– Group Theory whithout Groups –

"All in mathematics is a tale of ~~groups~~ symmetries." (H. Poincaré)

Goal: group theory = theory of symmetries

An abstract group is a set G together with

$$
\begin{aligned}
\_\cdot\_ &: G \to G \to G \\
1 &: G \\
\_^{-1} &: G \to G
\end{aligned}
$$

st
$$
\begin{aligned}
x \cdot 1 &= x && \text{forall } x : G \\
1 \cdot x &= x && \text{——} \\
x \cdot x^{-1} &= 1 && \text{——} \\
x^{-1} \cdot x &= 1 && \text{——} \\
x \cdot (y \cdot z) &= (x \cdot y) \cdot z && \text{for all } x, y, z : G
\end{aligned}
$$

① HoTT has built-in symmetries

The symmetries of $a : A$ are the elements of $a = a$.

Univalence axiom: A, B type

$$\begin{aligned} A = B &\longrightarrow A \simeq B \\ p &\longmapsto \tilde{p} \\ \mathrm{refl}_A &\longmapsto \mathrm{id}_A \end{aligned}$$

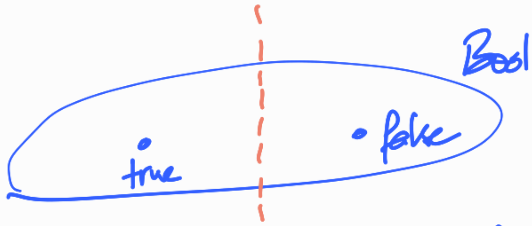
$\simeq$ is an equivalence. Write
$$\begin{aligned} A \simeq B &\longrightarrow A = B \\ \varphi &\longmapsto \overline{\varphi} \end{aligned}$$

1) $A \equiv \mathcal{U}$ $\quad a \equiv \mathrm{Bool}$

$$(\mathrm{Bool} = \mathrm{Bool}) \simeq (\mathrm{Bool} \simeq \mathrm{Bool})$$

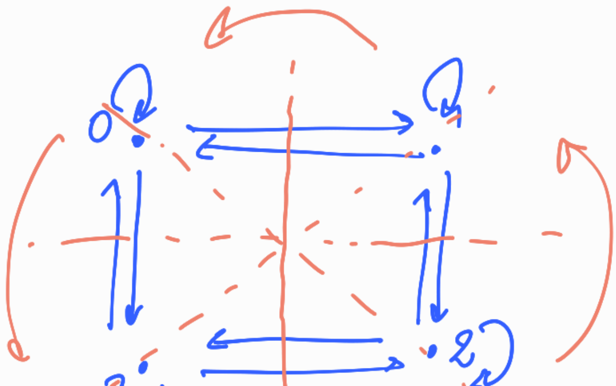
two equivalences $\mathrm{Bool} \xrightarrow{\sim} \mathrm{Bool}$

1. $\mathrm{id}_{\mathrm{Bool}}$
2. swap : true $\mapsto$ false
   false $\mapsto$ true



2) $\mathrm{Graph} :\equiv \sum_{V:\mathcal{U}} (V \to V \to \mathrm{Bool})$

$S : \mathrm{Graph}$



$V \equiv \mathrm{Fin}_4 \qquad e : V \to V \to \mathrm{Bool}$

$e\ i\ j :\equiv \mathrm{true}$ always except

$e\ 0\ 2 :\equiv e\ 2\ 0 :\equiv e\ 1\ 3 :\equiv e\ 3\ 1 :\equiv$ false

$$(S = S) \simeq \sum_{p : \mathrm{Fin}_4 = \mathrm{Fin}_4} \mathrm{trp}_p\, e = e$$

transport in $V \mapsto (V \to V \to \mathrm{Bool})$

$$(\mathrm{trp}_{q : V = V'}\, f)\ \underbrace{i'\ j'}_{\text{in } V'} = f(\tilde{q}^{-1} i')(\tilde{q}^{-1} j')$$

$$\simeq \sum_{p : \mathrm{Fin}_4 = \mathrm{Fin}_4} \prod_{i,j : \mathrm{Fin}_4} e(\tilde{p}^{-1} i)(\tilde{p}^{-1} j) = e\,i\,j$$

$$\simeq \sum_{\varphi : \mathrm{Fin}_4 \simeq \mathrm{Fin}_4} \prod_{i\,j : \mathrm{Fin}_4} e(\varphi^{-1} i)(\varphi^{-1} j) = e\,i\,j$$

