HoTTEST Summer School 2022 Lectures 10-12

by Egbert Rijke

What you will learn:

- propositional truncations
- univalence axiom
- univalent combinatorics.
- How to express yourself in type theory
- How to think carefully about concepts and getting it right.
- How to spot univalence in your daily life

Consider a map $f: A \to B$. If we would use the Curry-Howard interpretation to assert surjectivity, it would look like

$$\prod_{(b:B)} \sum_{(a:A)} f(a) = b.$$

If f is "surjective" in this way then we get

$$g: B \to A$$
$$H: f \circ g \sim id.$$

Let A be a type. To count the elements of A is to obtain a number $n : \mathbb{N}$ with an equivalence $e : \mathrm{Fin}(n) \simeq A$

$$\mathrm{count}(A) := \sum_{(n:\mathbb{N})} (\mathrm{Fin}(n) \simeq A).$$

We might want to express that A is inhabited. We'd like that "is-inhabited(A)" is a proposition.

For this purpose we will introduce

propositional truncations.

universal property of propositional trunc.

Let A be a type, $f : A \to P$, where P is a proposition.

Defn. We say that f is a <u>propositional truncation</u> if the map

$$- \circ f : (P \xrightarrow{g} Q) \to (A \xrightarrow{g \circ f} Q)$$

is an equivalence for all $Q : \mathrm{Prop}$.

Rmk. Since Q is a proposition, it follows that $(P \to Q)$ and $(A \to Q)$ are also propositions. So $(P \to Q) \to (A \to Q)$ is an equiv iff

$$(A \to Q) \to (P \to Q).$$

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, bend left, "g"] & \\
P \arrow[r, dashed, "\exists!"'] & Q
\end{tikzcd}$$

Proposition. Consider $f : A \to P$, $f' : A \to P'$, where P and P' are propositions.
Consider the following three conditions:

(i) f is a propositional truncation.

(ii) f' is a propositional truncation.

(iii) $P \simeq P'$.

Claim. If 2 out of 3 properties hold, then so does the 3rd.

Proof. Assume (iii) holds. We will show (i) $\Leftrightarrow$ (ii).
Let Q be a proposition.

$$\begin{tikzcd}
 & (A \to Q) \arrow[dl, "- \circ f"'] \arrow[dr, "- \circ f'"] & \\
(P \to Q) \arrow[rr, "\simeq"'] & & (P' \to Q)
\end{tikzcd}$$

If (i) and (ii) hold.

$$(P \to P') \simeq (A \xrightarrow{f'} P') \qquad (P' \to P) \simeq (A \xrightarrow{f} P)$$

$\square$

From now on we assume that for every type A there is a proposition $\|A\|$ equipped with a map $\eta : A \to \|A\|$ (the unit of the propositional truncation) such that

$$(\|A\| \to Q) \to (A \to Q)$$

is an equivalence for every proposition Q

Lemma. Let $f : A \to B$. Then we obtain

$$\|f\| : \|A\| \to \|B\|.$$

Proof.

$$(\|A\| \to \|B\|) \simeq (A \to \|B\|)$$

$$\eta \circ f$$

$$\eta : B \to \|B\|$$

Corollary If $A \simeq B$, then $\|A\| \simeq \|B\|$

$A \leftrightarrow B$

Recall that if P and Q are props. then $P+Q$ is a proposition iff $P \to \neg Q$.

$\mathrm{inl}(p) = \mathrm{inl}(p')$ because $p = p'$ in P

$\mathrm{inl}(p) = \mathrm{inr}(q)$ have $p : P$, $q : Q$, $P \to \neg Q$. $\lightning$

$\mathrm{inr}(q) = \mathrm{inl}(p')$ — " —

$\mathrm{inr}(q) = \mathrm{inr}(q')$ because $q = q'$ in Q.

Defn. Define $P \vee Q := \| P + Q \|$.

Lemma. $(P \vee Q \to R) \simeq (P \to R) \times (Q \to R)$

$$
\begin{aligned}
(P \vee Q \to R) &\simeq (\| P + Q \| \to R) \\
&\simeq (P + Q \to R) \\
&\simeq (P \to R) \times (Q \to R). \quad 13.8
\end{aligned}
$$

Consider $P : A \to \mathrm{Prop}$. Then

$\sum_{(x:A)} P(x)$ is a prop iff

$\prod_{(x,y:A)} P(x) \to P(y) \to x = y$. ex.

