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1. a) True by definition

b) False, this excludes $T(cx) = cT(x)$ property.

~~g) True, this is what [illegible] shows.~~

~~d) True by definition~~

c) False, this is True only if T is linear by Thm. 2.4

d) True by definition, $T(0_V) + 0_W = T(0 \cdot v) = 0 \cdot T(v) = 0_W$

e) False, $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

f) False, to be true this is guaranteed if T is also one-to-one.

g) True, since linear transformation is uniquely determined by its action on basis, by Thm. 2.6 corollary this must be true.

h) False, only true if x_1, x_2 are lin. ind., otherwise their images must be related in the same way which is not true for arbitrary $y_1, y_2 \in W$.

2. i) Let $a, b \in \mathbb{R}^3$ s.t. $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c \in \mathbb{R}$ arbitrary.

$$\begin{aligned} T(ca + b) &= T(ca_1 + b_1, ca_2 + b_2, ca_3 + b_3) \\ &= (ca_1 + b_1 - ca_2 - b_2, 2ca_3 + 2b_3) \\ &= c(a_1 - a_2, 2a_3) + (b_1 - b_2, b_3) \\ &= cT(a) + T(b) \end{aligned}$$

✓ T is linear

ii) $N(T) = \{ v \in V : T(v) = 0 \}$

$R(T) = \{ T(v) : v \in V \}$

Need all a_1, a_2, a_3 s.t. $(a_1 - a_2, 2a_3) = 0$

$\Longrightarrow a_1 - a_2 = 0 \Rightarrow a_1 = a_2$

$2a_3 = 0 \Rightarrow a_3 = 0$

$\Longrightarrow (a_1, a_2, a_3) = (a_1, a_1, 0)$

$= (1, 1, 0)\, a_1$

So, $N(T) = \text{span}(\{1, 1, 0\})$ ✓

Need all $T(v)$ s.t $v \in \mathbb{R}^3$, that is

$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = (1,0)a_1 + (-1,0)a_2 + (0,2)a_3$

Let $\{(1,0), (-1,0), (0,2)\}$ be candidate basis for $R(T)$,

since $(-1,0) = -1(1,0)$, we exclude $(-1,0)$ from the set

and get $\{(1,0), (0,2)\}$ which is clearly lin. ind. and spans $R(T)$ ✓

iii) Since $\text{span}(\{(1,1,0)\}) = N(T) \Rightarrow \text{nullity}(T) = 1$

$\text{span}(\{(1,0), (0,2)\}) = R(T) \Rightarrow \text{rank}(T) = 2$

$\dim \mathbb{R}^3 = 3$,

nullity T + rank T = $\dim V$ by dim. Thm. $\Rightarrow 1 + 2 = 3$ ✓

iv) Since T is linear and $N(T) \neq \{0\}$, T isn't one-to-one by 2.4 theorem.

Since rank$T$ $= \dim(R(T)) = \dim \mathbb{R}^2$

$\Rightarrow R(T) = \mathbb{R}^2$ (definition of T being onto).

3. $P_2(\mathbb{R}) \to P_3(\mathbb{R})$, $T(f(x)) = x f(x) + f'(x)$

i) Let $f, g \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$ arbitrary,

$$T(cf+g)(x) = x(cf+g)(x) + (cf' + g')(x)$$
$$= c(xf + f')(x) + (xg + g')(x)$$
$$= c(xf(x) + f'(x)) + (xg(x) + g'(x))$$
$$= cT(f) + T(g(x))$$ ✓ T is linear.

ii) Need all $f(x) \Rightarrow x f(x) + f'(x) = 0$.

$f(x) = ax^2 + bx + c$

$\Rightarrow x(ax^2 + bx + c) + 2ax + b = 0$

$\Rightarrow ax^3 + bx^2 + cx + 2ax + b = 0$

$\Rightarrow ax^3 + bx^2 + (2a + c)x + b = 0$

$\Rightarrow a = 0$

$b = 0$

$2a + c = 0 \Rightarrow 2(0) + c = 0 \Rightarrow c = 0$

$b = 0$

Thus, $N(T) = \{0\}$.

$T(f(x)) = xf(x) + f'(x)$

$\Rightarrow x(ax^2+bx+c) + 2ax + b$

$= a(x^3+2x) + b(x^2+1) + cx$ since highest degrees of x differ

Since $\{x^3+2x, x^2+1, x\}$ is clearly lin. ind and spans $R(T)$, it is its basis.

iii) $\underset{\text{nullity } T}{\dim(N(T))} = 0$, $\underset{\text{rank } T}{\dim(R(T))} = 3$, $\dim(P_2(\mathbb{R})) = 3$

By dim. Theorem (nullity T + rank T = dim V):

$0 + 3 = 3$ ✓

iv) By Thm. 2.4, since T is linear and $N(T) = \{0\}$, T is one-to-one.

