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Modelling the Furuta Pendulum

Magnus Gäfvert

Department of Automatic Control
Lund Institute of Technology
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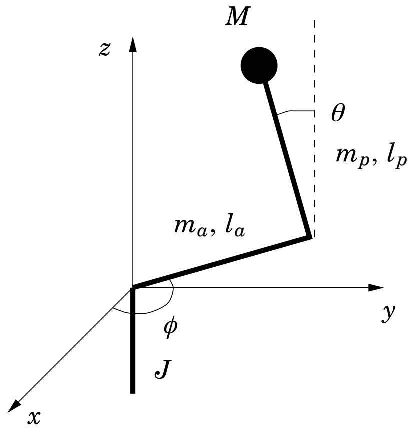

Figure 1 The furuta pendulum.

1. Introduction

This report contains derivations of the Furuta pendulum dynamics using the Euler-Lagrange equations.

The Furuta pendulum is shown in Figure 1. It consists of two connected inertial bodies: A center pillar with moment of inertia J, rigidly connected to a horizontal arm with length l_a and homogenously line distributed mass m_a. The pendulum arm with length l_p and homogenously line distributed mass m_p, and the balancing body with point distributed mass M.

2. Kinematics

The position of a point P on the pendulum can be described with the position vector

$$r(r_a, r_p) = (r_x(r_a, r_p), r_y(r_a, r_p), r_z(r_a, r_p))^\top \tag{1}$$

with

$$\begin{aligned} r_x(r_a, r_p) &= r_a \cos\phi - r_p \sin\phi \sin\theta, \\ r_y(r_a, r_p) &= r_a \sin\phi + r_p \cos\phi \sin\theta, \\ r_z(r_a, r_p) &= r_p \cos\theta. \end{aligned} \tag{2}$$

The variable r_a is the radial position on the horizontal arm, and r_p is the radial position on the pendulum arm. The radial distances are measured from the center of rotation for the bodies respectively. Taking time derivatives of (1) gives an expression for the velocity

$$v(r_a, r_p) = (v_x(r_a, r_p), v_y(r_a, r_p), v_z(r_a, r_p))^\top \tag{3}$$

of P on the pendulum, with

$$\begin{aligned} v_x(r_a, r_p) &= -r_a \sin\phi\dot{\phi} - r_p \cos\theta \sin\phi\dot{\theta} - r_p \sin\theta \cos\phi\dot{\phi}, \\ v_y(r_a, r_p) &= r_a \cos\phi\dot{\phi} + r_p \cos\theta \cos\phi\dot{\theta} - r_p \sin\theta \sin\phi\dot{\phi}, \\ v_z(r_a, r_p) &= -r_p \sin\theta\dot{\theta}. \end{aligned} \tag{4}$$

This is then used to express the square magnitude of the velocity for P:

$$v^2(r_a, r_p) = (r_a^2 + r_p^2 \sin^2\theta)\dot{\phi}^2 + 2r_a r_p \cos\theta\dot{\phi}\dot{\theta} + r_p^2\dot{\theta}^2 \tag{5}$$

3. Energy expressions

Expressions for kinetic and potiential energy is derived in this section. Kinetic energy is derived from solving the integral

$$T = \frac{1}{2}\int v^2 dm, \tag{6}$$

using (5), and potential energy from solving

$$V = g\int r_z dm \tag{7}$$

using (1). The derivations are done for each body separately.

Center pillar

$$\begin{aligned} 2T_c &= J\dot{\phi}^2 \\ V_c &= 0; \end{aligned} \tag{8}$$

Horizontal arm

$$\begin{aligned} 2T_a &= \int_0^{l_a} v^2(s,0) m_a/l_a ds \\ &= \frac{1}{3}m_a l_a^2\dot{\phi}^2 \\ V_a &= 0; \end{aligned} \tag{9}$$

Pendulum arm

$$\begin{aligned} 2T_p &= \int_0^{l_p} v^2(r_a,s) m_p/l_p ds \\ &= m_p(l_a^2 + \frac{1}{3}l_p^2\sin^2\theta)\dot{\phi}^2 + m_p l_a l_p\cos\theta\dot{\phi}\dot{\theta} + \frac{1}{3}m_p l_p^2\dot{\theta}^2 \\ V_p &= g\int_0^{l_p} r_z(l_a,s) m_p/l_p ds \\ &= \frac{1}{2}m_p g l_p\cos\theta \end{aligned} \tag{10}$$

