LINEAR STABILITY ANALYSIS

TURING SYSTEM

TURING EQUATIONS

$$\frac{\partial A}{\partial t} = \underbrace{5A - 6B}_{\text{reaction term}} + \underbrace{D_A \cdot \frac{\partial^2 A}{\partial x^2}}_{\text{diffusion term}} = f(A,B) + \frac{\partial^2 A}{\partial x^2} \qquad \frac{\partial B}{\partial t} = 6A - 7B + D_B \cdot \frac{\partial^2 B}{\partial x^2} = g(A,B) + \frac{\partial^2 B}{\partial x^2}$$

$\delta A \ll |A_*|$

$\delta B \ll |B_*|$

1. Studying System without Diffusion

$$\frac{\partial A}{\partial t} = f(A,B)$$

$$\frac{\partial B}{\partial t} = g(A,B)$$

- At steady states A^* and B^*

$$f(A_*, B_*) = 0$$

$$g(A_*, B_*) = 0$$

- Adding a small perturbation

$$A(t) = A_* + \delta A(t)$$

$$B(t) = B_* + \delta B(t)$$

- Re-evaluate differential equations

$$\frac{\partial A(t)}{\partial t} = \frac{\partial [A_* + \delta A(t)]}{\partial t} = f(A_* + \delta A(t), B_* + \delta B(t)) = \frac{\partial \delta A}{\partial t}$$

$$\frac{\partial B(t)}{\partial t} = \frac{\partial [B_* + \delta B(t)]}{\partial t} = g(A_* + \delta A(t), B_* + \delta B(t)) = \frac{\partial \delta B}{\partial t}$$

Taylor Expand: Linearising system around steady states

$$f(A + \delta A, B + \delta B) = f(A,B) + \frac{\partial f(A,B)}{\partial A} \cdot \delta A + \frac{\partial f(A,B)}{\partial B} \cdot \delta B +$$

$$\frac{1}{2!} \cdot \frac{\partial^2 f(A,B)}{\partial A^2} \cdot (\delta A)^2 + \frac{1}{2!} \cdot \frac{\partial^2 f(A,B)}{\partial B^2} \cdot (\delta B)^2 + \frac{\partial^2 f(A,B)}{\partial A \partial B} \cdot \delta A \cdot \delta B \ \ldots\ldots \text{ higher order terms.}$$

- If δA and δB are small enough, higher order terms can be ignored as $(\delta A)^2, (\delta A)^3 \ldots$ become very small. Therefore we assume:

$$\cancel{f(A_* + \delta A, B_* + \delta B)} \quad f(A_* + \delta A, B_* + \delta B) = \underbrace{f(A_*, B_*)}_{=0} + \frac{\partial f(A,B)}{\partial A} \cdot \delta A + \frac{\partial f(A,B)}{\partial B} \cdot \delta B$$

Because $\frac{\partial A}{\partial t} = \frac{\partial \delta A}{\partial t}$:

$$\frac{\partial \delta A}{\partial t} = \frac{\partial f(A,B)}{\partial A} \cdot \delta A + \frac{\partial f(A,B)}{\partial B} \cdot \delta B = a_{11} \cdot \delta A + a_{12} \cdot \delta B = \frac{\partial \delta A}{\partial t}$$

$$\frac{\partial \delta B}{\partial t} = \frac{\partial g(A,B)}{\partial A} \cdot \delta A + \frac{\partial g(A,B)}{\partial B} \cdot \delta B = a_{21} \cdot \delta A + a_{22} \cdot \delta B = \frac{\partial \delta B}{\partial t}$$

$a_{11} = \frac{\partial f(A,B)}{\partial A}$ $\quad a_{12} = \frac{\partial f(A,B)}{\partial B}$

$a_{21} = \frac{\partial g(A,B)}{\partial A}$ $\quad a_{22} = \frac{\partial g(A,B)}{\partial B}$

Analysis of perturbation without Diffusion

$$W = \begin{bmatrix} \delta A \\ \delta B \end{bmatrix}$$

- With this technique we study if perturbations grow or decay.

