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1. a) False, $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$, matrix multiplication is not commutative

b) True by Theorem 2.14

c) False, $[U(w)]_\gamma = [U]_\beta^\gamma [w]_\beta$

d) True, matrix of identity transformation w.r.s. to same bases for domain and codomain is the identity matrix

e) False, this is valid only if same basis is used for domain and codomain

f) False, let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $A^2 = I$, however $A \neq I$ and $A \neq -I$

g) False, this is true if V and W are n, m tuple vector spaces only as suggested by Theorem 2.15, d)

h) False, let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = 0$, but $A \neq 0$

i) True by theorem 2.15 $\Rightarrow L_{A+B} = (A+B)\bar{x}$
$= A\bar{x} + B\bar{x}$
$= L_A + L_B$

j) True, this is the definition of identity matrix which is square and its entries are given by δ_{ij} at ij

2. a) i) $2B = \begin{pmatrix} 2 & 0 & -6 \\ 8 & 2 & 4 \end{pmatrix}$, $3C = \begin{pmatrix} 3 & 3 & 12 \\ -3 & -6 & 0 \end{pmatrix}$

$2B + 3C = \begin{pmatrix} 5 & 3 & 6 \\ 5 & -4 & 4 \end{pmatrix}$

$\Rightarrow$

$A(2B+3C) = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 & 3 & 6 \\ 5 & -4 & 4 \end{pmatrix}$

$A\bar{b}_1 = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + \begin{pmatrix} 15 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$

$A\bar{b}_2 = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -12 \\ 4 \end{pmatrix} = \begin{pmatrix} -9 \\ 10 \end{pmatrix}$

$A\bar{b}_3 = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} + \begin{pmatrix} 12 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \end{pmatrix}$

$\Rightarrow A(2B+3C) = \begin{pmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{pmatrix}$

✓

ii) $AB = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}1&0&-3\\4&1&2\end{pmatrix}$

$A\bar{b}_1 = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}12\\-4\end{pmatrix} = \begin{pmatrix}13\\-2\end{pmatrix}$

$A\bar{b}_2 = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

$A\bar{b}_3 = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}-3\\2\end{pmatrix} = \begin{pmatrix}-3+6\\-6-2\end{pmatrix} = \begin{pmatrix}3\\-8\end{pmatrix}$

$\Rightarrow AB = \begin{pmatrix}13&3&3\\-2&-1&-8\end{pmatrix}$

$(AB)D = \begin{pmatrix}13&3&3\\-2&-1&-8\end{pmatrix}\begin{pmatrix}2\\-2\\3\end{pmatrix} = \begin{pmatrix}26+6+9\\-4+2-24\end{pmatrix} = \begin{pmatrix}29\\-26\end{pmatrix}$

iii) $\cancel{AB\bar{d}_1 = \begin{pmatrix}13&3&3\\-2&-1&-8\end{pmatrix}}$

By theorem 2.16, matrix multiplication is associative and thus $(AB)D = A(BD) = \begin{pmatrix}29\\-26\end{pmatrix}$

b) i) $A^t = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}$

ii) $A^tB = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}\begin{pmatrix}3&-2&0\\1&-1&4\\5&5&3\end{pmatrix}$

$A^t\bar{b}_1 = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}\begin{pmatrix}3\\1\\5\end{pmatrix} = \begin{pmatrix}6-3+20\\15+1+10\end{pmatrix} = \begin{pmatrix}23\\26\end{pmatrix}$

$A^t\bar{b}_2 = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}\begin{pmatrix}-2\\-1\\5\end{pmatrix} = \begin{pmatrix}-4+3+20\\-10-1+10\end{pmatrix} = \begin{pmatrix}19\\-1\end{pmatrix}$

$A^t\bar{b}_3 = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}\begin{pmatrix}0\\4\\3\end{pmatrix} = \begin{pmatrix}-12+12\\4+6\end{pmatrix} = \begin{pmatrix}0\\10\end{pmatrix}$

$\Rightarrow A^tB = \begin{pmatrix}23&19&0\\26&-1&10\end{pmatrix}$

iii) $BC^t = \begin{pmatrix}3&-2&0\\1&-1&4\\5&5&3\end{pmatrix}\begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}12\\4+12\\20+9\end{pmatrix} = \begin{pmatrix}12\\16\\29\end{pmatrix}$

iv) $CB = \begin{pmatrix}4&0&3\end{pmatrix}\begin{pmatrix}3&-2&0\\1&-1&4\\5&5&3\end{pmatrix}$

