Waves Session 1

Wednesday, November 2, 2022 11:58 AM

Welcome to Martin's Lab Book.
The work in the following section was completed on:

Tuesday, February 14th, 2023
9am – 12pm
in a synchronous manner becoming of a lab workbook.

Aims

- Complete Part I on Fourier Series.
- Examine the Fourier Series as specified in the lab script.
- Understand how a function can be decomposed into a series of sinusoidal functions with different frequencies.
- Perform numerical integration.
- Complete Part II on Thermal Waves.
- Familiarisation with quantitative ideas of thermal conduction.
- Familiarisation with quantitative use of Fourier and Bessel functions, as well as receiving experience in data analysis.
- Note that the overall objective is to obtain a best estimate of the thermal diffusivity of PTFE, and its associated uncertainty of the value.

Part I: Fourier Analysis of a Square Wave

1.2 Fourier Series

Task 1.1: Proving the Amplitude-Phase Form of Fourier Series

[13/02/2023]

A periodic function in time $T(t)$ with period τ can be expressed as a Fourier series, i.e. an infinite sum of sine and cosine functions:

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{\tau}t\right) + b_n \sin\left(\frac{2\pi n}{\tau}t\right)\right].$$

where a_n and b_n are complex coefficients (amplitudes) given by the expressions:

$$a_n = \frac{2}{\tau}\int_0^{\tau} T(t)\cos\left(\frac{2\pi n}{\tau}t\right)dt$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} T(t)\sin\left(\frac{2\pi n}{\tau}t\right)dt$$

Alternatively, Eq (1.1) can be rewritten in "amplitude-phase" form as:

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\beta_n \sin\left(\frac{2\pi n}{\tau}t - \Delta\phi_n\right).$$

where β_n and $\Delta\varphi_n$ are the amplitude and phase lag respectively:

$$\beta_n = \sqrt{{a_n}^2 + {b_n}^2}$$

$$\Delta\phi_n = -\arctan(a_n/b_n)$$

Each component of these series labelled *n* represents a harmonic mode, a sinusoidal function with angular frequency:

$$\omega_n = \frac{2\pi n}{\tau}.$$

Consequently, we begin by using the relation: $sin(x-y) = sin(x)cos(y) - cos(x)sin(y)$.

Therefore, $cos(y) = b_n$ and $sin(y) = a_n \Rightarrow \beta^2 = (cos\,y)^2 + (sin\,y)^2 = (a_n)^2 + (n_n)^2$.
As required, we have $\beta_n = \sqrt{a_n^2 + b_n^2}$.
By trigonometry, we consequently know that: $tan\ \Delta\varphi_n = \frac{y}{x} \Rightarrow \Delta\varphi_n = -arctan\left(\frac{a_n}{b_n}\right)$.
Putting all this together, we have:

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\beta_n \sin\left(\frac{2\pi n}{\tau}t - \Delta\phi_n\right).$$

Q.E.D.

1.3 Fourier Analysis of a Square Wave

Task 1.2: Fourier Series

[13/02/2023]

Assume that your square function has a period of τ = 240 with an amplitude between T = [0, 100]. Units are not strictly needed for this exercise, but you can choose to use [s] and [℃] if you want.

a. Calculate (with 'pen a paper'!) the amplitudes an and bn of the Fourier series up to n = 3 and make a table of your results for future reference.
b. Create a dataset that represents a square wave and plot it using e.g. Excel, Python, Google Sheets, etc. Make sure you choose an appropriate number of time divisions (timestep) in your plot.
c. Use your calculated coefficients to plot the Fourier series of the square function using Eq. (1.1) and add it to your plot (see Fig. 1.1). The n = 1 component is known as the 'fundamental frequency' - why?
d. Is your Fourier analysis able to reproduce the square wave? Explain.

$$a_n = \frac{2}{240}\int_0^{120} \cos\frac{2\pi n t}{240}dt = \frac{1}{120}\int_0^{120} \cos\frac{\pi n t}{120}dt = \frac{\sin(n\pi)}{n\pi}$$
$$b_n = \frac{2}{240}\int_0^{120} \sin\frac{2\pi n t}{240}dt = \frac{1}{120}\int_0^{120} \sin\frac{\pi n t}{120}dt = \frac{1-\cos(n\pi)}{n\pi}$$

n	a_n	b_n
1	0	$\frac{2}{\pi}$
2	0	0
3	0	$\frac{2}{3\pi}$
4	0	0
5	0	$\frac{2}{5\pi}$
6	0	0
7	0	$\frac{2}{7\pi}$

Table 1: Fourier coefficients up to $n = 7$.