$$\boxed{\frac{\partial W}{\partial t} = \begin{bmatrix} \frac{\partial f}{\partial A} & \frac{\partial f}{\partial B} \\ \frac{\partial g}{\partial A} & \frac{\partial g}{\partial B} \end{bmatrix} \begin{bmatrix} \delta A \\ \delta B \end{bmatrix}} \rightarrow \begin{aligned} \frac{\partial \delta A}{\partial t} &= \frac{\partial f}{\partial A} \cdot \delta A + \frac{\partial f}{\partial B} \cdot \delta B \\ \frac{\partial \delta B}{\partial t} &= \frac{\partial g}{\partial A} \cdot \delta A + \frac{\partial g}{\partial B} \cdot \delta B \end{aligned}$$

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2. Studying system WITH Diffussion

- System is defined from $x=0$ to $x=L$.
- At $x=0$ and $x=L$: $\frac{\partial^2 A}{\partial x^2} = 0$ and $\frac{\partial^2 B}{\partial x^2} = 0$

$$\frac{\partial A}{\partial t} = f(A,B) + \boxed{\frac{\partial^2 A}{\partial x^2}} \cdot D_A$$

$$\frac{\partial B}{\partial t} = g(A,B) + \boxed{\frac{\partial^2 B}{\partial x^2}} \cdot D_B$$

STEP 1: Reducing second order partial derivative term to ODE simpler term.

Fourier Definitions

$$F(k) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) \cdot e^{-ikx} \, dx \qquad f(x) = \mathcal{F}^{-1}(F(k)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \cdot e^{ikx} \, dk \; (*)_1$$

- $$\frac{\partial A(x,t)}{\partial x} = \mathcal{F}^{-1}\left(\frac{\partial A(k,t)}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{2\pi} \cdot \int_0^L A(k,t) \cdot e^{ikx} \, dk\right] = \frac{1}{2\pi} \cdot \int_0^L A(k,t) \cdot \frac{d}{dx} \cdot e^{ikx} \, dk =$$

$$= \frac{1}{2\pi} \cdot \int_0^L ik \cdot A(k,t) \cdot e^{ikx} \, dK = ik \cdot \underbrace{\frac{1}{2\pi} \int_0^L A(k,t) \cdot e^{ikx} \, dk}_{A(x,t):\ \text{following inverse Fourier definition. } (*)_1} \Rightarrow \frac{\partial (A(x,t))}{\partial x} = ik \cdot A(x,t)$$

- $$\frac{\partial^2 A(x,t)}{\partial x^2} = \left(\frac{\partial A}{\partial x}\right)^2 = \mathcal{F}^{-1}\left[\left(\frac{\partial (ik \cdot A(x,t))}{\partial t}\right)\right] = \frac{\partial}{\partial x} \cdot \left[\frac{1}{2\pi} \cdot \int_0^L ik \cdot A(k,t) \cdot e^{ikx} \, dk =\right.$$

$$= \frac{1}{2\pi} \int_0^L ik \cdot A(k,t) \cdot \frac{\partial e^{ikx}}{\partial x} \, dk = \frac{1}{2\pi} \int_0^L (ik)^2 \cdot A(k,t) \cdot e^{ikx} \, dk = (ik)^2 \cdot \underbrace{\frac{1}{2\pi} \int_0^L A(k,t) \cdot e^{ikx} \, dk}_{A(x,t):\ \text{following inverse fourier definition.}}$$

$$\frac{\partial^2 A(x,t)}{\partial x^2} = -k^2 \cdot A(x,t) \longrightarrow \boxed{\begin{aligned} \frac{\partial^2 A(x,t)}{\partial x^2} &= -k^2 \cdot A \\ \frac{\partial^2 B(x,t)}{\partial x^2} &= -k^2 \cdot B \end{aligned}}$$

Partial second order terms reduced to simpler ODE terms to use in linear stability Analysis

STEP 2 : Defining boundary conditions

System is defined from $x=0$ to $x=L$.

