MFE230Q: Assignment 1 - Solution

1. *Fundamental theorem*: In each of the following one-period economies (as defined in class): find an arbitrage, or prove that one does not exist. If there is an arbitrage, show which type it is (first or second). If there is no arbitrage, characterize *all* the state prices. For which which (if any) of these economies does the LOOP hold?

   a) $\overline{D} = \begin{bmatrix} -1 & 1 & 2 & 3 \end{bmatrix}$,

   b) $\overline{D} = \begin{bmatrix} -1 & 1.2 & 0.99 & 0.9 \\ -2 & 2.4 & 2 & 1.8 \end{bmatrix}$,

   c) $\overline{D} = \begin{bmatrix} -1 & 1.2 & 0.9 \\ -0.98 & 1.2 & 0.9 \end{bmatrix}$,

   d) $\overline{D} = \begin{bmatrix} -1 & 3 & 1 & 1 \\ -1 & 0 & 2 & 2 \\ -1 & 1 & 2 & 1 \end{bmatrix}$,

   e) $\overline{D} = \begin{bmatrix} -1 & 3 & 1 & 2 \\ -1 & 1 & 1.5 & 1.7 \\ -1 & 2.5 & 2 & 1.5 \end{bmatrix}$.

Answers:

- a) There's no arbitrage, since there exist $\psi \gg 0$. State prices: $\psi = \{1-2\psi_2-3\psi_3, \psi_2, \psi_3\}$ with $\{\psi_2, \psi_3\} \in R^2_{++}$ and $1-2\psi_2-3\psi_3 > 0$. Any $\{\psi_2, \psi_3\}$ in this set will give a strictly positive state price vector. LOOP holds since there exists a state price.
- b) Arbitrage portfolio: $h = (-2, 1)^T$; Payoffs: $s_0 = 0$, $V^1 = \{0, 0.02, 0\}$. This is an arbitrage type 1. LOOP holds since there exists a state price vector $\psi = \{\psi_1, 0, \frac{1-1.2\psi_1}{0.9}\}$.
- c) Arbitrage portfolio: $h = (1, -1)^T$; Payoffs: $s_0 = -0.02$, $V^1 = \{0, 0\}$. This is an arbitrage type 2. Also, the LOOP is violated, since the two assets have different prices but generate the same cash flows in each state.
- d) There's no arbitrage, since there exist $\psi \gg 0$. State Prices: $\psi = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{6}\}$. LOOP holds since there exists a vector of state prices.
- e) Example of arbitrage portfolio: $h = (1, -2, 1)^T$; Payoffs: $s_0 = 0$, $V^1 = \{3.5, 0, 0.1\}$. This is an arbitrage type 1. LOOP holds since there exists a vector of state prices, $\psi = \{-0.0119, 0.2024, 0.4167\}$. There is also a type 2 arbitrage. For example, the portfolio $h = (0.2738, -0.5952, 0.3095)^T$ costs $s_0 = -0.0119$, generates payoff $V^1 = (1, 0, 0)$, and is therefore both a type 1 and type 2 arbitrage.

2. *Stock valuation*: Ms. Brown, a wealthy industrialist, is considering investing in a project that requires an immediate nonrefundable investment of USD 100 Million. There are two states of the world, $\omega \in \{\text{Good}, \text{Bad}\}$ one year from now. The project will generate the following liquidating cash flows in these states:

State	Probability	Stock market return, $\tilde{r}_m$	Project cash flows, $\tilde{C}$
Good	70%	20%	USD 300 Million
Bad	30%	5%	USD 40 Million

Here we have also listed the probability for the two states to occur, as well as the stock market's return in the two states. The one-year risk-free rate is $r_f = 10\%$, i.e., one dollar deposited in the bank grows to $1 + r_f$ dollars in a year.