$C\bar{b}_1 = (27)$

$C\bar{b}_2 = (7)$

$C\bar{b}_3 = (9)$

$\Rightarrow CB = \begin{pmatrix}27&7&9\end{pmatrix}$

v) $CA = \begin{pmatrix}4&0&3\end{pmatrix}\begin{pmatrix}2&5\\-3&1\\4&2\end{pmatrix}$

$C\bar{a}_1 = (20)$

$C\bar{a}_2 = (26)$ $\Rightarrow CA = \begin{pmatrix}20&26\end{pmatrix}$

3. a) i) $U(1) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$U(x) = (1,0,-1) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1)$

$U(x^2) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$\Rightarrow [U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

ii) $T(1) = 2 = 2(1) + 0(x) + 0(x^2)$

$T(x) = 3 + x + 2x = 3 + 3x = 3(1) + 3(x) + 0(x^2)$

$T(x^2) = 2x(3+x) + 2x^2 = 6x + 4x^2 = 0(1) + 6(x) + 4(x^2)$

$\Rightarrow [T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$

iii) $UT(1) = U(T(1)) = U(2) = (2,0,2)$

$= 2(1,0,0) + 0(0,1,0) + 2(0,0,1)$

$UT(x) = U(T(x)) = U(3+3x) = (6,0,0)$

$= 6(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$UT(x^2) = U(T(x^2)) = U(6x + 4x^2) = (6,4,-6)$

$= 6(1,0,0) + 4(0,1,0) - 6(0,0,1)$

$\Rightarrow [UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

By Thm. 2.11 $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta^\beta$

That is, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} =$

$\begin{pmatrix} 2+0 & 3+3 & 0+6 \\ 0 & 0 & 4 \\ 2 & 3-3 & -6 \end{pmatrix}$

$= \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$ ✓

6) i) ~~$B = (x, x^2)$~~

$[h(x)]_\beta = 3(1) - 2(x) + 1(x^2) = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

ii) $U(h(x)) = U(3 - 2x + x^2) = (1, 1, 5)$

$= 1(1,0,0) + 1(0,1,0) + 5(0,0,1)$

$\Rightarrow [U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

iii) By thm 2.14, $[U(h(x))]_\gamma = [U]_\beta^\gamma [h(x)]_\beta$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-2+0 \\ 0+0+1 \\ 3+2+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ ✓

4) a) $T : M_{2\times2}(F) \to M_{2\times2}(F)$ s.t. $T(A) = A^t$.

Thm 2.14 $\Rightarrow [T(A)]_\alpha = [T]_\alpha^\alpha [A]_\alpha$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ 4 \\ 6 \end{pmatrix}$

b) ~~$[T(f(x))]_\alpha = [T]$~~ $T : P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$

$[T(f(x))]_\alpha = [T]_\beta^\alpha [f(x)]_\beta$

$= \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 0-6+0 \\ 8-12+6 \\ 0+0+0 \\ 0+0+6 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$ ✓

5) a) Suppose UT is 1-1, thus $UT(v) = 0_Z$ ~~[illegible]~~ only for $v = 0_V \in V$

Let v be any vector in $N(T)$, so $T(v) = 0_W$ then $U(T(v)) = U(0_W)$, by linearity of U, $U(0_W) = 0_Z$, Therefore $UT(v) = 0_Z$

Since UT is 1-1, then $v = 0_V$. Because any vector v in $N(T)$ must be a zero vector, it follows that $N(T) = \{0_V\}$, and hence T is one-to-one.

U doesn't have to be one-to-one. Consider $T: \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = (x, 0)$. ~~[illegible]~~ T is then one-to-one. Let $U: \mathbb{R}^2 \to \mathbb{R}$ defined by $U(x,y) = x$. ~~[illegible]~~ U is not 1-1 since $U(0,1) = 0$ for example. The composition is then $UT(x) = U(x,0) = x$ and it is one-to-one. So UT can be 1-1 if U is not.

b) Suppose UT is onto, ~~[illegible]~~ thus for any $z \in Z$, $\exists v \in V$ s.t. $UT(v) = z$.

$\Rightarrow U(T(v)) = z$. Let $w = T(v)$, then $w \in W$ and $U(w) = z$.

Thus for any $z \in Z$, there is $w \in W$ s.t. $U(w) = z$. Hence U is onto.

T doesn't have to be onto. Consider same transformations as in counter-example in a), then T is not onto as its range is only x-axis, but U is onto.

then $UT(x) = U(x,0) = x$ which is onto, therefore T doesn't have to be onto.

c) Suppose U and T are 1-1 and onto,
then let $v \in V$ s.t. $UT(v) = 0_Z$, therefore
$U(T(v)) = 0_Z$.
Since U is 1-1, $N(U) = \{0_W\}$, so $T(v) = 0_W$.
Since T is 1-1, $N(T) = \{0_V\}$ and $v = 0_V$.
Therefore $N(UT) = \{0_V\}$, hence UT is 1-1.

