SDEs are all you need for latent variable models

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June 19, 2024

Motivation and outline

- Hierarchical VAEs are great, but today $T \to \infty$ layers are preferred.
- Formally we can start from finite T and take the limit but
- I want to understand the continuous limit modelling.

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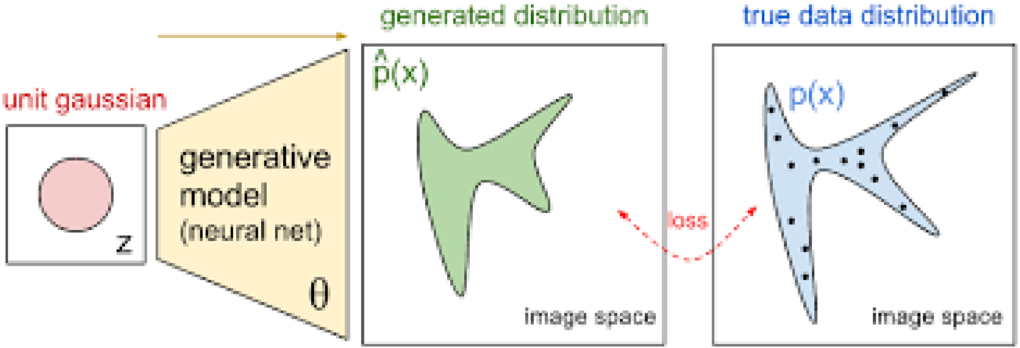
- The VAE and the Ladder VAE
- DDPM, VDM and DiffEnc (=diffusion with an encoder)
- Fokker-Planck - the master equation for the marginal distribution $p(\mathbf{z}(t); t)$
- SDE, ODE and reverse time SDE
- ODE and SDE for hierarchical generative models

Based upon Winther, Jeha and Dittadi, 2024, *in preparation*

Latent variable model - recap



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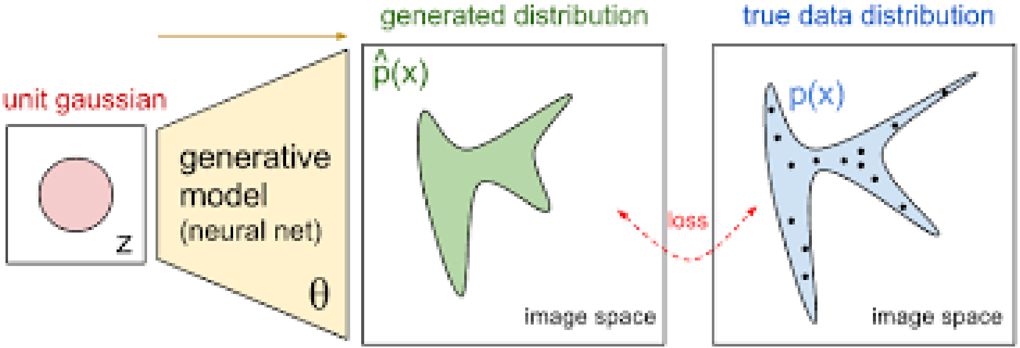

- Construct distribution $\hat{p}(\mathbf{x})$ from latent $\mathbf{z}$ + transformation:

$$\mathbf{z} \sim p(\mathbf{z})$$
$$\mathbf{x} \sim p(\mathbf{x}|\mathbf{z})$$

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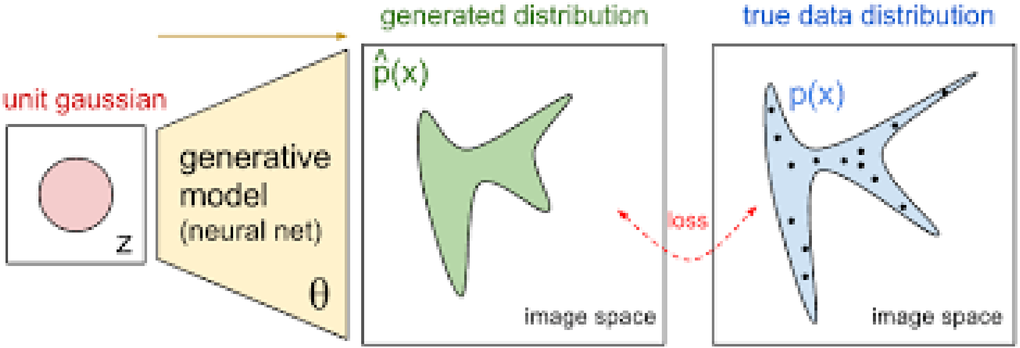

