - (264.214 \pm 0.079) = (335.441 \pm 0.0079)\, ds$$

$\omega 1 = \pi$

$$D_{\gamma} = \frac{\omega \Delta r^2}{2 ln(\gamma)^2} = \frac{(\pi)\left(18.07 \times 10^{-3}\right)^2}{2ln(0.0492)^2} = (1.1307 \pm 0.0065) \times 10^{-4} m^2 s^{-1}$$

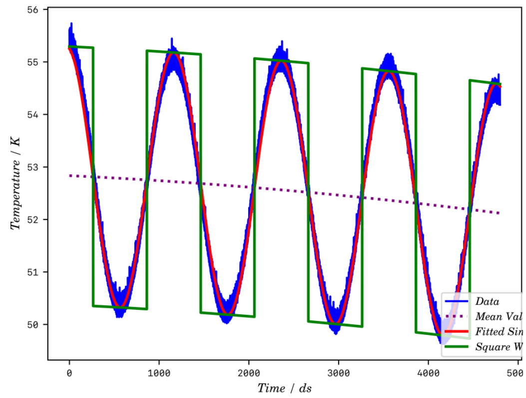
$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{(\pi)\left(18.07 \times 10^{-3}\right)^2}{2(33.5441)^2} = (9.117 \pm 0.052) \times 10^{-7} m^2 s^{-1}$$

Figure 7: The dataset A for τ = 2 mins. As can be seen, as we increase the time elapsed, the system slowly edges upwards, increasing in temperature. The system gains thermal energy.



Task 2.5: 'Back of the Envelope' Estimate of D

T = 1200 ds



Discarded Values: 2min_B

$$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)\, mm$$

$$\gamma = \frac{A_{inner}}{A_{outer}} = \frac{3.2086 \pm 0.01015}{50.0000 \pm 0.0000} = 0.0642 \pm 0.0002$$

$$\Delta\varphi = Phase_{inner} - Phase_{outer} = (610.301 \pm 0.000) - (172.801 \pm 16.777) = (437.502 \pm 1.678)\, ds$$

$\omega 1 = \pi$

$$D_{\gamma} = \frac{\omega \Delta r^2}{2 ln(\gamma)^2} = \frac{(\pi)\left(18.07 \times 10^{-3}\right)^2}{2ln(0.0642)^2} = (1.3602 \pm 0.0083) \times 10^{-4} m^2 s^{-1}$$

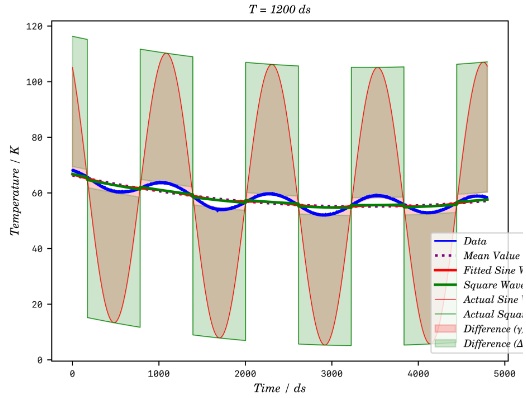
$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{(\pi)\left(18.07 \times 10^{-3}\right)^2}{2(436.502)^2} = (5.359 \pm 0.412) \times 10^{-7} m^2 s^{-1}$$

Divide denominator by 10

Figure 8: The dataset B for τ = *2 mins*. As can be seen, as we decrease the time elapsed, the system slowly edges upwards, increasing in temperature. The system loses thermal energy from a strong initial amount of energy. This decrease is significant to count as affected by a strong temperature transient. **DISCARDED**

