ELEC 442 HW 1

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ELEC 442 Homework 1

①
$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} Q & d \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

when Q is a rotation matrix, $Q^{-1} = Q^T$ is always true because Q is always a 3×3 matrix. The number of columns of Q must be 3 because we're transforming a 3×1 vector. The number of rows of Q must also be 3 so the dimensions of the transformed vector are also 3×1.

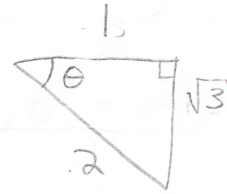
$$y = Qx + d$$
$$Q^T y = Q^T Q x + Q^T d$$
$$Q^T y = x + Q^T d$$
$$x = Q^T y - Q^T d$$

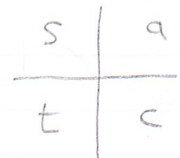
$$\boxed{\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Q^T & -Q^T d \\ 0^T & 1 \end{bmatrix}}$$

②
$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{bmatrix} \rightarrow R(i,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

So rotation about i by $120°$

$Q_2 = R(i, 120°)$





$\cos\theta = -\frac{1}{2}$ $\quad \theta = 120°, 240°$

$\sin\theta = \frac{\sqrt{3}}{2}$ $\quad \theta = 120°, 60°$

$\theta = 120°$

$$Q_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow R(k,\theta) \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cos\theta = \frac{1}{\sqrt{2}}$, $\theta = 45°, -45°$

$\sin\theta = -\frac{1}{\sqrt{2}}$, $\theta = -45°, -135°$

$\theta = -45°$

So, rotation about k by $-45°$

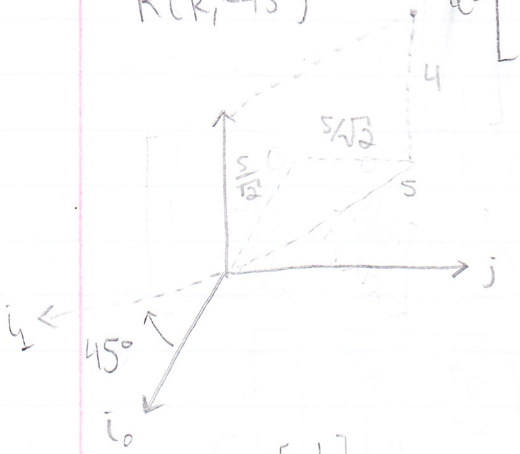
$Q_1 = R(k, -45°)$

(a)

$R(i, 120°)$
$R(k, -45°)$

$$d = \begin{bmatrix} -5/\sqrt{2} \\ 5/\sqrt{2} \\ 4 \end{bmatrix}$$

$$\boxed{\begin{aligned} \theta_1 &= -45° \\ d_1 &= -5 \\ a_1 &= 4 \\ \alpha_1 &= 120° \end{aligned}}$$



(b) $^1p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$^0p = {}^0T_1 {}^1p \longrightarrow \begin{bmatrix} ^0p \\ 1 \end{bmatrix} = \begin{bmatrix} Q & d \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} ^1p \\ 1 \end{bmatrix}$$

$$^0p = Q\,{}^1p + d \qquad Q = Q_1 Q_2$$

$$Q_2 {}^1p = {}^1p$$

^1p has only i components, and Q_2 is a rotation about i, so Q_2 would have no effect on ^1p

$$^0p = Q_1 {}^1p + d$$

$$Q_1 {}^1p = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$^0p = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} -5/\sqrt{2} \\ 5/\sqrt{2} \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} -4/\sqrt{2} \\ 4/\sqrt{2} \\ 4 \end{bmatrix} = {}^0p}$$

(c) To solve this, we can use the relation from Question 1:

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} Q & d \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Q^T & -Q^T d \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix}$$

$$^1p = Q^T \, {}^0p - Q^T d$$

$$Q^T = \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ -0.3536 & -0.3536 & 0.8660 \\ -0.6124 & -0.6124 & -0.50 \end{bmatrix}$$ done with matlab

$$\boxed{^1p = \begin{bmatrix} 5.7071 \\ -3.8177 \\ 1.3876 \end{bmatrix}}$$ done with matlab

