Q3.

a) $\frac{dS}{S} = (\hat{\mu} - \delta)dt + \sigma dW^P$

$\frac{dB}{B} = r dt$

Therefore, the corresponding PDE should be:

$(r-\delta) S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC = 0$

$C(t, S^*) = S^* - K$ for $\forall$ first hitting time t.

Which equals to $C(S^*) = S^* - K$

Here, since the option has no maturity date and the option payoff depends only on S^*, hence no explicit relationship with time.

b) Right now it's a simple ODE function:

$$\begin{cases} (r-\delta) S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS} - rC = 0 \\ C(S^*) = S^* - K \end{cases}$$

Let $C = S^\alpha$ $\quad \therefore (r-\delta)\alpha S^\alpha + \frac{1}{2}\sigma^2 \alpha(\alpha-1) S^\alpha - r S^\alpha = 0$

$\therefore \frac{1}{2}\sigma^2 \alpha^2 + (r - \delta - \frac{\sigma^2}{2})\alpha - r = 0 \quad (*)$

If x_1 and x_2 are the roots for $(*)$, then $C = A S^{x_1} + B S^{x_2}$

Here
$$\begin{cases} x_1 = \dfrac{\delta + \frac{\sigma^2}{2} - r + \sqrt{(r-\delta-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} > 0 \\ x_2 = \dfrac{\delta + \frac{\sigma^2}{2} - r - \sqrt{(r-\delta-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} < 0 \end{cases}$$

$$\begin{cases} C(S^*) = S^* - K \\ C(0) = 0 \end{cases} \quad \therefore B = 0 \quad A = \frac{S^* - K}{S^{*x_1}}$$

$\therefore C(S) = \frac{S^* - K}{S^{*x_1}} S^{x_1} = (S^* - K)\left(\frac{S}{S^*}\right)^{x_1} \quad x_1 = \frac{\delta + \frac{\sigma^2}{2} - r + \sqrt{(r-\delta-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2}$

c) We need to choose S^* s.t. $(S^*-K)\left(\frac{S}{S^*}\right)^{x_1}$ is maximized.

$$f(S^*) = \frac{S^*-K}{S^{*x_1}}$$

$$\text{Here } f'(S) = \frac{S^{x_1} - (S-K)x_1 S^{x_1-1}}{S^{2x_1}} = (1-x_1)S^{-x_1} + Kx_1 S^{-x_1-1}$$

Let $f'(S)=0$ $\quad (1-x_1)+Kx_1S^{-1}=0$

$\therefore S^* = \frac{Kx_1}{x_1-1} > K$ is the optimal S^*.

When $\delta \to 0$ $\quad x_1 = \frac{\delta + \frac{\sigma^2}{2} - r + \sqrt{(r-\delta-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2}$

$$\to \frac{\frac{\sigma^2}{2} - r + r + \frac{\sigma^2}{2}}{\sigma^2} = 1$$

$\therefore x_1 \to 1$ as $\delta \to 0$. Then optimal $S^* = K\frac{x_1}{x_1-1} \to \infty$

This means, if no dividends, then investor won't set barrier to maximize the value, hence it equals to holding a share of stock with no expiration, since right now $C(S)=S$.