Balancing mass

$$\begin{aligned} 2T_m &= M(l_a^2 + l_p^2\sin^2\theta)\dot{\phi}^2 + 2Ml_a l_p\cos\theta\dot{\phi}\dot{\theta} + Ml_p^2\dot{\theta}^2 \\ V_m &= Mgl_p\cos\theta \end{aligned} \tag{11}$$

The total kinetic energy of the pendulum is given by

$$T = T_c + T_a + T_p + T_m, \tag{12}$$

and the total potential energy by

$$V = V_c + V_a + V_p + V_m. \tag{13}$$

4. Equations of motion

Forming the Lagrangian

$$L = T - V \tag{14}$$

the equations of motion are given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} &= \tau_\phi \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \tau_\theta \end{aligned} \tag{15}$$

with τ_ϕ and τ_θ being external torques applied to the horizontal arm joint and the pendulum arm joint respectively. The partial derivatives are:

$$\begin{aligned} \frac{\partial L}{\partial \phi} &= 0 \\ \frac{\partial L}{\partial \dot{\phi}} &= \left(J + (M + \frac{1}{3}m_a + m_p)l_a^2 + (M + \frac{1}{3}m_p)l_p^2 \sin^2\theta\right)\dot{\phi} \\ &\quad + \left(M + \frac{1}{2}m_p\right) l_a l_p \cos\theta\dot{\theta} \\ \frac{\partial L}{\partial \theta} &= \left(M + \frac{1}{3}m_p\right) l_p^2 \cos\theta\sin\theta\dot{\phi}^2 - \left(M + \frac{1}{2}m_p\right) l_a l_p \sin\theta\dot{\phi}\dot{\theta} \\ &\quad + \left(M + \frac{1}{2}m_p\right) g l_p \sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= \left(M + \frac{1}{2}m_p\right) l_a l_p \cos\theta\dot{\phi} + \left(M + \frac{1}{3}m_p\right) l_p^2\dot{\theta} \end{aligned} \tag{16}$$

Inserting (16) into (15) and introducing

$$\begin{aligned} \alpha &\triangleq J + (M + \frac{1}{3}m_a + m_p)l_a^2 & \beta &\triangleq (M + \frac{1}{3}m_p)l_p^2 \\ \gamma &\triangleq (M + \frac{1}{2}m_p)l_a l_p & \delta &\triangleq (M + \frac{1}{2}m_p) g l_p \end{aligned} \tag{17}$$

yields the equations of motion for the pendulum:

$$\begin{aligned} \left(\alpha + \beta\sin^2\theta\right)\ddot{\phi} + \gamma\cos\theta\ddot{\theta} + 2\beta\cos\theta\sin\theta\dot{\phi}\dot{\theta} - \gamma\sin\theta\dot{\theta}^2 &= \tau_\phi \\ \gamma\cos\theta\ddot{\phi} + \beta\ddot{\theta} - \beta\cos\theta\sin\theta\dot{\phi}^2 - \delta\sin\theta &= \tau_\theta \end{aligned} \tag{18}$$

Equation (18) can be written in matrix form as

$$D(\phi,\theta)\begin{pmatrix}\ddot{\phi} \\ \ddot{\theta}\end{pmatrix} + C(\phi,\theta,\dot{\phi},\dot{\theta})\begin{pmatrix}\dot{\phi} \\ \dot{\theta}\end{pmatrix} + g(\phi,\theta) = \tau \tag{19}$$

with matrices defined by

$$\begin{aligned} D(\phi,\theta) &\triangleq \begin{pmatrix} \alpha + \beta\sin^2\theta & \gamma\cos\theta \\ \gamma\cos\theta & \beta \end{pmatrix}, \\ C(\phi,\theta,\dot{\phi},\dot{\theta}) &\triangleq \begin{pmatrix} \beta\cos\theta\sin\theta\dot{\theta} & \beta\cos\theta\sin\theta\dot{\phi} - \gamma\sin\theta\dot{\theta} \\ -\beta\cos\theta\sin\theta\dot{\phi} & 0 \end{pmatrix}, \quad g(\phi,\theta) \triangleq \begin{pmatrix} 0 \\ -\delta\sin\theta \end{pmatrix}. \end{aligned} \tag{20}$$