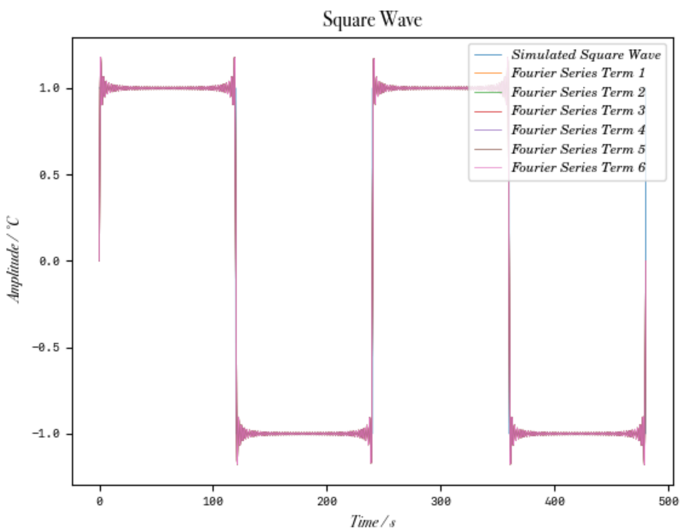

Figure 1: Fourier Series of Square Wave up to $n = 6$.

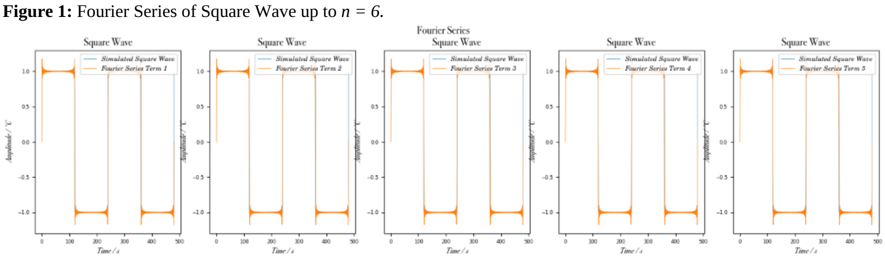

Figure 2: Fourier Series of Square Wave for n = *1, 2, 3, 4 and 5* individually. As can be seen from the Fourier series, there is an increasingly accurate reproduction of the square wave, with Gibbs' phenomenon, the name given to the overshoots at the jump discontinuities, but otherwise the linear combination of sinusoidal waves is accurate.

1.4 Numerical Integration of a Test Dataset

Task 1.3: Numerical Integration Test
[13/02/2023]

Download datasets (as .txt files) from Blackboard for either a semi-circle or a sine function. Each dataset has a ‘low resolution’ and ‘high resolution’ versions (see Fig. 1.2). Then:

a. Make a sketch of how to implement numerical integration graphically using ‘rectangles’ or ‘trapezoids’ for either of your chosen functions.
b. Apply either of your numerical integration methods to your chosen function with lower and higher resolution and show your numerical results. How different are these results to the expected ‘analytical’ answer?

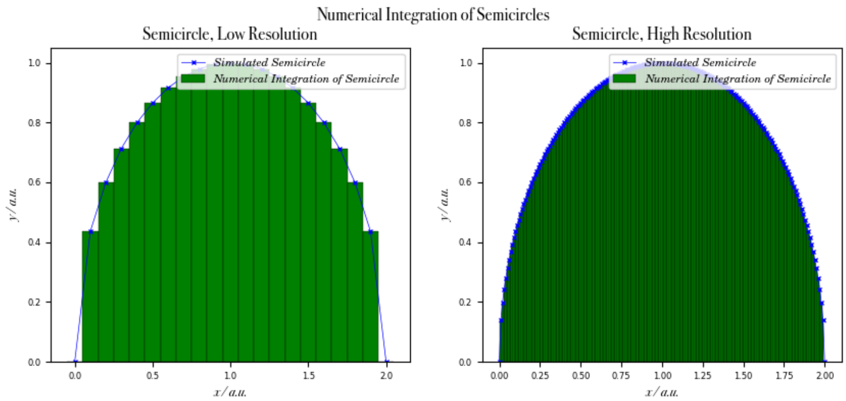

Figure 3: Semicircles with rectangle rule numerical integration method.
Area 1 (left): 1.5522591630000004 with percentage difference of 1.1801124995447336 %
Area 2 (right): 1.5702085159199997 with percentage difference of 0.037421202537205274 %

As can be seen, for the high resolution graph, there is a significantly reduced percentage difference as compared to the lower resolution wave, which is more poorly approximated with many overestimates throughout, and underestimates at the edges.

Part II: Thermal Waves Experiment

2.2 Experiment Overview

The thermal diffusivity of a material D depends on its thermal conductivity, specific heat and density. We will assume that the thermal diffusivity is constant over the duration of the experiment.