(d) $$^0\omega_{0,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{rad}{s}$$ translations don't affect angular velocity

$$^1\omega_{0,1} = Q^T \, {}^0\omega_{0,1}$$

$$= Q^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{^1\omega_{0,1} = \begin{bmatrix} 0.7071 \\ -0.3536 \\ -0.6124 \end{bmatrix} rad/s}$$

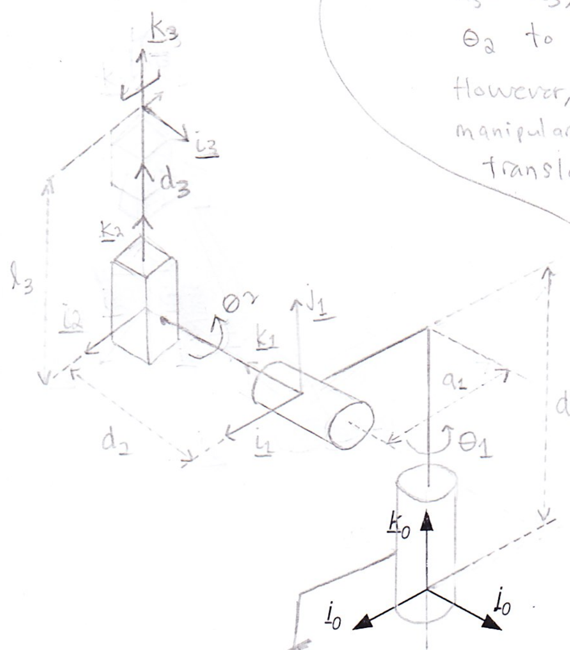
☆

4. Sketch the "home" position of the manipulator described by the table of D-H parameters below, starting from the base coordinate system shown. Label all coordinate systems (only need to label **i** and **k** vectors in frames), dimensions, and joint displacements (show polarity). In the table, joint variables are enclosed in parentheses. Find the abstract expression for the geometric Jacobian and discuss the existence of singularities.

		rotation about k	translation along k	translation along new i	rotation about new i
		θ_i	d_i	a_i	α_i
revolute	**Link 1**	(θ_1)	d_1	a_1	$\pi/2$
revolute	**Link 2**	(θ_2)	d_2	0	$-\pi/2$
prismatic	**Link 3**	$\pi/2$	$(d_3)+l_3$	0	0

$$\underset{\sim}{O_3} - \underset{\sim}{O_0} = (d_3+l_3)\underline{k_2} + d_2\underline{k_1} + a_1\underline{i_1} + d_1\underline{k_0}$$

$$\underset{\sim}{O_3} - \underset{\sim}{O_1} = (d_3+l_3)k_2 + d_2\underline{k_1}$$



There is a translational singularity when $d_3 = -l_3$, because we can no longer vary θ_2 to cause translation in the $\underline{k_2} \times \underline{i_2}$ plane. However, we can now use θ_2 to vary the manipulator orientation. While we lost a translation DOF, we gained an orientation DOF that we didn't previously have. We see this in the expression for the Jacobian below. The overall rank has stayed the same. So, there are no singularities.

Note: If there were no offsets, a_1 and d_2, there would be a singularity in the home position.

