MFE 230Q HW3 Solutions:

1a) We have $x(t) = e^{at}$, $x(t+dt) = e^{a(t+dt)}$. Thus

$$\begin{aligned} dx &\equiv x(t+dt) - x(t) \\ &= e^{at}\left(e^{a\,dt} - 1\right) \\ &\stackrel{dt\to 0}{=} e^{at}\left[(1 + a\,dt + \ldots) - 1\right] \\ &= e^{at} a\,dt. \end{aligned} \tag{1}$$

1b) $x(t) = \int_0^t g(s)\,dW(s)$, $x(t+dt) = \int_0^{t+dt} g(s)\,dW(s)$. Thus

$$\begin{aligned} dx &\equiv x(t+dt) - x(t) \\ &= g(t)\,dW(t). \end{aligned} \tag{2}$$

1c) $x(W(t)) = e^{\alpha W(t)}$, $x(W(t+dt)) = e^{\alpha W(t+dt)} \equiv e^{\alpha(W(t)+dW(t))}$. Thus

$$\begin{aligned} dx &\equiv x(t+dt) - x(t) \\ &= e^{\alpha W(t)}\left(e^{\alpha\,dW} - 1\right) \\ &= e^{\alpha W(t)}\left[\left(1 + \alpha\,dW + \frac{\alpha^2}{2} dW^2 + \ldots\right) - 1\right] \\ &= e^{\alpha W(t)}\left(\alpha\,dW + \frac{\alpha^2}{2}\,dt\right) \\ &= x(t)\left(\alpha\,dW + \frac{\alpha^2}{2}\,dt\right). \end{aligned} \tag{3}$$

Alternatively, since x is a function of W, we can use Ito's lemma

$$dx = x_W\,dW + \frac{1}{2} x_{WW}\,dW^2. \tag{4}$$

1d) $x(y(t)) = e^{\alpha y(t)}$, $x(y(t+dt)) = e^{\alpha y(t+dt)} \equiv e^{\alpha(y(t)+dy(t))}$. Thus

$$\begin{aligned} dx &\equiv x(t+dt) - x(t) \\ &= e^{\alpha y(t)}\left(e^{\alpha\,dy} - 1\right) \\ &= e^{\alpha y(t)}\left[\left(1 + \alpha\,dy + \frac{\alpha^2}{2} dy^2 + \ldots\right) - 1\right] \\ &= e^{\alpha y(t)}\left(\alpha\,(\mu\,dt + \sigma\,dW) + \frac{\alpha^2\sigma^2}{2}\,dt\right) \\ &= x(t)\left[\left(\alpha\mu + \frac{\alpha^2\sigma^2}{2}\right)\,dt + \alpha\sigma\,dW\right]. \end{aligned} \tag{5}$$

Alternatively, since x is a function of y, we can use Ito's lemma

$$dx = x_y \, dy + \frac{1}{2} x_{yy} \, dy^2. \tag{6}$$

1e) $x(y(t)) = y(t)^2$, $x(y(t+dt)) = y(t+dt)^2 \equiv (y(t) + dy(t))^2 = y(t)^2 + 2y(t)\,dy(t) + dy(t)^2$. Thus

$$\begin{aligned} dx &\equiv x(t+dt) - x(t) \\ &= 2y(t)\,dy(t) + dy(t)^2 \\ &= 2y(t)\left(\alpha y(t)\,dt + \sigma y(t)\,dW\right) + \sigma^2 y^2\,dt \\ &= \left(2\alpha + \sigma^2\right) x(t)\,dt + 2\sigma x(t)\,dW. \end{aligned} \tag{7}$$

Thus, both dx and dy follow geometric Brownian motion (GBM) processes. Alternatively, since x is a function of y, can use Ito's lemma

$$dx = x_y \, dy + \frac{1}{2} x_{yy} \, dy^2. \tag{8}$$

1f) $x(y(t)) = y(t)^{-1}$, $x(y(t+dt)) = y(t+dt)^{-1} \equiv (y(t) + dy(t))^{-1} = y(t)^{-1}\left(1 - \frac{dy}{y} + \left(\frac{dy}{y}\right)^2\right)$. Thus

$$\begin{aligned} dx &\equiv x(t+dt) - x(t) \\ &= y(t)^{-1}\left(-\frac{dy}{y} + \left(\frac{dy}{y}\right)^2\right) \\ &= y(t)^{-1}\left[-\left(\alpha\,dt + \sigma\,dW\right) + \sigma^2\,dt\right] \\ &= x(t)\left[\left(\sigma^2 - \alpha\right)\,dt - \sigma\,dW\right]. \end{aligned} \tag{9}$$