Defn. We define $\exists_{(x:A)} P(x) :\equiv \| \sum_{(x:A)} P(x) \|$

Lem. $\left(\left(\exists_{(x:A)} P(x) \right) \to R \right) \simeq \prod_{(x:A)} \left(P(x) \to R \right)$

$$
\begin{aligned}
\exists_{(x:A)} P(x) \to R &\doteq \| \textstyle\sum_{(x:A)} P(x) \| \to R \\
&\simeq \left(\textstyle\sum_{(x:A)} P(x) \right) \to R \\
&\simeq \textstyle\prod_{(x:A)} \left(P(x) \to R \right)
\end{aligned}
$$

Logical connective	Type theory
$\top$	$\mathbb{1}$
$\bot$	$\emptyset$
$P \Rightarrow Q$	$P \to Q$
$P \wedge Q$	$P \times Q$
$P \vee Q$	$- \quad \lVert P + Q \rVert$
$P \Longleftrightarrow Q$	$(P \to Q) \times (Q \to P)$
$\exists_{(x:A)} P(x)$	$- \quad \lVert \sum_{(x:A)} P(x) \rVert$
$\forall_{(x:A)} P(x)$	$\prod_{(x:A)} P(x)$.

Defn. Let $f : A \to B$. Define

$$\mathrm{im}(f) := \sum_{(b:B)} \| \mathrm{fib}_f(b) \| .$$

$$
\begin{aligned}
\sum_{(b:B)} \mathrm{fib}_f(b) &\doteq \sum_{(b:B)} \sum_{(a:A)} f(a) = b \\
&\simeq \sum_{a:A} \underbrace{\sum_{b:B} f(a) = b} \\
&\simeq A
\end{aligned}
$$

☹

$q_f : A \to \mathrm{im}(f) \qquad q_f(a) := (f(a), \eta(a, \mathrm{refl}))$

$i_f : \mathrm{im}(f) \to B \qquad i_f := \mathrm{pr}_1$

note i_f is an embedding.

q_f is surjective.

Defn. Let $f : A \to B$. The f is surjective if

$$\text{is-surj}(f) := \prod_{b:B} \| \mathrm{fib}_f(b) \|$$

To show $q_f : A \to \mathrm{im}(f)$ is surjective

let $b : B$ $\quad x : \| \mathrm{fib}_f(b) \|$. WTS $\| \mathrm{fib}_{q_f}(b, x) \|$

$$\left(\| \mathrm{fib}_f(b) \| \to \| \mathrm{fib}_{q_f}(b, x) \| \right) \simeq \left(\mathrm{fib}_f(b) \to \| \mathrm{fib}_{q_f}(b, x) \| \right)$$

Let $a : A$, $p : f(a) = b$. STS.

$$\| \mathrm{fib}_{q_f} \underbrace{(f(a), \eta(a, \mathrm{refl}))}_{q_f(a)} \| \qquad \square$$

Def. We say that A is finite if we have an elt of type

$$\text{is-finite}(A) := \left\| \sum_{(n:\mathbb{N})} \mathrm{Fin}(n) \simeq A \right\|$$

$$\text{Fin } \mathbb{F} := \sum_{(X:\mathcal{U}_0)} \text{is-finite}(X)$$

$$(\mathbb{F}_n :=)\ BS_n := \sum_{(X:\mathcal{U}_0)} \| \mathrm{Fin}(n) \simeq A \|$$

Lemma. If A and B are finite, then $A+B$ is finite.

Proof. $H : \text{is-finite}(A)$, $K : \text{is-finite}(B)$

may $n:\mathbb{N}$, $e : \mathrm{Fin}(n) \simeq A$, $m:\mathbb{N}$, $f : \mathrm{Fin}(m) \simeq B$.

$$\mathrm{Fin}_{n+m} \simeq \mathrm{Fin}_n + \mathrm{Fin}_m \simeq A+B. \quad \eta(n+m, \alpha)$$

$\square$

$$\text{is-finite}'(A) :\equiv \sum_{n:\mathbb{N}} \| \mathrm{Fin}(n) \simeq A \|$$

The univalence axiom

Definition. Consider $A, B : \mathcal{U}$. Then we define a map
$$\mathsf{equiv\text{-}eq} : (A = B) \longrightarrow (A \simeq B)$$
by $\mathsf{equiv\text{-}eq}(\mathsf{refl}) := \mathsf{id}$.

