Show $var(\hat{\beta}) = \sigma^2(X^TX)^{-1}$

As stated:

$$\hat{\beta} = (X^TX)^{-1}X^Ty$$

Substitute the linear regression model $y = X\beta + \epsilon$ in.

$$\hat{\beta} = (X^TX)^{-1}X^T(X\beta + \epsilon)$$

Expand:

$$\hat{\beta} = (X^TX)^{-1}X^TX\beta + (X^TX)^{-1}X^T\epsilon$$

Simplify: $(X^TX)^{-1}X^TX = I$

$$\Rightarrow \hat{\beta} = \beta + (X^TX)^{-1}X^T\epsilon$$

$$Var(\hat{\beta}) = Var(\overset{=0}{\beta} + (X^TX)^{-1}X^T\epsilon)$$

- $(X^TX)^{-1}X^T$ is a constant matrix and $Var(A\epsilon) = A\,var(\epsilon)A^T$ for any matrix A, so we factor out our expression.

$$Var(\hat{\beta}) = (X^TX)^{-1}X^T\,Var(\epsilon)\,X(X^TX)^{-1}$$

- $Var(\epsilon) = \sigma^2 I$

$$Var(\hat{\beta}) = \sigma^2\underbrace{(X^TX)^{-1}X^TX}_{I}(X^TX)^{-1}$$

$$Var(\hat{\beta}) = \sigma^2(X^TX)^{-1}$$