② Structure of $a = a$

Given $a : A$

$\mathrm{refl}_a : \boxed{a = a}$

$\_ \cdot \_ : (a = a) \to (a = a) \to (a = a)$

$^{-1} : (a = a) \to (a = a)$

$$\Rightarrow \quad \mathrm{linv} : \prod_{p : a = a} p^{-1} \cdot p = \mathrm{refl}_a$$

$$\mathrm{rinv} : \prod_{p : a = a} p \cdot p^{-1} = \mathrm{refl}_a$$

$$\mathrm{ass} : \prod_{p, q, r : a = a} (p \cdot q) \cdot r = p \cdot (q \cdot r)$$

$$\mathrm{lunit} : \prod_{p : a = a} \mathrm{refl}_a \cdot p = p$$

$$\mathrm{runit} : \prod_{p : a = a} p \cdot \mathrm{refl}_a = p$$

Sum up:

$a = a$ is a good notion of type of symmetries

if $a = a$ is a set, then it is an abstract group

3) Definition of groups

For A type, $\|A\|$ proposition

$$|-| : A \longrightarrow \|A\|$$

satisfying the property: for $P : \|A\| \to \mathrm{Prop}$,

$$\prod_{x : \|A\|} P_x \xrightarrow[\sim]{- \circ |-|} \prod_{x : A} P(|a|)$$

" To prove $P x$ from the data $x : \|A\|$,
it suffices to suppose $x \equiv |a|$ for some $a : A$ "

<u>Connectedness</u>:

A is connected when A is non empty

and $\prod_{x,x':A} \|x = x'\|$ holds

" Every element is <u>merely</u> equal to any other "

$$\neq \quad \prod_{x,x':A} x = x' \quad \rightsquigarrow \quad \text{isProp}(A)$$

" Every element is <u>effectively</u> equal to any other "

$$\mathcal{U}_*^{=1} :\equiv \sum_{A:\mathcal{U}} A \times \text{isConn}\, A \times \underbrace{\text{isGrpd}\, A}_{\prod_{x,y:A} \text{isSet}(x=y)}$$

≤ 1

The type of groups is denoted Group, and is a <u>unary sum</u> of $\mathcal{U}_*^{=1}$.

data Group : $\mathcal{U}$ where
$\Omega : \mathcal{U}_*^{=1} \to \text{Group}$

$$\underline{\Omega} : \mathcal{U}_*^{=1} \longrightarrow \text{Group}$$

equivalence

$B : \mathrm{Group} \longrightarrow \mathcal{U}_*^{=1}$ the inverse of $\underline{\Omega}$.

↑ classifying BG is the classifying type of G.

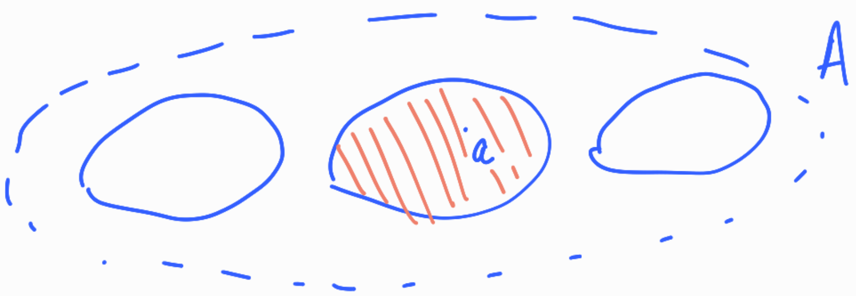
"G: Group is black box, but is entirely determined by BG, in the sense $\underline{\Omega} BG = G$"

we have an identification

Given G: Group, BG is a pointed type, denote sh_G the distinguished element of BG.

