

A SUBSPACE-CONJUGATE GRADIENT METHOD FOR LINEAR MATRIX EQUATIONS

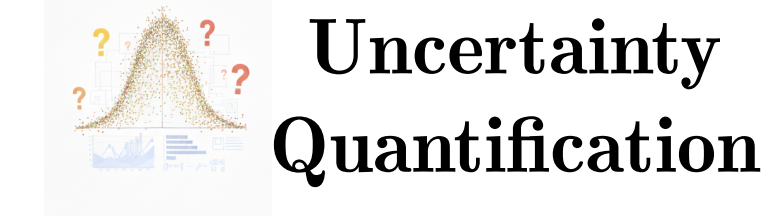
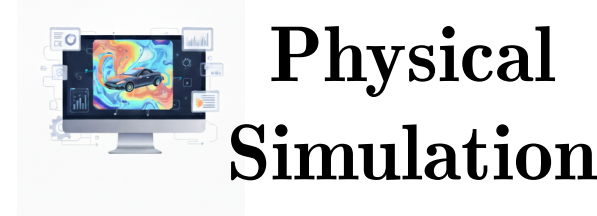
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The problem

Consider a **multiterm Sylvester equation**

$$\mathbf{A}_1 \mathbf{X} \mathbf{B}_1 + \cdots + \mathbf{A}_m \mathbf{X} \mathbf{B}_m = \mathbf{C} \quad (1)$$

that appears in



Finding the solution \mathbf{X} of (1) is

- for $m \leq 2$: **easy** (projection methods, ADI, Riemannian optimization methods, etc...)
- for $m > 2$: **more challenging** (matrix-oriented Krylov methods, ad-hoc projection methods, Riemannian optimization)

Matrix-oriented Conjugate Gradient

Assumptions

- \mathbf{A}_i and \mathbf{B}_i are symmetric matrices of size $(n \times n)$
- Right-hand side $\mathbf{C} = \mathbf{C}_1 \mathbf{C}_2^\top$ has low rank (s_C).
- $\mathcal{L}(\mathbf{X}) = \sum_{i=1}^m \mathbf{A}_i \mathbf{X} \mathbf{B}_i$ is SPD w.r.t. $\langle \mathbf{X}, \mathbf{Y} \rangle = \text{trace}(\mathbf{X}^\top \mathbf{Y})$.

Matrix-oriented CG is equivalent to standard CG, since

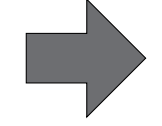
$$\sum_{i=1}^m \mathbf{A}_i \mathbf{X} \mathbf{B}_i = \mathbf{C} \iff \left(\sum_{i=1}^m \mathbf{B}_i^\top \otimes \mathbf{A}_i \right) \text{vec}(\mathbf{X}) = \text{vec}(\mathbf{C}).$$

Given an initial guess $\mathbf{X}_0 \in \mathbb{R}^{n \times n}$, the matrix-oriented CG iterates are

$$\begin{aligned} \mathbf{X}_{k+1} &= \mathbf{X}_k + \alpha_k \mathbf{P}_k & \text{where } \alpha_k \in \mathbb{R} & \quad (\text{solution}) \\ \mathbf{R}_{k+1} &= \mathbf{C} - \mathcal{L}(\mathbf{X}_{k+1}) & & \quad (\text{residual}) \\ \mathbf{P}_{k+1} &= \mathbf{R}_{k+1} + \beta_k \mathbf{P}_k & \text{where } \beta_k \in \mathbb{R} & \quad (\text{direction}) \end{aligned}$$

These matrices are in *factored form*, e.g., $\mathbf{P}_k = \mathbf{P}_k \mathbf{P}_k^\top$, and throughout the iterations, blocks get larger accumulating redundant information.

Low rank truncation



- delayed or stagnating convergence
- challenging to control the rank

Subspace-Conjugate Gradient

Key idea

Replace α, β scalars by α, β matrices

As in matrix-oriented CG, define $\Phi: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ such that

$$\Phi(\mathbf{X}) = \frac{1}{2} \langle \mathcal{L}(\mathbf{X}), \mathbf{X} \rangle - \langle \mathbf{C}, \mathbf{X} \rangle,$$

so that \mathbf{X}^* , the exact solution of (1), satisfies $\mathbf{X}^* = \arg \min_{\mathbf{X} \in \mathbb{R}^{n \times n}} \Phi(\mathbf{X})$.

