SES (Simple Exponential Smoothing):

Equation derivation:

- first-order linear differential equation (exponential decay – because the sign is negative)
- it is exponential because this kind of equation usually leads to an exponential solution when solved

$$\frac{dy_{(t)}}{dt} = -\lambda\left(y_{(t)} - x_{(t)}\right) \tag{1}$$

- we can also assume:

$$\frac{dy_{(t)}}{dt} \approx \frac{y_{(t)} - y_{(t-1)}}{\Delta t} \tag{2}$$

- we assume that the time step is:

$$\Delta t = 1 \tag{3}$$

- the we get:

$$\frac{dy_{(t)}}{dt} \approx y_{(t)} - y_{(t-1)} \tag{4}$$

- if we plug two equations together:

$$y_{(t)} - y_{(t-1)} = -\lambda\left(y_{(t)} - x_{(t)}\right) \tag{5}$$

- we try to isolate $y_{(t)}$:

$$\begin{gathered} y_{(t)} = -\lambda . y_{(t)} + \lambda . x_{(t)} + y_{(t-1)} \\ y_{(t)} + \lambda . y_{(t)} = \lambda . x_{(t)} + y_{(t-1)} \implies y_{(t)}(1+\lambda) = \lambda . x_{(t)} + y_{(t-1)} \\ y_{(t)} = \frac{\lambda . x_{(t)}}{(1+\lambda)} + \frac{y_{(t-1)}}{(1+\lambda)} \implies y_{(t)} = \frac{\lambda}{(1+\lambda)} x_{(t)} + \frac{y_{(t-1)}}{(1+\lambda)} . \frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda}} \end{gathered} \tag{6}$$

- define the coefficient alpha:

$$\alpha = \frac{\lambda}{1+\lambda} \implies \lambda = \frac{\alpha}{1-\alpha} \tag{7}$$

- substitute the coefficient alpha into the equation:

$$\begin{gathered} \alpha = \frac{\lambda}{1+\lambda} \implies \lambda = \frac{\alpha}{1-\alpha} \\ \alpha = \frac{\lambda}{1+\lambda} \implies \lambda = \frac{\alpha}{1-\alpha} \end{gathered} \tag{8}$$

$$y_{(t)} = \alpha . x_{(t)} + \alpha . y_{(t-1)} \frac{1}{\lambda} \tag{9}$$

- plug the coefficient lambda

$$y_{(t)} = \alpha . x_{(t)} + \cancel{\alpha} . y_{(t-1)} \frac{1}{\frac{\cancel{\alpha}}{1-\alpha}}$$

$$y_{(t)} = \alpha . x_{(t)} + y_{(t-1)}(1-\alpha) \tag{10}$$

Example:

- let's have these data:

i	x_i
1	5
2	15
3	10
4	20

- let's assume 2 scenarios:
    - α = 0.1
    - α = 0.9

- for α = 0.1 and $y_0 = 5$

$$y_{(t)} = 0.1x_{(t)} + 0.9 . y_{(t-1)}$$

$$y_{(t=1)} = 0.1x_{(1)} + 0.9 . y_{(0)} = 0.1 * 5 + 0.9 * 5 = 5$$

$$y_{(t=2)} = 0.1x_{(2)} + 0.9 . y_{(1)} = 0.1 * 15 + 0.9 * 5 = 6$$

$$y_{(t=3)} = 0.1x_{(3)} + 0.9 . y_{(2)} = 0.1 * 10 + 0.9 * 6 = 6.4$$

$$y_{(t=3)} = 0.1x_{(4)} + 0.9 . y_{(3)} = 0.1 * 20 + 0.9 * 6.4 = 7.76$$

- for α = 0.9 and $y_0 = 5$

$$y_{(t)} = 0.9x_{(t)} + 0.1 . y_{(t-1)}$$

$$y_{(t=1)} = 0.9x_{(1)} + 0.1y_{(0)} = 0.1 * 5 + 0.9 * 5 = 5$$

$$y_{(t=2)} = 0.9x_{(2)} + 0.1y_{(1)} = 0.9 * 15 + 0.1 * 5 = 14$$

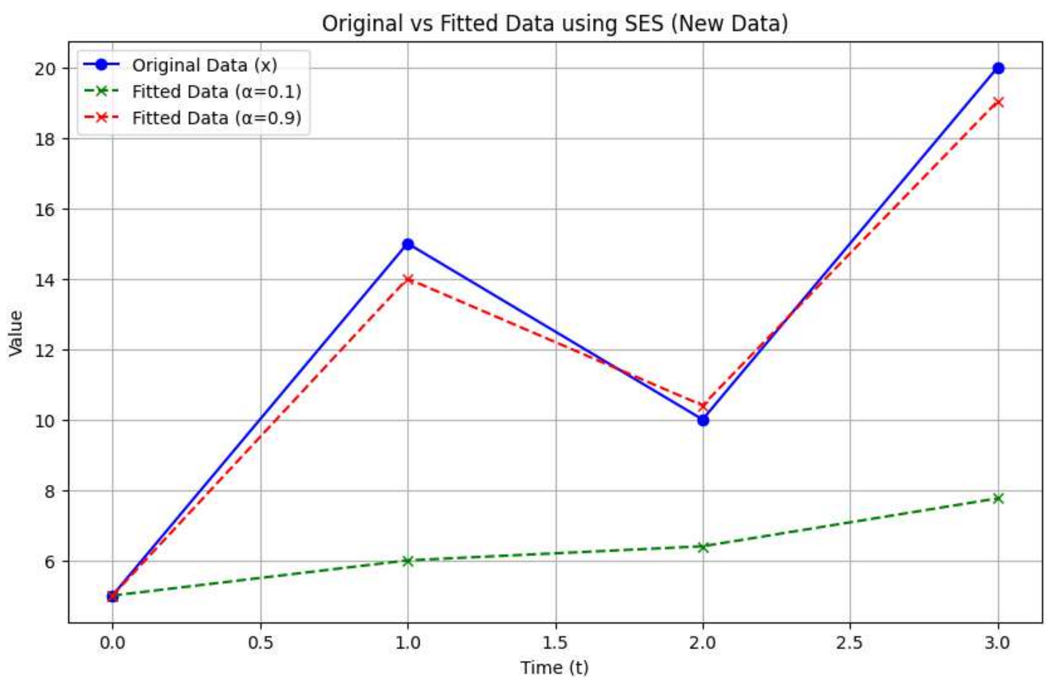
$$y_{(t=3)} = 0.9x_{(3)} + 0.1 . y_{(2)} = 0.9 * 10 + 0.1 * 14 = 10.4$$

$$y_{(t=3)} = 0.9x_{(4)} + 0.1 . y_{(2)} = 0.9 * 20 + 0.1 * 10.4 = 19.04$$

- interpretation of the results:

i	x_i	$y_{i\ (\alpha=0.1)}$	$y_{i\ (\alpha=0.9)}$
1	5	5	5
2	15	6	14
3	10	6.4	10.4
4	20	7.76	19.04

- if we choose **low** alpha we put more weights on **past observations**
- if we choose **high** alpha we put more weights on the **recent observations**



- from the fitted values we can see that:
    - **low alpha** damp/smooths the data **a lot**
    - **high alpha** damp/smooths the data **a bit**