On the Estimation of Security Price Volatilities from Historical Data

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Homework 4

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Topic:

Estimating capital asset price conditional volatility through public data

Data considered:
- opening prices (O)
- closing prices (C)
- high prices (H)
- low prices (L)

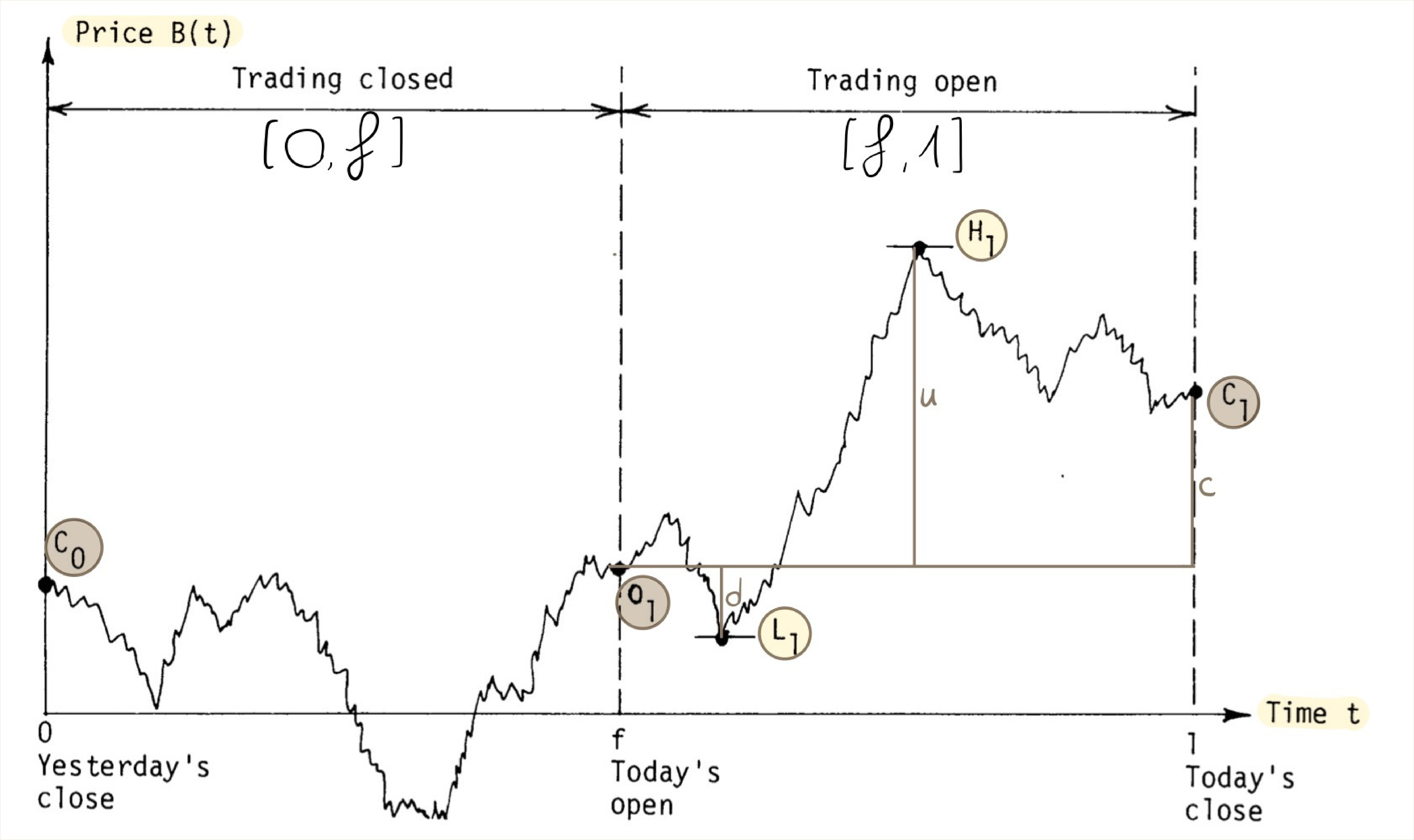
Model assumption

Security prices are governed by a diffusion process with differential representation

$P(t) = \phi(B(t)) \qquad dB = \sigma \, dz$

Transformed price series $B = \phi^{-1}(P)$

One trading day graph



Price versus Time

Classical estimator

Benchmark role

$$\hat{\sigma}_0^2 = (C_1 - C_0)^2 \qquad f = 0$$

Advantages:

- simplicity of use
- unbiased estimation

Disadvantages:

- ignores other information

Relative Efficiency measure:

Consider the classical estimator as benchmark for other estimators

$$Eff(\hat{y}) = \frac{var(\hat{\sigma}_0^2)}{var(\hat{y})}$$

Alternative estimators

Opening & closing prices estimator

$$\hat{\sigma}_1^2 = \frac{(O_1 - C_0)^2}{2f} + \frac{(C_1 - O_1)^2}{2(1-f)} \qquad 0 < f < 1$$

Disadvantages

- It considers only opening and closing prices, which are snapshot, whereas low and high prices can provide more information on the volatility

Alternative estimators

High & Low prices estimators

Disadvantages

- they ignore joint effects between u, d, c (normalized quantities of H, L and C with respect to O)

$$\hat{\sigma}_2^2 = \frac{(H_1 - L_1)^2}{4\ln 2} = \frac{(u-d)^2}{4\ln 2} \qquad f = 0$$

$$\hat{\sigma}_3^2 = a\,\frac{(O_1 - C_0)^2}{f} + (1-a)\,\frac{(u-d)^2}{(1-f)\,4\ln 2} \qquad 0<f<1$$

with a = 0.17 minimum variance of the estimator

Alternative estimators

Unbiased analytic scale-invariant estimators

Aim: to find $\hat{\sigma}^2 = D(u,d,c)$ minimum squared error estimator which is

