Hand-In Exercise: Admittance Controller

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1 System Modeling

This section contains a derivation of the equations of motion for a robot arm consisting of the three first links of a UR5e robot. The model is derived using Lagrange–D'Alembert's Principle.

An illustration of the robot is shown in Figure 1 that includes the kinematic parameters specified in Table 1. The figure also show position vectors for the center of mass of each joint; the coordinates of the center of masses are specified in (1).

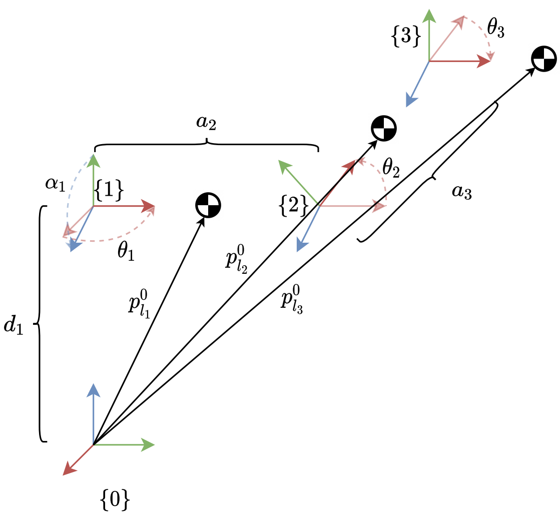


Figure 1: Illustration of considered robot, including kinematic parameters and reference frames.

Link	a_i	α_i	d_i	q_i
1	0	$\pi/2$	d_1	q_1
2	a_2	0	0	q_2
3	a_3	0	0	q_3

Table 1: DH-parameters for the 3-link robot arm.

$$\mathbf{p}^1_{l_1} = \begin{bmatrix} p^1_{l_1,x} \\ p^1_{l_1,y} \\ p^1_{l_1,z} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.02561 \\ 0.00193 \end{bmatrix}, \quad \mathbf{p}^2_{l_2} = \begin{bmatrix} p^2_{l_2,x} \\ p^2_{l_2,y} \\ p^2_{l_2,z} \end{bmatrix} = \begin{bmatrix} 0.2125 \\ 0 \\ 0.11336 \end{bmatrix}, \quad \mathbf{p}^3_{l_3} = \begin{bmatrix} p^3_{l_3,x} \\ p^3_{l_3,y} \\ p^3_{l_3,z} \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.0 \\ 0.0265 \end{bmatrix} \tag{1}$$

To derive a dynamical model of the robot, the mass and inertia tensor of each link are given in

$$\mathbf{I}^1_1 = \begin{bmatrix} 0.0084 & 0 & 0 \\ 0 & 0.0064 & 0 \\ 0 & 0 & 0.0084 \end{bmatrix}, \mathbf{I}^2_2 = \begin{bmatrix} 0.0078 & 0 & 0 \\ 0 & 0.21 & 0 \\ 0 & 0 & 0.21 \end{bmatrix}, \mathbf{I}^3_3 = \begin{bmatrix} 0.0016 & 0 & 0 \\ 0 & 0.0462 & 0 \\ 0 & 0 & 0.0462 \end{bmatrix} \tag{2}$$

Finally, the masses of the links are $m_1 = 3.761$ kg, $m_2 = 8.058$ kg, $m_3 = 2.846$ kg.

1.1 Robot Kinematics

The kinematics of the robot are given by the homogeneous transformations

$$\mathbf{A}_1^0 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1.0 & 0 & d_1 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}, \mathbf{A}_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2\,c_2 \\ s_2 & c_2 & 0 & a_2\,s_2 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}, \mathbf{A}_3^2 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3\,c_3 \\ s_3 & c_3 & 0 & a_3\,s_3 \\ 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \tag{3}$$

where $c_i = \cos(q_i)$ and $s_i = \sin(q_i)$ in addition to the transformations

$$\mathbf{T}_2^0 = \begin{bmatrix} c_1\,c_2 & -c_1\,s_2 & s_1 & a_2\,c_1\,c_2 \\ c_2\,s_1 & -s_1\,s_2 & -c_1 & a_2\,c_2\,s_1 \\ s_2 & c_2 & 0 & d_1 + a_2\,s_2 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \tag{4}$$

$$\mathbf{T}_3^0 = \begin{bmatrix} c_{23}\,c_1 & -s_{23}\,c_1 & s_1 & c_1\,(a_3\,c_{23} + a_2\,c_2) \\ c_{23}\,s_1 & -s_{23}\,s_1 & -c_1 & s_1\,(a_3\,c_{23} + a_2\,c_2) \\ s_{23} & c_{23} & 0 & d_1 + a_3\,s_{23} + a_2\,s_2 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \tag{5}$$

where $c_{ij} = \cos(q_i + q_j)$ and $s_{ij} = \sin(q_i + q_j)$.

