# Quantile on Quantiles*

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**Abstract**

Distributional effects provide insight into how a treatment impacts inequality. This paper extends this notion in two ways. First, it recognizes that inequality spans multiple dimensions—for example, within and between groups—with treatments potentially influencing both and creating trade-offs. Second, it addresses the challenge of ranking heterogeneous groups, which depends on the social welfare function. To this end, I introduce a model to simultaneously study distributional effects within and between groups while remaining agnostic about this function. The model consists of a quantile function with two indices: one capturing within-group heterogeneity, the other addressing between-group differences. I propose a two-step quantile regression estimator and show that it converges to a bivariate Gaussian process. In an empirical application, I find that business training in Kenya has large positive effects, mostly among successful firms in the best-performing markets, suggesting complementarities between business and market performance.

## 1 Introduction

Since Koenker and Bassett (1978), quantile regression has been widely used for policy evaluation. Quantile treatment effects are particularly appealing when policymakers care about distributional consequences, such as targeting low-income households or addressing inequality rather than maximizing aggregate outcomes. With grouped data, however, focusing solely on the unconditional distribution overlooks essential aspects of inequality: it has within- and between-group components, and policies can create a trade-off between them. For instance, if geographical areas define the groups, it is theoretically possible to reduce within-region inequality by moving people across space into a more segregated spatial allocation, while leaving the unconditional income distribution unchanged. Yet most applied work has predominantly focused on treatment effect heterogeneity along a single dimension, either within or between groups. For example,

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there is evidence that trade shocks have heterogeneous effects both within and between regions (see, e.g., Chetverikov et al., 2016; Antràs et al., 2017; Galle et al., 2023). Similarly, place-based policies have been found to stimulate local growth and employment (Becker et al., 2010; Busso et al., 2013; Ehrlich and Seidel, 2018), but at the same time they have increased within-region inequality (Lang et al., 2023; Albanese et al., 2023).[1] These findings make two points. First, both within- and between-group inequality matter for understanding policy impacts. Second, they highlight that the two dimensions are not independent: policies can improve outcomes along one dimension while worsening them along the other. This concern is also reflected in policy priorities: the United Nations' Sustainable Development Goals call for reducing inequality "within and among countries," and the European Union's cohesion policy explicitly targets both regional convergence and equity within regions. Understanding policy impacts, therefore, requires a framework that models both dimensions of inequality jointly.

This paper introduces a novel model and estimator to study treatment effect heterogeneity and inequality both within and between groups simultaneously, where groups may be defined by regions, firms, or industries. I propose a two-dimensional quantile model, where one dimension captures within-group heterogeneity through group-specific conditional quantile functions, while the other aggregates across groups by modeling the quantile function of these group-level quantiles. This yields a two-dimensional quantile function that, for instance, describes how group-level conditional medians vary across groups. The framework thus provides an econometric tool to jointly model within- and between-group heterogeneity, offering new insights into the dynamics of inequality.

Estimation proceeds in two stages. In the first stage, I run quantile regressions separately for each group, using covariates that vary within groups (see, e.g., Galvao and Wang, 2015; Chetverikov et al., 2016; Melly and Pons, 2025). In the second stage, the fitted values from the first step are regressed on all covariates using quantile regression, separately for each quantile. Intuitively, the first stage captures heterogeneity within groups, while the second aggregates these results to recover heterogeneity across groups at specific points of the within distribution. The estimator is flexible as coefficients may vary freely along both dimensions, and groups' relative ranks can change along the within-group distribution.

Establishing the asymptotic properties of the estimator involves three challenges: the non-smoothness of the objective function, the generated dependent variable in the second stage, and

[1] Place-based policies are policies implemented by government or supranational organizations with the aim to promote economic development within a given area (see Neumark and Simpson, 2015 for an overview on the topic).

the different convergence rates of the two steps. The first stage, which uses only observations within a group, converges at rate $\sqrt{n}$, where $n$ is the number of observations per group. By contrast, the second stage, which captures heterogeneity across groups, converges at rate $\sqrt{m}$, where $m$ is the number of groups. The setting is related to Chen et al. (2003), who analyze estimators with non-smooth objectives and nonparametric first steps, and to Volgushev et al. (2019) and Galvao et al. (2020), who provide detailed Bahadur representations for the first step quantile regression.[2] Building on this work and the process results in Angrist et al. (2006), I establish weak convergence in a framework where both $n$ and $m$ diverge, subject to the mild condition $\frac{\sqrt{m}\log n}{n} \to 0$. I also propose an inference procedure for uniform hypotheses.

Alongside these theoretical results, the paper demonstrates how the framework can be used for policy evaluation when trade-offs between inequality dimensions are central. The model provides a flexible tool for analyzing how policies affect the outcome distribution along multiple dimensions, and it also offers valuable insights for descriptive analyses of inequality within and between groups—which is a matter of considerable policy importance. Compared to variance decompositions or simple comparisons of mean or median outcomes across groups, the two-level quantile function provides a richer picture of multidimensional inequality. It can reveal which parts of the within-group distribution drive inequality between groups. Moreover, the framework helps address the challenge of ranking heterogeneous groups when there is no stochastic dominance relationship between them. For example, Chetty and Hendren (2018a,b) show that the same commuting zone may be associated with high mobility at one part of the parental distribution and low mobility at another, implying that the definition of a "good" or "bad" neighborhood depends on the within-group rank. In such cases, groups cannot be ordered unambiguously using standard approaches. The proposed method enables comparisons by ranking groups at multiple points of the distribution and, when combined with a welfare function, yields a welfare-based measure of group performance.

Furthermore, under the assumption of rank invariance across treatment states, the framework identifies individual treatment effects. This feature makes it particularly useful for optimal treatment assignment when policymakers maximize a rank-dependent social welfare function and baseline outcomes are not observed (see, e.g., Manski, 2004; Kitagawa and Tetenov, 2018, 2021). In such cases, the assignment rule exploits effect heterogeneity over the outcome distribution,

[2]See also Ma and Koenker (2006), Chen et al. (2021), and Bhattacharya (2020) on quantile regression with generated regressors or dependent variables. In contrast, my setting allows the dimension of the first stage to grow with the number of groups, features two stages converging at different rates, and establishes uniform asymptotic results.

both within and between groups. The model is relevant when treatment is assigned either at the group or individual level. Group-level interventions are common in economics—for instance, place-based policies and infrastructure projects such as highways, railways, and sanitation systems affect all individuals in a locality, and educational policies are often implemented at the jurisdiction, school, or classroom level.

As an empirical application, I extend the analysis of McKenzie and Puerto (2021) by assessing the impact of business training on income from work in Kenya, considering distributional effects both within and between markets. The objective of the experiment is to improve the outcomes of small businesses. A standard quantile regression of income on the treatment dummy captures heterogeneity across firms, but it does not account for whether a median firm (in the unconditional distribution) operates in a high- or low-performing market, even though the treatment may affect these firms differently. Poorly performing markets may face disadvantages such as location or low consumer traffic, and complementarities between individual ability and market quality may also be important—factors that a traditional quantile regression cannot capture. Allowing for heterogeneity along multiple dimensions with the two-dimensional quantile model provides additional insights. The results show larger treatment effects for firms that perform well within successful markets. Specifically, effects increase with both a firm's rank within its market and the market's overall rank, providing evidence of complementarities between individual and group ranks. This finding is useful for policymakers in deciding which firms and markets to target, depending on their objective. For example, if the goal is to maximize total income growth, reduce inequality, or balance the two, the treatment can be directed toward different parts of the distribution or groups.

Distributional effects and inequalities within groups have been widely studied in both theoretical and applied work. On the methodological side, Galvao and Wang (2015), Chetverikov et al. (2016), and Melly and Pons (2025) develop estimators to model heterogeneity in treatment effects along the within-group distribution.[3] On the applied side, Autor et al. (2021) and Friedrich (2022) analyze how import competition and trade shocks affect wage distributions within local labor markets and firms, respectively. Helpman et al. (2017) show that most of the rise in wage inequality occurs within sectors and occupations, largely driven by differences across firms. Further work, such as Autor et al. (2016) and Engbom and Moser (2022), examines how minimum wages affect within-state inequality in the US and Brazil.

By contrast, some studies emphasize inequality across groups. Haltiwanger et al. (2024)

[3] Galvao and Wang (2015) consider a panel data setting, where groups are individuals and time periods are the within-group units.

attribute most of the recent rise in earnings inequality to differences across industries. Research on place-based policies provides further examples, showing how interventions to support lagging or underdeveloped regions shape disparities across regions (see, e.g., Busso et al., 2013; Ehrlich and Seidel, 2018; Ehrlich and Overman, 2020).

Only a few papers consider both within- and between-group inequality, and these mostly take a descriptive approach. Typically, they rely on variance decompositions, separating overall inequality into within- and between-group components rather than examining both dimensions simultaneously. For example, Bourguignon and Morrisson (2002) analyze the historical evolution of income inequality both within and between countries, and Akerman et al. (2013) study wage inequality between and within firms, sectors, and occupations.

This paper also contributes to the theoretical literature on multidimensional unobserved heterogeneity, where coefficients can vary along more than one dimension. Fernández-Val, Gao, Liao, and Vella (2022) introduce a model that allows for both within- and between-group heterogeneity. Within-group heterogeneity is captured by letting coefficients vary over the outcome distribution in a distribution regression framework, while group-specific coefficients capture between-group heterogeneity. A limitation of their approach, however, is that identification of between-group heterogeneity relies on within-group variation in the variable of interest. Arellano and Bonhomme (2016) study a fixed-effects model in which group effects are modeled as latent variables using a correlated random-effects approach. Treatment effect heterogeneity can arise from dependence on an individual rank variable and from low-dimensional latent group effects. Other related contributions include Frumento, Bottai, and Fernández-Val (2021), who allow individual-level variables to affect the within-group distribution while group-level variables shift the between-group distribution, and Liu (2024), who study a panel data model where the effect of individual-level variables depends on a group-level rank variable, with the individual error entering additively. In contrast to these models, the framework proposed in this paper allows the effects of both individual- and group-level variables to vary freely along both dimensions, without imposing any restrictions on the dimensionality of group effects.

The remainder of the paper is structured as follows. Section 2 introduces the model and Section 3 discusses how it can be used for policy evaluation and optimal treatment assignment. Section 4 presents the estimator, and Section 5 develops its asymptotic properties. Section 6 analyzes the finite-sample performance in a Monte Carlo study. Section 7 illustrates the approach with an empirical application, and Section 8 concludes.

## 2 Model

Consider a dataset with two dimensions where $j = 1, \ldots, m$ indexes the groups and $i = 1, \ldots, n$ denotes the individuals.[4] I start by considering a naive version of the model imposing strong assumptions on the evolution of the group ranks over the within distribution. Later, I relax these assumptions and present the more general model considered in this paper.

**Simple model** - A naive attempt to construct a model specifies the following structural function for the outcome variable $y_{ij}$ given the individual-level variables $x_{1ij}$, and the group-level variables $x_{2j}$:

$$y_{ij} = q(x_{1ij}, x_{2j}, v_j, u_{ij}), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \tag{1}$$

where $q(\cdot)$ is strictly increasing in the rank variables $u_{ij}$ and $v_j$. The individual-level rank variable $u_{ij}$ is responsible for differences in outcomes between individuals with the same observable characteristics, including group membership. Conversely, $v_j$ is responsible for differences across groups. Further, for the sake of this illustration let

$$\begin{aligned} u_{ij}|x_{1ij}, x_{2j}, v_j &\sim U(0,1), \\ v_j|x_{1ij}, x_{2j} &\sim U(0,1). \end{aligned}$$

Since $v_j$ varies only between groups and $u_{ij}$ is standard uniform distributed within each group, $u_{ij}$ and $v_j$ are independent conditional on the covariates:

$$u_{ij} \perp\!\!\!\perp v_j | x_{1ij}, x_{2j}.$$

Conditional on $x_{ij} = (x'_{1ij}, x'_{2j})'$ and $v_j$, $q(x_{1ij}, x_{2j}, v_j, u_{ij})$ is strictly monotonic with respect to $u_{ij}$ so that

$$Q(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_j) = q(x_{1ij}, x_{2j}, v_j, \tau_1) \tag{2}$$

is the $\tau_1$-conditional quantile function of the outcome $y_{ij}$ conditional on $x_{1ij}, x_{2j}$, and $v_j$. If there are no $x_{1ij}$ variables, the $\tau_1$-conditional quantile function of $y_{ij}$ reduces to the unconditional percentiles of the outcome in group $j$. Further, as $q(\cdot)$ is strictly monotonic with respect to $v_j$, we can construct the $\tau_2$-conditional quantile function of the $\tau_1$-conditional quantile function of the outcome within each group:

$$Q(\tau_2, Q(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_j)|x_{1ij}, x_{2j}) = q(x_{1ij}, x_{2j}, \tau_2, \tau_1). \tag{3}$$

[4] For ease of notation, I assume a balanced dataset. However, the results generalize to heterogeneous group sizes.

Thus, $\tau_2$ ranks the groups (conditional on the covariates) according to their conditional quantiles. A caveat of this model is that it imposes strong restrictions on the evolution of the group ranks at different values of $\tau_1$. More precisely, the ranks are assumed to be constant over $\tau_1$. Suppose for a moment that there are no covariates and take groups $j = \{h, l\}$ with $v_h$ and $v_l$ such that $v_h > v_l$. Strict monotonicity of $q(v_j, \tau_1)$ with respect to $v_j$ implies

$$q(v_h, \tau_1) > q(v_l, \tau_1), \quad \forall \tau_1 \in (0, 1).$$

Hence, if a group has a higher first decile, it must also have a higher ninth decile. This would be satisfied, for example, if the outcome is generated by $y_{ij} = h(x_{1ij}, x_{2j}, u_{ij}) + f(x_{1ij}, x_{2j}, v_j)$, where $h(\cdot)$ and $f(\cdot)$ are strictly increasing in their third argument. That is, if conditional on the covariates, all groups share the same outcome distribution up to a location parameter. Essentially, this requires that $v_j$ enter as a pure location shifter conditional on $(x_{1ij}, x_{2j})$.[5]

The restriction on the evolution of the ranks over the distribution of $\tau_1$ is a consequence of the strict monotonicity assumption on $q(\cdot)$ with respect to the *scalar* rank variable $v_j$. Given that this assumption is not satisfied in most real-world scenarios, in this paper, I allow for the possibility that conditional on covariates, groups can differ in more moments than their mean. In this way, groups can be at different ranks at different values of $\tau_1$.

A straightforward extension would include a bivariate $v_j$, where one element determines the mean and the other the variance. This corresponds to $y_{ij} = h(x_{1ij}, x_{2j}, v_j^{(1)}, u_{ij}) + f(x_{1ij}, x_{2j}, v_j^{(2)})$. Hence, conditional on the covariates $x_{1ij}$ and $x_{2j}$, the outcome has the same distribution across groups, but different locations and variances. The heterogeneity in the variances arises due to the interaction between the individual rank variable $u_{ij}$ and the group rank variable $v_j^{(1)}$. In this example, it is not feasible to completely separate $u_{ij}$ and $v_j$, and the group rank can vary over $\tau_1$. The $\tau_1$-conditional quantile function in each group is $q(x_{1ij}, x_{2j}, v_j, \tau_1) = h(x_{1ij}, x_{2j}, v_j^{(1)}, \tau_1) + f(x_{1ij}, x_{2j}, v_j^{(2)})$. Yet, we can still construct a $\tau_2$-conditional quantile function by noting that for each $\tau_1$, there exist a scalar-valued function $v_j(\tau_1)$ such that $q(x_{1ij}, x_{2j}, v_j, \tau_1) = q(x_{1ij}, x_{2j}, v_j(\tau_1), \tau_1)$. With proper normalization and imposing monotonicity with respect to this scalar rank variable, we can construct the $\tau_2$-conditional quantile function. To give an illustration, let $y_{ij} = u_{ij}(v_j^{(1)} + \epsilon) + v_j^{(2)}$ for some scalar $\epsilon$. Then, $q(v_j, \tau_1) = \tau_1\epsilon + \tau_1 v_j^{(1)} + v_j^{(2)}$ is the $\tau_1$-conditional quantile function. It follows directly that $v_j(\tau_1) = \tau_1 v_j^{(1)} + v_j^{(2)}$ is the scalar-valued function that ranks groups at $\tau_1$. Clearly, the model can be further generalized. For instance, with a trivariate

[5] This assumption could also be satisfied if there was no overlap between groups. I rule out this possibility since this is not satisfied in most economic applications. A sufficient condition that guarantees a constant ranking over $\tau_1$ is the existence of a strict first-order stochastic dominance relationship between groups.

$v_j$, we could allow groups to be heterogeneous with respect to their skewness. Similarly, with an infinitely dimensional $v_j$, it is possible to allow for unrestricted heterogeneity between groups, which is the approach that I take in the paper.

**A more general model** - In this paper, I do not restrict the heterogeneity between groups and allow $v_j$ to be possibly infinite-dimensional. In this way, I allow the group-level conditional quantile functions to vary unrestricted with respect to $\tau_1$ so that groups can be at different ranks for different values of $\tau_1$. At the same time, I maintain the assumptions on the scalar $u_{ij}$, and the $\tau_1$-conditional quantile function in equation (2) remains unchanged.[6]

To make the problem concrete, I consider the following linear specification:

$$y_{ij} = x_{1ij}'\beta(u_{ij}, v_j) + x_{2j}'\gamma(u_{ij}, v_j) + \alpha(u_{ij}, v_j), \tag{4}$$

where $\alpha(u_{ij}, v_j)$ is the intercept. It follows that the $\tau_1$-conditional quantile function can be equivalently written as

$$Q(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_j) = x_{1ij}'\beta(\tau_1, v_j) + x_{2j}'\gamma(\tau_1, v_j) + \alpha(\tau_1, v_j), \tag{5}$$

where only the sum of the last two terms is identified since $x_{2j}$ does not exhibit variation within groups.

Modeling the heterogeneity between groups still requires restricting the relationship between the $\tau_1$-conditional quantile function and the possibly infinite dimensional vector $v_j = (v_j^{(1)}, v_j^{(2)}, \dots)$. As in the bivariate example, for each $\tau_1 \in (0,1)$, I assume that there exists a scalar-valued function $v_j(\tau_1)$ such that

$$q(x_{1ij}, x_{2j}, v_j, \tau_1) = q(x_{1ij}, x_{2j}, v_j(\tau_1), \tau_1).$$

With proper normalization and imposing strict monotonicity of $q(x_{1ij}, x_{2j}, v_j(\tau_1), \tau_1)$ with respect to $v_j(\tau_1)$, yields the $\tau_2$-conditional quantile function of $Q(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_j)$,

$$Q(\tau_2, Q(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_j)|x_{1ij}, x_{2j}) = x_{1ij}'\beta(\tau_1, \tau_2) + x_{2j}'\gamma(\tau_1, \tau_2) + \alpha(\tau_1, \tau_2), \tag{6}$$

which I refer to as the $(\tau_1, \tau_2)$-conditional quantile function. Model (6) allows for substantial heterogeneity. All coefficients have two quantile indices: one for the heterogeneity across groups ($\tau_2$) and one for the heterogeneity within groups ($\tau_1$). The outer quantile function is the

[6]With a multidimensional $u_{ij}$, it is not possible to identify the structural function. In such a case, the quantile function identifies the individuals that have an outcome equal to the corresponding quantile (see Hoderlein and Mammen, 2007).

conditional quantile function of the conditional quantile function of the outcome within each group, $Q(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_j)$. Importantly, the multidimensionality of $v_j$ implies that $\tau_2$ ranks the groups (conditional on the covariates) according to *their* $\tau_1$-conditional quantile function, and group ranks can change over the within distribution.[7]

It is important to note that the assumptions of the existence of the scalar-valued function $v_j(\tau_1)$ and the scalar rank variable $u_{ij}$, and strict monotonicity of $q(\cdot)$ are necessary for providing a structural interpretation of the conditional quantile function in equation (6) (see Matzkin, 2003; Torgovitsky, 2015). These assumptions imply rank invariance along both dimensions—that is, the ranking of units within groups and the ranking of groups at specific values of $\tau_1$ remain unaffected by changes in $x_{1ij}$ or $x_{2j}$. However, even without these assumptions, model (6) continues to identify well-defined parameters. More precisely, with rank invariance, the coefficients can be interpreted as individual effects, whereas without these assumptions, only the effects of the distribution are identified. Below, I discuss the interpretation of the parameters in both cases.

The main advantage of this way of ordering groups at different values of $\tau_1$ is that it remains agnostic with respect to the social welfare function of the policymaker. When groups contain heterogeneous agents and a total ordering is not feasible, ranking them is a non-trivial task without specifying a social welfare function. A utilitarian policymaker would rank the groups according to their mean. However, using the mean (or median) outcome to rank groups is unsatisfactory for at least two reasons. First, an equality-minded policymaker is not indifferent over two allocations with the same mean but different variances. On the contrary, it is possible to find an allocation with a smaller mean that is strictly preferred to an alternative assignment with a higher variance. Second, this ranking does not provide information about which part of the within distribution is driving the differences between groups, and a few outlying observations could have a large effect on this measure of between-group inequality. Later, I provide an example where comparing averages across regions shows substantial income differences across

[7] The two-dimensional condtional quantile function is directly related to the conditional cdf of the outcome $y$ by the following transformation:

$$F_{Y|X}(y|x) = \int_0^1 \int_0^1 1\{q(x, \tau_1, \tau_2) \leq y\} d\tau_2 d\tau_1. \tag{7}$$

Inverting the conditional cdf in equation (7) yields the one-dimensional conditional quantile function. Hence, no information is lost when modeling both dimensions. Further, this implies that it is always possible to present the results with a different and/or unified rank variable. For example, one might be interested in looking at treatment effect heterogeneity over two dimensions, where the second dimension ranks groups according to their median (or any other percentile). This requires estimating individual treatment effects. Then, the groups can be sorted by their median rank, and the treatment effect at different values of $\tau_1$ can be plotted against their group rank.

regions. However, a large part of these differences are driven by high top wages in a few regions. While comparing regional medians it more robust to outliers, it compares regions at a single point of the within distribution and might fail to capture differential labor market situations for a large portion of the workers. I show that this is the case mostly for low-income workers. These weaknesses also extend to other methods used to assess within and between heterogeneity, such as variance decompositions. Instead, with the two-dimensional quantile function, I provide information about which part of the within distribution is driving the between heterogeneity. Specifically, if groups were heterogenous only due to different location parameters, the group ranks would remain stable over the distribution of $\tau_1$. By contrast, if the shape of the conditional distribution varies over groups, we expect group ranks to change over $\tau_1$. This implies that the between heterogeneity depends on the within dimension $\tau_1$, which also implies that a decomposition is no longer possible. Clearly, a unified notion of group ranks can also be constructed. For instance, in Section 3, I show that a social welfare function can be used to assign welfare weights to each group, enabling the construction of a unified measure of group order.