The univalence axiom assert that
$$\mathsf{equiv\text{-}eq} : (A = B) \to (A \simeq B)$$
is an equivalence for all $A, B : \mathcal{U}$.

Remarks:

- By the fundamental theorem of identity types, it follows that
$$\sum_{(B:\mathcal{U})} A \simeq B$$
is contractible for all $A : \mathcal{U}$.

- Often people give the slogan version of univalence

$$(A = B) \simeq (A \simeq B)$$

Q. Is this statement equivalent to univalence or not?

A. $$\underbrace{\left(\sum_{B:\mathcal{U}} A = B\right)}_{\text{contractible}} \simeq \left(\sum_{(B:\mathcal{U})} A \simeq B\right)$$

- Univalence is due to Voevodsky (2009)

- Univalence implies function extensionality.

Propositional extensionality.

Recall that $\mathrm{Prop}_{\mathcal{U}} :\equiv \sum_{(X:\mathcal{U})} \underbrace{\text{is-prop}(X)}_{\text{is a proposition}}$.

$\mathrm{pr}_1 : \mathrm{Prop}_{\mathcal{U}} \to \mathcal{U}$

is an embedding, i.e.

$$\mathrm{ap}_{\mathrm{pr}_1} : (P = Q) \xrightarrow{\simeq} (\mathrm{pr}_1 P = \mathrm{pr}_1 Q)$$

$$\begin{aligned}(P = Q) &\simeq (\mathrm{pr}_1 P = \mathrm{pr}_1 Q) \\ &\overset{\mathrm{UA}}{\simeq} (\mathrm{pr}_1 P \simeq \mathrm{pr}_1 Q) \\ &\simeq (\mathrm{pr}_1 P \leftrightarrow \mathrm{pr}_1 Q) \\ &\text{“}= (P \leftrightarrow Q)\text{”} \quad \text{(convention)}\end{aligned}$$

Thm (propositional extensionality) For any P, Q

$$\text{iff-eq} : (P = Q) \to (P \leftrightarrow Q)$$

is an equivalence. $\square$

Cor. Consider two subtypes $P, Q : A \to \mathrm{Prop}$. Then

$$(P = Q) \simeq \prod_{a:A} P(a) \leftrightarrow Q(a)$$

Proof.

$$\sum_{Q : A \to \mathrm{Prop}} \prod_{a:A} P(a) \leftrightarrow Q(a)$$

$$\simeq \prod_{a:A} \underbrace{\sum_{Q : \mathrm{Prop}_{\mathcal{U}}} P(a) \leftrightarrow Q}$$

$$(A \to \mathrm{Prop}_{\mathcal{U}}) \simeq \sum_{X:\mathcal{U}} X \hookrightarrow A.$$

Thm. consider a type A. Then we have an equiv.

$$(A \underset{B}{\to} \mathcal{U}) \simeq \sum_{(X:\mathcal{U})} X \to A$$

$$\psi(B) :\equiv \left(\sum_{(x:A)} B(x), \quad pr_1 \right)$$

$$\varphi(X, f) :\equiv fib_f$$

- $\varphi(\psi(B)) = B$ What is equality in $A \to \mathcal{U}$?

Lemma Let $B, C : A \to \mathcal{U}$

$$(B = C) \simeq \prod_{(a:A)} B(a) \simeq C(a)$$

$$\sum_{C : A \to \mathcal{U}} \prod_{a:A} B(a) \simeq C(a)$$

$$\simeq \prod_{a:A} \underbrace{\sum_{C:\mathcal{U}} B(a) \simeq C}$$

$$\varphi(\psi(B))(a) \simeq B(a)$$

$$\doteq fib_{pr_1}(a) \overset{\sim}{\swarrow} 10.7$$

$$\psi(\varphi(X,f)) = (X,f)$$

Lemma. Let $(X,f), (Y,g) : \sum_{Z:\mathcal{U}} Z \to A$.

$$((X,f) = (Y,g)) \simeq \sum_{e : X \simeq Y} f \sim g \circ e$$

$$\begin{tikzcd} X \arrow[rr, "e", "\simeq"'] \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

$$\sum_{Y:\mathcal{U}} \sum_{g : Y \to A} \sum_{e : X \simeq Y} f \sim g \circ e$$

$$\simeq \underbrace{\sum_{Y:\mathcal{U}} \sum_{e : X \simeq Y}} \sum_{g : Y \to A} f \sim g \circ e$$