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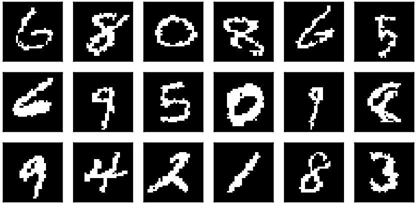
- Marginalisation:

$$\hat{p}(\mathbf{x}) = \int p(\mathbf{x}|\mathbf{z})p(\mathbf{z})d\mathbf{z}$$

OpenAI Generative models blog

Illustrative example of a (deep) generative model - recap

Training data $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ binary MNIST



Generative model for $\mathbf{z} \in \mathbb{R}^2$ and $\mathbf{x} \in \{0,1\}^{28\times 28}$

$$\begin{cases} \mathbf{z} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ x^{j,k} \sim \text{Bernoulli}(p = f^{j,k}(\mathbf{z})) \end{cases}$$

Decoder network

$$f(\mathbf{z}) = \text{Sigmoid}(\mathbf{V} \tanh(\mathbf{W}\mathbf{z} + \mathbf{b}) + \boldsymbol{\beta})$$

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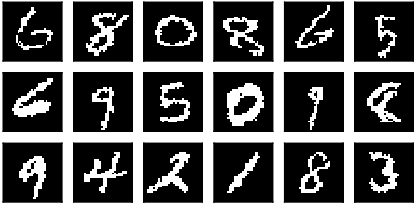
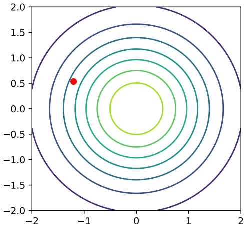


Generation

$$\mathbf{z} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\mathbf{z} = (-1.2033, 0.5340)$$



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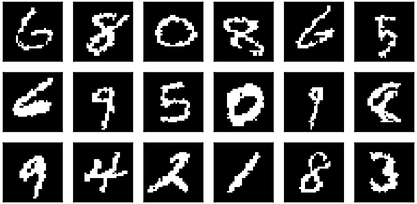

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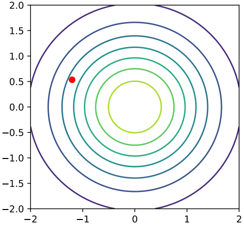
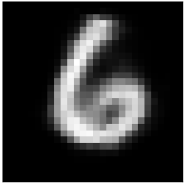


$f(\mathbf{z})$



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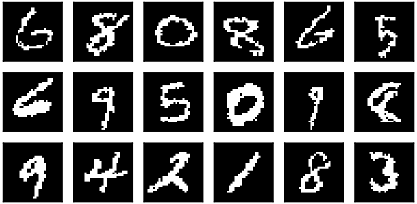

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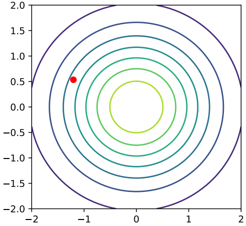

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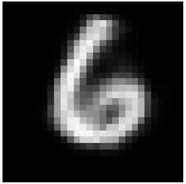
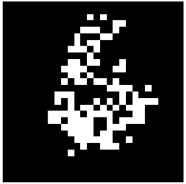


$x^{j,k} \sim \text{Bern}(f^{j,k}(\mathbf{z}))$



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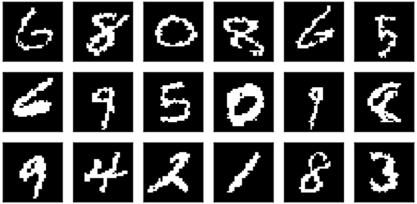

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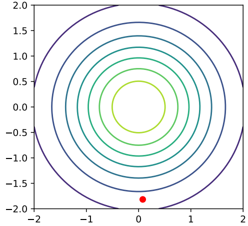
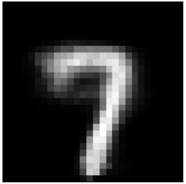
Generation

$$\mathbf{z} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

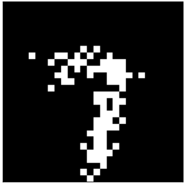
$$\mathbf{z} = (0.0791, -1.8165)$$



$f(z)$



$x^{j,k} \sim \text{Bern}(f^{j,k}(z))$



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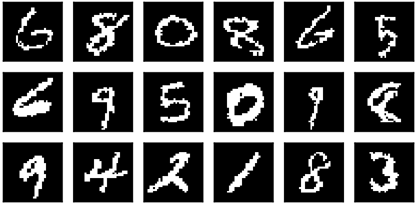

