MFE230Q: HW5 Solutions

1. *Black Scholes economy 1*: As noted in class, all derivatives whose CF's only depend upon $S(T)$ satisfy the same PDE, and thus the same expectation. Hence, for the put we have

$$P(S(t),\, t) = \mathrm{E}_t^Q \left[e^{-r(T-t)} \, \max(0, K - S(T)) \right]. \tag{1}$$

It is convenient to write this as a sum of two terms:

$$\begin{aligned} P(S(t),\, t) &= e^{-r(T-t)} \, \mathrm{E}_t^Q \left[K \, \mathbf{1}_{(K>S(T))} - S(T) \, \mathbf{1}_{(K>S(T))} \right] \\ &= K \, e^{-r(T-t)} \, \mathrm{E}_t^Q \left[\mathbf{1}_{(K>S(T))} \right] - e^{-r(T-t)} \, \mathrm{E}_t^Q \left[S(T) \, \mathbf{1}_{(K>S(T))} \right]. \end{aligned} \tag{2}$$

Let us start with the first term. It is convenient to first transform from S to $y = \log S$. From Ito's lemma, we have

$$dy = \left(r - \delta - \frac{\sigma^2}{2} \right) dt + \sigma \, dW^Q. \tag{3}$$

Integrating, we get:

$$y(T) - y(t) = \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma \left(W^Q(T) - W^Q(t) \right). \tag{4}$$

Define the random variable $X \equiv \frac{1}{\sqrt{T-t}} \left(W^Q(T) - W^Q(t) \right) \overset{Q}{\sim} N(0,1)$. We can thus write that the put option is in the money if

$$\log K - \left(y(t) + \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma \sqrt{T-t} \, X \right) > 0, \tag{5}$$

or equivalently, if

$$\log \left(\frac{S(t)}{K} \right) + \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) + \sigma \sqrt{T-t} \, X < 0, \tag{6}$$

which can be rewritten as

$$X < -d_4 \tag{7}$$

where we have defined

$$d_4 \equiv \frac{1}{\sigma \sqrt{T-t}} \left(\log \left(\frac{S(t)}{K} \right) + \left(r - \delta - \frac{\sigma^2}{2} \right) (T-t) \right). \tag{8}$$

Therefore, the expectation can be written as

$$\mathrm{E}_t^Q \left[\mathbf{1}_{(\log K > \log S(T))} \right] = \mathrm{E}_t^Q \left[\mathbf{1} \left(X < -d_4 \right) \right]. \tag{9}$$

Recalling the probability density of a normal variable, this equals

$$
\begin{aligned}
\mathrm{E}_t^Q\left[\mathbf{1}\left(X < -d_4\right)\right] &= \int_{-\infty}^{\infty} dX\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{X^2}{2}}\,\mathbf{1}\left(X < -d_4\right) \\
&= \int_{-\infty}^{-d_4} dX\,\frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}} \\
&= N(-d_4). \qquad (10)
\end{aligned}
$$

Hence, we find that the first term in the Black Scholes put option pricing formula for a dividend-paying stock is

$$
P_1 = K\,e^{-r(T-t)}N(-d_4). \qquad (11)
$$

To solve for the second term, there is a little more algebra involved. Now, from eq. (4) we can write

$$
S(T) = S(t)e^{\left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W^Q(T)-W^Q(t)\right)}. \qquad (12)
$$