Assume that Ms. Brown uses USD 100 Million of her own capital and sets up a firm with the one objective of investing in the project. She then sells the firm to the market (through an initial public offering, an IPO).

a) What will be the market price of the firm?[1]

b) Another model for asset pricing is the so-called Capital Asset Pricing Model, the CAPM. The CAPM is based on stronger assumptions than no-arbitrage theory and can therefore be used for in situations when there is not enough information to completely characterize an asset's price through noarbitrage arguments. The CAPM states that the expected returns on a stock, $r_s = E[\tilde{r}_s]$, can be calculated as

$$r_s = r_f + \beta_s(r_m - r_f),$$

where r_m is the expected return on the market, $r_m = E[\tilde{r}_m]$. Here, β_s, the stock's "beta", is defined as the covariance between the stock's and market's returns divided by the variance of market returns,

$$\beta_s = \frac{cov(\tilde{r}_s, \tilde{r}_m)}{\sigma_m^2}, \qquad \sigma_m^2 = var(\tilde{r}_m).$$

Given that the current value of the firm is P, the (random) return of the stock is defined as

$$\tilde{r}_s = \frac{\tilde{C} - P}{P}.$$

[1]Note that the price may be different from 100 Million, without constituting an "arbitrage." Noarbitrage arguments are valid in financial markets. However, firms may create value through their real investments, e.g., by innovation. Ms. Brown's real investment of USD 100 Million in the firm may therefore fall outside of the realm of noarbitrage theory, although the stock price dynamics of the firm in the market will satisfy noarbitrage princinples.

Verify that the CAPM leads to the same value for the firm as the noarbitrage argument in a).[2]

Answers:

- a) First compute risk neutral probabilities by solving

$$
\begin{aligned}
q \times r_m^G + (1-q) \times r_m^B &= r_f \\
q \times 20 + (1-q) \times 5 &= 10
\end{aligned}
$$

  you get $q = 1/3$. Since there exist risk neutral probabilities use the no arbitrage pricing formula

$$Price = \frac{E^Q[\tilde{C}]}{1+r_f} = 115.15$$

- b) Using the true probabilities compute all the moments that you need for CAPM

$$
\begin{aligned}
E[\tilde{r}_m] &= 0.155 \\
E[\tilde{C}] &= 222 \\
cov(\tilde{C}, \tilde{r}_m) &= 8.19 \\
var(\tilde{r}_m) &= 0.0047
\end{aligned}
$$

  Then solve for P

$$\frac{E[\tilde{C} - P]}{P} = r_f + \frac{1}{P} \frac{cov(\tilde{C}, \tilde{r}_m)}{var(\tilde{r}_m)} (E[\tilde{r}_m] - r_f)$$

  and get $P = 115.15$.

(a) *Option pricing and arbitrage*: A call option on a stock gives the buyer the right but not the obligation to buy the stock at a pre-specified price, K, at a future point in time (the exercise date). Similarly, a put option gives the owner the right but not the obligation to sell the stock at a pre-specified price, K. It follows immediately that the value of these options on the exercise date are

$$
\begin{aligned}
C &= \max(S-K, 0), \\
P &= \max(K-S, 0).
\end{aligned}
$$

[2] Note that the true probabilities for different outcomes are used in the CAPM calculations although they are not in a). For this specific case, you can verify that the probabilities actually cancel out, and the price of the firm is independent of them even when the CAPM is used.

Assume that the the following prices of $t = 1$ exercise date call options are observed (at $t = 0$).

Exercise price, K	Current price
45	$C = 21$
50	$C = 17$
55	$C = 13$
60	$C = 10$

Further, assume that each integer value of the time 1 stock price, S, between 1 and 100 is possible with positive probability, i.e., for each $n = 1, 2, \ldots, 100$, there is a positive probability that the stock price at $t = 1$ is n.[3] Is there an arbitrage? If so, show it. Otherwise, disprove it.

Answers: Arbitrage portfolio $C_1 - 2C_2 + C_3$. At $t = 0$ price is zero and payoff at $t = 1$ will be positive with positive probability when $S \in \{46, .., 54\}$ and zero elsewhere. (If this is not clear draw the payoffs of this portfolio as a function of S, you should get a hat shape payoff function.)

[3] More generally, we could assume a strictly positive probability density function for the stock price at $t = 1$, so that there is a strictly positive probability for the stock price at $t = 1$ to be within any interval, $[S, S + \epsilon]$ for all $S > 0$ and $\epsilon > 0$, but since this would imply an infinite state space (which we have not yet covered in class) we avoid this approach.