1.2 Center of Mass of Links

The center of mass for each link have the following coordinates

$$\mathbf{p}_{c1}^0 = \begin{bmatrix} \mathrm{pl}_{13}\,s_1 \\ -\mathrm{pl}_{13}\,c_1 \\ d_1 + \mathrm{pl}_{12} \end{bmatrix} \tag{6}$$

$$\mathbf{p}_{c2}^0 = \begin{bmatrix} \mathrm{pl}_{23}\,s_1 + a_2\,c_1\,c_2 + \mathrm{pl}_{21}\,c_1\,c_2 \\ a_2\,c_2\,s_1 - \mathrm{pl}_{23}\,c_1 + \mathrm{pl}_{21}\,c_2\,s_1 \\ d_1 + a_2\,s_2 + \mathrm{pl}_{21}\,s_2 \end{bmatrix} \tag{7}$$

$$\mathbf{p}_{c3}^0 = \begin{bmatrix} \mathrm{pl}_{31}\,(c_1\,c_2\,c_3 - c_1\,s_2\,s_3) + \mathrm{pl}_{33}\,s_1 + a_2\,c_1\,c_2 + a_3\,c_1\,c_2\,c_3 - a_3\,c_1\,s_2\,s_3 \\ a_2\,c_2\,s_1 - \mathrm{pl}_{33}\,c_1 - \mathrm{pl}_{31}\,(s_1\,s_2\,s_3 - c_2\,c_3\,s_1) + a_3\,c_2\,c_3\,s_1 - a_3\,s_1\,s_2\,s_3 \\ d_1 + a_3\,s_{23} + \mathrm{pl}_{31}\,s_{23} + a_2\,s_2 \end{bmatrix} \tag{8}$$

where $c_{ij} = \cos(q_i + q_j)$ and $s_{ij} = \sin(q_i + q_j)$.

1.3 Link Velocities

The rotational part of the three link Jacobians are

$$\mathbf{J}_O^{l_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{J}_O^{l_2} = \begin{bmatrix} 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1.0 & 0 & 0 \end{bmatrix}, \mathbf{J}_O^{l_3} = \begin{bmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1.0 & 0 & 0 \end{bmatrix} \tag{9}$$

The first two translational parts of the link Jacobians are

$$\mathbf{J}_P^{l_1} = \begin{bmatrix} \mathrm{pl}_{13}\,c_1 & 0 & 0 \\ \mathrm{pl}_{13}\,s_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\mathbf{J}_P^{l_2} = \begin{bmatrix} \mathrm{pl}_{23}\,c_1 - a_2\,c_2\,s_1 - \mathrm{pl}_{21}\,c_2\,s_1 & -c_1\,s_2\,(a_2 + \mathrm{pl}_{21}) & 0 \\ \mathrm{pl}_{23}\,s_1 + a_2\,c_1\,c_2 + \mathrm{pl}_{21}\,c_1\,c_2 & -s_1\,s_2\,(a_2 + \mathrm{pl}_{21}) & 0 \\ 0 & c_2\,(a_2 + \mathrm{pl}_{21}) & 0 \end{bmatrix} \tag{11}$$

1.4 Inertia-Tensors of Links

The inertia tensors in base frame are

$$\mathbf{I}_{l_1}(\mathbf{q}) = \begin{bmatrix} I_{11,1} - I_{11,1}\, {s_1}^2 + I_{13,3}\, {s_1}^2 & 0.5\, s_{2.0*1}\, (I_{11,1} - I_{13,3}) & 0 \\ 0.5\, s_{2.0*1}\, (I_{11,1} - I_{13,3}) & I_{13,3} + I_{11,1}\, {s_1}^2 - I_{13,3}\, {s_1}^2 & 0 \\ 0 & 0 & I_{12,2} \end{bmatrix} \tag{12}$$

$$\mathbf{I}_{l_2}(\mathbf{q}) = \begin{bmatrix} i_{11} & \sigma_1 & \sigma_3 \\ \sigma_1 & i_{22} & \sigma_2 \\ \sigma_3 & \sigma_2 & i_{33} \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} i_{11} &= I_{11,1}\, {c_1}^2\, {c_2}^2 + I_{12,2}\, {c_1}^2\, {s_2}^2 + I_{13,3}\, {s_1}^2 \\ i_{22} &= I_{13,3}\, {c_1}^2 + I_{11,1}\, {c_2}^2\, {s_1}^2 + I_{12,2}\, {s_1}^2\, {s_2}^2 \\ i_{33} &= I_{12,2}\, {c_2}^2 + I_{11,1}\, {s_2}^2 \\ \sigma_1 &= I_{11,1}\, c_1\, s_1\, {c_2}^2 + I_{12,2}\, c_1\, s_1\, {s_2}^2 - I_{13,3}\, c_1\, s_1 \\ \sigma_2 &= I_{11,1}\, c_2\, s_1\, s_2 - I_{12,2}\, c_2\, s_1\, s_2 \\ \sigma_3 &= I_{11,1}\, c_1\, c_2\, s_2 - I_{12,2}\, c_1\, c_2\, s_2 \end{aligned} \tag{14}$$

$$\tag{15}$$

$$\mathbf{I}_{l_3}(\mathbf{q}) = \begin{bmatrix} i_{11} & \sigma_1 & \sigma_3 \\ \sigma_1 & i_{22} & \sigma_2 \\ \sigma_3 & \sigma_2 & i_{33} \end{bmatrix} \tag{16}$$