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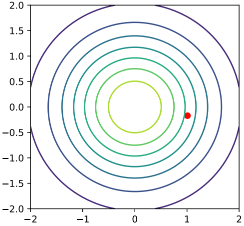
$$\mathbf{z} = (1.0097, -0.1711)$$



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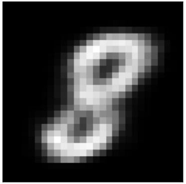

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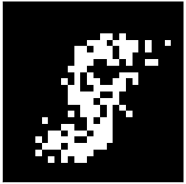

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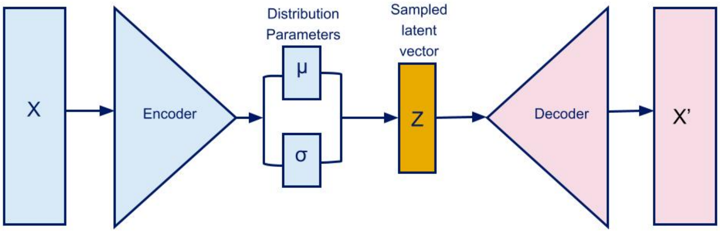
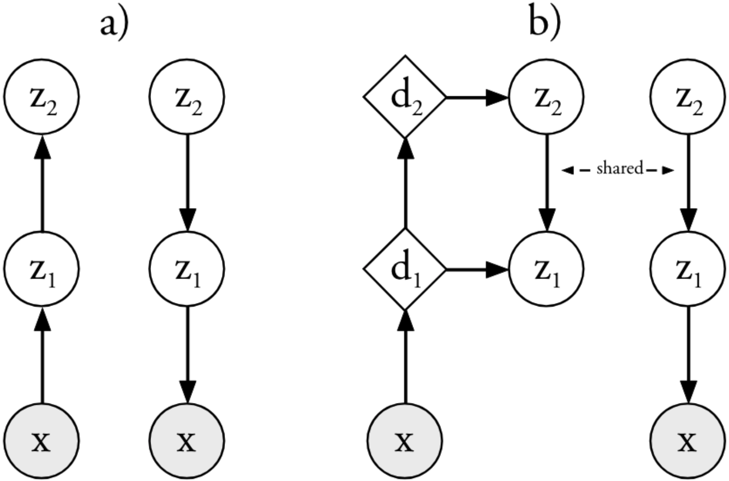
VAE - recap



- The log likelihood lower bound (aka the ELBO)

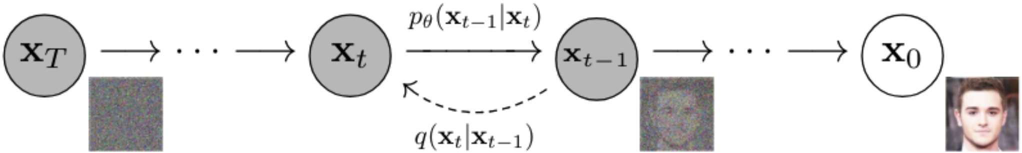
$$\log p(\mathbf{x}) \geq \mathbb{E}_{q(\mathbf{z}|\mathbf{x})} \left[\log \frac{p(\mathbf{x}|\mathbf{z})p(\mathbf{z})}{q(\mathbf{z}|\mathbf{x})} \right]$$

Hierarchical VAEs - recap



Ladder VAE, C. K. Sønderby et al, 2016

Diffusion models DDPM and VDM - recap



- May be viewed as a VAE with
  - simple noising encoder with marginal - $q(\mathbf{z}_t|\mathbf{x}) = \int q(\mathbf{z}_0, \ldots, \mathbf{z}_T|\mathbf{x})d\mathbf{z}_{-t}$:

$$q(\mathbf{z}_t|\mathbf{x}) = \mathcal{N}(\mathbf{z}_t|\alpha_t\mathbf{x}, \sigma_t^2\mathbf{I})$$

with

$$\mathrm{SNR}(t) = \frac{\alpha_t^2}{\sigma_t^2}$$

decreasing closer to the prior

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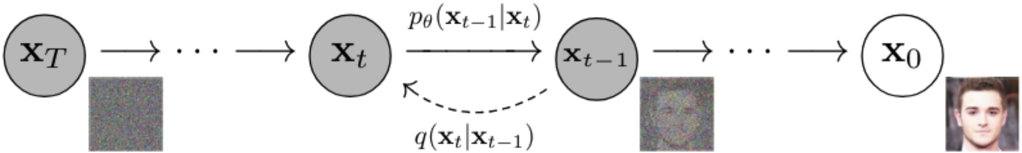