Therefore, we can write the expectation as

$$
\begin{aligned}
P_2(S(t),\,t) &= -S(t)\,e^{-\left(\delta+\frac{\sigma^2}{2}\right)(T-t)}\,\mathrm{E}_t^Q\left[e^{\sigma\left(W^Q(T)-W^Q(t)\right)}\,\mathbf{1}_{(S(T)<K)}\right] \\
&= -S(t)\,e^{-\left(\delta+\frac{\sigma^2}{2}\right)(T-t)}\,\mathrm{E}_t^Q\left[e^{\sigma\sqrt{T-t}\,X}\;\mathbf{1}_{(X<-d_4)}\right] \\
&= -S(t)\,e^{-\left(\delta+\frac{\sigma^2}{2}\right)(T-t)}\int_{-\infty}^{\infty} dX\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{X^2}{2}}\,e^{\sigma\sqrt{T-t}\,X}\;\mathbf{1}_{(X<-d_4)} \\
&= -S(t)\,e^{-\delta(T-t)}\int_{-\infty}^{\infty} dX\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(X-\sigma\sqrt{T-t}\right)^2}\;\mathbf{1}_{(X<-d_4)} \\
&= -S(t)\,e^{-\delta(T-t)}\,N(-d_3), \qquad (13)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
d_3 &= d_4 + \sigma\sqrt{T-t} \\
&= \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S(t)}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)\right]. \qquad (14)
\end{aligned}
$$

Combining, we get the B/S put price

$$
P = K\,e^{-r(T-t)}N(-d_4) - S(t)\,e^{-\delta(T-t)}\,N(-d_3). \qquad (15)
$$

2. *Black Scholes economy 2*

(a) Under the risk neutral measure, we have

$$dy = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dW^Q. \tag{16}$$

Hence, $Y(T)$ is normally distributed

$$Y(T) \overset{Q}{\sim} N\left(y(0) + \left(r - \frac{\sigma^2}{2}\right) T,\, \sigma^2 T\right) \tag{17}$$

Hence, the value of the security that pays $\frac{1}{\epsilon}$ if $Y(T) \in (y, y+\epsilon)$ is

$$\begin{aligned} AD^y(0, y(0)) &= \mathrm{E}_0^Q\left[e^{-rT}\left(\frac{1}{\epsilon}\right)\mathbf{1}\left(Y(T) \in (y, y+\epsilon)\right)\right] \\ &= \left(\frac{1}{\epsilon}\right) e^{-rT}\int_y^{y+\epsilon} dY(t)\, \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left[\left(\frac{1}{2\sigma^2 T}\right)\left(Y(T) - y(0) - \left(r - \frac{\sigma^2}{2}\right)T\right)^2\right] \\ &= \left(\frac{1}{\epsilon}\right) e^{-rT}\left[N\left(\frac{y + \epsilon - y(0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - N\left(\frac{y - y(0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right] \end{aligned} \tag{18}$$

As $\epsilon \to 0$, by Taylor expanding, we get

$$\begin{aligned} AD^y(0, y(0)) &= \left(\frac{1}{\sigma\sqrt{T}}\right) e^{-rT}\, n\left(\frac{y - y(0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\ &= e^{-rT}\, \pi_0^Q\left(Y(T) = y\right). \end{aligned} \tag{19}$$

(b) We can replicate the call price by purchasing, for each state ($Y(T) = y$), $\max(0,\, e^y - K)$ shares of the AD security at the price obtained in 3a. Thus, the call option price is

$$C^K(0, y(0)) = \int_{\log K}^{\infty} dy\, AD^y(0, y(0))\, \left(e^y - K\right). \tag{20}$$

It is convenient to break this into two terms. The second term is

$$\begin{aligned} C_2^K(0, y(0)) &= -Ke^{-rT}\int_{\log K}^{\infty} dy\, \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left\{\left(\frac{1}{2\sigma^2 T}\right)\left[y - y(0) - \left(r - \frac{\sigma^2}{2}\right)T\right]^2\right\} \\ &= -Ke^{-rT}\, N(d_2). \end{aligned} \tag{21}$$

The first term is

$$\begin{aligned} C_1^K(0, y(0)) &= e^{-rT}\int_{\log K}^{\infty} dy\, e^y\, \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left\{\left(\frac{1}{2\sigma^2 T}\right)\left[y - y(0) - \left(r - \frac{\sigma^2}{2}\right)T\right]^2\right\} \\ &= S(0)\, N(d_1). \end{aligned} \tag{22}$$

Combining, we see that the solution is equivalent to the Black-Scholes call option.