where

$$\begin{aligned} i_{11} &= I_{11,1}\, {\sigma_6}^2 + I_{12,2}\, {\sigma_7}^2 + I_{13,3}\, {s_1}^2 \\ i_{22} &= I_{11,1}\, {\sigma_4}^2 + I_{12,2}\, {\sigma_5}^2 + I_{13,3}\, {c_1}^2 \\ i_{33} &= I_{11,1}\, {\sigma_9}^2 + I_{12,2}\, {\sigma_8}^2 \\ \sigma_1 &= I_{12,2}\, \sigma_7\, \sigma_5 - I_{11,1}\, \sigma_6\, \sigma_4 - I_{13,3}\, c_1\, s_1 \\ \sigma_2 &= -I_{11,1}\, \sigma_4\, \sigma_9 - I_{12,2}\, \sigma_5\, \sigma_8 \\ \sigma_3 &= I_{11,1}\, \sigma_6\, \sigma_9 - I_{12,2}\, \sigma_7\, \sigma_8 \\ \sigma_4 &= s_1\, s_2\, s_3 - c_2\, c_3\, s_1 \\ \sigma_5 &= c_2\, s_1\, s_3 + c_3\, s_1\, s_2 \\ \sigma_6 &= c_1\, c_2\, c_3 - c_1\, s_2\, s_3 \\ \sigma_7 &= c_1\, c_2\, s_3 + c_1\, c_3\, s_2 \\ \sigma_8 &= c_2\, c_3 - s_2\, s_3 \\ \sigma_9 &= c_2\, s_3 + c_3\, s_2 \end{aligned} \tag{17}$$

1.5 Potential Energy

The potential energy of the robot is

$$E_{\mathrm{pot}}(\mathbf{q}) = -g_3\, m_2\, (d_1 + a_2\, s_2 + \mathrm{pl}_{21}\, s_2) - g_3\, m_1\, (d_1 + \mathrm{pl}_{12}) - g_3\, m_3\, (d_1 + a_3\, s_{23} + \mathrm{pl}_{31}\, s_{23} + a_2\, s_2) \tag{18}$$

1.6 Kinetic Energy

The kinetic energy of the robot is

$$E_{\mathrm{kin}}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{B}(\mathbf{q})\dot{\mathbf{q}} \tag{19}$$

where the mass matrix is given by

$$\mathbf{B}(\mathbf{q}) = \begin{bmatrix} b_{11} & \sigma_1 & \sigma_4 \\ \sigma_1 & b_{22} & \sigma_2 \\ \sigma_4 & \sigma_2 & m_3 {a_3}^2 + 2.0 m_3 a_3 \mathrm{pl}_{31} + m_3 {\mathrm{pl}_{31}}^2 + I_{13,3} \end{bmatrix} \tag{20}$$

where

$b_{11} = I_{11,1} + 2.0 I_{12,2} - 0.5 I_{11,1}\sigma_3 + 0.5 I_{12,2}\sigma_3 - 0.5 I_{11,1} c_{2.0*2} +$

$0.5 I_{12,2} c_{2.0*2} + 0.5 {a_2}^2 m_2 + 0.5 {a_2}^2 m_3 + 0.5 {a_3}^2 m_3 + m_1 {\mathrm{pl}_{13}}^2 +$

$0.5 m_2 {\mathrm{pl}_{21}}^2 + m_2 {\mathrm{pl}_{23}}^2 + 0.5 m_3 {\mathrm{pl}_{31}}^2 +$

$m_3 {\mathrm{pl}_{33}}^2 + 0.5 m_3 {\mathrm{pl}_{31}}^2 \sigma_3 + 0.5 {a_2}^2 m_2 c_{2.0*2} +$

$0.5 {a_2}^2 m_3 c_{2.0*2} + 0.5 m_2 {\mathrm{pl}_{21}}^2 c_{2.0*2} + a_2 m_2 \mathrm{pl}_{21} + a_3 m_3 \mathrm{pl}_{31} + 0.5 {a_3}^2 m_3 \sigma_3 +$

$a_3 m_3 \mathrm{pl}_{31} \sigma_3 + a_2 m_2 \mathrm{pl}_{21} c_{2.0*2} + a_2 a_3 m_3 c_3 + a_2 m_3 \mathrm{pl}_{31} c_3 + a_2 a_3 m_3 \sigma_5 + a_2 m_3 \mathrm{pl}_{31} \sigma_5$

$b_{22} = 2.0 I_{13,3} + {a_2}^2 m_2 + {a_2}^2 m_3 + {a_3}^2 m_3 + m_2 {\mathrm{pl}_{21}}^2 +$

$m_3 {\mathrm{pl}_{31}}^2 + 2.0 a_2 m_2 \mathrm{pl}_{21} + 2.0 a_3 m_3 \mathrm{pl}_{31} + 2.0 a_2 a_3 m_3 c_3 + 2.0 a_2 m_3 \mathrm{pl}_{31} c_3$

$\sigma_1 = -a_3 m_3 \mathrm{pl}_{33} s_{23} - m_3 \mathrm{pl}_{31} \mathrm{pl}_{33} s_{23} - a_2 m_2 \mathrm{pl}_{23} s_2 - a_2 m_3 \mathrm{pl}_{33} s_2 - m_2 \mathrm{pl}_{21} \mathrm{pl}_{23} s_2$

$\sigma_2 = m_3 {a_3}^2 + 2.0 m_3 a_3 \mathrm{pl}_{31} + a_2 m_3 c_3 a_3 + m_3 {\mathrm{pl}_{31}}^2 + a_2 m_3 c_3 \mathrm{pl}_{31} + I_{13,3}$

$\sigma_3 = c_{2*2+2*3}$

$\sigma_4 = -m_3 \mathrm{pl}_{33} s_{23} (a_3 + \mathrm{pl}_{31})$

$\sigma_5 = c_{2.0*2+3}$

1.7 Robot Dynamics including External Forces

We use Lagrange–D'Alembert's Principle to setup a dynamical model of the robot from

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q} \tag{21}$$

where $\mathbf{Q}$ is an n-dimensional vector of generalized forces and $\mathcal{L} = E_{\text{kin}} - E_{\text{pot}}$.