(X, id)

$$\simeq \sum_{g : X \to A} f \sim g$$

$$\begin{tikzcd} X \arrow[rr, "x \mapsto (f(x),(x,\mathrm{refl}))", "\simeq"'] \arrow[dr, "f"'] & \circlearrowleft & \sum_{a:A} \mathrm{fib}_f(a) \arrow[dl, "\mathrm{pr}_1"] \\ & A & \end{tikzcd}$$

$$\simeq \mathbb{1}$$

$$\psi(\varphi(X,f)) \equiv \left(\sum_{a:A} \mathrm{fib}_f(a), \mathrm{pr}_1\right) = (X,f)$$

$\square$

$$BS_n := \sum_{X:\mathcal{U}} \underbrace{\| Fin_n \simeq X \|}$$

$BS_n \xrightarrow{pr_1} \mathcal{U}$ is an embedding.

$$(X =_{BS_n} Y) \simeq (X \simeq Y)$$

$$(Fin_n = Fin_n) \simeq (\underbrace{Fin_n \simeq Fin_n}_{\text{permutations}})$$

Prop. $\sum_{(X:BS_2)} X$ is contractible.

Claim. $(Fin_2 \simeq X) \xrightarrow{ev_0} X$ is an equivalence.

Note: Being an equivalence is a property.
$H : \| Fin_2 \simeq X \|$. We may assume $Fin_2 \simeq X$.
STS $(Fin_2 \simeq Fin_2) \xrightarrow[ev_0]{} Fin_2$ is an equiv

$\square$

Cor. There is no dependent function

$$\prod_{X : BS_2} X$$

Proof. $(Fin_2 = X) \simeq X$, so

$$\prod_{X : BS_2} X \simeq \prod_{X : BS_2} Fin_2 = X \quad : H$$

$(Fin_2, H) : \underline{\text{is-contr}}\ BS_2$

Cor. There is no dependent function

$$H : \prod_{X : \mathcal{U}} \|X\| \to X$$ (global choice)
(it is inconsistent w
univalence)

$$H : \prod_{X : BS_2} \underbrace{\|X\|} \to X$$

$$\simeq \prod_{X : BS_2} X \quad ↯$$

Defn. The axiom of choice asserts that

for every set! A and every fam. B of sets? over A,

$$\left(\prod_{a:A} \|B(a)\|\right) \to \left\|\prod_{a:A} B(a)\right\|$$

Cor There is no dep. fn $X + \neg X$

$$\neg \prod_{X:\mathcal{U}} \text{is-decidable}(X)$$

$$\prod_{X:BS_2} X + \underbrace{\neg X} \simeq \prod_{X:BS_2} X$$

$$\|X\| \to \underbrace{\neg\neg X}$$

Def. Lem asserts that for all $P:\mathrm{Prop}$

$(P \vee \neg P)$ holds

A is k-truncated

iff $\Sigma_A : (A \to \mathcal{U}) \to \mathcal{U}$ is k-trunc.

Univalent combinatorics

$$\text{is-surjective}(f) := \prod_{(b:B)} \|\mathrm{fib}_f(b)\|$$

$$\text{is-finite}(A) := \left\| \sum_{(n:\mathbb{N})} \mathrm{Fin}(n) \simeq A \right\|$$

$$\mathrm{im}(f) := \sum_{(b:B)} \|\mathrm{fib}_f(b)\|$$

$$BS_n := \sum_{(X:\mathcal{U})} \|\mathrm{Fin}(n) \simeq X\|$$

- $\sum_{(X:\mathcal{U})} A \simeq X$ is contractible.
- Univalence implies prop ext

$$(P =_{\mathrm{Prop}} Q) \simeq (P \leftrightarrow Q)$$

- Univalence gives a duality between type families over A and maps into A.

$$(A \to \mathcal{U}) \simeq \sum_{(X:\mathcal{U})} X \to A$$

$$(A \to \mathrm{Prop}) \simeq \sum_{(X:\mathcal{U})} X \hookrightarrow A$$

- Univalence is used in lots of useful characterizations of identity types.

Often in combinatorics there are multiple ways to consider certain objects to be the same.

Eg. - An inclusion $\mathrm{Fin}(n) \hookrightarrow \mathrm{Fin}(m)$ determines an n-element subtype of $\mathrm{Fin}(m)$.

To define n-element subtypes of $\mathrm{Fin}(m)$, we should consider two embeddings the same if they are the same up to permutation on $\mathrm{Fin}(n)$.