The price to pay for this flexibility is that the interpretation of the coefficients becomes more involved as the groups' ranks vary over $\tau_1$. Further, with individual-level covariates, the ranks may vary even within the groups.[8] Yet, this last point is common with quantile models. The coefficient vectors $\beta(\tau_1, \tau_2)$ and $\gamma(\tau_1, \tau_2)$ tell how the $(\tau_1, \tau_2)$-conditional quantile function responds to a change in $x_{1ij}$ or $x_{2j}$ by one unit. To facilitate the interpretation, it is helpful to fix $\tau_1$. For example, $\beta(0.5, \tau_2)$ gives the effect of $x_{1ij}$ on the $\tau_2$-conditional quantile function of the group (conditional) medians. Hence, it allows us to assess the effect of $x_{1ij}$ on the distribution of group medians, with groups with the highest medians positioned at the top and those with the lowest medians at the bottom of the distribution.

Interpreting these coefficients as the effects for individuals at a specific point of the distribution requires rank invariance over treatment states.[9] Given the multi-dimensionality of the model, rank invariance must hold both within groups and between groups at a given within rank. Rank invariance within groups requires that within-group ranks do not change over treatment states. Instead, rank invariance between groups requires that for each $\tau_1$, the ranks between groups remain stable over treatment states. While this is a strong assumption, there are cases

[8] For example, a group (e.g., region) might have a different rank for high- and low-educated individuals.

[9] A rank invariance (or rank preservation) assumption is used, for example, in Heckman et al. (1997) and Chernozhukov and Hansen (2005). Without a rank stability assumption, individual treatment effects are not identified. Chernozhukov et al. (2023) suggest conditional prediction intervals that can be obtained with a relaxation of this assumption.

where unconditional rank invariance is violated but still holds within and between groups. For example, effect heterogeneity over the distribution of groups could violate rank invariance in the population. With rank invariance, the coefficients can be interpreted as individual effects, and $\beta(\tau_1, \tau_2)$ (or $\gamma(\tau_1, \tau_2)$) gives the quantile effects for individuals at the $\tau_1$th percentile of their groups, belonging to a group at the $\tau_2$th percentile, where this second distribution is viewed from their perspective. Clearly, conditional on being in the lower tail of the within-group distribution, it is better to be in groups with relatively high low wages and a compressed wage distribution. Differently, individuals at the top of the within-group wage distribution will favor groups with high top wages.

**Example 1. Without covariates**

I now consider a special case of model (6) where there are no regressors, and the model provides a quantile function of the outcome over two dimensions. The $\tau_1$-conditional quantile function in group $j$ simplifies to

$$Q(\tau_1, y_{ij}|v_j) = \alpha(\tau_1, v_j),$$

where $Q(\tau_1, y_{ij}|v_j)$ is $\tau_1$th percentile of the outcome $y_{ij}$ in group $j$. It follows directly that

$$Q(\tau_2, Q(\tau_1, y_{ij}|v_j)) = \alpha(\tau_1, \tau_2)$$

is the $\tau_2$th percentile, over all groups, of the $\tau_1$th group percentiles.

Let groups be defined by geographical regions, and let the outcome $y_{ij}$ be the income earned by individual $i$ in region $j$. This model enhances our understanding of inequality within and between these regions and sheds light on the variation of the within percentiles of the outcome over groups. For example, if differences are predominantly within regions, we would observe significant variations along the $\tau_1$ dimension and relatively smaller differences along the $\tau_2$ dimension. Additionally, this model enables us to determine whether heterogeneity between regions becomes more pronounced for higher values of $\tau_1$, providing insights into how the higher end of the wage distribution varies across groups. It thus offers a nuanced perspective on the dynamics of inequality within and between geographical regions.

These heterogeneous coefficients are identified by two-step quantile regression. (i) The conditional quantile function in each group is identified by $\tau_1$ quantile regressions of $y_{ij}$ on $x_{1ij}$ for each group separately. (ii) The second dimension is identified by $\tau_2$ quantile regressions of the fitted values from the first-stage on $x_{1ij}$ and $x_{2j}$.

**Remark 1 (Within versus between distributions).** The model discussed in this paper focuses on simultaneously estimating the effect on the distribution of the outcome within and between groups. Melly and Pons (2025) consider a similar model where the heterogeneity arises from the individual rank variable $u_{ij}$ and the focus is on the within distribution. Starting from equation (6) and assuming that $(x_{1ij}, x_{2j}) \perp\!\!\!\perp v_j$, it is possible to obtain their model by integrating over $v_j$:[10]

$$\begin{aligned}\mathbb{E}\left[Q\left(\tau_1, y_{ij}|x_{1ij}, x_{2j}, v_i\right)|x_{1ij}, x_{2j}\right] =& x_{1ij}' \int \beta(\tau_1, v) dF_V(v) + x_{2j}' \int \gamma(\tau_1, v) dF_V(v) \\ &+ \int \alpha(\tau_1, v) dF_V(v) \\ =& x_{1ij}' \bar{\beta}(\tau_1) + x_{2j}' \bar{\gamma}(\tau_1) + \bar{\alpha}(\tau_1).\end{aligned}$$

Hence, when model (6) holds, they identify the average effects over groups at the $\tau_1$ quantile of the within distribution.

If only the heterogeneity of group averages is of interest, one could consider the conditional quantile function of the conditional expectation function in each group. Starting from equation (4) and assuming that $(x_{1ij}, x_{2j}) \perp\!\!\!\perp u_{ij}$, we attain

$$Q(\tau_2, \mathbb{E}_{i|j}[y_{ij}|x_{1ij}, x_{2j}]|x_{1ij}, x_{2j}) = x_{1ij}' \bar{\beta}(\tau_2) + x_{2j}' \bar{\gamma}(\tau_2) + \bar{\alpha}(\tau_2),$$

with

$$\begin{aligned}\mathbb{E}_{i|j}[y_{ij}|x_{1ij}, x_{2j}] =& x_{1ij}' \mathbb{E}_{i|j}[\beta(u_{ij}, v_j)|x_{1ij}, x_{2j}] + x_{2j}' \mathbb{E}_{i|j}[\gamma(u_{ij}, v_j)|x_{1ij}, x_{2j}] + \mathbb{E}_{i|j}[\alpha(u_{ij}, v_j)|x_{1ij}, x_{2j}] \\ =& x_{1ij}' \bar{\beta}(v_j) + x_{2j}' \bar{\gamma}(v_j) + \bar{\alpha}(v_j),\end{aligned}$$

where the notation $\mathbb{E}_{i|j}$ stresses that the expectation is taken conditional on the group. This setting is common in empirical research where only aggregated data is available.

If the primary focus is heterogeneity between groups, one may prefer to study heterogeneities in group medians rather than averages. This choice aligns with the framework suggested in this paper, where the specific quantile of $\tau_1 = 0.5$ is considered.

# 3 Application to Policy Evaluation and Optimal Treatment Assignment

In this section, I explore two conceptual frameworks that align with the model proposed in this paper. These examples parallel the different interpretations of quantile regression results. The first framework focuses on the impact on the distribution, while the second requires rank invariance to recover individual effects.

[10] Note that Melly and Pons (2025) include the intercept in $x_{2j}$.

### 3.1 Distributional Effects: Policy Evaluation

In settings with an inequality-minded policymaker (see e.g. Kitagawa and Tetenov, 2021), researchers often consider a rank-dependent welfare function where welfare is a function of a weighted average of the outcomes with weights that depend on the rank of an individual in the population:

$$W := \int \int Y_{ij} \cdot w(\text{Rank}(Y_{ij}))di\ dj. \tag{8}$$

Such a welfare function can be equivalently written as a weighted average of the outcomes:

$$W = \int_0^1 q(\theta)w(\theta)d\theta, \tag{9}$$

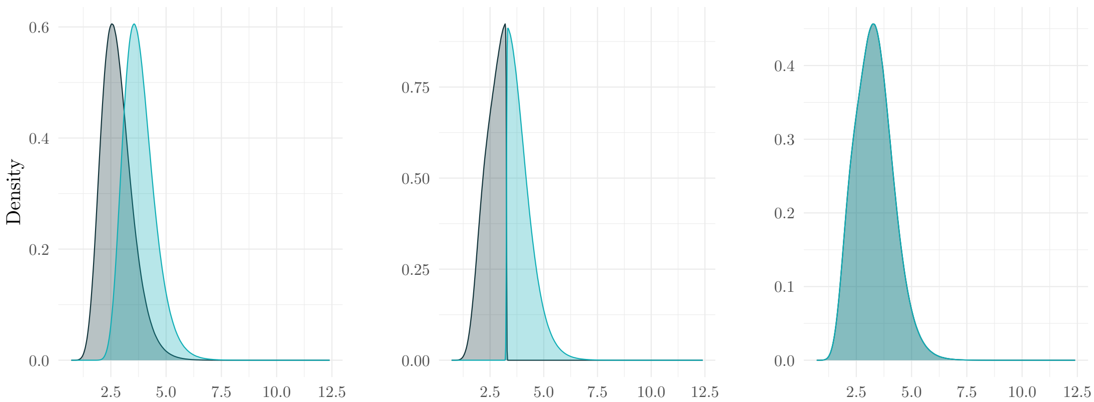
where $q(\theta)$ is the unconditional quantile function of $y$ and the weights $w(\theta)$ depend on the population quantiles $\theta$. This social welfare function is quite general and comprises, for example, the extended Gini family. However, with grouped data, this welfare function does not consider the structure of the data and ignores the role of inequality within and between groups. For instance, all allocations with the same marginal distribution of the outcome yield identical welfare, irrespective of the distribution within and between the groups; thus, the policymaker should be indifferent between any such allocations. Figure 1 shows three hypothetical allocations for two groups. In Allocation A, both groups have the same outcome distribution, with the exception that the blue group has a higher mean. Allocation B considers a scenario with high segregation where high-outcome individuals belong to the blue group, and low-outcome individuals are in the grey group. This allocation yields a lower within-group inequality, yet between-group inequality is larger. Allocation C eliminates inequality between groups by randomly assigning individuals to either group. However, this allocation has the highest within-group inequality. Despite the visible difference in within and between inequality, these allocations have the same marginal distribution of the outcome; hence, if welfare is measured by equation (9), these allocations are welfare equivalent.

To allow for potential trade-offs over different dimensions of inequality, I consider a welfare function that is a weighted average of the outcomes with weights that depend on both within and between ranks:

$$W = \int_0^1 \int_0^1 q(\tau_1, \tau_2) \cdot w(\tau_1, \tau_2)d\tau_2 d\tau_1, \tag{10}$$

where $q(\tau_1, \tau_2)$ is the two-level quantile function and $w(\tau_1, \tau_2)$ are the welfare weights assigned to units at the $\tau_1$ percentile in their group belonging to a group at the $\tau_2$ percentile.

Figure 1: Allocations with the Same Marginal Outcome Distribution




(a) Allocation A (b) Allocation B (c) Allocation C

*Notes*: The figure shows the kernel densities of two groups in three different allocations with the same outcome distribution but different redistribution between the groups.

The main advantage of this welfare function is its ability to account for the interdependencies between within-group and between-group inequality. For instance, when regions define groups, "*local inequality is actually the inverse of area-level income segregation*" (Glaeser et al., 2009). Whether reducing inequality in one dimension at the cost of increasing it in the other is desirable from a policy perspective and how to evaluate changes in both dimensions depends on the welfare function. The social welfare in equation (10) allows us to consider these trade-offs explicitly.

Both dimensions of inequality are directly relevant to welfare. Within-group inequality can have welfare effects through multiple channels. First, individuals might include a relative component in their utility functions, comparing themselves with their peers, neighbors, and co-workers (see, for example, Galí, 1994; Luttmer, 2005; Card et al., 2012). As a result, their relative rank within the group matters. Using this welfare function, policymakers can assign a higher weight to the lower tail of the within-group distribution to account for these externalities. Second, within-group inequality might cause other externalities. To give an illustration, Breza et al. (2018) find that pay inequality among production workers does, in some circumstances, reduce output, and Fehr et al. (2020) document that inequality reduces trust.[11] Additionally, inequality within groups is associated with negative outcomes. For instance, Glaeser et al. (2009) document that more unequal cities have higher murder rates, and Chetty and Hendren (2018b) show that areas with higher income inequality are linked to lower outcomes for children from

[11] Støstad and Cowell (2024) model inequality as an externality. However, inequality is modeled only using the unconditional distribution.

low-income families.[12]

Between-group inequality, or regional inequality, is central to the goals of many political institutions. Federal states or supranational entities (e.g., the European Union) rely on fiscal federalism to transfer resources across jurisdictions. For instance, the EU Regional Development fund aims at *correcting imbalances between regions* and after the German reunification, the concept of *equivalent living conditions*[13] played a major role in German regional policy, aiming to mitigate regional disparities. Further, between-group inequality can have adverse effects as the quality of the neighborhood in which children grow up significantly impacts their future outcomes (Chetty et al., 2016; Chetty and Hendren, 2018a,b).[14]

I now show that, depending on the weights $w(\tau_1, \tau_2)$, equation (10) simplifies to a welfare function where only the mean of the outcome or only the within or between component matters. For example, with $w(\tau_1, \tau_2) = 1$, we obtain a Benthamite (or utilitarian) welfare function:

$$W = \mathbb{E}[y_{ij}].$$

If $w(\tau_1, \tau_2) = w(\tau_1)$, welfare simplifies to a weighted average of the expectation of the group quantiles:

$$W = \int_0^1 \mathbb{E}_j[q(\tau_1, v_j)]w(\tau_1)d\tau_1,$$

where only the within distribution matters. This welfare function would be relevant if the focus were solely on studying (the effects on) the within distribution (see, for example, Autor et al., 2021; Friedrich, 2022; Autor et al., 2016; Engbom and Moser, 2022; Lang et al., 2023).

Differently, if $w(\tau_1, \tau_2) = w(\tau_2)$, the welfare function (10) simplifies to a weighted average of the conditional expectation in each group:

$$W = \mathbb{E}_j \left[ \int_0^1 q(\tau_1, v_j)d\tau_1 w(v_j) \right] = \mathbb{E}_j \left[ \mathbb{E}_{i|j}[y_{ij}]w(v_j) \right].$$

In this case, the welfare function assigns different weights to different groups based on their mean, and the outcome distribution within the group does not matter. This welfare function is relevant if the aim is to study inequality in average outcomes or regional GDP per capita (see, for example, Becker et al., 2010; Busso et al., 2013). An argument for considering only inequality between regions is that, in theory, it is possible to redistribute within the region.

[12] Conversely, a certain degree of local inequality or heterogeneity might be beneficial by providing employment opportunities for individuals in the lower tail of the distribution (Mazzolari and Ragusa, 2013) or fostering empathy for the poor (Glaeser, 2000).

[13] Gleichwertige Lebensverhältnisse.

[14] Poor regional or labor market performance is associated with different adverse outcomes such as an increase in fatal drug overdoses (Pierce and Schott, 2016), and single-mother families (Autor et al., 2021).

However, in practice, this is often unfeasible, and redistribution is costly. Further, as long as a policymaker is not indifferent between two allocations where the second is a mean-preserving spread of the within-group distribution of the first, a function considering both components of inequality better captures welfare.

To capture any trade-off between different dimensions of inequality, we have to choose weights that depend on both dimensions. The weighting function determines the extent to which a policymaker is willing to accept an increase in within-group inequality to decrease between-group inequality while keeping welfare constant. A wide range of weighting functions can be considered. For instance, a possible extension of the Gini social welfare function to two dimensions uses $w(\tau_1, \tau_2) = 2(1 - \omega\tau_1 - (1-\omega)\tau_2)$ with $\omega \in [0, 1]$. The higher $\omega$, the more important within-group inequality becomes to the policymaker. In the extreme case where $\omega = 1$, the problem reduces to[15]

$$W = \int_0^1 E_j[q(\tau_1, v_j)] 2(1-\tau_1) d\tau_1 = E[y] \left[ \frac{\int_0^1 E_j[q(\tau_1, v_j)] 2(1-\tau_1) d\tau_1}{E[y]} \right] = E[y](1 - I_{Gini}),$$

where $\int_0^1 E_j[q(\tau_1, v_j)] d\tau_1 = E[y]$ and $I_{Gini} = 1 - \frac{\int_0^1 E_j[q(\tau_1, v_j)] 2(1-\tau_1) d\tau_1}{E[y]}$ is the Gini index in the average group (see Kitagawa and Tetenov (2021) for more details on the one-dimensional case). With this weighting function, the welfare weights only decay linearly in the within and between ranks. Alternatively, a weighting function that is not additively separable in $\tau_1$ and $\tau_2$ allows for a more complex evolution of the weights where the effect of one rank on the welfare weights depends on the other dimension.

### 3.2 Individual Effects: Optimal Treatment Assignment

This subsection considers a setting that requires the stronger assumption of rank invariance over treatment states. Under this additional assumption, the individual treatment effects are identified and can be used to pinpoint optimal treatment rules. I use the conventional potential outcome framework with a treatment variable. Let $y_{ij}(d)$ denote the potential outcome under treatment state $d \in \{0, 1\}$. The object of interest in this paper is the two-level quantile function of the potential outcomes, conditional on observed characteristics $x_{ij} = (x'_{1ij}, x'_{2j})'$:

$$q(d, x_{ij}, \tau_1, \tau_2),$$

[15]For the one-dimensional Gini social welfare function see, for example, Blackorby and Donaldson (1978) and Weymark (1981).

which can be used to compute the conditional quantile treatment effects over both dimensions:[16]

$$q(1, x_{ij}, \tau_1, \tau_2) - q(0, x_{ij}, \tau_1, \tau_2).$$

Consider a policymaker who observes data from a *sample* population with a given group structure and has to decide whom to treat in a *target* population (subject to some capacity/budget constraint) by maximizing a rank-dependent social welfare function. I consider a static setting where the policy-maker chooses whom to treat out of a pool of individuals or groups based on their *unobserved* ranks. This is in contrast to a dynamic setting (e.g., Adusumilli et al., 2019), where the policymaker has to make sequential decisions, as well as to the one in Kitagawa and Tetenov (2021), where the goal is to optimally assign individuals to treatment based on *observable* covariates. Baseline outcomes can be in the set of covariates; however, these are not always available (see, e.g., Tarozzi et al., 2015). Further, this setting also differs from the one considered in Kaji and Cao (2023), which allows for heterogeneity only across one dimension. Instead, with grouped data, one might want to exploit treatment effect heterogeneity within and between groups to more efficiently allocate the treatment. As in the one-dimensional case, targeting the most deprived individuals or groups is not necessarily optimal (see Haushofer et al., 2022). Instead, there could be a trade-off between targeting the most deprived units and targeting the units that profit the most from the treatment. This trade-off is particularly relevant in this setting as group membership could be an important determinant of impact.

For simplicity, I consider a case without covariates; however, the framework can be easily extended to include other variables.[17] The policymaker maximizes social welfare over a class of feasible policies $\mathcal{G} \in \{u_{ij}, v_j(u_{ij}) \in (0,1) \times (0,1)\}$ that assigns individuals to treatment based on their ranks $(u_{ij}, v_j)$.[18] If these ranks were observed, it would be possible to include them in the set of covariates, and the problem would coincide with the one in Kitagawa and Tetenov (2021). Since these ranks are unobserved, we rely on distributional methods to estimate them.

In this example, the model discussed in the paper remains relevant even if the welfare function

[16]Integrating the conditional quantile treatment effects over both $\tau_1$ and $\tau_2$ yields average treatment effects:

$$ATE = \int_0^1 \int_0^1 [q(1, x_{ij}, \tau_1, \tau_2) - q(0, x_{ij}, \tau_1, \tau_2)]\, d\tau_2 d\tau_1.$$

[17]If the inclusion of additional variables is necessary to identify the distribution of potential outcomes, it is straightforward to recover the unconditional distribution by integrating out the covariates.

[18]In the following, I write $v_j$ for ease of notation. However, $v_j$ should be regarded as the rank variable ranking groups at a specific point of the within distribution. A version of the problem could consider a treatment rule of the type $G = 1(u_{ij} \leq \tilde{u}, v_j \leq \tilde{v})$ for some fixed $\tilde{u}$ and $\tilde{v}$. A similar setting is considered in Kaji and Cao (2023). Yet, this class of decision rules might be restrictive in the presence of a trade-off between deprivation and impact, as discussed in Haushofer et al. (2022).

depends only on the unconditional distribution of the outcome.[19] Hence, for this illustration, I consider a welfare function as in equation (9). However, the results can be generalized to other welfare functions. When a treatment rule $G$ is applied to the target population, the social welfare is proportional to:

$$W(q_G) = \int_0^1 q_G(\theta) w(\theta) d\theta, \tag{11}$$

where $q_G$ is the quantile function of the outcome $y_{ij}$ under treatment rule $G$:

$$y_{ij} = 1\{(u_{ij}, v_j) \in G\} y_{ij}(1) + 1\{(u_{ij}, v_j) \notin G\} y_{ij}(0),$$

and the optimal treatment rule solves[20]

$$G^* \in \arg\max_{G \in \mathcal{G}} W(G). \tag{12}$$

To make the problem operational, we need to identify individual treatment effects and assign welfare weights to each observation under each policy rule. Under rank invariance, $q(1, \tau_1, \tau_2) - q(0, \tau_1, \tau_2)$ gives the treatment effect for an individual at quantiles $(\tau_1, \tau_2)$. Further, using equation (7), we can identify the conditional quantile function of the potential outcomes under a given policy rule $G$. These objects can then be used to assign ranks and welfare weights $w_{ij}$ to each observation. Notably, individuals at different $\tau_1$th percentiles may share the same welfare weight due to their placement in different groups. However, individuals with the same $y_{ij}$ have identical welfare weights. Summing the welfare weights within groups gives the weights assigned to group $j$:[21]

$$w_j = \sum_{i=1}^{n} w_{ij}.$$

While the rank of a group changes over $\tau_1$, these groups' weights can offer a unified and welfare-based measure of a group's rank or deprivation.

To find the optimal treatment assignment rule that maximizes the social welfare function in the target population, Kitagawa and Tetenov (2018) assume that the joint distribution of the potential outcome and covariates is the same in the sample and target populations. In

[19] Recall that the advantage of the method in this setting is that it allows the exploitation of treatment effect heterogeneity over both dimensions simultaneously and, therefore, can more efficiently allocate the treatment.

[20] Solving problem (12) is nontrivial as it lacks a closed-form solution even if we knew the distribution of the potential outcomes (Kitagawa and Tetenov, 2021). One difficulty arises because the welfare weights assigned to an individual might depend on the treatment assignment of other agents. Intuitively, the welfare weight assigned to an individual is weakly increasing in the outcomes of the other individuals.

