Continuous and algebraic domains in univalent foundations

Tom de Jong[a,b,*], Martín Hötzel Escardó[a]

[a] *School of Computer Science, University of Birmingham, Birmingham, United Kingdom*
[b] *School of Computer Science, University of Nottingham, Nottingham, United Kingdom*

Abstract

We develop the theory of continuous and algebraic domains in constructive and predicative univalent foundations, building upon our earlier work on basic domain theory in this setting. That we work predicatively means that we do not assume Voevodsky's propositional resizing axioms. Our work is constructive in the sense that we do not rely on excluded middle or the axiom of (countable) choice. To deal with size issues and give a predicatively suitable definition of continuity of a dcpo, we follow Johnstone and Joyal's work on continuous categories. Adhering to the univalent perspective, we explicitly distinguish between data and property. To ensure that being continuous is a property of a dcpo, we turn to the propositional truncation, although we explain that some care is needed to avoid needing the axiom of choice. We also adapt the notion of a domain-theoretic basis to the predicative setting by imposing suitable smallness conditions, analogous to the categorical concept of an accessible category. All our running examples of continuous dcpos are then actually examples of dcpos with small bases which we show to be well behaved predicatively. In particular, such dcpos are exactly those presented by small ideals. As an application of the theory, we show that Scott's D_∞ model of the untyped λ-calculus is an example of an algebraic dcpo with a small basis. Our work is formalised in the Agda proof assistant and its ability to infer universe levels has been invaluable for our purposes.

Keywords: constructivity, predicativity, univalent foundations, homotopy type theory, propositional resizing, domain theory, continuous domain, algebraic domain

*Corresponding author

Email addresses: `tom.dejong@nottingham.ac.uk` (Tom de Jong), `m.escardo@cs.bham.ac.uk` (Martín Hötzel Escardó)

URL: `https://www.tdejong.com` (Tom de Jong), `https://www.cs.bham.ac.uk/~mhe` (Martín Hötzel Escardó)

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1. Introduction

Domain theory [1] is a well-established subject in mathematics and theoretical computer science with applications to programming language semantics [61, 64, 51], higher-type computability [43], topology, and more [17]. We explore the development of domain theory from the univalent point of view [75, 73]. This means that we work with the stratification of types as singletons, propositions, sets, 1-groupoids, etc. Our work does not require any higher inductive types other than the propositional truncation, and the only consequences of univalence needed here are function extensionality and propositional extensionality. Additionally, we work constructively and predicatively, as described below.

1.1. Constructivity

That we work constructively means that we do not assume excluded middle, or weaker variants, such as Bishop's LPO [7], or the axiom of choice (which implies excluded middle), or its weaker variants, such as the axiom of countable choice. An advantage of working constructively and not relying on these additional logical axioms is that our development is valid in every $(\infty, 1)$-topos [66] and not just those in which the logic is classical.

Our commitment to constructivity has the particular consequence that we cannot simply add a least element to a set to obtain the free pointed dcpo. Instead of adding a single least element representing an undefined value, we must work with a more complex type of partial elements (Section 3.4). Similarly, the booleans under the natural ordering fail to be a dcpo, so we use the type of (small) propositions, ordered by implication, instead. Finally, we mention two other domain-theoretic aspects in this work that require particular attention when working constructively. Firstly, it is well-known that the are several inequivalent notions of a finite subset in constructive mathematics and to characterise the compact elements of a powerset we need to use the *Kuratowski* finite subsets [40, 26, 10, 16]. Secondly, *single step function* are classically defined by a case distinction (using excluded middle) on the ordering of elements. Constructively, we cannot, in general, make this case distinction, so we use subsingleton suprema to define single step functions instead.

1.2. Predicativity

Our work is predicative in the sense that we do not assume Voevodsky's *resizing* rules [74, 75] or axioms. In particular, powersets of small types are large.

There are several (philosophical, model-theoretic, proof-theoretic, etc.) arguments for keeping the type theory predicative, see for instance [72, 69, 68], and [32, Section 1.1] for a brief overview, but here we only mention one that we consider to be amongst the most interesting. Namely, the existence of a computational interpretation of propositional impredicativity axioms for univalent foundations is an open problem.

A common approach to deal with domain-theoretic size issues in a predicative foundation is to work with information systems [62, 63], abstract bases [1] or formal topologies [57, 58, 9] rather than dcpos, and approximable relations rather than *Scott continuous functions*. Instead, we work directly with dcpos and Scott continuous functions. In dealing with size issues, we draw inspiration from category theory and make crucial use of type universes and type equivalences to capture *smallness*. For example, in our development of the Scott model of PCF [27, 19], the dcpos have carriers in the second universe $\mathcal{U}_1$ and least upper bounds for directed families indexed by types in the first universe $\mathcal{U}_0$. Moreover, up to equivalence of types, the order relation of the dcpos takes values in the lowest universe $\mathcal{U}_0$. Seeing a poset as a category in the usual way, we can say that these dcpos are large, but locally small, and have small filtered colimits. The fact that the dcpos have large carriers is in fact unavoidable and characteristic of predicative settings, as proved in [32].

Because the dcpos have large carriers it is a priori not clear that complex constructions of dcpos, involving countably infinite iterations of exponentials for example, do not result in a need for ever-increasing universes and are predicatively possible. We show that they are possible through a careful tracking of type universe parameters, and this is illustrated by the construction of Scott's D_∞.

Since keeping track of these universes is prone to mistakes, we have formalised our work in Agda (see Section 1.6); its ability to infer and keep track of universe levels has been invaluable.

1.3. Contributions

In previous work [27] we developed domain theory in constructive and predicative univalent foundations and considered basic applications in the semantics of programming languages, such as the Scott model of PCF [51, 64]. However, we did not discuss a rich and deep topic in domain theory: *algebraic* and *continuous* dcpos [17]. We present a treatment of their theory including several examples in our constructive and predicative approach, where we deal with size issues by taking direct inspiration from category theory and the work of Johnstone and Joyal on continuous categories [24] in particular.

Classically, a dcpo D is said to be *continuous* if for every element x of D the set of elements *way below* it is directed and has supremum x. The problem with this definition in our foundational setup is that the type of elements way below x is not necessarily small. Although this does not stop us from asking it to be directed and having supremum x, this still poses a problem: for example, there would be no guarantee that its supremum is preserved by a Scott continuous function, as it is only required to preserve suprema of directed families indexed by small types.

Our solution is to use the ind-completion to give a predicatively suitable definition of continuity of a dcpo, following the category theoretic work by Johnstone and Joyal [24]. Some care is needed to ensure that the resulting definition expresses a property of a dcpo, rather than additional structure. This is of course where the propositional truncation comes in useful, but there are two natural ways of using the truncation. We show that one of them yields a well behaved notion that serves as our definition of continuity, while the other, which we call *pseudocontinuity*, is problematic in a constructive context. In a classical setting where the axiom of choice is assumed, the two notions (continuity and pseudocontinuity) are equivalent.

Another approach is to turn to the notion of a *basis* [1, Section 2.2.2], but to include smallness conditions. While we cannot expect the type of elements way below an element x to be small, in many examples it is the case that the type of basis elements way below x is small. We show that if a dcpo has a small basis, then it is continuous. In fact, all our running examples of continuous dcpos are actually examples of dcpos with small bases. Moreover, dcpos with small bases are better behaved. For example, they are locally small and so are their exponentials. Furthermore, we show that having a small basis is equivalent to being presented by ideals. For algebraic dcpos, bases work especially well constructively, at least in the presence of set quotients and univalence, as explained in Section 7.3. In particular, we show that Scott's D_∞, as originally conceived in [61], and recalled in the setting of predicative univalent foundations in Section 3.7, is algebraic and that it has a small compact basis.

1.4. Related work

In short, the distinguishing features of our work are: (i) the adoption of homotopy type theory as a foundation, (ii) a commitment to predicatively and constructively valid reasoning, (iii) the use of type universes to avoid size issues concerning large posets.

The standard works on domain theory, e.g. [1, 17], are based on traditional impredicative set theory with classical logic. A constructive, topos valid, and hence impredicative, treatment of some domain theory can be found in [70, Chapter III].

Domain theory has been studied predicatively in the setting of formal topology [57, 58, 9] in [45, 47, 59] and the more recent categorical papers [35, 36]. In this predicative setting, one avoids size issues by working with information systems [62, 63], abstract bases [1] or formal topologies, rather than dcpos, and approximable relations rather than Scott continuous functions. Hedberg [21] presented some of these ideas in Martin-Löf Type Theory and formalised them in the proof assistant ALF [44], a precursor to Agda. A modern formalisation in Agda based on Hedberg's work was recently carried out in Lidell's master thesis [41].

Our development differs from the above line of work in that it studies dcpos directly and uses type universes to account for the fact that dcpos may be large. An advantage of this approach is that we can work with (Scott continuous) functions rather than the arguably more involved (approximable) relations.

Another approach to formalising domain theory in type theory can be found in [6, 11]. Both formalisations study ω-chain complete preorders, work with setoids, and make use of Coq's impredicative sort `Prop`. A setoid is a type equipped with an equivalence relation that must be respected by all functions. The particular equivalence relation given by equality is automatically respected of course, but for general equivalence relations this must be proved explicitly. The aforementioned formalisations work with preorders, rather than posets, because they are setoids where two elements x and y are related if $x \leq y$ and $y \leq x$. Our development avoids the use of setoids thanks to the adoption of the univalent point of view. Moreover, we work predicatively and we work with the more general directed families rather than ω-chains, as we intend the theory to also be applicable to topology and algebra [17].

There are also constructive accounts of domain theory aimed at program extraction [5, 49]. Both these works study ω-chain complete posets (ω-cpos) and define notions of ω-continuity for them. The former [5] is notably predicative, but makes use of additional logical axioms: countable choice, dependent choice and Markov's Principle, which are validated by a realisability interpretation. The latter [49] uses constructive logic to extract witnesses but employs classical logic in the proofs of correctness by phrasing them in the double negation fragment of constructive logic. By contrast, we study (continuous) dcpos rather than (ω-continuous) ω-cpos and is fully constructive without relying on additional principles such as countable choice or Markov's Principle.

Yet another approach is the field of *synthetic domain theory* [55, 56, 22, 52, 53]. Although the work in this area is constructive, it is still impredicative, as it is based on topos logic; but more importantly it has a focus different from that of regular domain theory. The aim is to isolate a few basic axioms and find models in (realisability) toposes where every object is a domain and every morphism is continuous. These models often validate additional axioms, such as Markov's Principle and countable choice, both of which are crucially used in the theory, as well as anti-classical axioms which contradict excluded middle. We have a different goal, namely to develop regular domain theory constructively and predicatively, but in a foundation compatible with excluded middle and choice, while not relying on them.

Our treatment of continuous (and algebraic) dcpos is based on the work of Johnstone and Joyal [24] which is situated in category theory where attention must be paid to size issues even in an impredicative setting. In the categorical context, a smallness criterion similar to our notion of having a small basis appears in [24, Proposition 2.16]. In contrast to Johnstone and Joyal [24], we use the propositional truncation to ensure that the type of continuous dcpos is a subtype of the type of dcpos. This, together with the related notion of pseudocontinuity, are discussed in Section 6.2. The particular case of a dcpo with a small compact basis is analogous to the notion of an accessible category [46].

In constructive set theory, our approach corresponds to working with partially ordered classes as opposed to sets [2]. Our notion of a small basis for a dcpo (Section 7) is similar, but different from Aczel's notion of a set-generated dcpo [2, Section 6.4]. While Aczel requires the set $\{b \in B \mid b \sqsubseteq x\}$ to be directed, we instead require the set of elements in B that are way-below x to be directed in line with the usual definition of a basis [1, Section 2.2.6].

Finally, abstract bases were introduced by Smyth as "R-structures" [67]. Our treatment of them and the round ideal completion is closer to that of Abramsky and Jung in the aforementioned [1, Section 2.2.6], although ours is based on families and avoids impredicative constructions.

1.5. Departures from our previous work

This paper presents a revised and expanded treatment of continuous and algebraic domains compared to our conference paper [31]. The presentation of basic domain theory (Section 3) and the construction of Scott's D_∞ in particular has been abridged for brevity, but full details can be found in Chapter 3 of the first author's PhD thesis [29] as well as the accompanying formalisation (Section 1.6). In [31] (and also [27]) the definition of a poset included the requirement that the carrier is a set, because we only realised later that this was redundant (Lemma 3.2).

With the notable exception of Example 7.19 and Section 7.3, the results of this paper can all be found in the aforementioned PhD thesis [29]. Compared to the thesis, we also mention two terminological changes:

- Instead of writing "α is cofinal in β", we now say that "β exceeds α" (see Definition 5.1). The issue with saying "cofinal" was that this word is ordinarily used for two subsets where one is already contained in the other. In particular, two cofinal subsets have the same least upper bound (if it exists) and this was not the case with our usage of the word "cofinal".

- We have swapped "continuity structure" for "continuity data" when discussing continuous dcpos to reflect that the morphisms do not preserve the data (cf. Remark 6.3). Accordingly, we no longer say that a dcpo is "structurally continuous"; instead writing that a dcpo is "equipped with continuity data". Similar terminological changes apply to the algebraic case.

The present treatment of continuous and algebraic dcpos and small (compact) bases is significantly different from that of our earlier work [31]. There, the definition of continuous dcpo was an amalgamation of pseudocontinuity and having a small basis, although it did not imply local smallness. In this work we have disentangled the two notions and based our definition of continuity on Johnstone and Joyal's notion of a continuous category [24] without making any reference to a basis. The current notion of a small basis is simpler and slightly stronger than that of our conference paper [31], which allows us to prove that having a small basis is equivalent to being presented by ideals.

1.6. Formalisation

All of our results are formalised in Agda, building on Escardó's TypeTopology development [14]. Hart's previously cited work [19] was also ported to the current TypeTopology development by Escardó [20]. The reference [30] precisely links each numbered environment (including definitions, examples and remarks) in this paper to its implementation. The HTML rendering has clickable links and so is particularly suitable for exploring the development. But this paper is self-contained and can be read independently from the formalisation.

1.7. Organisation

Section 2: A brief introduction to univalent foundations with a particular focus on type universes and the propositional truncation, as well as a discussion of impredicativity in the form of Voevodsky's propositional resizing axioms.

Section 3: An abridged overview of basic domain theory in constructive and predicative univalent foundations, including directed complete posets (dcpos), Scott continuous maps, the lifting of a set, exponentials and bilimits of dcpos, and Scott's D_∞ model of the untyped λ-calculus.

Section 4: Definition and examples of the way-below relation and compact elements.

Section 5: The ind-completion of a preorder: a tool used to discuss continuity and pseudocontinuity of dcpos.

Section 6: Definitions of continuous and algebraic dcpos accompanied by a discussion on pseudocontinuity and issues concerning the axiom of choice.

Section 7: The notion of a small (compact) basis: strengthening continuity (resp. algebraicity) by imposing smallness conditions.

