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1. Let $T : V \to \mathbb{R}^3$ defined by: $T\left(\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}\right) = (a, b, c)$.

We must show that this map is isomorphism.
To prove it is linear, let $X, Y \in V$ and $k \in \mathbb{R}$ arbitrary. Then $X = \begin{pmatrix} a_1 & a_1+b_1 \\ 0 & c_1 \end{pmatrix}$ and $Y = \begin{pmatrix} a_2 & a_2+b_2 \\ 0 & c_2 \end{pmatrix}$

Then:

$$T(kX + Y) = T\begin{pmatrix} ka_1 + a_2 & ka_1 + a_2 + kb_1 + b_2 \\ 0 & kc_1 + c_2 \end{pmatrix}$$

$$= (ka_1 + a_2, kb_1 + b_2, kc_1 + c_2)$$

$$= \cancel{kT(X) + T}$$

$$= k(a_1, b_1, c_1) + (a_2, b_2, c_2)$$

$$= kT(X) + T(y)$$

✓.

We need to show that T is 1-1 and onto to be isomorphism.

(1-1) $\Rightarrow$ $T\left(\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}\right) = (a, b, c) = (0, 0, 0)$

$\Rightarrow$ $a = b = c = 0$, so $N(T) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

and hence T is one to one.

~~Since T is linear that is both 1-1 and onto, it is an isomorphism from V to $\mathbb{R}^3$~~

(Onto) $\Rightarrow$ Let $(x, y, z) \in \mathbb{R}^3$ arbitrary, then we need a matrix in V that maps to it.

$\Rightarrow T\left(\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}\right) = (a, b, c) = (x, y, z)$. Choose $a = x$, $b = y$, and $c = z$ we have $\begin{pmatrix} x & x+y \\ 0 & z \end{pmatrix} \in V$. Since this pre-image exists for every vector in $\mathbb{R}^3$, T is onto.

Since T is linear and one-to-one and onto, it is an isomorphism from V to $\mathbb{R}^3$.

2. φ is isomorphism if it is linear and bijective.

Linear: Let $A, D \in M_{n\times n}(\mathbb{F})$ and $k \in \mathbb{F}$ arbitrary then: $\varphi(A + kD) = B^{-1}(A + kD)B = B^{-1}(AB + kDB)$
$= B^{-1}AB + kB^{-1}DB = \varphi(A) + k\varphi(D)$

(1-1): if $\varphi(A) = B^{-1}AB = 0$, then $A = B0B^{-1} = 0$, so φ is one-to-one.

(Onto): Let $D \in M_{n\times n}(\mathbb{F})$ co-domain. arbitrary, then $\varphi(BDB^{-1}) = D$ and so φ is onto.

Thus, φ is isomorphism.

3) a) False, it should be $[x'_j]_\beta$
b) True by Thm. 2.22.
c) True by Thm. 2.23.
d) False, it should be $B = Q^{-1}AQ$
e) True, follows from thm 2.23 and defn. of similar, matrix representation of the same linear operation w.r.t. to diff bases are related by conjugation with invertible change of coordinate matrix.

4. ~~a)~~ $Q = [I_V]_{\beta'}^{\beta}$

a) $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta}$

$\Rightarrow I_V(a_1, a_2) = a_1(1,0) + a_2(0,1) = a_1 e_1 + a_2 e_2$
$I_V(b_1, b_2) = b_1(1,0) + b_2(0,1) = b_1 e_1 + b_2 e_2$,

so $Q = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \rightarrow \left([(a_1,a_2)]_\beta = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, [(b_1,b_2)]_\beta = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right)$

b) $I_V(0,10) = a(-1,3) + b(2,-1)$

$-a + 2b = 0$
$3a - b = 10$ $\sim$ $a - 2b = 0$, $3a - b = 10$ $\sim$ $a - 2b = 0$, $5b = 10$

$\rightarrow b = 2$
$a = 4$

$I_V(5,0) = a(-1,3) + b(2,-1)$

$-a + 2b = 5$
$3a - b = 0$ $\sim$ $a - 2b = -5$, $3a - b = 0$ $\sim$ $a - 2b = -5$, $5b = 15$

