Мартин

1 patition

A partition on an rela line interval [a,b] is a sequence $x_1, ..x_k$, such that:

$$a = t_1 < .. < t_k = b$$

2 variation

given operator $f : [a, b]-> R$, with partition P, define variation of f on P as:

$$V(f, P) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

3 Total Variation

Considering a function f with values in range $[a, b] \in R$

$$V_a^b = \sup_p \sum_{i=0}^{n_p-1} |f(x_{i+1}) - f(x_i)| = \sup_p V(f, P)$$

4 Bounded variation

a continuous function f is said to be a of bounded variation(BV) on an interval [a,b] $\subset$ R if the total variation on it if finite:

$$f \in BV([a, b]) <-> V_a^b(f) < \infty$$

$$BV([a, b]) := \{f \in C[a, b] | V_a^b(f) < \infty\}$$

5 refine

: not needed ?

6 monotone examples

not needed ?

7 Banach space

We have to see that $BV[a,b]$ is a Banch Space

7.1 Lemma for sup-Norm

if $f : [a,b] -> R$ is of bounded variation, then f is bounded and:

$$||f||_\infty \leq |f(a)| + V_a^b f$$

proof let $a \leq x \leq b$, then:

$$|f(x) - f(a)| \leq V(f, P) \leq V_a^b(f)$$

$$|f(x)| \leq |f(a)| + V_a^b(f)$$

7.2

We will show that $V_a^b(f)$ is not quite a norm:

Lemma Let $f, g \in BV[a,b]$ and $c \in R$, then:

1. $V_a^b(f) = 0 <-> f$ is a constant
2. $V_a^b(cf) = |c|V_a^b$
3. $V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$

proof

1. $V_a^b(f) = 0 <-> f$ is a constant
   obvious
   Question: we can't consider f=0 ?
2. $V_a^b(cf) = |c|V_a^b$ Let P be a partition of [a,b]
   $$V(cf, P) \leq |c|V(f, P) \leq |c|V_a^b(f)$$
3. $V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$
   Let P be a partition
   $$V(f+g, P) \leq V(f, P) + V(g, P)$$

Lets supreum both sides

$$V_a^b(f+g) \leq V_a^b(f) + V_a^b(g)$$

8 BV norm

We will show that:

$$||f||_{BV} = f(a) + V_a^b(f)$$

is a norm on BV[a,b]

9 Theorem

BV[a,b] is complete under $||f||_{BV}$

10 Question

B[a,b] - set of all bounded functions ?

proof Let (f_n) is a Cauchy seq in BV[a,b], then it's also a Cauchy seq in B[a,b].

(f_n) converges to $f \in B[a,b]$

Lets show that $f \in BV[a,b]$ and $||f - f_n||_{BV} -> 0$

Let P be a partiotion in [a,b]

Let $\varepsilon > 0$, there exists an N, such that for all m,n > N:

$$(f_n(a) - f_m(a)) + (V(f_n) - V(f_m), P) < \varepsilon$$

$$|f_n(a) - f(a)| + V(f_n - f, P) = \lim_{m->\infty} \{|f_n(a) - f_m(a)| + V(f_n - f_m, P)\}$$

$$|f_n(a) - f(a)| + V(f_n - f, P) \leq \varepsilon, \forall n \geq N$$

This stands true for every partition P in [a,b], so $f_n - f \in BV[a,b]$

$$||f_n - f||_{BV} = |f_n(a) - f(a)| + V_a^b(f_n - f) \leq \varepsilon, \forall n \geq N$$

meaning (f_n) converges to $f \in BV[a,b]$ with respect to the $||.||_{BV}$ norm

11 BV[a,b] is a Banach Space

$BV[a,b]$ is a Banach Space with respect to the $||.||_{BV}$ norm:

Considering previous statement:

$$V_a^b = 0 <-> f = const$$

Implies that:

$$||f||_{BV} = 0 <-> f = 0$$

let $c \in R$ and $f \in BV[a,b]$

$$||cf||_{BV} = f(ca) + V_a^b(cf) = |c|f(a) + |c|V_a^b(f) = |c|||f||_{BV}$$

$$||f-g||_{BV} = (f-g)(a)+V_a^b(f-g) \leq f(a)-g(a)+V_a^b(f)-V_a^b(g) = ||g||_{BV}+||f||_{BV}$$

Considering the completeness property from the previous theorem, BV[a,b] is a Banach Space in regards to $||.||_{BV}$

Дали следното твърдение е нужно?

12 Theorem

Let $f \in BV[a,b]$ is a real valued function. Then there exists a monotonically increasing functiong $g : [a,b]-> R$ such that both g and $g - f$ are increasing. Consequently, f=g-(g-f) is the difference of two increasing functions.

proof For $[a,d] \subseteq [a,b]$ Let $V_c^d(f)$ be the total variation of f on [a,d]. Then it can be verified that for any $t \in [a,b]$:

$$V_a^b(f) = V_a^t(f) + V_t^b(f)$$

$$\sum_{i=a}^{b} |f(x_i) - f(x_{i-1})| = \sum_{i=a}^{t} |f(x_i) - f(x_{i-1})| + \sum_{i=t}^{b} |f(x_i) - f(x_{i-1})|$$

We apply $\sup_P$ function:

$$\sup_P \sum_{i=a}^{b} |f(x_i) - f(x_{i-1})| = \sup_P \sum_{i=a}^{t} |f(x_i) - f(x_{i-1})| + \sup_P \sum_{i=t}^{b} |f(x_i) - f(x_{i-1})|$$

$$V_a^b(f) = V_a^t(f) + V_t^b(f)$$

Let $a \leq t \leq s \leq b$

$$V_a^s(f) - V_a^t(f) = V_t^s(f) \geq |f(s) - f(t)| \geq f(s) - f(t)$$

Then $g(t) := V_a^t(f)$ is monotonically increasing. Then:

$$g(s) - g(t) = V_a^s(f) - V_a^t(f) = V_t^s(f) \geq |f(s) - f(t)| \geq f(s) - f(t)$$

Дали ще учим за Банахови Алгебри, дали да го включа?