230Q: Problem Set #3

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Problem 1

Solution

Part A

$$
\begin{aligned}
X(t) &= e^{at} \\
dX &= \alpha e^{\alpha t} dt
\end{aligned}
$$

Part B

$$
\begin{aligned}
X(t) &= \int_0^t g(s) dW(s) \\
dX &= g(t) dW(t)
\end{aligned}
$$

Part C

$$
\begin{aligned}
X(t) &= e^{\alpha W(t)} \\
dX &= \alpha e^{\alpha W(t)} dW + \frac{1}{2}\alpha^2 e^{\alpha W(t)} dt
\end{aligned}
$$

Part D

$$
\begin{aligned}
X(t) &= e^{\alpha Y(t)} \\
dX &= \alpha e^{\alpha Y(t)} dY + \frac{1}{2}\alpha^2 e^{\alpha Y(t)} (dY)^2 \\
&= \alpha e^{\alpha Y(t)} (\mu dt + \sigma dW) + \frac{1}{2}\alpha^2 e^{\alpha Y(t)} \cdot \sigma^2 dt \\
&= \alpha e^{\alpha Y(t)} (\mu + \frac{1}{2}\alpha\sigma^2) dt + \alpha e^{\alpha Y(t)} \sigma dW
\end{aligned}
$$

Part E

$$
\begin{aligned}
x(t) &= y^2(t) \\
dx &= 2y dy + (dy)^2 \\
&= 2y(\alpha y dt + \sigma y dW) + \sigma^2 y^2 dt \\
&= y^2 \left(2\alpha + \sigma^2\right) dt + 2\sigma y^2 dW
\end{aligned}
$$

Part F

$$
\begin{aligned}
x(t) &= \frac{1}{y(t)} \\
dx &= -\frac{1}{y^2} dy + 2\frac{1}{y^3} (dy)^2 \\
&= -\frac{1}{y^2} (\alpha y dt + \sigma y dW) + \frac{2}{y^3} \cdot \sigma^2 y^2 dt \\
&= \frac{1}{y} \left(-\alpha + 2\sigma^2\right) dt - \frac{\sigma}{y} dW
\end{aligned}
$$

Problem 2

Solution

First, we calculate the derivatives of X as follows

$$X(t) = \int_0^t \sigma_s dw_s \Rightarrow \begin{cases} dx = \sigma_t dw \\ (dx)^2 = \sigma_t^2 dt \end{cases}$$

Then, Let $Y(t) = e^{iuX(t)}$. We know $Y(0) = 1$.
Applying Ito lemma, we get

$$\begin{aligned} dY &= iue^{iux(t)}dx + \frac{1}{2}(iu)^2 e^{iux(t)}(dx)^2 \\ &= iue^{iux(t)}\sigma_t dw - \frac{1}{2}u^2 e^{iux(t)}\sigma_t^2 dt \end{aligned}$$

Take an integral on both sides

$$Y(t) - 1 = \int_0^t iue^{iux(t)}\sigma_e dW - \frac{1}{2}\int_0^t u^2 e^{iux(t)}\sigma_t^2 dt$$

Take the expectation of the formula. By the martingale properties, we know $\int_0^t iue^{iux(t)}\sigma_t dw = 0$. So, we have

$$E[Y(t)] = 1 - \frac{1}{2}u^2 \int_0^t \sigma^2(s) E\left[e^{iux(t)}\right] dt$$

Let $m_t = E[Y(t)] = E\left[e^{iux(t)}\right]$. We know that $m(0) = 1$
Then, solve the ODE question as below

$$\begin{aligned} m_t &= 1 - \frac{1}{2}u^2 \int_0^t \sigma^2(s) m_s ds \\ m_t' &= -\frac{1}{2}u^2 \sigma^2(t) m_t \\ m_t &= e^{-\frac{1}{2}u^2 \int_0^t \sigma^2(s) ds} \end{aligned}$$

So, we have

$$E\left[e^{iux(t)}\right] = e^{-\frac{u^2}{2}\int_0^t \sigma^2(s) ds}$$

Problem 3

Solution

$$
\begin{aligned}
&dX = \alpha X dt + \sigma_t dW \\
&\int_0^t dX = \int_0^t \alpha X dt + \int_0^t \sigma_t dw \\
&E\left[X_t - X_0\right] = E\left[\int_0^t \alpha X_s ds\right] + E\left[\int_0^t \sigma_s dW_s\right] \\
&E\left[X_t\right] = E\left[X_0\right] + E\left[\int_0^t \alpha X_s ds\right] \\
&E\left[X_t\right] = E\left[X_0\right] + \int_0^t \alpha E\left[X_s\right] ds
\end{aligned}
$$