3. *Black Scholes economy 3*

(a) Under the risk-neutral measure, the stock price process follows

$$dS = (r - \delta)\, S\, dt + \sigma\, S\, dz.$$

Since the call pays no dividend, we have

$$\begin{aligned} rC &= \mathrm{E}_t^Q\,[dC] \\ &= C_t + (r-\delta)\, SC_S + \frac{\sigma^2}{2} S^2 C_{SS}. \end{aligned} \tag{23}$$

However, since there is no explicit time-dependence in the state variable dynamics, and no explicit time-dependence in the payoff, it follows that this call will have the same value each time the same value of S is reached. It thus follows that $C_t = 0$, implying that its dynamics reduce to

$$rC = (r-\delta)\, SC_S + \frac{\sigma^2}{2} S^2 C_{SS}. \tag{24}$$

(b) Assuming $C(S) \sim S^\alpha$, we find

$$rS^\alpha = (r-\delta)\,\alpha S^\alpha + \alpha(\alpha-1)\frac{\sigma^2}{2} S^\alpha \tag{25}$$

or equivalently that

$$0 = \left(\frac{\sigma^2}{2}\right)\alpha^2 + \alpha(r - \delta - \frac{\sigma^2}{2}) - r \tag{26}$$

with solutions

$$\alpha_\pm = \left(\frac{1}{\sigma^2}\right)\left[-(r-\delta-\frac{\sigma^2}{2}) \pm \sqrt{(r-\delta-\frac{\sigma^2}{2})^2 + 2r\sigma^2}\right]. \tag{27}$$

Note that, assuming $r > 0$, the term inside the square root is larger than the term outside. As such, we have $\alpha_+ > 0$, $\alpha_- < 0$. Furthermore, $\alpha_+ = 1$ when the dividend payout $\delta = 0$. Moreover, it is straightforward to demonstrate that α_+ is increasing in δ in that $\frac{\partial \alpha_+}{\partial \delta} > 0$.

Thus, we know that the call price is of the form

$$C(S) = AS^{\alpha_+} + BS^{\alpha_-} \tag{28}$$

The boundary conditions are

$$C(S=0) = 0 \tag{29}$$

$$C(S=S^*) = S^* - K \tag{30}$$

Solving for A and B, I find

$$C(S) = (S^* - K)\left(\frac{S}{S^*}\right)^{\alpha_+}. \tag{31}$$

(c) One can either use smooth pasting or, more easily in this case:

$$\begin{aligned} 0 &= \frac{\partial C}{\partial S^*} \\ &= (1-\alpha_+)(S^*)^{-\alpha_+} + \alpha_+ K (S^*)^{-\alpha_+ - 1} \end{aligned} \tag{32}$$

implying that

$$S^* = K\left(\frac{\alpha_+}{\alpha_+ - 1}\right) \tag{33}$$

Recall that $\alpha_+ > 1$ when $\delta > 0$, but approaches one as $\delta \Rightarrow 0$. Thus, as $\delta \Rightarrow 0$, we find $S^* \Rightarrow \infty$, implying that it is always better to wait. This is consistent with the fact that, for finite maturity American call options, it is never optimal to exercise early if the dividend is zero.

4. Bond pricing:

(a) Because there is no explicit time-dependence in the payoff or the dynamics of the model, we expect the claim to depend only on V and not explicitly on t. Since this is an asset with no dividend, we know that

$$\begin{aligned} rP &= \frac{1}{dt}\mathrm{E}_t^Q\left[dP\right] \\ &= rVP_V + \frac{\sigma^2}{2}V^2 P_{VV}. \end{aligned} \tag{34}$$

We look for a solution of the form $P = V^\alpha$. Plugging in, we find

$$0 = -r + r\alpha + \frac{\sigma^2}{2}\alpha(\alpha - 1). \tag{35}$$

Using the quadratic formula, we get

$$\begin{aligned} \alpha_\pm &= \left(\frac{1}{\sigma^2}\right)\left[-\left(r - \frac{\sigma^2}{2}\right) \pm \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] \\ &= 1,\ -\frac{2r}{\sigma^2}. \end{aligned} \tag{36}$$