Section 8: The (round) ideal completion of an abstract basis as a continuous dcpo with a small basis.

Section 9: Bilimits and exponentials of structurally continuous (or algebraic) dcpos (with small bases), including a proof that Scott's D_∞ is algebraic with a small compact basis.

2. Foundations

We work within intensional Martin-Löf Type Theory and we include $+$ (binary sum), Π (dependent product), Σ (dependent sum), Id (identity type), and inductive types, including $\mathbf{0}$ (empty type), $\mathbf{1}$ (type with exactly one element $\star : \mathbf{1}$) and $\mathbb{N}$ (natural numbers). In general we adopt the same conventions of [73]. In particular, we simply write $x = y$ for the identity type $\mathrm{Id}_X(x, y)$ and use $\equiv$ for the judgemental equality, and for dependent functions $f, g : \Pi_{x:X} A(x)$, we write $f \sim g$ for the pointwise equality $\Pi_{x:X} f(x) = g(x)$.

2.1. Universes

We assume a universe $\mathcal{U}_0$ and two operations: for every universe $\mathcal{U}$, a successor universe $\mathcal{U}^+$ with $\mathcal{U} : \mathcal{U}^+$, and for every two universes $\mathcal{U}$ and $\mathcal{V}$ another universe $\mathcal{U} \sqcup \mathcal{V}$ such that for any universe $\mathcal{U}$, we have $\mathcal{U}_0 \sqcup \mathcal{U} \equiv \mathcal{U}$ and $\mathcal{U} \sqcup \mathcal{U}^+ \equiv \mathcal{U}^+$. Moreover, $(-) \sqcup (-)$ is idempotent, commutative, associative, and $(-)^+$ distributes over $(-) \sqcup (-)$. We write $\mathcal{U}_1 :\equiv \mathcal{U}_0^+, \mathcal{U}_2 :\equiv \mathcal{U}_1^+, \ldots$ and so on. If $X : \mathcal{U}$ and $Y : \mathcal{V}$, then $X + Y : \mathcal{U} \sqcup \mathcal{V}$ and if $X : \mathcal{U}$ and $Y : X \to \mathcal{V}$, then the types $\Sigma_{x:X} Y(x)$ and $\Pi_{x:X} Y(x)$ live in the universe $\mathcal{U} \sqcup \mathcal{V}$; finally, if $X : \mathcal{U}$

and $x, y : X$, then $\mathrm{Id}_X(x, y) : \mathcal{U}$. The type of natural numbers $\mathbb{N}$ is assumed to be in $\mathcal{U}_0$ and we postulate that we have copies $\mathbf{0}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{U}}$ in every universe $\mathcal{U}$. This has the useful consequence that while we do not assume cumulativity of universes, embeddings that lift types to higher universes are definable. For example, the map $(-) \times \mathbf{1}_{\mathcal{V}}$ takes a type in any universe $\mathcal{U}$ to an equivalent type in the higher universe $\mathcal{U} \sqcup \mathcal{V}$. All our examples go through with just two universes $\mathcal{U}_0$ and $\mathcal{U}_1$, but the theory is more easily developed in a general setting.

2.2. The univalent point of view

Within this type theory, we adopt the univalent point of view [73]. A type X is a *proposition* (or *truth value* or *subsingleton*) if it has at most one element, i.e. we have an element of the type $\text{is-prop}(X) :\equiv \prod_{x,y:X} x = y$. A major difference between univalent foundations and other foundational systems is that we *prove* that types are propositions or properties. For instance, we can show (using function extensionality) that the axioms of directed complete poset form a proposition. A type X is a *set* if any two elements can be identified in at most one way, i.e. we have an element of the type $\prod_{x,y:X} \text{is-prop}(x = y)$.

2.3. Extensionality axioms

The univalence axiom [73] is not needed for our development, although we do pause to point out its consequences in a few places, namely in Sections 2.5 and 7.3 and Proposition 6.5.

We assume function extensionality and propositional extensionality, often tacitly:

(i) *Propositional extensionality*: if P and Q are two propositions, then we postulate that $P = Q$ holds exactly when we have both $P \to Q$ and $Q \to P$.

(ii) *Function extensionality*: if $f, g : \prod_{x:X} A(x)$ are two (dependent) functions, then we postulate that $f = g$ holds exactly when $f \sim g$.

Function extensionality has the important consequence that the propositions form an exponential ideal, i.e. if X is a type and $Y : X \to \mathcal{U}$ is such that every $Y(x)$ is a proposition, then so is $\Pi_{x:X} Y(x)$ [73, Example 3.6.2]. In light of this, universal quantification is given by Π-types in our type theory.

2.4. Propositional truncation

In Martin-Löf Type Theory, an element of $\prod_{x:X} \sum_{y:Y} \phi(x, y)$, by definition, gives us a function $f : X \to Y$ such that $\prod_{x:X} \phi(x, f(x))$. In some cases, we wish to express the weaker "for every $x : X$, there exists some $y : Y$ such that $\phi(x, y)$" without necessarily having an assignment of x's to y's. A good example of this is when we define directed families later (see Definition 3.3). This is achieved through the notion of propositional truncation.

Given a type $X : \mathcal{U}$, we postulate that we have a proposition $\|X\| : \mathcal{U}$ with a function $|-| : X \to \|X\|$ such that for every proposition $P : \mathcal{V}$ in any universe $\mathcal{V}$, every function $f : X \to P$ factors (necessarily uniquely, by function extensionality) through $|-|$. Diagrammatically,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "|-|"'] & & P \\ & \|X\| \arrow[ur, dashed] & \end{tikzcd}$$

Notice that the induction and recursion principles automatically hold up to an identification: writing $\bar{f}$ for the dashed map above, we have an identification $\bar{f}(|x|) = f(x)$ for every $x : X$ because P is assumed to be a proposition. This is sufficient for our purposes and we do not require these equalities to hold judgementally.

Existential quantification $\exists_{x:X} Y(x)$ is given by $\|\Sigma_{x:X} Y(x)\|$. One should note that if we have $\exists_{x:X} Y(x)$ and we are trying to prove some proposition P, then we may assume that we have $x : X$ and $y : Y(x)$ when constructing our element of P. Similarly, we can define disjunction as $P \vee Q :\equiv \|P + Q\|$.

We assume throughout that every universe is closed under propositional truncations, meaning that if $X : \mathcal{U}$ then $\|X\| : \mathcal{U}$ as well. We also stress that propositional truncation is the only higher inductive type used in our work.

Finally we recall a useful result due to Kraus et al. [39, Theorem 5.4] which has several applications in this paper.

Lemma 2.1. *Every constant map to a set factors through the truncation of its domain. Here, a map is constant if any two of its values are equal.*

2.5. Size and impredicativity

We introduce the notion of smallness and use it to define propositional resizing axioms, which we take to be the definition of impredicativity in univalent foundations.

Definition 2.2 (Smallness)**.** A type X in any universe is said to be **$\mathcal{U}$-small** if it is equivalent to a type in the universe $\mathcal{U}$. That is, X is $\mathcal{U}$-small $:\equiv \Sigma_{Y:\mathcal{U}}(Y \simeq X)$.

Here, the symbol $\simeq$ refers to Voevodsky's notion of equivalence [73]. Notice that the type that expresses the $\mathcal{U}$-smallness of X is a proposition if and only if the univalence axiom holds, see [12, Sections 3.14 and 3.36.3].

Definition 2.3 (Type of propositions $\Omega_{\mathcal{U}}$)**.** The type of propositions in a universe $\mathcal{U}$ is $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \text{is-prop}(P) : \mathcal{U}^+$.

Observe that $\Omega_{\mathcal{U}}$ itself lives in the successor universe $\mathcal{U}^+$. We often think of the types in some fixed universe $\mathcal{U}$ as *small* and accordingly we say that $\Omega_{\mathcal{U}}$ is *large*. Similarly, the powerset of a type $X : \mathcal{U}$ is large. Given our predicative setup, we must pay attention to universes when considering powersets:

Definition 2.4 ($\mathcal{V}$-powerset $\mathcal{P}_{\mathcal{V}}(X)$, $\mathcal{V}$-subsets)**.** Let $\mathcal{V}$ be a universe and $X : \mathcal{U}$ type. We define the **$\mathcal{V}$-powerset** $\mathcal{P}_{\mathcal{V}}(X)$ as $X \to \Omega_{\mathcal{V}} : \mathcal{V}^+ \sqcup \mathcal{U}$. Its elements are called **$\mathcal{V}$-subsets** of X.

Definition 2.5 ($\in, \subseteq$)**.** Let x be an element of a type X and let A be an element of the powerset $\mathcal{P}_{\mathcal{V}}(X)$. We write $x \in A$ for the type $\mathrm{pr}_1(A(x))$. The first projection pr_1 is needed because $A(x)$, being of type $\Omega_{\mathcal{V}}$, is a pair. Given two $\mathcal{V}$-subsets A and B of X, we write $A \subseteq B$ for $\prod_{x:X}(x \in A \to x \in B)$.

Function extensionality and propositional extensionality imply that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.6 (Total space of a subset, $\mathbb{T}$)**.** The **total space** of a $\mathcal{T}$-valued subset S of a type X is defined as $\mathbb{T}(S) :\equiv \Sigma_{x:X}(x \in S)$.

One could ask for a *resizing axiom* asserting that $\Omega_{\mathcal{U}}$ has size $\mathcal{U}$, which we call *the propositional impredicativity of* $\mathcal{U}$. A closely related axiom is *propositional resizing*, which asserts that every proposition $P : \mathcal{U}^+$ has size $\mathcal{U}$. Without the addition of such resizing axioms, the type theory is said to be *predicative*. As an example of the use of impredicativity in mathematics, we mention that the powerset has unions of arbitrary subsets if and only if propositional resizing holds [12, Section 3.36.6].

We note that the resizing axioms are actually theorems when classical logic is assumed. This is because if $P \vee \neg P$ holds for every proposition in $P : \mathcal{U}$, then the only propositions (up to equivalence) are $\mathbf{0}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{U}}$, which have equivalent copies in $\mathcal{U}_0$, and $\Omega_{\mathcal{U}}$ is equivalent to a type $\mathbf{2}_{\mathcal{U}} : \mathcal{U}$ with exactly two elements.

3. Basic domain theory in univalent foundations

We review basic domain theory in constructive and predicative univalent foundations, laying the foundations for developing the theory of continuous and algebraic domains. For brevity, we only include proofs when they deviate from their classical counterparts, paying special attention to universe parameters, the distinction between data and property, and the use of the propositional truncation. We note that full details may be found in Chapter 3 of the first author's PhD thesis [29] or the accompanying formalisation [30].

3.1. Introduction to constructive and predicative domain theory

We offer the following overture in preparation of our development, especially if the reader is familiar with domain theory in a classical, set-theoretic setting.

The basic object of study in domain theory is that of a *directed complete poset* (dcpo). In (impredicative) set-theoretic foundations, a dcpo can be defined to be a poset that has least upper bounds of all directed subsets. A naive translation of this to our foundation would be to proceed as follows. Define a poset in a universe $\mathcal{U}$ to be a type $P : \mathcal{U}$ with a reflexive, transitive and antisymmetric relation $- \sqsubseteq - : P \times P \to \mathcal{U}$. Since we wish to consider posets and not categories we require that the values $p \sqsubseteq q$ of the order relation are *subsingletons*. Then we could say that the poset $(P, \sqsubseteq)$ is *directed complete* if every directed family $I \to P$ with indexing type $I : \mathcal{U}$ has a least upper bound (supremum). The problem with this definition is that there are no interesting examples in our constructive and predicative setting. For instance, assume that the poset $\mathbf{2}$ with two elements $0 \sqsubseteq 1$ is directed complete, and consider a proposition $A : \mathcal{U}$ and the directed family $A + \mathbf{1} \to \mathbf{2}$ that maps the left component to 0 and the right component to 1. By case analysis on its hypothetical supremum, we conclude that the negation of A is decidable. This amounts to weak excluded middle (which is equivalent to De Morgan's Law) and is constructively unacceptable.

To try to get an example, we may move to the poset $\Omega_{\mathcal{U}_0}$ of propositions in the universe $\mathcal{U}_0$, ordered by implication. This poset does have all suprema of families $I \to \Omega_{\mathcal{U}_0}$ indexed by types I in the *first universe* $\mathcal{U}_0$, given by existential quantification. But if we consider a directed family $I \to \Omega_{\mathcal{U}_0}$ with I in the *same universe* as $\Omega_{\mathcal{U}_0}$ lives, namely the *second universe* $\mathcal{U}_1$, existential quantification gives a proposition in the *second universe* $\mathcal{U}_1$ and so doesn't give its supremum. In this example, we get a poset such that

(i) the carrier lives in the universe $\mathcal{U}_1$,

(ii) the order has truth values in the universe $\mathcal{U}_0$, and

(iii) suprema of directed families indexed by types in $\mathcal{U}_0$ exist.

Regarding a poset as a category in the usual way, we have a large, but locally small, category with small filtered colimits (directed suprema). This is typical of all the concrete examples that we consider, such as the dcpos in the Scott model of PCF [27] and Scott's D_∞ model of the untyped λ-calculus (Section 3.7). We may say that the predicativity restriction increases the universe usage by one. However, for the sake of generality, we formulate our definition of dcpo with the following universe conventions:

(i) the carrier lives in a universe $\mathcal{U}$,

(ii) the order has truth values in a universe $\mathcal{T}$, and

(iii) suprema of directed families indexed by types in a universe $\mathcal{V}$ exist.

So our notion of dcpo has three universe parameters $\mathcal{U}, \mathcal{V}$ and $\mathcal{T}$. We will say that the dcpo is *locally small* when $\mathcal{T}$ is not necessarily the same as $\mathcal{V}$, but the order has $\mathcal{V}$-small truth values. Most of the time we mention $\mathcal{V}$ explicitly and leave $\mathcal{U}$ and $\mathcal{T}$ to be understood from the context.

3.2. Directed complete posets indexed by universe parameters

We now define directed complete poset in constructive and predicative univalent foundations. We carefully explain our use of the propositional truncation in our definitions and, as mentioned above, the type universes involved.

Definition 3.1 (Preorder and poset)**.** A **preorder** $(P, \sqsubseteq)$ is a type $P : \mathcal{U}$ together with a proposition-valued binary relation $\sqsubseteq : P \to P \to \Omega_{\mathcal{T}}$ that is reflexive and transitive. A **poset** is a preorder $(P, \sqsubseteq)$ that is antisymmetric: if $p \sqsubseteq q$ and $q \sqsubseteq p$, then $p = q$ for every $p, q : P$.

Lemma 3.2. *If $(P, \sqsubseteq)$ is a poset, then P is a set.*

Proof. For every $p, q : P$, the composite

$$(p = q) \xrightarrow{\text{by reflexivity}} (p \sqsubseteq q) \times (q \sqsubseteq p) \xrightarrow{\text{by antisymmetry}} (p = q)$$

is constant since $(p \sqsubseteq q) \times (q \sqsubseteq p)$ is a proposition. By [39, Lemma 3.11] it therefore follows that P must be a set. $\square$

From now on, we will simply write "let P be a poset" leaving the partial order $\sqsubseteq$ implicit. We will often use the symbol $\sqsubseteq$ for partial orders on different carriers when it is clear from the context which one it refers to.