- $\mathrm{Fin}(n) \hookrightarrow \mathrm{Fin}(m)$ has $(m)_n$ - falling factorial number of elements

- The type of n-ell subtypes of $\mathrm{Fin}(m)$ is $\binom{m}{n}$

- A partition on $\mathrm{Fin}(n)$ into k nonempty blocks determines a way to write

$$m_1 + \cdots + m_k = n$$

1. The number of partitions of n is the Bell number B_n
2. The partition number of n is something else.

- Latin squares.
  - up to isotopy
  - up to paratopy.

- Necklaces vs Bracelets.

$\mathbb{Z}_n$ D_n

When you introduce a concept in type theory, you do that by defining the type of all objects in that concept. Univalence then characterizes the identity type of this type.

Binomial types. (Section 17 of the book)

Defn. A <u>decidable embedding</u> is a map

$f : A \to B$ such that $\mathrm{fib}_f(b)$ is a decidable proposition for all $b : B$.

$$\mathrm{dProp} := \sum_{P : \mathrm{Prop}} P + \neg P$$

$$\simeq \sum_{P : \mathrm{Prop}} P + \sum_{P : \mathrm{Prop}} \neg P$$

$$\simeq \sum_{P : \mathrm{Prop}} P = 1 + \sum_{P : \mathrm{Prop}} P = \emptyset \simeq 2$$

Let's write $A \hookrightarrow_d B$ for the type of decidable embs.

$\binom{A}{B}$ "the type of decidable subtypes of A of size B"

Defn. define $\binom{A}{B} := \sum_{X : \mathcal{U}_{(B)}} X \hookrightarrow_d A$

where $\mathcal{U}_{(B)} := \sum_{X : \mathcal{U}} \| B \simeq X \|$

Lemma. $(X, f), (Y, g) : \binom{A}{B}$ then

$$((X,f) = (Y,g)) \simeq \sum_{e : X \simeq Y} f \sim g \circ e$$

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

Proof. $$\sum_{Y : \mathcal{U}_{(B)}} \sum_{g : Y \hookrightarrow_d A} \sum_{e : X \simeq Y} f \sim g \circ e$$

$$\simeq \sum_{g : X \hookrightarrow_d A} f \sim g \simeq 1$$

What goes on in $\binom{A}{B}$

$$X : \mathcal{U}_{(B)} \longmapsto X \hookrightarrow_d A$$

$$\ddot{B} \qquad\qquad B \hookrightarrow_d A$$

equivalences give an action on $B \hookrightarrow_d A$ by

$\mathrm{Aut}(B) := (B \simeq B)$ $\qquad f \cdot e := f \circ e$

$\binom{A}{B}$ is the type of orbits of this action

Proposition. We have equivs

$$\binom{\emptyset}{\emptyset} \simeq 1 \qquad \binom{A+1}{\emptyset} \simeq 1$$

$$\binom{\emptyset}{B+1} \simeq \emptyset \qquad \binom{A+1}{B+1} \simeq \binom{A}{B+1} + \binom{A}{B}$$

Proof. $\mathcal{U}_{(\emptyset)}$ is contractible, because types that are merely equiv to $\emptyset$ are empty.

$$\binom{A}{\emptyset} \doteq \sum_{X : \mathcal{U}_\emptyset} X \hookrightarrow_d A \simeq \emptyset \hookrightarrow_d A \simeq 1$$

$\binom{\emptyset}{B+1}$ is empty because there are

no maps $X \to \emptyset$ if

$\| B+1 \simeq X \|$

$$\binom{A+1}{B+1} \simeq \binom{A}{B+1} + \binom{A}{B}$$

$$\sum_{X : \mathcal{U}_{(B+1)}} X \hookrightarrow_d A+1 \simeq \sum_{P : A+1 \to d\mathrm{Prop}} \| B+1 \simeq \sum_{x : A+1} P(x) \|$$

$$\simeq \sum_{Q : d\mathrm{Prop}} \sum_{P : A \to d\mathrm{Prop}} \| B+1 \simeq \left(\left(\sum_{a : A} P(a) \right) + Q \right) \|$$

$$\simeq \left(\sum_{P : A \to d\mathrm{Prop}} \| B+1 \simeq \sum_{a : A} P(a) \| \right) + \left(\sum_{P : A \to d\mathrm{Prop}} \| B+1 \simeq \underbrace{\sum_{a : A} P(a)}_{} + 1 \| \right)$$