The matrices $D(\phi,\theta)$ and $C(\phi,\theta,\dot{\phi},\dot{\theta})$ satisfies the fundamental property

$$N(\phi,\theta,\dot{\phi},\dot{\theta}) = \dot{D}(\phi,\theta) - 2C(\phi,\theta,\dot{\phi},\dot{\theta}) \tag{21}$$

with the skew symmetric matrix

$$N(\phi,\theta,\dot{\phi},\dot{\theta}) = \begin{pmatrix} 0 & \gamma\sin\theta\dot{\theta} - 2\beta\cos\theta\sin\theta\dot{\phi} \\ -\gamma\sin\theta\dot{\theta} + 2\beta\cos\theta\sin\theta\dot{\phi} & 0 \end{pmatrix}. \tag{22}$$

The external torques τ can be divided into a driving torque on the ϕ-joint and dissipation terms as

$$\tau = \tau_u - \tau_F. \tag{23}$$

5. Integration model

The equations of motion (18) can be rewritten on a form suitable for integration:

$$\begin{aligned}
\frac{d}{dt}\phi &= \dot{\phi} \\
\frac{d}{dt}\dot{\phi} &= \frac{1}{\alpha\beta - \gamma^2 + (\beta^2+\gamma^2)\sin^2\theta}\Big\{\beta\gamma(\sin^2\theta - 1)\sin\theta\dot{\phi}^2 - 2\beta^2\cos\theta\sin\theta\dot{\phi}\dot{\theta} + \beta\gamma\sin\theta\dot{\theta}^2 \\
&\quad -\gamma\delta\cos\theta\sin\theta + \beta\tau_\phi - \gamma\cos\theta\tau_\theta\Big\} \\
\frac{d}{dt}\theta &= \dot{\theta} \\
\frac{d}{dt}\dot{\theta} &= \frac{1}{\alpha\beta - \gamma^2 + (\beta^2+\gamma^2)\sin^2\theta}\Big\{\beta(\alpha + \beta\sin^2\theta)\cos\theta\sin\theta\dot{\phi}^2 + 2\beta\gamma(1 - \sin^2\theta)\sin\theta\dot{\phi}\dot{\theta} \\
&\quad -\gamma^2\cos\theta\sin\theta\dot{\theta}^2 + \delta(\alpha + \beta\sin^2\theta)\sin\theta - \gamma\cos\theta\tau_\phi + (\alpha + \beta\sin^2\theta)\tau_\theta\Big\}
\end{aligned} \tag{24}$$

6. Equilibrium points

It follows from inserting $\ddot{\phi} = \ddot{\theta} = \dot{\theta} \equiv 0$, $\theta \equiv \theta_0$ and $\dot{\phi} \equiv \dot{\phi}_0$ in (18) that

$$\sin\theta_0\left(\beta\cos\theta_0\dot{\phi}_0^2 + \delta\right) = 0 \tag{25}$$

holds in stationarity. Solving for θ_0 the following equilibrium points are obtained:

$$\begin{aligned}
\theta_0 &= k\pi \quad \text{with} \quad k \in \mathcal{Z} \quad \text{for all} \quad \dot{\phi}_0 \in \mathcal{R} \\
\theta_0 &= \pi - \arccos\left(\frac{\delta}{\beta\dot{\phi}_o^2}\right), \quad \text{for} \quad \dot{\phi}_0 \neq 0
\end{aligned} \tag{26}$$

7. Linearization

Rewriting (19) as

$$\frac{d}{dt}\begin{pmatrix}\dot{\phi} \\ \dot{\theta}\end{pmatrix} = D^{-1}(\phi,\theta)\left(\tau - C(\phi,\theta,\dot{\phi},\dot{\theta})\begin{pmatrix}\dot{\phi} \\ \dot{\theta}\end{pmatrix} - g(\phi,\theta)\right) \tag{27}$$

and introducing the state variable

$$x \triangleq \begin{pmatrix} \phi \\ \dot{\phi} \\ \theta \\ \dot{\theta} \end{pmatrix} \tag{28}$$

we get the state equation

$$\frac{dx}{dt} = f(x, \tau) \tag{29}$$

with f defined appropriately. The linearized model at the equilibrium point $x_0 = (\phi_0, \dot{\phi}_0, \theta_0, \dot{\theta}_0)$, $\tau_0 = (0, 0)$ is obtained from