We have $UG :\equiv (sh_G = sh_G)$.

What if A is 1-truncated, and $a : A$ but $\neg \mathrm{isConn} A$?



$$A_{(a)} :\equiv \sum_{x : A} \|a = x\|$$

$$\sum_{X : \mathcal{U}} \|X = \mathrm{Fin}_2\|$$

$$A_{(a)} \xhookrightarrow[pr_1]{} A \quad \text{injection}$$

Def: If A is 1-truncated, $a : A$,

$$\mathrm{Aut}_A\, a :\equiv \underline{\Omega}\left(A_{(a)}, (a, |\mathrm{refl}_a|)\right)$$

Ex

1) Set is 1-truncated.

$S : \mathrm{Set}$ $\quad$ $\mathrm{Aut}_{\mathrm{Set}}\, S$ is the groups of permutations of elements of S.

2) When $S \equiv \mathrm{Fin}_n$, $\quad \Sigma_n :\equiv \mathrm{Aut}_{\mathrm{Set}}\, \mathrm{Fin}_n$

the group of permutations on n elements.

with better, $\quad B\Sigma_2 :\equiv \sum_{X:\mathcal{U}} \|X = \mathrm{Fin}_2\|$

$B\Sigma_2 =$ the connected component of Fin_2 in Set pointed at Fin_2.

$$= \sum_{X:\mathrm{Set}} \|X = \mathrm{Fin}_2\|$$

3) $\mathrm{Aut}_{S^1} \bullet$ is the group $\quad |\, S^1 : \mathcal{U}$ $\quad$ has an element $\bullet : S^1$ / has a path $\circlearrowleft : \bullet = \bullet$

of integers,

$$\mathbb{Z} :\equiv \mathrm{Aut}_{S^1} \bullet$$

4) $S \equiv 1$, $\quad \Sigma_1 :\equiv \mathrm{Aut}_{\mathrm{Set}}\, 1$ $\quad$ trivial group

$$\Rightarrow \Sigma_1 = \underline{\Omega}(\mathbb{1}, *)$$

(4) Construction of groups from other groups

Def: The type of G-sets is $G\text{-Set} :\equiv BG \longrightarrow \mathrm{Set}$

For any $X : BG \to \mathrm{Set}$,

$$\mathrm{ap}\, X : \underbrace{(sh_G = sh_G)}_{UG} \longrightarrow \underbrace{X(sh_G) = X(sh_G)}_{U \ldots}$$

In G-Set, there is

$$\begin{aligned} Pr_G : BG &\longrightarrow \mathrm{Set} \\ z &\longmapsto (sh_G = z) \end{aligned}$$

G-Set is 1-truncated

Thm: We can construct a path

$$p_G : G = \mathrm{Aut}_{G\text{-Set}}\, Pr_G$$

Rk:

$$\mathrm{Group} \xrightarrow{\ U\ } \mathrm{AbsGroup}$$

$$\downarrow$$

$$\sum_{G:\mathrm{Set}} \sum_{m:G\to G\to G} \sum_{i:G\to G} \cdots$$

$sh : \prod BG$

G : Group

$sh_G = pr_1\, pr_2\, B\, G$

$sh = pr_1 \circ pr_2 \circ B$

$$G \longmapsto UG \equiv (sh_G = sh_G)$$

U is an equivalence.

WIP https://unimath.github.io/SymmetryBook/book.pdf

$$\underbrace{a \overset{p}{\rightsquigarrow} a' \overset{q}{\rightsquigarrow} a''}$$ $\quad p \cdot q$

$\hookrightarrow q \cdot p$

$$\widetilde{q \cdot p} = \tilde{q} \circ \tilde{p}$$