Definition 3.3 ((Semi)directed family)**.** A family $\alpha : I \to P$ of elements of a poset P is **semidirected** if whenever we have $i, j : I$, there exists some $k : I$ such that $\alpha_i \sqsubseteq \alpha_k$ and $\alpha_j \sqsubseteq \alpha_k$. We frequently use the shorthand $\alpha_i, \alpha_j \sqsubseteq \alpha_k$ to denote the latter requirement. Such a family is **directed** if it is semidirected and its domain I is inhabited.

Remark 3.4. Note our use of the propositional truncation in defining when a family is *directed*. To make this explicit, we write out the definition in type-theoretic syntax: a family $\alpha : I \to P$ is directed if

(i) we have an element of $\|I\|$, and

(ii) $\Pi_{i,j:I} \|\Sigma_{k:I}(\alpha_i \sqsubseteq \alpha_k) \times (\alpha_j \sqsubseteq \alpha_k)\|$.

The use of the propositional truncation ensures that the types (i) and (ii) are propositions and hence that being (semi)directed is a property of a family. The type (ii) without truncation would instead express an assignment of a chosen $k : I$ for every $i, j : I$ instead.

Least upper bounds or *suprema* (of families) are defined as usual.

Definition 3.5 ($\mathcal{V}$-directed complete poset, $\mathcal{V}$-dcpo, $\bigsqcup \alpha$, $\bigsqcup_{i:I} \alpha_i$)**.** For a universe $\mathcal{V}$, a **$\mathcal{V}$-directed complete poset** (or **$\mathcal{V}$-dcpo**, for short) is a poset D such that every directed family $\alpha : I \to D$ with $I : \mathcal{V}$ has a supremum in D that we denote by $\bigsqcup \alpha$ or $\bigsqcup_{i:I} \alpha_i$.

Remark 3.6. Explicitly, we ask for an element of the type

$$\Pi_{I:\mathcal{V}} \Pi_{\alpha:I \to D}(\text{is-directed}\,\alpha \to \Sigma_{x:D}(x \text{ is-sup-of } \alpha)),$$

where (x is-sup-of α) is the type expressing that x is the supremum of α. Even though we used Σ and not $\exists$ in this expression, this type is still a proposition: By [73, Example 3.6.2], it suffices to prove that the type $\Sigma_{x:D}(x \text{ is-sup-of } \alpha)$ is a proposition. So suppose that we have $x, y : D$ with $p : x$ is-sup-of α and $q : y$ is-sup-of α. Being the supremum of a family is a property because the partial order is proposition-valued. Hence, by [73, Lemma 3.5.1], to prove that $(x, p) = (y, q)$, it suffices to prove that $x = y$. But this follows from antisymmetry and the fact that x and y are both suprema of α.

We will sometimes leave the universe $\mathcal{V}$ implicit, and simply speak of a dcpo. On other occasions, we need to carefully keep track of universe levels. To this end, we make the following definition.

Definition 3.7 ($\mathcal{V}$-DCPO$_{\mathcal{U},\mathcal{T}}$)**.** Let $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{T}$ be universes. We write $\mathcal{V}$-DCPO$_{\mathcal{U},\mathcal{T}}$ for the type of $\mathcal{V}$-dcpos with carrier in $\mathcal{U}$ and order taking values in $\mathcal{T}$. We often leave the parameters $\mathcal{U}$ and $\mathcal{T}$ implicit.

Remark 3.8. In particular, it is very important to keep track of the universe parameters of the lifting (Section 3.4) and of exponentials (Section 3.5) in order to ensure that it is possible to construct Scott's D_∞ (Section 3.7) and the Scott model of PCF [27] in our predicative setting.

In many examples and applications, we deal with dcpos with a least element, denoted by $\perp_D$ or simply $\perp$, and in which case we speak of *pointed* dcpos.

Definition 3.9 (Local smallness)**.** A $\mathcal{V}$-dcpo D is **locally small** if $x \sqsubseteq y$ is $\mathcal{V}$-small for every $x, y : D$.

Lemma 3.10. *A $\mathcal{V}$-dcpo D is locally small if and only if we have a relation $\sqsubseteq_{\mathcal{V}} : D \to D \to \mathcal{V}$ such that $x \sqsubseteq y$ holds precisely when $x \sqsubseteq_{\mathcal{V}} y$ does.*

Proof. The $\mathcal{V}$-dcpo D is locally small exactly when we have an element of

$$\Pi_{x,y:D}\Sigma_{T:\mathcal{V}}(T \simeq x \sqsubseteq y).$$

But this type is equivalent to

$$\Sigma_{R:D\to D\to\mathcal{V}}\Pi_{x,y:D}(R(x,y) \simeq x \sqsubseteq y)$$

by distributivity of Π over Σ [73, Theorem 2.5.17]. ∎

Nearly all examples of $\mathcal{V}$-dcpos in this paper will be locally small. We now introduce two fundamental examples of dcpos: the type of subsingletons and powersets.

Example 3.11 (The type of subsingletons as a pointed dcpo)**.** For any type universe $\mathcal{V}$, the type $\Omega_{\mathcal{V}}$ of subsingletons in $\mathcal{V}$ is a poset if we order the propositions by implication. Note that antisymmetry holds precisely because of propositional extensionality. Moreover, $\Omega_{\mathcal{V}}$ has a least element, namely $\mathbf{0}_{\mathcal{V}}$, the empty type in $\mathcal{V}$, and suprema for all (not necessarily directed) families indexed by a type in $\mathcal{V}$. Finally, paying attention to the universe levels we observe that $\Omega_{\mathcal{V}} : \mathcal{V}$-DCPO$_{\mathcal{V}^+,\mathcal{V}}$, hence it is locally small.

Example 3.12 (Powersets as pointed dcpos)**.** Recalling our treatment of subset and powersets from Section 2.5, we show that powersets give examples of pointed dcpos. Specifically, for every type $X : \mathcal{U}$ and every type universe $\mathcal{V}$, the subset inclusion $\subseteq$ makes $\mathcal{P}_{\mathcal{V}}(X)$ into a poset, where antisymmetry holds by function extensionality and propositional extensionality. Moreover, $\mathcal{P}_{\mathcal{V}}(X)$ has a least element of course: the empty set $\emptyset$. We also claim that $\mathcal{P}_{\mathcal{V}}(X)$ has suprema for all (not necessarily directed) families $\alpha : I \to \mathcal{P}_{\mathcal{V}}(X)$ with $I : \mathcal{V}$. Given such a family α, its least upper bound is given by $\bigcup \alpha :\equiv \lambda x \,.\, \exists_{i:I}\, x \in \alpha_i$, the set-theoretic union, which is well-defined as $(\exists_{i:I}\, x \in \alpha_i) : \mathcal{V}$. Finally, paying attention to the universe levels we observe that $\mathcal{P}_{\mathcal{V}}(X) : \mathcal{V}\text{-DCPO}_{\mathcal{V}^+ \sqcup \mathcal{U}, \mathcal{V} \sqcup \mathcal{U}}$. In the case that $X : \mathcal{U} \equiv \mathcal{V}$, we obtain the simpler, locally small $\mathcal{P}_{\mathcal{V}}(X) : \mathcal{V}\text{-DCPO}_{\mathcal{V}^+, \mathcal{V}}$.

Of course, $\Omega_{\mathcal{V}}$ is easily seen to be equivalent to $\mathcal{P}_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}})$, so Example 3.12 subsumes Example 3.11, but it is instructive to understand Example 3.11 first.

3.3. Scott continuous maps

Definition 3.13 (Scott continuity)**.** A function $f : D \to E$ between two $\mathcal{V}$-dcpos is **(Scott) continuous** if it preserves directed suprema, i.e. if $I : \mathcal{V}$ and $\alpha : I \to D$ is directed, then $f(\bigsqcup \alpha)$ is the supremum in E of the family $f \circ \alpha$.

Remark 3.14. When we speak of a Scott continuous function between D and E, then we will always assume that D and E are both $\mathcal{V}$-dcpos for the same arbitrary but fixed type universe $\mathcal{V}$. Notice that Scott continuity is a property of a map and that any Scott continuous function is monotone.

Remark 3.15. In constructive mathematics it is not possible to exhibit a discontinuous function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$, because sheaf [71, Chapter 15] and realizability models [48, e.g. Proposition 3.1.6] imply that it is consistent to assume that all such functions are continuous. This does not mean, however, that we cannot exhibit a discontinuous function between dcpos. In fact, the negation map $\neg : \Omega \to \Omega$ is not monotone and hence not continuous. If we were to preclude such examples, then we can no longer work with the full type Ω of all propositions, but instead we must restrict to a subtype of propositions, for example by using dominances [55]. Indeed, this approach is investigated in the context of topos theory in [50, 42] and for computability instead of continuity in univalent foundations in [13].

Definition 3.16 (Strictness)**.** A Scott continuous function $f : D \to E$ between pointed dcpos is **strict** if $f(\bot_D) = \bot_E$.

Lemma 3.17. *A poset D is a pointed $\mathcal{V}$-dcpo if and only if it has suprema for all semidirected families indexed by types in $\mathcal{V}$ that we will denote using the $\bigvee$ symbol. In particular, a pointed $\mathcal{V}$-dcpo has suprema of all families indexed by propositions in $\mathcal{V}$.*

Moreover, if f is a Scott continuous and strict map between pointed $\mathcal{V}$-dcpos, then f preserves suprema of semidirected families.

Classically, this can be proved by case distinction on whether the domain of a semidirected family is inhabited, but here we avoid this as follows:

Proof. If D is complete with respect to semidirected families indexed by types in $\mathcal{V}$, then it is clearly a $\mathcal{V}$-dcpo and it is pointed because the supremum of the family indexed by the empty type is the least element. Conversely, if D is a pointed $\mathcal{V}$-dcpo and $\alpha : I \to D$ is a semidirected family with $I : \mathcal{V}$, then

$$\begin{aligned} \hat{\alpha} : I + \mathbf{1}_{\mathcal{V}} &\to D \\ \mathrm{inl}(i) &\mapsto \alpha_i \\ \mathrm{inr}(\star) &\mapsto \bot \end{aligned}$$

is directed and hence has a sup in D which is also the least upper bound of α.

A pointed $\mathcal{V}$-dcpo must have suprema for all families indexed by propositions in $\mathcal{V}$, because any such family is semidirected. Finally, suppose that $\alpha : I \to D$ is semidirected and that $f : D \to E$ is Scott continuous and strict. Using the $\widehat{(-)}$-construction from above, as well as Scott continuity and strictness of f, we get

$$f(\bigvee \alpha) \equiv f(\bigsqcup \hat{\alpha}) = \bigsqcup f \circ \hat{\alpha} = \bigsqcup \widehat{f \circ \alpha} \equiv \bigvee f \circ \alpha,$$

finishing the proof. □

Definition 3.18 (Isomorphism)**.** A Scott continuous map $f : D \to E$ is an **isomorphism** if we have a Scott continuous inverse $g : E \to D$.

Definition 3.19 (Scott continuous retract)**.** A dcpo D is a **(Scott continuous) retract** of E if we have Scott continuous maps $s : D \to E$ and $r : E \to D$ such that s is a section of r. We denote this situation by $D \underset{r}{\overset{s}{\rightleftarrows}} E$.

Lemma 3.20. *If D is a retract of E and E is locally small, then so is D.*

Proof. We claim that $x \sqsubseteq_D y$ and $s(x) \sqsubseteq_E s(y)$ are equivalent, which proves the lemma as E is assumed to be locally small. One direction of the equivalence is given by the fact that s is monotone. In the other direction, assume that $s(x) \sqsubseteq s(y)$ and note that $x = r(s(x)) \sqsubseteq r(s(y)) = y$, as r is monotone and s is a section of r. □

3.4. Lifting

We now turn to constructing pointed $\mathcal{V}$-dcpos from sets. First of all, every discretely ordered set is a $\mathcal{V}$-dcpo, where discretely ordered means that we have $x \sqsubseteq y$ exactly when $x = y$. In fact, ordering X discretely yields the free $\mathcal{V}$-dcpo on the set X in the categorical sense.

With excluded middle, the situation for *pointed* $\mathcal{V}$-dcpos is also straightforward. Simply order the set X discretely and add a least element. However, in [27, Lemma 17], we showed, by considering $X \equiv \mathbb{N}$ and a reduction to the constructive taboo LPO [7], that this approach is constructively unsatisfactory. Moreover, in [32] we proved a general constructive no-go theorem showing that there is a nontrivial dcpo with decidable equality if and only if weak excluded middle holds.

Our solution to the above will be to work with the lifting monad, sometimes known as the partial map classifier monad from topos theory [25, 55, 56, 38], which has been extended to constructive type theory by Reus and Streicher [53] and recently to univalent foundations by Escardó and Knapp [13, 37].

Definition 3.21 (Lifting, partial element, $\mathcal{L}_{\mathcal{V}}(X)$; [13, Section 2.2])**.** We define the type of **partial elements** of a type $X : \mathcal{U}$ with respect to a universe $\mathcal{V}$ as

$$\mathcal{L}_{\mathcal{V}}(X) :\equiv \Sigma_{P:\Omega_{\mathcal{V}}}(P \to X)$$

and we also call it the **lifting** of X with respect to $\mathcal{V}$.

Every (total) element of X gives rise to a partial element of X through the following map:

Definition 3.22 (η_X)**.** The map $\eta_X : X \to \mathcal{L}_{\mathcal{V}}(X)$ is defined by mapping x to the tuple $(\mathbf{1}_{\mathcal{V}}, \lambda\, u \,.\, x)$, where we have omitted the witness that $\mathbf{1}_{\mathcal{V}}$ is a subsingleton. We sometimes omit the subscript in η_X.

Besides these total elements, the lifting has another distinguished element that will be the least in the order with which we shall equip the lifting.

Definition 3.23 ($\perp$)**.** For every type $X : \mathcal{U}$ and universe $\mathcal{V}$, we denote the element $(\mathbf{0}_{\mathcal{V}}, \varphi) : \mathcal{L}_{\mathcal{V}}(X)$ by $\perp$. (Here φ is the unique map from $\mathbf{0}_{\mathcal{V}}$ to X.)

Proposition 3.24 ([31, Lemma 18])**.** *The $(\mathcal{V}^+ \sqcup \mathcal{U})$-valued binary relation on $\mathcal{L}_{\mathcal{V}}(X)$ given by*

$$(P, \varphi) \sqsubseteq (Q, \psi) :\equiv P \to (P, \varphi) = (Q, \psi)$$

is a partial order on $\mathcal{L}_{\mathcal{V}}(X)$ for every set $X : \mathcal{U}$. Moreover, it is equivalent to the relation

$$(P, \varphi) \sqsubseteq' (Q, \psi) :\equiv \Sigma_{f:P\to Q}(\varphi \sim \psi \circ f)$$

that is valued in $\mathcal{V} \sqcup \mathcal{U}$.

