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1. a) False, it's basis is empty set.
b) True since every finite generating set can be refined down to a linearly independent generating set.
c) False, some vector spaces like $P(x)$ have infinite basis
d) False, vector spaces can have more than one basis, but all of them must have same number of vectors.
e) True by 1st Corollary of Thm. 1.10 (Replacement Thm)
f) False, it is $n+1$
g) False, it's $m \cdot n$
h) True, this is exactly what Replacement Theorem shows.
i) False, this is only true if generating set is a basis, which is not specified.
j) True, by Thm 1.11, subspace of a finite-dimensional vector space has a dimension at most of this vector space implying that it must too be finite-dimensional.
k) True, they are $\{0\}$ and $V$ respectively.
l) True by Corollary 2 of Replacement Theorem

2) A set is a basis if it's lin. ind. and spans the corresponding vector space.

a) To check lin. ind., ~~set each entry~~ set lin. combination of each entry equal to zero.

$a(2,-4,1) + b(0,3,-1) + c(6,0,-1) = 0$, convert to system of lin. eqns.

$$\begin{cases} 2a + 6c = 0 \\ -4a + 3b = 0 \\ a - b - c = 0 \end{cases} \xrightarrow{\text{convert to augmented matrix}} \begin{pmatrix} 2 & 0 & 6 & 0 \\ -4 & 3 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & -1 & -1 & 0 \\ -4 & 3 & 0 & 0 \\ 2 & 0 & 6 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 4R_1} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the augmented matrix is in REF and has 3rd column without a leading entry, this indicates the system has inf. many solutions so the set is linearly dependent and not a basis for $\mathbb{R}^3$.

b) Same solving logic applies here:

$a(-1,3,1) + b(2,-4,-3) + c(-3,8,2) = 0$.

Convert to system of lin. eqns:

$$\begin{cases} -a + 2b - 3c = 0 \\ 3a - 4b + 8c = 0 \\ a - 3b + 2c = 0 \end{cases} \xrightarrow{\text{convert to augmented matrix}} \begin{pmatrix} -1 & 2 & -3 & 0 \\ 3 & -4 & 8 & 0 \\ 1 & -3 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 3R_1]{R_3 \to R_3 + R_1} \begin{pmatrix} -1 & 2 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_1 \to -1R_1} \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \end{pmatrix}$$

Since this augmented matrix is in REF without columns that don't have a leading entry, there is only one solution to this system which is for coefficients to all be 0, so the set is lin. ind.

Next we need to check that the set spans $\mathbb{R}^3$;

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That is an arbitrary 3-tuple $(a_1, a_2, a_3)$ in $\mathbb{R}^3$ can be written as a linear combination of the vectors in the set $\Rightarrow$

$$(a_1, a_2, a_3) = x(-1,3,1) + y(2,-4,-3) + z(-3,8,2)$$

Convert to system of lin. eqns:

$$\begin{cases} -x + 2y - 3z = a_1 \\ 3x - 4y + 8z = a_2 \\ x - 3y + 2z = a_3 \end{cases} \xrightarrow{\text{convert to aug. matr.}} \begin{pmatrix} -1 & 2 & -3 & a_1 \\ 3 & -4 & 8 & a_2 \\ 1 & -3 & 2 & a_3 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 3R_1]{R_3 \to R_3 + R_1} \begin{pmatrix} -1 & 2 & -3 & a_1 \\ 0 & 2 & -1 & 3a_1 + a_2 \\ 0 & -1 & -1 & a_1 + a_3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} -1 & 2 & -3 & a_1 \\ 0 & -1 & -1 & a_1 + a_3 \\ 0 & 2 & -1 & 3a_1 + a_2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} -1 & 2 & -3 & a_1 \\ 0 & -1 & -1 & a_1 + a_3 \\ 0 & 0 & -3 & 5a_1 + a_2 + 2a_3 \end{pmatrix}$$

$$\xrightarrow[R_3 \to -\frac{1}{3}R_3]{R_1 \to -1R_1,\ R_2 \to -1R_2} \begin{pmatrix} 1 & -2 & 3 & -a_1 \\ 0 & 1 & 1 & -a_1 - a_3 \\ 0 & 0 & 1 & -\frac{5}{3}a_1 - \frac{1}{3}a_2 - \frac{2}{3}a_3 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - 3R_3} \begin{pmatrix} 1 & -2 & 0 & 4a_1 + a_2 + 2a_3 \\ 0 & 1 & 0 & \frac{2}{3}a_1 + \frac{1}{3}a_2 - \frac{1}{3}a_3 \\ 0 & 0 & 1 & -\frac{5}{3}a_1 - \frac{1}{3}a_2 - \frac{2}{3}a_3 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 0 & \frac{16}{3}a_1 + \frac{5}{3}a_2 + \frac{4}{3}a_3 \\ 0 & 1 & 0 & \frac{2}{3}a_1 + \frac{1}{3}a_2 - \frac{1}{3}a_3 \\ 0 & 0 & 1 & -\frac{5}{3}a_1 - \frac{1}{3}a_2 - \frac{2}{3}a_3 \end{pmatrix}$$

convert back to system

$$\xrightarrow{GS} \begin{aligned} x &= \frac{16}{3}a_1 + \frac{5}{3}a_2 + \frac{4}{3}a_3 \\ y &= \frac{2}{3}a_1 + \frac{1}{3}a_2 - \frac{1}{3}a_3 \\ z &= -\frac{5}{3}a_1 - \frac{1}{3}a_2 - \frac{2}{3}a_3 \end{aligned}$$

Thus, the set spans $\mathbb{R}^3$ and thus is its basis!