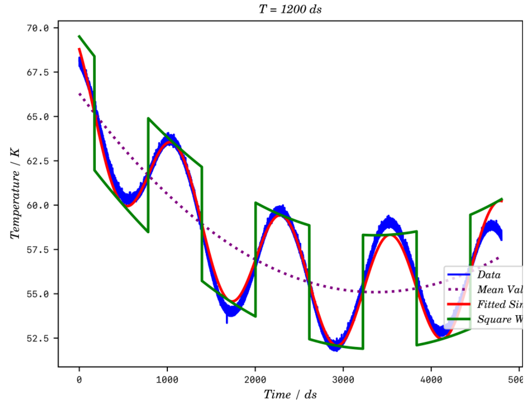
Task 2.5: 'Back of the Envelope' Estimate of D

T = 1200 ds



Task 2.5: 'Back of the Envelope' Estimate of D

T = 1200 ds



4

Values: 4min_A

$$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)\, mm$$

$$\gamma = \frac{A_{inner}}{A_{outer}} = \frac{9.361 \pm 0.004}{50.000 \pm 0.000} = 0.1872 \pm 0.0001$$

$$\Delta\varphi = Phase_{inner} - Phase_{outer} = (1213.546 \pm 1.946) - (1200.000 \pm 0.000) = (13.55 \pm 0.19)\, ds$$

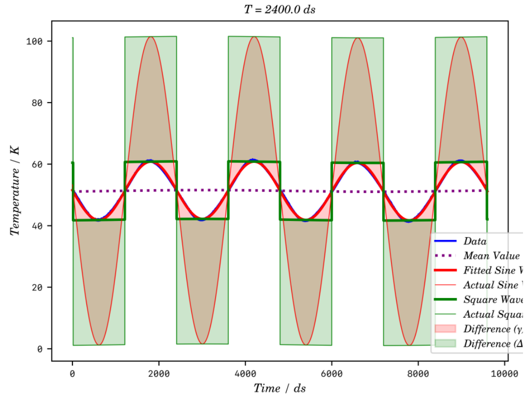
$\omega 1 = \frac{\pi}{2}$

$$D_{\gamma} = \frac{\omega \Delta r^2}{2 ln(\gamma)^2} = \frac{\left(\frac{\pi}{2}\right)\left(18.07 \times 10^{-3}\right)^2}{2ln(0.1872)^2}$$

$$= (9.136 \pm 0.052) \times 10^{-5} m^2 \cdot s^{-1}$$

Task 2.3: First 'Back of the Envelope' Estimate of D

T = 2400.0 ds



Task 2.3: First 'Back of the Envelope' Estimate of D

$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{\left(\frac{\pi}{2}\right)\left(18.07 \times 10^{-3}\right)^2}{2(1.355)^2}$$

$$= (1.398 \pm 0.040) \times 10^{-4} m^2 \cdot s^{-1}$$

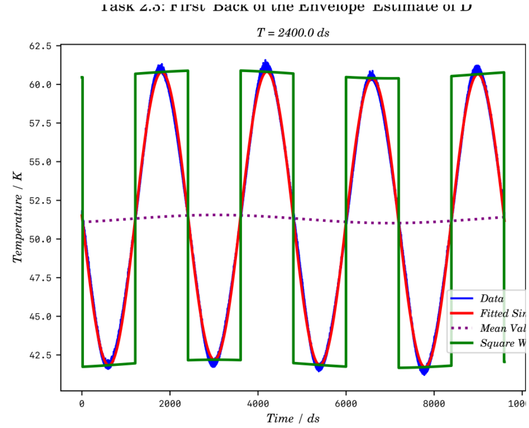


Figure 9: The dataset A for τ = *4 mins*. There is no significant increase or decrease in the system's temperature range, as it stays relatively stable, which is quite interesting to note, as opposed to most of the other graphs.