Let $z \in Z$, then since U is onto, $\exists w \in W$ s.t.
$U(w) = z$. Since T is also onto, then for the
same w, there exists $v \in V$ s.t. $T(v) = w$.
Substituting to $U(w)$ we have $U(T(v)) = z$

$\Rightarrow UT(v) = z$.

Therefore, for any $z \in Z$, there is a pre-image $v \in V$
and so UT is onto. ✓

6. We must show that ~~(AB)~~ $(AB)C = A(BC)$ ✓
to prove that multiplication of matrices is associative, and
hence the entry in ith row and lth column of $(AB)C$
is equal to entry in ith row and lth column of $A(BC)$.
Let A, B, C be $m \times n$, $n \times p$, $p \times q$ matrices respectively,
~~with~~ with entries A_{ij}, B_{jk}, C_{kl} respectively.
By definition of matrix multiplication, ~~$(AB)_{ik} = \sum_{k=1}^{n} A_{ij} B_{kj}$~~
$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$, using this we find entry in
i-th row and l-th column of $(AB)C$

$$\Rightarrow ((AB)C)_{il} = \sum_{k=1}^{p} (AB)_{ik} C_{kl}$$
$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk}\right) C_{kl}$$

$$(BC)_{jl} = \sum_{k=1}^{p} B_{jk} C_{kl}$$
$$\Rightarrow (A(BC))_{il} = \sum_{j=1}^{n} A_{ij} (BC)_{jl} = \sum_{j=1}^{n} A_{ij}\left(\sum_{k=1}^{p} B_{jk} C_{kl}\right)$$

Since these are finite sums, the order of summation can be exchanged and thus

$$\sum_{k=1}^{p} \sum_{j=1}^{n} A_{ij} B_{jk} C_{kl} = \sum_{j=1}^{n} \sum_{k=1}^{p} A_{ij} B_{jk} C_{kl}.$$

Since i-th row and l-th column is equivalent for both $(AB)C$ and $A(BC)$, and it holds for any arbitrary entry, the matrices are equal, hence we've shown that matrix multiplication is associative. ∎

7. a) False, $([T]_{\alpha}^{\beta})^{-1} = [T^{-1}]_{\beta}^{\alpha}$

b) True by definition

c) False, L_A can map only F^n to F^m

d) False, two finite dimensional vector spaces are isomorphic only if they have the same dimension which isn't the case here

e) True, spaces are isomorphic iff their dimensions are equal $\Rightarrow n+1 = m+1 \Rightarrow n = m$

f) False, only guaranteed if A and B are square matrices.

g) True, by definition

h) True, L_A has inverse iff A does, so we have $L_{A^{-1}}$ is the inverse of L_A

i) True by definition.

8) a) Not invertible since dimensions of domain and codomain are not equal $(4 \neq 3)$

b) Not invertible, same reason as in a)

9) a) Not, $\dim \mathbb{F}^3 \neq \dim P_3(\mathbb{F})$

b) Isomorphic since $\dim \mathbb{F}^4 = \dim P_3(\mathbb{F}) = 4$

c) Isomorphic since same dimension of 4

d) Not isomorphic, $A \in V$ has a form of $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Since span $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ is lin. ind. and spans V, it is its basis.

But, $\dim V \neq \mathbb{R}^4 \Rightarrow 3 \neq 4$.

10. a) Suppose A and B are $n \times n$ matrices and are invertible:

To prove invertibility of AB, we need a matrix that acts as its inverse. Let such matrix be $B^{-1}A^{-1}$, then:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= A(I)A^{-1}$$
$$= AA^{-1}$$
$$= I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$
$$= B^{-1}(I)B$$
$$= B^{-1}B$$
$$= I$$

thus, AB is invertible and its inverse $(AB)^{-1}$ is $B^{-1}A^{-1}$

b) Suppose AB is invertible. Since it is invertible, its determinant is not 0.

Since $\det(AB) = \det(A)\det(B)$, we have $\det(A)\det(B) \neq 0$, so neither $\det A$ or $\det B$ are zero, hence A and B are square matrices with non-zero determinants and so are invertible.

c) Suppose A is invertible. Consider inverse of the transpose as $(A^{-1})^t$, then:

$$A^t (A^{-1})^t = (A^{-1}A)^t$$
$$= I^t = I$$

and

$$(A^{-1})^t A^t = (AA^{-1})^t$$
$$= I^t = I.$$

Thus, A^t is invertible and its inverse $(A^t)^{-1}$ is $(A^{-1})^t$.