The dynamics of the 3-DOF link robot can be written as

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} + \mathbf{J}^T(\mathbf{q})\mathbf{h}_e \tag{22}$$

where $\mathbf{h}_e$ is an external wrench applied to the end-effector, the generalized coordinate $\mathbf{q}$, $\mathbf{B}(\mathbf{q})$ is given in (21), and the remaining parameter matrices are

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3]^T \tag{23}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & -a_2\dot{q}_3 m_3 s_3(a_3 + \mathrm{pl}_{31}) & -a_2 m_3 s_3(\dot{q}_2 + \dot{q}_3)(a_3 + \mathrm{pl}_{31}) \\ C_{31} & a_2\dot{q}_2 m_3 s_3(a_3 + \mathrm{pl}_{31}) & 0 \end{bmatrix} \tag{24}$$

where

$C_{11} = -\dot{q}_2(0.5I_{12,2}\sigma_{12} - 0.5I_{11,1}\sigma_{12} - 0.5I_{11,1}s_{2.0*2} + 0.5I_{12,2}s_{2.0*2} +$
$0.5{a_3}^2 m_3\sigma_{12} + 0.5m_3{\mathrm{pl}_{31}}^2\sigma_{12} + 0.5{a_2}^2 m_2 s_{2.0*2} + 0.5{a_2}^2 m_3 s_{2.0*2} +$
$0.5m_2{\mathrm{pl}_{21}}^2 s_{2.0*2} + a_3 m_3\mathrm{pl}_{31}\sigma_{12} + a_2 m_2\mathrm{pl}_{21}s_{2.0*2} + a_2 a_3 m_3\sigma_1$
$+a_2 m_3\mathrm{pl}_{31}\sigma_1) - \dot{q}_3(0.5I_{12,2}\sigma_{12} - 0.5I_{11,1}\sigma_{12} +$
$0.5m_3(2a_3 + 2\mathrm{pl}_{31})(0.5a_2\sigma_1 + 0.5a_3\sigma_{12} + 0.5\mathrm{pl}_{31}\sigma_{12} + 0.5a_2 s_3))$

$C_{12} = 0.5I_{11,1}\dot{q}_1\sigma_{12} - 0.5I_{12,2}\dot{q}_1\sigma_{12} + 0.5I_{11,1}\dot{q}_1 s_{2.0*2} - 0.5I_{12,2}\dot{q}_1 s_{2.0*2}$
$-\sigma_7 - \sigma_6 - 0.5{a_2}^2\dot{q}_1 m_2 s_{2.0*2} - 0.5{a_2}^2\dot{q}_1 m_3 s_{2.0*2} - 0.5\dot{q}_1 m_2{\mathrm{pl}_{21}}^2 s_{2.0*2} -$
$a_2 a_3\dot{q}_1 m_3\sigma_1 - a_2\dot{q}_1 m_3\mathrm{pl}_{31}\sigma_1 - \sigma_8 - a_2\dot{q}_1 m_2\mathrm{pl}_{21}s_{2.0*2} -$
$\sigma_5 - \sigma_4 - \sigma_3 - \sigma_2 - a_2\dot{q}_2 m_2\mathrm{pl}_{23}c_2 - a_2\dot{q}_2 m_3\mathrm{pl}_{33}c_2 -$
$\dot{q}_2 m_2\mathrm{pl}_{21}\mathrm{pl}_{23}c_2$

$C_{13} = 0.5I_{11,1}\dot{q}_1\sigma_{12} - 0.5I_{12,2}\dot{q}_1\sigma_{12} - \sigma_7 - \sigma_6 - 0.5a_2 a_3\dot{q}_1 m_3\sigma_1 -$
$0.5a_2\dot{q}_1 m_3\mathrm{pl}_{31}\sigma_1 - \sigma_8 - \sigma_5 - \sigma_4 - \sigma_3 - \sigma_2 - 0.5a_2 a_3\dot{q}_1 m_3 s_3 -$
$0.5a_2\dot{q}_1 m_3\mathrm{pl}_{31}s_3$

$C_{21} = 0.5\dot{q}_1(I_{12,2}\sigma_{12} - I_{11,1}\sigma_{12} - I_{11,1}s_{2.0*2}$
$+I_{12,2}s_{2.0*2} + \sigma_{10} + \sigma_9 + {a_2}^2 m_2 s_{2.0*2}$
$+{a_2}^2 m_3 s_{2.0*2} + m_2{\mathrm{pl}_{21}}^2 s_{2.0*2} + \sigma_{11} + 2.0a_2 m_2\mathrm{pl}_{21}s_{2.0*2} + 2.0a_2 a_3 m_3\sigma_1 +$
$2.0a_2 m_3\mathrm{pl}_{31}\sigma_1)$

$C_{31} = 0.5\dot{q}_1(I_{12,2}\sigma_{12} - I_{11,1}\sigma_{12} + \sigma_{10} + \sigma_9 + \sigma_{11} + a_2 a_3 m_3 s_3 + a_2 m_3\mathrm{pl}_{31}s_3 +$