Discarded Values: 4min_B

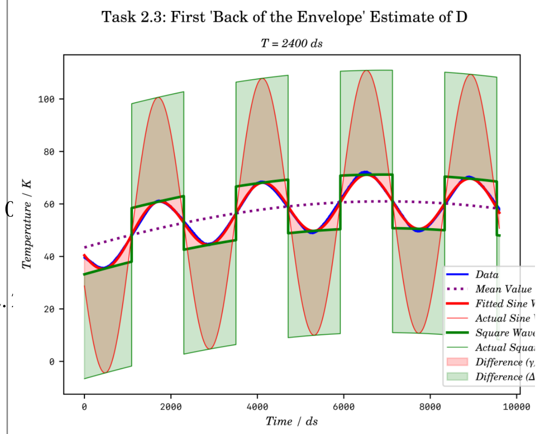
$$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)\,mm$$

$$\gamma = \frac{A_{inner}}{A_{outer}} = \frac{10.211 \pm 0.008}{50.000 \pm 0.000} = 0.2042 \pm 0.0002$$

$$\Delta\varphi = Phase_{inner} - Phase_{outer} = 1200.000 - 1093.012$$
$$= (106.99 \pm 0.30)\,ds$$

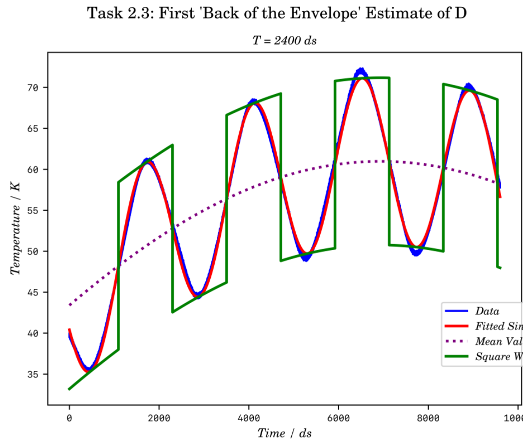
$$\omega 1 = \frac{\pi}{2}$$

$$D_{\gamma} = \frac{\omega \Delta r^2}{2ln(\gamma)^2} = \frac{\left(\frac{\pi}{2}\right)\left(18.07 \times 10^{-3}\right)^2}{(2ln(0.2042))^2} = (1.0163 \pm 0.0058) \times 10^{-4} m^2 s^{-1} \quad = (1.$$



$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{\left(\frac{\pi}{2}\right)\left(18.07 \times 10^{-3}\right)^2}{2(10.6986)^2} = (2.2405 \pm 0.0178) \times 10^{-6} m^2 s^{-1} \quad = (2.$$

Figure 10: The dataset for τ = *4 mins*. As can be seen, as we increase the time elapsed, the system slowly edges upwards, increasing in temperature. The system gains thermal energy. **DISCARDED**



6

Values: 6min

$$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)\,mm$$

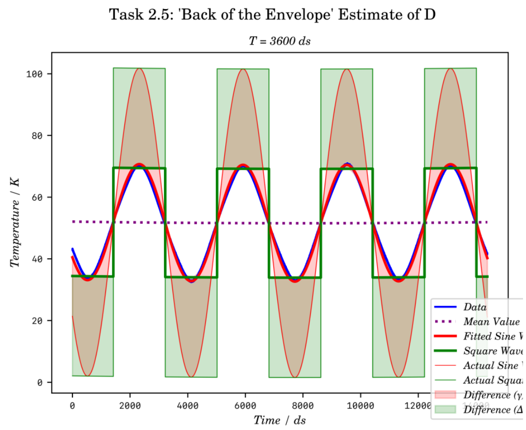
$$\gamma = \frac{A_{inner}}{A_{outer}} = \frac{17.6305 \pm 0.0088}{(50.000 \pm 0.000)} = 0.3526 \pm 0.0002$$

$$\Delta\varphi = Phase_{inner} - Phase_{outer} = (1800.000 \pm 0.000) - (1421.635 \pm 4.923)$$
$$= (378.365 \pm 4.923)\,ds$$

$$\omega 1 = \frac{\pi}{3}$$

$$D_{\gamma} = \frac{\omega \Delta r^2}{2ln(\gamma)^2} = \frac{\left(\frac{\pi}{3}\right)\left(18.07 \times 10^{-3}\right)^2}{2ln(0.3526)^2}$$