$\rightarrow b = 3$
$a = 1$

Therefore, $Q = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \rightarrow \left([(0,10)]_\beta = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, [(5,0)]_\beta = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right)$

c) $I_V(1,0) = a(2,5) + b(-1,-3)$

$2a - b = 1$
$5a - 3b = 0$ $\sim$ $2a - b = 1$, $-a = -3$ $\rightarrow a = 3$

$b = 5$

$I_V(0,1) = a(2,5) + b(-1,-3)$

$$\begin{aligned} 2a - b &= 0 \\ 5a - 3b &= 1 \end{aligned} \sim \begin{aligned} 2a - b &= 0 \\ -a &= 1 \end{aligned} \rightarrow \begin{aligned} a &= -1 \\ b &= -2 \end{aligned}$$

So, $Q = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \rightarrow \left([(1,0)]_\beta = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, [(0,1)]_\beta = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right)$

d) $I_V(2,1) = a(-4,3) + b(2,-1)$

$$\begin{aligned} -4a + 2b &= 2 \\ +3a - b &= 1 \end{aligned} \sim \begin{aligned} -2a + b &= 1 \\ +3a - b &= 1 \end{aligned} \sim \begin{aligned} -2a + b &= 1 \\ a &= 2 \end{aligned}$$

$\rightarrow b = 5$

$I_V(-4,1) = a(-4,3) + b(2,-1)$

$$\begin{aligned} -4a + 2b &= -4 \\ 3a - b &= 1 \end{aligned} \sim \begin{aligned} -2a + b &= -2 \\ 3a - b &= 1 \end{aligned} \sim \begin{aligned} -2a + b &= -2 \\ a &= -1 \end{aligned}$$

$\rightarrow b = -4$

So, $Q = \begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix} \rightarrow \left([(2,1)]_\beta = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, [(-4,1)]_\beta = \begin{pmatrix} -1 \\ -4 \end{pmatrix} \right)$

5) $Q = [I_V]_{\beta'}^{\beta}$

a) $I_V(1) = 0 \cdot (2x^2 - x) + 1(3x^2 + 1) - 3x^2$

$I_V(x) = -1 \cdot (2x^2 - x) + 0 \cdot (3x^2 + 1) + 2x^2$

$I_V(x^2) = 0 \cdot (2x^2 - x) + 0(3x^2 + 1) + 1x^2$

thus, $Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \rightarrow \left([1]_\beta = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, [x]_\beta = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, [x^2]_\beta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

b) $I_V(x^2 + x + 4) = 2 \cdot (x^2 - x + 1) + 3(x+1) - 1(x^2 + 1)$

$I_V(4x^2 - 3x + 2) = 1 \cdot (x^2 - x + 1) - 2(x+1) + 3(x^2 + 1)$

$I_V(2x^2 + 3) = 1 \cdot (x^2 - x + 1) + 1(x+1) + 1(x^2 + 1)$

So, $Q = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix} \rightarrow \left([x^2 + x + 4]_\beta = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, [4x^2 - 3x + 2]_\beta = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, [2x^2 + 3]_\beta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$

6)

The change of basis formula is $[L_A]_\beta = Q^{-1} A Q$ where A is representation of L_A in standard basis and Q is change of basis matrix whose columns are vectors of basis β.