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  - generative model parameter sharing

    $$p(\mathbf{z}_{t-1}|\mathbf{z}_t) = \mathcal{N}(\mathbf{z}_{t-1}|\mu_\theta(\mathbf{z}_t, t), \Sigma_\theta(t)) \ .$$

DDPM, Ho et al, 2020, VDM, Kingma et al, 2021

VDM

- $T+1$ layers and indices $t, s \in [0,1]$: $s(i) = \frac{i-1}{T}$ and $t(i) = \frac{i}{T}$
- Hierarchical generative model

$$p_\theta(\mathbf{x}, \mathbf{z}) = p(\mathbf{x}|\mathbf{z}_0)p(\mathbf{z}_1)\prod_{i=1}^{T} p_\theta(\mathbf{z}_{s(i)}|\mathbf{z}_{t(i)})$$

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- ELBO

$$\mathrm{ELBO}(\mathbf{x}) = \underbrace{\mathbb{E}_{q(\mathbf{z}_0|\mathbf{x})}\left[\log p(\mathbf{x}|\mathbf{z}_0)\right]}_{\text{reconstruction}} - \underbrace{\mathcal{D}_{\mathrm{KL}}(q(\mathbf{z}_1|\mathbf{x})||p(\mathbf{z}_1))}_{\text{prior}} -$$
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- Use Bayes to reverse the direction of diffusion, $s < t$, for details VDM, Kingma et al, 2021

$$q(\mathbf{z}_s|\mathbf{z}_t, \mathbf{x}) = \mathcal{N}(\mathbf{z}_s|\mu_Q, \sigma_Q^2\mathbf{I})$$

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- We can choose the generative model to have the same functional form:

$$p(\mathbf{z}_s|\mathbf{z}_t) = \mathcal{N}(\mathbf{z}_s|\mu_P, \sigma_P^2\mathbf{I})$$

- Make even closer: $\mu_Q = \mu_Q(\mathbf{z}_t, \mathbf{x}_t, t)$, $\mu_P = \mu_Q(\mathbf{z}_t, \hat{\mathbf{x}}(\mathbf{z}_t, t), t)$ and $\sigma_P = \sigma_Q = \sigma_Q(t)$.

Diffusion loss and the $T \to \infty$ limit

- The diffusion term now becomes

$$-\sum_{i=1}^{T} \mathbb{E}_{q(\mathbf{z}_{t(i)}|\mathbf{x})} \left[\frac{1}{2\sigma_Q^2} \|\mu_P - \mu_Q\|_2^2 \right] = -\sum_{i=1}^{T} \frac{\Delta \mathrm{SNR}(\mathrm{t(i)})}{2} \mathbb{E}_{q(\mathbf{z}_{t(i)}|\mathbf{x})} \left[\|\hat{\mathbf{x}}(\mathbf{z}_{t(i)}, t(i)) - \mathbf{x}\|_2^2 \right]$$

with $\Delta \mathrm{SNR}(t) = \mathrm{SNR}(t) - \mathrm{SNR}(s)$.

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  with $\Delta\mathrm{SNR}(t) = \mathrm{SNR}(t) - \mathrm{SNR}(s)$.
- The $T \to \infty$ limit is well-defined with $\Delta t = t - s = 1/T$ and

$$-\frac{1}{2}\sum_{i=1}^{T} \frac{\Delta\mathrm{SNR}(t(i))}{\Delta t}\mathbb{E}_{q(\mathbf{z}_{t(i)}|\mathbf{x})}\left[\|\hat{\mathbf{x}}(\mathbf{z}_{t(i)}, t(i)) - \mathbf{x}\|_2^2\right]\Delta t \to$$

$$-\frac{1}{2}\int_0^1 \frac{d\mathrm{SNR}(t)}{dt}\mathbb{E}_{q(\mathbf{z}_t|\mathbf{x})}\left[\|\hat{\mathbf{x}}(\mathbf{z}_t, t) - \mathbf{x}\|_2^2\right] dt$$

  using

$$\sum_{i=1}^{T} \ldots \Delta t \to \int_0^1 \ldots dt \ \text{ for } \ T \to \infty \ .$$

DiffEnc - Can we make the encoder more powerful?