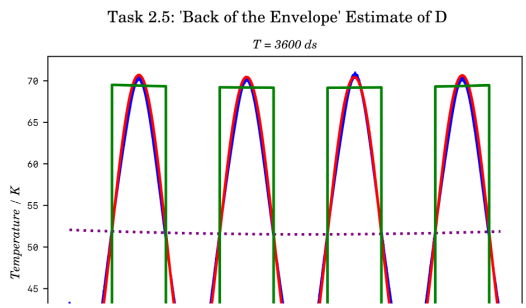
$$= (9.4408 \pm 0.0540) \times 10^{-4} m^2 s^{-1}$$

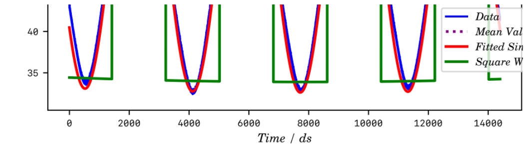


$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{\left(\frac{\pi}{3}\right)\left(18.07 \times 10^{-3}\right)^2}{2(37.8365)^2}$$

$$= (7.1655 \pm 0.1908) \times 10^{-7} m^2 s^{-1}$$

Figure 11: The dataset for τ = *6 mins*. This graph also stays stable interestingly enough, like.





8

Values: 8min

$$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)\, mm$$

$$\gamma = \frac{A_{inner}}{A_{outer}} = \frac{28.1370 \pm 0.0057}{(50.000 \pm 0.000)} = 0.5627 \pm 0.00428$$

$$\Delta\varphi = Phase_{inner} - Phase_{outer} = (2400.000 \pm 0.000) - (1417.621 \pm 18.452)$$
$$= (98.2379 \pm 1.8452)\, ds$$

$$\omega 1 = \frac{\pi}{4}$$

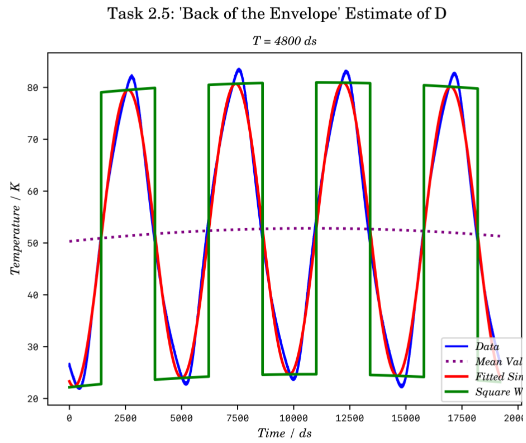
$$D_\gamma = \frac{\omega \Delta r^2}{2 ln(\gamma)^2} = \frac{\left(\frac{\pi}{4}\right)\left(18.07 \times 10^{-3}\right)^2}{2 ln(0.5627)^2}$$

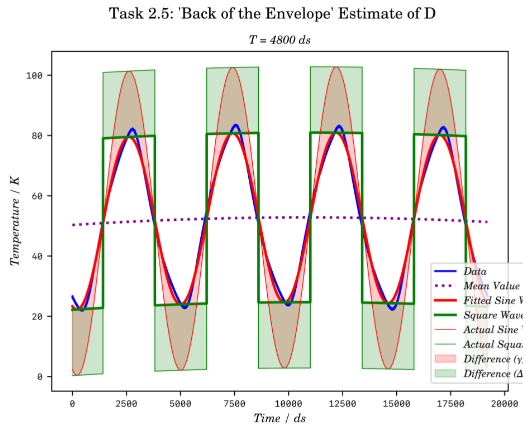
$$= (3.1033 \pm 0.0289) \times 10^{-3} m^2 s^{-1}$$

$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = (\pi/4)(18.07 \times 10^{-3})\text{\textasciicircum}2/(2(9.8238)\text{\textasciicircum}2)$$

$$= (1.0629 \pm 0.0404) \times 10^{-7} m^2 s^{-1}$$

Figure 12: The dataset for τ = *8 mins*. As can be seen, as we increase the time elapsed, the system very slowly edges upwards, increasing in temperature, before decreasing slightly. Each wave cycle demonstrates the *"shark-fin"* shape.