C

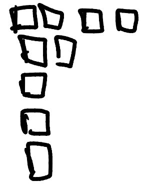
$$\simeq \sum_{X : \mathcal{U}_{(B+1)}} X \hookrightarrow_d A + \sum_{X : \mathcal{U}_{(B)}} X \hookrightarrow_d A \doteq \binom{A}{B+1} + \binom{A}{B}$$

$\square$

Thm. If A and B are finite types of size n and k, then $\binom{A}{B}$ is finite of size $\binom{n}{k}$

Proof. $\binom{\mathrm{Fin}(n)}{\mathrm{Fin}(k)} \overset{\checkmark}{\simeq} \mathrm{Fin}\binom{n}{k}$ $\square$

Partitions and Ferrers diagrams (Young)



$9 \quad 4 + 2 + 1 + 1 + 1$

Def. decidable

- An equivalence relation R consists of

$$R : A \to A \to \mathrm{dProp}$$

$$r : \prod_{(x:A)} R\,x\,x$$

$$s : \prod_{(x,y:A)} R\,x\,y \to R\,y\,x$$

$$t : \prod_{(x,y,z:A)} R(y,z) \to R(x,y) \to R(x,z)$$

$\mathrm{EqRel}(A)$ is the type of all equiv rels.

- A partition of A consists of

$$P : (A \to \mathrm{dProp}) \to \mathrm{dProp}$$

s.t. for all $Q : A \to \mathrm{dProp}$, if $P(Q)$ then Q is nonempty i.e $\left\| \sum_{x:A} Q(x) \right\|$

and for every $x : A$

$$\text{is-contr}\left(\sum_{Q : A \to \mathrm{Prop}} P(Q) \times Q(x)\right)$$

- A Σ-decomposition of A consists of

$X : \mathbb{F}$ — quotient A/R

$Y : X \to \sum_{Z : \mathbb{F}} \| Z \|$ — fibers of $q : A \to A/R$

$e : A \simeq \sum_{x : X} Y(x)$ — explicit.

- A surjection out of A consists of

$X : \mathbb{F}$

$f : A \to X$

$s : \text{is-surj}(f)$

Defn. A <u>stirling</u> type of the second kind.

$$\left\{ {A \atop B} \right\} := \sum_{X : \mathcal{U}_{(B)}} A \twoheadrightarrow X$$

$$(A \twoheadrightarrow B) /\!\!/ \mathrm{Aut}(B)$$

We want to describe the type of ways to decompose A as Σ-type where the order in which we sum the elts of A is irrelevant.

$$\text{Ferrers-Diagram}(n) : \sum_{X : \mathbb{F}} \sum_{Y : X \to \sum_{Z : \mathbb{F}} \|Z\|} \left\| \mathrm{Fin}(n) \simeq \sum_{x : X} Y(x) \right\|$$

$$\big((X, Y) = (X', Y')\big) \simeq \sum_{e : X \simeq X'} \prod_{x : X} Y(x) \simeq Y'(e(x))$$

$$m_1 + \cdots + m_k = n = m'_1 + \cdots + m'_{k'}$$

$$k = k'$$

$$e : \mathrm{Fin}(k) \simeq \mathrm{Fin}(k')$$

$$m_i = m'_{e(i)}$$

(according to 3rd def)

Thm $\underbrace{\text{Ferrers-Diags}(n)}_{\text{not going to be a set.}} \simeq \sum_{A : BS_n} \text{Partition}(A) \quad \leftarrow$

$$A \mapsto \text{Partition}(A)$$

action of S_n on $\text{Partition}(\mathrm{Fin}(n))$

$\|\text{Ferrers\_Diag}(n)\|_0$ is finite and # elts is nth partition number.

Proof.

$$\sum_{A : BS_n} \sum_{X : \mathbb{F}} \sum_{Y : X \to \sum_{Z : \mathbb{F}} \|Z\|} A \simeq \sum_{(x : X)} Y(x)$$

$$BS_n = \sum_{A : \mathcal{U}} \|\mathrm{Fin}(n) \simeq A\|$$

$$\simeq \sum_{(X:\mathbb{F})} \sum_{Y : X \to \sum_{z:\mathbb{F}} \|z\|} \left\| \mathrm{Fin}(n) \simeq \sum_{(x:X)} Y(x) \right\| \qquad \square$$