To measure D, a regular temperature fluctuation is established at one point in a solid and the variation of temperature is measured at another point. The solid in this case is a cylinder of PTFE (poly-tetra-fluoro-ethylene). A regular temperature wave is launched at the outer surface of the cylinder by dipping it alternately in boiling and icy water. Equal times (of the order of minutes) are spent in the hot and cold sources. Thus, a “square” temperature wave going from ~0°C to ~100°C is established at the outer surface of the cylinder. The period of this wave corresponds to the sum of the times spent in the hot and cold baths over one full cycle.

The cylinder takes some time to heat and cool as the wave of temperature variation propagates radially inwards. A concentric central hole in the cylinder contains a small electronic thermometer which has a low heat capacity, thus responding rapidly to the temperature on the walls of the central hole. This thermometer is connected to a computer so that the inner wall temperature can be measured during the cyclic temperature changes.

2.4 Plane Slab Model

We note that the flow of heat in a medium is described by the heat equation, which for a plane slab (i.e. in 1-dimension) is given by:

$$\frac{\partial T(x,\ t)}{\partial t} = D\frac{\partial^2 (T(x,\ t))}{\partial x^2},\ (2.1)$$

where $T(x,\ t)$ is the temperature in the medium as a function of space and time and D is the thermal diffusivity of the medium. Eq. (2.1) is a diffusion equation, which is a linear differential equation like Schrodinger's equation and the wave equation in the Electrical Wave experiment. A solution for $T(x,\ t)$ decays in space and oscillates in time with angular frequency ω:

$$T(x,\ t) = Ce^{-\sqrt{(\frac{\omega}{2D})}x} sin\left(\sqrt{\frac{\omega}{2D}}x - \omega t\right),\ (2.2)$$

where C is a constant dependent on initial and boundary conditions.

Task 2.1: Heat Equation
[13/02/2023]

a. From Eq. (2.1), what are the units of thermal diffusivity?
b. Look up examples of typical materials and their thermal diffusivities. Remember to record your sources of information.
c. Prove that Eq. (2.2) is a solution to the heat equation.

The units of thermal diffusivity are **square metres per second**.

In the plane slab model, the cylinder is crudely treated as a 1-dimensional slab of thickness $\Delta r = r_{outer} - r_{inner}$ with the x-axis pointing radially inwards. We define $x = 0$ at the outer boundary of the cylinder, which implies $x = \Delta r$ at the inner boundary.

Thermal diffusivity of selected materials and substances

Material	Thermal diffusivity (mm^2/s)
Steel, stainless 304A at 27 °C	4.2
Pyrolytic graphite, normal to layers	3.6
Steel, stainless 310 at 25 °C	3.352
Inconel 600 at 25 °C	3.428
Quartz	1.4
Sandstone	1.15
Ice at 0 °C	1.02
Brick, common	0.52
Glass, window	0.34
Water at 25 °C	0.143
PTFE (Polytetrafluorethylene) at 25 °C	0.124
PP (polypropylene) at 25 °C	0.096
Nylon	0.09
Rubber	0.089 - 0.13
Wood (yellow pine)	0.082
Paraffin at 25 °C	0.081
PVC (polyvinyl chloride)	0.08
Oil, engine (saturated liquid, 100 °C)	0.0738
Alcohol	0.07

Table 2: Thermal diffusivity of selected materials and substances.

To prove that: $T(x,\ t) = Ce^{-\sqrt{\left(\frac{\omega}{2D}\right)x}} sin\left(\sqrt{\frac{\omega}{2D}}x - \omega t\right)$ is a solution to the heat equation, we need to show that it satisfies the equation: $\frac{\partial T}{\partial t} = D\,\nabla^2 T$

where D is the thermal diffusion coefficient and $\nabla^2 T$ is the Laplacian of T. Taking the partial derivative of $T(x,\ t)$ with respect to t, we get:

$$\frac{\partial T}{\partial t} = -C\omega e^{-\sqrt{\left(\frac{\omega}{2D}\right)x}} cos\left(\sqrt{\left(\frac{\omega}{2D}\right)x} - \omega t\right)$$

Next, we'll find the Laplacian of $T(x,\ t)$:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \text{ (in 3 dimensions)}$$

But in our case, we're only considering one dimension, so the Laplacian is just:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2}$$

Taking the second partial derivative of $T(x,\ t)$ with respect to x, we get:

$$-\sqrt{\left(\frac{\omega}{2D}\right)} x sin\left(\sqrt{\left(\frac{\omega}{2D}\right)x} - \omega t\right)\Bigg)$$

Finally, we can substitute these results into the heat equation and check if it's satisfied: $\frac{\partial T}{\partial t} = D\,\nabla^2 T$

$$-C\omega e^{-\frac{\sqrt{\frac{\omega}{2D}}}{x}} cos\left(\frac{\sqrt{\frac{\omega}{2D}}}{x} - \omega t\right) = D\,C\left(\frac{\left(\frac{\omega}{2D}\right)^2}{x^2} + \frac{\sqrt{\frac{\omega}{2D}}}{x}\right) e - \sqrt{\frac{\omega}{2D}} \Big/ x\, sin\left(\sqrt{\left(\frac{\omega}{2D}\right)x} - \omega t\right)$$

Since both sides of the equation are equal, we can conclude that $T(x,\ t)$ is a solution to the heat equation.