[21] In the case of unbalanced groups, larger groups are more likely to have a higher welfare weight. This feature can be desirable if the cost of assigning a group to the treatment does not depend on the number of observations in this group. Alternatively, it is possible to compute the average weights.

this setting, I need the joint distribution of $(y_{ij}(1), y_{ij}(0), v_j, u_{ij})$ is the same in both populations. Since the ranks are normalized, $u_{ij}$ follows the same distribution in both populations by construction. Therefore, one can equivalently assume that for all $u_{ij}$, the joint distribution of $y_{ij}(1), y_{ij}(0), v_j(u_{ij})|u_{ij}$ is the same in the sample and target populations.

In summary, the two-level conditional quantile function of potential outcomes in the sample population enables the identification of treatment effects for an individual at a given individual and group rank. With this information, we can identify $y_{ij}(1)$ for all $i$ and $j$. Subsequently, for each $G \in \mathcal{G}$, we can construct the counterfactual outcome $y_{ij} = 1\{(u_{ij}, v_j) \in G\}y_{ij}(1) + 1\{(u_{ij}, v_j) \notin G\}y_{ij}(0)$ along with the corresponding outcome distribution and welfare. The optimal treatment rule then solves equation (12).

# 4 Estimator

In this section, I propose a two-step quantile regression estimator for model (6). For simplicity of notation, I consider the same set of quantiles to model both dimensions, although this is not a requirement. For each dimension, I approximate the function using $\#\tau$ different quantiles. The first stage consists of group-by-group quantile regressions. For each group $j$ and quantile $\tau_1$, the outcome is regressed on the individual level variables $x_{1ij}$ using quantile regression. Then, for each group and $\tau_1$, the fitted values are saved. In the second stage, for each $\tau_1$, the first-stage fitted values $\hat{y}_{ij}(\tau_1)$ are regressed on all variables using all observations. This is again done with quantile regression for each $\tau_2$. Thus, estimation comprises $\#\tau \times m$ first stage regression and $\#\tau \times \#\tau$ second stages. Formally, the first-stage quantile regression solves the following minimization problem for each group $j$ and quantile $\tau_1$ separately:

$$\hat{\beta}_j(\tau_1) := \left(\hat{\beta}_{1,j}(\tau_1), \hat{\beta}_{2,j}(\tau_1)'\right)' = \underset{(b_1,b_2)\in\mathbb{R}^{dim(x_1)+1}}{\arg\min} \frac{1}{n}\sum_{i=1}^{n} \rho_{\tau_1}(y_{ij} - b_1 - x_{1ij}' b_2), \tag{13}$$

where $\rho_\tau(x) = (\tau - 1\{x < 0\})x$ for $x \in \mathbb{R}$ is the check function. For group $j$, the true vector of first stage coefficients is given by $\beta_j(\tau_1) := \beta(\tau_1, v_j) = (\alpha(\tau_1, v_i) + x_{2j}'\gamma(\tau_1, v_j), \beta(\tau_1, v_j)')'$ and the fitted values $\hat{y}_{ij}(\tau_1) = \hat{\beta}_{1,j}(\tau_1) + x_{1ij}'\hat{\beta}_{2,j}(\tau_1)$ are estimators of the $\tau_1$-conditional quantile function $Q(\tau_1, y_{ij}|x_{ij}, v_j)$.

The second stage quantile regression then solves for all $(\tau_1, \tau_2)$:

$$\hat{\delta}(\hat{\beta}(\tau_1), \tau_2) = \underset{(a,b,g)\in\mathbb{R}^{dim(x)+1}}{\arg\min} \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} \rho_{\tau_2}(\hat{y}_{ij}(\tau_1) - x_{2j}' g - x_{1ij}' b - a), \tag{14}$$

where the notation stresses the dependence on the first step and $\delta = (\alpha, \beta', \gamma')'$.

Implementing the estimator is straightforward, requiring only programs for quantile regression. The lack of a closed-form solution for quantile regression might increase computing time, but recent algorithms enable simultaneous estimation of numerous quantiles, significantly improving computational speed. Moreover, the first stage is easily parallelizable, as all first-stage quantile regressions run independently across the groups, and the second stage is also parallelizable with respect to $\tau_1$.[22]

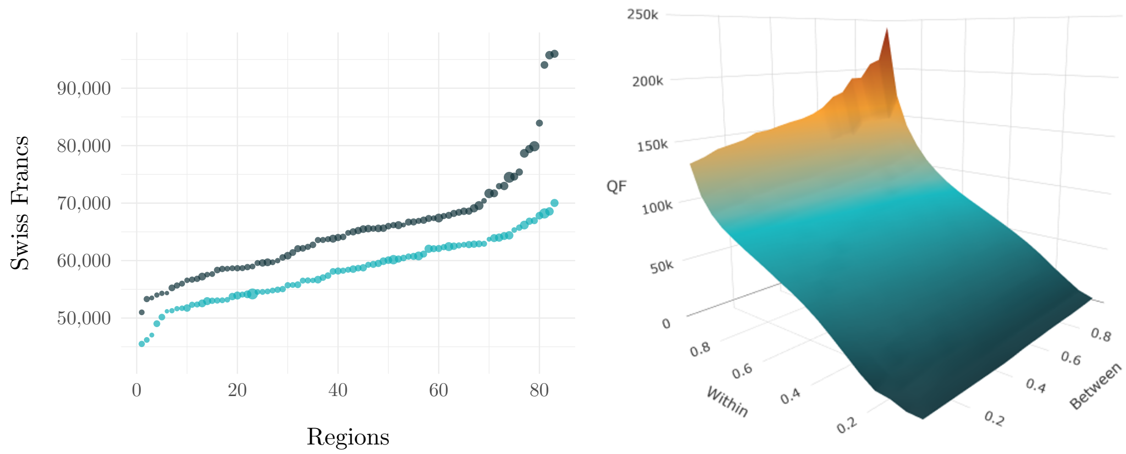
**Remark 2** (**Alternative estimators - instrumental variables**)**.** Model (6) assumes that the variation of both the $x_{1ij}$ and $x_{2j}$ are exogenous so that quantile regression in both stages yields consistent estimates. If this is not the case, the framework can be easily extended to accommodate instrumental variables. Depending on which variables are assumed to be endogenous, either the second stage or both stages could be estimated using an instrumental variable quantile regression estimator (see, e.g., Chernozhukov and Hansen, 2005). Future research should explore this possibility.

**Example 2** (Continuation of Example 1)**.** Consider the setting of Example 1, where the goal is to analyze income heterogeneity between and within geographical regions, and there are no covariates. One possibility for analyzing income heterogeneity across regions is to consider differences in median or average wages. Using administrative data from the Federal Statistical Office of Switzerland, I show that these two measures fail to capture important features of income heterogeneity between regions. Groups are defined by 2-digit ZIP codes. These groups are on a finer resolution than Swiss cantons and offer a more precise measure of labor markets. The dataset comprises information on 4.2 million individuals aged between 30 and 63, divided into 83 groups in the year 2021. Since there are no covariates, the first-step estimation consists of calculating the $\tau_1$ sample quantiles in the groups. Subsequently, in the second stage, I compute the $\tau_2$ quantiles, over groups, of the group-level quantiles. I consider the set of quantiles $\{0.01, 0.02, \ldots, 0.99\}$ in both stages. Without individual-level covariates, the first-stage fitted values are constant within the groups, so one can collapse the dataset at the group level after the first step to increase computation speed. In this case, weights should be included in the second step.

Figure 2 shows the regional averages and medians of yearly income in panel (a) and the two-dimensional quantile function of the same variable in panel (b). Both regional averages

[22] Ensuring the monotonicity of the estimated two-level quantile functions across both dimensions might require a rearrangement operation, as suggested in Chernozhukov et al. (2009, 2010). Due to the nested structure of the problem, rearrangement along the $\tau_1$ dimension should be performed after the first stage. Monotonicity of the first stage in all groups guarantees that the second stage quantile regression remains monotonic along the $\tau_1$ dimension. Rearrangement along the $\tau_2$ dimension can be implemented subsequent to the second stage.

Figure 2: Income Heterogeneity Within and Between Regions




(a) Average and Median Income by Region (b) Two-Dimensional Quantile Function of Income

*Notes*: Figure 2a shows the heterogeneity in average (dark blue) and median (light blue) yearly income across regions defined by 2-digit ZIP codes. Figure 2b shows the two-dimensional quantile function of yearly income within and between regions.

and medians are independently arranged from low to high. The darker dots in Figure 2a reveal substantial differences in average income across regions. However, Figure 2b shows that a large portion of these differences in mean income can be attributed to high top incomes in a few regions. Across most of the $\tau_1$ distribution, the differences across regions are substantially smaller compared to the right tail of the within distribution. Thus, the differences in average wages shown in Figure 2a, not only mask substantial within-region income heterogeneity but are predominantly driven by differences in top incomes.

The lighter dots in Figure 2a show that the heterogeneity in median wages across regions is substantially smaller than the heterogeneity in average income. However, this measure solely reflects the heterogeneity at one point of the distribution, potentially overlooking the labor market situation of a considerable portion of workers. More specifically, median wages within a region might poorly relate to the labor market situation of low earners. To see this, we can look at the correlation of the group ranks for different values of $\tau_1$. Table 1 shows that a group's rank at the very low end of the within distribution exhibits only a small correlation with the ranks at other deciles of the distribution. On the other hand, there is a high correlation between the ranks at the middle and the top of the distribution. This suggests the presence of a different mechanism influencing the lower tail of the within distribution.

Table 1: Correlation of Regions' Ranks over $\tau_1$

| | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 1.00 | | | | | | | | |
| 0.2 | 0.73 | 1.00 | | | | | | | |
| 0.3 | 0.54 | 0.95 | 1.00 | | | | | | |
| 0.4 | 0.48 | 0.90 | 0.98 | 1.00 | | | | | |
| 0.5 | 0.44 | 0.85 | 0.95 | 0.98 | 1.00 | | | | |
| 0.6 | 0.37 | 0.80 | 0.91 | 0.95 | 0.98 | 1.00 | | | |
| 0.7 | 0.26 | 0.73 | 0.86 | 0.90 | 0.94 | 0.97 | 1.00 | | |
| 0.8 | 0.18 | 0.67 | 0.81 | 0.86 | 0.89 | 0.94 | 0.99 | 1.00 | |
| 0.9 | 0.09 | 0.58 | 0.73 | 0.78 | 0.81 | 0.87 | 0.94 | 0.98 | 1.00 |

*Note:*
The table shows the correlation matrix of the regions' ranks at different values of $\tau_1$.

# 5 Asymptotic Theory

**Notation** - Let $\tau = (\tau_1, \tau_2)$ and denote the true parameter vectors $\beta_{j,0}(\tau_1)$ and $\delta_0(\beta_0, \tau) := \delta_0(\tau_2, \beta_0(\tau_1))$. To simplify notation, I suppress the dependency of $\delta$ and $\beta_j$ on $\tau_1$ and $\tau_2$, unless necessary. For a random variable $h_{ij}$, $\mathbb{E}_{i|j}[h_{ij}]$ denotes the expectation over $i$ in group $j$. Let $K_1$ be the dimension of $x_{1ij}$ and $K_2$ be the number of regressors in $x_2$. Furthermore, let $K = K_1 + K_2 + 1$ be the total number of regressors. Finally, denote the $(K_1 + 1)$-dimensional vector of first stage regressors as $\tilde{x}_{ij} = (1, x'_{1ij})'$. I prove weak convergence of the whole quantile regression process for $\tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T}$, where $\mathcal{T}$ is a compact set of included in $(0, 1)$. The symbol $\ell^\infty(\mathcal{T} \times \mathcal{T})$ denotes the set of component-wise bounded vector-valued functions of $\mathcal{T} \times \mathcal{T}$ and $\rightsquigarrow$ denotes weak convergence.

## 5.1 Consistency and Asymptotic Normality

The derivation of asymptotic results faces two primary challenges: the non-smoothness of the quantile regression objective function and the increasing dimension of the first stage as the number of groups diverges to infinity. Several studies have addressed the asymptotic properties of estimators with non-smooth objective functions, leveraging the smoothness of the limiting objective function (see, for example, Newey and McFadden, 1994). Notably, Pakes and Pollard (1989) study the properties of Z-estimators without imposing smoothness conditions on the sample equations. Building on this work, Chen et al. (2003) broadens the scope to two-step estimators, where the parameter of interest depends on an infinite-dimensional preliminary parameter.

To derive the asymptotic results, I rely on results in Chen et al. (2003). Similarly to their paper, my second stage parameter vector depends on a preliminary first stage whose dimension increases with the sample size. To this end, I start by making the assumptions necessary to ensure that the first-stage quantile regression is well-behaved. For this first analysis, I build on the work of Volgushev et al. (2019) and Galvao et al. (2020) and make the following assumptions:

**Assumption 1** (**Sampling**). *(i) The processes* $\{(y_{ij}, x_{ij}) : i = 1, \dots, n\}$ *are i.i.d. across* $j$*. (ii) For each* $j$*, the observations* $(y_{ij}, x_{ij})$ *are i.i.d. across* $i$*.*

**Assumption 2** (**Covariates**). *(i) For all* $j = 1, \dots, m$ *and all* $i = 1, \dots, n$*,* $\|x_{ij}\| \leq C$ *almost surely. (ii) The eigenvalues of* $\mathbb{E}_{i|j}[\tilde{x}_{ij}\tilde{x}'_{ij}]$ *and* $\mathbb{E}[x_{ij}x'_{ij}]$ *are bounded away from zero and infinity uniformly across* $j$*.*

**Assumption 3** (**Individual-level heterogeneity**). *The conditional distribution* $F_{y_{ij}|x_{1ij},v_j}(y|x,v)$ *is twice differentiable w.r.t.* $y$*, with the corresponding derivatives* $f_{y_{ij}|x_{1ij},v_j}(y|x,v)$ *and* $f'_{y_{ij}|x_{1ij},v_j}(y|x,v)$*. Further, assume that*

$$f_y^{max} := \sup_j \sup_{y \in \mathbb{R},\ x \in \mathcal{X}_1} |f_{y_{ij}|x_{1ij},v_j}(y|x,v)| < \infty,$$

*and*

$$\bar{f}'_y := \sup_j \sup_{y \in \mathbb{R},\ x \in \mathcal{X}_1} |f'_{y_{ij}|x_{1ij},v_j}(y|x,v)| < \infty.$$

*where* $\mathcal{X}_1$ *is the support of* $x_{1ij}$*.*

**Assumption 4** (**Bounded density I**). *There exists a constant* $f_y^{min} < f_y^{max}$ *such that*

$$0 < f_{min} \leq \inf_j \inf_{\tau_1 \in \mathcal{T}} \inf_{x \in \mathcal{X}_1} f_{y_{ij}|x_{1ij},v_j}(Q(\tau_1, y_{ij}|x_{ij}, v_j)|x,v).$$

These are standard assumptions in the quantile regression literature. Assumption 1, assumes that the observations are i.i.d. within and between groups. Assumption 2 requires that the regressors are bounded and that both matrices $\mathbb{E}_{i|j}[\tilde{x}_{ij}\tilde{x}'_{ij}]$ and $\mathbb{E}[x_{ij}x'_{ij}]$ are invertible. Assumptions 3 and 4 require smoothness and boundedness of the conditional distribution of the outcome variable $y_{ij}$ given $(x_{ij}, v_j)$, the density, and its derivatives. This first set of assumptions ensures that the first-stage estimator is well-behaved and allows to apply Lemma 3 in Galvao et al. (2020).

To ensure that the second-step quantile regression is well-behaved, I make the following assumptions:

**Assumption 5** (**Group-level heterogeneity**). *The conditional distribution* $F_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(q|x)$ *is twice continuously differentiable w.r.t.* $q$*, with the corresponding derivatives* $f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(q|x)$ *and* $f'_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(q|x)$*. Further, assume that*

$$f_Q^{max} := \sup_{\tau_1\in\mathcal{T},\ q\in\mathbb{R},\ x\in\mathcal{X}} |f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(q|x)| < \infty$$

*and*

$$\bar{f}'_Q := \sup_{\tau_1\in\mathcal{T},\ q\in\mathbb{R},\ x\in\mathcal{X}} |f'_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(q|x)| < \infty.$$

*where* $\mathcal{X}$ *is the support of* $x_{ij}$*.*

**Assumption 6** (**Bounded density II**). *There exists a constant* $f_Q^{min} < f_Q^{max}$ *such that*

$$0 < f_{min} \leq \inf_{\tau_1,\tau_2\in\mathcal{T}\times\mathcal{T}} \inf_{x\in\mathcal{X}} f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x'_{ij}\delta_0(\tau)|x).$$

**Assumption 7** (**Compact parameter space**). *For all* $\tau$*,* $\beta_{j,0}(\tau_1) \in int(\mathcal{B}_j)$ *and* $\delta_0(\beta_0,\tau) \in int(\mathcal{D})$*, where* $\mathcal{B}_j$ *and* $\mathcal{D}$ *are compact subsets of* $\mathbb{R}^{K_1+1}$ *and* $\mathbb{R}^K$*, respectively.*

**Assumption 8** (**Coefficients**). *For all* $\tau_1,\tau'_1 \in \mathcal{T}$ *and* $j = 1,\ldots,m$*,* $\|\beta_j(\tau_1) - \beta_j(\tau'_1)\| \leq C|\tau_1 - \tau'_1|$*. Further, for all* $\tau,\tau' \in \mathcal{T}\times\mathcal{T}$ *and* $\|\delta(\tau) - \delta(\tau')\| \leq C|\tau_1 - \tau'_1| + \leq C|\tau_2 - \tau'_2|$*.*

Assumptions 5 and 6 are the second stage counterpart of of assumptions 3 and 4, with the difference that the conditional distribution $F_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(q|x)$ is required to be *continuously* differentiable. This additional assumption on the distribution of the second stage dependent variable is sufficient to ensure that its second derivative is Lipschitz continuous. An implication of this assumption is that groups must be sufficiently heterogeneous, ruling out the possibility that, conditional on the covariates, only a few group types exist at each $\tau_1$. Assumption 7 requires the parameter spaces to be compact. Compactness of the parameter space is a common assumption in the quantile regression literature, see e.g., Honoré et al. (2002); Chernozhukov and Hansen (2006), and Zhang et al. (2019). Compactness of $\mathcal{D}$ is necessary to use the results in Chen et al. (2003), while compactness of $\mathcal{B}_j$ is useful as it directly implies that the covering integral is finite, but could easily be relaxed. Finally, assumption 8 ensures that the coefficients are continuous functions of the quantile indices.

Since quantile regression is consistent but not unbiased, we need the number of observations per group to diverge to infinity. At the same time, the second-stage quantile regression exploits the heterogeneity between groups, which is determined by the heterogeneity of the group-level quantile functions, a group-specific term. Thus, also the number of groups must diverge. The following assumption states two different growth rates of the number of observations per group relative to the number of groups:

**Assumption 9 (Growth rates).** *As $m \to \infty$, we have*

*(a)* $\frac{\log m}{n} \to 0$,

*(b)* $\frac{\sqrt{m} \log n}{n} \to 0$.

I show that the relative growth rate in Assumption 9(a) is sufficient for consistency of the estimator. This condition is exceptionally weak, as the number of observations per group can increase at an almost arbitrarily slow rate. Differently, weak convergence requires assumption 9(b). This second growth rate is relatively mild as the number of observations per group must grow faster than the square root of the number of groups.[23] Closely related, it is worth mentioning that in empirical applications, groups might be defined by industries and geographical regions, such as counties, which tend to contain many individuals, so that the results in this section provide a useful approximation. Similarly, one might ask whether it would be possible to relax the required growth rate by using smoothed quantile regression and/or bias correction after the first stage. Given the results in the literature, smoothing does not help relaxing the assumptions for the results. It could help improve the finite sample performance at the cost of choosing a smoothing parameter. At the same time, the bias-corrected smoothed quantile regression estimator for panel data suggested in Galvao and Kato (2016) is not applicable in this setting, as it assumes homogeneity of the coefficients over groups. In ongoing work, Franguridi et al. (2024) derive an explicit formula for the bias of the leading term of the expansion. However, making the bias correction feasible remains a major challenge since it requires estimating some components involving higher-order derivatives. Furthermore, choosing tuning parameters is even more difficult. Exploring these possibilities is left for future research.

The first result of this paper states weak uniform consistency of the two-step estimator.

**Theorem 1** (Uniform Consistency). *Let assumptions 1-8 and 9(a) be satisfied. Then as $m \to \infty$*

$$\sup_{\tau \in \mathcal{T} \times \mathcal{T}} ||\hat{\delta}(\hat{\beta}, \tau) - \delta_0(\beta_0, \tau)|| \xrightarrow{p} 0.$$

[23] Until recently, there has been a substantial gap in the required rate of growth of $n$ relative to $m$ of nonlinear estimators with a smooth objective function and those with a non-smooth one, such as quantile regression. For unbiased asymptotic normality, the bias has to decrease more quickly than $1/\sqrt{m}$. Using new results in Volgushev et al. (2019) and Galvao et al. (2020), I show that $m(\log n)/\sqrt{n} \to 0$ is enough to ensure unbiased asymptotic normality of the estimator. Galvao et al. (2020) require the stronger condition $m(\log(n))^2/n \to 0$ because, in their case, the bias converges more quickly than $1/\sqrt{mn}$ to establish unbiased asymptotic normality of their estimator.

To establish weak convergence, I begin by showing that $\hat{\delta}(\hat{\beta},\tau) - \delta_0(\beta_0,\tau)$ can be approximated by a linear function of two terms, each accounting for the estimation errors arising from different steps of the estimation. If the first stage parameter vector $\beta_0(\tau_1) = (\beta_{0,1}(\tau_1)', \ldots, \beta_{0,m}(\tau_1)')'$ was known, the true second-stage parameter vector $\delta_0(\beta_0,\tau)$ uniquely[24] satisfies:

$$\mathbb{E}[m(\delta_0,\beta_0,\tau)] = 0 \tag{15}$$

with

$$m(\delta,\beta,\tau) = x'_{ij}[\tau_2 - 1(\tilde{x}'_{ij}\beta_j(\tau_1) \leq x'_{ij}\delta(\beta,\tau))]. \tag{16}$$

Let $M(\delta,\beta,\tau) = \mathbb{E}[m(\delta,\beta,\tau)]$, denote the sample counterpart $M_{mn}(\delta,\beta,\tau) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} m(\delta,\beta,\tau)$. While $M(\delta,\beta,\tau)$ is a smooth function, this property does not extend to $M_{mn}(\delta,\beta,\tau)$.
Two expressions are central to establishing asymptotic normality. (i) The sample moment evaluated at the true parameters:

$$M_{mn}(\delta_0,\beta_0,\tau) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} m(\delta_0,\beta_0,\tau), \tag{17}$$

(ii) and the pathwise derivative of $M(\delta,\beta_0,\tau)$ in the direction $(\beta - \beta_0)$:

$$\Gamma_2(\delta,\beta_0,\tau)[\beta-\beta_0] = \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2j}(\delta,\beta_0,\tau)[\beta_j - \beta_{j,0}], \tag{18}$$

where $\Gamma_2(\delta,\beta_0,\tau)$ is $K \times ((K_1+1)\cdot m)$, $\Gamma_{2j}(\tau,\delta,\beta_0)$ is the $j$th $K \times (K_1+1)$ submatrix of $\Gamma_2(\delta,\beta_0,\tau)$ and $\frac{1}{m}\bar{\Gamma}_{2j}(\tau,\delta,\beta_0) := \Gamma_{2j}(\tau,\delta,\beta_0)$ with

$$\Gamma_{2j}(\delta,\beta_0,\tau) = \frac{\partial}{\partial\beta_j}M(\delta,\beta_0,\tau) = -\frac{1}{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x'_{ij}\delta(\beta_0,\tau)|x_i)x_{ij}\tilde{x}'_{ij}\right]. \tag{19}$$

The expression in equation (17) is directly related to the leading term of a Bahadur expansion of the unfeasible estimator $\hat{\delta}(\beta_0,\tau)$:

$$\hat{\delta}(\beta_0,\tau) - \delta(\beta_0,\tau) = \Gamma_1(\delta_0,\beta_0,\tau)^{-1}\cdot\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} m(\delta_0,\beta_0,\tau) + o_p\left(m^{-1/2}\right), \tag{20}$$

where $\Gamma_1(\delta_0,\beta_0,\tau) = \frac{\partial M(\delta,\beta_0,\tau_2)}{\partial\delta} = \mathbb{E}[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x'_{ij}\delta_0(\beta_0,\tau)|x_{ij})x_{ij}x'_{ij}]$. Thus, equation (17) captures the estimation error that would arise due to random variation in the second stage if we knew the true first stage and equation (18) captures the effect of the first estimation error on the second-step estimates $\hat{\delta}(\hat{\beta},\tau)$.