In light of Remark 3.8, we carefully keep track of the universe parameters of the lifting in the following proposition.

Proposition 3.25 (cf. [13, Theorem 1])**.** *For a set $X : \mathcal{U}$, the lifting $\mathcal{L}_{\mathcal{V}}(X)$ ordered as in Proposition 3.24 is a pointed $\mathcal{V}$-dcpo. In full generality we have $\mathcal{L}_{\mathcal{V}}(X) : \mathcal{V}\text{-DCPO}_{\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}^+\sqcup\mathcal{U}}$, but if $X : \mathcal{V}$, then $\mathcal{L}_{\mathcal{V}}(X)$ is locally small.*

3.5. Exponentials

Exponentials will be crucial in Scott's D_∞ construction (Section 3.7).

Definition 3.26 (Exponential of (pointed) dcpos, E^D)**.** The **exponential** of two $\mathcal{V}$-dcpos D and E is given by the poset E^D defined as follows. Its carrier is the type of Scott continuous functions from D to E. The order is given pointwise, i.e. $f \sqsubseteq_{E^D} g$ holds if $f(x) \sqsubseteq_E g(x)$ for every $x : D$. Notice that if E is pointed, then so is E^D with least element given the constant function at the least element of E. Finally, it is straightforward to show that E^D is $\mathcal{V}$-directed complete, so that E^D is another $\mathcal{V}$-dcpo.

Note that the exponential E^D is a priori not locally small even if E is because the partial order quantifies over all elements of D. But if D has a small basis then E^D will be locally small when E is (Proposition 7.9).

Remark 3.27. Recall from Remark 3.8 that it is necessary to carefully keep track of the universe parameters of the exponential. In general, the universe levels of E^D can be quite large and complicated. For if $D : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{T}}$ and $E : \mathcal{V}\text{-DCPO}_{\mathcal{U}',\mathcal{T}'}$, then the exponential E^D has a carrier in the universe

$$\mathcal{V}^+ \sqcup \mathcal{U} \sqcup \mathcal{T} \sqcup \mathcal{U}' \sqcup \mathcal{T}'$$

and an order relation that takes values in $\mathcal{U} \sqcup \mathcal{T}'$.

Even if $\mathcal{V} = \mathcal{U} \equiv \mathcal{T} \equiv \mathcal{U}' \equiv \mathcal{T}'$, the carrier of E^D still lives in the larger universe $\mathcal{V}^+$, because the type expressing Scott continuity for $\mathcal{V}$-dcpos quantifies over all types in $\mathcal{V}$. Actually, the scenario where $\mathcal{U} = \mathcal{U}' = \mathcal{V}$ cannot happen in a predicative setting unless D and E are trivial, in a sense made precise in [32].

Even so, in many applications such as those in [27] or Section 3.7, if we take $\mathcal{V} \equiv \mathcal{U}_0$ and all other parameters to be $\mathcal{U} \equiv \mathcal{T} \equiv \mathcal{U}' \equiv \mathcal{T}' \equiv \mathcal{U}_1$, then the situation is much simpler and D, E and the exponential E^D are all elements of $\mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ with all of them being locally small (remember that this is defined up to equivalence). This turns out to be a very favourable situation for both the Scott model of PCF [27] and Scott's D_∞ model of the untyped λ-calculus (Section 3.7). In summary, if we take $\mathcal{V} \equiv \mathcal{U}_0$ and all other parameters to be $\mathcal{U}_1$, then the iterated exponentials all remain in $\mathcal{U}_1$.

After defining products of dcpos as usual (which we omit here for the sake of brevity), we can state and prove a universe parametric version of the universal property of exponentials. In the proposition below we can have $D : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{T}}$ and $E : \mathcal{V}\text{-DCPO}_{\mathcal{U}',\mathcal{T}'}$ for arbitrary universes $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}'$ and $\mathcal{T}'$. In particular, the universe parameters of D and E, apart from the universe of indexing types, need not be the same.

Proposition 3.28. *The exponential defined above satisfies the appropriate universal property: the* evaluation map $\text{ev} : E^D \times D \to E, (g, x) \mapsto g(x)$ *is Scott continuous and if* $f : D' \times D \to E$ *is a Scott continuous function, then there is a unique Scott continuous map* $\bar{f} \colon D' \to E^D$ *such that the diagram*

$$\begin{tikzcd}
D' \times D \arrow[d, dashed, "\bar{f} \times \mathrm{id}_D"'] \arrow[dr, "f"] & \\
E^D \times D \arrow[r, "\mathrm{ev}"'] & E
\end{tikzcd}$$

commutes.

Proof. As in the classical case. □

3.6. Bilimits

In Section 3.7, we give a predicative account of Scott's D_∞ model of the untyped λ-calculus [61]. Here, we describe the general machinery underlying Scott's construction.

A priori one might expect that iterative constructions of dcpos, such as Scott's D_∞, may result in a need for ever-increasing universes and are predicatively impossible. We show, through a careful tracking of type universe parameters, that this is not the case. Secondly, differences arise from proof relevance and these complications are tackled with techniques in univalent foundations and Lemma 2.1 in particular, as discussed right before Lemma 3.36, for example. Compared to Scott's original paper [61], we also generalise from sequential bilimits to directed bilimits.

Definition 3.29 (Deflation)**.** A continuous endofunction $f : D \to D$ is a **deflation** if $f(x) \sqsubseteq x$ for all $x : D$.

Definition 3.30 (Embedding-projection pair)**.** An **embedding-projection pair** from a $\mathcal{V}$-dcpo D to a $\mathcal{V}$-dcpo E consists of two Scott continuous functions $\varepsilon : D \to E$ (the **embedding**) and $\pi : E \to D$ (the **projection**) such that ε is a section of π and $\varepsilon \circ \pi$ is a deflation.

For the remainder of this section, fix the following setup, where we try to be as general as possible regarding universe levels. We fix a directed preorder $(I, \sqsubseteq)$ with $I : \mathcal{V}$ and such that $\sqsubseteq$ takes values in some universe $\mathcal{W}$. Now suppose that $(I, \sqsubseteq)$ indexes a family of $\mathcal{V}$-dcpos with embedding-projection pairs between them, i.e. we have

- for every $i : I$, a $\mathcal{V}$-dcpo $D_i : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{T}}$, and
- for every $i, j : I$ with $i \sqsubseteq j$, an embedding-projection pair $(\varepsilon_{i,j}, \pi_{i,j})$ from D_i to D_j.

Moreover, we require that the following compatibility conditions hold:

$$\text{for every } i : I, \text{ we have } \varepsilon_{i,i} = \pi_{i,i} = \mathrm{id}; \tag{1}$$

$$\text{for every } i \sqsubseteq j \sqsubseteq k \text{ in } I, \text{ we have } \varepsilon_{i,k} \sim \varepsilon_{j,k} \circ \varepsilon_{i,j} \text{ and } \pi_{i,k} \sim \pi_{i,j} \circ \pi_{j,k}. \tag{2}$$

The goal is now to construct another $\mathcal{V}$-dcpo D_∞ with embedding-projections pairs $(\varepsilon_{i,\infty} : D_1 \hookrightarrow D_\infty, \pi_{i,\infty} : D_\infty \to D_i)$ for every $i : I$, such that $\left(D_\infty, (\varepsilon_{i,\infty})_{i:I}\right)$ is the colimit of the diagram given by $(\varepsilon_{i,j})_{i \sqsubseteq j \text{ in } I}$ and $\left(D_\infty, (\pi_{i,\infty})_{i:I}\right)$ is the limit of the diagram given by $(\pi_{i,j})_{i \sqsubseteq j \text{ in } I}$. In other words, $\left(D_\infty, (\varepsilon_{i,\infty})_{i:I}, (\pi_{i,\infty})_{i:I}\right)$ is both the colimit and the limit in the category of $\mathcal{V}$-dcpos with embedding-projections pairs between them. We say that it is the *bilimit*.

Definition 3.31 (D_∞)**.** We define a poset D_∞ as follows. Its carrier is given by the type of elements σ of the product $\Pi_{i:I} D_i$ satisfying $\pi_{i,j}(\sigma_j) = \sigma_i$ whenever $i \sqsubseteq j$. That is, the carrier is the type

$$\sum_{\sigma : \Pi_{i:I} D_i} \prod_{i,j:I, i \sqsubseteq j} \pi_{i,j}(\sigma_j) = \sigma_i.$$

Note that this defines a subtype of $\Pi_{i:I} D_i$ as the condition $\prod_{i,j:I,i\sqsubseteq j} \pi_{i,j}(\sigma_j) = \sigma_i$ is a property by [73, Example 3.6.2] and the fact that each D_i is a set. These functions are ordered pointwise, i.e. if $\sigma, \tau : \Pi_{i:I} D_i$, then $\sigma \sqsubseteq_{D_\infty} \tau$ exactly when $\sigma_i \sqsubseteq_{D_i} \tau_i$ for every $i : I$.

The proof of the following is as in the classical case, but we pay attention to the universe levels.

Lemma 3.32. *The poset D_∞ is $\mathcal{V}$-directed complete with suprema calculated pointwise, and we have $D_\infty : \mathcal{V}\text{-DCPO}_{\mathcal{U}\sqcup\mathcal{V}\sqcup\mathcal{W},\mathcal{U}\sqcup\mathcal{T}}$.*

Remark 3.33. We allow for general universe levels here, which is why D_∞ lives in the relatively complicated universe $\mathcal{U}\sqcup\mathcal{V}\sqcup\mathcal{W}$. In concrete examples, the situation often simplifies. E.g., in Section 3.7 we find ourselves in the favourable situation described in Remark 3.27 where $\mathcal{V} \equiv \mathcal{W} \equiv \mathcal{U}_0$ and $\mathcal{U} \equiv \mathcal{T} \equiv \mathcal{U}_1$, so that we get $D_\infty : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$, as the bilimit of a diagram of dcpos $D_n : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ indexed by natural numbers.

Definition 3.34 ($\pi_{i,\infty}$)**.** For every $i : I$, we define the Scott continuous function $\pi_{i,\infty} : D_\infty \to D_i$ by $\sigma \mapsto \sigma_i$.

While we could closely follow [61] up until this point, we will now need a new idea to proceed. Our goal is to define maps $\varepsilon_{i,\infty} : D_i \to D_\infty$ for every $i : I$ so that $\varepsilon_{i,\infty}$ and $\pi_{i,\infty}$ form an embedding-projection pair. We give an outline of the idea for defining this map $\varepsilon_{i,\infty}$. For an arbitrary element $x : D_i$, we need to construct $\sigma : D_\infty$ at component $j : I$, say. If we had $k : I$ such that $i, j \sqsubseteq k$, then we could define $\sigma_j : D_j$ by $\pi_{j,k}(\varepsilon_{i,k}(x))$. Now semidirectedness of I tells us that there exists such a $k : I$, so the point is to somehow make use of this propositionally truncated fact. This is where Lemma 2.1 comes in. We recall that it says that a constant map to a set factors through the propositional truncation of its domain. We define a map $\kappa^x_{i,j} : (\Sigma_{k:I}\,(i \sqsubseteq k) \times (j \sqsubseteq k)) \to D_j$ by sending k to $\pi_{j,k}(\varepsilon_{i,k}(x))$ and show it to be constant, so that it factors through the truncation of its domain. In the special case that $I \equiv \mathbb{N}$, as in [61], we could simply take k to be the sum of the natural numbers i and j, but this does not work in the more general directed case, of course.

Definition 3.35 ($\kappa^x_{i,j}$)**.** For every $i, j : I$ and $x : D_i$ we define the function

$$\kappa^x_{i,j} : (\Sigma_{k:I}\,(i \sqsubseteq k) \times (j \sqsubseteq k)) \to D_j$$

by mapping $k : I$ with $i, j \sqsubseteq k$ to $\pi_{j,k}(\varepsilon_{i,k}(x))$.

Lemma 3.36. *The function $\kappa^x_{i,j}$ is constant for ever $i, j : I$ and $x : D_i$. Hence, $\kappa^x_{i,j}$ factors through $\exists_{k:I}\,(i \sqsubseteq k) \times (j \sqsubseteq k)$ by Lemma 2.1.*

Proof. If we have $k_1, k_2 : I$ with $i \sqsubseteq k_1, k_2$ and $j \sqsubseteq k_1, k_2$, then by semidirectedness of I, there exists some $k : K$ with $k_1, k_2 \sqsubseteq k$ and hence,

$$\begin{aligned}
&(\pi_{j,k_1} \circ \varepsilon_{i,k_1})(x) \\
&= (\pi_{j,k_1} \circ \pi_{k_1,k} \circ \varepsilon_{k_1,k} \circ \varepsilon_{i,k_1})(x) && \text{(as } \varepsilon_{k_1,k} \text{ is a section of } \pi_{k_1,k}) \\
&= (\pi_{j,k} \circ \varepsilon_{i,k})(x) && \text{(by Equation (2))} \\
&= (\pi_{j,k} \circ \pi_{k_2,k} \circ \varepsilon_{k_2,k} \circ \varepsilon_{i,k_2})(x) && \text{(as } \varepsilon_{k_2,k} \text{ is a section of } \pi_{k_2,k}) \\
&= (\pi_{j,k_2} \circ \varepsilon_{i,k_2})(x) && \text{(by Equation (2)),}
\end{aligned}$$

proving that $\kappa^x_{i,j}$ is constant. □

Definition 3.37 ($\rho_{i,j}$)**.** For every $i, j : I$, the type $\exists_{k:I}\,(i \sqsubseteq k) \times (j \sqsubseteq k)$ has an element since $(I, \sqsubseteq)$ is directed. Thus, Lemma 3.36 tells us that we have a function $\rho_{i,j} : D_i \to D_j$ such that if $i, j \sqsubseteq k$, then the equation

$$\rho_{i,j}(x) = \kappa^x_{i,j}(k) \equiv \pi_{j,k}(\varepsilon_{i,k}(x)) \tag{3}$$

holds for every $x : D_i$.

Definition 3.38 ($\varepsilon_{i,\infty}$)**.** The map ρ induces a map $\varepsilon_{i,\infty} : D_i \to D_\infty$ by sending $x : D_i$ to the function $\lambda\, j : I\,.\,\rho_{i,j}(x)$. To see that this is well-defined, assume that we have $j_1 \sqsubseteq j_2$ in J and $x : D_i$. We have to show that $\pi_{j_1,j_2}\Big(\left(\varepsilon_{i,\infty}(x)\right)_{j_2}\Big) = \left(\varepsilon_{i,\infty}(x)\right)_{j_1}$. By semidirectedness of I and the fact that are looking to prove a proposition, we may assume to have $k : I$ with $i \sqsubseteq k$ and $j_1 \sqsubseteq j_2 \sqsubseteq k$. Then,

$$\begin{aligned}
\pi_{j_1,j_2}\Big(\left(\varepsilon_{i,\infty}(x)\right)_{j_2}\Big) &\equiv \pi_{j_1,j_2}(\rho_{i,j_2}(x)) \\
&= \pi_{j_1,j_2}(\pi_{j_2,k}(\varepsilon_{i,k}(x))) && \text{(by Equation (3))} \\
&= \pi_{j_1,k}(\varepsilon_{i,k}(x)) && \text{(by Equation (2))} \\
&= \rho_{i,j_1}(x) && \text{(by Equation (3))} \\
&\equiv \left(\varepsilon_{i,\infty}(x)\right)_{j_1},
\end{aligned}$$

as desired.