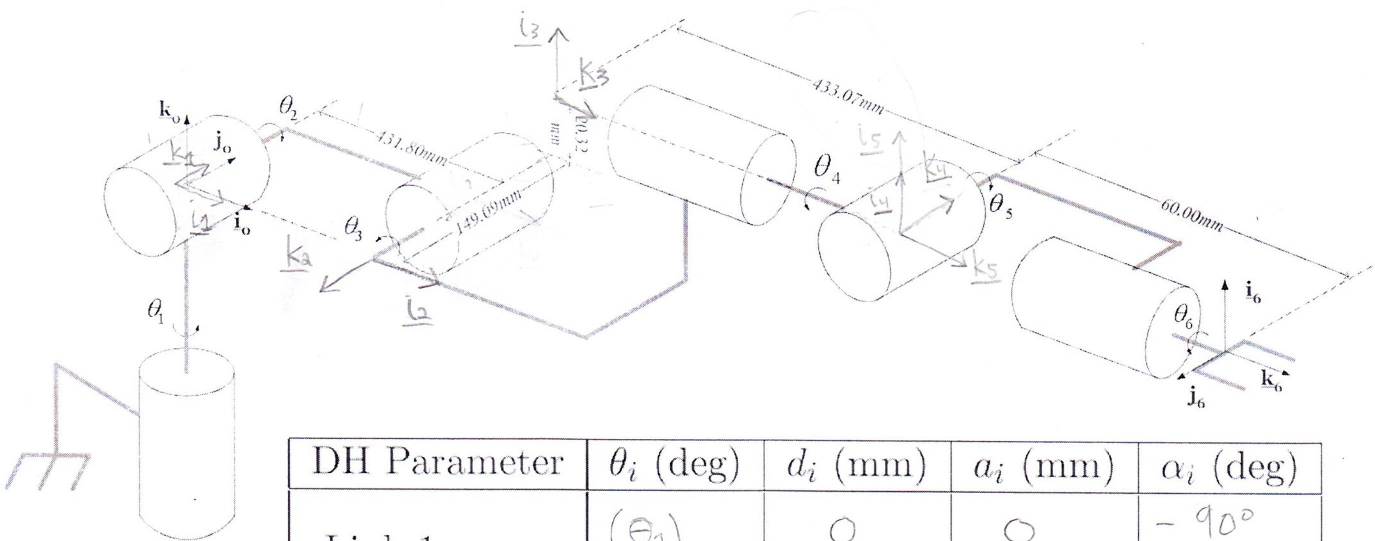
$$\underline{J} = \begin{bmatrix} \underline{k_0} \times (\underset{\sim}{O_3} - \underset{\sim}{O_0}) & \underline{k_1} \times (\underset{\sim}{O_3} - \underset{\sim}{O_1}) & \underline{k_2} \\ \underline{k_0} & \underline{k_1} & 0 \end{bmatrix}$$

$$\underline{J} = \begin{bmatrix} \underline{k_0} \times ((d_3+l_3)\underline{k_2} + d_2\underline{k_1} + a_1\underline{i_1} + \cancel{d_1\underline{k_0}}) & \underline{k_1} \times ((d_3+l_3)k_2 + \cancel{d_2\underline{k_1}}) & \underline{k_2} \\ \underline{k_0} & \underline{k_1} & 0 \end{bmatrix}$$

when $d_3 = -l_3$ $\quad \underline{k_0} \times d_1\underline{k_0} = 0$ $\quad$ when $d_3 = -l_3$ $\quad \underline{k_1} \times d_2\underline{k_1} = 0$

$$\boxed{\underline{J} = \begin{bmatrix} d_1\underline{k_0} \times \underline{k_1} + a_1\underline{k_0} \times \underline{i_1} & 0 & \underline{k_2} \\ \underline{k_0} & \underline{k_1} & 0 \end{bmatrix}}$$

5. The Puma 560 has a reach of 0.92m and a payload capacity of 2.3kg, making it ideal for medium-to-lightweight assembly, welding, materials handling, packaging and inspection applications. Using the schematic on the next page, do the following:
   a. Directly on the schematic, assign coordinate frames according to the D-H convention (only need to label **i** and **k** vectors for each frame). Assume $\underline{C_0}$ and $\underline{C_6}$ as illustrated are in the ``home" position. Fill in Table 1 the values of the DH-parameters. For each joint, consider the positive rotation to be in the *right-handed sense*. (NB: This was not always the case in the notes).
   b. Compose a chain of transformations that give the relationship between the base ($\{o_0, \underline{C_0}\}$) and end-effector ($\{o_6, \underline{C_6}\}$) coordinate systems (use notation from Salcudean notes as was done for example 2.5 on p 31).
   c. Determine the manipulator Jacobian symbolically and discuss when singularities occur.
   d. Write a Matlab m-file which prompts the user for the sequence of 6 joint angles in degrees (e.g., "45,-45,45,0,-30,90"), then outputs the resulting homogenous transformation matrix (relating the base and end effector) and the manipulator Jacobian (relating joint velocities to end-effector velocities). As well, graphically plot the location of each link origin (e.g., using Matlab's ``plot3" function, indicate each origin with an "*") for the given joint angles.
   e. Use your code to compute end-effector transformations and manipulator Jacobians for the joint vector sets: $q_A=[0°;0°;0°;0°;0°;0°]^T$, $q_B=[0°;0°;-90°;0°;0°;180°]^T$, $q_C=[45°;-45°;45°;0°;-30°;90°]^T$.