Heuristically, the idea is to approximate $\hat{\delta}(\hat{\beta},\tau) - \delta_0(\beta_0,\tau)$ with the linear function $\Gamma_1(\delta_0,\beta_0,\tau)^{-1}$ $\left(\sum_{j=1}^{m}\Gamma_{2j}(\delta_0,\beta_0,\tau)\,[\hat{\beta}_j(\tau_1) - \beta_{j,0}(\tau_1)] + M_{mn}(\delta_0,\beta_0,\tau)\right)$. To this end, I show that the two expressions are asymptotically equivalent, up to a remainder term that converges to zero sufficiently

[24] Under weak regularity conditions.

fast, uniformly in $\tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T}$. Then, if we can show asymptotic normality for the linear function, asymptotic normality of the estimator follows.

The following Lemma establishes the asymptotic properties of equations (17) and (18).

**Lemma 1.** *Let the model in equation (6) and assumptions 1-7 hold. Then*

*(i) Under assumption 9(b), as $m \to \infty$:*

$$\sup_{\tau \in \mathcal{T} \times \mathcal{T}} \left\| \frac{1}{m} \sum_{j=1}^{m} \bar{\Gamma}_{2,j}(\delta_0, \beta_0, \tau) \left( \hat{\beta}_j(\tau_1) - \beta_{j,0}(\tau_1) \right) \right\| = o_p\left( \frac{1}{\sqrt{m}} \right), \tag{21}$$

*where* $\bar{\Gamma}_{2j}(\delta_0, \beta_0, \tau) = \mathbb{E}_{i|j}\left[ f_{Q(\tau_1, y_{ij}|x_{ij}, v_j)|x_{ij}}(x'_{ij}\delta_0(\tau)|x_{ij}) x_{ij} \tilde{x}'_{ij} \right]$.

*(ii) As $m \to \infty$:*

$$\sqrt{m}\left( M_{mn}(\delta_0, \beta_0, \tau) \right) \rightsquigarrow \mathbb{G}(\cdot), \ \ in\ \ell^\infty(\mathcal{T} \times \mathcal{T}), \tag{22}$$

*where $\mathbb{G}$ is a mean-zero Gaussian process with a uniformly continuous sample path and covariance function*

$$\begin{aligned}\Omega_2(\tau, \tau') &= \mathbb{E}\left[ [\tau_2 - 1(\tilde{x}'_{ij}\beta_{j,0}(\tau_1) \leq x'_{ij}\delta_0(\beta_0, \tau)][\tau'_2 - 1(\tilde{x}'_{ij}\beta_{j,0}(\tau'_1) \leq x'_{ij}\delta_0(\beta_0, \tau')] x_{ij} x'_{ij} \right] \\ &= (\min(\tau_2, \tau'_2) - \tau_2 \tau'_2) \mathbb{E}[x_{ij} x'_{ij}].\end{aligned}$$

*(iii) Under assumption 9(b), as $m \to \infty$:*

$$\sup_{\tau \in \mathcal{T} \times \mathcal{T}} \left\| \mathrm{Cov}\left( M_{mn}(\delta_0, \beta_0, \tau), \frac{1}{m} \sum_{j=1}^{m} \bar{\Gamma}_{2,j}(\delta_0, \beta_0, \tau) \left( \hat{\beta}_j(\tau_1) - \beta_{j,0}(\tau_1) \right) \right) \right\| = o_p\left( \frac{1}{\sqrt{m}} \right).$$

Lemma 1 shows that the first-stage error converges to zero at a faster rate than the standard deviation of the second stage. This is true as long as $n$ diverges, no matter how slowly. Hence, the asymptotic distribution of the estimator can be approximated by a linear function of the sum of two terms converging at different rates, so that the asymptotic behavior will be determined by the term converging at a slower rate, and in the first-order asymptotic distribution, only the second stage matters. Similar results are documented in Chetverikov et al. (2016) and in Melly and Pons (2025).

To show weak convergence, I first show that the linearization holds uniformly in $\tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T}$. Then, it remains to show that uniformly over $\tau_1, \tau_2$ the first stage error is $o_p(m^{-1/2})$ so that the estimator $\hat{\delta}(\hat{\beta}, \tau)$ has the same asymptotic distribution as the unfeasible estimator $\hat{\delta}(\beta_0, \tau)$. Hence, weak converges of $\sqrt{m}\left( \hat{\delta}(\beta_0, \tau) - \delta_0(\beta_0, \tau) \right)$ is sufficient to show weak converges of $\sqrt{m}\left( \hat{\delta}(\hat{\beta}, \tau) - \delta_0(\beta_0, \tau) \right)$.

**Theorem 2** (**Weak Convergence**)**.** *Let assumptions 1-8 and 9(b) be satisfied. Then*

$$\sqrt{m}\left(\hat{\delta}(\hat{\beta},\cdot)-\delta_0(\beta_0,\cdot)\right) \rightsquigarrow \Gamma_1^{-1}(\cdot)\mathbb{G}(\cdot) \quad in\ \ell^\infty(\mathcal{T}\times\mathcal{T}),$$

$\Gamma_1 = \Gamma_1(\delta_0, \beta_0, \cdot)$*, where* $\mathbb{G}(\cdot)$ *is a mean-zero Gaussian process with uniformly continuous sample paths and covariance function*

$$\begin{aligned}\Omega_2(\tau,\tau') &= \mathbb{E}\left[[\tau_2 - 1(\tilde{x}_{ij}'\beta_{j,0}(\tau_1) \le x_{ij}'\delta_0(\beta_0,\tau)][\tau_2' - 1(\tilde{x}_{ij}'\beta_{j,0}(\tau_1') \le x_{ij}'\delta_0(\beta_0,\tau')]x_{ij}x_{ij}'\right] \\ &= (\min(\tau_2,\tau_2') - \tau_2\tau_2')\mathbb{E}[x_{ij}x_{ij}'].\end{aligned}$$

Theorem 2 shows that the entire coefficient vector converges at the $1/\sqrt{m}$ rate despite $mn$ observations being used for the estimation. This is neither specific nor surprising to this quantile regression method. Instead, it is a consequence of modeling heterogeneities between groups. Imposing equality of $\beta(\tau_1,\tau_2)$ over groups, it would be possible to estimate this coefficient at the $1/\sqrt{mn}$ rate. However, since $\beta(\tau_1,\tau_2)$ is allowed to vary over groups through the dependency on $\tau_2$, variation between groups is necessary for identification. Similarly, in the least squares case, it is always possible to estimate the coefficients on the individual-level variables at the $1/\sqrt{mn}$ rate, for instance, by implementing a fixed effects estimator. However, by exploiting only the within-group variation, it cannot identify heterogeneities between groups. Ultimately, the between-group variation, which slows down the convergence rate, has to be used to identify between-group heterogeneity.

**Remark 3** (Degree of heterogeneity and growth condition)**.** Assumption 5 implies that groups are heterogeneous. Without group-level heterogeneity, the second step quantile regression using the true first stage would be deterministic (degenerate). This suggests a faster rate of convergence with an asymptotic distribution where all the variance would come from the first stage. With a linear second step regression, Melly and Pons (2025) derive the adaptive asymptotic distribution of their estimator under the stronger growth condition that $m(\log n)^2/n \to 0$ and suggest an adaptive inference procedure that remains valid under the different rates of convergence; that is, regardless of the presence or absence of group level heterogeneity. This growth condition would be sufficient to adapt the proof of Lemma 1 to ensure that $\sqrt{mn}\frac{1}{m}\sum_{j=1}^m \bar{\Gamma}_{2,j}(\delta_0,\beta_0,\tau)$ $\left(\hat{\beta}_j(\tau_1) - \beta_{j,0}(\tau_1)\right) \xrightarrow{d} N(0,\Omega_1(\tau))$ with $\Omega_1(\tau) = \mathbb{E}_j\left[\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)V_j(\tau_1)\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)'\right]$, where $V_j(\tau_1)$ is the asymptotic covariance matrix of $\hat{\beta}_j(\tau_1)$. It is likely that the inference procedure suggested here is valid adaptively. For instance, simulations without group-level heterogeneity suggest that the confidence intervals have the correct coverage even without group-level heterogeneity. In this case, however, it is not possible to use the same proof strategy because quantile regression is nonlinear, and the proof relies on a linearization which holds only under heterogeneity. Intuitively, I need strong separation of the groups since without heterogeneity between groups, the estimated group-level conditional quantile functions are identical up to the first stage error, and the estimator should converge at the faster $\sqrt{mn}$ rate.

### 5.2 Inference

To perform inference, I suggest a clustered bootstrap procedure, where entire groups are resampled with replacement. In a similar setting, Liao and Yang (2018); Lu and Su (2023) and Fernández-Val et al. (2022) show that the procedure is uniformly valid in the rate of convergence of the estimator,[25] and Melly and Pons (2025) show adaptive validity of the clustered variance estimator. Given the better performance of the bootstrap estimator in simulations for this setting, I suggest using the bootstrap in this paper. Since entire groups are resampled, the first stage is unaffected and does not need to be recomputed. Consequently, the procedure is equivalent to resampling the first-stage fitted values. The following algorithm describes how to perform inference using the clustered bootstrap. Since the randomness comes from which group are sampled, I denote $\hat{\beta}_j(z^*, \tau)$ the $(K_1 + 1)$-dimensional vector containing first-stage estimates of group $j$ in the bootstrap sample.

**Algorithm 1** (Bootstrap Variance Estimator)**.** *Draw a random sample with replacement* $\{(\hat{y}^*_{1j}(\tau_1), \ldots, \hat{y}^*_{nj}(\tau_1)), (x^*_{1j}, \ldots, x^*_{nj}) : j = 1, \ldots, m,\ \tau_1 \in \mathcal{T}\}$ *from* $\{(\hat{y}_{1j}(\tau_1), \ldots, \hat{y}_{nj}(\tau_1)), (x_{1j}, \ldots, x_{nj}) : j = 1, \ldots, m,\ \tau_1 \in \mathcal{T}\}$*, and run the second step estimator (Equation (14)) using the resampled data to obtain* $\hat{\delta}^*(\hat{\beta}(z^*), \tau)$*. Repeat the previous step for each bootstrap replication* $b = 1, \ldots, B$*, to obtain* $\{\hat{\delta}^*(\hat{\beta}(z^*), \tau)\}_{b=1}^{B}$ *for each* $\tau$*. Compute a bootstrap estimate of* $\Gamma_1^{-1}\Omega_2(\tau)\Gamma_1^{-1}$*, such as the variance of the bootstrap estimates or the interquartile range rescaled by the normal distribution; that is,* $\hat{\Gamma}_1^{*-1}\hat{\Omega}_2^*(\tau)\hat{\Gamma}_1^{*-1} = (q_{0.75}(\tau) - q_{0.25}(\tau))/(z_{0.75} - z_{0.25})$ *for* $\tau \in \mathcal{T} \times \mathcal{T}$*, where* $q_p(\tau)$ *is the pth percentile of the bootstrap estimates.*[26]

The following theorem states that the asymptotic distribution of $\hat{\delta}(\hat{\beta}, \tau) - \delta_0(\beta_0, \tau)$ can be approximated with the distribution of $\hat{\delta}^*(\hat{\beta}(z^*), \tau) - \hat{\delta}(\hat{\beta}, \tau)$.

**Theorem 3 (Validity of the Bootstrap).** *Assume that the condition for Theorem 2 are satisfied. Then,*

$$\sqrt{m}\left(\hat{\delta}^*(\hat{\beta}(z^*), \cdot) - \hat{\delta}(\hat{\beta}, \cdot)\right) \rightsquigarrow^* \Gamma_1^{-1}(\cdot)\mathbb{G}(\cdot) \quad \text{in } \ell^\infty(\mathcal{T} \times \mathcal{T}).$$

[25] These papers refer to this bootstrap procedure as cross-sectional bootstrap since the focus is on traditional panel data models.

[26] The interquartile range rescaled with the normal distribution is used, for example, in Chernozhukov et al. (2013).

Given the large number of coefficients estimated in the model, researchers might be interested in testing hypotheses involving multiple parameters. For instance, one might want to test whether a subvector of $\delta(\tau_1, \tau_2)$, denoted by $\delta_k(\tau_1, \tau_2)$ for a given index set $k$, is constant across quantiles:

$$H_0 : \delta_k(\tau) = \bar{\delta}_k, \ \forall \ \tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T},$$

with $\bar{\delta}_k = \int_{\tau_2} \int_{\tau_1} \hat{\delta}(\tau_1, \tau_2) d\tau_1 d\tau_2$. Kolmogorov-Smirnov and Cramér-von Mises type tests are suitable in these settings. For instance, the Kolmogorov-Smirnov test statistics can be constructed by:

$$t^{KS} = \sup_{\tau \in \mathcal{T} \times \mathcal{T}} \sqrt{\left(\hat{\delta}_k(\tau) - \bar{\delta}_k\right)' \hat{V}_k(\tau)^{-1} \left(\hat{\delta}_k(\tau) - \bar{\delta}_k\right)}, \tag{23}$$

where $\hat{V}_k(\tau)$ is a bootstrap estimate of the asymptotic variance of $\hat{\delta}_k(\tau)$. To obtain the critical values, I follow Chernozhukov and Fernández-Val (2005) and use the bootstrap to mimic the test statistic. However, instead of recentering, I impose the null hypothesis using the parametric bootstrap based on the estimated quantile regression process.[27] For each bootstrap iteration, the corresponding test statistic is computed as follows:

$$t_b^{KS} = \sup_{\tau \in \mathcal{T} \times \mathcal{T}} \sqrt{\left(\hat{\delta}_k^{*b}(\tau) - \hat{\delta}_k^{*b}\right)' \hat{V}_k(\tau)^{-1} \left(\hat{\delta}_k^{*b}(\tau) - \hat{\delta}_k^{*b}\right)}, \tag{24}$$

where $\hat{\delta}_k^{*b} = \int_{\tau_2} \int_{\tau_1} \hat{\delta}^{*b}(\tau_1, \tau_2) d\tau_1 d\tau_2$. The critical values of a test with size $\alpha$ are the $(1 - \alpha)$ quantile of $\{t_b^{KS} : 1 \leq b \leq B\}$.

Following Chernozhukov et al. (2013), it is possible to construct functional confidence intervals that cover the entire function with a pre-specified rate by inverting the acceptance region of the KS statistics

$$t_b^{KS} = \sup_{\tau \in \mathcal{T} \times \mathcal{T}} \sqrt{\left(\hat{\delta}_k^{*b}(\tau) - \hat{\delta}_k(\tau)\right)' \hat{V}_k(\tau)^{-1} \left(\hat{\delta}_k^{*b}(\tau) - \hat{\delta}_k(\tau)\right)}. \tag{25}$$

The $(1 - \alpha)$ functional confidence bands for a coefficient $\hat{\delta}_k(\tau)$ can be constructed by

$$\hat{\delta}_k(\tau) \pm \hat{t}_{1-\alpha}^* \cdot \sqrt{\hat{V}_k(\tau)},$$

where $\hat{t}_{1-\alpha}^*$ is the $1 - \alpha$ quantile of $\{t_b^{KS} : 1 \leq b \leq B\}$. For more information, see Chernozhukov et al. (2013).

Table 2: Bias and Standard Deviation

| $\tau_1 \setminus \tau_2$ | $\beta$ 0.25 | $\beta$ 0.5 | $\beta$ 0.75 | $\gamma$ 0.25 | $\gamma$ 0.5 | $\gamma$ 0.75 |
|---|---|---|---|---|---|---|
| | | | (m, n) = (25,25) | | | |
| 0.25 | -0.023 | 0.004 | 0.034 | -0.030 | -0.006 | 0.018 |
| | (0.119) | (0.110) | (0.117) | (0.243) | (0.222) | (0.239) |
| 0.5 | -0.021 | -0.001 | 0.027 | -0.029 | -0.010 | 0.014 |
| | (0.114) | (0.106) | (0.111) | (0.240) | (0.219) | (0.235) |
| 0.75 | -0.029 | -0.005 | 0.024 | -0.031 | -0.012 | 0.014 |
| | (0.114) | (0.112) | (0.119) | (0.246) | (0.222) | (0.236) |
| | | | (m, n) = (25,200) | | | |
| 0.25 | -0.010 | 0.000 | 0.007 | -0.004 | 0.006 | 0.019 |
| | (0.071) | (0.067) | (0.072) | (0.237) | (0.215) | (0.232) |
| 0.5 | -0.010 | -0.002 | 0.005 | -0.004 | 0.004 | 0.018 |
| | (0.067) | (0.066) | (0.070) | (0.237) | (0.215) | (0.235) |
| 0.75 | -0.010 | -0.004 | 0.006 | -0.007 | 0.004 | 0.017 |
| | (0.070) | (0.069) | (0.072) | (0.237) | (0.217) | (0.238) |
| | | | (m, n) = (200,25) | | | |
| 0.25 | -0.023 | 0.004 | 0.030 | -0.018 | 0.003 | 0.022 |
| | (0.043) | (0.040) | (0.042) | (0.082) | (0.072) | (0.078) |
| 0.5 | -0.024 | -0.001 | 0.023 | -0.018 | 0.001 | 0.019 |
| | (0.041) | (0.037) | (0.040) | (0.078) | (0.072) | (0.077) |
| 0.75 | -0.032 | -0.007 | 0.020 | -0.020 | -0.002 | 0.018 |
| | (0.043) | (0.038) | (0.042) | (0.079) | (0.072) | (0.078) |
| | | | (m, n) = (200,200) | | | |
| 0.25 | -0.005 | 0.001 | 0.006 | -0.004 | 0.001 | 0.003 |
| | (0.028) | (0.026) | (0.028) | (0.076) | (0.073) | (0.079) |
| 0.5 | -0.005 | 0.000 | 0.006 | -0.004 | 0.000 | 0.003 |
| | (0.028) | (0.025) | (0.028) | (0.076) | (0.073) | (0.079) |
| 0.75 | -0.006 | 0.000 | 0.006 | -0.005 | 0.001 | 0.002 |
| | (0.028) | (0.026) | (0.028) | (0.077) | (0.073) | (0.079) |
| | | | (m, n) = (200,400) | | | |
| 0.25 | -0.003 | 0.000 | 0.003 | -0.004 | -0.003 | 0.002 |
| | (0.026) | (0.023) | (0.026) | (0.077) | (0.073) | (0.079) |
| 0.5 | -0.003 | 0.000 | 0.003 | -0.004 | -0.003 | 0.002 |
| | (0.025) | (0.023) | (0.025) | (0.077) | (0.073) | (0.079) |
| 0.75 | -0.004 | -0.001 | 0.003 | -0.005 | -0.004 | 0.002 |
| | (0.026) | (0.024) | (0.026) | (0.077) | (0.073) | (0.079) |

*Notes:* Results based on 2,000 Monte Carlo simulations. The table provides bias and standard deviation.

# 6 Simulations

To analyze the small sample performance of the estimator, I perform a Monte Carlo simulation with the following data generating process:

$$y_{ij} = 1 + \beta \cdot x_{1ij} + \gamma \cdot x_{2j} + \eta_j(1 - 0.1 \cdot x_{1ij} - 0.1 \cdot x_{2j}) + \nu_{ij}(1 + 0.1 \cdot x_{1ij} + 0.1 \cdot x_{2j}),$$

with $x_{1ij} = 1 + h_j + w_{ij}$, where $h_j \sim U[0,1]$ and $w_{ij}, x_{2j}, \eta_j, \nu_{ij}$ are $N(0,1)$. This is a

[27] Compared to the nonparametric bootstrap, the parametric one shows better performance in simulations as the latter was conservative.

Table 3: Coverage Probability

| $\tau_1 \setminus \tau_2$ | $\beta$ 0.25 | $\beta$ 0.5 | $\beta$ 0.75 | $\gamma$ 0.25 | $\gamma$ 0.5 | $\gamma$ 0.75 |
|---|---|---|---|---|---|---|
| | (m, n) = (25,25) | | | | | |
| 0.25 | 0.970 | 0.973 | 0.969 | 0.948 | 0.954 | 0.953 |
| 0.5 | 0.972 | 0.973 | 0.970 | 0.949 | 0.951 | 0.948 |
| 0.75 | 0.971 | 0.968 | 0.972 | 0.949 | 0.958 | 0.946 |
| | (m, n) = (25,200) | | | | | |
| 0.25 | 0.985 | 0.987 | 0.985 | 0.957 | 0.959 | 0.965 |
| 0.5 | 0.986 | 0.985 | 0.981 | 0.956 | 0.956 | 0.964 |
| 0.75 | 0.988 | 0.988 | 0.987 | 0.955 | 0.953 | 0.954 |
| | (m, n) = (200,25) | | | | | |
| 0.25 | 0.916 | 0.948 | 0.899 | 0.929 | 0.943 | 0.928 |
| 0.5 | 0.905 | 0.955 | 0.925 | 0.936 | 0.954 | 0.932 |
| 0.75 | 0.878 | 0.952 | 0.931 | 0.941 | 0.959 | 0.943 |
| | (m, n) = (200,200) | | | | | |
| 0.25 | 0.964 | 0.965 | 0.954 | 0.948 | 0.938 | 0.940 |
| 0.5 | 0.955 | 0.961 | 0.956 | 0.945 | 0.940 | 0.944 |
| 0.75 | 0.961 | 0.963 | 0.961 | 0.947 | 0.942 | 0.947 |
| | (m, n) = (200,400) | | | | | |
| 0.25 | 0.957 | 0.958 | 0.961 | 0.948 | 0.936 | 0.939 |
| 0.5 | 0.963 | 0.961 | 0.961 | 0.946 | 0.938 | 0.934 |
| 0.75 | 0.959 | 0.963 | 0.959 | 0.944 | 0.940 | 0.931 |

*Notes:* Results based on 2,000 Monte Carlo simulations. The table provides the coverage probability of the 95% confidence intervals.

location-scale-shift model over both quantile indices. I set $\beta = \gamma = 1$. The true coefficients are $\beta(\tau_1, \tau_2) = \gamma(\tau_1, \tau_2) = 1 + 0.1 \cdot F^{-1}(\tau_1) - 0.1 \cdot F^{-1}(\tau_2)$, where $F$ is the standard normal cdf. I consider the sample sizes $(m, n) = \{(25, 25), (200, 25), (25, 200), (200, 200), (200, 400)\}$ and focus on the set of quantiles $\{0.25, 0.5, 0.75\}$ using 2,000 Monte Carlo simulations.

