$$\nabla^2 \psi(r,\vartheta,\varphi) + V(r)\,\psi(r,\vartheta,\varphi) = E\,\psi(r,\vartheta,\varphi)$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial r^2} r^2 R(r) \cdot Y(\vartheta,\varphi) + \frac{1}{r^2}\hat{l}^2 R(r)\,Y(\vartheta,\varphi) +$$

$$\frac{1}{r^2} Y(\vartheta,\varphi)\frac{\partial^2}{\partial r^2} r^2 R(r) + \frac{1}{r^2} R(r)\,\hat{l}^2 Y(\vartheta,\varphi) + V(r)R(r)Y(\vartheta,\varphi) =$$

$$\frac{1}{R(r)}\frac{\partial^2}{\partial r^2} r^2 R(r) + r^2 V(r) - r^2 E + \frac{1}{Y(\vartheta,\varphi)}\hat{l}^2 Y(\vartheta,\varphi) = 0$$

Potentiale für die Separation in Kugelkoordinaten

$V(r,\varphi) = \ldots$ $\quad V(x,y) = \ldots (x\,y)$

$V(r)\,\psi(r,\vartheta,\varphi) = E\,\psi(r,\vartheta,\varphi)$

$E\,R(r)\,Y(\vartheta,\varphi) \quad \Big| : R(r)\,Y(\vartheta,\varphi) \quad \cdot r^2$

Ξ: $\tilde{Y}_{1,1}(\theta,\varphi) = \frac{1}{\sqrt{2}}\left(-iY_{1,0} + \frac{1}{\sqrt{2}}\left(Y_{1,1} - Y_{1,-1}\right)\right)$

$Y_{1,1}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}$ $\quad \tau := \sqrt{\frac{3}{8\pi}}$

$r = \sqrt{x^2+y^2+z^2} = 1$

$\tilde{x} = x$

$Y_{1,1}(x,y,z) = -\tau\left(\sqrt{x^2+y^2}\,\frac{iy+x}{\sqrt{x^2+y^2}}\right) = -\tau(iy+x)$ $\quad \tilde{y} = -z$

$\tilde{z} = y$

$\tilde{Y}_{1,1}(x,y,z) = -\tau(iz+x)$

$= \tau\left(-iz - \frac{1}{2}\left(\frac{x+x+iy-iy}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2}\right)\right)$

$$= \tau\left(-iz - \frac{1}{2}\left(\sqrt{x^2+y^2}\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right) + \sqrt{x^2+y^2}\left(\frac{x-iy}{\sqrt{x^2+y^2}}\right)\right)\right)$$

$$\tilde{Y}_{1,1}(\theta,\varphi) = \tau\left(-i\cos\theta - \frac{1}{2}\left(\underbrace{2\sin\theta\cos\varphi}_{=e^{i\varphi}+e^{-i\varphi}}\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(i\sqrt{\frac{3}{8\pi}}\cos\right) - \frac{1}{2}\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi} - \frac{1}{2}\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$$

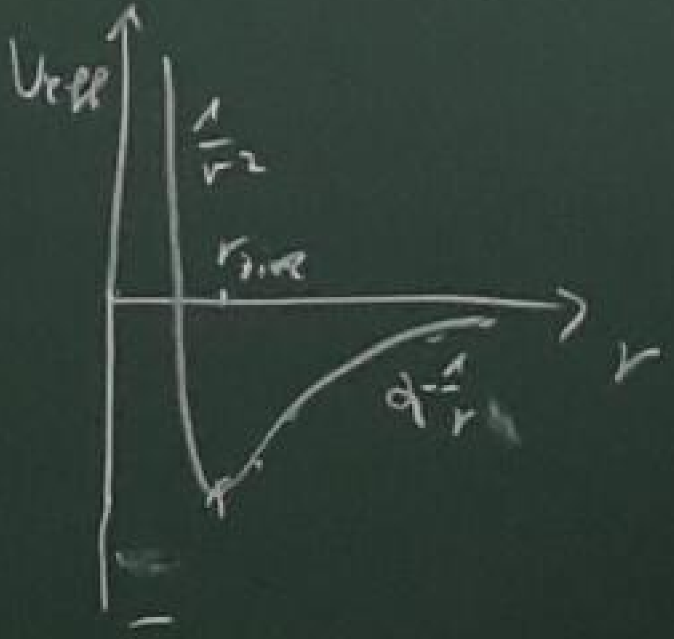
$$R_x^T(\omega) = R_x^{-1}(\omega) = R_x(-\omega)$$

$$\tilde{x} = x$$
$$\tilde{y} = -z$$
$$\tilde{z} = y$$

$$R_x^{-1}(90^\circ)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -z \\ y \end{pmatrix}$$

