MFE 230Q: Stochastic Calculus with Applications to Asset Pricing

Assignment 4

Authors:

Zhongze Gong (id: 3037579285),
Jin Kim (id: 3037487232),
Divya Singh (id: 3037504185),
Anirudh Tunoori (id: 3037573405)

Instructor: Johan Walden

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Problem 1

Solution

According to the BS methodology, we have the following formula:

$$V_t = e^{-r(T-t)} E_t^Q[\Phi(S(T)) \mid s(t) = s]$$

In this case, the final value function is $\Phi(S_T) = \mathbf{1}_{S_T \geqslant K}$
So, we have

$$\begin{aligned} & E_t^Q[\Phi(S(T)) \mid S(t) = S] \\ = & E_t^Q\left[1_{S_T \geqslant k} \mid S(t) = S\right] \end{aligned}$$

Because the price follows a log normal distribution, we know that $S_T = S \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^Q\right)$

So, the condition $S_T \geqslant K$ is equal to

$$\begin{aligned} & S_T \geqslant K \\ & \Rightarrow S \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^Q\right) \geqslant K \\ & \Rightarrow \left(r - \frac{\sigma^2}{2}\right) t + \sigma \omega_t^Q \geqslant \ln(K/S) \\ & \Rightarrow W_t^Q \geqslant \frac{1}{\sigma}\left[\ln(K/S) - \left(r - \sigma^2/2\right) t\right] \end{aligned}$$

Then the price of the option is

$$V_t = e^{-r(T-t)} \cdot \left[1 - \varphi\left(\frac{1}{\sigma}\left(\ln(K/S) - \left(r - \sigma^2/2\right) t\right)\right] = 0.4341$$

Problem 2

Solution

According to the result, we need 1244 steps to nail the price down to an error of less than one cent.

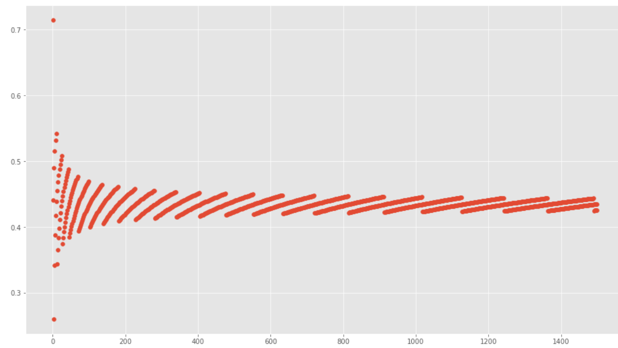

Figure 1: Estimated value as a function of number of steps

Problem 3

Solution

For each method, I let the M ranges from 5000 to 15000 with an increment of 100 for each step. Then, for each M, I simulate the M paths for 100 times, and calculate the accuracy which is defined by the times of more than 95 simulation gives a close enough solution.

Part A

Without antithetic path, we can see that when M is greater than 10000, the accuracy becomes more stable and doesn't go under the 0.95 line.

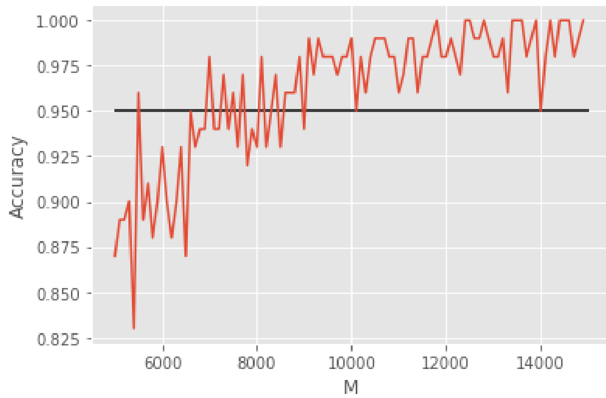

Figure 2: Without antithetic path

Part B

With antithetic path, we can see that when M is larger 10000, the accuracy becomes more stable and doesn't go under the 0.95 line.

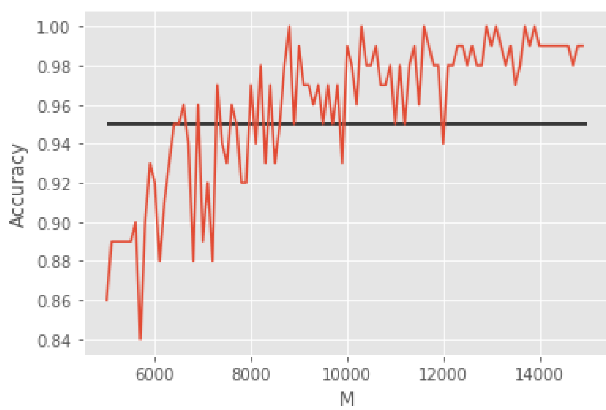

Figure 3: Without antithetic path

In conclusion, the antithetic method doesn't have a big impact on the result or the computational cost.

Problem 4

Solution

There isn't a superior one. We can compare these two methods in two dimensions.
First, the computational complexity. The binomial tree takes more time to compute because when we simulate more periods, we will see a explosion in calculating the node probability. So, we need to use the decimal package to do the calculation, which will slow down the calculation process. But the Monte-Carlo method doesn't need so much calculation.
Second, the error. Using the binomial tree model, we will get a deterministic error so we can easily know after which N can we get a reliable result. However, in Monte-Carlo method, the error is random so we can only give the M with 95% likelihood to nail the price down to an error of less than a cent.
In conclusion, the binomial method gives a deterministic result but is much slower. The Monte-Carlo method gives a random result, but is much faster.