$a_2 a_3 m_3\sigma_1 + a_2 m_3\mathrm{pl}_{31}\sigma_1) \quad \sigma_1 = s_{2.0*2+3} \quad \sigma_2 = \dot{q}_3 m_3\mathrm{pl}_{31}\mathrm{pl}_{33}c_{23}$

$\sigma_3 = \dot{q}_2 m_3\mathrm{pl}_{31}\mathrm{pl}_{33}c_{23} \quad \sigma_4 = a_3\dot{q}_3 m_3\mathrm{pl}_{33}c_{23} \quad \sigma_5 = a_3\dot{q}_2 m_3\mathrm{pl}_{33}c_{23}$

$\sigma_6 = 0.5\dot{q}_1 m_3{\mathrm{pl}_{31}}^2\sigma_{12} \quad \sigma_7 = 0.5{a_3}^2\dot{q}_1 m_3\sigma_{12} \quad \sigma_8 = a_3\dot{q}_1 m_3\mathrm{pl}_{31}\sigma_{12}$

$\sigma_9 = m_3{\mathrm{pl}_{31}}^2\sigma_{12} \quad \sigma_{10} = {a_3}^2 m_3\sigma_{12} \quad \sigma_{11} = 2.0a_3 m_3\mathrm{pl}_{31}\sigma_{12} \quad \sigma_{12} = s_{2*2+2*3}$

$$\boldsymbol{g}(\mathbf{q}) = \begin{bmatrix} 0 \\ -g_3 m_3(\sigma_1 + a_2 c_2 + a_3 c_2 c_3 - a_3 s_2 s_3) - g_3 m_2(a_2 c_2 + \mathrm{pl}_{21} c_2) \\ -g_3 m_3(\sigma_1 + a_3 c_2 c_3 - a_3 s_2 s_3) \end{bmatrix} \tag{25}$$

where

$$\sigma_1 = \mathrm{pl}_{31}(c_2 c_3 - s_2 s_3) \tag{26}$$

$$\tag{27}$$

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -s_1\sigma_3 & -c_1\sigma_1 & -a_3 s_{23} c_1 \\ c_1\sigma_3 & -s_1\sigma_1 & -a_3 s_{23} s_1 \\ 0 & \sigma_3 & \sigma_4 \\ 0 & s_1 & s_1 \\ 0 & \sigma_2 & \sigma_2 \\ 1.0 & 0 & 0 \end{bmatrix} \tag{28}$$

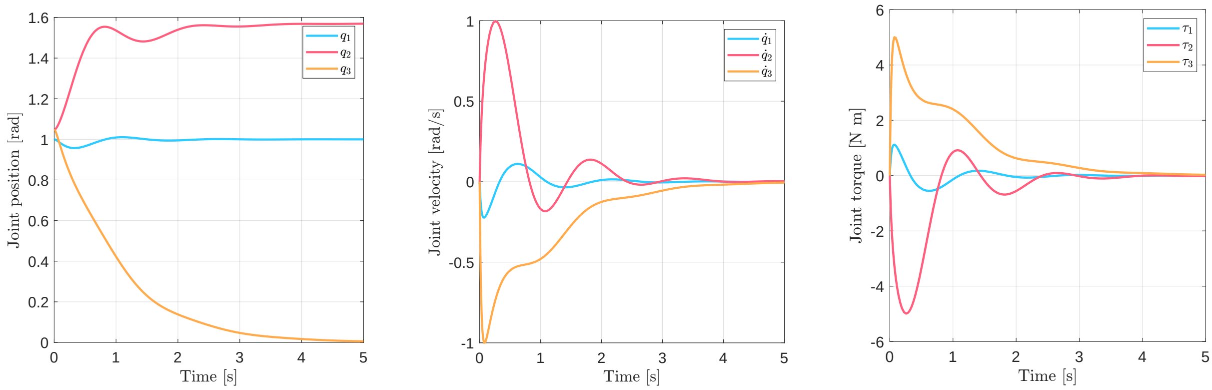
where

$$\begin{aligned} \sigma_1 &= a_3 s_{23} + a_2 s_2 \\ \sigma_2 &= -c_1 \\ \sigma_3 &= \sigma_4 + a_2 c_2 \\ \sigma_4 &= a_3 c_{23} \end{aligned} \tag{29}$$

$$\tag{30}$$

1.8 Verification of Model

Simulation of the robot arm with input $\mathbf{Q} = [\ \tau_1\ \ \tau_2\ \ \tau_3\ ]^T = -\mathbf{D}\dot{\mathbf{q}}$, where $\mathbf{D} \in \mathbb{R}^{3\times3}$ is a diagonal damping matrix. The system is simulated with the initial condition $\mathbf{q} = [\ q_1\ q_2\ q_3\ ]^T = [\ 1\ \pi/3\ \pi/3\ ]^T$, and $\mathbf{D} = 5 \cdot \mathbf{I}$.



(a) Joint positions $\mathbf{q}$. (b) Joint velocities $\dot{\mathbf{q}}$. (c) Joint torques $\boldsymbol{\tau}$.

Figure 2: Simulation of robot arm.