DH Parameter	θ_i (deg)	d_i (mm)	a_i (mm)	α_i (deg)
Link 1	(θ_1)	0	0	$-90°$
Link 2	(θ_2)	0	431.8mm	180°
Link 3	$(\theta_3)+90°$	-149.09	20.32	90°
Link 4	(θ_4)	433.07	0	90°
Link 5	(θ_5)	0	0	$-90°$
Link 6	(θ_6)	60	0	0

$\boxed{5b}$

$$\begin{bmatrix} \vec{x} \\ 1 \end{bmatrix} \qquad {}^0X = {}^0T_1 {}^1X$$

3×3

$$\begin{bmatrix} Q\vec{x} + \vec{d} \\ 1 \end{bmatrix} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \begin{bmatrix} 1/\sqrt{2} & & \\ & \cdot & \cdot \\ & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & \cdot & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{5}}{\sqrt{2}} \\ \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \underline{C}_6 & \underset{\sim}{\Omega}_6 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \underline{C}_0 & \underset{\sim}{\Omega}_0 \\ 0^T & 1 \end{bmatrix} \overset{{}^0T_1}{\begin{bmatrix} e^{\theta_1 k\times} e^{-\pi/2\, i\times} & 0 \\ 0^T & 1 \end{bmatrix}} \overset{{}^1T_2}{\begin{bmatrix} e^{\theta_2 k\times} e^{\pi i\times} & e^{\theta_2 k\times}(431.8i) \\ 0^T & 1 \end{bmatrix}}$$

$${}^2T_3 \begin{bmatrix} e^{(\theta_3+\frac{\pi}{2})k\times} e^{\pi/2\, i\times} & -149.09\,k + e^{(\theta_3+\frac{\pi}{2})k\times}(20.32\,i) \\ 0^T & 1 \end{bmatrix}$$

$${}^3T_4 \begin{bmatrix} e^{\theta_4 k\times} e^{\pi/2\, i\times} & 433.07k \\ 0^T & 1 \end{bmatrix}$$

$$
{}^{4}T_{5} \qquad\qquad\qquad {}^{5}T_{6}
$$

$$
\begin{bmatrix} e^{\theta_5 k\times} e^{-\pi/2\, i\times} & 0 \\ 0^T & 1 \end{bmatrix}
\begin{bmatrix} e^{\theta_6 k\times} & 60k \\ 0^T & 1 \end{bmatrix}
$$

$$
\theta_3 = \tan^{-1}\left(\frac{A_3}{D_4}\right)
$$

5c

$$\mathbb{J} = \begin{bmatrix} \mathbb{k}_0 \times (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_0) & \mathbb{k}_1 \times (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_0) & \mathbb{k}_2 \times (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_2) & \mathbb{k}_3 \times (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_3) & \mathbb{k}_4 \times (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_4) & \mathbb{k}_5 \times (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_4) \\ \mathbb{k}_0 & \mathbb{k}_1 & \mathbb{k}_2 & \mathbb{k}_3 & \mathbb{k}_4 & \mathbb{k}_5 \end{bmatrix}$$

$\underline{\mathbb{Q}}_1 = \underline{\mathbb{Q}}_0$, $\underline{\mathbb{Q}}_5 = \underline{\mathbb{Q}}_4$

Premultiply by $\begin{bmatrix} I & (\underline{\mathbb{Q}}_6 - \underline{\mathbb{Q}}_4)\times \\ 0 & I \end{bmatrix}$