This completes the definition of $\varepsilon_{i,\infty}$. From this point on, we can typically work with it using Equation (3) and the fact that $\left(\varepsilon_{i,\infty}(x)\right)_j$ is defined as $\rho_{i,j}(x)$. Adapting [61] to the directed case, we can then prove the following theorems.

Theorem 3.39. *For every $i : I$, the pair $(\varepsilon_{i,\infty}, \pi_{i,\infty})$ is an embedding-projection pair from D_i to D_∞.*

Theorem 3.40. *The $\mathcal{V}$-dcpo D_∞ with the maps $\left(\pi_{i,\infty}\right)_{i:I}$ is the limit of the diagram $\Big(\left(D_i\right)_{i:I}, \left(\pi_{i,j}\right)_{i \sqsubseteq j}\Big)$. That is, given a $\mathcal{V}$-dcpo $E : \mathcal{V}\text{-DCPO}_{\mathcal{U}',\mathcal{T}'}$ and Scott continuous functions $f_i : E \to D_i$ for every $i : I$ such that the diagram*

$$\begin{tikzcd}
E \arrow[rr, "f_i"] \arrow[dr, "f_j"'] & & D_i \\
& D_j \arrow[ur, "\pi_{i,j}"'] &
\end{tikzcd}$$

commutes for every $i \sqsubseteq j$, we have a unique continuous function $f_\infty : E \to D_\infty$ making the diagram

$$\begin{tikzcd}
E \arrow[rr, "f_i"] \arrow[dr, dashed, "f_\infty"'] & & D_i \\
& D_\infty \arrow[ur, "\pi_{i,\infty}"'] &
\end{tikzcd}$$

commute for every $i : I$.

Similarly, the $\mathcal{V}$-dcpo D_∞ with the maps $\left(\varepsilon_{i,\infty}\right)_{i:I}$ is the colimit of the diagram $\Big(\left(D_i\right)_{i:I}, \left(\varepsilon_{i,j}\right)_{i \sqsubseteq j}\Big)$.

It should be noted that in the above universal property, E can have its carrier in any universe $\mathcal{U}'$ and its order taking values in any universe $\mathcal{T}'$, even though we required all D_i to have their carriers and orders in two fixed universes $\mathcal{U}$ and $\mathcal{T}$, respectively.

The proof of the colimit property relies on the following lemma which is also useful later on.

Lemma 3.41. *Every element $\sigma : D_\infty$ is the directed supremum of $\bigsqcup_{i:I} \varepsilon_{i,\infty}(\sigma_i)$.*

Proposition 3.42. *The bilimit of locally small dcpos is locally small, i.e. if every $\mathcal{V}$-dcpo D_i is locally small for all $i : I$, then so is D_∞.*

Proof. If every D_i is locally small, then for every $i : I$, we have a *specified* $\mathcal{V}$-valued partial order $\sqsubseteq^i_{\mathcal{V}}$ on D_i such that for every $i : I$ and every $x, y : D_i$, we have an equivalence $(x \sqsubseteq_{D_i} y) \simeq (x \sqsubseteq^i_{\mathcal{V}} y)$. Hence, $(\sigma \sqsubseteq_{D_\infty} \tau) \equiv (\Pi_{i:I}(\sigma_i \sqsubseteq_{D_i} \tau_i)) \simeq (\Pi_{i:I}(\sigma_i \sqsubseteq^i_{\mathcal{V}} \tau_i))$, but the latter is small, because $I : \mathcal{V}$ and $\sqsubseteq^i_{\mathcal{V}}$ is $\mathcal{V}$-valued. ☐

3.7. Scott's D_∞ model of the untyped λ-calculus

We are finally in a position to construct Scott's D_∞ [61] predicatively. Formulated precisely, we construct a pointed $D_\infty : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ such that D_∞ is isomorphic to its self-exponential $D_\infty^{D_\infty}$, employing the machinery from Section 3.6.

Definition 3.43 (D_n)**.** We inductively define pointed dcpos $D_n : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ for every natural number n by setting $D_0 :\equiv \mathcal{L}_{\mathcal{U}_0}(\mathbf{1}_{\mathcal{U}_0})$ and $D_{n+1} :\equiv D_n^{D_n}$.

In light of Remark 3.8 we highlight the fact that every D_n is a $\mathcal{U}_0$-dcpo with carrier in $\mathcal{U}_1$ by the discussion of universe parameters of exponentials in Remark 3.27.

Definition 3.44 (ε_n, π_n)**.** We inductively define for every natural number n, two Scott continuous maps $\varepsilon_n : D_n \to D_{n+1}$ and $\pi_n : D_{n+1} \to D_n$:

(i)
- $\varepsilon_0 : D_0 \to D_1$ is given by mapping $x : D_0$ to the continuous function that is constantly x,
- $\pi_0 : D_1 \to D_0$ is given by evaluating a continuous function $f : D_0 \to D_0$ at $\perp$ which is itself continuous by Proposition 3.28,

(ii)
- $\varepsilon_{n+1} : D_{n+1} \to D_{n+2}$ takes a continuous function $f : D_n \to D_n$ to the continuous composite $D_{n+1} \xrightarrow{\pi_n} D_n \xrightarrow{f} D_n \xrightarrow{\varepsilon_n} D_{n+1}$, and
- $\pi_{n+1} : D_{n+2} \to D_{n+1}$ takes a continuous function $f : D_{n+1} \to D_{n+1}$ to the continuous composite $D_n \xrightarrow{\varepsilon_n} D_{n+1} \xrightarrow{f} D_{n+1} \xrightarrow{\pi_n} D_n$.

The maps ε_n and π_n form an embedding-projection pair for each natural number n, and, by taking compositions, we obtain embedding-projection pairs $(\varepsilon_{n,m}, \pi_{n,m})$ from D_n to D_m whenever $n \leq m$.

Definition 3.45 (D_∞)**.** Applying Definitions 3.31, 3.34 and 3.38 to the above diagram yields $D_\infty : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ with embedding-projection pairs $(\varepsilon_{n,\infty}, \pi_{n,\infty})$ from D_n to D_∞ for every natural number n.

Following [61], we can show the following.

Theorem 3.46. *The pointed $\mathcal{U}_0$-dcpos D_∞ and $D_\infty^{D_\infty}$ are isomorphic.*

Moreover, by (for instance) embedding $\eta(\star) : D_0$ into D_∞, we see that D_∞ is not the trivial pointed dcpo.

4. The way-below relation and compactness

The way-below relation is the fundamental ingredient in the development of continuous dcpos. Following Scott [60], a computational intuition of $x \ll y$ says that every computation of y has to produce x, or something better than x, at some stage.

Definition 4.1 (Way-below relation, $x \ll y$)**.** An element x of a $\mathcal{V}$-dcpo D is **way below** an element y of D if whenever we have a directed family $\alpha : I \to D$ indexed by $I : \mathcal{V}$ such that $y \sqsubseteq \bigsqcup \alpha$, then there exists $i : I$ such that $x \sqsubseteq \alpha_i$ already. We denote this situation by $x \ll y$.

Lemma 4.2. *The way-below relation enjoys the following properties.*

(i) it is proposition-valued;

(ii) if $x \ll y$, then $x \sqsubseteq y$;

(iii) if $x \sqsubseteq y \ll v \sqsubseteq w$, then $x \ll w$;

(iv) it is antisymmetric;

(v) it is transitive.

Proof. (i) Using that a dependent product of propositions (over an arbitrary type) is again a proposition together with the fact that we propositionally truncated the existence of $i : I$ in the definition. (ii) Simply take $\alpha : \mathbf{1}_{\mathcal{V}} \to D$ to be $u \mapsto y$. (iii) Suppose that $\alpha : I \to D$ is directed with $w \sqsubseteq \bigsqcup \alpha$. Then $v \sqsubseteq \bigsqcup \alpha$, so by assumption that $y \ll v$ there exists $i : I$ with $y \sqsubseteq \alpha_i$ already. But then $x \sqsubseteq \alpha_i$. (iv) Follows from (ii). (v) Follows from (ii) and (iii). ☐

In general, the way below relation is not reflexive. The elements for which it is have a special status and are called compact. We illustrate this notion by a series of examples.

Definition 4.3 (Compactness)**.** An element of a dcpo is **compact** if it is way below itself.

Example 4.4. The least element of a pointed dcpo is always compact.

We recall the $\mathcal{V}$-dcpo of propositions $\Omega_{\mathcal{V}}$ with designated elements $\mathbf{0}_{\mathcal{V}}$ (the empty type) and $\mathbf{1}_{\mathcal{V}}$ (the unit type) from Example 3.11.

Example 4.5 (Compact elements in $\Omega_{\mathcal{V}}$)**.** The compact elements of $\Omega_{\mathcal{V}}$ are exactly $\mathbf{0}_{\mathcal{V}}$ and $\mathbf{1}_{\mathcal{V}}$. In other words, the compact elements of $\Omega_{\mathcal{V}}$ are precisely the decidable propositions.

Proof. By Example 4.4 we know that $\mathbf{0}_{\mathcal{V}}$ must be compact. For $\mathbf{1}_{\mathcal{V}}$, suppose that we have $Q_{(-)} : I \to \Omega_{\mathcal{V}}$ directed such that $\mathbf{1}_{\mathcal{V}} \sqsubseteq \exists_{i:I} Q_i$. Then there exists $i : I$ such that Q_i holds, and hence, $\mathbf{1}_{\mathcal{V}} \sqsubseteq Q_i$. Now suppose that $P : \Omega_{\mathcal{V}}$ is compact. We show that P is decidable. The family $\alpha : (P + \mathbf{1}_{\mathcal{V}}) \to \Omega_{\mathcal{V}}$ given by $\mathrm{inl}(p) \mapsto \mathbf{1}_{\mathcal{V}}$ and $\mathrm{inr}(\star) \mapsto \mathbf{0}_{\mathcal{V}}$ is directed and $P \sqsubseteq \bigsqcup \alpha$. Hence, by compactness, there exists $i : P + \mathbf{1}_{\mathcal{V}}$ such that $P \sqsubseteq \alpha_i$ already. Since being decidable is a property of a proposition, we actually get such an i and by case distinction on it we get decidability of P. ☐

We recall the lifting $\mathcal{L}_{\mathcal{V}}(X)$ of a set X as a $\mathcal{V}$-dcpo from Section 3.4.

Example 4.6 (Compact elements in the lifting)**.** An element (P, φ) of the lifting $\mathcal{L}_{\mathcal{V}}(X)$ of a set $X : \mathcal{V}$ is compact if and only if P is decidable. Hence, the compact elements of $\mathcal{L}_{\mathcal{V}}(X)$ are exactly $\bot$ and $\eta(x)$ for $x : X$.

Proof. To see that compactness implies decidability of the domain of the partial element, we proceed as in the proof of Example 4.5, but for a partial element (P, φ), we consider the family $\alpha : (P + \mathbf{1}_{\mathcal{V}}) \to \mathcal{L}_{\mathcal{V}}(X)$ given by $\mathrm{inl}(p) \mapsto \eta(\varphi(p))$ and $\mathrm{inr}(\star) \mapsto \bot$. Conversely, if we have a partial element (P, φ) with P decidable, then either P is false in which case $(P, \varphi) = \bot$ which is compact by Example 4.4, or P holds. So suppose that P holds and let $\alpha : I \to \mathcal{L}_{\mathcal{V}}(X)$ be directed with $P \sqsubseteq \bigsqcup \alpha$. Since P holds, the element $\bigsqcup \alpha$ must be defined, so there exists $i : I$ such that α_i is defined. But for this $i : I$ we also have $\bigsqcup \alpha = \alpha_i$ by construction of the supremum, and hence, $P \sqsubseteq \alpha_i$, proving compactness of (P, φ). ☐

For characterising the compact elements of the powerset, we introduce a lemma, as well as the notion of Kuratowski finiteness and the induction principle for Kuratowski finite subsets.

Lemma 4.7. *The compact elements of a dcpo are closed under (existing) binary joins.*

Proof. Suppose that x and y are compact elements of a $\mathcal{V}$-dcpo D with z as their least upper bound and suppose that we have $\alpha : I \to D$ directed with $z \sqsubseteq \bigsqcup \alpha$. Then $x \sqsubseteq \bigsqcup \alpha$ and $y \sqsubseteq \bigsqcup \alpha$, so by compactness there exist $i, j : I$ such that $x \sqsubseteq \alpha_i$ and $y \sqsubseteq \alpha_j$. By semidirectedness of α, there exists $k : I$ with $\alpha_i, \alpha_j \sqsubseteq \alpha_k$, so that $x, y \sqsubseteq \alpha_k$. But z is the join of x and y, so $z \sqsubseteq \alpha_k$, as desired. ☐

Kuratowski finiteness is investigated in [40, 26, 10, 16] among other places.

Definition 4.8 (Kuratowski finiteness)**.**

(i) A type X is **Kuratowski finite** if there exists some natural number $n : \mathbb{N}$ and a surjection $e : \mathrm{Fin}(n) \twoheadrightarrow X$, where $\mathrm{Fin}(n)$ is the standard finite type with exactly n elements [54, Section 7.3].

(ii) A subset is **Kuratowski finite** if its total space (recall Definition 2.6) is a Kuratowski finite type.

Thus, a type X is *Kuratowski finite* if its elements can be finitely enumerated, possibly with repetitions, although the repetitions can be removed when X has decidable equality.

Lemma 4.9. *The Kuratowski finite subsets of a set are closed under finite unions and contain all singletons.*

Proof. The empty set and any singleton are clearly Kuratowski finite. Moreover, if A and B are Kuratowski finite subsets, then we may assume to have natural numbers n and m and surjections $\sigma : \mathrm{Fin}(n) \twoheadrightarrow \mathbb{T}(A)$ and $\tau : \mathrm{Fin}(m) \twoheadrightarrow \mathbb{T}(B)$. We can then patch these together to obtain a surjection $\mathrm{Fin}(n+m) \twoheadrightarrow \mathbb{T}(A \cup B)$, as desired. ☐

The following induction principle appears as [26, Definition 5.4.1] and is closely related to the higher inductive presentation in [16].

Lemma 4.10 (Induction for Kuratowski finite subsets)**.** *A property of subsets of a type X holds for all Kuratowski finite subsets of X as soon as*

(i) it holds for the empty set,

(ii) it holds for any singleton subset, and

(iii) it holds for $A \cup B$, whenever it holds for A and B.

