3.

(a) $(r-\delta)SC_S+\frac{1}{2}\sigma^2C_{SS}S^2-rC=0$

let $C=S^\alpha$ , $C'=\alpha S^{\alpha-1}$ , $C''=\alpha(\alpha-1)S^{\alpha-2}$

(b) $\Rightarrow (r-\delta)\alpha \cancel{S^\alpha}+\frac{1}{2}\sigma^2\alpha(\alpha-1)\cancel{S^\alpha}-r\cancel{S^\alpha}=0$

$$(r-\delta)\alpha+\frac{1}{2}\sigma^2\alpha(\alpha-1)-r=0$$

$$\frac{1}{2}\sigma^2\alpha^2+(r-\delta-\frac{1}{2}\sigma^2)\alpha-r=0$$

$$\alpha=\frac{-(r-\delta-\frac{1}{2}\sigma^2)\pm\sqrt{(r-\delta-\frac{1}{2}\sigma^2)^2+4\cdot\frac{1}{2}\sigma^2\cdot r}}{\sigma^2}$$

let $w=r-\delta-\frac{1}{2}\sigma^2$

$$\alpha=\frac{-w\pm\sqrt{w^2+2\sigma^2r}}{\sigma^2} \qquad \alpha_+=\frac{-w+\sqrt{w^2+2\sigma^2r}}{\sigma^2}>0$$

$$\alpha_-=\frac{-w-\sqrt{w^2+2\sigma^2r}}{\sigma^2}<0$$

$\therefore$ $C=AS^{\alpha+}+BS^{\alpha-}$

$C(\infty)=0 \Rightarrow A=0$

$C(S^*)=S^*-K \Rightarrow BS^{*\alpha-}=S^*-K$

$$B=\frac{(S^*-K)}{S^{*\alpha-}}$$

$$\Rightarrow C=(S^*-K)\left(\frac{S}{S^*}\right)^{\alpha-}=(S^*-K)\left(\frac{S}{S^*}\right)^{\frac{-(r-\delta-\frac{1}{2}\sigma^2)-\sqrt{(r-\delta-\frac{1}{2}\sigma^2)^2+2\sigma^2r}}{\sigma^2}}$$

(c) $$\frac{\partial C}{\partial S^*}=\frac{\partial}{\partial S^*}\left(\frac{S^*-K}{S^{*\alpha-}}\cdot S^{\alpha-}\right)=0 \qquad (S^*-K)S^{*-\alpha}$$

$$\Rightarrow S^{*-\alpha}+(-\alpha)S^{*-\alpha-1}(S^*-K)=0$$

$$\Rightarrow \quad S^* - \alpha(S^* - K) = 0$$

$$\Rightarrow \quad (1-\alpha) S^* = -\alpha K$$

$$\Rightarrow \quad S^* = \frac{-\alpha}{1-\alpha} K$$

$$\boxed{\alpha_- = \frac{-w - \sqrt{w^2 + 2\sigma^2 r}}{\sigma^2} < 0}$$

$$w = r - \delta - \frac{1}{2}\sigma^2$$

if $\delta \to 0$, $\alpha \to -\frac{2r}{\sigma^2}$

$$S^* \to \frac{\frac{2r}{\sigma^2}}{1 + \frac{2r}{\sigma^2}} K = \frac{2r}{\sigma^2 + 2r} K$$