Table 2 shows the bias and standard deviation. Table 3 shows the coverage probability of the 95% confidence intervals. Bootstrap standard errors are computed using 200 repetitions.

While $\hat{\beta}$ and $\hat{\gamma}$ share the same asymptotic properties, their finite sample behavior differs. The simulations reveal that the bias of $\hat{\beta}$ decreases with both larger $n$ and $m$, whereas the bias of $\hat{\gamma}$ decreases primarily with increasing $m$. Similarly, the variance of $\hat{\gamma}$ is only marginally affected by a larger number of observations per group, while the variance of $\hat{\beta}$ improves more noticeably with increasing $n$. Nonetheless, the most substantial reduction in variance is achieved by increasing the number of groups. As $n$ grows, further increases in within-group observations have little impact on the variance of $\hat{\beta}$. Specifically, the initial decline in the bias and variance of $\hat{\beta}$ – primarily driven by a reduction in first-stage error – declines quickly with increasing $n$, and the difference between $n = 200$ and $n = 400$ is minimal.

The coverage probabilities of the 95% confidence intervals in Table 3 are close to 95%.

There are some minor discrepancies which, however, disappear as the number of groups and observations per group increase. In some instances with $(m, n) = (200, 25)$, the confidence interval tends to undercover, but this is likely driven by the large bias arising in the first stage (see Table 2).

Table 4 shows the rejection probabilities of 5% Kolmogorov-Smirnov and Cramér-von Mises

Table 4: Rejection Probability of the KS and CvM Tests

| $(\phi, \psi)$ | (0, 0) | (0, 0.1) | (0.1, 0) | (0.1, 0.1) | (0.2, 0.2) |
|---|---|---|---|---|---|
| Panel (a): Kolmogorov-Smirnov Test | | | | | |
| $H_0 : \gamma(\tau) = \bar{\gamma}$ | | | | | |
| (m, n) = (25,25) | 0.007 | 0.005 | 0.007 | 0.009 | 0.034 |
| (m, n) = (25,200) | 0.015 | 0.013 | 0.020 | 0.032 | 0.173 |
| (m, n) = (200,25) | 0.026 | 0.209 | 0.251 | 0.469 | 0.996 |
| (m, n) = (200,200) | 0.046 | 0.307 | 0.397 | 0.826 | 1.000 |
| $H_0 : \beta(\tau) = \bar{\beta}$ | | | | | |
| (m, n) = (25,25) | 0.026 | 0.108 | 0.101 | 0.156 | 0.537 |
| (m, n) = (25,200) | 0.056 | 0.536 | 0.548 | 0.885 | 1.000 |
| (m, n) = (200,25) | 0.026 | 0.767 | 0.822 | 0.970 | 1.000 |
| (m, n) = (200,200) | 0.057 | 1.000 | 1.000 | 1.000 | 1.000 |
| Panel (b): Cramér-von Mises Test | | | | | |
| $H_0 : \gamma(\tau) = \bar{\gamma}$ | | | | | |
| (m, n) = (25,25) | 0.014 | 0.026 | 0.022 | 0.027 | 0.165 |
| (m, n) = (25,200) | 0.023 | 0.030 | 0.035 | 0.047 | 0.381 |
| (m, n) = (200,25) | 0.044 | 0.381 | 0.414 | 0.789 | 1.000 |
| (m, n) = (200,200) | 0.061 | 0.446 | 0.430 | 0.895 | 1.000 |
| $H_0 : \beta(\tau) = \bar{\beta}$ | | | | | |
| (m, n) = (25,25) | 0.038 | 0.223 | 0.231 | 0.373 | 0.921 |
| (m, n) = (25,200) | 0.068 | 0.728 | 0.844 | 0.988 | 1.000 |
| (m, n) = (200,25) | 0.048 | 0.937 | 0.995 | 1.000 | 1.000 |
| (m, n) = (200,200) | 0.056 | 1.000 | 1.000 | 1.000 | 1.000 |

*Notes:* The table shows the rejection probabilities of a 5% test testing the null hypothesis of effects homogeneity over both dimensions. The test is performed using the Kolmogorov-Smirnov and the Cramér von Mises tests statistics. The results are based on 1,000 Monte Carlo simulations using 100 bootstrap replications.

tests for the null hypotheses that $\beta(\tau) = \bar{\beta}$ and that $\gamma(\tau) = \bar{\gamma}$.[28] For this set of simulations, I consider a variation of the data-generating process above. The data is generated by:

$$y_{ij} = 1 + x_{1ij} + x_{2j} + \eta_j(1 - \psi(x_{1ij} + x_{2j})) + \nu_{ij}(1 + \phi(x_{1ij} + x_{2j})),$$

where all variables' distribution are unchanged. The parameter $\phi$ regulate effect heterogeneity over $\tau_1$ and $\psi$ determines heterogeneity over $\tau_2$. I consider five different combinations of values of $\phi$ and $\psi$ and test the null hypothesis of effect homogeneity of both $x_{1ij}$ and $x_{2j}$. The rejection

[28] The Cramér-von Mises test statistics is $t^{CvM} = \int_{\mathcal{T}} \int_{\mathcal{T}} \left(\hat{\delta}_k(\tau) - \bar{\delta}_k\right)' \hat{V}_k(\tau) \left(\hat{\delta}_k(\tau) - \bar{\delta}_k\right) d\tau_1 d\tau_2$, and the critical value are computed by bootstrapping the test statistic as with the Kolmogorov-Smirnov test.

probabilities are computed using 400 simulations, and each of these simulations includes 100 bootstrap replications. I conduct the simulations on the set of quantiles $0.1, 0.2, \ldots, 0.9$. In the first column, the null hypothesis is true, and we should observe a rejection probability of 5% (empirical size). In all other cases, the rejection probability indicates the power of the test. In smaller samples, the empirical size tends to be conservative. Additionally, with little heterogeneity and small sample sizes, the test seems to have difficulty detecting the heterogeneity. However, columns (2)-(5) demonstrate that as the degree of heterogeneity or the sample size increases, the test becomes more powerful. There are also notable differences in the performance of the test for $\beta$ compared to $\gamma$. When the number of groups is small, the test on $\gamma$ shows substantially lower power. Nonetheless, this discrepancy diminishes as $m$ increases, leading to a substantial improvement in the power of the test. In contrast, heterogeneity in $\beta$ appears easier to detect in small samples, with the power of this test increasing as the number of observations per group grows. In comparison, for $\gamma$, increases in $n$ yield only marginal improvements in power.

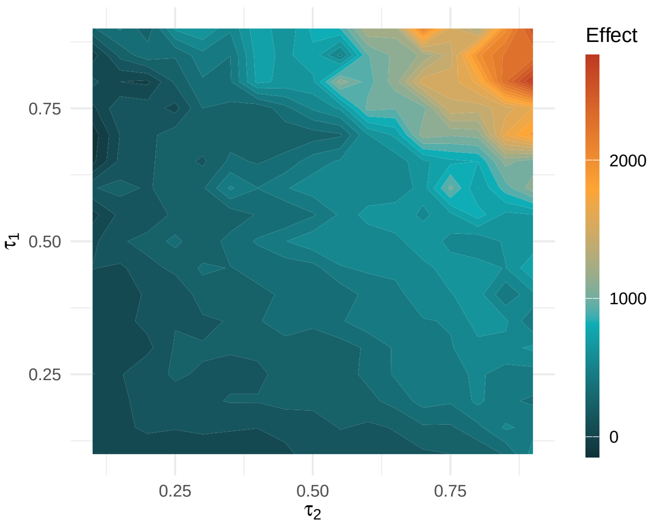
In the simulations, the Cramér-von Mises test demonstrates greater power than the Kolmogorov–Smirnov test. However, it is important to note that different data-generating processes may lead to different results, as each test is better at detecting certain types of deviations. Consequently, there is no uniformly more powerful test, as one may outperform the other depending on the nature of the deviation.

# 7 Empirical Application: Distributional Impacts of Business Training within and between Markets

In an empirical application, I complement the findings of McKenzie and Puerto (2021) by offering new insights into distributional effects. Their study aims to analyze the impact of business training on the outcomes of female-owned businesses and the spillover effects of the program.[29] The sample comprises 3,537 female-owned businesses operating in 157 different rural markets in Kenya. The training program is randomly assigned to firms through a two-stage randomization process. The first stage involves market-level randomization, where 93 markets are treatment markets, and the remaining 64 serve as control markets. In the second stage, individual-level randomization assigns businesses in the treatment markets to training or

[29] The training program spans five days and covers topics such as bookkeeping, recordkeeping, marketing, financial concepts, costing and pricing, and the development of new business ideas. Moreover, it specifically addresses challenges faced by women in business. For more detailed information about the program's structure or the experimental setting, refer to McKenzie and Puerto (2021) and their appendix.

Figure 3: Effect of Training Assignment on Income from Work

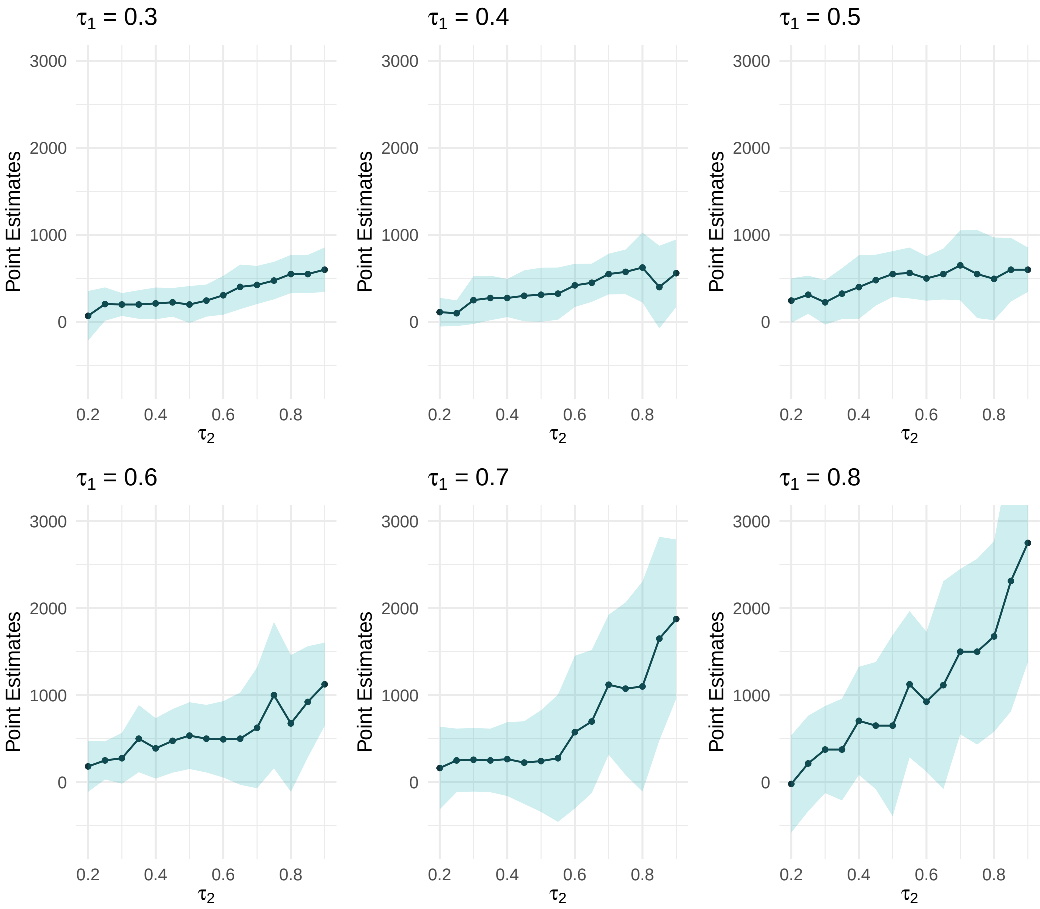



*Notes*: The figure shows the effect of the treatment on the within market $\tau_1$ and the between market distribution $\tau_2$ of income from work.

control. Randomization is stratified by geographical region, market size, and quartiles of weekly profits to ensure a balanced sample. This results in 1,172 individual firms assigned to training and 2,365 firms assigned to the control group. The firms in the sample are primarily engaged in retail activities, such as selling fruits and vegetables or grains. A significant share also operate as restaurants or tailoring businesses. Most of these businesses are small-scale operations without employees–only about 20% report having any employees.

McKenzie and Puerto (2021) find a positive effect of training on the business survival after three years. Further, the training increases average weekly sales and profits by 18 and 15 percent, respectively, and firm owners assigned to the training report better mental health and a higher subjective standard of living. However, the spillover effects on businesses in treatment markets not assigned to the program remain unclear, with point estimates being small and not statistically significant. Further, in the original paper, they estimate the distributional effects of training on profits and sales. This analysis uses data collected in two waves three years after the training program, and they document larger effects in the upper tail of the outcome distribution. In my analysis, I use data from the same two waves, and if a business is observed in both waves, I average the outcomes over the two waves. Further, I define groups based on markets, and to ensure that I have enough observations for the estimation, I drop markets with fewer than 15 businesses, though results remain similar with different cutoffs. The final dataset includes 2,922 firms operating in 116 markets. On average, there are 27 observations per market. I use income

Figure 4: Effect of Training Assignment on Income From Work




*Notes*: The figure shows the effect of the treatment on the between market distribution $\tau_2$ of income from work for selected quantiles of the within market distribution $\tau_1$. The shaded areas show the 95% confidence intervals estimated using clustered bootstrap standard errors computed with 1000 replications.

from work for the business owner as the outcome of interest.[30] In the sample, the average income from work is 2,300 Kenyan shillings.[31]

The dependent variable has a mass point at zero. One reason is that these variables are coded to zero for firms that exit the market. More precisely, around 11% of the firms in the final dataset did not survive after three years and 10% of the owners report having no income. Due to this censoring issue, I refrain from computing the effects too far in the lower tail. Since these mass points could invalidate inference, I do not report confidence intervals for quantiles affected by the problem.[32] Further away from the lower tail, this problem neither affects the results nor invalidates inference.

[30] Using weekly sales or weekly profits as outcome variables gives similar results.

[31] In April 2023, 1,000 Kenyan Shillings are around 7.5 USD.

[32] If the second stage fitted values for at least one observation equal zero, I will consider the cell as being affected by the mass point. Fitted values of zero suggest a perfect fit, at least for some observations.

I estimate the following model:

$$y_{ij} = \beta_1(u_{ij}, v_j) \cdot d_{ij} + \beta_2(u_{ij}, v_j) \cdot s_{ij} + \alpha(u_{ij}, v_j), \tag{26}$$

where $y_{ij}$ is the income of the owner of firm $i$ operating in market $j$, $d_{ij}$ is the treatment dummy, and $s_{ij}$ is a binary variable that accounts for potential spillover effects and takes value 1 for firms in the treatment markets that are assigned to the control group.

Table 5: $P$-Values of Cramér-von Mises and Kolmogorov-Smirnov Tests

| | $P$-Value |
|---|---|
| Cramér-von Mises | 0.024 |
| Kolmogorov-Smirnov | 0.006 |

*Notes:* The table shows the p-values of the Cramér-von Mises and Kolmogorov-Smirnov tests for the null hypothesis that the coefficients are homogeneous over both dimensions. The test is performed with a parametric bootstrap using 1000 replications.

Figures 3 shows the treatment effects estimates over the two dimensions for the quantiles indices $\{0.2, 0.3, \dots, 0.9\}$. Figures 4 plots the point estimates and confidence intervals over the distribution of markets $\tau_2$ when fixing the within-market quantile $\tau_1$. The results suggest that both within-group and between-group heterogeneities play essential roles, resulting in larger positive treatment effects in the upper tail of both distributions. For instance, at $\tau_1 = 0.7$, the effect increases from 200 Kenyan Shillings in the lower tail of the distribution to well over 1,000 in the upper tail. Simultaneously, the within-market rank plays a major role, even for firms operating in the most prosperous markets, where the effect goes from 600 Kenyan Shillings at $\tau_1 = 0.3$ to well over 2,000 at $\tau_1 = 0.8$. The larger effect in the upper tail of both distributions could indicate the presence of complementarities between individual ability and market quality, suggesting that both are necessary to benefit from the treatment.

I perform a Kolmogorov-Smirnov and Carmér-von-Mises tests to test for treatment effect heterogeneity over $(\tau_1, \tau_2)$ and report the $p$-value of the test statistic in Table 5. As shown, we reject the null of treatment effect homogeneity at the 5% level, hence providing evidence of treatment effect heterogeneity.

To assess the extent to which groups in the upper tail of the between distribution at low values of $\tau_1$ are also in the upper tail for high values of $\tau_1$, Table 6 presents the correlation matrix of group ranks across the $\tau_1$ dimension. The table provides the correlation of group ranks at any two points of the within distribution. If heterogeneity between groups is solely due to a location shift, the ranks do not change over $\tau_1$, and we would observe a correlation of 1 for any two values

Table 6: Correlation of Ranks over $\tau_1$ for Income from Work

| | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 1 | | | | | | | |
| 0.3 | 0.74 | 1 | | | | | | |
| 0.4 | 0.65 | 0.87 | 1 | | | | | |
| 0.5 | 0.53 | 0.76 | 0.85 | 1 | | | | |
| 0.6 | 0.49 | 0.66 | 0.72 | 0.82 | 1 | | | |
| 0.7 | 0.42 | 0.6 | 0.66 | 0.69 | 0.83 | 1 | | |
| 0.8 | 0.36 | 0.51 | 0.58 | 0.62 | 0.77 | 0.88 | 1 | |
| 0.9 | 0.32 | 0.44 | 0.42 | 0.47 | 0.59 | 0.6 | 0.69 | 1 |

*Notes:* The table shows the correlation matrix of the ranks at different values of $\tau_1$.

of $\tau_1$. A lower correlation suggests that groups rank differently at different points of the within distribution, indicating that there is no unique notion of poor-performing or good-performing markets across $\tau_1$. The correlation matrix shows that the ranks vary across the $\tau_1$, providing evidence that a univariate rank variable would miss important characteristics of the data. Yet, it is also noticeable that the rank at $\tau_1 = 0.5$ is correlated with the ranks at different points in the distribution, indicating the presence of some underlying mechanism affecting the entire group distribution.

Finally, to assess the impact of the training program on welfare, I compute welfare under both the realized outcome and a counterfactual scenario without the intervention. I then calculate the percentage improvement in welfare attributable to the treatment. Welfare is measured using the function defined in equation (10), considering four different weighting functions. First, I use a utilitarian specification with $w(\tau_1, \tau_2) = 1$. Then, I apply a two-dimensional extension of the Gini social welfare function, given by $w(\tau_1, \tau_2) = 2(1 - \omega\tau_1 - (1 - \omega)\tau_2)$, with $\omega \in \{0.2, 0.5, 0.8\}$, to capture different trade-offs between within- and between-group inequality. The treatment leads to higher welfare, as the effect is positive and there is no evidence of negative spillovers. Moreover, the welfare gain is larger when the policymaker assigns more weight to within-market inequality, likely because the training program increases the disparity between markets.

# 8 Conclusion

Distributional effects are particularly interesting when analyzing treatment effect heterogeneity. In economics, heterogeneity manifests itself across various dimensions, encompassing within and between groups, where groups can be, among others, geographical regions, industries, or firms. I argue that both these dimensions are welfare relevant and might be interdependent in many settings. At the same time, papers looking at both dimensions are rare and often implic-

Table 7: Welfare gains under different weighting schemes

| Weighting Scheme | Welfare Gain (%) |
|---|---|
| $\omega = 0.2$ | 11.53 |
| $\omega = 0.5$ | 13.14 |
| $\omega = 0.8$ | 15.16 |
| Utilitarian ($w = 1$) | 15.33 |

*Notes*: The table reports the percentage increase in welfare under the treatment scenario relative to the counterfactual without treatment. Welfare is computed using equation (10) with different weighting schemes. The utilitarian case sets $w(\tau_1, \tau_2) = 1$, while the other cases use $w(\tau_1, \tau_2) = 2(1 - \omega\tau_1 - (1 - \omega)\tau_2)$, with $\omega \in \{0.2, 0.5, 0.8\}$.

itly impose strong restrictions on the social welfare function. This paper provides a method to analyze heterogeneity and distributional effects within and between groups simultaneously while at the same time remaining agnostic about the objective function of the policymaker. To this end, I introduce a quantile model with two quantile indices: one capturing heterogeneity within groups and the other addressing heterogeneity between groups. The conditional quantile function of each group models the within-group heterogeneity. Then, to aggregate the results over the distribution of groups, I model the conditional quantile function of these group-level quantile functions. I show that constructing the two-level quantile function involves a trade-off between a simple model with a unique group rank and a more flexible model that allows for unrestricted heterogeneity between groups, and where group ranks can change over the within distribution. This paper follows the second approach as this offers a more realistic model. I show that the method can be used for policy evaluation when the interest is on the effect on the distribution and the policymaker wants to consider a trade-off between different dimensions of inequality. Further, under the stronger condition of rank invariance, the model identifies individual effects and can, therefore, be used for optimal treatment assignment, where the optimal policy rule exploits multiple dimensions of treatment effect heterogeneity to assign the treatment.

I suggest estimating the model using a two-step quantile regression estimator with within-group regressions in the first stage and between-group regressions in the second stage. I show uniform consistency and weak convergence when the number of observations per group and the number of groups grows to infinity. The estimator can provide new insights about inequalities in grouped data. For instance, a descriptive illustration studies income heterogeneity within and between the labor markets in Switzerland. The results show that a large portion of the group-level heterogeneities are driven by high top wages in a few regions, whereas for most of the within distribution, differences between regions are less marked. A result that conventional methods, such as variance decompositions or simple comparisons of mean or median wages, would fail to capture. Furthermore, the data show that group ranks change substantially over the within distribution, suggesting that differences in median wages between regions do not provide a meaningful picture of the labor market situation of low-income individuals.