$\mathbb{k}_3$ always // to $\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_3$

$$\mathbb{J} = \begin{bmatrix} \mathbb{k}_0 \times (\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_0) & \mathbb{k}_1 \times (\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_0) & \mathbb{k}_2 \times (\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_2) & \cancel{\mathbb{k}_3 \times (\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_3)} & 0 & 0 \\ \mathbb{k}_0 & \mathbb{k}_1 & \mathbb{k}_2 & \mathbb{k}_3 & \mathbb{k}_4 & \mathbb{k}_5 \end{bmatrix}$$

① Singularity when $\mathbb{k}_3 // \mathbb{k}_5$. $\mathbb{k}_3$, $\mathbb{k}_4$, $\mathbb{k}_5$ are cooplanar. This happens when $\theta_5 = 0$

Let $(\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_0) = a\mathbb{k}_0 + b\mathbb{k}_1 + c\underline{i}_1$, $(\underline{\mathbb{Q}}_4 - \underline{\mathbb{Q}}_2) = a\mathbb{k}_0 + b\mathbb{k}_1 + d\underline{i}_1$

$\mathbb{k}_0 \times a\mathbb{k}_0 = 0$

$$\mathbb{J} = \begin{bmatrix} b\mathbb{k}_0 \times \mathbb{k}_1 + c\mathbb{k}_0 \times \underline{i}_1 & a\mathbb{k}_1 \times \mathbb{k}_0 + c\mathbb{k}_1 \times \underline{i}_1 & a\mathbb{k}_2 \times \mathbb{k}_0 + \cancel{b\mathbb{k}_2 \times \mathbb{k}_1} + d\mathbb{k}_2 \times \underline{i}_1 & 0 & 0 & 0 \\ \mathbb{k}_0 & \mathbb{k}_1 & \mathbb{k}_2 & \mathbb{k}_3 & \mathbb{k}_4 & \mathbb{k}_5 \end{bmatrix}$$

$\mathbb{k}_2$ always // to $\mathbb{k}_1$

$$\mathbb{J} = \begin{bmatrix} \mathbb{k}_0 \times (b\mathbb{k}_1 + c\underline{i}_1) & \mathbb{k}_1 \times (a\mathbb{k}_0 + c\underline{i}_1) & \mathbb{k}_2 \times (a\mathbb{k}_0 + d\underline{i}_1) & 0 & 0 & 0 \\ \mathbb{k}_0 & \mathbb{k}_1 & \mathbb{k}_2 & \mathbb{k}_3 & \mathbb{k}_4 & \mathbb{k}_5 \end{bmatrix}$$

$$\mathbb{J}_{11} = \begin{bmatrix} \mathbb{k}_0 \times (b\mathbb{k}_1 + c\underline{i}_1) & \mathbb{k}_1 \times (a\mathbb{k}_0 + c\underline{i}_1) & \mathbb{k}_2 \times (a\mathbb{k}_0 + d\underline{i}_1) \end{bmatrix}$$

② → $\mathbb{k}_1 // (b\mathbb{k}_1 + c\underline{i}_1)$ and $\mathbb{k}_0 // (a\mathbb{k}_0 + c\underline{i}_1)$: when $c = 0$

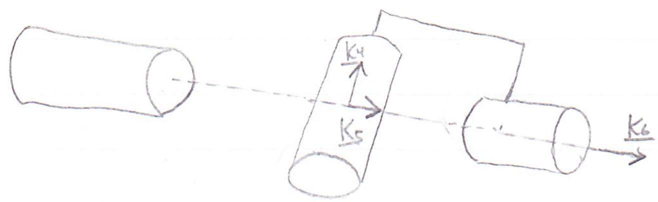
→ $\mathbb{k}_2 // (b\mathbb{k}_1 + c\underline{i}_1)$ $\mathbb{k}_0 // (a\mathbb{k}_0 + d\underline{i}_1)$: never occurs