$$\frac{d(\delta x)}{dt} = \left.\frac{\partial f}{\partial x}\right|_0 \delta x + \left.\frac{\partial f}{\partial \tau}\right|_0 \tau \triangleq A\delta x + B\tau \tag{30}$$

with $\delta x \triangleq x - x_0$. For $x_0 = (0, 0, 0, 0)$ that gives us

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\delta\gamma}{\alpha\beta-\gamma^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\alpha\delta}{\alpha\beta-\gamma^2} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \frac{\beta}{\alpha\beta-\gamma^2} & -\frac{\gamma}{\alpha\beta-\gamma^2} \\ 0 & 0 \\ -\frac{\gamma}{\alpha\beta-\gamma^2} & \frac{\alpha}{\alpha\beta-\gamma^2} \end{pmatrix} \tag{31}$$

with eigenvalues

$$\left\{ 0, 0, \pm\sqrt{\frac{\alpha\delta}{\alpha\beta - \gamma^2}} \right\}. \tag{32}$$

For $x_0 = (0, 0, \pi, 0)$ we get

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\delta\gamma}{\alpha\beta-\gamma^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\alpha\delta}{\alpha\beta-\gamma^2} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \frac{\beta}{\alpha\beta-\gamma^2} & \frac{\gamma}{\alpha\beta-\gamma^2} \\ 0 & 0 \\ \frac{\gamma}{\alpha\beta-\gamma^2} & \frac{\alpha}{\alpha\beta-\gamma^2} \end{pmatrix} \tag{33}$$

with eigenvalues

$$\left\{ 0, 0, \pm i\sqrt{\frac{\alpha\delta}{\alpha\beta - \gamma^2}} \right\}. \tag{34}$$

In the limit case $J, m_a \to \infty$ and $m_p \to 0$ the modes of a simple pendulum are restored since

$$\sqrt{\frac{\alpha\delta}{\alpha\beta - \gamma^2}} \to \sqrt{\frac{g}{l_p}}. \tag{35}$$

8. Linear state feedback control

The linearized model (30) can be used to derive a continuous time state feedback controller on the form

$$\tau_u = -Lx \tag{36}$$

with $L \triangleq (l_\phi, l_{\dot\phi}, l_\theta, l_{\dot\theta})$. The linear dynamics of the equilibrium point $x_0 = (0,0,0,0)$ yields the closed loop characteristic equation

$$s^4 - \frac{\gamma l_{\dot\theta} - \beta l_{\dot\phi}}{\alpha\beta - \gamma^2}s^3 - \frac{\gamma l_\theta - \beta l_\phi + \alpha\delta}{\alpha\beta - \gamma^2}s^2 - \frac{\delta l_{\dot\phi}}{\alpha\beta - \gamma^2}s - \frac{\delta l_\phi}{\alpha\beta - \gamma^2} = 0, \tag{37}$$

and the dynamics of $x_0 = (0,0,\pi,0)$ yields

$$s^4 + \frac{\gamma l_{\dot\theta} - \beta l_{\dot\phi}}{\alpha\beta - \gamma^2}s^3 + \frac{\gamma l_\theta - \beta l_\phi + \alpha\delta}{\alpha\beta - \gamma^2}s^2 + \frac{\delta l_{\dot\phi}}{\alpha\beta - \gamma^2}s + \frac{\delta l_\phi}{\alpha\beta - \gamma^2} = 0. \tag{38}$$

Equating the coefficients in (37) and (38) with the coefficients of the desired closed loop characteristic equation

