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Proof:

1. We will prove this statement by induction on n.

Base case: For $n=1$, we have $w_1 \in W$ and a_1 scalar. we need to show $a_1 v_1 \in W$ Since W is a subspace, it is closed under scalar multiplication, so $a_1 w_1 \in W$. Thus, base case holds.

Inductive hypothesis: Assume the statement holds for some $k \geq 1$, so any linear combination of k vectors of W is also in W $\Rightarrow$ $w_1, \ldots, w_k \in W$, $a_1 w_1 + \ldots + a_k w_k \in W$ for any scalars $a_1, \ldots, a_k$.

Inductive step: We must show the statement is true for $n = k+1$. Let $w_1, \ldots, w_k, w_{k+1} \in W$, then the linear combination of these vectors is given by $a_1 w_1 + \ldots + a_k w_k + a_{k+1} w_{k+1}$.

$\Rightarrow (a_1 w_1 + \ldots + a_k w_k) + a_{k+1} w_{k+1}$. Let $a_1 w_1 + \ldots + a_k w_k$ be denoted as U, then $U \in W$ by inductive hypothesis.

Let $a_{k+1} w_{k+1}$ be denoted as V, then since W is a subspace, $w_{k+1} \in W$ and V must also be in W since it's closed under scalar multiplication.

$a_1 w_1 + a_k w_k + a_{k+1} w_{k+1} \Rightarrow U + V \in W$ since W is closed under vector addition as it is a subspace.

Thus, by mathematical induction on n, the statement is true for all integers $n \geq 1$.

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2. a) True, let all coefficients be 0, then zero vector is a lin. comb. of any nonempty set of vectors.

b) False, span($\emptyset$) $= 0$.

c) True, span(S) is smallest subspace containing S, which is exactly what's constructed from taking intersection of all subspaces containing S.

d) False, one may not multiply an equation by 0, else change of set of solutions.

e) True, this is true by definition.

f) False, some systems are inconsistent, where an equation is in the form $0 = c$ for $c \neq 0$.

Need $a, b \in \mathbb{R}$ s.t.:

3. $(-2,2,2) = a(1,2,-1) + b(-3,-3,3)$ | convert to system of lin. eqns.

$$\begin{cases} a - 3b = -2 \\ 2a - 3b = 2 \\ -a + 3b = 2 \end{cases} \xrightarrow{\text{convert to augmented matrix}} \begin{pmatrix} 1 & -3 & -2 \\ 2 & -3 & 2 \\ -1 & 3 & 2 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -3 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{convert back to system of lin eqns}} \quad a = 4, \quad 3b = 6 \Rightarrow b = 2$$

Thus, the first vector can be expressed as a lin. combination of the other two vectors, for $a = 4$ and $b = 2$.

Need $a, b \in \mathbb{R}$ s.t.:

4. $6x^3 - 3x^2 + x + 2 = a(x^3 - x^2 + 2x + 3) + b(2x^3 - 3x + 1)$

Convert to system of lin. eqns.

$$\begin{cases} a + 2b = 6 \\ -a = -3 \\ 2a - 3b = 1 \\ 3a + b = 2 \end{cases}$$

Per second equation, $a = 3$, then per first equation b must be 1.5, however for third, $2(3) - 3(1.5) = 1.5 \neq 1$, so first polynomial can't be expressed as a lin. combination of the other two.

5. Let A be an arbitrary ~~m~~ symmetric 2×2 matrix from the set. Then its general form is $\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$.

To show that the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices, we need to show that they generate A, so:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{M_1} + a_{22}\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{M_2} + a_{12}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{M_3}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Thus, A can be written as a linear combination of $\{M_1, M_2, M_3\}$. Given A is arbitrary, $\text{span}(\{M_1, M_2, M_3\})$ thus is the set of all symmetric 2×2 matrices.

6. a) False, not every vector need to be a linear combination of other vectors in S, just at least one needs to per definition of lin. dependence.

b) True, since any scalar times 0 will result in 0, thus, a zero vector may have a nontrivial representation as linear combination of the elements of that set.

c) False, for linearly dependent set, there must be vectors in the set, so empty set is lin. independent.

d) False, Thm. 1.7 shows that lin. independent subsets can be extended to lin. dep. sets, contradicting the statement.

e) True, per corollary of Theorem 1.6

f) True, this is the definition of lin. independence.