Since $\dim(P_3(\mathbb{R})) = 4$, rank $T \neq \dim(P_3(\mathbb{R}))$

$\Rightarrow R(T) \neq P_3(\mathbb{R})$, so T is not onto

4. a) Let $a, b \in \mathbb{R}^2$ s.t. $a = (a_1, a_2)$, $b = (b_1, b_2)$, and let $c \in \mathbb{R}$ arbitrary

$T(ca+b) = cT(a) + T(b)$ ← Need to show is false to prove T isn't linear

$ca+b = (ca_1+b_1, ca_2+b_2)$

$T(ca+b) = (ca_1+b_1, (ca_1+b_1)^2)$

$cT(a) = c(a_1, a_1^2) = (ca_1, ca_1^2)$

$T(b) = (b_1, b_1^2)$

Since $T(ca+b) = (ca_1+b_1, c^2a_1^2+b_1^2)$

$\neq cT(a) + T(b) = (ca_1+b_1, ca_1^2+b_1^2)$

$T(a_1, a_2) = (a_1, a_1^2)$ is not linear.

b) Let $a, b \in \mathbb{R}^2$, $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c \in \mathbb{R}$ arbitrary.

$T(ca+b) = cT(a) + T(b)$

$ca+b = (ca_1+b_1, ca_2+b_2)$

$T(ca+b) = (ca_1+b_1+1, ca_2+b_2)$

$cT(a) = c(a_1+1, a_2) = (ca_1+c, ca_2)$

$T(b) = (b_1 + 1, b_2)$.
$cT(a) + T(b) = (ca_1 + b_1 + c + 1, ca_2 + b_2)$
Since $T(ca + b) \neq cT(a) + T(b)$, $T(a_1, a_2) = (a_1 + 1, a_2)$ is not linear.

5) To prove that such a linear transformation exists, we will use Theorem 2.6., which states for a given basis of V in $T: V \to W$, there exists exactly one linear transformation s.t. $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$, where n is # of basis vectors of V.

With such statement, we will need to show that $(1,1)$ and $(2,3)$ are basis of $\mathbb{R}^2$ and use results of the Theorem to find $T(8, 11)$.

$\{(1,1), (2,3)\}$ is clearly lin ind. since neither of the two vectors is a scalar multiple of the other and since this is lin. ind. ~~subset~~ set of 2 vectors, it spans $\mathbb{R}^2$ by and Corollary of Replacement Theorem.

Now, by Theorem 2.6, T must exist and $(8, 11)$ can be expressed as $2(1,1) + 3(2,3)$.

By linearity of T, we can find image of $(8, 11)$ based on the images of respective basis vectors, so:

$T(2(1,1) + 3(2,3)) = 2(1,0,2) + 3(1,-1,4)$
$= (2,0,4) + (3,-3,12) = (5,-3,16) = T(8,11)$.

6. Let the candidate linear transformation be:
$T: \mathbb{R}^2 \to \mathbb{R}^2$: $T(x,y) = (y, 0)$. We will verify T is linear and $N(T) = R(T)$.

Let $x, y \in \mathbb{R}^2$ s.t. $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $c \in \mathbb{R}$ arbitrary.

$$T(cx + y) = T(cx_1 + y_1, cx_2 + y_2)$$
$$= (cx_2 + y_2, 0)$$
$$= c(x_2, 0) + (y_2, 0)$$
$$= cT(x) + T(y)$$

✓, T is linear.

$N(T) \Rightarrow$ all x, y s.t. $T(x,y) = 0$

$\Rightarrow (y, 0) = (0,0) \Rightarrow y = 0$

So, $(x,y) = (x, 0) = (1,0)x$, thus $N(T) = \text{span}(\{(1,0)\})$

$R(T) \Rightarrow (y,0) = (1,0)(y)$

$\Rightarrow R(T) = \text{Span}(\{(1,0)\})$.

Thus, $N(T) = R(T)$ and so we found the lin transformation

7. a) $[T(e_1)]_\gamma = T(1,0) = (2,3,1) = 2e_1 + 3e_2 + e_3$

$[T(e_2)]_\gamma = T(0,1) = (-1, 4, 0) = -1e_1 + 4e_2 + 0e_3$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

b) $[T(e_1)]_\gamma$ $T(1,0,0) = (2,1) = 2e_1 + e_2$

$[T(e_2)]_\gamma$ $T(0,1,0) = (3,0) = 3e_1 + 0e_2$

$[T(e_3)]_\gamma = T(0,0,1) = (-1, 1) = -1e_1 + e_2$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$.