2 Admittance Control

2.1 Control Law

This section presents the admittance controller, where the translational part of the admittance controller is given by

$$\mathbf{M}_p \Delta \ddot{\mathbf{p}}_{cd} + \mathbf{D}_p \Delta \dot{\mathbf{p}}_{cd} + \mathbf{K}_p \Delta \mathbf{p}_{cd} = \mathbf{f} \tag{31}$$

where $\mathbf{M}_p, \mathbf{D}_p, \mathbf{K}_p$ are 3×3 matrices, $\mathbf{f} \in \mathbb{R}^3$ is force given in Base frame, and $\Delta \mathbf{p}_{cd} = \mathbf{p}_c - \mathbf{p}_d$.

The rotational part of the admittance controller can be expressed in different ways depending on the representation use for orientation. Two representations are

$$\mathbf{M}_o \Delta \dot{\boldsymbol{\omega}}_{cd}^d + \mathbf{D}_o \Delta \boldsymbol{\omega}_{cd}^d + \mathbf{K}_o' \boldsymbol{\epsilon}_{cd}^d = \boldsymbol{\mu}^d \tag{32}$$

$$\mathbf{M}_o \Delta \ddot{\boldsymbol{\phi}}_{cd} + \mathbf{D}_o \Delta \dot{\boldsymbol{\phi}}_{cd} + \mathbf{K}_o \Delta \boldsymbol{\phi}_{cd} = \mathbf{T}^T(\boldsymbol{\phi}_c)\boldsymbol{\mu} \tag{33}$$

where $\mathbf{M}_o, \mathbf{D}_o, \mathbf{K}_o \in \mathbb{R}^{3\times3}$, and $\boldsymbol{\mu}^d, \boldsymbol{\mu} \in \mathbb{R}^3$ is the torque applied to the end-effector given in desired frame and base frame, respectively. Furthermore, $\Delta \boldsymbol{\omega}_{cd}^d = \mathbf{R}_0^d(\boldsymbol{\omega}_c - \boldsymbol{\omega}_d)$, $\Delta \boldsymbol{\phi}_{cd} = \boldsymbol{\phi}_c - \boldsymbol{\phi}_d$ and $\boldsymbol{\epsilon}_{cd}^d = \eta_d \boldsymbol{\epsilon}_c - \eta_c \boldsymbol{\epsilon}_d - S(\boldsymbol{\epsilon}_c)\boldsymbol{\epsilon}_d$. The rotational stiffness matrix is

$$\mathbf{K}_o' = 2\mathbf{E}^T(\eta_{cd}, \boldsymbol{\epsilon}_{cd}^d)\mathbf{K}_o$$

where $\mathbf{K}_o$ is the stiffness in Euler angle representation, and

$$\mathbf{E}(\eta, \boldsymbol{\epsilon}) = \eta \mathbf{I} - S(\boldsymbol{\epsilon})$$

The Euler angles are represented as XYZ Tait-Bryan angles, given by

$$\boldsymbol{\phi} = [\varphi \quad \vartheta \quad \psi]^T \tag{34}$$

with the transformation

$$\mathbf{T}(\boldsymbol{\phi}) = \begin{bmatrix} 1 & 0 & \sin(\vartheta) \\ 0 & \cos(\varphi) & -\cos(\varphi)\cdot\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi)\cdot\cos(\vartheta) \end{bmatrix} \tag{35}$$

2.2 Gain Selection

Given a second order system on the form

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{36}$$

the poles that characterize the system's behavior are parameterized in terms of the damping ratio ζ, and natural frequency ω_n, where

$$\zeta = \frac{b}{2\sqrt{km}} \quad \wedge \quad \omega_n = \sqrt{\frac{k}{m}} \tag{37}$$

For a critically damped system, the poles must be on the real axis, such that

$$b^2 - 4mk = 0 \quad \therefore \quad \zeta = 1 \tag{38}$$

To select the impedance parameters, some choices must be made; for example, the stiffness k can be defined in terms of the maximum desired displacement $\tilde{x}_d$ in steady state for a given force f, as

$$k = \frac{f}{\tilde{x}_d} \tag{39}$$

Then, given some desired mass $m = m_d$ the damping b can be computed as

$$b = \sqrt{4m_d k} \tag{40}$$

The gains used for the admittance controller, satisfying (41), are chosen as

$$\mathbf{M}_p = \mathbf{M}_o = 5\cdot\mathbf{I} \qquad \mathbf{K}_p = 10\cdot\mathbf{I} \qquad \mathbf{K}_d = 14.142\cdot\mathbf{I} \quad , \quad \mathbf{I} \in \mathbb{R}^{3\times3} \tag{41}$$

2.3 Implementation

The controllers are implemented in Simulink.

Given the end-effector wrench $\mathbf{h}_e^0$ (in base frame) with transformation $\mathbf{h}_e$

$$\mathbf{h}_e^d = \mathbf{Ad}\left(\mathbf{T}_0^d\right)\mathbf{h}_e^0 = \begin{bmatrix}\mathbf{f}^d \\ \boldsymbol{\mu}^d\end{bmatrix} \quad , \quad \mathbf{Ad}(\mathbf{T}_B^A) = \begin{bmatrix}\mathbf{R}_B^A & \mathbf{0} \\ \mathbf{S}(\mathbf{t}_B^A)\mathbf{R}_B^A & \mathbf{R}_B^A\end{bmatrix} \tag{42}$$

where $\mathbf{S}(\cdot)$ denotes the skew-matrix operator, and $\mathbf{Ad}(\mathbf{T}_B^A)$ is the adjoint matrix for some transformation

$$\mathbf{T}_B^A = \begin{bmatrix}\mathbf{R}_B^A & \mathbf{t}_B^A \\ \mathbf{0} & 1\end{bmatrix} \tag{43}$$

The moment in desired frame is computed from the wrench $\mathbf{h}_e$ as

$$\boldsymbol{\mu}^d = \mathbf{Ad}\left(\mathbf{T}_0^d\right)\mathbf{h}_e^0 \tag{44}$$

where $\mathbf{h}_e$ is an external wrench applied to the end-effector.