- Let the encoder be depth dependent - marginal:

$$q(\mathbf{z}_t|\mathbf{x}) = \mathcal{N}(\mathbf{z}_t|\alpha_t \mathbf{x}_t, \sigma_t^2 \mathbf{I})$$

with $\mathbf{x}_t = F_\phi(\mathbf{x}, t)$ learned encoder.

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- Yes, we can with a small modification of the diffusion term:

$$-\frac{1}{2}\sum_{i=1}^{T} \frac{\Delta \mathrm{SNR}(t(i))}{\Delta t} \mathbb{E}_{q(\mathbf{z}_{t(i)}|\mathbf{x})} \left[\left\| \hat{\mathbf{x}}(\mathbf{z}_{t(i)}, t(i)) - \mathbf{x}_{t(i)} - \frac{\Delta \mathbf{x}_{t(i)}}{\Delta \log \mathrm{SNR}(t(i))} \right\|_2^2 \right] \Delta t \rightarrow$$

$$-\frac{1}{2}\int_0^1 \frac{d\mathrm{SNR}(t)}{dt} \mathbb{E}_{q(\mathbf{z}_t|\mathbf{x})} \left[\left\| \hat{\mathbf{x}}(\mathbf{z}_t, t) - \mathbf{x}_t - \frac{d\mathbf{x}_t}{d \log \mathrm{SNR}(t)} \right\|_2^2 \right] dt$$

  with $\Delta \mathbf{x}_t = \mathbf{x}_t - \mathbf{x}_s$.

DiffEnc - Can we make the encoder more powerful?

- Let the encoder be depth dependent - marginal:

$$q(\mathbf{z}_t|\mathbf{x}) = \mathcal{N}(\mathbf{z}_t|\alpha_t \mathbf{x}_t, \sigma_t^2 \mathbf{I})$$

  with $\mathbf{x}_t = F_\phi(\mathbf{x}, t)$ learned encoder.

- Yes, we can with a small modification of the diffusion term:

$$-\frac{1}{2}\sum_{i=1}^{T} \frac{\Delta \mathrm{SNR}(t(i))}{\Delta t} \mathbb{E}_{q(\mathbf{z}_{t(i)}|\mathbf{x})} \left[\left\| \hat{\mathbf{x}}(\mathbf{z}_{t(i)}, t(i)) - \mathbf{x}_{t(i)} - \frac{\Delta \mathbf{x}_{t(i)}}{\Delta \log \mathrm{SNR}(t(i))} \right\|_2^2 \right] \Delta t \to$$

$$-\frac{1}{2}\int_0^1 \frac{d\mathrm{SNR}(t)}{dt} \mathbb{E}_{q(\mathbf{z}_t|\mathbf{x})} \left[\left\| \hat{\mathbf{x}}(\mathbf{z}_t, t) - \mathbf{x}_t - \frac{d\mathbf{x}_t}{d \log \mathrm{SNR}(t)} \right\|_2^2 \right] dt$$

  with $\Delta \mathbf{x}_t = \mathbf{x}_t - \mathbf{x}_s$.

- Must be a better way to find the continuous limit! This is unsatisfactory!
- Use continuous time formulation to see if we can make formulation more flexible

DiffEnc, Nielsen et al, ICLR, 2024

Encoder helps a bit

Model	Type	CIFAR-10	ImageNet 32×32
Flow Matching OT (Lipman et al., 2022)	Flow	2.99	3.53
Stochastic Int. (Albergo & Vanden-Eijnden, 2022)	Flow	2.99	3.48*
NVAE (Vahdat & Kautz, 2020)	VAE	2.91	3.92*
Image Transformer (Parmar et al., 2018)	AR	2.90	3.77*
VDVAE (Child, 2020)	VAE	2.87	3.80*
ScoreFlow (Song et al., 2021)	Diff	2.83	3.76*
Sparse Transformer (Child et al., 2019)	AR	2.80	–
Reflected Diffusion Models (Lou & Ermon, 2023)	Diff	2.68	3.74*
VDM (Kingma et al., 2021) *(10M steps)*	Diff	2.65	3.72*
ARDM (Hoogeboom et al., 2021)	AR	2.64	–
Flow Matching TN (Zheng et al., 2023)	Flow	2.60	3.45
Our experiments (8M and 1.5M steps, 3 seed avg)			
VDM with **v**-parameterization	Diff	2.64	3.46
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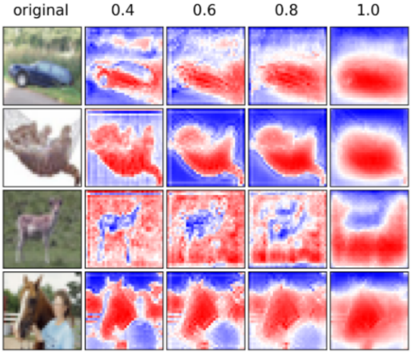

DiffEnc, Nielsen et al, ICLR, 2024 and more recently
Neural Flow Diffusion Models, Bartosh et al, 2024

Time dependent stochastic variables - setting the stage

- Let us learn how to formulate models directly in the continuous setting!