16

Values: 16min

$$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)\, mm$$

$$\gamma = \frac{A_{inner}}{A_{outer}} = \frac{38.9089 \pm 0.0441}{(50.000 \pm 0.000)} = 0.7782 \pm 0.0009$$

$$\Delta\varphi = Phase_{inner} - Phase_{outer} = (4800.000 \pm 0.000) - (1852.415 \pm 152.329)$$
$$= (2947.585 \pm 152.329)\, ds$$

$$\omega 1 = \frac{\pi}{8}$$

$$D_\gamma = \frac{\omega \Delta r^2}{2 ln(\gamma)^2} = \frac{\left(\frac{\pi}{8}\right)\left(18.07 \times 10^{-3}\right)^2}{\left(2 ln(0.7782)^2\right)} = (1.6308 \pm 0.1739) \times 10^{-2} m^2 s^{-1}$$

$$D_{\Delta\varphi} = \frac{\omega \Delta r^2}{2\Delta\varphi^2} = \frac{\left(\frac{\pi}{8}\right)\left(18.07 \times 10^{-3}\right)^2}{2(294.7585)^2} = (1.1807 \pm 0.1222) \times 10^{-8} m^2 s^{-1}$$

Figure 13: The dataset for τ = *16 mins*. This is the aforementioned *"shark-fin shape"* very clearly, for each wave cycle.