2.3.1 Quaternion-Based Controller

Given a unit quaternion

$$\mathbf{q}_{cd}^d = \left(\eta_{cd}^d, \boldsymbol{\epsilon}_{cd}^d\right) \quad , \quad \eta = \cos\left(\frac{\vartheta}{2}\right) \in \mathbb{R} \quad \boldsymbol{\epsilon} = \sin\left(\frac{\vartheta}{2}\right)\mathbf{r} \in \mathbb{R}^3 \tag{45}$$

for some orientation of angle ϑ around a unit vector $\mathbf{r}$. Furthermore, $\eta^2 + \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = 1$, meaning it is normalized.

The quaternion $\mathbf{q}_{cd}^d = (\eta_{cd}, \boldsymbol{\epsilon}_{cd}^d)$ is obtained from integration of $\boldsymbol{\omega}_{cd}^d$ as

$$(\mathbf{q}_{cd}^d + dt) = \exp\left\{\frac{dt}{2}\boldsymbol{\omega}_{cd}^d(t+dt)\right\} \otimes \mathbf{q}_{cd}^d(t) \tag{46}$$

where $\otimes$ refers to a quaternion product, and $\exp\{\mathbf{r}\}$ is defined as

$$\exp\{\mathbf{r}\} = (\eta, \boldsymbol{\epsilon}) = \left(\cos(\|\mathbf{r}\|), \frac{\mathbf{r}}{\|\mathbf{r}\|}\sin(\|\mathbf{r}\|)\right) \tag{47}$$

Given the quaternion $\mathbf{q}_{cd}^d = \left(\eta_{cd}^d, \boldsymbol{\epsilon}_{cd}^d\right)$, the orientation of the compliant frame $\mathbf{q}_c^0$ (in the base frame) is obtained as

$$\mathbf{q}_c^0 = \mathbf{q}_d^0 \otimes \mathbf{q}_{cd}^d \tag{48}$$

2.3.2 Euler Angle-Based Controller

The Euler angle $\Delta\boldsymbol{\phi}_{cd}$ is obtained from integration of $\Delta\dot{\boldsymbol{\phi}}_{cd}$ as

$$\Delta\boldsymbol{\phi}_{cd}(t+dt) = \Delta\boldsymbol{\phi}_{cd}(t) + dt \cdot \Delta\dot{\boldsymbol{\phi}}_{cd}(t+dt) \tag{49}$$

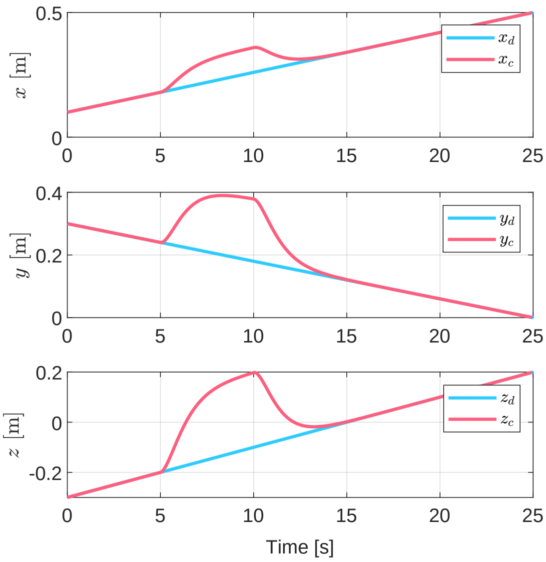
The orientation $\boldsymbol{\phi}_c$ of the compliant frame is obtained from the Euler angle $\Delta\boldsymbol{\phi}_{cd}$ as

$$\boldsymbol{\phi}_c = \Delta\boldsymbol{\phi}_{cd} + \boldsymbol{\phi}_d \tag{50}$$

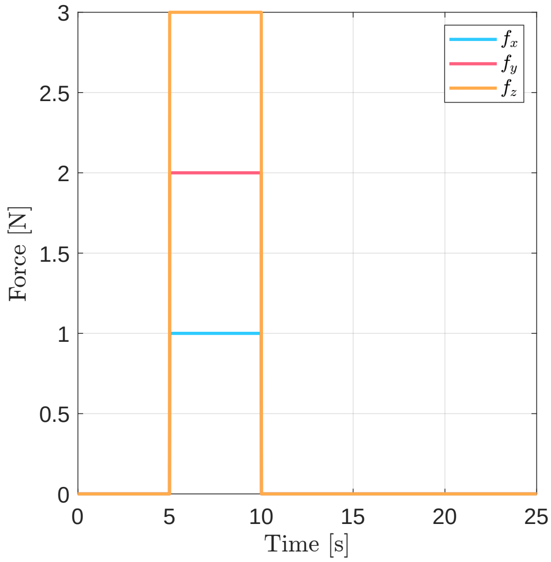
3 Simulation

Three types of admittance controller simulations are demonstrated: translational (Figure 3), Euler angles (Figure 4), and quaternions (Figure 5). These follow the same point-to-point motion for a duration of $t = 25$ seconds. An end-effector wrench is applied during the simulation; a force $\mathbf{f} = [\,1\ 2\ 3\,]^T$ N for five seconds at $t = 5$, and a torque $\boldsymbol{\mu} = [\,1\ 0.5\ 1\,]^T$ N m for five seconds at $t = 15$.