3. We will construct a basis (a subset of given set that is lin. indep. and generates $\mathbb{R}^3$) recursively.

Let $v_1$ be in this new set, it is clear that $a(2,-3,1)=0$, $a$ must be $0$, thus $\{v_1\}$ is lin. ind.

Let $v_3$ be in this set with $v_1$, since it is $-4$ times $v_1$, it shouldn't be part of the set because it will make it lin. dependent.

Let $v_2$ be in this set with $v_1$, then:
$a(2,-3,1)+b(1,4,-2)=0$. If $a=b=0$ is the only solution, then $\{v_1, v_2\}$ is linearly ind., let's check:

$$\begin{aligned} 2a+b&=0\\ -3a+4b&=0\\ a-2b&=0 \end{aligned} \xrightarrow{E_1 \leftrightarrow E_3} \begin{aligned} a-2b&=0\\ -3a+4b&=0\\ 2a+b&=0 \end{aligned} \xrightarrow[E_3 \to E_3-2E_1]{E_2 \to E_2+3E_1} \begin{aligned} a-2b&=0\\ -2b&=0\\ 5b&=0. \end{aligned}$$

Clearly $a=b=0$ is the only solution, so $\{v_1, v_2\}$ is linearly independent set, continue...

Let $v_4$ be in the set with $v_1$ and $v_2$, same check is required:

$a(2,-3,1)+b(1,4,-2)+c(1,37,-17)=0$

$$\begin{aligned} 2a+b+c&=0\\ -3a+4b+37c&=0\\ a-2b-17c&=0 \end{aligned} \xrightarrow{E_1 \leftrightarrow E_3} \begin{aligned} a-2b-17c&=0\\ -3a+4b+37c&=0\\ 2a+b+c&=0 \end{aligned} \xrightarrow[E_3 \to E_3-2E_1]{E_2 \to E_2+3E_1} \begin{aligned} a-2b-17c&=0\\ -2b-14c&=0\\ 5b+35c&=0 \end{aligned}$$

$$\xrightarrow{E_2 \to -\frac{1}{2}E_2} \begin{aligned} a-2b-17c&=0\\ b+7c&=0\\ 5b+35c&=0 \end{aligned} \xrightarrow{E_3 \to E_3-5E_2} \begin{aligned} a-2b-17c&=0\\ b+7c&=0\\ 0&=0 \end{aligned}$$

$$\xrightarrow{E_1 \to E_1+2E_2} \begin{aligned} a-3c&=0\\ b+7c&=0\\ 0&=0 \end{aligned} \longrightarrow \begin{aligned} a&=3c\\ b&=-7c, \end{aligned}$$

Thus, there are infinetly many solutions to this system, so $v_4$ shouldn't be a part of the set.

Let $v_5$ be in a set with $v_1$ and $v_2$, same check is req.

$a(2,-3,1)+b(1,4,-2)+c(-3,-5,8)=0$

$$\begin{aligned} 2a+b-3c&=0\\ -3a+4b-5c&=0\\ a-2b+8c&=0 \end{aligned} \xrightarrow{E_1 \leftrightarrow E_3} \begin{aligned} a-2b+8c&=0\\ -3a+4b-5c&=0\\ 2a+b-3c&=0 \end{aligned} \xrightarrow[E_3 \to E_3-2E_1]{E_2 \to E_2+3E_1} \begin{aligned} a-2b+8c&=0\\ -2b+19c&=0\\ 5b-19c&=0 \end{aligned}$$

$$\xrightarrow{E_2 \to -\frac{1}{2}E_2} \begin{aligned} a - 2b + 8c &= 0 \\ b - 9.5c &= 0 \\ 5b - 19c &= 0 \end{aligned} \quad \xrightarrow{E_3 \to E_3 - 5E_2} \begin{aligned} a - 2b + 8c &= 0 \\ b - 9.5c &= 0 \\ 28.5c &= 0 \end{aligned}$$

Since $c = 0 \Rightarrow b = 0 \Rightarrow a = 0$, thus the solution is for $a, b, c$ to be all $0$, so the set is lin. ind.
Since we reduced generating set to its subset with lin. ind. vectors, by Thm 1.9 its a basis for $\mathbb{R}^3$