Example Output Data:

```
> Given: a(bx + c) + d(ex + f) + g, we have:

a :  38.90885963235642 ± 0.04413803226569785
b :  -0.0006480498473992863 ± 3.1599888754987045e-07
c :  -1.9411351274787774 ± 0.003221397068933095
d :  232054.37124694407 ± 1549205.5754588493
e :  -1.2047565378081985e-06 ± 4.021560178888501e-06
f :  -1.55568359924627 ± 0.0504473731264641
g :  232105.5362198636 ± 1549205.5689815972
γ: 0.778177+/-0.000883 unitless
φ₁: 1852.415000+/-152.329000 ds
Δφ: 294.758500+/-15.232900 s
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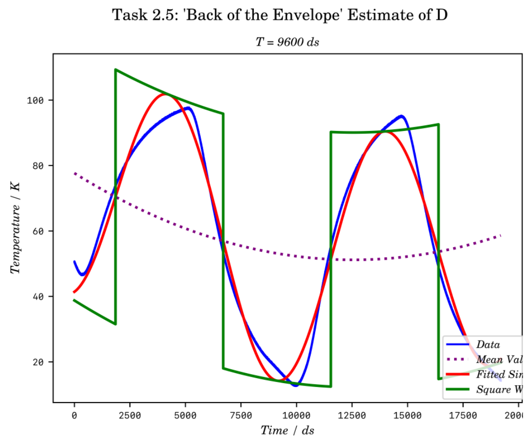

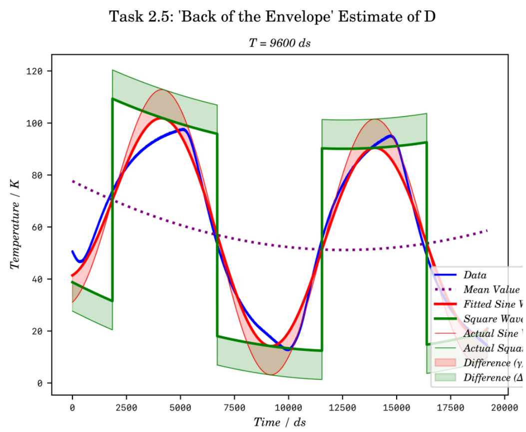

	D_γ: 0.016308257412+/-0.000173882968 $m^2 \cdot s^{-1}$ D_φ: 0.000000011807+/-0.000000001222 $m^2 \cdot s^{-1}$	

Table 1: Summary of values of Δr, γ, $\Delta\varphi$, ω_1, D_γ, $D_{\Delta\varphi}$.

Task 2.6: Fourier analysis of the 4 min data – Obtaining a better estimate for *D* using the 1-D plane slab model.

a. Perform Fourier analysis of the τ = 4 min dataset to obtain an amplitude-phase Fourier Series of the data, truncated at n = 3. Write down the amplitudes and phase lags for n = 1 (fundamental), 2 and 3.
b. Calculate values of D for each harmonic, using both the Transmission Factor, Diffusivity, DTF , and Phase Lag Diffusivity, DPL .
c. Compare your diffusivity value from Fourier analysis to your diffusivity value from the 'back of the envelope' analysis and comment on it.

Task 2.7: Fourier analysis of the remaining datasets – comparing different values for *D*

a. Repeat the Fourier analysis procedure in Task 2.7 for other datasets and obtain the respective values of D for different harmonics. Feel free to extend the truncation to n > 3 if and when you judge necessary.
b. Plot the values of DTF and DPL from Fourier analysis for all the datasets on the same plot. Can you decide on a unique value for D? Compare this to your diffusivity from the 'back of the envelope' analysis and comment on it.

Task 2.6a
Evidently, we wish to improve on our *back-of-the-envelope* estimation which we performed in Task 2.5, which can be accomplished through taking the Fourier analysis of our datasets, starting with our τ *= 4 mins*, which has been truncated into *n = 3*, where having done the calculations we found, the even harmonics disappear and only odd ones remain. Therefore, in actuality, we are looking at *1, 3, 5*. We can summarise the values of *D* below in **Table 2**:

τ / min	Values	Plots
1	$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06) mm$ $D_\gamma = \dfrac{\omega \Delta r^2}{2ln(\gamma)^2}$ = (2π)(18.07 × 10^-3)^2/(2 ln()^2) $D_{\Delta\varphi} = \dfrac{\omega \Delta r^2}{2\Delta\varphi^2}$ = (2π)(18.07 × 10^-3)^2/(2()^2)	
2	$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06) mm$ $D_\gamma = \dfrac{\omega \Delta r^2}{2ln(\gamma)^2}$ = (π)(18.07 × 10^-3)^2/(2 ln()^2) $D_{\Delta\varphi} = \dfrac{\omega \Delta r^2}{2\Delta\varphi^2}$ = (π)(18.07 × 10^-3)^2/(2()^2)	
4	$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06) mm$ $D_\gamma = \dfrac{\omega \Delta r^2}{2ln(\gamma)^2}$ = (π/2)(18.07 × 10^-3)^2/(2 ln()^2) $D_{\Delta\varphi} = \dfrac{\omega \Delta r^2}{2\Delta\varphi^2}$ = (π/2)(18.07 × 10^-3)^2/(2()^2)	
6	$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06) mm$ $D_\gamma = \dfrac{\omega \Delta r^2}{2ln(\gamma)^2}$ = (π/3)(18.07 × 10^-3)^2/(2 ln()^2) $D_{\Delta\varphi} = \dfrac{\omega \Delta r^2}{2\Delta\varphi^2}$ = (π/3)(18.07 × 10^-3)^2/(2()^2)	
8	$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06) mm$ $D_\gamma = \dfrac{\omega \Delta r^2}{2ln(\gamma)^2}$ = (π/4)(18.07 × 10^-3)^2/(2 ln()^2)	

