MFE 230Q: Stochastic Calculus
with Applications to Asset Pricing

Assignment 2

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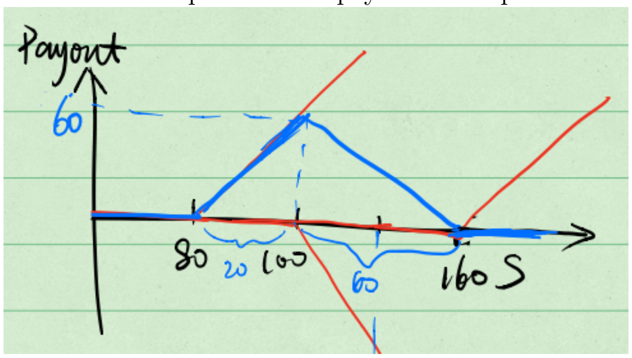
Spring 2022

Problem 1

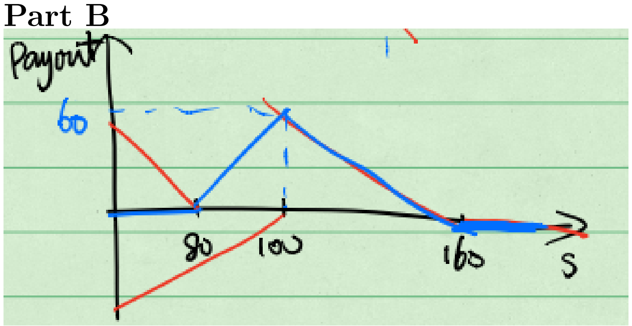
Solution

Part A

The blue line represents the payout of the portfolio. The red lines represent options' payout.



Part B



By choosing $h = (3, -4, 1)^T$, we can generate the same payout as what we have in 1(a).

Part C

Proof. Put call parity describes the relationship between the price of call option and put option of the same underlying assets and expiration dates.

$$C_0 + Ke^{-rT} = p + S$$

The value of the two portfolios are

$$V_1 = 3C_{0,1} - 4C_{0,2} + C_{0,3}$$
$$V_2 = 3P_{0,1} - 4P_{0,2} + P_{0,3}$$

By applying put call parity, we get

$$c_{0,i} = p_{0,i} + S - K_i e^{-r}$$

Plug in the numbers and we get

$$\begin{aligned} V_1 &= 3c_{0,1} - 4c_{0,2} + c_{0,3} \\ &= 3(p_{0,1} + S - 80e^{-r}) - 4(p_{0,2} + S - 100e^{-r}) + (p_{0,3} + S - 160e^{-r}) \\ &= 3p_{0,1} - 4p_{0,2} + p_{0,3} \end{aligned}$$

□

Problem 2

Part A

The D matrix is as follows

$$D = \begin{bmatrix} 120 & 105 & 100 \\ 105 & 105 & 105 \end{bmatrix}$$

The rank of the D matrix is $rank(D) = 2 < M$, so the market is not complete

Part B

By definition, we can formulate the following linear equation systems

$$\begin{bmatrix} 105 \\ 100 \end{bmatrix} = \begin{bmatrix} 120 & 105 & 100 \\ 105 & 105 & 105 \end{bmatrix} \begin{bmatrix} \psi_\uparrow \\ \psi_\rightarrow \\ \psi_\downarrow \end{bmatrix}$$

Simplify it, we get

$$\begin{cases} 120\psi_\uparrow + 105\psi_\rightarrow = 105 - 100\psi_\downarrow \\ 105\psi_\uparrow + 105\psi_\rightarrow = 100 - 105\psi_\downarrow \end{cases}$$

In the end, we have

$$\begin{cases} \psi_\uparrow = \frac{1}{3} + \frac{1}{3}\psi_\downarrow \\ \psi_\rightarrow = \frac{13}{21} - \frac{4}{3}\psi_\downarrow \end{cases}$$

Part C

$$\begin{cases} \frac{1}{3} + \frac{1}{3}\psi_\downarrow > 0 \\ \frac{13}{21} - \frac{4}{3}\psi_\downarrow > 0 \\ \psi_\downarrow > 0. \end{cases}$$

These inequalities directly relate to the following results

$$0 < \phi_\downarrow < 0.4643$$

Because there exists a strictly positive state price vector, the market admits no arbitrage.