The new iterate for the solution is

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mathbf{P}_k \alpha_k \mathbf{P}_k^\top \quad \text{where} \quad \alpha_k \in \mathbb{R}^{s_k \times s_k}.$$

Let $\phi: \mathbb{R}^{s_k \times s_k} \rightarrow \mathbb{R}$ be $\phi(\alpha) = \Phi(\mathbf{X}_k + \mathbf{P}_k \alpha \mathbf{P}_k^\top)$, then

$$\alpha_k = \arg \min_{\alpha \in \mathbb{R}^{s_k \times s_k}} \phi(\alpha) \quad (2)$$

Proposition

The matrix $\alpha_k \in \mathbb{R}^{s_k \times s_k}$ minimizer of (2) is the unique solution of

$$\mathbf{P}_k^\top \mathcal{L}(\mathbf{P}_k \alpha \mathbf{P}_k^\top) \mathbf{P}_k = \mathbf{P}_k^\top \mathbf{R}_k \mathbf{P}_k. \quad (3)$$

α_k solution of (3) is equivalent to imposing the **local orthogonality condition**

$$\text{vec}(\mathbf{R}_{k+1}) \perp \text{range}(\mathbf{P}_k \otimes \mathbf{P}_k). \quad (4)$$

The new iterate for the direction matrix is

$$\mathbf{P}_{k+1} = \mathbf{R}_{k+1} + \mathbf{P}_k \beta_k \mathbf{P}_k^\top \quad \text{where} \quad \beta_k \in \mathbb{R}^{s_k \times s_k}.$$

Proposition

The matrix $\beta_k \in \mathbb{R}^{s_k \times s_k}$ is obtained imposing the **\mathcal{L} -orthogonality condition**

$$\text{vec}(\mathcal{L}(\mathbf{P}_{k+1})) \perp \text{range}(\mathbf{P}_k \otimes \mathbf{P}_k). \quad (5)$$

β_k satisfying (5) is the unique solution of

$$\mathbf{P}_k^\top \mathcal{L}(\mathbf{P}_k \beta \mathbf{P}_k^\top) \mathbf{P}_k = -\mathbf{P}_k^\top \mathcal{L}(\mathbf{R}_{k+1}) \mathbf{P}_k. \quad (6)$$

Notice that (3) and (6) are smaller size multiterm Sylvester equations, that is

$$\sum_{i=1}^m \tilde{\mathbf{A}}_i \alpha \tilde{\mathbf{B}}_i = \mathbf{P}_k^\top \mathbf{R}_k \mathbf{P}_k \quad \text{and} \quad \sum_{i=1}^m \tilde{\mathbf{A}}_i \beta \tilde{\mathbf{B}}_i = -\mathbf{P}_k^\top \mathcal{L}(\mathbf{R}_{k+1}) \mathbf{P}_k$$

where $\tilde{\mathbf{A}}_i = \mathbf{P}_k^\top \mathbf{A}_i \mathbf{P}_k$ and $\tilde{\mathbf{B}}_i = \mathbf{P}_k^\top \mathbf{B}_i \mathbf{P}_k$ have size $(s_k \times s_k)$.

Why matrix coefficients?

Matrix-oriented CG

$\alpha_k \in \mathbb{R}$ satisfies the local orthogonality condition

$$\text{vec}(\mathbf{R}_k) \perp \text{range}(\mathbf{P}_k),$$

where $\text{range}(\mathbf{P}_k)$ is a subspace of \mathbb{R}^{n^2} of dimension 1.

The j -th column of \mathbf{X}_{k+1} is equal to the j -th column of \mathbf{X}_k updated by

$$\mathbf{u} = \alpha_k (p_{j,1} \mathbf{p}_1 + \cdots + p_{j,s_k} \mathbf{p}_{s_k})$$

where $\mathbf{P}_k = [\mathbf{p}_1, \dots, \mathbf{p}_{s_k}]$ and $P_k(i, j) = p_{ij}$.

Subspace-CG

$\alpha_k \in \mathbb{R}^{s_k \times s_k}$ satisfies the local orthogonality condition

$$\text{vec}(\mathbf{R}_k) \perp \text{range}(\mathbf{P}_k \otimes \mathbf{P}_k) \subseteq \mathbb{R}^{n^2},$$

where $\text{range}(\mathbf{P}_k \otimes \mathbf{P}_k)$ is a subspace of \mathbb{R}^{n^2} of dimension s_k^2 .