	$D_{\Delta\varphi} = \dfrac{\omega \Delta r^2}{2\Delta\varphi^2} =$ (π/4)(18.07 × 10^-3)^2/(2()^2)	
16	$\Delta r = 20.57 - 2.50 = (18.07 \pm 0.06)mm$ $D_\gamma = \dfrac{\omega \Delta r^2}{2ln(\gamma)^2} =$ (π/8)(18.07 × 10^-3)^2/(2 ln()^2) $D_{\Delta\varphi} = \dfrac{\omega \Delta r^2}{2\Delta\varphi^2} =$ (π/8)(18.07 × 10^-3)^2/(2()^2)	

Table 2: Summary table of improved D values, using Fourier analysis method.

We can extend some of our plots to a higher number of harmonics for some of the plots to gauge a better understanding of the trends and dynamic at play. The chosen number of harmonics that we decided upon was n = ___. This is summarised in the following plot if we are to consider all of them in one:

Clearly from the above plots, we can conclude that as the period τ increases, we have that D values decrease, and converge upon their respective theoretical values. Next up, we attempt to obtain even more accurate values of D, by means of Bessel function analysis, through the γ and $\Delta\varphi$ previously obtained from Fourier analysis.

2.5 Cylindrical Model – Bessel Functions

2.5.1. Introduction and Theory

What follows is an introduction to the maths of heat conduction in axisymmetric polar coordinates, which might not be as straightforward as the 1-D plane slab model presented before. Even if you cannot follow the maths in detail, it is useful to get a ‘flavour’ of the maths/physics involved in solving the problem. In the end, values for *D* will be obtained using a program that works with Bessel functions. A more accurate modelling of heat conduction in the PTFE must consider its actual geometrical shape, i.e. not a plane slab but a cylinder instead.

In this case, the heat/diffusion equation to solve involves the radial coordinate *r*:

$$\frac{\partial T(r,\, t)}{\partial t} = D\, \frac{(\partial^2)\; T(r,\, t)}{\partial r^2} + D\, \frac{1}{r}\frac{\partial T(r,\, t)}{\partial r},\; (2.5)$$

The solution is then given by:

$$T(r,\, t) = \; constant\; \bullet\; \Re e\left[(J_0)\left(r\sqrt{\frac{i\omega}{D}}\right)\right],\; (2.6)$$

where ω is the angular frequency of the wave, and *J*0 is a Bessel function of first kind.
Because J_0 is complex, *T*(*r*, *t*) can be written in amplitude-phase form as:

$$T(r,\, t) = \; constant\; \bullet\; \Re e\Bigg[M_0.\left(r\sqrt{\frac{i\omega}{D}}\right)\;\; \exp(i\left(\omega t + \Phi_0\left(r\sqrt{\frac{i\omega}{D}}\right)\right)],\; (2.6)$$

where M_0 and Φ_0 are the modulus and phase factor of *J*0, respectively.
Both have complex argument: $r\sqrt{\frac{i\omega}{D}}$.

For a detailed derivation of the heat equation for a cylinder and its solution involving Bessel functions, please refer to the Appendix.

2.5.3. Bessel transmission factor and phase lag calculations

Previously, we have defined the transmission factor as the ratio between the plane slab temperature amplitudes evaluated at the inner and outer radius of the PTFE cylinder. Following the same logic as before, but now with an improved mathematical model, we can write the transmission factor for the *nth* mode in terms of the Bessel temperature amplitudes as:

$$\gamma_n = \frac{M_{0\left(\sqrt{\frac{i\omega_n}{D}}\, r_{inner}\right)}}{M_{0\left(\sqrt{\frac{i\omega_n}{D}}\, r_{outer}\right)}},\; (2.7)$$

Similarly, the phase lag can be written in terms of Bessel phase factors as:

$$\Delta\phi_n = \; \Phi_0\left(\sqrt{\frac{\omega_n}{D}}\, r_{inner}\right) - \; \Phi_0\left(\sqrt{\frac{\omega_n}{D}}\, r_{outer}\right),\; (2.8)$$