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  - Time dependent 1d stochastic variable $x(t)$ and data point y
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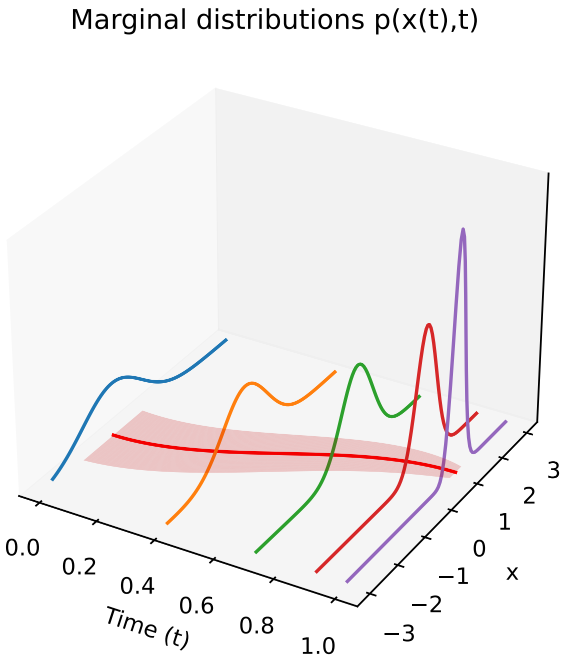
- Can also be a stochastic differential equation (SDE)

$$dx = a(x, t)dt + b(x, t)dW$$

- Interpolant

$$x(t) = t\, y_1 + (1 - t)\, y_0 + \gamma(t)\epsilon, \ \ \epsilon \sim \mathcal{N}(0, 1) \ .$$

The marginal



Fokker-Planck equation - setting the stage

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$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V(x,t)p(x,t)\right] + \left(\frac{\partial}{\partial x}\right)^2\left[D(x,t)p(x,t)\right] \ .$$

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- Derivation: Use definition of partial derivative, integrate it over an arbitrary function $h(x)$, use conditional distribution, integration by parts and truncation of Taylor expansion to second order.

SDE and Wiener process

- SDE

$$dx = a(x,t)dt + b(x,t)dW$$

- Wiener process $W(t)$ with independent Gaussian increment: $dW \sim \mathcal{N}(0, dt)$

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- Derivation of Fokker-Planck for SDE: Same as above and derive the stochastic process $h(x)$ where h is an arbitrary function:
$$\begin{aligned} dh &= h(x+dx) - h(x) \\ &= h'(x)dx + \frac{1}{2}h''(x)(dx)^2 + \mathcal{O}((dx)^3) \end{aligned}$$

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- In infinitesimal limit keep lowest power only (Ito's lemma):
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- Fokker-Planck for SDE

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[a(x,t)p(x,t)\right] + \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 \left[b^2(x,t)p(x,t)\right] \; .$$

- Drift: $V(x,t) = a(x,t)$ and diffusion: $D(x,t) = \frac{b^2(x,t)}{2}$.

Shifting diffusion into drift

- Fokker-Planck again:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V(x,t)p(x,t)\right] + \left(\frac{\partial}{\partial x}\right)^2 \left[D(x,t)p(x,t)\right] \ .$$

- We can shift the diffusion into the drift

$$\left(\frac{\partial}{\partial x}\right)^2 \left[D(x)p(x,t)\right] = \frac{\partial}{\partial x}\left[\frac{\partial D(x)}{\partial x}p(x,t) + D(x)\frac{\partial \log p(x,t)}{\partial x}p(x,t)\right] \ .$$

- using $\frac{\partial p}{\partial x} = p\frac{\partial \log p}{\partial x}$.