a) $V_C = -\frac{Z Z' e^2}{4\pi\varepsilon_0 r}$ $\quad V_Z = \frac{L^2}{2mr^2}$

$$V_{eff} = \frac{-e^2}{4\pi\varepsilon_0 r} + \frac{L^2}{2mr^2}$$



b) $V_{eff}' = \frac{e^2}{4\pi\varepsilon_0 r^2} - \frac{2L^2}{2mr^3} \stackrel{!}{=} 0$

$$\frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{L^2}{mr^3} \Rightarrow r_{Kreis} = \frac{4L^2\pi\varepsilon_0}{e^2 m} \qquad L^2 = l(l+1)\hbar^2$$

$$r_{kreis} = \frac{4\pi\varepsilon_0 \hbar^2}{e^2 m} l(l+1) \qquad l = n-1$$

$$= \frac{4\pi\varepsilon_0 \hbar^2}{e^2 m} n^2 = a_0 n^2$$

$$V_{eff}(r_{kreis}) = -\frac{e^2}{8\pi\varepsilon_0 n^2 a_0} = -\frac{E_{ryd}}{n^2}$$

c) $$m\,\omega_{kep}^2\, r_{kreis} = \frac{e^2}{4\pi\varepsilon_0 r_{kreis}} \Rightarrow \omega_{kep} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m r_{kreis}^3}} = \frac{m e^4}{(4\pi\varepsilon_0)^2 \hbar^3 n^3}$$

ψ_n, ψ_{n-1}

$$\psi_n(r,\theta,\varphi,t) = R_n(r) P_n(\theta) e^{im\varphi} e^{-i\frac{E_n}{\hbar}t} \quad m = n-1$$

$$\psi_{n-1}(r,\theta,\varphi,t) = R_{n-1}(r) P_{n-1}(\theta) e^{im'\varphi} e^{-i\frac{E_{n-1}}{\hbar}t} \quad m' = (n-1)-1$$

$$|\psi_n + \psi_{n-1}|^2 = |R_n(r) P_n(\theta)|^2 \left| e^{i((n-1)\varphi - \frac{E_n}{\hbar}t)} + e^{i((n-2)\varphi - \frac{E_{n-1}}{\hbar}t)} \right|^2$$

$$= \quad || \quad \cdot 2\left(1 + \cos\left(\varphi + \frac{1}{\hbar}(E_{n-1} - E_n)\right)\right)$$

$$\omega = -\frac{E_{n-1} - E_n}{\hbar} \qquad E_{n-1} - E_n = -\frac{E_{ryd}}{(n-1)^2} + \frac{E_{ryd}}{n^2} \approx -2\frac{E_{ryd}}{n^3}$$

$$\frac{1}{(n-1)^2} - \frac{1}{n^2} = \frac{2n-1}{(n-1)^2 n^2} \approx \frac{2}{n^3}$$

$$r_{zirk} = \frac{4\pi\varepsilon_0 \hbar^2}{e^2 m} l(l+1) \qquad l = n-1$$

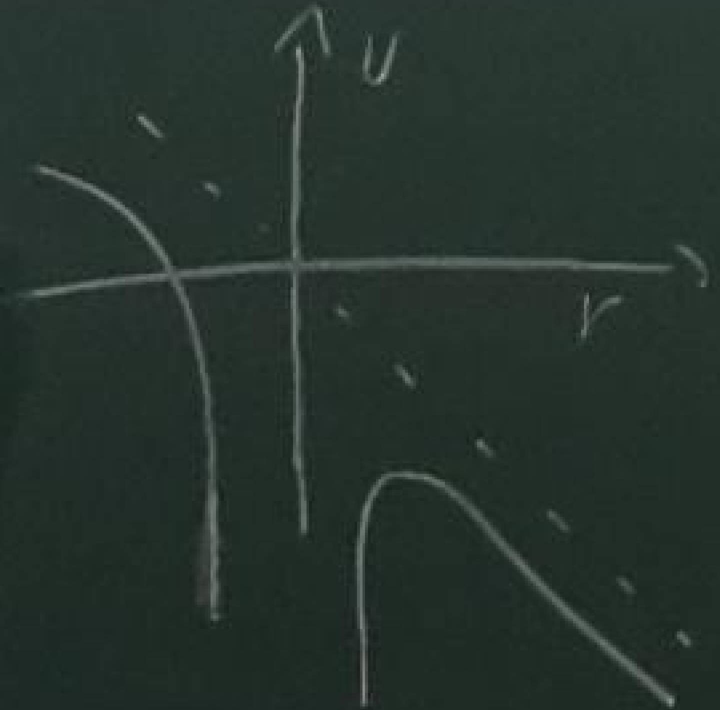
$$= \frac{4\pi\varepsilon_0 \hbar^2}{e^2 m} n^2 = a_0 n^2$$