Part D

he price of the put option is calculated according to AD securities

$$\begin{aligned} P_{105} &= [15, 0, 0] \begin{bmatrix} \psi_\uparrow \\ \psi_\rightarrow \\ \psi_\downarrow \end{bmatrix} \\ &= 15\psi_\uparrow \\ &= 15 \times \frac{1}{3}(1 + \psi_\downarrow) \\ &= 5(1 + \psi_\downarrow) \end{aligned}$$

So, the range of P_{105} is $P_{105} \in (5, 7.3214)$

Part E

The price of the put option is calculated according to AD securities

$$
\begin{aligned}
P_{100} &= [20, 5, 0] \begin{bmatrix} \psi_{\uparrow} \\ \psi_{\rightarrow} \\ \psi_{\downarrow} \end{bmatrix} \\
&= 20\psi_{\uparrow} + 5\psi_{\rightarrow} \\
&= 20 \times \frac{1}{3} \times (1 + \phi_{\downarrow}) + 5 \times (\frac{13}{21} - \frac{4}{3}\phi_{\downarrow}) \\
&= \frac{205}{21} \\
&= 9.7619
\end{aligned}
$$

Construct a replicating portfolio for this option

$$
\begin{cases}
120\Delta + 105B = 20 \\
105\Delta + 105B = 5 \\
100\Delta + 105B = 0
\end{cases}
$$

We get that

$$
\begin{cases}
\Delta = 1 \\
B = -\frac{20}{21}
\end{cases}
$$

The cash flow of the option can be directly replicated by existing stocks and bonds. So, in an arbitrage free market, the price of the option should be

$$
\begin{aligned}
C &= 105 \times 1 + 100 \times (-\frac{20}{21}) \\
&= 9.7619
\end{aligned}
$$

Problem 3

Part A

irst, we calculate the risk neutral probability $q = \frac{1+R-d}{u-d} = 0.5$
Then, we can calculate the option prices backward

$$C(1,1) = \frac{1}{1.05} \times (0.5 \times 49 + 0.5 \times 13) = 29.52$$
$$C(1,0) = \frac{1}{1.05} \times (0.5 \times 13 + 0.5 \times 0) = 6.19$$
$$C(0) = \frac{1}{1.05} \times (0.5 \times 29.52 + 0.5 \times 6.19) = 17.01$$

Part B

$$\begin{aligned} C(0) &= \left(\frac{1}{1+R_F}\right)^2 E_0^Q[C(2)] \\ &= \left(\frac{1}{1.05}\right)^2 \times (0.25 \times 49 + 0.5 \times 13 + 0.25 \times 0) \\ &= 17.01 \end{aligned}$$

Part C

$$P(1,1) = max(\frac{1}{1.05} \times (0 \times 0.5 + 2 \times 0.5), 0) = 0.9524$$
$$P(1,0) = max(\frac{1}{1.05} \times (2 \times 0.5 + 29 \times 0.5), 20) = 20$$
$$P(0) = max(\frac{1}{1.05} \times (0.9524 \times 0.5 + 20 \times 0.5), 10) = 10$$

Part D

For the first question, if it is exercised, the cash flow is what the investor get from exercising $CF = S(t) - K$. If it is not exercised, the investor short a stock and lend K dollars, so the cash flow is $CF = (S(t) - K)$. So, the cash flows are equivalent.
For the second question, if it is exercised, the cash flow in the future is 0. If is is not exercised, there exists two different situations.

1. $S(T) > K$: $CF = S(T) - K + Ke^{r(T-t)} - S(T) = K(e^{r(T-t)}) > 0$
2. $S(T) < K$: $CF = 0 + Ke^{r(T-t) - S(T)}$. Because $K > S(T), e^{r(T-t)} > 1$, the above cash flow is greater than zero.

In conclusion, the investor is better off if he doesn't exercise the American call option early.

Problem 4

Part A

$$\begin{aligned}
&E_0\left[\tilde{x}_2\right] \\
&= 0.6 \times (0.7 \times 100 + 0.3 \times 80) + 0.4 \times (0.5 \times 60 + 0.5 \times 40) \\
&= 76.4
\end{aligned}$$

Part B

$$\begin{aligned}
&E_H\left[\tilde{x}_2\right] \\
&= 0.7 \times 100 + 0.3 \times 80 \\
&= 94
\end{aligned}$$

Part C

$$\begin{aligned}
&E_T\left[\tilde{x}_2\right] \\
&= 0.5 \times 60 + 0.5 \times 40 \\
&= 50
\end{aligned}$$

Part D

$$\begin{aligned}
&E_0\left[E_1\left[\tilde{x}_2\right]\right] \\
=&0.6 \times E_H\left[\tilde{x}_2\right] + 0.4 \times E_T\left[\tilde{x}_2\right] \\
=&0.6 \times 94 + 0.4 \times 50 \\
=&76.4
\end{aligned}$$

It gives the same result as part A

Part E

Proof.

$$\begin{aligned}
\text{LHS } &= E_0\left[\tilde{x}_2\right] = \sum_{w_1, w_2} \pi\left(w_1, \omega_2\right) x_2\left(w_1, w_2\right). \\
RHS &= E_0\left[E\left[\tilde{x}_2\right]\right] \\
&= E_0\left[\sum_{w_2} \pi\left(w_2 \mid w_1\right) x_2\left(w_1, w_2\right)\right] \\
&= E_0\left[\sum_{w_2} \frac{\pi\left(w_1, w_2\right)}{\pi\left(w_1\right)} x_2\left(w_1, w_2\right)\right] \\
&= \sum_{w_1, w_2} \pi\left(w_1, w_2\right) x_2\left(w_1, w_2\right) \\
&= \text{LHS}
\end{aligned}$$

□