$x=0$ ——— $x=L$

- Zero-flux or Neumann Boundary conditions are applied:

$$\boxed{\begin{aligned} &\frac{\partial^2 A}{\partial x^2}(x=0,t) = \frac{\partial^2 A}{\partial x^2}(x=L,t) = 0 \\ &\frac{\partial^2 B}{\partial x^2}(x=0,t) = \frac{\partial^2 B}{\partial x^2}(x=L,t) = 0 \end{aligned}}$$

- $\frac{\partial^2 A}{\partial x^2}$ is defined as:

$$\frac{\partial^2 A(x,t)}{\partial x^2} = -k^2 \cdot A(x,t) \quad \text{or}$$

$$\frac{\partial^2 A}{\partial x^2}(x,t) = -k^2 \cdot \frac{1}{2\pi} \int_0^L A(k,t) \cdot e^{ikx}\, dk.$$

To define boundary conditions we will use: this expression.

$\int u \cdot dv = uv - \int v\, du$

$$\frac{\partial^2 A(x,t)}{\partial x^2} = -k^2 \cdot \frac{1}{2\pi} \int_0^L A(k,t) \cdot e^{ikx} dk = -k^2 \cdot \frac{1}{2\pi} \int_0^L A(k,t) \cdot (\cos(kx) + i \sin(kx))\, dk =$$

$$= \frac{-k^2}{2\pi} \left(\int_0^L A(k,t) \cdot \cos(kx)\, dk + i \int_0^L A(k,t) \cdot \sin(kx)\, dk \right) =$$

(we integrate the 1st integral by parts)

$$\frac{\partial^2 A}{\partial x^2} = \frac{-k^2}{2\pi} \left(\left[\frac{\sin(kx)}{k} \cdot A(k,t) \right]_0^L - \int_0^L \frac{\sin(kx)}{k} \cdot \frac{\partial A(k,t)}{\partial k} \cdot dk + i \int_0^L A(k,t) \cdot \sin(kx)\, dk \right)$$

To define boundary conditions $\longrightarrow \frac{\partial^2 A}{\partial x^2}(0,t) = \frac{\partial^2 A}{\partial x^2} = (L,t) = 0$

function is defined with $\sin(kx)$: If $\sin(kx=0) \rightarrow \frac{\partial^2 A}{\partial x^2} = 0$

- for $x=0 \longrightarrow \sin(k \cdot 0) = \sin(0) = 0 \longrightarrow \frac{\partial^2 A(0,t)}{\partial x^2} = 0$
- for $x = L \longrightarrow \sin(k \cdot L) \rightarrow$ if $k = \frac{n \cdot \pi}{L} \rightarrow \sin(kL) = \sin\left(\frac{n \cdot \pi \cdot \cancel{L}}{\cancel{L}}\right) = \sin(n\pi)$

$\sin(n\pi) = 0$ for all $n \in \{0, N\}$

Therefore if $k = \frac{n\pi}{L}\ \forall n \in \{0, N\} \rightarrow \sin(0,t) = \sin(L,t) = 0$

Therefore if $k = \frac{n\pi}{L}\ \forall_{n \in \{0,N\}} \rightarrow \frac{\partial^2 A}{\partial x^2}(x=0,t) = \frac{\partial^2 A}{\partial x^2}(x=L,t) = 0$

Same applies to $\frac{\partial^2 B}{\partial x^2}$.

for $k = \frac{n\pi}{L}\ \forall_{n \in \{0,N\}} \rightarrow \frac{\partial^2 B}{\partial x^2}(x=0,t) = \frac{\partial^2 B}{\partial x^2}(x=L,t) = 0$

Zero flux boundary condition is applied when k is restricted to $k = \frac{n\pi}{L}\ \forall n \in \{0, N\}$