7. We will use augmented matrix to show if set is lin. dep. or lin. indep. If lin. indep., system has only one solution which is 0 for all coeff. $\Longrightarrow$ no columns without a leading entry in RREF else the set is lin. dep.

~~If lin. ind. ⟹ 0 only trivial representation, so all scalars in a lin. comb. must be 0, lin. dep. otherwise~~

a) So, $a(x^3+2x^2)+b(-x^2+3x+1)+c(x^3-x^2+2x-1)=0$

Convert to system of lin. equations.

$$\begin{cases} a \quad + c = 0 \\ 2a - b - c = 0 \\ 3b + 2c = 0 \\ b - c = 0 \end{cases} \xrightarrow{\text{convert to augmented matrix}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & -3 & 0 \end{pmatrix} \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -4 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{1}{5}R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{pmatrix} \xrightarrow{R_4 \to R_4 + 4R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the augmented matrix is in REF now and there are no columns without a leading entry, this is a lin. independent set since there is only one solution, which is for all coeff to be 0.

b) $a(x^3-x)+b(2x^2+4)+c(-2x^3+3x^2+2x+6)=0$

Convert to system of lin. eqns:

$$\begin{cases} a \quad - 2c = 0 \\ 2b + 3c = 0 \\ -a \quad + 2c = 0 \\ 4b + 6c = 0 \end{cases} \xrightarrow{\text{convert to augmented matrix}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 6 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The augmented matrix is in REF now, since 3rd column doesn't have leading entry, this indicates there are inf. many solutions, so the set is linearly dependent.

c) $a(1,-1,2)+b(1,-2,1)+c(1,1,4)=0$ | convert to sys. of lin. eqns.

$$\begin{cases} a+b+c=0 \\ -a-2b+c=0 \\ 2a+b+4c=0 \end{cases} \xrightarrow{\text{convert to augmented matrix.}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 \\ 2 & 1 & 4 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_2+R_1]{R_3 \to R_3-2R_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3-R_2} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The augmented matrix is in REF now. Since 3rd column doesn't have a leading entry, the set is lin. dependent.

8.

a) Need to show W is a subspace of $\mathcal{F}(\mathbb{R})$. Thus, we must show $0 \in W$ and W is closed under addition and multiplication.

1. Zero function is one that maps all numbers to 0, since $f(t)=0$ for $\forall t \in \mathbb{R}$, since $f(1)=0$, zero function is in W ✓.
2. Let $f, g \in W$ s.t. $f(1)=0$ and $g(1)=0$. Then $(f+g)(t)=f(t)+g(t)=f(1)+g(1)=0+0=0$. Thus, $f+g \in W$ ✓
3. Let $c \in \mathbb{R}$ arbitrary. Then $(cf)(t)=cf(t)$ $=cf(1)=c0=0 \Rightarrow cf \in W$ ✓.

Thus, W is a subspace of $\mathcal{F}(\mathbb{R})$.

b). To show W is not a subspace of $\mathcal{F}(\mathbb{R})$, either of the 3 properties in a) must be false.

1. Zero function $\Rightarrow f(t)=0$ $\forall t \in \mathbb{R}$. Since zero function doesn't equal to 1 at 0 for t, it is not in W, so W is not a subspace of $\mathcal{F}(\mathbb{R})$.

9. Let v_1, v_2, v_3 be $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ respectively.
Clearly, none of these vectors is a scalar multiple of the other, so the hypothesis is satisfied.

However, v_3 is a linear combination of v_1 and v_2, that is $v_3 = av_1 + bv_2$ for $a = b = 1$

$\Rightarrow 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ or $v_3 = v_1 + v_2$.

Rearranging the equation we get $1 \cdot v_1 + 1 \cdot v_2 - 1 \cdot v_3 = 0$, hence the set is linearly dependent, contradicting the statement.