- unbiased $\quad E(D(u,d,c)) = \sigma^2$
- analytic $\quad D(u,d,c) = a_{200}u^2 + a_{020}d^2 + a_{002}c^2 + a_{110}ud + a_{101}uc + a_{011}dc$
- scale invariant $\quad D(\lambda u, \lambda d, \lambda c) = \lambda^2 D(u,d,c) \qquad \lambda > 0$

Suppose $f = 0$

$$D(u,d,c) = a_{200}u^2 + a_{020}d^2 + a_{002}c^2 + a_{110}ud + a_{101}uc + a_{011}dc =$$

$$\left.\begin{array}{l} a_{200} = a_{020} \\ a_{101} = a_{011} \end{array}\right]$$ by simmetry property $B(t)$ and $-B(t)$ have same distribution

$2a_{200} + a_{110} + 2a_{101} = 0$] by invariant property $B(t)$ and $B(1-t) - B(1)$ have same distribution

$$= a_{200}(u^2 + d^2) + a_{002}c^2 - 2(a_{200} + a_{101})ud + a_{101}c(u+d) =$$

$$E(u^2) = E(d^2) = E(c^2) = E(c(u+d)) = \sigma^2$$

$$E(ud) = (1 - 2\ln 2)\sigma^2$$

$$= a_1(u-d)^2 + a_2(c(u+d) - 2ud) + (1 - (a_1 + a_2)4\ln 2 + a_2)c^2$$

$$V(a_1, a_2) = E\Big[\Big(a_1 \underbrace{((u-d) - (4\ln 2)c^2)}_{X} + a_2 \underbrace{(c(u+d) - 2ud + (1-4\ln 2)c^2)}_{Y} + \underbrace{c^2}_{Z}\Big)^2\Big]$$

is minimum for a_1 a_2 which satisfy

$$E[(a_1X + a_2Y + Z)X] = E[(a_1X + a_2Y + Z)Y] = 0$$

$$a_1^* = \frac{E(XY)E(YZ) - E(Y^2)E(XZ)}{E(X^2)E(Y^2) - (E(XY))^2} = 0.511$$

$$a_2^* = \frac{E(XY)E(XZ) - E(Y^2)E(YZ)}{E(X^2)E(Y^2) - (E(XY))^2} = -0.019$$

Substituting a_1^* a_2^* in $D(u,d,c)$

$$D(u,d,c) = 0.511(u-d)^2 - 0.019(c(u+d) - 2ud) - 0.383c^2 = \hat{\sigma}_4^2$$

Alternative estimators

Unbiased analytic scale-invariant estimators

Eliminating cross product terms:

$$\hat{\sigma}_5^2 = 0.5(u-d)^2 - (2\ln 2 - 1)c^2$$

Suppose $0 < f < 1$

$$\hat{\sigma}_6^2 = a\,\frac{(O_1 - C_0)^2}{f} + (1-a)\,\frac{\hat{\sigma}_4^2}{1-f}$$

with a = 0.12 minimum variance of the estimator

Summary estimators

estimator of σ^2	relative efficiency
$\hat{\sigma}_1^2 = \dfrac{(O_1 - C_0)^2}{2f} + \dfrac{(C_1 - O_1)^2}{2(1-f)}$	$Eff(\hat{\sigma}_1^2) = 2$
$\hat{\sigma}_2^2 = \dfrac{(H_1 - L_1)^2}{4\ln 2} = \dfrac{(u-d)^2}{4\ln 2}$	$Eff(\hat{\sigma}_2^2) \cong 5.2$
$\hat{\sigma}_3^2 = a\dfrac{(O_1 - C_0)^2}{f} + (1-a)\dfrac{(u-d)^2}{(1-f)4\ln 2}$	$Eff(\hat{\sigma}_3^2) \cong 6.2$
$\hat{\sigma}_4^2 = 0.511(u-d)^2 - 0.019(c(u+d) - 2ud) - 0.383c^2$	$Eff(\hat{\sigma}_4^2) \cong 7.4$
$\hat{\sigma}_5^2 = 0.5(u-d)^2 - (2\ln 2 - 1)c^2$	$Eff(\hat{\sigma}_5^2) \cong 7.4$
$\hat{\sigma}_6^2 = a\dfrac{(O_1 - C_0)^2}{f} + (1-a)\dfrac{\hat{\sigma}_4^2}{1-f}$	$Eff(\hat{\sigma}_6^2) \cong 8.4$

Conclusions:

Satisfying improvements in terms of relative efficiency

The new **unbiased analytic scale-invariant estimators** are more efficient than the classical ones: about 8 times better!

They exploit all infromation captured by the four data "available on newspaper's pages".

Together scale invariance and analyticity provides a *space reduction*: from an infinite dimensional problem to a six parameter formula for D(u,d,c) which is further simplified.

Limitations

Unpredictable factors:

Exhogenous shocks (e.g. oil crisis) influence stock prices

Predictable factors:

Using a continuous process on discrete time points provides biased statistics (finite transaction volume)

Further developments

1. Consider more than one security (multivariate models, Sharpe 1970)

2. Seek for better models (since the one used is assumed to be good for price volatility)

3. Estimate additional quantities despite conditional volatility

Possible application

- Compute the conditional variance using tick-by-tick data as "true" variance
- Estimate prices through Arima models with external regressors
- Compute GK measures based on Arima's predictions
- Compare GK and Garch predictions with respect to true variance