Finally, in an empirical application, I extend the findings of McKenzie and Puerto (2021) by assessing the impact of business training on firm performance in Kenya, allowing for treatment effect heterogeneity within and between markets. I find large positive effects on income from work of the business owners with stronger effects among good-performing firms (in their markets) operating in thriving markets, indicating that there might be complementarities between individual and group ranks.

As pointed out in Remark 2, it would be possible to extend these results to include instrumental variables. Future researchers could combine the computational advancement with new estimation methods enabling the extension of these results to account for endogenous treatments and alternative identification strategies.

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# A Preliminary Lemmas

Let $\mathcal{B}$ is a vector space endowed with a pseudo-metric $||\cdot||_{\mathcal{B}}$, which is a sup-norm metric in the sense that $||\beta-\beta'||_{\mathcal{B}}=\sup_j||\beta_j-\beta'_j||$.

**Lemma 2** (Uniform consistency of $\hat{\beta}_j(\tau_1)$)**.** *Under Assumptions 1-4 and 9(a), we have*

$$\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\max_{1\leq j\leq m}\|\hat{\beta}_j(\tau_1)-\beta_{j,0}(\tau_1)\|=o_p(1).$$

*Proof.* The proof follows directly by the proof of Lemma 3 in Melly and Pons (2025) after noting that $\hat{\beta}(\tau_1)$ does not depend on $\tau_2$. ∎

**Lemma 3** (Bahadur representation of the first stage estimator)**.** *Let assumption 1-4 be satisfied. Then,*

$$\hat{\beta}_j(\tau_1)-\beta_j(\tau_1)=\frac{1}{n}\sum_{i=1}^{n}\phi_{j,\tau_1}(\tilde{x}_{ij},y_{ij})+R^{(1)}_{nj}(\tau_1)+R^{(2)}_{nj}(\tau_1),\tag{27}$$

*where*

$$\phi_{j,\tau_1}(\tilde{x}_{ij},y_{ij})=-B^{-1}_{j,\tau_1}\tilde{x}_{ij}(1(y_{ij}\leq\tilde{x}_{ij}\beta_j(\tau_1))-\tau_1),\tag{28}$$

*with* $B_{j,\tau_1}=\mathbb{E}_{i|j}[f_{y|x}(Q_{y|x}(\tau_1|\tilde{x}'_{ij}\beta_j)|\tilde{x}_{ij})\tilde{x}_{ij}\tilde{x}'_{ij}]$ *and*

$$\sup_j\sup_{\tau_1\in\mathcal{T}}\left\|R^{(2)}_{nj}(\tau_1)\right\|=O_p\left(\frac{\log n}{n}\right)\tag{29}$$

$$\sup_j\sup_{\tau_1\in\mathcal{T}}\left\|\mathbb{E}_{i|j}\left[R^{(1)}_{nj}(\tau_1)\right]\right\|=O\left(\frac{\log n}{n}\right)\tag{30}$$

$$\sup_j\sup_{\tau\in\mathcal{T}}\left\|\mathbb{E}\left[\left(R^{(1)}_{nj}(\tau_1)-\mathbb{E}_{i|j}[R^{(1)}_{nj}(\tau_1)]\right)\left(R^{(1)}_{nj}(\tau)-\mathbb{E}_{i|j}[R^{(1)}_{nj}(\tau_1)]\right)'\right]\right\|=O\left(\left(\frac{\log n}{n}\right)^{3/2}\right).\tag{31}$$

*Proof.* See Lemma 3 in Galvao et al. (2020). ∎

**Lemma 4.** *Under assumptions 1-2 and 7 ,*

$$\sup_{||\beta-\beta_0||_{\mathcal{B}}\leq\zeta_m,\ ||\delta-\delta_0||\leq\zeta_m}||M_{mn}(\delta,\beta,\tau)-M(\delta,\beta,\tau)-M_{mn}(\delta_0,\beta_0,\tau)||=o_p(m^{-1/2}),\tag{32}$$

*for all positive sequences* $\zeta_m=o(1)$.

*Proof.* This result is implied by Theorem 3 in Chen et al. (2003). Hence, I show now that the conditions to apply the theorem are satisfied. First, recall that $\mathbb{E}[m(\delta_0,\beta_0,\tau)]=0$ and that by Assumption 1 the data is i.i.d. To check condition (3.1), note that $m(\delta,\beta,\tau)=\rho_{\tau_2}(\tilde{x}'_{ij}\beta_j-x'_{ij}\delta)$. By the properties of the check function $\rho_{\tau_2}(y+z)-\rho_{\tau_2}(y)\leq 2\cdot||z||$. Hence,

$$\begin{aligned}
&m(\delta',\beta',\tau)-m(\delta'',\beta'',\tau)\\
&\quad=\rho_{\tau_2}(\tilde{x}'_{ij}\beta'_j-x'_{ij}\delta')-\rho_{\tau_2}(\tilde{x}'_{ij}\beta''_j-x'_{ij}\delta')+\rho_{\tau_2}(\tilde{x}'_{ij}\beta''_j-x'_{ij}\delta')-\rho_{\tau_2}(\tilde{x}'_{ij}\beta''_j-x'_{ij}\delta'')\\
&\quad\leq 2||\tilde{x}'_{ij}(\beta'_j-\beta''_j)||_{\mathcal{B}}+2||x'_{ij}(\delta'-\delta'')||.
\end{aligned}\tag{33}$$

It follows that $m(\delta, \beta, \tau)$ is Hölder continuous because

$$|m(\delta', \beta', \tau) - m(\delta'', \beta'', \tau)| \leq C_1 ||\beta_j' - \beta_j''||_{\mathcal{B}} + C_2 ||\delta' - \delta''||$$

with $C_1 = 2 \cdot ||\tilde{x}_{ij}||$ and $C_2 = 2 \cdot ||x_{ij}||$. which are bounded by assumption 2. This implies that condition (3.1) in Chen et al. (2003) is satisfied.

Condition (3.2) is satisfied as

$$\begin{aligned}
\mathbb{E}\Bigg[ \sup_{||\beta_j' - \beta_j|| \leq \zeta,\ ||\delta' - \delta|| \leq \zeta} & |m(\delta, \beta, \tau) - m(\delta', \beta', \tau)|^2 \Bigg] \\
& \leq \mathbb{E}\big[ ||x_{ij}||^2 |1(\tilde{x}_{ij}' \beta_j(\tau_1) \leq x_{ij}' \delta) - 1(\tilde{x}_{ij}' \beta_j(\tau_1) \leq x_{ij}'(\delta + \zeta))| \\
& \quad + ||x_{ij}||^2 |1(\tilde{x}_{ij}' \beta_j(\tau_1) \leq x_{ij}'(\delta + \zeta)) - 1(\tilde{x}_{ij}'(\beta_j(\tau_1) + \zeta) \leq x_{ij}'(\delta + \zeta))| \big] \\
& \leq \mathbb{E}\big[ ||x_{ij}||^2 |F_{Q(\tau_1, y|x,\nu)|x}(x_{ij}' \delta) - F_{Q(\tau_1, y|x,\nu)|x}(x_{ij}'(\delta + \zeta))| \\
& \quad + ||x_{ij}||^2 |F_{Q(\tau_1, y|x,\nu)|x}(x_{ij}' \delta) - F_{Q(\tau_1, y|x,\nu)|x}(x_{ij}'(\delta + \zeta) - \tilde{x}_{ij}' \zeta))| \big] \\
& \leq K \cdot \zeta
\end{aligned}$$

for some $K < \infty$, since $x_{ij}$ is bounded by assumption 2.
To check condition (3.3), I start by noting that by assumption 7, $\mathcal{D}$ is a compact subset of $\mathbb{R}^K$. Further $\beta_j \in \mathcal{B}_j$ for all $j$, where $\mathcal{B}_j$ is a compact set of $\mathbb{R}^{K_1}$. It follows by Tychonoff's Theorem that $\beta \in \mathcal{B}$, where $\mathcal{B} = \prod_{j=1}^m \mathcal{B}_j$ is also compact. Since both sets are compact, the covering numbers of $\mathcal{B}$ and $\mathcal{D}$ are known, and the condition is satisfied.

∎

# B Proofs of Asymptotic Results

## B.1 Pointwise Results

### B.1.1 Consistency

**Theorem 4** (**Consistency**). *Let assumptions 1-7 and 9(a) be satisfied. Then,* $\hat{\delta}(\hat{\beta}, \tau) - \delta_0(\beta_0, \tau) \xrightarrow{p} 0$.

*Proof of Theorem 4.* To prove the results, I apply Theorem 1 in Chen et al. (2003) and start by showing that the conditions to apply the theorem are satisfied. First, by definition $M(\delta_0, \beta_0, \tau) = 0$ and $||M_{mn}(\hat{\delta}, \hat{\beta}, \tau)|| \leq \inf_{\delta \in \mathcal{D}_\zeta} ||M_{mn}(\delta, \hat{\beta}, \tau)|| + o_p(1)$ so that condition (1.1) is satisfied. Condition (1.4) is implied by Lemma 2 and (1.5) is implied by Lemma 4. Condition (1.3) is satisfied since $M(\delta, \beta, \tau)$ is Lipschitz-continuous over $\beta_j$ at $\beta_j = \beta_{j,0}$ with respect to the metric $|| \cdot ||_{\mathcal{B}}$. Condition (1.2) is satisfied as $M(\delta, \beta_0)$ is uniquely minimized at $M(\delta_0, \beta_0)$, since $\mathbb{E}[x_{ij} x_{ij}']$ is full rank (Assumption 2) and by Assumption 6. Since all the conditions are satisfied, the result follows by Theorem 1 in Chen et al. (2003).

∎

### B.1.2 Asymptotic Normality

*Proof of Lemma 1.* **Part (i)** Inserting equation (27) from Lemma 3 in equation (18) gives:

$$\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)(\hat{\beta}_j-\beta_{j,0}) \tag{34}$$

$$=\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]\left(\frac{1}{n}\sum_{i=1}^{n}\phi_{j,\tau_1}(\tilde{x}_{ij},y_{ij})\right) \tag{35}$$

$$+\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]R_{nj}^{(1)}(\tau_1) \tag{36}$$

$$+\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]R_{nj}^{(2)}(\tau_1) \tag{37}$$

First, note that by Assumption 2, $x_{ij}$ is bounded by a constant $C$ such that $x_{ij}\tilde{x}_{ij}'$ is also bounded. Further, by Assumption 5, $f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})$ is also bounded uniformly over $\tau$. It follows directly that the conditional expectation $\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]$ is bounded uniformly over $\tau$.

Next, consider the third term (37). Together with equation (29), it implies that

$$\sup_{\tau_2\in\mathcal{T}}\sup_{\tau_1\in\mathcal{T}}\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]R_{nj}^{(2)}(\tau_1)=O_p\left(\frac{\log n}{n}\right). \tag{38}$$

For the second term (36), Since $\mathrm{Var}\left(R_{nj}^{(1)}(\tau)\right)=o\left(\frac{1}{n}\right)$ by (31), the conditional expectation is bounded and since the observations are independent across groups, we have that

$$\mathrm{Var}\left(\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]R_{nj}^{(1)}(\tau_1)\right)=o_p\left(\frac{1}{mn}\right).$$

In addition, by (30), $\sup_j\sup_{\tau_1\in\mathcal{T}}\mathbb{E}_{i|j}\left[R_{nj}^{(1)}(\tau_1)\right]=O\left(\frac{\log n}{n}\right)$ such that

$$\sup_{\tau_2\in\mathcal{T}}\sup_{\tau_1\in\mathcal{T}}\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]R_{nj}^{(1)}(\tau_1)=O\left(\frac{\log n}{n}\right)$$

where the uniformity over $\tau_2$ follows since $R_{nj}^{(1)}(\tau_1)$ does not depend on $\tau_2$. Putting this together, by the Chebyshev inequality and under Assumption 9(b), we have that

$$\sup_{\tau_2\in\mathcal{T}}\sup_{\tau_1\in\mathcal{T}}\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]R_{nj}^{(1)}(\tau_1)=o_p\left(\frac{1}{\sqrt{m}}\right). \tag{39}$$

It follows that both (36) and (37) are $o_p\left(\frac{1}{\sqrt{m}}\right)$ uniformity over $\tau$.

Consider now the first term (35):

$$
\begin{aligned}
&\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]\left(\frac{1}{n}\sum_{i=1}^{n}\phi_{j,\tau_1}(\tilde{x}_{ij},y_{ij})\right)\\
&\qquad=\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]\\
&\qquad\quad\cdot\left(-\frac{B_{j,\tau_1}^{-1}}{n}\sum_{i=1}^{n}\tilde{x}_{ij}(1(y_{ij}\le\tilde{x}_{ij}\beta_j(\tau_1))-\tau_1)\right)\\
&\qquad=\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}s_{ij}(\tau).
\end{aligned}
$$

This is a sample mean over $mn$ i.i.d. observations denoted by $s_{ij}(\tau)$. The model in equation (2) implies that $\mathbb{E}_{i|j}\left[1(y_{ij}\le\tilde{x}_{ij}\beta_j(\tau))|\tilde{x}_{ij}\right]=\tau_1$, which together with Assumption 2(iii) gives $\mathbb{E}[s_{ij}(\tau)]=0$. In addition,

$$
\begin{aligned}
\operatorname{Var}(s_{ij}(\tau_1))=&\mathbb{E}_j\left[\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)\operatorname{Var}(\phi_{j,\tau_1})\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)'\right]\\
=&\mathbb{E}_j[\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)B_{j,\tau}^{-1}\tau(1-\tau)\mathbb{E}_{i|j}[\tilde{x}_{ij}\tilde{x}_{ij}']B_{j,\tau}^{-1}\bar{\Gamma}_{2j}(\tau,\delta_0,\beta_0)'],
\end{aligned}
\tag{40}
$$

where $\bar{\Gamma}_{2j}(\delta_0,\beta_0,\tau)=\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]$.

It follows by the Lindeberg-Lévy Central Limit Theorem that

$$
\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}s_{ij}(\tau)=O_p\left(\frac{1}{\sqrt{mn}}\right).
\tag{41}
$$

This last results implies that the first term (34) is $o_p\left(\frac{1}{\sqrt{m}}\right)$ pointwise.

To get uniform results, note that

$$
\left\{\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]\left(\frac{-B_{j,\tau_1}^{-1}}{n}\sum_{i=1}^{n}\tilde{x}_{ij}(1(y_{ij}\le\tilde{x}_{ij}\beta(\tau_1))-\tau_1)\right),\tau_1\in\mathcal{T},\tau_2\in\mathcal{T},\beta\in\mathcal{B}\right\}
$$

is a Donsker class for any compact set $\mathcal{B}$. This follows by noting that $\{1(y_{ij}\le\tilde{x}_{ij}\beta_j(\tau)),\tau_1\in\mathcal{T},\tau_2\in\mathcal{T},\beta\in\mathcal{B}\}$ is a VC subgraph class and hence a bounded Donsker class. Hence,

$$
\left\{\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_{ij}(1(y_{ij}\le\tilde{x}_{ij}\beta)-\tau),\tau_1\in\mathcal{T},\tau_2\in\mathcal{T},\beta\in\mathcal{B}\right\}
$$

is also bounded Donsker with a square-integrable envelope $2\cdot\max_{i\in1,\dots,n}|\tilde{x}_{ij}|\le 2\cdot C$. The whole function is then Donsker by the boundedness of $\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'\delta_0|x_{ij})x_{ij}\tilde{x}_{ij}'\right]$ and $B_{j,\tau_1}^{-1}$. Hence, it follows that the equation (34) is $o_p\left(\frac{1}{\sqrt{m}}\right)$ uniformly in $\tau_1$ and $\tau_2$.

**Part (ii)** This part of the proof is implied by the proof of Theorem 2.

**Part (iii)** Note that $\sum_{j=1}^{m} \frac{1}{m}\bar{\Gamma}_{2,j}(\beta_0,\tau)\left(\hat{\beta}_j - \beta_{j,0}\right)$ is asymptotically equivelent to (up to a $o_p\left(\frac{1}{\sqrt{m}}\right)$ term)

$$\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x'_{ij}\delta_0|x_{ij})x_{ij}\tilde{x}'_{ij}\right]\left(-\frac{B_{j,\tau_1}^{-1}}{n}\sum_{i=1}^{n}\tilde{x}_{ij}(1(y_{ij}\le \tilde{x}_{ij}\beta_j(\tau_1))-\tau_1)\right)$$

Since the observations are independent over $j$ and $i$ (Assumption 1) we only need to analyze the covariance for a given $i$ and $j$:

$$\begin{aligned}
&\mathrm{Cov}\left(\tilde{x}_{ij}(1(y_{ij}\le \tilde{x}_{ij}\beta_j(\tau_1))-\tau_1), x'_{ij}1(\tilde{x}'_{ij}\beta_{j,0}(\tau_1)\le x'_{ij}\delta_0(\beta_0,\tau)-\tau_2)\right)\\
&\quad= \mathrm{Cov}\left(\tilde{x}_{ij}(1(y_{ij}\le \tilde{x}_{ij}\beta_{j,0}(\tau_1))-\tau_1), x'_{ij}1(x'_{ij}[\delta(\tau_1,v_j)-\delta_0(\beta_0,\tau)]+\alpha(\tau_1,v_j)\le 0)-\tau_2)\right)\\
&\quad=\mathbb{E}\left[\tilde{x}_{ij}(1(y_{ij}\le \tilde{x}_{ij}\beta_{j,0}(\tau_1))-\tau_1)x'_{ij}1(x'_{ij}[\delta(\tau_1,v_j)-\delta_0(\beta_0,\tau)]+\alpha(\tau_1,v_j)\le 0)-\tau_2]\right]\\
&\quad=\mathbb{E}_j\left[\mathbb{E}_{i|j}\left[\tilde{x}_{ij}(1(y_{ij}\le \tilde{x}_{ij}\beta_{j,0}(\tau_1))-\tau_1)|x_{ij}\right]x'_{ij}1(x'_{ij}[\delta(\tau_1,v_j)-\delta_0(\beta_0,\tau)]+\alpha(\tau_1,v_j)\le 0)-\tau_2]\right]=0.
\end{aligned}$$

Where the second line follows since both terms have mean zero.

This implies that uniformly in $\tau \in \mathcal{T}\times\mathcal{T}$,

$$\mathrm{Cov}\left(M_{mn}(\delta_0,\beta_0,\tau), \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2,j}(\beta_0,\tau)\left(\hat{\beta}_j-\beta_{j,0}\right)\right)=o_p\left(\frac{1}{\sqrt{m}}\right)$$

■

**Theorem 5** (**Asymptotic Normality**)**.** *Let assumptions 1-7 and 9(b) be satisfied. Then*

$$\sqrt{m}\left(\hat{\delta}(\hat{\beta},\tau)-\delta_0(\beta_0,\tau)\right)\xrightarrow{d} N(0,\Gamma_1^{-1}\Omega_2(\tau)\Gamma_1'^{-1}) \tag{42}$$

*with* $\Gamma_1 = \Gamma_1(\delta_0,\beta_0,\tau)$.

*Proof of Theorem 5.* To prove the results, I apply Theorem 2 in Chen et al. (2003) and start by showing that the conditions to apply the theorem are satisfied. First, assumption 9(b) implies 9(a) so that by Theorem 4, $\hat{\delta}(\hat{\beta},\tau)-\delta_0(\beta_0,\tau)\xrightarrow{p} 0$. Therefore, following Chen et al. (2003), I can replace the parameter space with a small or shrinking ball around the true parameter. Let $\mathcal{D}_{\zeta,\tau}=\{\delta\in\mathcal{D}: ||\delta-\delta_0(\tau)||\le\zeta_m\}$ and $\mathcal{B}_{\zeta,\tau_1}=\{\beta\in\mathcal{B}: ||\beta-\beta_0(\tau_1)||\le\zeta_m\}$.

Next, by definition

$$||M_{mn}(\hat{\delta},\hat{\beta},\tau)|| \le \inf_{\delta\in\mathcal{D}_\zeta}||M_{mn}(\delta,\hat{\beta},\tau)||+o_p(m^{-1/2})$$

so that condition (2.1) is trivially satisfied.

Recall the matrix

$$\Gamma_1(\delta, \beta_0, \tau_2) = \frac{\partial M(\delta, \beta_0, \tau_2)}{\partial \delta} = \mathbb{E}[f_{Q(\tau_1, y_{ij}|x_{ij}, v_j)|x_{ij}}(x_{ij}'\delta|x_{ij})x_{ij}x_{ij}']. \tag{43}$$

By assumption 5, $\Gamma_1(\delta, \beta_0, \tau_2)$ exist, is continuous at $\delta = \delta_0$. Further, $\Gamma_1(\delta_0, \beta_0, \tau_2)$ is full rank by assumptions 2 and 6. Hence, condition (2.2) is satisfied.

Denote $\Gamma_2(\delta, \beta_0, \tau)[\beta(\tau_1) - \beta_0(\tau_1)] = \sum_{j=1}^{m} \Gamma_{2j}(\delta, \beta_0, \tau)[\beta_j(\tau_1) - \beta_{j,0}(\tau_1)]$ the pathwise derivative of $M(\delta, \beta_0, \tau_1)$ in the direction $(\beta - \beta_0, \tau_1)$, where

$$\begin{aligned}
\Gamma_{2j}(\delta, \beta_0, \tau) =& \frac{\partial}{\partial \beta_j}\left[\mathbb{E}[\tau_2 - F_{Q(\tau_1, y_{ij}|x_{ij}, v_j)|x_{ij}}(x_{ij}'\delta(\beta_0, \tau)|x_{ij})x_{ij}]\right] \\
=& \frac{\partial}{\partial \beta_j}\left[\mathbb{E}_j\left[\mathbb{E}_{i|j}\left[\tau_2 - F_{Q(\tau_1, y_{ij}|x_{ij}, v_j)|x_{ij}}(x_{ij}'\delta(\beta_0, \tau)|x_{ij})x_{ij}\right]\right]\right] \\
=& -\frac{1}{m}\mathbb{E}_{i|j}\left[f_{Q(\tau_1, y_{ij}|x_{ij}, v_j)|x_{ij}}(x_{ij}'\delta(\beta_0, \tau)|x_i)x_{ij}\tilde{x}_{ij}'\right].
\end{aligned}$$

By assumption 5 the pathwise derivative will exist in all directions $(\beta_j - \beta_{j,0}) \in \mathcal{B}_j$.