Thus, the value of the claim takes the form:

$$P(V) = AV + BV^{-\left(\frac{2r}{\sigma^2}\right)}. \tag{37}$$

where the parameters A and B are determined from boundary conditions. These conditions are:

$$\begin{aligned} P(V \Rightarrow \infty) &= 0 \\ P(V \Rightarrow V_B) &= 1. \end{aligned} \tag{38}$$

Plugging these in, we find $A = 0$ and $B = V_B^{\left(\frac{2r}{\sigma^2}\right)}$. Thus, we find

$$P(V) = \left(\frac{V}{V_B}\right)^{-\left(\frac{2r}{\sigma^2}\right)}. \tag{39}$$

(b) Because there is no explicit time-dependence in the payoff or the dynamics of the model, we expect the first claim to depend only on V and not explicitly on t. Since this is an asset with a dividend C, we know that

$$\begin{aligned} rD &= \frac{1}{dt}\mathrm{E}_t^Q\left[dD + C\,dt\right] \\ &= rVD_V + \frac{\sigma^2}{2}V^2 D_{VV} + C. \end{aligned} \tag{40}$$

The *general solution* to this ODE is the same as above: $D(V) = AV + BV^{-(\frac{2r}{\sigma^2})}$ for some constants A and B. One *particular solution* is $D = \frac{C}{r}$, which should be intuitively understood as the present value of a perpetuity on a non-defaultable bond. Now, this first claim stops receiving CF's the first time V reaches V_B. Hence, $D(V = V_B) = 0$. Further, as $V \Rightarrow \infty$, the debt becomes riskless, and therefore its value should approach the value of the riskless perpetuity: $D(V \Rightarrow \infty) = \frac{C}{r}$. Putting these together, we find the value of the first debt claim to be

$$\begin{aligned} D_1(V) &= \frac{C}{r}\left[1 - \left(\frac{V}{V_B}\right)^{-(\frac{2r}{\sigma^2})}\right] \\ &= \frac{C}{r}\left[1 - P(V)\right]. \end{aligned} \tag{41}$$

The interpretation is that debtholders have to relinquish their claim to a riskless perpetuity, whose value is $\frac{C}{r}$, the first time V reaches V_B. We know that the claim to \$1 the first time V reaches V_B is $P(V)$, so the claim to $\frac{C}{r}$, the first time V reaches V_B is clearly $\frac{C}{r}P(V)$.

The second debt claim is a claim to the firm the first time V reaches V_B. By definition, the value of the claim at that time is V_B. Hence, the second part of the debt claim is

$$\begin{aligned} D_2(V) &= V_B\left(\frac{V}{V_B}\right)^{-(\frac{2r}{\sigma^2})} \\ &= V_B\, P(V). \end{aligned} \tag{42}$$

Combining, we get

$$\begin{aligned} D(V) &= D_1(V) + D_2(V) \\ &= \frac{C}{r} + \left(V_B - \frac{C}{r}\right)\left(\frac{V}{V_B}\right)^{-(\frac{2r}{\sigma^2})}. \end{aligned} \tag{43}$$

(c) We have

$$\begin{aligned} E(V) &= V - D(V) \\ &= V - \frac{C}{r} - \left(V_B - \frac{C}{r}\right)\left(\frac{V}{V_B}\right)^{-(\frac{2r}{\sigma^2})}. \end{aligned} \tag{44}$$

To identify the optimal default boundary V_B, we can use smooth pasting, or alternatively

$$\begin{aligned} 0 &= \frac{\partial E(V, V_B)}{\partial V_B} \\ &= V^{-(\frac{2r}{\sigma^2})}\left[-\left(1 + (\frac{2r}{\sigma^2})\right)V_B^{(\frac{2r}{\sigma^2})} + \left(\frac{C}{r}\right)(\frac{2r}{\sigma^2})V_B^{(\frac{2r}{\sigma^2})-1}\right] \end{aligned} \tag{45}$$

Hence, we find

$$V_B = \left(\frac{C}{r}\right)\left(\frac{2r}{2r+\sigma^2}\right). \tag{46}$$