

SES (Simple Exponential Smoothing):

Equation derivation:

- first-order linear differential equation (exponential decay – because the sign is negative)
- it is exponential because this kind of equation usually leads to an exponential solution when solved

$$\frac{dy_{(t)}}{dt} = -\lambda(y_{(t)} - x_{(t)}) \quad (1)$$

- we can also assume:

$$\frac{dy_{(t)}}{dt} \approx \frac{y_{(t)} - y_{(t-1)}}{\Delta t} \quad (2)$$

- we assume that the time step is:

$$\Delta t = 1 \quad (3)$$

- then we get:

$$\frac{dy_{(t)}}{dt} \approx y_{(t)} - y_{(t-1)} \quad (4)$$

- if we plug two equations together:

$$y_{(t)} - y_{(t-1)} = -\lambda(y_{(t)} - x_{(t)}) \quad (5)$$

- we try to isolate $y_{(t)}$:

$$\begin{aligned} y_{(t)} &= -\lambda \cdot y_{(t)} + \lambda \cdot x_{(t)} + y_{(t-1)} \\ y_{(t)} + \lambda \cdot y_{(t)} &= \lambda \cdot x_{(t)} + y_{(t-1)} \Rightarrow y_{(t)}(1 + \lambda) = \lambda \cdot x_{(t)} + y_{(t-1)} \\ y_{(t)} &= \frac{\lambda \cdot x_{(t)}}{(1 + \lambda)} + \frac{y_{(t-1)}}{(1 + \lambda)} \Rightarrow y_{(t)} = \frac{\lambda}{(1 + \lambda)} x_{(t)} + \frac{y_{(t-1)}}{(1 + \lambda)} \cdot \frac{\lambda}{\lambda} \end{aligned} \quad (6)$$

- define the coefficient alpha:

$$\alpha = \frac{\lambda}{1 + \lambda} \Rightarrow \lambda = \frac{\alpha}{1 - \alpha} \quad (7)$$

- substitute the coefficient alpha into the equation:

$$\begin{aligned} \alpha &= \frac{\lambda}{1 + \lambda} \Rightarrow \lambda = \frac{\alpha}{1 - \alpha} \\ \alpha &= \frac{\lambda}{1 + \lambda} \Rightarrow \lambda = \frac{\alpha}{1 - \alpha} \end{aligned} \quad (8)$$

$$y_{(t)} = \alpha \cdot x_{(t)} + \alpha \cdot y_{(t-1)} \frac{1}{\lambda} \quad (9)$$

- plug the coefficient lambda

$$y_{(t)} = \alpha \cdot x_{(t)} + \alpha \cdot y_{(t-1)} \frac{1}{\frac{\alpha}{1-\alpha}} \quad (10)$$

$$y_{(t)} = \alpha \cdot x_{(t)} + y_{(t-1)}(1 - \alpha)$$

Example:

- let's have these data:

i	x _i
1	5
2	15
3	10
4	20

- let's assume 2 scenarios:

- $\alpha = 0.1$

- $\alpha = 0.9$

- for $\alpha = 0.1$ and $y_0 = 5$

$$y_{(t)} = 0.1x_{(t)} + 0.9 \cdot y_{(t-1)}$$

$$y_{(t=1)} = 0.1x_{(1)} + 0.9 \cdot y_{(0)} = 0.1 * 5 + 0.9 * 5 = 5$$

$$y_{(t=2)} = 0.1x_{(2)} + 0.9 \cdot y_{(1)} = 0.1 * 15 + 0.9 * 5 = 6$$

$$y_{(t=3)} = 0.1x_{(3)} + 0.9 \cdot y_{(2)} = 0.1 * 10 + 0.9 * 6 = 6.4$$

$$y_{(t=3)} = 0.1x_{(4)} + 0.9 \cdot y_{(3)} = 0.1 * 20 + 0.9 * 6.4 = 7.76$$

- for $\alpha = 0.9$ and $y_0 = 5$

$$y_{(t)} = 0.9x_{(t)} + 0.1 \cdot y_{(t-1)}$$

$$y_{(t=1)} = 0.9x_{(1)} + 0.1y_{(0)} = 0.9 * 5 + 0.1 * 5 = 5$$

$$y_{(t=2)} = 0.9x_{(2)} + 0.1y_{(1)} = 0.9 * 15 + 0.1 * 5 = 14$$

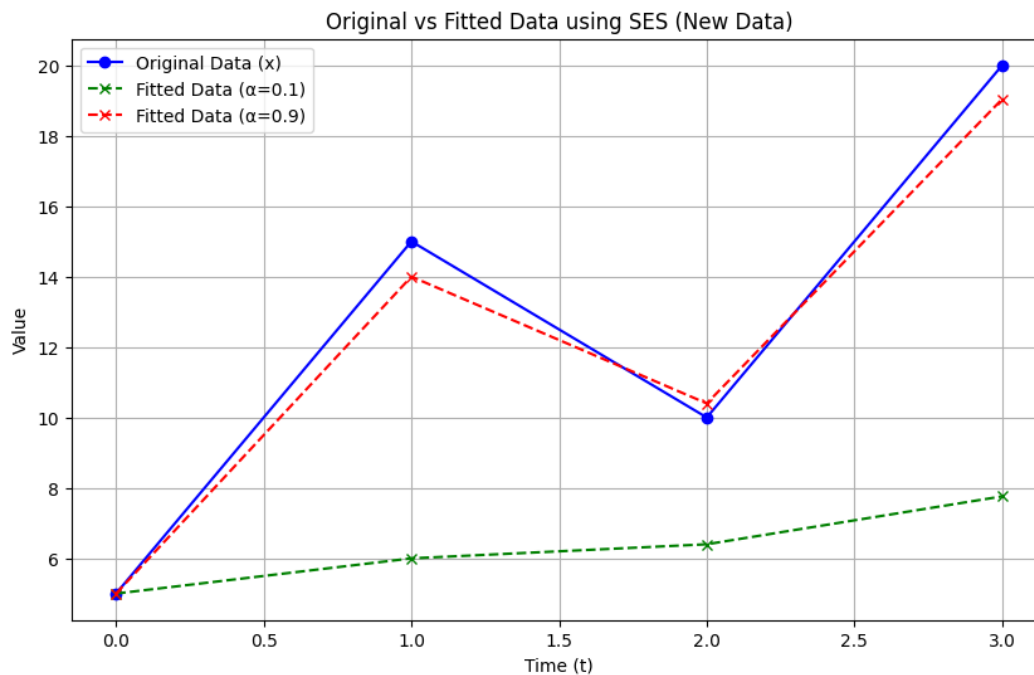
$$y_{(t=3)} = 0.9x_{(3)} + 0.1 \cdot y_{(2)} = 0.9 * 10 + 0.1 * 14 = 10.4$$

$$y_{(t=3)} = 0.9x_{(4)} + 0.1 \cdot y_{(2)} = 0.9 * 20 + 0.1 * 10.4 = 19.04$$

- interpretation of the results:

i	x_i	$y_i (\alpha=0.1)$	$y_i (\alpha=0.9)$
1	5	5	5
2	15	6	14
3	10	6.4	10.4
4	20	7.76	19.04

- if we choose **low** alpha we put more weights on **past observations**
- if we choose **high** alpha we put more weights on the **recent observations**



- from the fitted values we can see that:
 - **low alpha** damp/smooths the data **a lot**
 - **high alpha** damp/smooths the data **a bit**