Condition (2.3) requires that for all $(\beta_j, \delta) \in \mathcal{B}_{\zeta_m} \times \mathcal{D}_{\zeta_m}$ with a positive sequence $\zeta_m = o_p(1)$, (i) $||M(\delta, \beta, \tau) - M(\delta, \beta_0, \tau) - \sum_{j=1}^{m} \Gamma_{2j}(\delta, \beta_0, \tau)[\beta_j - \beta_{j,0}]|| \leq c \cdot \sup_j ||\beta_j - \beta_{j,0}||^2$ for some constant $c \geq 0$; and (ii) $|| \sum_{j=1}^{m} \Gamma_{2j}(\delta, \beta_0, \tau)[\beta_j - \beta_{j,0}] - \sum_{j=1}^{m} \Gamma_{2j}(\delta_0, \beta_0, \tau)[\beta_j - \beta_{j,0}]|| \leq o(1)\zeta_m$. Note that sequence $\zeta_m$ is also defined in terms of the radius of the ball around $\beta_0$ and $\delta_0$. Hence, we need this sequence converge to zero at a rate weakly slower than $(\beta - \beta_0)$ and $(\delta - \delta_0)$. Using a Taylor approximation and since Assumption 5 implies that $M(\delta, \beta, \tau)$ is twice continuously differentiable we have that

$$M(\delta, \beta, \tau) - M(\delta, \beta_0, \tau) = \sum_{j=1}^{m} \Gamma_{2j}(\delta, \beta_0, \tau)[\beta_j - \beta_{j,0}] + O_p(||\beta - \beta_0||_{\mathcal{B}}^2) \tag{44}$$

which implies (i). Condition (2.3ii) is trivially satisfied by Assumption 5.

Condition (2.4), is satisfied if $||\hat{\beta}_j - \beta_{j,0}||_{\mathcal{B}} = o_p(m^{-1/4})$. The proof of Lemma 1 in Galvao and Wang (2015) implies that

$$\mathrm{P}\left\{\max_{1 \leq j \leq m} ||\hat{\beta}_j(\tau) - \beta_{j,0}(\tau)|| > \zeta\right\} \leq O(m \exp{(-n)}).$$

If $\frac{\sqrt{m}\log n}{n} \to 0$ (Assumption 9(b)), $\sup_j ||\hat{\beta}_j - \beta_{j,0}|| = o_p(m^{-1/4})$, so that condition (2.4) in Chen et al. (2003) is satisfied.

Condition (2.5) is implied by Lemma 4.

Under these conditions, it follows by the proof of Theorem 2 in Chen et al. (2003) that

$$\begin{aligned}
&\sqrt{m}\left(\hat{\delta}(\hat{\beta},\tau)-\delta_0(\beta_0,\tau)\right)\\
&=-\Gamma_1(\delta_0,\beta_0,\tau)^{-1}\sqrt{m}\left(\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2j}(\tau,\delta_0,\beta_0)[\hat{\beta}_j(\tau_1)-\beta_{j,0}(\tau_1)]+M_{mn}(\delta_0,\beta_0,\tau)\right)\\
&\quad+o_p(1) \qquad (45)
\end{aligned}$$

Then, adding condition (2.6), which follows directly by Lemma 1 with

$$\sqrt{m}\left(M_{mn}(\delta_0,\beta_0,\tau)+\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2,j}(\beta_0,\tau)\left(\hat{\beta}_j-\beta_{j,0}\right)\right)\xrightarrow{d}N(0,\Omega_2(\tau)). \qquad (46)$$

The final result follows by Theorem 2 in Chen et al. (2003). ∎

## B.2 Uniform Results

### B.2.1 Uniform Consistency

*Proof of Theorem 1.* Note that for all $\zeta>0$ there exist $\epsilon(\zeta)$ such that

$$\inf_{\tau\in\mathcal{T}\times\mathcal{T}}\inf_{||\delta-\delta_0(\tau)||>\zeta}\|M(\delta,\beta_0,\tau)\|>\epsilon(\zeta).$$

This implies that uniformly over $\tau\in\mathcal{T}\times\mathcal{T}$, if $\delta$ is more than $\zeta$ away from $\delta_0(\tau)$, then $M(\delta,\beta,\tau)$ is at least $\epsilon(\zeta)$. Hence, for any $\tau\in\mathcal{T}$, $||\delta-\delta_0(\tau)||>\zeta$ implies $\|M(\delta,\beta_0,\tau)\|>\epsilon(\zeta)$. Since, it must also hold for the supremum over $\tau\in\mathcal{T}\times\mathcal{T}$, it follows that

$$\left\{\sup_{\tau\in\mathcal{T}\times\mathcal{T}}||\hat{\delta}(\hat{\beta},\tau)-\delta_0(\tau)||>\zeta\right\}\subseteq\left\{\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M(\hat{\delta}(\hat{\beta},\tau),\beta_0,\tau)\|>\epsilon(\zeta)\right\}$$

and therefore that

$$P\left(\sup_{\tau\in\mathcal{T}\times\mathcal{T}}||\hat{\delta}(\hat{\beta},\tau)-\delta_0(\tau)||>\zeta\right)\leq P\left(\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M(\hat{\delta}(\hat{\beta},\tau),\beta_0,\tau)\|>\epsilon(\zeta)\right)$$

.

I need to show that $\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M(\hat{\delta},\beta_0,\tau)\|=o_p(1)$.

Note that by triangle inequality

$$\begin{aligned}
\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M(\hat{\delta},\beta_0,\tau)\|\leq&\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M(\hat{\delta},\beta_0,\tau)-M(\hat{\delta},\hat{\beta},\tau)\|\\
&+\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M(\hat{\delta},\hat{\beta},\tau)-M_{mn}(\hat{\delta},\hat{\beta},\tau)\|\\
&+\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\|M_{mn}(\hat{\delta},\hat{\beta},\tau)\|
\end{aligned}$$

The first term is $o_p(1)$ by continuity and uniform consistency of $\hat{\beta}(\tau)$. Next, note that $\mathcal{M} = \{m(\delta, \beta, \tau) : \delta \in \mathcal{D}, \beta \in \mathcal{B}, \tau \in \mathcal{T} \times \mathcal{T}\}$ is Lipschitz continuous, hence by Theorem 2.7.11 in van der Vaart and Wellner (1996) we can directly bound $N_{[\,]}(\varepsilon, \mathcal{M}, \|\cdot\|_{L_2(P)})$ from above by the covering number of the class $\{\delta, \in \mathcal{D}, \beta \in \mathcal{B}, \tau \in \mathcal{T} \times \mathcal{T}\}$ which is finite for any $\varepsilon > 0$ by assumption 7. It follows directly by Theorem 19.4 in van der Vaart (1998), that the class is Glivenko-Cantelli. Hence, the second term is also $o_p(1)$. The third term is also $o_p(1)$ by construction. ∎

### B.2.2 Weak Convergence

*Proof of Theorem 2.* The proof consists of three parts. First, I show that the lineralization in equation (45) holds uniformly over $\tau_1, \tau_2$. Second, I show that uniformly over $\tau_1, \tau_2$, $\hat{\delta}(\hat{\beta}, \cdot) - \hat{\delta}(\beta_0, \cdot) = o_p\left(m^{-1/2}\right)$. With this result, weak convergence of $\sqrt{m}\left(\hat{\delta}(\beta_0, \cdot) - \delta_0(\beta_0, \cdot)\right)$ directly implies weak convergence of $\sqrt{m}\left(\hat{\delta}(\hat{\beta}, \cdot) - \delta_0(\beta_0, \cdot)\right)$. Third, I show that $\Gamma_1(\cdot)\sqrt{m}\left(\hat{\delta}(\beta_0, \cdot) - \delta_0(\beta_0, \cdot)\right) \rightsquigarrow \mathbb{G}(\cdot)$ in $\ell^\infty(\mathcal{T} \times \mathcal{T})$.

**Part 1 – Linearization**

By the proof of Theorem 2 in Chen et al. (2003) we have

$$||M_{mn}(\hat{\delta}, \hat{\beta}) - \mathcal{L}_{mn}(\hat{\delta})|| = o_p(m^{-1/2})$$

with

$$\mathcal{L}_{mn}(\hat{\delta}) = M_{mn}(\delta_0, \beta_0) + \Gamma_1(\delta_0, \beta_0)(\hat{\delta} - \delta_0) + \frac{1}{m}\sum_{j=1}^{m} \bar{\Gamma}_{2,j}(\delta_0, \beta_0)(\hat{\beta}_j - \beta_{j,0})$$

Now, I show that this linearization holds uniformly in $\tau \in \mathcal{T} \times \mathcal{T}$. If I can show that all pieces are asymptotically equicontinuous, then the entire thing is asymptotically equicontinuous, and the approximation is uniform by Newey (1991).

I start by noting that

$$m(\delta', \beta', \tau') - m(\delta'', \beta'', \tau'') = m(\delta', \beta', \tau') - m(\delta'', \beta'', \tau') + m(\delta'', \beta'', \tau') - m(\delta'', \beta'', \tau'').$$

From the proof of Lemma 4, we know that $m(\delta', \beta', \tau') - m(\delta'', \beta'', \tau') \leq 2||\tilde{x}_{ij}|| \cdot ||(\beta'_j - \beta''_j)||_{\mathcal{B}} + 2||x_{ij}|| \cdot ||(\delta' - \delta'')||$. Next, by the properties of the check function

$$\begin{aligned} m(\delta'', \beta'', \tau') - m(\delta'', \beta'', \tau'') \leq& |\tau''_2 - \tau'_2| \cdot ||\tilde{x}'_{ij}\beta''_j - x'_{ij}\delta''|| \\ \leq& |\tau''_2 - \tau'_2| \cdot ||x_{ij}|| \left(||\beta''_j||_{\mathcal{B}} + ||\delta''||\right), \end{aligned}$$

where $x_{ij}$ is bounded by assumption 2. Hence, $(\delta, \beta, \tau) \mapsto m(\delta, \beta, \tau)$ is asymptotically equicontinuous.

By Taylor expansion

$$M(\delta,\beta,\tau)|_{\delta=\hat{\delta}(\tau)} = \mathbb{E}\left[f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'d(\tau)|x)x_{ij}x_{ij}'\right]|_{d(\tau)=\delta^*(\tau)}\left(\hat{\delta}(\tau)-\delta_0(\tau)\right)$$

where $\delta^*(\tau)$ is on the line connecting $\hat{\delta}(\tau)$ with $\delta(\tau)$ for each $\tau \in \mathcal{T}\times\mathcal{T}$ and is allowed to vary for each row of the Jacobian matrix. Then, by uniform consistency of $\hat{\delta}(\tau_1,\tau_2)$ (Theorem 1) and since by Assumption 5, $f_{Q(\tau_1,y_{ij}|x_{ij},v_j)|x_{ij}}(x_{ij}'d(\tau)|x)$ is uniformly continuous and bounded it follows that

$$M(\delta,\beta,\tau)|_{\delta=\hat{\delta}(\tau)} = [\Gamma_1(\delta_0,\beta_0,\tau)+o_p(1)]\,(\hat{\delta}(\tau)-\delta_0(\tau)) \tag{47}$$

uniformly over $\tau_1,\tau_2$. Hence, $\tau \mapsto \Gamma_1(\delta_0,\beta_0)(\hat{\delta}-\delta_0)$ is stochastically equicontinuous.
Similarly, by Taylor expansion

$$M(\delta_0,\beta,\tau)|_{\beta=\hat{\beta}} = \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_2(\delta_0,\beta,\tau)|_{\beta=\hat{\beta}^*}(\hat{\beta}_j-\beta_{j,0}),$$

where $\beta^*(\tau_1)$ is on the line connecting $\hat{\beta}(\tau_1)$ with $\beta_0(\tau_1)$ for each $\tau\in\mathcal{T}\times\mathcal{T}$. Using the same argument as above, and by uniform consistency of $\hat{\beta}_j$ over $j$ and $\tau$ (Theorem 2), it follows that

$$M(\delta_0,\beta,\tau)|_{\beta=\hat{\beta}} = \frac{1}{m}\sum_{j=1}^{m}\left[\bar{\Gamma}_2(\delta_0,\beta,\tau)+o_p(1)\right](\hat{\beta}_j-\beta_{j,0})$$

uniformly in $\tau\in\mathcal{T}\times\mathcal{T}$. Hence, $\tau\mapsto\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_2(\delta_0,\beta,\tau)(\hat{\beta}_j-\beta_{j,0})$ is asymptotically equicontinuous.

Since all three terms of $\mathcal{L}(\delta)$ as well as $M_{mn}(\delta,\beta,\tau)$ are asymptotically equicontinuous, it follows that the linearization holds uniformly over $\tau_1,\tau_2$. Hence, uniformly over $\tau_1,\tau_2$, the asymptotic behavior of $\sqrt{m}\left(\hat{\delta}(\hat{\beta},\tau)-\delta_0(\beta_0,\tau)\right)$ is determined by

$$-\Gamma_1(\delta_0,\beta_0,\tau)^{-1}\sqrt{m}\left(\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2j}(\tau,\delta_0,\beta_0)[\hat{\beta}_j(\tau_1)-\beta_{j,0}(\tau_1)]+M_{mn}(\delta_0,\beta_0,\tau)\right) \tag{48}$$

**Part 2**

From the proof of Lemma 1, we know that the first stage error is $o_p\left(\frac{1}{\sqrt{m}}\right)$ uniformly over $\tau_1,\tau_2\in\mathcal{T}\times\mathcal{T}$. Hence, the limiting distribution of $\hat{\delta}(\hat{\beta},\tau)$ is the same as the one of the infeasible estimator $\hat{\delta}(\beta_0,\tau)$. Formally,

$$\sup_{\tau_1,\tau_2\in\mathcal{T}\times\mathcal{T}}\|\hat{\delta}(\hat{\beta},\tau)-\hat{\delta}(\beta_0,\tau)\| = o_p\left(m^{-1/2}\right).$$

**Part 3 – Weak Convergence of $\sqrt{m}\left(\hat{\delta}(\beta_0,\cdot)-\delta_0(\hat{\beta},\cdot)\right)$**

This part of the proof closely follows the work of Angrist et al. (2006). Let $\mathbb{Q}_{mn}(\delta, \beta, \tau_1, \tau_2) = \frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n}\rho_{\tau_2}(\tilde{x}'_{ij}\beta_{j,0} - x_{ij}\delta) - \rho_{\tau}(\tilde{x}'_{ij}\beta_{j,0}(\tau_1) - x_{ij}\delta_0(\tau))$ and $Q(\tau_1, \tau_2, \delta, \beta) = \mathbb{E}[\rho_{\tau_2}(\tilde{x}'_{ij}\beta_{j,0} - x_{ij}\delta) - \rho_{\tau}(\tilde{x}'_{ij}\beta_{j,0}(\tau_1) - x_{ij}\delta_0(\tau))]$. Similarly to Angrist et al. (2006), the empirical processes $(\beta, \delta, \tau_1, \tau_2) \mapsto \mathbb{Q}_{mn}(\delta, \beta, \tau_1, \tau_2)$ are stochastically equicontinuous because

$$\left|\mathbb{Q}_{mn}(\delta', \beta', \tau'_1, \tau'_2) - \mathbb{Q}_{mn}(\delta'', \beta'', \tau''_1, \tau''_2)\right| \leq C_1 \cdot |\tau'_1 - \tau''_1| + C_2 \cdot |\tau'_2 - \tau''_2| + C_3 \cdot \|\beta' - \beta''\|_{\mathcal{B}} + C_3 \cdot \|\delta' - \delta''\|,$$

where $C_1 = 2 \cdot C \cdot \sup_{\beta \in \mathcal{B}} \|\beta\|_{\mathcal{B}}$, $C_2 = 2 \cdot C \cdot \sup_{\delta \in \mathcal{D}} \|\delta\|$ and $C_3 = 2 \cdot C$. The constant $C$ is defined in Assumption 2. Note that $C_1$, $C_2$ and $C_3$ are neither functions of $\tau_1$ nor $\tau_2$. It follows that, $\mathbb{Q}_{mn}(\delta', \beta', \tau'_1, \tau'_2)$ is Glivenko-Cantelli so that uniformly in $(\tau, \beta, \delta) \in \{\mathcal{T} \times \mathcal{T}, \mathcal{B}, \mathcal{D}\}$,

$$\mathbb{Q}_{mn}(\delta, \beta, \tau_1, \tau_2) = Q(\delta, \beta, \tau_1, \tau_2) + o_p(1). \tag{49}$$

Next consider a collection of closed balls $B_M(\delta(\tau))$ with radius $M$ centered around $\delta(\tau)$, and let $\delta_M(\tau) = \delta(\tau) + \xi_M(\tau) \cdot \nu(\tau)$, where $\nu(\tau)$ is a direction vector with $||\nu(\tau)|| = 1$ and $\xi(\tau)$ is a positive scalar such that $\xi_M(\tau) \geq M$. Then uniformly in $\tau \in \mathcal{T} \times \mathcal{T}$:

$$\begin{aligned}
\left(\frac{M}{\xi_M(\tau)}\right) &\cdot (\mathbb{Q}_{mn}(\delta_M(\tau), \beta, \tau_1, \tau_2) - \mathbb{Q}_{mn}(\delta(\tau), \beta, \tau_1, \tau_2)) \\
&\geq \mathbb{Q}_{mn}(\delta^*_M(\tau), \beta, \tau_1, \tau_2) - \mathbb{Q}_{mn}(\delta(\tau), \beta, \tau_1, \tau_2) \\
&\geq Q(\delta^*_M(\tau), \beta, \tau_1, \tau_2) - Q(\delta(\tau), \beta, \tau_1, \tau_2) + o_p(1) \\
&> \epsilon_M + o_p(1),
\end{aligned}$$

for some $\epsilon_M > 0$, where the first inequality follows by convexity in $\delta$ for $\delta^*_M(\tau)$ (i.e. the point on the boundary of $B_M(\delta(\tau))$, that lies on the line connecting $\delta_M(\tau)$ and $\delta(\tau)$). The second inequality is implied by equation (49); and the third follows because by convexity and by Assumption 2 and Assumption 6, $\delta(\tau)$ is the unique minimizer of $Q(\delta, \tau_1, \tau_2)$ uniformly in $\tau \in \mathcal{T}$. It then follows that,

$$\sup_{\tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T}} \left\|\hat{\delta}(\beta_0, \tau_1, \tau_2) - \delta_0(\beta_0, \tau_1, \tau_2)\right\| \xrightarrow{p} 0. \tag{50}$$

Next, note that

$$\|\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}[\rho_{\tau_2}(\tilde{x}'_{ij}\beta_{j,0}(\tau_1) - x'_{ij}\hat{\delta}(\beta_0, \tau))]\| \leq C \cdot \sup_i \sup_j \|x_{ij}\|/m.$$

By assumption 2, $\sup_i \sup_j \|x_{ij}\| = O_p(1)$. Hence, uniformly in $\tau \in \mathcal{T}$

$$\sqrt{m}\frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left[\rho_{\tau_2}(\tilde{x}'_{ij}\beta_{j,0}(\tau_1) - x'_{ij}\hat{\delta}(\beta_0, \tau))\right] = \sqrt{m}M_{mn}(\hat{\delta}, \beta_0, \tau_1, \tau_2) = o_p(1), \tag{51}$$

Note that,

$$(\tau_1,\tau_2,\delta,\beta)\mapsto\sqrt{m}\left[M_{mn}(\hat{\delta},\beta_0,\tau_1,\tau_2)-M(\hat{\delta},\beta_0,\tau_1,\tau_2)\right]$$

is stochastic equicontinuous over $\mathcal{T}\times\mathcal{T}\times\mathcal{D}\times\mathcal{B}$, with respect to the $L_2(P)$ pseudometric

$$\xi((\delta',\beta',\tau_1',\tau_2'),(\delta'',\beta'',\tau_1'',\tau_2''))^2=\max_{k=1,\dots,K}\mathbb{E}\left[\left(m_k(\delta',\beta',\tau_1',\tau_2')-m_k(\delta'',\beta'',\tau_1'',\tau_2'')\right)^2\right]$$

where $k=1,\dots,K$ indexes the elements of $m(\cdot)$.

Observe that $\{1(\tilde{x}_{ij}'\beta_j(\tau_1)\leq x_{ij}'\delta(\beta,\tau)),\tau_1\in\mathcal{T},\tau_2\in\mathcal{T},\beta\in\mathcal{B},\delta\in\mathcal{D}\}$ is a VC subgraph class and hence a bounded Donsker class for any compact sets $\mathcal{B}$ and $\mathcal{D}$, where the sets are compact by assumption. Hence, it follows that the function class

$$\left\{x_{ij}'[\tau_2-1(\tilde{x}_{ij}'\beta_j(\tau_1)\leq x_{ij}'\delta(\beta,\tau))],\tau_1\in\mathcal{T},\tau_2\in\mathcal{T},\beta\in\mathcal{B},\delta\in\mathcal{D}\right\}$$

is also Donsker with a square-integrable envelope $2\cdot\max_{i,j}\|x_{ij}\|$, and stochastic equicontinuity follows.

Next, stochastic equicontinuity of $(\tau_1,\tau_2,\delta)\mapsto\sqrt{m}\left[M_{mn}(\hat{\delta},\beta_0,\tau_1,\tau_2)-M(\hat{\delta},\beta_0,\tau_1,\tau_2)\right]$ together with uniform consistency of $\hat{\delta}(\beta_0,\tau_1,\tau_2)$ and the resulting convergence with respect to the pseudometric $\sup_{\tau_1,\tau_2\in\mathcal{T}\times\mathcal{T}}\xi\left((\hat{\delta}(\tau),\beta_0(\tau_1),\tau_1,\tau_2)),(\delta(\tau),\beta_0(\tau_1),\tau_1,\tau_2))\right)=o_p(1)$ imply

$$\begin{aligned}\sqrt{m}&\left[M_{mn}(\hat{\delta},\beta_0,\tau_1,\tau_2)-M(\hat{\delta},\beta_0,\tau_1,\tau_2)\right]\\&=\sqrt{m}\left[M_{mn}(\delta_0,\beta_0,\tau_1,\tau_2)-M(\delta_0,\beta_0,\tau_1,\tau_2)\right]+o_p(1),\quad\text{in }\ell^\infty(\mathcal{T}\times\mathcal{T}).\end{aligned}\tag{52}$$

Note that convergence with respect to the pseudometric follows from

$$\begin{aligned}&\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\max_{k=1,\dots,K}\xi\left(d(\tau),\beta_0(\tau_1),\tau_1,\tau_2)),(\delta(\tau),\beta_0(\tau_1),\tau_1,\tau_2))\right)^2\\&=\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\max_{k=1,\dots,K}\mathbb{E}\left[\left(x_{ijk}\left(1(\tilde{x}_{ij}'\beta_{j,0}(\tau_1)\leq x_{ij}'d(\tau))-1(\tilde{x}_{ij}'\beta_{j,0}(\tau_1)\leq x_{ijk}'(\delta(\tau)))\right)\right)^2\right]\\&\leq\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\max_{k=1,\dots,K}\mathbb{E}\left[\left(x_{ijk}\left(F_{Q(\tau_1,y|x,\nu)|x}(x_{ij}'d(\tau))-F_{Q(\tau_1,y|x,\nu)|x}(x_{ij}'(\delta))\right)\right)^2\right]\\&\leq\sup_{\tau\in\mathcal{T}\times\mathcal{T}}\max_{k=1,\dots,K}\mathbb{E}\left[x_{ijk}^2\right]\mathbb{E}\left[\left(F_{Q(\tau_1,y|x,\nu)|x}(x_{ij}'d(\tau))-F_{Q(\tau_1,y|x,\nu)|x}(x_{ij}'(\delta))\right)^2\right]\\&\leq C^2\mathbb{E}\left[f_Q^{max}\cdot||x_{ij}||^2\right]\sup_{\tau\in\mathcal{T}\times\mathcal{T}}||d(\tau)-\hat{\delta}(\tau)||^2,\end{aligned}$$

where $f_Q^{max}$ is an uniform upper bound on the density of $Q(\tau_1,y|x,\eta)$ given $x$ defined in Assumption 5. The first inequality follows by the law of iterated expectation, the second by Cauchy-Schwarz, and the third by a Taylor expansion.