Thus, dx also follows a GBM process. Alternatively, since x is a function of y, can use Ito's lemma

$$dx = x_y \, dy + \frac{1}{2} x_{yy} \, dy^2. \tag{10}$$

2) From its definition, we have $dX(t) = \sigma_t \, dW(t)$. We wish to calculate $\mathrm{E}_0\left[e^{iuX(t)}\right]$. It is convenient to define

$$F(X(t)) \equiv e^{iuX(t)}. \tag{11}$$

Thus, $F_t = 0$, $F_X = iue^{iuX(t)}$, $F_{XX} = -u^2 e^{iuX(t)}$. Therefore, from Ito's lemma, we have

$$\begin{aligned} dF &= F_t\,dt + F_X\,dX + \frac{1}{2}F_{XX}\,dX^2 \\ &= iue^{iuX(t)}\sigma_t\,dW(t) - \frac{u^2}{2}e^{iuX(t)}\sigma_t^2\,dt. \end{aligned} \tag{12}$$

Formally integrating, we get

$$F(T) - F(0) \;=\; iu \int_0^T e^{iuX(t)} \sigma_t \, dW(t) - \frac{u^2}{2} \int_0^T e^{iuX(t)} \sigma_t^2 \, dt. \tag{13}$$

Taking the expectations of both sides, and using $X(0) = 0$, and hence, $F(0) = e^{iu(0)} = 1$, we find

$$\begin{aligned} \mathrm{E}_0 \left[F(T) \right] &\equiv \mathrm{E}_0 \left[e^{iuX(T)} \right] \\ &= 1 - \frac{u^2}{2} \int_0^T dt \, \sigma_t^2 \, \mathrm{E}_0 \left[e^{iuX(t)} \right]. \end{aligned} \tag{14}$$

It is convenient to define

$$m(t) \equiv \mathrm{E}_0 \left[e^{iuX(t)} \right], \tag{15}$$

We can then rewrite eq. (14) as

$$m(T) \;=\; 1 - \frac{u^2}{2} \int_0^T dt \, \sigma_t^2 \, m(t). \tag{16}$$

Taking a derivative wrt T we find

$$\frac{dm(T)}{dT} \;=\; -\frac{u^2}{2} \sigma_T^2 \, m(T), \tag{17}$$

which can be re-written as

$$\frac{dm}{m} \;=\; -\frac{u^2}{2} \sigma_t^2 \, dt. \tag{18}$$

The solution is

$$\log \left(\frac{m(T)}{m(0)} \right) \;=\; -\frac{u^2}{2} \int_0^T \sigma_t^2 \, dt. \tag{19}$$

From eq. (16) we see that $m(0) = 1$. Thus, we find

$$m(T) \;\equiv\; \mathrm{E}_0 \left[e^{iuX(T)} \right] = e^{-\frac{u^2}{2} \int_0^T \sigma_t^2 \, dt} \tag{20}$$

3) Formally integrating the SDE, we find

$$X(s) - X(0) \;=\; \alpha \int_0^s X(t) \, dt + \int_0^s \sigma_t \, dW(t). \tag{21}$$

Taking the expectation of both sides, we get

$$\mathrm{E}_0 \left[X(s) \right] = X(0) + \alpha \int_0^s dt \, \mathrm{E}_0 \left[X(t) \right] \tag{22}$$

regardless of the functional form of the stochastic process σ_t. At this point, it is convenient to define

$$m(s) \equiv \mathrm{E}_0\left[X(s)\right]. \tag{23}$$

Eq. (22) can the be rewritten as

$$m(s) = X(0) + \alpha \int_0^s dt\, m(t) \tag{24}$$

Taking a derivative wrt time-s, we find

$$\frac{dm(s)}{ds} = \alpha\, m(s). \tag{25}$$

whose solution is

$$m(T) \equiv \mathrm{E}_0\left[X(T)\right] = m(0)\, e^{\alpha T} = X(0)\, e^{\alpha T}. \tag{26}$$

