1. (a) $x(t) = e^{\alpha t}$

$$dx = \alpha e^{\alpha t} dt$$

(b) $x(t) = \int_0^t g(s)\, dW(s)$

$$dx = g(t)\, dW(t)$$

(c) $x(t) = e^{\alpha W(t)}$

$$dx = \alpha e^{\alpha W(t)} dW + \frac{1}{2}\alpha^2 e^{\alpha W(t)} dt$$

(d) $x(t) = e^{\alpha y(t)}$

$$\begin{aligned} dx &= \alpha e^{\alpha y(t)} dy + \frac{1}{2}\alpha^2 e^{\alpha y(t)} (dy)^2 \\ &= \alpha e^{\alpha y(t)} (\mu dt + \sigma dW) + \frac{1}{2}\alpha^2 e^{\alpha y(t)} \cdot \sigma^2 dt \\ &= \alpha e^{\alpha y(t)} (\mu + \frac{1}{2}\alpha\sigma^2) dt + \alpha e^{\alpha y(t)} \sigma dW \end{aligned}$$

(e) $x(t) = y^2(t)$

$$\begin{aligned} dx &= 2y\,dy + (dy)^2 \\ &= 2y(\alpha y dt + \sigma y dW) + \sigma^2 y^2 dt \\ &= y^2(2\alpha + \sigma^2) dt + 2\sigma y^2 dW \end{aligned}$$

(f) $x(t) = \frac{1}{y(t)}$

$$\begin{aligned} dx &= -\frac{1}{y^2} dy + 2\frac{1}{y^3} (dy)^2 \\ &= -\frac{1}{y^2}(\alpha y dt + \sigma y dW) + \frac{2}{y^3} \cdot \sigma^2 y^2 dt \\ &= \frac{1}{y}(-\alpha + 2\sigma^2) dt - \frac{\sigma}{y} dW \end{aligned}$$

2. $X(t) = \int_0^t \sigma_s \, dW_s \Rightarrow \begin{cases} dX = \sigma_t dW \\ (dX)^2 = \sigma_t^2 dt \end{cases}$

let $Y(t) = e^{iuX(t)}$ , $Y(0) = 1$

$$dY = iue^{iuX(t)} dX + \frac{1}{2}(iu)^2 e^{iuX(t)} (dX)^2$$
$$= iue^{iuX(t)} \sigma_t dW - \frac{1}{2}u^2 e^{iuX(t)} \sigma_t^2 dt$$

$$Y(t) - 1 = \int_0^t iue^{iuX(t)} \sigma_t dW - \frac{1}{2}\int_0^t u^2 e^{iuX(t)} \sigma_t^2 dt$$

$$E[Y(t)] = 1 - \frac{1}{2} u^2 \int_0^t \sigma^2(s) E[e^{iuX(t)}] dt$$

let $m_t = E[Y(t)] = E[e^{iuX(t)}]$

$$m_t = 1 - \frac{1}{2}u^2 \int_0^t \sigma^2(s) m_s ds$$

$$m_t' = -\frac{1}{2} u^2 \sigma^2(t) m_t \quad , \quad m(0) = 1$$

$$m_t = e^{-\frac{1}{2}u^2 \int_0^t \sigma^2(s) ds}$$

$$\Rightarrow E[e^{iuX(t)}] = e^{-\frac{u^2}{2}\int_0^t \sigma^2(s) ds}$$

3. $dx = \alpha X dt + \sigma_t dW$

$$\int_0^t dX = \int_0^t \alpha X dt + \int_0^t \sigma_t dW$$

$$E[X_t - X_0] = E[\int_0^t \alpha X_s ds] + E[\int_0^t \sigma_s dW_s]$$

$$E[X_t] = E[X_0] + E[\int_0^t \alpha X_s ds]$$

$$E[X_t] = E[X_0] + \int_0^t \alpha E[X_s] ds$$

let $E[X_t] = m_t$ , $m_0 = E[X_0]$

$$m_t = E[X_0] + \int_0^t \alpha m_s ds$$

$$m_t' = \alpha m_t \Rightarrow m_t = E[X_0] e^{\alpha t}$$

$$E[X_t] = E[X_0]\cdot e^{\alpha t}$$

4.

$$R(X,Y) = X^2 + Y^2$$

$$dR = 2X\,dX + 2Y\,dY + \tfrac{1}{2}[2(dX)^2 + 2(dY)^2 + 0(dX)(dY)]$$

$$= 2X dX + 2Y dY + (dX)^2 + (dY)^2$$

$$= 2X(\alpha X dt - Y dW) + 2Y(\alpha Y dt + X dW) + Y^2 dt + X^2 dt$$

$$= (2\alpha X^2 + 2\alpha Y^2 + X^2 + Y^2)dt + (2XY - 2XY)dW$$

$$= (X^2 + Y^2)(2\alpha + 1)dt$$

$$\Rightarrow dR = (2\alpha + 1) R\,dt$$

5. (a) $dX = \alpha X dt + \sigma dW$

$$Y = e^{-\alpha t} X_t, \quad Y_t = -\alpha Y \quad Y_X = e^{-\alpha t} \quad Y_{XX} = 0$$

$$dY = -\alpha Y dt + e^{-\alpha t} dX$$

$$= -\alpha Y dt + e^{-\alpha t}(\alpha X dt + \sigma dW)$$

$$= (-\alpha Y + \alpha X e^{-\alpha t})dt + \sigma e^{-\alpha t} dW$$

$$= \sigma e^{-\alpha t} dW$$

(b) $$\int_0^t dY = \int_0^t \sigma e^{-\alpha s} dW_s$$

$$Y_t - Y_0 = \int_0^t \sigma e^{-\alpha s} dW_s$$

$$e^{-\alpha t} X_t - X_0 = \int_0^t \sigma e^{-\alpha s} dW_s$$

$$e^{-\alpha t} X_t = X_0 + \int_0^t \sigma e^{-\alpha s} dW_s$$

$$X_t = e^{\alpha t} X_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s$$

(c) $X(t) \mid \mathcal{F}_0$ is normally distributed since each increment since $t=0$ is independent and normally distributed. $\alpha X dt$ is a constant and $\sigma dW$ is normally distributed.

$$E[X(T) \mid X(0)] = E[e^{\alpha T} X_0] + E[\sigma \int_0^T e^{\alpha(t-s)} dW_s]$$
$$= e^{\alpha T} X_0 + \sigma e^{\alpha T} E[\int_0^T e^{-\alpha s} dW_s]$$
$$= e^{\alpha T} X_0$$

$$Var[X(T) \mid X(0)] = Var(e^{\alpha T} X_0 + \sigma \int_0^T e^{\alpha(T-s)} dW_s)$$
$$= \sigma^2 Var(\int_0^T e^{\alpha(T-s)} dW_s)$$
$$= \sigma^2 ( E[(\int_0^T e^{\alpha(T-s)} dW_s)^2] - E[\int_0^T e^{\alpha(T-s)} dW_s]^2 )$$
$$= \sigma^2 \int_0^T E[e^{2\alpha(T-s)}] ds$$
$$= \sigma^2 (-\tfrac{1}{2\alpha}) \cdot e^{2\alpha(T-s)} \Big|_0^T$$
$$= -\frac{\sigma^2}{2\alpha} (1 - e^{2\alpha T})$$
$$= \frac{\sigma^2}{2\alpha} (e^{2\alpha T} - 1)$$