$$(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) = 0, \tag{39}$$

and solving for the feedback gains gives

$$\begin{aligned}
l_\phi &= -\frac{\alpha\beta - \gamma^2}{\delta}\omega_1^2\omega_2^2 \\
l_{\dot\phi} &= -2\frac{\alpha\beta - \gamma^2}{\delta}\omega_1\omega_2(\omega_1\zeta_2 + \omega_2\zeta_1) \\
l_\theta &= -\frac{\alpha\delta}{\gamma} - \frac{\alpha\beta - \gamma^2}{\gamma}(\frac{\beta}{\delta}\omega_1^2\omega_2^2 + \omega_1^2 + \omega_2^2 + 4\omega_1\omega_2\zeta_1\zeta_2) \\
l_{\dot\theta} &= -2\frac{\alpha\beta - \gamma^2}{\gamma}(\frac{\beta}{\delta}\omega_1^2\omega_2\zeta_2 + \frac{\beta}{\delta}\omega_1\omega_2^2\zeta_1 + \omega_1\zeta_1 + \omega_2\zeta_2)
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
l_\phi &= \frac{\alpha\beta - \gamma^2}{\delta}\omega_1^2\omega_2^2 \\
l_{\dot\phi} &= 2\frac{\alpha\beta - \gamma^2}{\delta}\omega_1\omega_2(\omega_1\zeta_2 + \omega_2\zeta_1) \\
l_\theta &= -\frac{\alpha\delta}{\gamma} + \frac{\alpha\beta - \gamma^2}{\gamma}(-\frac{\beta}{\delta}\omega_1^2\omega_2^2 + \omega_1^2 + \omega_2^2 + 4\omega_1\omega_2\zeta_1\zeta_2) \\
l_{\dot\theta} &= 2\frac{\alpha\beta - \gamma^2}{\gamma}(-\frac{\beta}{\delta}\omega_1^2\omega_2\zeta_2 - \frac{\beta}{\delta}\omega_1\omega_2^2\zeta_1 + \omega_1\zeta_1 + \omega_2\zeta_2)
\end{aligned} \tag{41}$$

respectively.

With a sampling period of 1 ms it is verified numerically that the feedback gains of the discrete time controller differ less than 1 % from the gains of the continuous time controller. With such fast sampling it is thus sound to use the continuous time design in a discrete controller.

9. Friction

The real pendulum exhibits significant friction in the ϕ-joint. The friction can be modeled in several ways.

Coulomb and viscous friction

$$\tau_F = \tau_C \operatorname{sgn} \dot{\phi} + \tau_v \dot{\phi} \tag{42}$$

Coulomb friction with stiction

$$\tau_F = \begin{cases} \tau_C \operatorname{sgn} \dot{\phi} & \text{if } \dot{\phi} \neq 0, \\ \tau_u & \text{if } \dot{\phi} = 0 \text{ and } |\tau_u| < \tau_S, \\ \tau_S \operatorname{sgn} \tau_u & \text{otherwise.} \end{cases} \tag{43}$$

In simulations the zero condition on the velocity is replaced by $|\dot{\phi}| < \dot{\phi}_\varepsilon$, with $\dot{\phi}_\varepsilon$ chosen appropriately.

10. Model Parameters

The pendulum state equations on integrable form (24) can be coded into a Simulink S-function. Simulations of the free pendulum dynamics reveals that stability is critically dependent on the choice of parameters. Simply setting $\alpha = \beta = \gamma = \delta \equiv 1$ leads to instability. Physically sound parameters can be found from measuring a real pendulum or from identification experiments.

Measured Parameters

Examples of physical parameters and model parameters are shown in Tables 1 and 2. Examples of friction model parameters for Coulomb friction with stiction

m_p [kg]	l_p [m]	m_a [kg]	l_a [m]	M [kg]	J [kg·m^2]
0.00775	0.4125	0.072	0.250	0.02025	0.0000972

Table 1 Real pendulum parameters

α [kg·m^2]	β [kg·m^2]	γ [kg·m^2]	δ [kg^2·m^2/s^2]
0.0033472	0.0038852	0.0024879	0.097625

Table 2 Real pendulum model parameters

(43), are given in Table 3.

Parameter Identification

The equations of motion (18) together with the Coulomb and viscous friction (42) can be written on regressor form as

$$y = \phi^T \theta \tag{44}$$

τ_S [Nm]	τ_C [Nm]	$\dot{\phi}_\varepsilon$ [rad/s]
0.015	0.01	0.02

Table 3 Friction model parameters

with

$$\phi^T = \begin{pmatrix} \ddot{\phi} & \sin^2\theta\ddot{\phi} + 2\cos\theta\sin\theta\dot{\phi}\dot{\theta} & \cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2 & 0 & \operatorname{sgn}\dot{\phi} & \dot{\phi} \\ 0 & \ddot{\theta} - \cos\theta\sin\theta\dot{\phi}^2 & \cos\theta\ddot{\phi} & -\sin\theta & 0 & 0 \end{pmatrix}, \tag{45}$$

$$y = \begin{pmatrix} \tau_u \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \tau_C \\ \tau_v \end{pmatrix}. \tag{46}$$

With suitable low-pass or band-pass filtering the least-squares solution for θ provides a set of model parameters. If the measured velocity and acceleration signals are used, the corresponding scaling constants must be taken into account.