Shifting diffusion into drift

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- using $\frac{\partial p}{\partial x} = p\frac{\partial \log p}{\partial x}$.
- Define

$$V_D(x,t) \equiv \frac{\partial D(x)}{\partial x} + D(x)\frac{\partial \log p(x,t)}{\partial x}$$

- Fokker-Planck again:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V_k(x,t)p(x,t)\right] + k\left(\frac{\partial}{\partial x}\right)^2 \left[D(x)p(x,t)\right] \tag{1}$$

$$V_k(x,t) = V_0(x) + kV_D(x,t) \ , \tag{2}$$

- Any choice of k will give the same model.

SDE, ODE and reverse time SDE

Probability	State
$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V_1(x,t)p(x,t)\right] + \left(\frac{\partial}{\partial x}\right)^2 \left[D(x,t)p(x,t)\right]$ $V_k(x,t) = V_0(x,t) + kV_D(x,t)$ $V_D(x,t) = \frac{\partial D(x,t)}{\partial x} + D(x,t)\frac{\partial \log p(x,t)}{\partial x}$	$dx = V_1(x)dt + b(x,t)dW$
$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V_0(x,t)p(x,t)\right]$	$dx = V_0(x,t)dt$
$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V_{-1}(x,t)p(x,t)\right] - \left(\frac{\partial}{\partial x}\right)^2 \left[D(x,t)p(x,t)\right]$	$dx = V_{-1}(x,t)dt + b(x,t)d\overline{W}$ $dt < 0$

- Diffusion coefficient: $D(x,t) = b^2(x,t)/2$.
- $W(t)$ Wiener process
- $\overline{W}(t)$ reverse time Wiener process, $dt < 0$ increments independent (no proof)

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$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[V_0(x,t)p(x,t)\right]$	$dx = V_0(x,t)dt$
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- Reverse time diffusion, Anderson, 1982 - prove that $x(t)$ and $\overline{W}(s)$, $s \leq t$ are independent with $\overline{W}(t)$ defined by

$$d\overline{W} = \frac{1}{p(x,t)}\frac{\partial}{\partial x}\left[b(x,t)p(x,t)\right]dt + dW$$

- The we can have run time in reverse, $dt < 0$ and have independent increments $d\overline{W}$.

ODE - Generative model

- Likelihood $p(y|x(1))$ and prior $p(x, 0)$
- (Neural) ODE, Chen et al, 2018:

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$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[f_0(x,t)p(x,t)\right]$$

- Full derivative

$$\begin{aligned}
\frac{dp(x,t)}{dt} &= \frac{\partial p(x,t)}{\partial t} + \frac{\partial p(x,t)}{\partial x}\frac{dx}{dt} \\
&= -\frac{\partial}{\partial x}\left[f_0(x,t)p(x,t)\right] + \frac{\partial p(x,t)}{\partial x}\frac{dx}{dt} \\
&= -\frac{\partial f_0(x,t)}{\partial x}p(x,t) + \left(\frac{dx}{dt} - f_0(x,t)\right) \\
&= -\frac{\partial f_0(x,t)}{\partial x}p(x,t)
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- Solution using $\frac{dp}{dt} = p\frac{d\log p}{dt}$ and $x(t)$, $t \in [0,1]$ being the solution to the ODE:

$$\log p(x(1),1) = \log p(x(0),0) - \int_0^1 \frac{\partial f_0(x(t),t)}{\partial x(t)}dt$$

SDE - Generative model

- Generative model

$$p : \qquad dx = f_1(x,t)dt + b(x,t)dW \ .$$

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$$q : \qquad dx = g_1(x,y,t)dt + b(x,t)dW' \ .$$

- Same noise $b(x,t)$ required (no proof) Archambeau et al, 2007

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- Let us compute the ELBO - continuous version Song et al, NeurIPS, 2021
- Discretization $T+1$ "layers", $i = 0, \dots, T$ setting $t(i) = \frac{i}{T}$:

$$\Delta x_t = x_{t+\Delta t} - x_t = \underbrace{f_1(x(t),t)}_{\equiv f_1(t)} \Delta t + \underbrace{b(x(t),t)}_{\equiv b(t)} \Delta W_i$$

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- Gaussian conditionals (changing the notation slightly here):

$$p(\mathbf{x}) = p(x_0) \prod_{i=0}^{T-1} p(x_{t(i+1)}|x_{t(i)}) = p(x_0) \prod_{i=0}^{T-1} \mathcal{N}(x_{t(i+1)}|x_{t(i)} + f_1(t(i))\Delta t, b^2(t(i))\Delta t)$$

$$q(\mathbf{x}|y) = q(x_0|y) \prod_{i=0}^{T-1} q(x_{t(i+1)}|x_{t(i)}, y) = q(x_0|y) \prod_{i=0}^{T-1} \mathcal{N}(x_{t(i+1)}|x_{t(i)} + g_1(t(i))\Delta t, b^2(t(i))\Delta t)$$