3.1 Translational



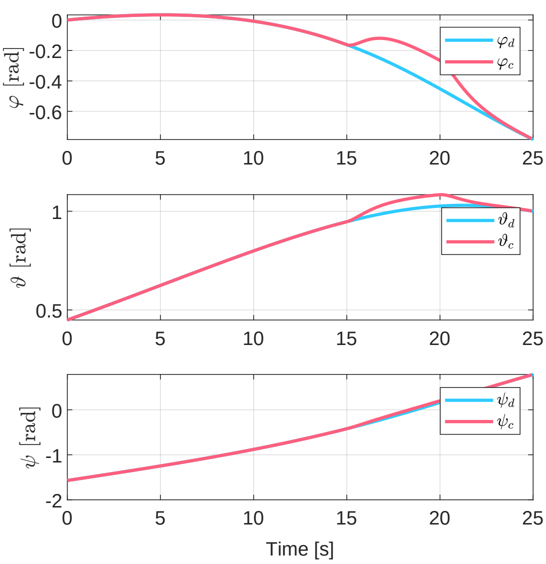
(a) Position



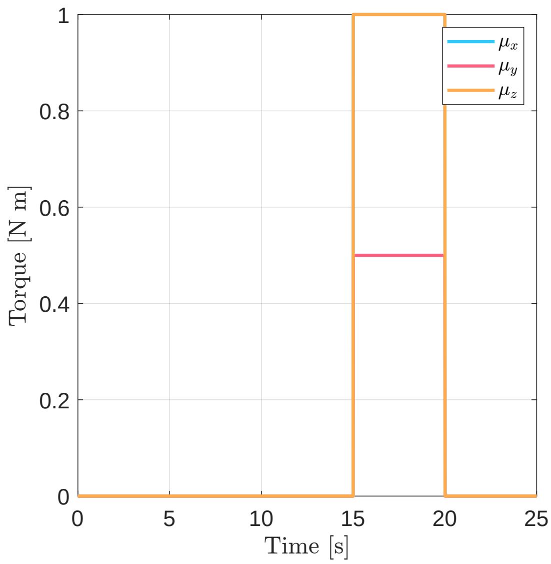
(b) End-effector force $\mathbf{f}$.

Figure 3: Admittance controller simulation for translational motion.

3.2 Rotational (Euler angles)



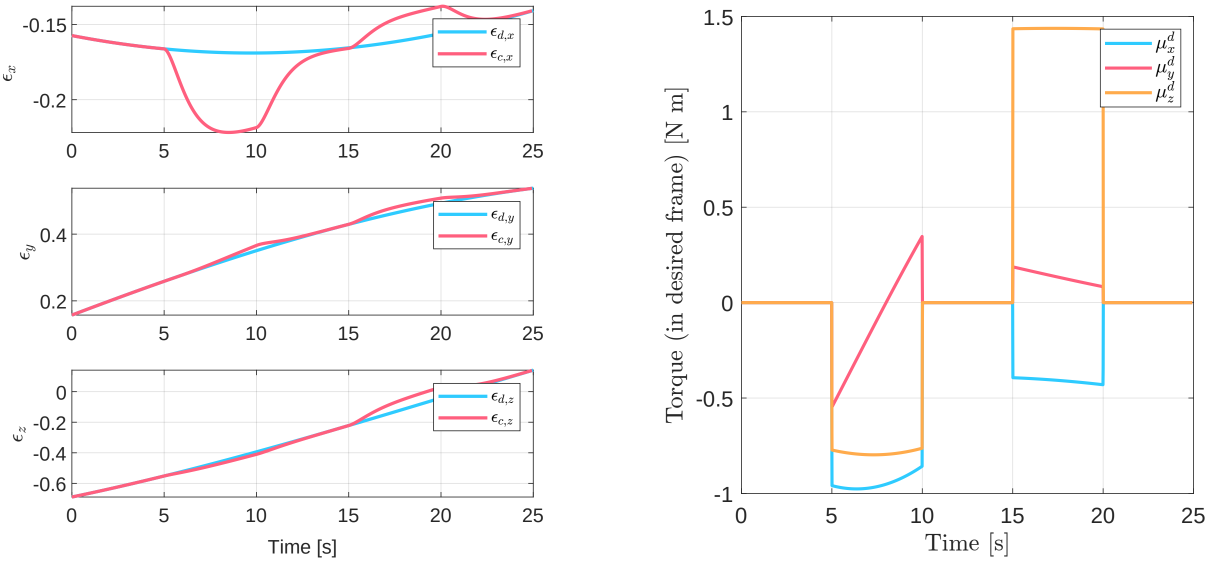
(a) Euler angles orientation $\boldsymbol{\phi}$.



(b) End effector moment $\mathbf{u}$.

Figure 4: Admittance controller simulation for Euler angles.

3.3 Rotational (Quaternion)



(a) Quaternion orientation $\boldsymbol{\epsilon}$.

(b) End-effector moment in desired frame $\boldsymbol{\mu}^d$.

Figure 5: Admittance controller simulation for quaternions.