Poisson Distribution: Expected Value and Variance:

Assume we have a random variable Y, such that $Y \sim Poisson(\lambda)$, then we can find its expected value and variance the following way. Recall that the expected value for any discrete random variable is a sum of all possible values multiplied by their likelihood of occurring P(y). Thus,

$$E(Y) = \sum_{y=0}^{\infty} y \frac{e^{-\lambda}\lambda^{y}}{y!}$$

Now, when y=0, the entire product is 0, so we can start the sum from y=1 instead. Additionally, we can divide the numerator and denominator by "y", since "y" will be non-zero in every case.

$$= \sum_{y=1}^{\infty} y \frac{e^{-\lambda}\lambda^{y}}{y!} = \sum_{y=1}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{(y-1)!}$$

Since lambda is a constant number, we can take out $\lambda e^{-\lambda}$, in front of the sum.

$$= \lambda e^{-\lambda} \sum_{y=1}^{\infty} \frac{\lambda^{y-1}}{(y-1)!}$$

Since we have "y-1" in both the numerator and denominator and the sum starts from 1, this is equivalent to starting the sum from 0 and using y instead.

$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!}$$

Calculus dictates that for any constant "c", $\sum_{x=0}^{\infty} \frac{c^{x}}{x!} = e^{c}$. We use this to simply the expression to:

$$= \lambda e^{-\lambda} e^{\lambda}$$

Lastly, since any value to the negative power is the same as 1 divided by that same value, then $e^{-\lambda} e^{\lambda} = 1$.

$$= \lambda$$

Now, let's move on to the variance. We are first going to express it in terms of expected values and then we are going to apply a similar approach to the one we used for the expected value.

We start off with the well-known relationship between the expected value and the variance, the variance is equal to the expected value of the squared variable, minus the expected value of the variable, squared.

$$Var(Y) = E(Y^2) - E(Y)^2$$

This next step might seem rather unintuitive, but we are simply expressing the squared variable in a way which makes it easier to manipulate. Knowing how to do proper operations with expected values, allows us to simplify the expression and plug in values we already know:

$$= E\big((Y)(Y-1)+Y\big) - E(Y)^2$$

$$= E\big((Y)(Y-1)\big) + E(Y) - E(Y)^2$$

$$= E\big((Y)(Y-1)\big) + (\lambda - \lambda^2)$$

We turn to the definition of the expected value once again.

$$= \sum_{y=0}^{\infty} (y)(y-1)\frac{e^{-\lambda}\lambda^y}{y!} + (\lambda - \lambda^2)$$

From here on out, the steps are pretty much the same once we took for the expected value:

1) we change the starting value of the sum, since the first 2 are zeroes

2) we cross our repeating values in the numerator and denominator

3) we take out the constant factors in front of the sum

4) adjust the starting value of the sum once again

5) substitute the sum with e^{λ}

$$= \sum_{y=2}^{\infty} (y)(y-1)\frac{e^{-\lambda}\lambda^y}{y!} + (\lambda - \lambda^2)$$

$$= \sum_{y=2}^{\infty} \frac{e^{-\lambda}\lambda^y}{(y-2)!} + (\lambda - \lambda^2)$$

$$= \lambda^2 e^{-\lambda} \sum_{y=2}^{\infty} \frac{\lambda^{y-2}}{(y-2)!} + (\lambda - \lambda^2)$$

$$= \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} + (\lambda - \lambda^2)$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + (\lambda - \lambda^2)$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

Therefore, both the mean and variance for a Poisson Distribution are equal to lambda (λ).