The j -th column of \mathbf{X}_{k+1} is equal to the j -th column of \mathbf{X}_k updated by

$$\left(\sum_{i=1}^{s_k} \alpha_{1,i} p_{j,i} \right) \mathbf{p}_1 + \cdots + \left(\sum_{i=1}^{s_k} \alpha_{s_k,i} p_{j,i} \right) \mathbf{p}_{s_k}$$

where $\alpha_k(i, j) = \alpha_{i,j}$.

Numerical example

Consider the stationary diffusion equation $-\nabla \cdot (\kappa \nabla u) = 0$ in $(0, 1) \times (0, 1)$ with Dirichlet boundary conditions and semiseparable diffusion coefficient:

$$\kappa(x, y) = \sum_{j=0}^m \delta_j \kappa_{x,j}(x) \kappa_{y,j}(y) = 1 + \sum_{j=1}^{m-1} \frac{10^j}{j!} x^j y^j.$$

The resulting multiterm linear equation is

$$\sum_{j=1}^{m_k} \delta_j (\mathbf{A}_{j,x} \mathbf{X} \mathbf{D}_{j,y} + \mathbf{D}_{j,x} \mathbf{X} \mathbf{A}_{j,y}) = \mathbf{C},$$

where \mathbf{C} has rank 4, $m_k = 4$ and for a total of 8 terms.

Performances of the Subspace-CG (Ss-CG), the Riemannian-nonlinear CG (R-NLCG) [1] and the matrix-oriented truncated preconditioned CG (TPCG) [2, 3] are compared.

n	Precond. type	maxrank	R-NLCG	TPCG	Ss-CG determ.	Ss-CG rand.
10000	\mathcal{P}_1	20	– (100)	– (100)	– (100)	– (100)
	\mathcal{P}_1	40	– (100)	– (100)	1.08 (5)	0.92 (5)
	\mathcal{P}_1	60	– (100)	– (100)	2.47 (5)	2.34 (5)
	\mathcal{P}_2	20	11.25 (36)	11.42 (38)	– (100)	– (100)
	\mathcal{P}_2	40	*42.97 (36)	15.54 (33)	– (100)	– (100)
	\mathcal{P}_2	60	*98.62 (35)	32.39 (28)	9.59 (5)	8.37 (5)
102400	\mathcal{P}_1	20	– (100)	– (100)	– (100)	– (100)
	\mathcal{P}_1	40	†	– (100)	18.17 (6)	8.74 (6)
	\mathcal{P}_1	60	†	– (100)	23.50 (5)	16.93 (5)
	\mathcal{P}_2	20	183.44 (41)	– (100)	– (100)	– (100)
	\mathcal{P}_2	40	†	446.94 (47)	– (100)	– (100)
	\mathcal{P}_2	60	†	884.20 (26)	115.73 (3)	101.91 (3)

– no convergence * Lower final residual norm than other methods † Out of Memory

Table 1: Running time in seconds, and in parenthesis the number of iterations. Stopping tolerance $\text{tol1} = 5 \cdot 10^{-6}$. Truncation tolerance $\text{tolrank} = 10^{-12}$. \mathcal{P}_1 : one-term precondition, \mathcal{P}_2 : two-term precondition, expensive.

References

- [1] I. Bioli, D. Kressner, and L. Robol. Preconditioned Low-Rank Riemannian Optimization for Symmetric Positive Definite Linear Matrix Equations. *SIAM J. Sci. Comput.*, 47(2):A1091–A1116, 2025.
- [2] D. Kressner and C. Tobler. Low-Rank Tensor Krylov Subspace Methods for Parametrized Linear Systems. *SIAM J. Matrix Anal. & Appl.*, 32(4):1288–1316, 2011.
- [3] V. Simoncini and Y. Hao. Analysis of the Truncated Conjugate Gradient Method for Linear Matrix Equations. *SIAM J. Matrix Anal. & Appl.*, 44(1):359–381, 2023.

Further details

D. Palitta, M. Iannacito, and V. Simoncini. A Subspace-Conjugate Gradient Method for Linear Matrix Equations. *SIAM J. Matrix Anal. & Appl.*, 46(4):2197–2225, 2025.