Proof. Let Q be a such a property and let A be an arbitrary Kuratowski finite subset of X. Since Q is proposition-valued, we may assume to have a natural number n and a surjection $\sigma : \mathrm{Fin}(n) \twoheadrightarrow \mathbb{T}(A)$. Then the subset A must be equal to the finite join of singletons $\{\sigma_0\} \cup \{\sigma_1\} \cup \cdots \cup \{\sigma_{n-1}\}$, which can be shown to satisfy Q by induction on n, and hence, so must A. ☐

Definition 4.11 (β)**.** For a set $X : \mathcal{U}$, we write $\beta : \mathrm{List}(X) \to \mathcal{P}_{\mathcal{U}}(X)$ for the canonical map which takes a list to its set of elements. (For the inductive definition of the type $\mathrm{List}(X)$, recall e.g. [73, Section 5.1].)

Lemma 4.12. *A subset $A : \mathcal{P}_{\mathcal{U}}(X)$ of a set $X : \mathcal{U}$ is Kuratowski finite if and only if it is in the image of β.*

Proof. The left to right direction follows from Lemma 4.9, while the converse follows easily from the induction principle for Kuratowski finite subsets where we use list concatenation in case (iii). ☐

Example 4.13 (Compact elements in $\mathcal{P}_{\mathcal{U}}(X)$)**.** The compact elements of $\mathcal{P}_{\mathcal{U}}(X)$ for a set $X : \mathcal{U}$ are exactly the Kuratowski finite subsets of X.

Proof. Suppose first that $A : \mathcal{P}_{\mathcal{U}}(X)$ is a compact element. The family

$$\left(\Sigma_{l:\mathrm{List}(X)}\, \beta(l) \subseteq A\right) \xrightarrow{\beta \circ \mathrm{pr}_1} \mathcal{P}_{\mathcal{U}}(X)$$

is directed, as it contains $\emptyset$ and we can concatenate lists to establish semidirectedness. Moreover, $\left(\Sigma_{l:\mathrm{List}(X)}\, \beta(l) \subseteq A\right)$ lives in $\mathcal{U}$ and we clearly have $A \subseteq \bigsqcup \beta \circ \mathrm{pr}_1$. So by compactness, there exists $l : \mathrm{List}(X)$ with $\beta(l) \subseteq A$ such that $A \subseteq \beta(l)$ already. But this says exactly that A is Kuratowski finite by Lemma 4.12.

For the converse we use the induction principle for Kuratowski finite subsets: the empty set is compact by Example 4.4, singletons are easily shown to be compact, and binary unions are compact by Lemma 4.7. $\square$

We end this section by presenting a few lemmas connecting the way-below relation and compactness to Scott continuous sections (Definition 3.19).

Lemma 4.14. *If we have a Scott continuous retraction* $D \underset{r}{\overset{s}{\rightleftarrows}} E$*, then* $y \ll s(x)$ *implies* $r(y) \ll x$ *for every* $x : D$ *and* $y : E$.

Proof. Suppose that $y \ll s(x)$ and that $x \sqsubseteq \bigsqcup \alpha$ for a directed family $\alpha : I \to D$. Then $s(x) \sqsubseteq s(\bigsqcup \alpha) = \bigsqcup (s \circ \alpha)$ by Scott continuity of s, so there exists $i : I$ such that $y \sqsubseteq s(\alpha_i)$ already. Now monotonicity of r implies $r(y) \sqsubseteq r(s(\alpha_i)) = \alpha_i$ which completes the proof that $r(y) \ll x$. $\square$

We also recall embedding-projection pairs from Definition 3.30.

Lemma 4.15. *The embedding in an embedding-projection pair* $D \underset{\pi}{\overset{\varepsilon}{\rightleftarrows}} E$ *preserves and reflects the way-below relation, i.e.* $x \ll y \iff \varepsilon(x) \ll \varepsilon(y)$*. In particular, an element* x *is compact if and only if* $\varepsilon(x)$ *is.*

Proof. Suppose that $x \ll y$ in D and let $\alpha : I \to E$ be directed with $\varepsilon(y) \sqsubseteq \bigsqcup \alpha$. Then $y = \pi(\varepsilon(y)) \sqsubseteq \bigsqcup \pi \circ \alpha$ by Scott continuity of π. Hence, there exists $i : I$ such that $x \sqsubseteq \pi(\alpha_i)$. But then $\varepsilon(x) \sqsubseteq \varepsilon(\pi(\alpha_i)) \sqsubseteq \alpha_i$ by monotonicity of ε and the fact that $\varepsilon \circ \pi$ is a deflation. This proves that $x \ll y$. Conversely, if $\varepsilon(x) \ll \varepsilon(y)$, then $x = \pi(\varepsilon(x)) \ll y$ by Lemma 4.14. $\square$

5. The ind-completion

The ind-completion will be a useful tool for phrasing and proving results about directed complete *posets* and is itself a directed complete *preorder*, cf. Lemma 5.2. It was introduced by Grothendieck and Verdier in [18, Section 8] in the context of category theory, but its role in order theory is discussed in [24, Section 1]. We will also use it in the context of order theory, but our treatment will involve a careful consideration of the universes involved, very similar to the original treatment in [18].

Definition 5.1 ($\mathcal{V}$-ind-completion $\mathcal{V}$-Ind(X), exceed, $\lesssim$)**.** The **$\mathcal{V}$-ind-completion** $\mathcal{V}$-Ind(X) of a preorder X is the type of directed families in X indexed by types in the universe $\mathcal{V}$. Such a family $\beta : J \to X$ **exceeds** another family $\alpha : I \to X$ if for every $i : I$, there exists $j : J$ such that $\alpha_i \sqsubseteq \beta_j$, and we denote this relation by $\alpha \lesssim \beta$.

Lemma 5.2.

(i) *The relation $\lesssim$ defines a preorder on the ind-completion.*

(ii) *The $\mathcal{V}$-ind-completion $\mathcal{V}$-*Ind(X) *of a preorder X is $\mathcal{V}$-directed complete.*

Proof. The first item is proved straightforwardly. For the second, suppose that we have a directed family $\alpha : I \to \mathcal{V}\text{-Ind}(X)$ with $I : \mathcal{V}$. Then each α_i is a directed family in X indexed by a type $J_i : \mathcal{V}$. We define the family $\hat{\alpha} : (\Sigma_{i:I} J_i) \to X$ by $(i, j) \mapsto \alpha_i(j)$. It is clear that $\hat{\alpha}$ exceeds each α_i, and that β exceeds $\hat{\alpha}$ if β exceeds every α_i. So it remains to show that $\hat{\alpha}$ is in fact an element of $\mathcal{V}$-Ind(X), i.e. that it is directed. Because α and each α_i are directed, I and each J_i are inhabited. Hence, so is the domain of $\hat{\alpha}$. It remains to show that $\hat{\alpha}$ is semidirected. Suppose we have $(i_1, j_1), (i_2, j_2)$ in the domain of $\hat{\alpha}$. By directedness of α, there exists $i : I$ such that α_i exceeds both α_{i_1} and α_{i_2}. Hence, there exist $j_1', j_2' : J_i$ with $\alpha_{i_1}(j_1) \sqsubseteq \alpha_i(j_1')$ and $\alpha_{i_2}(j_2) \sqsubseteq \alpha_i(j_2')$. Because the family α_i is directed in X, there exists $j : J_i$ such that $\alpha_i(j_1'), \alpha_i(j_2') \sqsubseteq \alpha_i(j)$. Hence, we conclude that $\hat{\alpha}(i_1, j_1) \equiv \alpha_{i_1}(j_1) \sqsubseteq \alpha_i(j_1') \sqsubseteq \alpha_i(j) \equiv \hat{\alpha}(i, j)$, and similarly for (i_2, j_2), which proves semidirectedness of $\hat{\alpha}$. ☐

Lemma 5.3. *Taking directed suprema defines a monotone map from a $\mathcal{V}$-dcpo to its $\mathcal{V}$-ind-completion, denoted by $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$.*

Proof. We have to show that $\bigsqcup \alpha \sqsubseteq \bigsqcup \beta$ for directed families α and β such that β exceeds α. Note that the inequality holds as soon as $\alpha_i \sqsubseteq \bigsqcup \beta$ for every i in the domain of α. For this, it suffices that for every such i, there exists a j in the domain of β such that $\alpha_i \sqsubseteq \beta_j$. But the latter says exactly that β exceeds α. ☐

Johnstone and Joyal [24] generalise the notion of continuity from posets to categories and do so by phrasing it in terms of $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ having a left adjoint. We follow their approach and now work towards this. It turns out to be convenient to use the following two notions, which are in fact equivalent by Lemma 5.7:

Definition 5.4 (Approximate, left adjunct)**.** For an element x of a dcpo D and a directed family $\alpha : I \to D$, we say that

(i) α **approximates** x if the supremum of α is x and each α_i is way below x, and

(ii) α is **left adjunct** to x if $\alpha \lesssim \beta \iff x \sqsubseteq \bigsqcup \beta$ for every directed family β.

Remark 5.5. For a $\mathcal{V}$-dcpo D, a function $L : D \to \mathcal{V}\text{-Ind}(D)$ is a left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ precisely when $L(x)$ is left adjunct to x for every $x : D$. Of course, we need to know that L is monotone and this is shown in the next lemma.

Lemma 5.6. *A function $L : D \to \mathcal{V}\text{-Ind}(D)$ is monotone and order-reflecting if $L(x)$ is left adjunct to x for every $x : D$.*

Proof. Suppose we are given elements $x, y : D$. By assumption, we know that $L(x) \lesssim L(y) \iff x \sqsubseteq \bigsqcup L(y)$, but $L(y)$ approximates y, so $\bigsqcup L(y) = y$ and hence $L(x) \lesssim L(y) \iff x \sqsubseteq y$, so L preserves and reflects the order. □

Lemma 5.7. *A directed family α approximates an element x if and only if it is left adjunct to it.*

Proof. Suppose first that α approximates x. If $\alpha \lesssim \beta$, then $x = \bigsqcup \alpha \sqsubseteq \bigsqcup \beta$, by Lemma 5.3. Conversely, if $x \sqsubseteq \bigsqcup \beta$, then β exceeds α: for if i is in the domain of α, then $\alpha_i \ll x$, so there exists j such that $\alpha_i \sqsubseteq \beta_j$ already.

In the other direction, suppose that α is left adjunct to x. We show that each α_i is way below x. If β is a directed family with $x \sqsubseteq \bigsqcup \beta$, then β exceeds α as α is assumed to be left adjunct to x. Hence, for every i, there exists j with $\alpha_i \sqsubseteq \beta_j$, proving that $\alpha_i \ll x$. Since α exceeds itself, we get $x \sqsubseteq \bigsqcup \alpha$ by assumption. For the other inequality, we note that $x \sqsubseteq \bigsqcup \hat{x}$, where $\hat{x} : \mathbf{1} \to D$ is the directed family that maps to x. Hence, as α is left adjunct to x, we must have that $\hat{x}$ exceeds α, which means that each α_i is below x. Thus, $\bigsqcup \alpha \sqsubseteq x$ and $x = \bigsqcup \alpha$ hold, as desired. □

Proposition 5.8. *For a $\mathcal{V}$-dcpo D, a function $L : D \to \mathcal{V}\text{-Ind}(D)$ is a left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ if and only if $L(x)$ approximates x for every $x : D$.*

Proof. Immediate from Lemma 5.7 and Remark 5.5. □

6. Continuous and algebraic dcpos

Using the ind-completion from the previous section, we turn to defining continuous and algebraic dcpos, paying special attention to size and constructivity issues regarding the axiom of choice. This second issue is discussed in Section 6.2 from two perspectives: type-theoretically, via a discussion on the placement of the propositional truncation, and categorically, via left adjoints.

6.1. Continuous dcpos

We define what it means for a $\mathcal{V}$-dcpo to be continuous and prove the fundamental interpolation properties for the way-below relation. Examples are postponed (see Sections 7.2 and 8.2) until we have developed the theory further.

Definition 6.1 (Continuity data, I_x, α_x)**.** **Continuity data** for a $\mathcal{V}$-dcpo D assigns to every $x : D$ a type $I_x : \mathcal{V}$ and a directed **approximating family** $\alpha_x : I \to D$ such that α_x has supremum x and each $\alpha_x(i)$ is way below x.

The notion of continuity data can be understood categorically as follows.

Proposition 6.2. *The type of continuity data for a $\mathcal{V}$-dcpo D is equivalent to the type of left adjoints to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$.*

Proof. Given continuity data for D, we define a function $D \to \mathcal{V}\text{-Ind}(D)$ by sending $x : D$ to the directed family α_x. This is indeed a left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ because of Proposition 5.8. Conversely, given a left adjoint $L : D \to \mathcal{V}\text{-Ind}(D)$, the assignment $x \mapsto L(x)$ is continuity data for D, again by Proposition 5.8. That these two mapping make up a type equivalence can be checked directly, using that the type expressing that a map is a left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ is a proposition. ☐

Remark 6.3. It should be noted that having continuity data is *not* property of a dcpo, i.e., the type of continuity data for a dcpo is not a subsingleton. Indeed an element $x : D$ can have several different approximating families, e.g. if $\alpha : I \to D$ approximates x, then so does $[\alpha, \alpha] : (I + I) \to D$. In other words, the left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ is not unique, although it is unique up to isomorphism (of the preorder $\mathcal{V}\text{-Ind}(D)$). In category theory this is typically sufficient, and often the best one can do. Johnstone and Joyal follow this philosophy in [24], but we want the type of continuous $\mathcal{V}$-dcpos to be a subtype of the $\mathcal{V}$-dcpos. One reason that property is preferred is that we only get a univalent category [3] of continuous dcpos if we consider maps that preserve imposed structure. In the case of continuity data, this would imply preservation of the way-below relation, but this excludes many Scott continuous maps, e.g. if the value of a constant map is not compact, then the map does not preserve the way-below relation.

It is natural to ask whether the univalence axiom can be used to identify these isomorphic objects. However, this is not the case because the ind-completion $\mathcal{V}\text{-Ind}(D)$ is *not* a univalent category in the sense of [3], as it is a preorder and not a poset. One way to obtain a subtype is to propositionally truncate the notion of continuity data and this is indeed the approach that we will take. However, another choice that would yield a property is to identify isomorphic elements of $\mathcal{V}\text{-Ind}(D)$. This approach is discussed at length in Section 6.2 and in particular it is explained to be inadequate in a constructive setting.

Definition 6.4 (Continuity of a dcpo)**.** A $\mathcal{V}$-dcpo is **continuous** if it has unspecified continuity data, i.e. if the type of continuity data is inhabited.

Thus, a dcpo is continuous if we have an *unspecified* function assigning an approximating family to every element of the dcpo.