③ → $\mathbb{k}_2 \times (a\mathbb{k}_0 + d\underline{i}_1)$ and $\mathbb{k}_1 \times (a\mathbb{k}_0 + c\underline{i}_1)$ are linearly dependant when $a = 0$

see next page for description of singularities

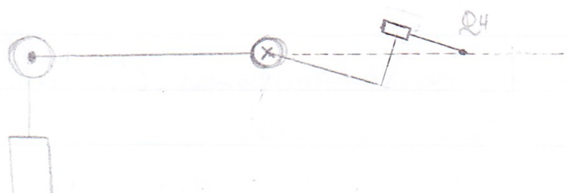
5c continued

① This singularity is a "wrist" singularity. When $\theta_5 = 0$, $\mathbb{k}_3$, $\mathbb{k}_4$, and $\mathbb{k}_5$ are coplanar $\rightarrow$ $\boxed{\text{wrist singularity when } \theta_5 = 0}$



② $J_{11} = \left[\, \mathbb{k}_0 \times (b\mathbb{k}_1 + c\,\mathbb{i}_1) \;\middle|\; \mathbb{k}_1 \times (a\mathbb{k}_0 + c\,\mathbb{i}_1) \;\middle|\; \mathbb{k}_2 \times (a\mathbb{k}_0 + d\,\mathbb{i}_1) \right]$

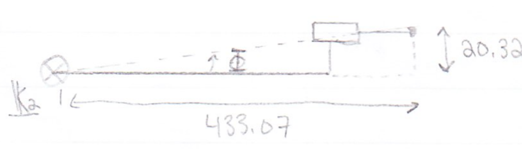
$a = 0$ occurs when we are in the orientation below: ℓ_4 is in the plane spanned by $\mathbb{k}_0$ and $\mathbb{i}_0$, and the arm is fully extended



in that case $(a = 0)$:

$J_{11} = \left[\, \mathbb{k}_0 \times (b\mathbb{k}_1 + c\,\mathbb{i}_1) \;\middle|\; \mathbb{k}_1 \times (c\,\mathbb{i}_1) \;\middle|\; \mathbb{k}_2 \times (d\,\mathbb{i}_1) \right]$

$\mathbb{k}_2 \parallel \mathbb{k}_1$ $\therefore$ these terms are linearly dependant.



$\Phi = \arctan\left(\frac{20.32}{433.07}\right) = 2.69°$

$\theta_3 = -\Phi$ when ℓ_4 is in plane spanned by $\mathbb{k}_1$ and $\mathbb{i}_1$

$\rightarrow$ $\boxed{\text{elbow singularity when } \theta_3 = -2.69°}$

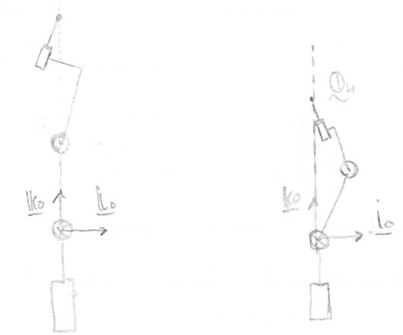
And theta 2 = 90n, n=0,1,2…

③ $J_{11} = \left[\mathbb{k}_0 \times (b\mathbb{k}_1 + c\mathbb{i}_1) \;\vdots\; \mathbb{k}_1 \times (a\mathbb{k}_0 + c\mathbb{i}_1) \;\vdots\; \mathbb{k}_2 \times (a\mathbb{k}_0 + d\mathbb{i}_1) \right]$

Shoulder singularity occurs when $\mathbb{Q}_4$ is in the plane spanned by $\mathbb{k}_0$ and $\mathbb{k}_1$
In that case,

$J_{11} = \left[\mathbb{k}_0 \times b\mathbb{k}_1 \;\vdots\; \mathbb{k}_1 \times a\mathbb{k}_0 \;\vdots\; \mathbb{k}_2 \times (a\mathbb{k}_0 + d\mathbb{i}_1) \right]$

these two terms are linearly dependant



for $\mathbb{Q}_4$ to be on the plane, there must must be no horizontal componets.

$$\boxed{\text{Shoulder singularity when: } 431.8\cos(\theta_2) + 433.07\cos\left(\theta_2 + \theta_3 + \tan^{-1}\left(\frac{20.32}{433.07}\right)\right) = 0}$$