4) Applying Ito's lemma to $R = X^2 + Y^2$ we find

$$\begin{aligned} dR &= R_X\, dX + R_Y\, dY + \frac{1}{2} R_{XX}\, dX^2 + \frac{1}{2} R_{YY}\, dY^2 + R_{XY}\, dX\, dY \\ &= 2X\, dX + 2Y\, dY + dX^2 + dY^2 \\ &= 2X\left(\alpha X\, dt - Y\, dW\right) + 2Y\left(\alpha Y\, dt + X\, dW\right) + Y^2\, dt + X^2\, dt \\ &= (2\alpha + 1)\left(X^2 + Y^2\right)\, dt \\ &= (2\alpha + 1) R\, dt. \end{aligned} \tag{27}$$

The solution is deterministic, namely:

$$R(T) = R(0)\, e^{(2\alpha+1)\, T}. \tag{28}$$

5a) From the definition $Y(t, X) = e^{-\alpha t} X$, we see that $Y_t = -\alpha X e^{-\alpha t}$, $Y_X = e^{-\alpha t}$, and $Y_{XX} = 0$. Applying Ito's lemma, we therefore get:

$$\begin{aligned} dY &= Y_t\, dt + Y_X\, dX + \frac{1}{2} Y_{XX}\, dX^2 \\ &= e^{-\alpha t}\left[-\alpha X\, dt + (\alpha X\, dt + \sigma\, dW)\right] \\ &= e^{-\alpha t} \sigma\, dW. \end{aligned} \tag{29}$$

5b) Formally integrating eq. (29), we get

$$Y(s) - Y(0) = \sigma \int_0^s e^{-\alpha t}\, dW(t). \tag{30}$$

Using the definition $Y(t, X) = e^{-\alpha t} X$ for all dates-t, we thus get

$$e^{-\alpha s} X(s) \quad = \quad e^{-\alpha(0)} X(0) + \sigma \int_0^s e^{-\alpha t}\, dW(t), \tag{31}$$

or equivalently,

$$X(s) \quad = \quad e^{\alpha s} X(0) + \sigma \int_0^s e^{\alpha(s-t)}\, dW(t). \tag{32}$$

5c) Intuitively, eq. (32) can be interpreted as a sum (ie. integral) of normals, which is normal itself. The mean is

$$\mathrm{E}_0\left[X(s)\right] \quad = \quad e^{\alpha s} X(0) + 0. \tag{33}$$

The variance is

$$\begin{aligned}
\mathrm{Var}_0\left[X(s)\right] &= \mathrm{E}_0\left[\left(X(s) - \mathrm{E}_0\left[X(s)\right]\right)^2\right] \\
&= \mathrm{E}_0\left[\left(\sigma \int_0^s e^{\alpha(s-t)}\, dW(t)\right)^2\right] \\
&= \sigma^2\, \mathrm{E}_0\left[\left(\int_0^s e^{\alpha(s-t)}\, dW(t)\right)\left(\int_0^s e^{\alpha(s-u)}\, dW(u)\right)\right] \\
&= \sigma^2 \int_0^s e^{2\alpha(s-t)}\, dt \\
&= \frac{\sigma^2}{2\alpha}\left[e^{2\alpha s} - 1\right].
\end{aligned} \tag{34}$$

There are two interesting limits: $s \to dt$ and $s \to \infty$. In the first case, we find

$$\begin{aligned}
\mathrm{Var}_0\left[X(s)\right] &\overset{s \to dt}{=} \frac{\sigma^2}{2\alpha}\left[e^{2\alpha\, dt} - 1\right] \\
&= \frac{\sigma^2}{2\alpha}\left[\left(1 + 2\alpha\, dt + \ldots\right) - 1\right] \\
&= \sigma^2\, dt,
\end{aligned} \tag{35}$$

which is consistent with the SDE.

In the second case, <u>if</u> $\alpha < 0$, then we find

$$\begin{aligned}
\mathrm{Var}_0\left[X(s)\right] &\overset{s \to \infty}{=} \frac{\sigma^2}{2\alpha}\left[e^{2\alpha\, \infty} - 1\right] \\
&= \frac{\sigma^2}{2|\alpha|}.
\end{aligned} \tag{36}$$

In contrast to Brownian motions, where the variance increases linearly with time, this process has a finite variance even over infinite time. Why? Because when $\alpha < 0$, this process is a mean-reverting process, so $X(t)$ never strays "too far" from its long term mean, which from eq. (33) is zero in the long-run.