Since the physical meanings of the transmission and phase lag factors are the same as the ones adopted throughout the experiment, (2.7) and (2.8) are directly comparable to γ_n and $\Delta\phi_n$ as calculated from the square wave Fourier modes and the Fourier decomposed data at r_{inner}.

Because *D* is contained within the argument of the Bessel functions, we should be able to evaluate transmission diffusivity, *DTF*, by solving for the parameter $\alpha_n = \sqrt{\frac{\omega_n}{D}}\, r_{inner}$ in the following equation:

$$\gamma_{n(calculated)} - \frac{M_{0\,(\sqrt{i}\alpha_n)}}{M_0\left(\sqrt{i} r_{outer\; r_{inner}\alpha_n}\right)} = \; 0\,,\; (2.9)$$

Similarly, for phase lag diffusivity DPL , we have:

$$\Delta\phi_{n(calculated)} - \left[\Phi_{0(\sqrt{i}\alpha_n)} - \Phi_0\left(\sqrt{i}r_{outer}\left(r_{inner}\right)\ \alpha_n\right)\right] = 0\,, \quad (2.10)$$

Equations (2.9) and (2.10) can be solved using the program BESSEL, which relies on an iterative process and is available in both Excel and Python versions. It handles the Bessel functions and uses root-finding algorithms to obtain the best value for α_n . You simply have to provide your transmission and phase lag data, their associated periods and run the program as instructed. Once BESSEL has calculated α_n , it will return the corresponding value for *D*, given by:

$$D = \frac{r_{inner}^2\omega_n}{\alpha_n^2}, \quad (2.11)$$

from the definition of α_n . Depending on the data you enter, it will return you the transmission diffusivities *DTF*, the phase lag diffusivities *DPL*, or both. In summary, the same set of results for γ_n and $\Delta\phi_n$ can be analysed using two different models to estimate diffusivity: one is a 1-D plane slab approximation, and the other is a more realistic cylindrical model which uses Bessel functions. They are two independent interpretations of the same problem and can give different values for *D* and different systematic errors.

Task 2.8: Bessel Analysis of Thermal Data
You have access to the program BESSEL in either Excel or Python formats – both do exactly the same thing. Indeed, this is a ‘black box’, however it is one that you can look into if you want to!

a. Choose one temperature dataset (e.g. τ = 8 min).
   I. Input the Fourier transmission factor, phase lag, and associated period for all calculated harmonic modes into the program BESSEL to output values for *DTF* and *DPL.*
   II. Plot your values of *DTF* and *DPL* for this dataset on the same plot.
b. Repeat (a) for all the other datasets.
c. Plot the values of *DTF* and *DPL* from Fourier analysis for all the datasets on the same plot. Can you decide on a unique value for *D*? Compare this to your diffusivity from the plane slab Fourier analysis and comment on it. Make sure to include the associated errors in your values of *D*!

The most important equation in question is from the Bessel calculator in the Python and Excel files used, which determines the value of D. This is (2.11). An iterative process using root-finding algorithms is used to determine the best value for α_n .

$$D = \frac{r_{inner}^2\omega_n}{\alpha_n^2}, \quad (2.11)$$

Consequently, from the use of our values of γ and $\Delta\varphi$, which was found by means of Fourier Analysis in earlier tasks, we are able to comprehensively calculate D. This has been tabulated below in **Table 3**:

τ (min)	D_γ	$D_{\Delta\varphi}$	*Plots*
1			
2			
4			
6			
8			
16			

Table 3: Final summary table of the data, using the Bessel function method.

We can clearly see from our estimation of the percentage difference between the literature value and the calculated value, therefore, we are able to conclude.