Task 2.2: Consider a sinusoidal temperature of angular frequency ω and phase lag 0 starting from r_{outer} and propagating inwards through the cylinder.
[Handwritten: 13/02/2023, Digitally Parsed: 26/02/2023]

a. Evaluate $T(x,\ t)$ at the inner and outer boundaries of the cylinder. Which one has greater amplitude? Is the thermal wave at r_{inner} leading or lagging the one at r_{outer}?
b. The amplitude transmission factor is defined as $\gamma = \frac{(Amplitude|r_{inner})}{(Amplitude|r_{outer})}$ and the phase lag is defined $\Delta\phi = Phase|r_{inner} - Phase|\,r_{outer}$. Show that in the case considered here

$\gamma = e^{-\sqrt{\left(\frac{\omega}{2D}\right)}\Delta r}$ and $\Delta\phi = \Delta r\sqrt{\left(\frac{\omega}{2D}\right)}$. Comment on their angular frequency dependence.

c. Hence show that the thermal diffusivity is given by: $D = \frac{\omega \Delta r^2}{2ln(\gamma)^2} = \frac{\omega \Delta r^2}{2 \Delta \phi^2}$. This suggests that if either the attenuation or phase lag are measured for a known value of Δr and ω, then D can be found. This is the principle of the analysis.

Task 2.2a

We have that: $T = Cexp\left(-x\sqrt{\frac{\omega}{2D}}\right)sin\left(\omega t - x\sqrt{\frac{\omega}{2D}}\right)$, where

$\Delta r = |r_{in} - r_{out}|,\ x_{r_{in}} = \Delta r,\ x_{r_{out}} = 0$.

Consequently, we have: $T = Ce^0 sin(-\omega t) = sin(-\omega t),\ for\ x_{r_{out}} = 0.$

$T = Ce^{-\Delta r\sqrt{\frac{\omega}{2D}}} sin\left(\Delta r\sqrt{\frac{\omega}{2D}} - \omega t\right),\ for\ x_{r_{out}} = \Delta r.$

$A_{r_{out}} = -C.\Delta r\sqrt{\left(\frac{\omega}{2D}\right)}$

$A_{r_{in}} = -C.e$

This means that the $A_{TW} at\ r_{out} > A_{TW} at\ r_{in}$, since it lacks an exponential decay element.
We have that:

For r_{out} peaks at: $-\omega t = \frac{\pi}{2} \Rightarrow t = -\frac{\pi}{2\omega}$

For r_{in} peaks at: $\Delta r\sqrt{\frac{\omega}{2D}} - \omega t = \frac{\pi}{2} \Rightarrow t = \Delta r\sqrt{\frac{1}{2\omega D}} - \frac{\pi}{2\omega}$

Therefore, we can conclude that the r_{in} wave is leading the r_{out} wave.

Task 2.2b

We have that: $\gamma = \frac{\varphi e^{-\Delta r\sqrt{\frac{\omega}{2D}}}}{4} = e^{-\Delta r\sqrt{\frac{\omega}{2D}}}$

Upon further simplification, and cancelling of terms, we have: $\Delta\varphi = \Delta r\sqrt{\frac{\omega}{2D}}$

Again, we have that amplitude is characterised by an exponential decay dependent on angular frequency, and that this dependence is the same as the phase difference, s.t. $\gamma = e^{-\Delta\varphi}$

Task 2.2c

$From\ \gamma = e^{-\Delta\varphi} = e^{\Delta r - \sqrt{\frac{\omega}{2D}}} \Rightarrow ln(\gamma)^2 = \left(\frac{\omega}{2D}\right)(\Delta r)^2$

$D = \frac{\omega(\Delta r)^2}{2ln(\gamma)^2} and\ \gamma = e^{-\Delta\varphi} \Rightarrow ln(\gamma) = -\Delta\varphi$

$Therefore,\ D = \frac{\omega(\Delta r)^2}{2(-\Delta\varphi)^2} = \frac{\omega(\Delta r)^2}{\left(2(\Delta\varphi)^2\right)}$ as expected.