$$\omega = \frac{2E_{ryd}}{\hbar n^3} = \frac{2e^2}{8\pi\varepsilon_0 a_0 \hbar n^3}$$

$$= \frac{me^4}{(4\pi\varepsilon_0)^2 \hbar^3 n^3}$$

$$V_{eff}(r_{zirk}) = -\frac{e^2}{8\pi\varepsilon_0 n^2 a_0} = -\frac{E_{ryd}}{n^2}$$

c) $$m\,\omega_{Kep}^2 r_{zirk} = \frac{e^2}{4\pi\varepsilon_0 r_{zirk}} \Rightarrow \omega_{Kep} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m r_{zirk}^3}} = \frac{me^4}{(4\pi\varepsilon_0)^2 \hbar^3 n^3}$$



$$E = -\frac{E_{n,ryd}}{n^2}$$

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r} - eEr$$

$$\frac{dV}{dr} = \frac{e^2}{4\pi\varepsilon_0 r_{max}^2} - eE \overset{!}{=} 0 \Rightarrow r_{max} = \sqrt{\frac{e}{4\pi\varepsilon_0 E}}$$

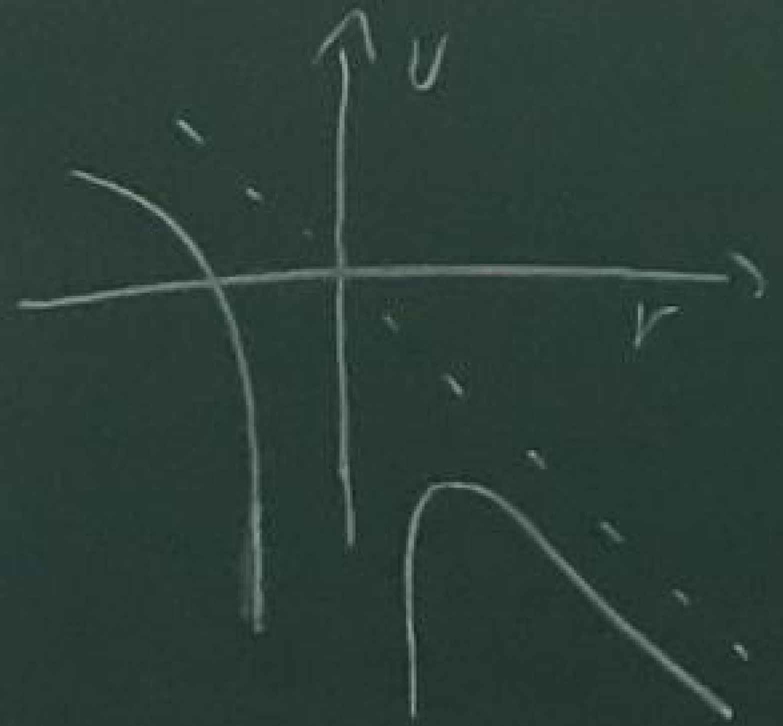
$$V(r_{max}) = -2\sqrt{\frac{e^3 E}{4\pi\varepsilon_0}}$$

$$E = -\frac{E_{ryd}}{n^2} \Rightarrow V(r_{max}) = -2\sqrt{\frac{e^3 E}{4\pi}}$$

$$\frac{E_{krit}^2}{4n^4} \leq \frac{1}{4n^4}\frac{e^4}{4\pi\varepsilon_0 a_0^2} = \frac{E_0}{16n^4} \qquad E_0 = \frac{e}{4\pi\varepsilon_0 a_0^2} = 5{,}1 \cdot 10^{11}\,\frac{V}{m}$$

$n = 1 \qquad E \geq 3{,}2\; 10^{10}\,\frac{V}{m}$

$n = 100 \qquad E \geq 320\,\frac{V}{m}$



$$E = -\frac{E_{Ryd}}{n^2}$$

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r} - eEr$$

$$\frac{dV}{dr} = \frac{e^2}{4\pi\varepsilon_0 r_{max}^2} - eE \stackrel{!}{=} 0 \Rightarrow r_{max} = \sqrt{\frac{e}{4\pi\varepsilon_0 E}}$$

$$V(r_{max}) = -2\sqrt{\frac{e^3 E}{4\pi\varepsilon_0}}$$

$$E = -\frac{E_{Ryd}}{n^2} \Rightarrow V(r_{max}) = -2\sqrt{\frac{e^3 E}{4\pi\varepsilon_0}} \Rightarrow \boxed{\frac{E_{Ryd}}{n^2} \leq 2\sqrt{\frac{e^3 E}{4\pi\varepsilon_0}}}$$