a) $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$, so $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$Q^{-1} = \frac{1}{1\cdot 2 - 1\cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 1 \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Or, $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \to [Q|I]$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$, where $\underbrace{\begin{matrix} 2 & -1 \\ -1 & 1 \end{matrix}}_{Q^{-1}}$

$$[L_A]_\beta = Q^{-1} A Q$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$$

b) $\beta = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \right\}$, so $Q = \begin{pmatrix}1&1&1\\1&0&1\\1&1&2\end{pmatrix}$

$$Q^{-1} = \frac{1}{-1}\begin{pmatrix}-1&-1&1\\-1&1&0\\1&0&-1\end{pmatrix}^T = \begin{pmatrix}1&1&-1\\1&-1&0\\-1&0&1\end{pmatrix}$$

$$[L_A]_\beta = Q^{-1} A Q$$

$$= \begin{pmatrix}1&1&-1\\1&-1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&1&-1\\2&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&1\\1&1&2\end{pmatrix}$$

$$= \begin{pmatrix}2&0&0\\-1&1&-2\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&1\\1&1&2\end{pmatrix}$$

$$= \begin{pmatrix}2&2&2\\-2&-3&-4\\1&1&2\end{pmatrix}$$

or, $[Q|I_n]$

$$\to \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&0&1&0&1&0\\1&1&2&0&0&1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&1&2&0&0&1\\1&0&1&0&1&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&0&1&-1&0&1\\0&-1&0&-1&1&0\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&1&-1&0\\0&0&1&-1&0&1\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc}1&0&0&1&1&-1\\0&1&0&1&-1&0\\0&0&1&-1&0&1\end{array}\right]$$

$\underbrace{\qquad}_{Q^{-1}}$

7. a) Let β be the standard basis and $\beta' = \{(1,m),(-m,1)\}$ be another basis for $\mathbb{R}^2$.

Then $[T]_{\beta'} = \left[[T((1,m))]_{\beta'}, [T((-m,1))]_{\beta'} \right]$

$$= \cancel{\left([(1,-m)]_{\beta'}, [(-m,-1)]_{\beta'} \right)}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q^{-1} = [I]_{\beta'}^{\beta}$$

$I_V(1,m) = 1(1,0) + m(0,1)$

$I_V(-m,1) = -m(1,0) + 1(0,1)$

$$\Rightarrow Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

$$Q = [I_V]_{\beta}^{\beta'} = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ -\frac{m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$$

$$[T]_{\beta} = Q^{-1}[T]_{\beta'}Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ -\frac{m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ -\frac{m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{pmatrix}$$

Then $T(x,y) = \begin{pmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$= \left(\frac{x+2ym-xm^2}{m^2+1}, \frac{-y+2xm+ym^2}{m^2+1} \right)$$

b) Similarly to a), ~~we~~ we have:

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ and

$Q^{-1} = [I_V]_{\beta'}^{\beta'}$ which from a) is $\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$,

then Q also follows from a) $\Rightarrow \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$

$$[T]_\beta = Q^{-1}[T]_{\beta'}Q$$

$$= \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{-m}{m^2+1} & \frac{1}{m^2+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{pmatrix}$$

Then, $T(x,y) = \begin{pmatrix} \frac{1}{m^2+1} & \frac{m}{m^2+1} \\ \frac{m}{m^2+1} & \frac{m^2}{m^2+1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \left(\frac{x+my}{1+m^2}, \frac{mx+m^2y}{1+m^2}\right)$$

8. We will use RREF(A|B) to set matrix to Identity matrix and use the following property to find inverse matrix:

$$(A \mid I_n) \Rightarrow (RREF(A) \mid A^{-1}) \text{ if } RREF(A) = I_n$$

a)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\overset{R_2 \to R_2 + R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{R_3 \to R_3 + R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right)$$

$$\overset{R_3 \to \frac{1}{3}R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right) \overset{\substack{R_2 \to R_2 - 2R_3 \\ R_1 \to R_1 - R_3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

b)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{R_3 \to R_3 - R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{R_3 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 3 & 3 & 1 & 1 & 0 \end{array}\right) \overset{R_3 \to \frac{1}{3}R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 2 & -\frac{1}{3} & \frac{2}{3} & 1 \end{array}\right)$$

$$\xrightarrow{\substack{R_1 \to R_1 - \frac{1}{2}R_3 \\ R_2 \to R_2 - \frac{1}{2}R_3 \\ R_3 \to \frac{1}{2}R_3}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7/6 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

c) $\det(C) = 1\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$= 0$

Since $\det(C) = 0$, the matrix is singular and thus is not invertible.