Proposition 6.5. *Assuming univalence, the identity type of two continuous $\mathcal{V}$-dcpos D and E is equivalent to the type of dcpo isomorphisms from D to E.*

Proof. By an application of the structure identity principle [12, Section 33.14], the identity type of D and E is equivalent to the type of order preserving and reflecting equivalences from D to E. But it is straightforward to show that an order preserving and reflecting equivalence is precisely a dcpo isomorphism. ☐

If we are working with small dcpos, which requires propositional resizing, then it is possible to extract continuity data from knowing that a dcpo is continuous because we can simply consider all elements way below a given element, see [8].

Proposition 6.6. *Continuity of a $\mathcal{V}$-dcpo D is equivalent to having an unspecified left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$.*

Proof. By Proposition 6.2 and functoriality of the propositional truncation. ☐

Lemma 6.7. *For elements x and y of a dcpo with continuity data, the following are equivalent:*

(i) $x \sqsubseteq y$;

(ii) $\alpha_x(i) \sqsubseteq y$ *for every* $i : I_x$;

(iii) $\alpha_x(i) \ll y$ *for every* $i : I_x$.

Proof. Note that (iii) implies (ii) and (ii) implies (i), because if $\alpha_x(i) \sqsubseteq y$ for every $i : I_x$, then $x = \bigsqcup \alpha_x \sqsubseteq y$, as desired. So it remains to prove that (i) implies (iii), but this holds, because $\alpha_x(i) \ll x$ for every $i : I_x$. ☐

Lemma 6.8. *For elements x and y of a dcpo with continuity data, x is way below y if and only if there exists $i : I_y$ such that $x \sqsubseteq \alpha_y(i)$.*

Proof. The left-to-right implication holds, because α_y is a directed family with supremum y, while the converse holds because $\alpha_y(i) \ll y$ for every $i : I_y$. ☐

We now prove three interpolation lemmas for dcpos with continuity data. Because the conclusions of the lemmas are propositions, the results will follow for continuous dcpos immediately.

Lemma 6.9 (Nullary interpolation for the way-below relation)**.** *For every $x : D$ of a continuous dcpo D, there exists $y : D$ such that $y \ll x$.*

Proof. The approximating family α_x is directed, so there exists $i : I_x$ and hence we can take $y :\equiv \alpha_x(i)$ since $\alpha_x(i) \ll x$. ☐

Although there are constructive proofs of the following in the literature, e.g. [1, Proposition 2.12], they are impredicative. Instead, we develop a predicative proof inspired by [24, Proposition 2.12].

Lemma 6.10 (Unary interpolation for the way-below relation)**.** *If $x \ll y$ in a continuous dcpo D, then there exists an* interpolant *$d : D$ such that $x \ll d \ll y$.*

Proof. Since we are proving a proposition, we may assume to be given continuity data for D. Thus, we can approximate every approximant $\alpha_y(i)$ of y by an approximating family $\beta_i : J_i \to D$. This defines a map $\hat{\beta}$ from I_y to $\mathcal{V}\text{-Ind}(D)$, the ind-completion of the $\mathcal{V}$-dcpo D, by sending $i : I_y$ to the directed family β_i. We claim that $\hat{\beta}$ is directed in $\mathcal{V}\text{-Ind}(D)$. Since α_y is directed, I_y is inhabited, so it remains to prove that $\hat{\beta}$ is semidirected. So suppose we have $i_1, i_2 : I_y$.

Because α_y is semidirected, there exists $i : I_y$ such that $\alpha_y(i_1), \alpha_y(i_2) \sqsubseteq \alpha_y(i)$. We claim that β_i exceeds β_{i_1} and β_{i_2}, which would prove semidirectedness of $\hat{\beta}$. We give the argument for i_1 only as the case for i_2 is completely analogous. We have to show that for every $j : J_{i_1}$, there exists $j' : J_i$ such that $\beta_{i_1}(j) \sqsubseteq \beta_i(j')$. But this holds because $\beta_{i_1}(j) \ll \bigsqcup \beta_i$ for every such j, as we have $\beta_{i_1}(j) \ll \alpha_y(i_1) \sqsubseteq \alpha_y(i) \sqsubseteq \bigsqcup \beta_i$.

Thus, $\hat{\beta}$ is directed in $\mathcal{V}$-Ind(D) and hence we can calculate its supremum in $\mathcal{V}$-Ind(D) to obtain the *directed* family $\gamma : (\Sigma_{i:I} J_i) \to D$ given by $(i, j) \mapsto \beta_i(j)$.

We now show that y is below the supremum of γ. By Lemma 6.7, it suffices to prove that $\alpha_y(i) \sqsubseteq \bigsqcup \gamma$ for every $i : I_y$, and, in turn, to prove this for an $i : I_y$ it suffices to prove that $\beta_i(j) \sqsubseteq \bigsqcup \gamma$ for every $j : J_i$. But this is immediate from the definition of γ. Thus, $y \sqsubseteq \bigsqcup \gamma$. Because $x \ll y$, there exists $(i, j) : \Sigma_{i:I} J_i$ such that $x \sqsubseteq \gamma(i, j) \equiv \beta_i(j)$.

Finally, for our interpolant, we take $d :\equiv \alpha_y(i)$. Then, indeed, $x \ll d \ll y$, because $x \sqsubseteq \beta_i(j) \ll \alpha_y(i) \equiv d$ and $d \equiv \alpha_y(i) \ll y$, completing the proof. □

The proof of the following is a straightforward application of unary interpolation as in the classical case.

Lemma 6.11 (Binary interpolation for the way-below relation)**.** *If $x \ll z$ and $y \ll z$ in a continuous dcpo D, then there exists an interpolant $d : D$ such that $x, y \ll d$ and $d \ll z$.*

Continuous dcpos are closed under retracts. Keeping track of universes, it holds in the following generality, where we recall (Definition 3.7) that we write $\mathcal{V}$-DCPO$_{\mathcal{U},\mathcal{T}}$ for the type of $\mathcal{V}$-dcpos with carriers in $\mathcal{U}$ and order relations taking values in $\mathcal{T}$.

Theorem 6.12. *If we have dcpos $D : \mathcal{V}$-DCPO$_{\mathcal{U},\mathcal{T}}$ and $E : \mathcal{V}$-DCPO$_{\mathcal{U}',\mathcal{T}'}$ such that D is a retract of E, then D is continuous if E is. Moreover, we can give continuity data for D if we have such data for E.*

Proof. We prove the result in case we are given continuity data for E, as the other will follow from that and the fact that the propositional truncation is functorial. So suppose that we have a Scott continuous section $s : D \to E$ and retraction $r : E \to D$. We claim that for every $x : D$, the family $r \circ \alpha_{s(x)}$ approximates x. Firstly, it is directed, because $\alpha_{s(x)}$ is and r is Scott continuous. Secondly,

$$\begin{aligned}
\bigsqcup r \circ \alpha_{s(x)} &= r\big(\bigsqcup \alpha_{s(x)}\big) && \text{(by Scott continuity of } r) \\
&= r(s(x)) && \text{(as } \alpha_{s(x)} \text{ is the approximating family of } s(x)) \\
&= x && \text{(because } s \text{ is a section of } r),
\end{aligned}$$

so the supremum of $r \circ \alpha_{s(x)}$ is x. Finally, we must prove that $r\big(\alpha_{s(x)}(i)\big) \ll x$ for every $i : I_x$. By Lemma 4.14, this is implied by $\alpha_{s(x)}(i) \ll s(x)$, which holds as $\alpha_{s(x)}$ is the approximating family of $s(x)$. □

Recall from Definition 3.9 that a $\mathcal{V}$-dcpo is locally small if $x \sqsubseteq y$ is equivalent to a type in $\mathcal{V}$ for all elements x and y.

Proposition 6.13. *A continuous dcpo is locally small if and only if its way-below relation has small truth values.*

Proof. By Lemmas 6.7 and 6.8, we have

$$x \sqsubseteq y \iff \forall_{i:I_x}(\alpha_x(i) \ll y) \quad \text{and} \quad x \ll y \iff \exists_{i:I_y}(x \sqsubseteq \alpha_y(i)),$$

for every two elements x and y of a dcpo with continuity data. But the types I_x and I_y are small, finishing the proof. The result also holds for continuous dcpos, because what we are proving is a proposition. ☐

Proposition 6.13 is significant because the definition of the way-below relation for a $\mathcal{V}$-dcpo D quantifies over all families into D indexed by types in $\mathcal{V}$.

6.2. Pseudocontinuity

In light of Proposition 6.2, we see that a $\mathcal{V}$-dcpo D can have continuity data in more than one way: the map $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ can have two left adjoints L_1, L_2 such that for some $x : D$, the directed families $L_1(x)$ and $L_2(x)$ exceed each other, yet are unequal. In order for the left adjoint to be truly unique (and not just up to isomorphism), the preorder $\mathcal{V}\text{-Ind}(D)$ would have to identify families that exceed each other. Of course, we could enforce this identification by passing to the poset reflection $\mathcal{V}\text{-Ind}(D)/{\approx}$ of $\mathcal{V}\text{-Ind}(D)$ and this section studies exactly that.

Another perspective on the situation is the following: The type-theoretic definition of having continuity data for a $\mathcal{V}$-dcpo D is of the form $\Pi_{x:D}\Sigma_{I:\mathcal{V}}\Sigma_{\alpha:I\to D}\ldots$, while continuity is defined as its propositional truncation $\|\Pi_{x:D}\Sigma_{I:\mathcal{V}}\Sigma_{\alpha:I\to D}\ldots\|$. Yet another way to obtain a property is by putting the propositional truncation on the *inside* instead: $\Pi_{x:D}\|\Sigma_{I:\mathcal{V}}\Sigma_{\alpha:I\to D}\ldots\|$. We study what this amounts to and how it relates to continuity and the poset reflection. Our results are summarised in Table 1 below.

Definition 6.14 (Pseudocontinuity)**.** A $\mathcal{V}$-dcpo D is **pseudocontinuous** if for every $x : D$ there exists an unspecified directed family that approximates x.

Notice that continuity data $\Rightarrow$ continuity $\Rightarrow$ pseudocontinuity, but reversing the first implication is an instance of global choice [12, Section 3.35.6], while reversing the second amounts to an instance of the axiom of choice that we do not expect to be provable in our constructive setting. We further discuss this point in Remark 6.16.

For a $\mathcal{V}$-dcpo D, the map $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$ is monotone, so it induces a unique monotone map $\bigsqcup_{\approx} : \mathcal{V}\text{-Ind}(D)/{\approx} \to D$ such that the diagram

$$\begin{tikzcd} \mathcal{V}\text{-Ind}(D)/{\approx} \arrow[rr, "\bigsqcup_{\approx}"] \arrow[dr, "{[-]}"'] & & D \\ & \mathcal{V}\text{-Ind}(D) \arrow[ur, "\bigsqcup"'] & \end{tikzcd} \tag{4}$$

commutes.

Proposition 6.15. *A $\mathcal{V}$-dcpo D is pseudocontinuous if and only if the map of posets $\bigsqcup_{\approx} : \mathcal{V}\text{-Ind}(D)/{\approx} \to D$ has a necessarily unique left adjoint.*

Observe that the type of left adjoints to $\bigsqcup_{\approx} : \mathcal{V}\text{-Ind}(D)/{\approx} \to D$ is a proposition, precisely because $\mathcal{V}\text{-Ind}(D)/{\approx}$ is a poset, cf. [3, Lemma 5.2], and hence the above uniqueness condition amounts to the contractibility of the type of left adjoints.

Proof. Suppose that $\bigsqcup_{\approx} : \mathcal{V}\text{-Ind}(D)/{\approx} \to D$ has a left adjoint L and let $x : D$ be arbitrary. We have to prove that there exists a directed family $\alpha : I \to D$ that approximates x. By surjectivity of the universal map $[-]$, there exists a directed family $\alpha : I \to D$ such that $L(x) = [\alpha]$. Moreover, α approximates x by virtue of Lemma 5.7, since for every $\beta : \mathcal{V}\text{-Ind}(D)$, we have

$$\begin{aligned} \alpha \lesssim \beta &\iff L(x) \leq [\beta] && \text{(since } L(x) = [\alpha]) \\ &\iff x \sqsubseteq \textstyle\bigsqcup_{\approx} [\beta] && \text{(since } L \text{ is a left adjoint to } \textstyle\bigsqcup_{\approx}) \\ &\iff x \sqsubseteq \textstyle\bigsqcup \beta && \text{(by Equation (4)).} \end{aligned}$$

The converse is more involved and we apply Lemma 2.1 which we recall says that every constant map with values in a set factors through the propositional truncation of its domain. Assume that D is pseudocontinuous. We start by constructing the left adjoint, so let $x : D$ be arbitrary. Writing $\mathcal{A}_x$ for the type of directed families that approximate x, we have an obvious map $\varphi_x : \mathcal{A}_x \to \mathcal{V}\text{-Ind}(D)$ that forgets that the directed family approximates x.

We claim that all elements in the image of φ_x exceed each other. For if α and β are directed families both approximating x, then for every i in the domain of α we know that $\alpha_i \ll x = \bigsqcup \beta$, so that there exists j with $\alpha_i \sqsubseteq \beta_j$. Hence, passing to the poset reflection, the composite $[-] \circ \varphi_x$ is constant. Thus, by Lemma 2.1 we have a (necessarily unique) map ψ_x making the diagram

$$\begin{tikzcd} \mathcal{A}_x \arrow[rr, "{[-] \circ \varphi_x}"] \arrow[dr, "|-|"'] & & \mathcal{V}\text{-Ind}(D)/{\approx} \\ & \|\mathcal{A}_x\| \arrow[ur, dashed, "\psi_x"'] & \end{tikzcd}$$

commute. Since D is assumed to be pseudocontinuous, we have that $\|\mathcal{A}_x\|$ holds for every $x : D$, so together with ψ_x this defines a map $L : D \to \mathcal{V}\text{-Ind}(D)/{\approx}$ by $L(x) :\equiv \psi_x(p)$, where p witnesses pseudocontinuity at x.

Lastly, we prove that L is indeed a left adjoint to $\bigsqcup_{\approx}$. So let $x : D$ be arbitrary. Since we're proving a property, we can use pseudocontinuity at x to specify a directed family α that approximates x. We now have to prove $[\alpha] \leq \beta' \iff x \sqsubseteq \bigsqcup_{\approx} \beta'$ for every $\beta' : \mathcal{V}\text{-Ind}(D)/{\approx}$. This is a proposition, so using quotient induction, it suffices to prove $[\alpha] \leq [\beta] \iff x \sqsubseteq \bigsqcup_{\approx} [\beta]$ for every

$\beta : \mathcal{V}\text{-Ind}(D)$. Indeed, for such β we have

$$\begin{aligned}
[\alpha] \leq [\beta] &\iff \alpha \lesssim \beta \\
&\iff x \sqsubseteq \textstyle\bigsqcup \beta \qquad \text{(by Lemma 5.7 and because } \alpha \text{ approximates } x\text{)} \\
&\iff x \sqsubseteq \textstyle\bigsqcup_{\approx} [\beta] \quad \text{(by Equation (4))},
\end{aligned}$$

finishing the proof. □

Thus, the explicit type-theoretic formulation and the formulation using left adjoints in each row of Table 1 (which summarises our findings) are equivalent.