8. $[T(e_1)]_\gamma$ $T(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)$

$[T(e_2)]_\gamma$ $T(0,1) = (-1,0,1) =$ ~~[illegible]~~ $-1(1,1,0) + 1(0,1,1) + 0(2,2,3)$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{pmatrix}$

$T[(\alpha_1)]_\gamma \Rightarrow T(1,2) = (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$

$T[(\alpha_2)]_\gamma \Rightarrow T(2,3) = (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$

$\Rightarrow [T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ 2/3 & 4/3 \end{pmatrix}$

9. $\alpha = \left\{ \overset{M_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{M_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{M_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \overset{M_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

a) $T\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1M_1 + 0M_2 + 0M_3 + 0M_4$

$T\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0M_1 + 0M_2 + 1M_3 + 0M_4$

$T\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0M_1 + 1M_2 + 0M_3 + 0M_4$

$T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0M_1 + 0M_2 + 0M_3 + 1M_4$

$\Rightarrow [T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b) $T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0M_1 + 2M_2 + 0M_3 + 0M_4$

$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1M_1 + 2M_2 + 0M_3 + 0M_4$

$T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0M_1 + 2M_2 + 0M_3 + 2M_4$

$\Rightarrow [T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = 1M_1 - 2M_2 + 0M_3 + 4M_4$

$\Rightarrow [A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}$

10. a) To prove $N(T)$ is a subspace of V for a linear map, we must show it's closed under addition and scalar multiplication and 0_V is in it.

1. Need $0_V \in N(T)$, since T is linear, $T(0_V) = 0_W$, thus $0_V \in N(T)$

2. Let $x, y \in N(T)$, then $T(x) = 0_W$, $T(y) = 0_W$
   $\Rightarrow$ ~~then~~ ~~x+y~~. $T(x+y) = T(x) + T(y) = \bar{0}_W + \bar{0}_W = \bar{0}_W$
   Thus, $x+y \in N(T)$

3. Let $c \in \mathbb{R}$ arbitrary, since $T(x) = 0_W$,
   $T(cx) = cT(x) = c\bar{0}_W = \bar{0}_W$ so $cx \in N(T)$.

Thus, $N(T)$ is a subspace of V.

6. ~~[scribble]~~ T is 1-1 iff $N(T) = \{\bar{0}_V\}$

$(\longrightarrow)$ Let T be 1-1 and linear and suppose $T(x) = 0$ for $x \in N(T)$, then $T(x) = 0 = T(\bar{0}_V) \Rightarrow x = 0_V$ since T is 1-1, thus $N(T) = \{\bar{0}_V\}$

$(\longleftarrow)$ Let $N(T) = \{\bar{0}_V\}$, suppose $T(x) = T(y)$ for $x, y \in V$, then $T(x) - T(y) = 0 = T(x-y)$ since T is linear.

$\Rightarrow (x-y) \in N(T) \Rightarrow x - y = 0 \Rightarrow x = y$, so T is 1-1.

We've proven the statement in both directions, so T is 1-1 iff $N(T) = \{\bar{0}_V\}$.

■

11. a) Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$ arbitrary, then we must show $L_A(cx+y) = cL_A(x) + L_A(y)$

$$\Rightarrow L_A(cx+y) = A(cx+y)$$
$$= cA(x) + A(y)$$
$$= cL_A(x) + L_A(y),$$ so L_A is linear.

b) To find basis for $N(L_A)$ we need $\bar{x}$ s.t. $A\bar{x} = 0$.

Thus, we will convert A to RREF and ~~[scribble]~~ add a column of 0s so that we can set the ~~variables~~ entries to 0, finding $\bar{x} = 0$.

A with 0 column $= \begin{pmatrix} 2 & 3 & 1 & 4 & -9 & 0 \\ 1 & 1 & 1 & 1 & -3 & 0 \\ 1 & 1 & 1 & 2 & -5 & 0 \end{pmatrix}$

We will convert A^* to RREF and find solutions.

$$\xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 2 & -4.5 & 0 \\ 1 & 1 & 1 & 1 & -3 & 0 \\ 1 & 1 & 1 & 2 & -5 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 2 & -4.5 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 1.5 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -0.5 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 2 & -4.5 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -1 & 1.5 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - 2R_3} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & -0.5 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 3R_2} \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{pmatrix} \xrightarrow{R_2 \to -2R_2} \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 0 \\ 0 & +1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{pmatrix}$$

Converting to system of linear equations, we have:

$$\begin{aligned} x_1 + 2x_3 - 2x_5 &= 0 \\ x_2 - x_3 + x_5 &= 0 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

, Let $x_3 = t_1$, $x_5 = t_2$, then:

$$\begin{aligned} x_1 &= -2t_1 + 2t_2 \\ x_2 &= t_1 - t_2 \\ x_4 &= 2t_2 \end{aligned}$$

Thus our $\bar{x}$ is given by $(-2t_1 + 2t_2, t_1 - t_2, t_1, 2t_2, t_2)$

$= t_1(-2, 1, 1, 0, 0) + t_2(2, -1, 0, 2, 1)$.

Thus, $N(L_A)$ is given by $\text{span}\{(-2,1,1,0,0), (2,-1,0,2,1)\}$, which is its basis, since these two vectors are lin. ind. and span $N(L_A)$.