STEP 3 : LINEAR STABILITY ANALYSIS WITH DIFFUSSION

$$\frac{\partial A(x,t)}{\partial t} = \frac{\partial \delta A}{\partial t} = \frac{\partial f}{\partial A}\cdot \delta A + \frac{\partial f}{\partial B}\cdot \delta B - D_A k^2 \cdot \delta A$$

$$\frac{\partial B(x,t)}{\partial t} = \frac{\partial \delta B}{\partial t} = \frac{\partial g}{\partial A}\cdot \delta A + \frac{\partial g}{\partial B}\cdot \delta B - D_B \cdot k^2 \cdot \delta B$$

$\Bigg\}\ k = \frac{n\pi}{L}\ \forall_{n \in \{0, \mathbb{N}\}}$

⚠ Boundary condition.

$$\delta A = A_0 \cdot e^{\sigma t} \cdot e^{ikx}$$

$$\delta B = \underbrace{B_0 \cdot e^{\sigma t}}_{\text{Amplitude term}} \cdot \underbrace{e^{ikx}}_{\text{Oscillations in space term.}}$$

If $\sigma > 0$: Perturbation grows $\rightarrow \frac{\partial \delta A}{\partial t} > 0$ → Unstable system

If $\sigma < 0$ Perturbation decays $\rightarrow \frac{\partial \delta A}{\partial t} < 0$ → Stable system.

$$\text{Jacobian} = \begin{bmatrix} \partial\left(\frac{\partial A}{\partial t}\right)/\partial A & \partial\left(\partial A/\partial t\right)/\partial B \\ \partial\left(\partial B/\partial t\right)/\partial A & \partial\left(\partial B/\partial t\right)\partial B \end{bmatrix} = \begin{bmatrix} \frac{\partial \delta A}{\partial A} - D_A k^2 & \partial \delta A/\partial B \\ \partial \delta B/\partial A & \partial \delta B/\partial B - D_B k^2 \end{bmatrix}$$

$$I\sigma = \text{Jacobian} \rightarrow \text{Jacobian} - \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} = 0$$

$$\det \begin{bmatrix} \frac{\partial \delta A}{\partial A} - D_A k^2 - \sigma & \partial \delta A/\partial B \\ \partial \delta B/\partial A & \partial \delta B/\partial B - D_B k^2 - \sigma \end{bmatrix} = 0$$

for $k = \frac{n\pi}{L}$ $\forall_{n \in \{0, \mathbb{N}\}}$

if $\sigma > 0$: Turing Instability (Pattern forms)

If $\sigma < 0$ Stable system (no pattern)

LINEAR STABILITY ANALYSIS ON TURING EXAMPLE (2EQ SYSTEM)

$$\left.\begin{aligned} \frac{dA}{dt} &= 5A - 6B + D_A \cdot \frac{\partial^2 A}{\partial x^2} \\ \frac{dB}{dt} &= 6A - 7B + D_B \cdot \frac{\partial^2 B}{\partial x^2} \end{aligned}\right\}$$

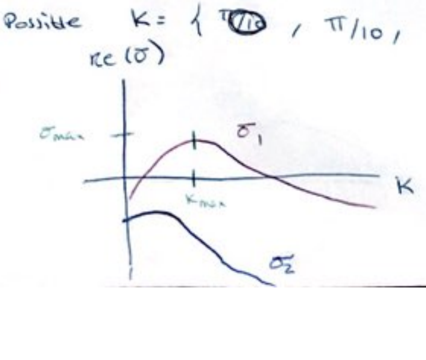
$$I\sigma = \text{Jacobian} \Rightarrow \det\left(\begin{bmatrix} 5 - D_A k^2 - \sigma & -6 \\ 6 & -7 - D_B k^2 - \sigma \end{bmatrix}\right) = 0$$

$$(5 - D_A k^2 - \sigma)(-7 - D_B k^2 - \sigma) - (-36) = 0$$

Solve this equation for σ for values of K when $K = \frac{n\pi}{L}$

In this case assume $L = 10$:

→ Possible $K = \{ 0, \pi/10, \pi/5, 3\pi/10 \cdots \}$



• K_{max} will be the wavenumber of the pattern formed.