Combining equations (51), (52) and (47) yields

$$o_p(1)=\left[\Gamma_1(\delta_0,\beta_0,\cdot)+o_p(1)\right]\sqrt{m}\left(\hat{\delta}(\cdot)-\delta_0(\cdot)\right)+\sqrt{m}\left[M_{mn}(\hat{\delta},\beta_0,\cdot)-M(\hat{\delta}_0,\beta_0,\cdot)\right]\Bigg].\tag{53}$$

Since the Eigenvalues of $\Gamma_1$ are bounded away from zero by assumption, we obtain that uniformly in $\tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T}$

$$\begin{aligned}\sup_{\tau_1,\tau_2 \in \mathcal{T}\times\mathcal{T}} &\|\sqrt{m}\,[M_{mn}(\delta_0,\beta_0,\tau_1,\tau_2) - M(\delta_0,\beta_0,\tau_1,\tau_2)] + o_p(1)\| \\ &\geq \left(\sqrt{\lambda} + o_p(1)\right) \cdot \sup_{\tau_1,\tau_2 \in \mathcal{T}\times\mathcal{T}} \sqrt{m}\|\hat{\delta}(\tau) - \delta_0(\tau)\|. \end{aligned} \tag{54}$$

where $\lambda$ is the minimum eigenvalue of $\Gamma_1(\delta_0, \beta_0, \tau)$. Observe that $\lambda > 0$ by Assumptions 2 and 6.

Assumption 8 implies that $\delta(\beta_0, \tau)$ is continuous with respect to $\tau$. It follows that $\tau \mapsto \sqrt{m}M_{mn}(\delta_0, \beta_0, \tau_1, \tau_2) - M(\delta_0, \beta_0, \tau_1, \tau_2)$ is stochastic equicontinuous over $\mathcal{T} \times \mathcal{T}$ with respect to the pseudometric, so that the functional central limit theorem implies

$$\sqrt{m}\,(M_{mn}(\delta_0,\beta_0,\cdot) - M(\delta_0,\beta_0,\cdot)) \rightsquigarrow \mathbb{G}(\cdot) \quad \text{in } \ell^\infty(\mathcal{T}\times\mathcal{T}) \tag{55}$$

where $\mathbb{G}(\cdot)$ is a Gaussian process with covariance function $\Omega_2(\cdot,\cdot)$ defined in Lemma 1. Hence, the left-hand side of (54) is $O_p(1)$ such that $\sup_{\tau_1,\tau_2\in\mathcal{T}\times\mathcal{T}}\|\sqrt{m}(\hat{\delta}(\beta_0,\tau) - \delta_0(\beta_0,\tau))\| = O_p(1)$. Together with equations (53)-(55) this implies

$$\begin{aligned}&\Gamma_1(\cdot)\sqrt{m}\left(\hat{\delta}(\beta_0,\cdot) - \delta_0(\beta_0,\cdot)\right) \\ &\quad = -\sqrt{m}\,(M_{mn}(\delta_0,\beta_0,\cdot) - M(\delta_0,\beta_0,\cdot)) + o_p(1) \rightsquigarrow \mathbb{G}(\cdot) \quad \text{in } \ell^\infty(\mathcal{T}\times\mathcal{T})\end{aligned}$$

■

### B.3 Inference

*Proof of Theorem 3.* This proof uses a similar strategy to the proof of Theorem 5.4 in Fernández-Val et al. (2022) The idea is to prove the result in two steps. First, show that $\hat{\delta}^* - \delta_0$ can be approximated by a linear function with an error of order $o_p^*(m^{-1/2})$. Then, show that the $\hat{\delta}^* - \hat{\delta}$ follow the same distribution as $\hat{\delta} - \delta_0$.

***Part 1 - Linearization*** Since the bootstrap algorithm that I consider samples entire groups, the first stage is the same in all bootstrap replications. Instead, the source of randomness is which groups are sampled. In this section, I make the dependency of $\beta$ on the data $z$ explicit and denote the vector containing first-stage estimates in the bootstrap world $\hat{\beta}(z^*)$.

It can be shown that

$$\hat{\delta}^*(\hat{\beta}(z^*),\tau) - \hat{\delta}(\hat{\beta},\tau) = O_p^*\left(m^{-1/2}\right)$$

which together with Theorem 5 implies

$$\hat{\delta}^*(\hat{\beta}(z^*),\tau) - \delta_0(\beta_0,\tau) = O_p^*\left(m^{-1/2}\right). \tag{56}$$

Next, the idea is to approximate $\sqrt{m}(\hat{\delta}^*(\hat{\beta}(z^*),\tau) - \delta_0(\beta_0,\tau))$ with a linear function. Hence, the goal is to show that

$$\begin{aligned}\sqrt{m}(\hat{\delta}^*(\hat{\beta}(z^*),\tau) - \delta_0(\beta_0,\tau)) &= \Gamma_1(\delta_0,\beta_0,\tau)^{-1} \\ \times\sqrt{m}\left(\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2j}(\delta_0,\beta_0,\tau)[\hat{\beta}_j(\tau_1,z^*) - \beta_{j,0}(\tau_1,z^*)] + M^*_{mn}(\delta_0,\beta_0,\tau)\right) &+ o_p^*(1),\end{aligned} \tag{57}$$

where $M^*_{mn}(\delta_0,\beta_0) = \frac{1}{m}\sum_{j=1}^{m}\bar{m}^*_j(\delta_0,\beta_0)$ and $\bar{m}_j(\delta,\beta,\tau) = \frac{1}{n}\sum_{i=1}^{n}x'_{ij}[\tau_2 - 1(\tilde{x}'_{ij}\beta_j(\tau_1) \leq x'_{ij}\delta(\beta),\tau)]$. For this part of the proof, I rely on the results from Chen et al. (2003). Define the linearization where the dependencies on $\tau$ are suppressed for ease of notation:

$$\mathcal{L}^*_{mn}(\delta) = M^*_{mn}(\delta_0,\beta_0) + \Gamma_1(\delta_0,\beta_0)(\delta-\delta_0) + \frac{1}{m}\sum_{j}\bar{\Gamma}^*_{2,j}(\delta_0,\beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*)). \tag{58}$$

The first step is to show that we can approximate $M^*_{mn}(\hat{\delta}^*,\hat{\beta}(z^*))$ by $\mathcal{L}^*_{mn}(\hat{\delta}^*)$ with an error of order $o_p^*(m^{-1/2})$ within a $O_p(m^{-1/2})$ neighborhood of $\delta_0$. Hence, I want to show that

$$||M^*_{mn}(\hat{\delta}^*,\hat{\beta}(z^*)) - \mathcal{L}^*_{mn}(\hat{\delta}^*)|| = o_p^*(m^{-1/2}).$$

By the triangle inequality:

$$\begin{aligned}||M^*_{mn}(\hat{\delta}^*,\hat{\beta}(z^*)) - \mathcal{L}_{mn}(\hat{\delta}^*)|| \leq & ||M(\hat{\delta}^*,\hat{\beta}(z^*)) - M(\hat{\delta}^*,\beta_0(z^*)) - \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2,j}(\delta_0,\beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*))|| \\ &+ ||M(\hat{\delta}^*,\beta_0(z^*)) - \Gamma_1(\delta_0,\beta_0)(\hat{\delta}^*-\delta_0)|| \\ &+ ||M^*_{mn}(\hat{\delta}^*,\hat{\beta}(z^*)) - M(\hat{\delta}^*,\hat{\beta}(z^*)) - M^*_{mn}(\delta_0,\beta_0)||.\end{aligned}$$

Where for the first term we have:

$$\begin{aligned}||M(\hat{\delta}^*,\hat{\beta}(z^*)) - & M(\hat{\delta}^*,\beta_0(z^*)) - \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2,j}(\delta_0,\beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*))|| \\ \leq & ||M(\hat{\delta}^*,\hat{\beta}(z^*)) - M(\hat{\delta}^*,\beta_0(z^*)) - \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2,j}(\hat{\delta}^*,\beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*))|| \\ &+ ||\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2,j}(\hat{\delta}^*,\beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*)) - \frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2,j}(\delta_0,\beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*))|| \\ = & O_p^*(||\hat{\beta}(z^*) - \beta_0||^2_{\mathcal{B}}) + o_p^*(1) \times ||\hat{\delta}^* - \delta_0|| = o_p^*(m^{-1/2}).\end{aligned}$$

since $\sup_j ||\hat{\beta}_j - \beta_{j,0}|| = o_p(m^{-1/4})$ as shown in the proof of Theorem 5.

For the second term, a Taylor approximation combined with $(\hat{\delta}^* - \delta_0) = O_p^*(m^{-1/2})$ implies:

$$||M(\hat{\delta}^*, \beta_0(z^*)) - \Gamma_1(\delta_0, \beta_0)(\hat{\delta}^* - \delta_0)|| = o_p^*(m^{-1/2}).$$

For the third term, we have:

$$||M_{mn}^*(\hat{\delta}^*, \hat{\beta}(z^*)) - M(\hat{\delta}^*, \hat{\beta}(z^*)) - M_{mn}^*(\delta_0, \beta_0)|| = o_p^*(m^{-1/2})$$

by condition 2.5 in Chen et al. (2003). Hence,

$$||M_{mn}^*(\hat{\delta}^*, \hat{\beta}(z^*)) - \mathcal{L}_{mn}^*(\hat{\delta}^*)|| = o_p^*(m^{-1/2}).$$

Similarly, I now shown that $||M_{mn}^*(\bar{\delta}^*, \hat{\beta}(z^*)) - \mathcal{L}_{mn}^*(\bar{\delta}^*)|| = o_p^*(m^{-1/2})$, where $\bar{\delta}^*$ is the value of $\delta$ that minimizes $\mathcal{L}_{mn}^*(\delta)$.

First, for $\bar{\delta}^*$ to be the value of $\delta$ that minimizes $\mathcal{L}^*(\delta)$ it must be equal to:

$$\begin{aligned}
\bar{\delta}^* - \delta_0 =& \Gamma_1(\delta_0, \beta_0)^{-1} \left( \frac{1}{m} \sum_{j=1}^{m} \bar{\Gamma}_{2j}^*(\delta_0, \beta_0)[\hat{\beta}_j(z^*) - \beta_{j,0}(z^*)] + \frac{1}{m} \sum_{j=1}^{m} \bar{m}_j^*(\delta_0, \beta_0) \right) \qquad (59)\\
=& \Gamma_1(\delta_0, \beta_0)^{-1} \left( \frac{1}{m} \sum_{j=1}^{m} \bar{\Gamma}_{2j}(\delta_0, \beta_0)[\hat{\beta}_j - \beta_{j,0}] + \frac{1}{m} \sum_{j=1}^{m} \bar{m}_j(\delta_0, \beta_0) \right) + O_p^*(m^{-1/2})\\
=& \hat{\delta} - \delta_0 + O_p^*(m^{-1/2})\\
=& O_p^*(m^{-1/2}). \qquad (60)
\end{aligned}$$

where the third line is implied by Theorem 5.

By the triangle inequality

$$\begin{aligned}
||M_{mn}^*(\bar{\delta}^*, \hat{\beta}(z^*)) - \mathcal{L}_{mn}^*(\bar{\delta}^*)|| \leq & ||M(\bar{\delta}^*, \hat{\beta}(z^*)) - M(\bar{\delta}^*, \beta_0(z^*)) - \sum_{j=1}^{m} \Gamma_{2,j}^*(\delta_0, \beta_0)(\hat{\beta}_j(z^*) - \beta_{j,0}(z^*))||\\
& + ||M(\bar{\delta}^*, \beta_0^*) - \Gamma_1(\delta_0, \beta_0)(\bar{\delta}^* - \delta_0)||\\
& + ||M_{mn}^*(\bar{\delta}^*, \hat{\beta}(z^*)) - M(\bar{\delta}^*, \hat{\beta}(z^*)) - M_{mn}^*(\delta_0, \beta_0)||.
\end{aligned}$$

For the first term, we have:

$$\begin{aligned}
||M(\bar{\delta}^*, \hat{\beta}(z^*)) - & M(\bar{\delta}^*, \beta_0(z^*)) - \sum_{j=1}^{m} \Gamma_{2j}(\delta_0, \beta_0)[\hat{\beta}_j(z^*) - \hat{\beta}_{j,0}(z^*)]||\\
\leq & ||M(\bar{\delta}^*, \hat{\beta}(z^*)) - M(\bar{\delta}^*, \beta_0^*) - \sum_{j=1}^{m} \Gamma_{2j}(\bar{\delta}, \beta_0)[\hat{\beta}_j(z^*) - \beta_{j,0}(z^*)]||\\
& + ||\sum_{j=1}^{m} \Gamma_{2j}(\bar{\delta}^*, \beta_0)[\hat{\beta}_j(z^*) - \beta_{j,0}(z^*)] - \sum_{j=1}^{m} \Gamma_{2j}(\delta_0, \beta_0)[\hat{\beta}_j(z^*) - \beta_{j,0}(z^*)]||\\
& + O_p(||\hat{\beta}_j - \beta_{j,0}||_{\mathcal{B}}^2) + o_p^*(1) \times ||\bar{\delta}^* - \delta_0|| = o_p^*(m^{-1/2})
\end{aligned}$$

For the second term, by differentiability of $M(\delta, \beta_0, \tau)$ and using equation (60) yields

$$||M(\bar{\delta}^*, \beta_0) - \Gamma_1(\bar{\delta}^* - \delta_0)|| = ||o(\bar{\delta}^* - \delta_0)|| = o_p^*(m^{-1/2}).$$

For the third term, by condition 2.5 in Chen et al. (2003), we have

$$||M_{mn}^*(\bar{\delta}^*, \hat{\beta}) - M(\bar{\delta}^*, \hat{\beta}) - M_{mn}^*(\delta_0, \beta_0)|| = o_p(m^{-1/2}).$$

Hence, it follows that

$$||M_{mn}^*(\bar{\delta}^*, \hat{\beta}(z^*)) - \mathcal{L}_{mn}^*(\bar{\delta}^*)|| = o_p^*(m^{-1/2}).$$

so that if I can show that $\bar{\delta}^* - \hat{\delta}^* = o_p^*(m^{-1/2})$, it follows from equation (59) that equation (57) holds. Following Pakes and Pollard (1989), we know that $M_{mn}^*(\delta, \beta_0)$ and $\mathcal{L}^*(\delta)$ are close at both $\hat{\delta}^*$ which almost minimizes $||M_{mn}^*(\delta, \hat{\beta}(z^*))||$ and at $\bar{\delta}^*$ which minimizes $\mathcal{L}^*(\delta)$. This means that $\hat{\delta}^*$ has to be close to minimizing $\mathcal{L}^*(\delta)$:

$$\begin{aligned} ||\mathcal{L}(\hat{\delta}^*)|| - o_p^*(m^{-1/2}) \leq & ||M_{mn}(\hat{\delta}^*, \hat{\beta}(z^*))|| \\ \leq & ||M_{mn}(\bar{\delta}^*, \hat{\beta}(z^*))|| + o_p^*(m^{-1/2}) \\ \leq & ||\mathcal{L}(\bar{\delta}^*)|| + o_p^*(m^{-1/2}). \end{aligned}$$

This implies

$$||\mathcal{L}(\hat{\delta}^*)|| = ||\mathcal{L}(\bar{\delta}^*)|| + o_p^*(m^{-1/2}),$$

and squaring both sides

$$||\mathcal{L}(\hat{\delta}^*)||^2 = ||\mathcal{L}(\bar{\delta}^*)||^2 + o_p^*(m^{-1}), \tag{61}$$

where the cross product is also $o_p(m^{-1})$ because $||\mathcal{L}(\bar{\delta}^*)||$ is of order $O_p^*(m^{-1/2})$.
The term $||\mathcal{L}(\delta)||^2$ has the simple expansion

$$||\mathcal{L}(\delta)||^2 = ||\mathcal{L}(\bar{\delta}^*)||^2 + ||\Gamma_1(\delta - \bar{\delta}^*)||^2 \tag{62}$$

around its global minimum. The cross-product term vanished because the residual vector $\mathcal{L}(\bar{\delta}^*)$, must be orthogonal to the columns of $\Gamma_1$. Let $\delta = \bar{\delta}^*$, and equations (61) and (62) give that

$$||\Gamma_1(\hat{\delta}^* - \bar{\delta}^*)|| = o_p^*(m^{-1/2}).$$

which implies

$$||(\hat{\delta}^* - \bar{\delta}^*)|| = o_p^*(m^{-1/2}).$$

as $\Gamma_1$ is full rank. Hence, equation (57) holds.

Similarly, to part 1 of the proof of Theorem 2, we can show that uniformly over $\tau_1, \tau_2 \in \mathcal{T} \times \mathcal{T}$,

$$\begin{aligned}
&\sqrt{m}(\hat{\delta}^*(\hat{\beta}(z^*), \tau) - \delta_0(\beta_0, \tau)) \\
=&\Gamma_1(\delta_0, \beta_0, \tau)^{-1}\sqrt{m}\left(\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2j}(\delta_0, \beta_0, \tau)[\hat{\beta}_j(\tau_1, z^*) - \beta_{j,0}(\tau_1, z^*)] + M^*_{mn}(\delta_0, \beta_0, \tau)\right) + o^*_p(m^{-1/2}),
\end{aligned} \tag{63}$$

***Part 2 - Asymptotic distribution of $\hat{\delta}^* - \hat{\delta}$***

For this last part of the proof, I borrow from the proof of Proposition H.1. in Fernández-Val et al. (2022). First, denote

$$\theta^*_{mn}(\tau) = \Gamma_1(\delta_0, \beta_0, \tau)^{-1}\frac{1}{m}\sum_{j=1}^{m}\left(\bar{\Gamma}^*_{2j}(\delta, \beta_0, \tau)[\hat{\beta}_j(z^*, \tau) - \beta_{j,0}(z^*, \tau)] + \bar{m}^*_j(\delta_0, \beta_0, \tau)\right).$$

Since the bootstrap algorithm samples entire groups, we can write:

$$\mathbb{E}^*\left[\theta^*_{mn}(\tau)\right] = \Gamma_1(\delta_0, \beta_0, \tau)^{-1}\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2j}(\delta, \beta_0)[\hat{\beta}_j(\tau) - \beta_{j,0}(\tau)] + \Gamma_1(\delta_0, \beta_0, \tau)^{-1}M_{mn}(\delta_0, \beta_0, \tau).$$

Combining the expressions for $\hat{\delta}^*(\hat{\beta}(z^*), \tau)$ (Equation 63) and $\hat{\delta}(\hat{\beta}, \tau)$ (Equation 48) yields that uniformly in $\tau \in \mathcal{T} \times \mathcal{T}$:

$$\hat{\delta}^*(\hat{\beta}(z^*), \tau) - \hat{\delta}(\hat{\beta}, \tau) = (\theta^*_{mn}(\tau) - \mathbb{E}^*[\theta^*_{mn}(\tau)]) + o^*_p(m^{-1/2})$$

For any $\tau^{(1)}, \ldots, ^{(T)}$, let $\Theta^*_{mn} = \left(\theta^*_{mn}(\tau^{(1)}) - \mathbb{E}^*[\theta^*_{mn}(\tau^{(1)})], \ldots, \theta^*_{mn}(\tau^{(T)}) - \mathbb{E}^*[\theta^*_{mn}(\tau^{(T)})]\right)$.

Let $\Sigma(\tau, \tau') = \Gamma_1(\tau)^{-1}\Omega_2(\tau, \tau')\Gamma_1(\tau')^{\prime -1}$. and $\Sigma = (\Sigma(\tau, \tau'))_{T \times T}$ and note that, $\text{Var}^*(\sqrt{m}\Theta^*_{mn}) = \Sigma$. Then by the central limit theorem for i.i.d. data:

$$\sqrt{m}\Theta^* \xrightarrow{d^*} N(0, \Sigma).$$

Hence, $\hat{\delta}^* - \hat{\delta}$ has the same asymptotic distribution as $\hat{\delta} - \delta_0$.

***Part 3 - Weak Convergence of the Bootstrap***

By Lemma 1 we know that uniformly in $\tau \in \mathcal{T} \times \mathcal{T}$,

$$\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}_{2,j}(\delta_0, \beta_0, \tau)\left(\hat{\beta}_j(\tau_1) - \beta_{j,0}(\tau_1)\right) = o_p\left(\frac{1}{\sqrt{m}}\right),$$

and it directly follows that

$$\frac{1}{m}\sum_{j=1}^{m}\bar{\Gamma}^*_{2,j}(\delta_0, \beta_0, \tau)\left(\hat{\beta}_j(z^*, \tau_1) - \beta_{j,0}(z^*, \tau_1)\right) = o^*_p\left(\frac{1}{\sqrt{m}}\right).$$

Hence, uniformly over $\tau \in \mathcal{T} \times \mathcal{T}$

$$\hat{\delta}^*(\hat{\beta}(z^*), \tau) - \hat{\delta}(\hat{\beta}, \tau) = \Gamma_1(\delta_0, \beta_0, \tau)^{-1} \left( \frac{1}{m} \sum_{j=1}^{m} \bar{m}_j^*(\delta_0, \beta_0, \tau) - \bar{m}_j(\delta_0, \beta_0, \tau) \right) + o_p^* \left( \frac{1}{\sqrt{m}} \right)$$

From part (i) of the proof of Theorem 2, we know that $\tau \mapsto M_{mn}(\delta_0, \beta_0, \tau)$ is asymptotically equicontinuous. Given that $\frac{1}{m} \sum_{j=1}^{m} \bar{m}_j^*(\delta_0, \beta_0, \tau)$ is constructed by resampling with replacement elements from $\bar{m}_j(\delta_0, \beta_0, \tau)$ it follow that $\tau \mapsto \frac{1}{m} \sum_{j=1}^{m} \bar{m}_j^*(\delta_0, \beta_0, \tau)$ is also asymptotically equicontinuous. Hence, by Theorem 18.14 in van der Vaart and Wellner (1996) weak convergence follow:

$$\sqrt{m} \left( \hat{\delta}^*(\hat{\beta}(z^*), \cdot) - \hat{\delta}(\hat{\beta}, \cdot) \right) \rightsquigarrow^* \Gamma_1^{-1}(\cdot) \mathbb{G}(\cdot) \quad in\ \ell^\infty(\mathcal{T} \times \mathcal{T}).$$

■