4) Let $S$ be a set of 4 polynomials s.t $S = \{p_1(x), p_2(x), p_3(x), p_4(x)\}$.
Let their degrees be distincts and w.l.g. in descending order s.t. $d_1 > d_2 > d_3 > d_4 \geq 0$ respectively.
$S$ is lin. ind. if:
$\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 p_4(x) = 0$ is trivial solution where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.
Since the degrees are ordered in the way s.t $d_1 > d_2 > d_3 > d_4$, thus the highest power of $x$ in the sum is $x^{d_1}$ which can only come from $\alpha_1 p_1(x)$ polynomial.
Let the leading coefficient of $p_1(x)$ be $c_1$ where $c_1 \neq 0$, thus the coefficient in the total sum is $\alpha_1 c_1$. In order for the sum to be equal to $0$ polynomial, $\alpha_1 c_1$ must be $0$, since $c_1$ can't be $0$, this implies that $\alpha_1 = 0$.
This eliminates $p_1(x)$ from lin. comb. sum such that:
$\alpha_2 p_2(x) + \alpha_3 p_3(x) + \alpha_4 (p_4(x)) = 0$.
Repeating same procedure as to $p_1$, we find that $\alpha_2 = \alpha_3 = \alpha_4$ must all be $0$. Since there is only one solution to this equation where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, $S$ is linearly independent set.

5) Let $v \in \text{span}(S)$. Then there exist $s_1, \dots, s_n \in S$ and $a_1, \dots, a_n$ scalars s.t. $v = a_1 s_1 + \dots + a_n s_n$

We must show that $\text{span}(S \cup \{v\}) = \text{span}(S)$, that is $\text{span}(s_1, \dots, s_n, v) = \text{span}(s_1, \dots, s_n)$.

To show this is true, we must show that a linear combination of span(S) must be in $\text{span}(s_1, \dots, s_n, v)$ and vice-versa.

~~$\text{span}(S) \subseteq \text{span}(S \cup \{v\})$:~~

~~$a_1 s_1 + \dots + a_n s_n$ is in $\text{span}(S \cup \{v\})$ since $v$ is in that span and $v = a_1 s_1 + \dots + a_n s_n$~~

~~$\text{span}(S \cup \{v\}) \subseteq \text{span}(S)$~~

~~$a_1 s_1 + \dots + a_n s_n +$~~

$\text{span}(S) \subseteq \text{span}(S \cup \{v\})$:

Let $w \in \text{span}(S)$. Then $w$ is a lin. comb. of vectors in $S$. Since $S \subset S \cup \{v\}$, any lin. comb. of vectors in $S$ is also a lin. comb. of vectors in $S \cup \{v\}$ for where $v$'s coefficient is zero.

Hence $w \in \text{span}(S \cup \{v\})$, proving $\text{span}(S) \subseteq \text{span}(S \cup \{v\})$.

$\text{span}(S \cup \{v\}) \subseteq \text{span}(S)$:

Let $u \in \text{span}(S \cup \{v\})$, then $u = b_1 s_1 + \dots + b_n s_n + c \cdot v$ for some scalar $c$.

Since $v = a_1 s_1 + \dots + a_n s_n$ bcs. $v \in \text{span}(S)$, substituting the expression for $u$ gives $u = b_1 s_1 + \dots + b_n s_1 + c(a_1 + \dots + a_n s_n$. Rearranging and combining terms gives:

$u = (a_1 + cb_1)s_1 + \dots + (a_n + cb_n)s_n$.

Since $u$ is lin. comb. of vectors from $S$, $u \in \text{span}(S)$. Thus $\text{span}(S \cup \{v\}) \subseteq \text{span}(S)$.

Thus, since subset inclusion held in both directions, $\text{span}(S \cup \{v\}) = \text{span}(S)$

$\blacksquare$

6. Let $A$ be an arbitrary $3\times3$ symmetric matrix, then its general form is $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$, that is $a_{ij} = a_{ji}$ for $1 \le i \le 3$, $1 \le j \le 3$

To compute its dimension, we must find its basis since # vectors in basis is the dimension of that vector space.

Let $\beta = \left\{ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} \right.$ be the candidate basis. It is clear that no vector is a linear combination of the other in this set. We thus must show that this lin. ind. set spans the vector space.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = a_{11}\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} + a_{12}\begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix} + a_{22}\begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}$$

$$+ a_{13}\begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} + a_{23}\begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix} + a_{33}\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Thus the set spans the vector space of $3\times3$ sym. matrices since it generates an arbitrary matrix $A$ in the vector space

7. Given set is lin. dep. subset of $\mathcal{F}(\mathbb{R})$ if for scalars $a, b, c$ not all $0$, $a\sin^2(x) + b\sin^2(x) + c \cdot 1 = 0$, for all $x \in \mathbb{R}$

Using pythagorean trig. identity, $\sin^2 x + \cos^2 x = 1$, thus $1 \cdot \sin^2(x) + 1(\cos^2(x)) - 1 \cdot 1 = 0$ for $a = b = 1$ and $c = -1$.

Since scalars are not all 0 and the non-trivial combination holds for all $x$, the set is lin. dependent.