Let $E\left[x_t\right] = m_t, \quad m_0 = E\left[x_0\right]$. We solve the following ODE problem

$$
\begin{aligned}
&m_t = E\left[X_0\right] + \int_0^t \alpha m_s dS \\
&m_t' = \alpha m_t \Rightarrow m_t = E\left[X_0\right] \cdot e^{\alpha t}
\end{aligned}
$$

To conclude, $E\left[X_t\right] = E\left[X_0\right] \cdot e^{\alpha t}$

Problem 4

Solution

By applying Taylor expansion, we have

$$\begin{aligned}
dR &= 2XdX + 2YdY + \frac{1}{2}\left[2(dX)^2 + 2(dY)^2 + 0(dX)(dY)\right] \\
&= 2XdX + 2YdY + (dX)^2 + (dY)^2 \\
&= 2X(\alpha X dt - Y dW) + 2Y(\alpha Y dt + X dW) + Y^2 dt + X^2 dt \\
&= \left(2\alpha X^2 + 2\alpha Y^2 + X^2 + Y^2\right) dt + (2XY - 2XY)dw \\
&= \left(X^2 + Y^2\right)(2\alpha + 1)dt
\end{aligned}$$

To conclude, we have $dR = (2\alpha + 1)Rdt$

Problem 5

Solution

Part A

First, calculate the derivatives of Y

$$Y = e^{-\alpha t}X_t, Y_t = -\alpha Y, Y_x = e^{-\alpha t}, Y_{xx} = 0$$

Then, apply Ito lemma

$$\begin{aligned} dY &= -\alpha Y dt + e^{-\alpha t} dX \\ &= -\alpha Y dt + e^{-\alpha t}(\alpha x dt + \sigma dw) \\ &= \left(-\alpha Y + \alpha X e^{-\alpha t}\right) dt + \sigma e^{-\alpha t} dW \\ &= \sigma e^{-\alpha t} dW \end{aligned}$$

Part B

$$\begin{aligned} \int_0^t dY &= \int_0^t \sigma e^{-\alpha s} dW_s \\ Y_t - Y_0 &= \int_0^t \sigma e^{-\alpha s} dW_s \\ e^{-\alpha t}X_t - X_0 &= \int_0^t \sigma e^{-\alpha s} dW_s \\ e^{-\alpha t}X_t &= X_0 + \int_0^t \sigma e^{-\alpha s} dW_s \\ X_t &= e^{\alpha t}x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s \end{aligned}$$

Part C

$X(t) \mid \tilde{F}_0$ is normally distributed since each increment is independent and normally distributed. The increment comprises $\alpha X dt$, which is a constant. The increment also comprises σdW, which is independent normally distributed. So, $X(t) \mid \tilde{F}_0$ is normally distributed.

For the expected value,

$$\begin{aligned} E[x(T) \mid x(0)] &= E\left[e^{\alpha T}x_0\right] + E\left[\sigma \int_0^T e^{\alpha(t-s)} dW_s\right] \\ &= e^{\alpha T}x_0 + \sigma e^{\alpha T} E\left[\int_0^T e^{-\alpha s} dW_s\right] \\ &= e^{\alpha T}X_0 \end{aligned}$$

For the variance,

$$
\begin{aligned}
\operatorname{Var}[X(T) \mid X(0)] &= \operatorname{Var}\left(e^{\alpha T} X_0 + \sigma \int_0^T e^{\alpha(T-s)} dW_s\right) \\
&= \sigma^2 \operatorname{Var}\left(\int_0^T e^{\alpha(T-s)} dW_s\right) \\
&= \sigma^2 \left(E\left[\left(\int_0^T e^{\alpha(T-s)} dW_s\right)^2\right] - E\left[\int_0^T e^{\alpha(T-s)} dW_s\right]^2\right) \\
&= \sigma^2 \int_0^T E\left[e^{2\alpha(T-s)}\right] ds \\
&= \sigma^2 \left(-\frac{1}{2\alpha}\right) \cdot e^{2\alpha(T-s)}\Big|_0^T \\
&= -\frac{\sigma^2}{2\alpha}\left(1 - e^{2\alpha T}\right) \\
&= \frac{\sigma^2}{2\alpha}\left(e^{2\alpha T} - 1\right)
\end{aligned}
$$