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- $\mathrm{ELBO}(y) =$

$$\mathbb{E}_{q(\mathbf{x}|y)}\left[\log \frac{p(y|x_1)p(\mathbf{x})}{q(\mathbf{x}|y)}\right] = -\mathcal{D}_{\mathrm{KL}}(q(\mathbf{x}|y)||p(\mathbf{x})) + \mathbb{E}_{q(x_1|y)}\left[\log p(y|x_1)\right]$$

$$\begin{aligned}\mathcal{D}_{\mathrm{KL}}(q(\mathbf{x}|y), p(\mathbf{x})) =& \mathcal{D}_{\mathrm{KL}}(q(x_0|y), p(x_0)) + \\ & \sum_{i=0}^{T-1} \int q(x_{t(i)}|y)\mathcal{D}_{\mathrm{KL}}(q(x_{t(i+1)}|x_{t(i)}, y), p(x_{t(i+1)}|x_{t(i)}))dx_{t(i)}\end{aligned}$$

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- Tractable KL-divergence between Gaussian distributions:

$$\mathcal{D}_{\mathrm{KL}}(q(x_{t(i+1)}|x_{t(i)}, y), p(x_{t(i+1)}|x_{t(i)})) = \frac{||f_1(t(i)) - g_1(t(i))||^2}{2b^2(t(i))}\Delta t \ .$$

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$$\mathcal{D}_{\mathrm{KL}}(q(\mathbf{x}|y), p(\mathbf{x})) = \mathcal{D}_{\mathrm{KL}}(q(x_0|y), p(x_0)) + \sum_{i=0}^{T-1} \int q(x_{t(i)}|y)\mathcal{D}_{\mathrm{KL}}(q(x_{t(i+1)}|x_{t(i)}, y), p(x_{t(i+1)}|x_{t(i)}))dx_{t(i)}$$

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$$\mathcal{D}_{\mathrm{KL}}(q(x_{t(i+1)}|x_{t(i)}, y), p(x_{t(i+1)}|x_{t(i)})) = \frac{||f_1(t(i)) - g_1(t(i))||^2}{2b^2(t(i))}\Delta t\ .$$

- Take $T \to \infty$ limit and changing notation again: $p(x_t) \to p(x, t)$ for marginal:

$$\mathrm{ELBO}(y) = \underbrace{-\mathcal{D}_{\mathrm{KL}}(q(x, 0|y), p(x, 0))}_{\text{Prior}} + \underbrace{\mathbb{E}_{q(x,1|y)}\left[\log p(y|x)\right]}_{\text{Likelihood}} \underbrace{- \int_0^1 \mathbb{E}_{q(x,t|y)}\left[\frac{||f_1(x,t) - g_1(x,y,t)||^2}{2b^2(x,t)}\right] dt}_{\text{Diffusion}}\ .$$

Closer look at the diffusion term and requirements on $q(x, t)$

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- Gaussian marginal

$$q(x,t|y) = \mathcal{N}(x|\alpha(y,t), \beta^2(y,t))$$

- (VDM: $\alpha(y,t) = \alpha_t y_t$ and $\beta(y,t) = \sigma_t$)
- Drift term

$$\begin{aligned} g_0(x,y,t) &= \frac{\partial \beta(y,t)}{\partial t} \frac{x - \alpha(y,t)}{\beta(y,t)} + \frac{\partial \alpha(y,t)}{\partial t} \\ g_1(x,y,t) &= g_0(x,y,t) + \frac{\partial}{\partial x} \frac{b^2(x,t)}{2} + \frac{b^2(x,t)}{2} \frac{\partial \log q(x,t|y)}{\partial x} \quad . \end{aligned}$$

Summary and what was omitted

- Diffusion models are really hierarchical VAEs
- Connection even more apparent when we introduce more advanced encoders
- Continuous formulation more natural, but there is a learning curve.

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- Diffusion models are really hierarchical VAEs
- Connection even more apparent when we introduce more advanced encoders
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- What I didn't have time to talk about in detail:
  - Derivation master equation (Fokker-Planck)
  - Wiener process and derivation of Fokker-Planck for SDE
  - Derivation of reverse time SDE
  - Derivation of drift from marginal
  - Score/matching and stochastic interpolants.
- Watch out for Winther, Jeha and Dittadi, 2024, *in preparation*
- Learn to love calculus!

ODE and constant cumulative distribution path