	Type-theoretic formulation	Formulation with adjoints	Prop.
Cont. data	$\Pi_{x:D}\Sigma_{I:\mathcal{V}}\Sigma_{\alpha:I\to D}\,\delta(\alpha, x)$	Specified left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$	×
Continuity	$\lVert\Pi_{x:D}\Sigma_{I:\mathcal{V}}\Sigma_{\alpha:I\to D}\,\delta(\alpha, x)\rVert$	Unspecified left adjoint to $\bigsqcup : \mathcal{V}\text{-Ind}(D) \to D$	✓
Pseudocont.	$\Pi_{x:D}\lVert\Sigma_{I:\mathcal{V}}\Sigma_{\alpha:I\to D}\,\delta(\alpha, x)\rVert$	Specified left adjoint to $\bigsqcup_{\approx} : \mathcal{V}\text{-Ind}(D)/{\approx} \to D$	✓

Table 1: Continuity (data) and pseudocontinuity of a dcpo D, where $\delta(\alpha, x)$ abbreviates that α is directed and approximates x.

Remark 6.16. The issue with pseudocontinuity is that taking the quotient introduces a dependence on instances of the axiom of choice when it comes to proving properties of pseudocontinuous dcpos. An illustrative example is the proof of unary interpolation (Lemma 6.10), where we used the continuity data to first approximate an element y by α_y and then, in turn, approximate every approximant $\alpha_y(i)$. With pseudocontinuity this argument would require *choosing* an approximating family for every i. Another example is that while the preorder $\mathcal{V}\text{-Ind}(D)$ is $\mathcal{V}$-directed complete, a direct lifting of the proof of this fact to the poset reflection $\mathcal{V}\text{-Ind}(D)/{\approx}$ requires the axiom of choice. Hence, the Rezk completion [3], of which the poset reflection is a special case, does not necessarily preserve (filtered) colimits. The same issues concerning the axiom of choice occur in [24, pp. 260–261] and the notion of continuity data follows their solution precisely. We then truncate this to get a property of dcpos (recall Remark 6.3), resulting in our definition of a continuity.

6.3. Algebraic dcpos

Many of our examples of dcpos are not just continuous, but satisfy the stronger condition of being algebraic, meaning their elements can be approximated by compact elements only.

Definition 6.17 (Algebraicity data, I_x, κ_x)**. Algebraicity data** for a $\mathcal{V}$-dcpo D assigns to every $x : D$ a type $I_x : \mathcal{V}$ and a directed family $\kappa_x : I \to D$ of **compact** elements such that κ_x has supremum x.

Definition 6.18 (Algebraicity)**.** A $\mathcal{V}$-dcpo is **algebraic** if it has some unspecified algebraicity data, i.e. if the type of algebraicity data is inhabited.

Lemma 6.19. *Every algebraic dcpo is continuous.*

Proof. We prove that algebraicity data for a dcpo yields continuity data. The claim for algebraic and continuous then follows by functoriality of the propositional truncation. It suffices to prove that $\kappa_x(i) \ll x$ for every $i : I_x$. By assumption, $\kappa_x(i)$ is compact and has supremum x. Hence, $\kappa_x(i) \ll \kappa_x(i) \sqsubseteq \bigsqcup \kappa_x = x$, so $\kappa_x(i) \ll x$. $\square$

7. Small bases

Recall that the traditional, set-theoretic definition of a dcpo D being continuous says that for every element $x \in D$, the subset $\{y \in D \mid y \ll x\}$ is directed with supremum x. As explained in the introduction, the problem with this definition in a predicative context is that the subset $\{y \in D \mid y \ll x\}$ is not small in general. But, as is well-known in domain theory, it is sufficient (and in fact equivalent) to instead ask that D has a subset B, known as a *basis*, such that the subset $\{b \in B \mid b \ll x\} \subseteq B$ is directed with supremum x, see [1, Section 2.2.2] and [17, Definition III-4.1]. The idea developed in this section is that in many examples we can find a *small* basis giving us a predicative handle on the situation.

If a dcpo has a small basis, then it is continuous. In fact, all our running examples of continuous dcpos are actually examples of dcpos with small bases. Moreover, dcpos with small bases are better behaved. For example, they are all locally small and so are their exponentials, which also have small bases (Section 9.2). In Section 8.3 we also show that having a small basis is equivalent to being presented by ideals.

Following the flow of Section 6, we first consider small bases for continuous dcpos, before turning to small compact bases for algebraic dcpos (Section 7.1). After presenting examples of dcpos with small compact bases in Section 7.2, we describe the canonical small compact basis for an algebraic dcpo and the role that the univalence axiom and a set replacement principle play in Section 7.3.

Definition 7.1 (Small basis)**.** For a $\mathcal{V}$-dcpo D, a map $\beta : B \to D$ with $B : \mathcal{V}$ is a **small basis** for D if the following conditions hold:

(i) for every $x : D$, the family $(\Sigma_{b:B}(\beta(b) \ll x)) \xrightarrow{\beta \circ \mathrm{pr}_1} D$ is directed and has supremum x;

(ii) for every $x : D$ and $b : B$, the proposition $\beta(b) \ll x$ is $\mathcal{V}$-small.

We will write $\downarrow_\beta x$ for the type $\Sigma_{b:B}(\beta(b) \ll x)$ and conflate this type with the canonical map $\downarrow_\beta x \xrightarrow{\beta \circ \mathrm{pr}_1} D$.

Item (ii) ensures not only that the type $\Sigma_{b:B}(\beta(b) \ll x)$ is $\mathcal{V}$-small, but also that a dcpo with a small basis is locally small (Proposition 7.5).

Remark 7.2. If $\beta : B \to D$ is a small basis for a $\mathcal{V}$-dcpo D, then the type $\downarrow_\beta x$ is small. Hence, we have a type $I : \mathcal{V}$ and an equivalence $\varphi : I \simeq \downarrow_\beta x$ and we see that the family $I \xrightarrow{\varphi} \downarrow_\beta x \xrightarrow{\beta \circ \mathrm{pr}_1} D$ is directed and has the same supremum as $\downarrow_\beta x \to D$. We will use this tacitly and write as if the type $\downarrow_\beta x$ is actually a type in $\mathcal{V}$.

Lemma 7.3. *If a dcpo comes equipped with a small basis, then it can be equipped with continuity data. Hence, if a dcpo has an unspecified small basis, then it is continuous.*

Proof. For every element x of a dcpo D, the family $\downarrow_\beta x \to D$ approximates x, so the assignment $x \mapsto \downarrow_\beta x$ is continuity data for D. $\square$

Lemma 7.4. *In a dcpo D with a small basis $\beta : B \to D$, we have $x \sqsubseteq y$ if and only if $\forall_{b:B}(\beta(b) \ll x \to \beta(b) \ll y)$ for every $x, y : D$.*

Proof. If $x \sqsubseteq y$ and $\beta(b) \ll x$, then $\beta(b) \ll y$, so the left-to-right implication is clear. For the converse, suppose that the condition of the lemma holds. Because $x = \bigsqcup \downarrow_\beta x$, the inequality $x \sqsubseteq y$ holds as soon as $\beta(b) \sqsubseteq y$ for every $b : B$ with $\beta(b) \ll x$, but this is implied by the condition. $\square$

Proposition 7.5. *A dcpo with a small basis is locally small. Moreover, the way-below relation on all of the dcpo has small values.*

Proof. The first claim follows from Lemma 7.4 and the second follows from the first and Proposition 6.13. $\square$

A notable feature of dcpos with a small basis is that interpolants for the way-below relation, cf. Lemmas 6.9 to 6.11, can be found in the basis. Using Lemma 7.3 which constructs continuity data from a small basis, the proofs are as in the classical case.

Lemma 7.6 (Interpolation in the basis for the way-below relation)**.** *Suppose D is a dcpo with a small basis $\beta : B \to D$.*

(i) For every $x : D$, there exists $b : B$ with $\beta(b) \ll x$.

(ii) If $x \ll y$, then there exists an interpolant $b : B$ such that $x \ll \beta(b) \ll y$.

(iii) If $x \ll z$ and $y \ll z$, then there exists an interpolant $b : B$ such that $x, y \ll \beta(b) \ll z$.

Before proving the analogue of Theorem 6.12 (closure under retracts) for small bases, we need a type-theoretic analogue of [1, Proposition 2.2.4] and [17, Proposition III-4.2], which essentially says that it is sufficient for a "subset" of $\downarrow_\beta x$ (given by σ in the lemma) to be directed and have supremum x.

Lemma 7.7. *Suppose that we have an element x of a $\mathcal{V}$-dcpo D together with two maps $\beta : B \to D$ and $\sigma : I \to \Sigma_{b:B}(\beta(b) \ll x)$ with $I : \mathcal{V}$. If $\downarrow_\beta x \circ \sigma$ is directed and has supremum x, then $\downarrow_\beta x$ is directed with supremum x too.*

Proof. Suppose that $\downarrow_\beta x \circ \sigma$ is directed and has supremum x. Obviously, x is an upper bound for $\downarrow_\beta x$, so we are to prove that it is the least. If y is an upper bound for $\downarrow_\beta x$, then it is also an upper bound for $\downarrow_\beta x \circ \sigma$ which has supremum x, so that $x \sqsubseteq y$ follows. So the point is directedness of $\downarrow_\beta x$. Its domain is inhabited, because σ is directed. Now suppose that we have $b_1, b_2 : B$ with $\beta(b_1), \beta(b_2) \ll x$. Since $x = \bigsqcup\left(\downarrow_\beta x \circ \sigma\right)$, there exist $i_1, i_2 : I$ such that $\beta(b_1) \sqsubseteq \beta(\mathrm{pr}_1(\sigma(i_1)))$ and $\beta(b_2) \sqsubseteq \beta(\mathrm{pr}_1(\sigma(i_2)))$. Since $\downarrow_\beta x \circ \sigma$ is directed, there exists $i : I$ with $\beta(\mathrm{pr}_1(\sigma(i_1))), \beta(\mathrm{pr}_1(\sigma(i_2))) \sqsubseteq \beta(\mathrm{pr}_1(\sigma(i)))$. Hence, writing $b :\equiv \mathrm{pr}_1(\sigma(i))$, we have $\beta(b) \ll x$ and $\beta(b_1), \beta(b_2) \sqsubseteq \beta(b)$. Thus, $\downarrow_\beta x$ is directed, as desired. $\square$

Theorem 7.8. *If we have a retract $D \underset{r}{\overset{s}{\rightleftarrows}} E$ and a small basis $\beta : B \to E$ for E, then $r \circ \beta$ is a small basis for D.*

Proof. First of all, note that E is locally small by Proposition 7.5. But being locally small is closed under retracts by Lemma 3.20, so D is locally small too. Moreover, we have continuity data for D by virtue of Theorem 6.12 and Lemma 7.3. Hence, the way-below relation is small-valued by Proposition 6.13. In particular, the proposition $r(\beta(b)) \ll x$ is small for every $b : B$ and $x : D$.

We are going to use Lemma 7.7 to show that $\downarrow_{r \circ \beta} x$ is directed and has supremum x for every $x : D$. By Lemma 4.14, the identity on B induces a well-defined map $\sigma : (\Sigma_{b:B}(\beta(b) \ll s(x))) \to (\Sigma_{b:B}(r(\beta(b)) \ll y))$. Now Lemma 7.7 tells us that it suffices to prove that $r \circ \downarrow_\beta s(x)$ is directed with supremum x. But $\downarrow_\beta s(x)$ is directed with supremum x, so by Scott continuity of r, the family $r \circ \downarrow_\beta s(x)$ is directed with supremum $r(s(x)) = x$, as desired. $\square$

Finally, a useful property of dcpos with small bases is that they yield locally small exponentials, as we can restrict the quantification in the pointwise order to elements of the small basis.

Proposition 7.9. *If D is a dcpo with an unspecified small basis and E is a locally small dcpo, then the exponential E^D is locally small too.*

Proof. Being locally small is a proposition, so in proving the result we may assume that D comes equipped with a small basis $\beta : B \to D$. For arbitrary Scott continuous functions $f, g : D \to E$, we claim that $f \sqsubseteq g$ precisely when $\forall_{b:B}(f(\beta(b)) \sqsubseteq g(\beta(b)))$, which is a small type using that E is locally small. The left-to-right implication is obvious, so suppose that $f(\beta(b)) \sqsubseteq g(\beta(b))$ for every $b : B$ and let $x : D$ be arbitrary. We are to show that $f(x) \sqsubseteq g(x)$. Since $x = \bigsqcup \downarrow_\beta x$, it suffices to prove $f\left(\bigsqcup \downarrow_\beta x\right) \sqsubseteq g\left(\bigsqcup \downarrow_\beta x\right)$ and in turn, that $f(\beta(b)) \sqsubseteq g\left(\bigsqcup \downarrow_\beta x\right)$ for every $b : B$. But is easily seen to hold, because $f(\beta(b)) \sqsubseteq g(\beta(b))$ for every $b : B$ by assumption. $\square$

7.1. Small compact bases

Similarly to the progression from continuous dcpos (Section 6.1) to algebraic ones (Section 6.3), we now turn to small *compact* bases.

Definition 7.10 (Small compact basis)**.** For a $\mathcal{V}$-dcpo D, a map $\beta : B \to D$ with $B : \mathcal{V}$ is a **small compact basis** for D if the following conditions hold:

(i) for every $b : B$, the element $\beta(b)$ is compact in D;

(ii) for every $x : D$, the family $(\Sigma_{b:B}(\beta(b) \sqsubseteq x)) \xrightarrow{\beta \circ \mathrm{pr}_1} D$ is directed and has supremum x;

(iii) for every $x : D$ and $b : B$, the proposition $\beta(b) \sqsubseteq x$ is $\mathcal{V}$-small.

We will write $\downarrow_\beta x$ for the type $\Sigma_{b:B}(\beta(b) \sqsubseteq x)$ and conflate this type with the canonical map $\downarrow_\beta x \xrightarrow{\beta \circ \mathrm{pr}_1} D$.

Remark 7.11*.* If $\beta : B \to D$ is a small compact basis for a $\mathcal{V}$-dcpo D, then the type $\downarrow_\beta x$ is small. Similarly to Remark 7.2, we will use this tacitly and write as if the type $\downarrow_\beta x$ is actually a type in $\mathcal{V}$.

Lemma 7.12. *If a dcpo comes equipped with a small compact basis, then it can be equipped with algebraicity data. Hence, if a dcpo has an unspecified small compact basis, then it is algebraic.*

Proof. For every element x of a dcpo D with a small compact basis $\beta : B \to D$, the family $\downarrow_\beta x \to D$ consists of compact elements and approximates x, so the assignment $x \mapsto \downarrow_\beta x$ is algebraicity data for D. $\square$

Actually, with suitable assumptions, we can get canonical algebraicity data from an unspecified small compact basis, as discussed in detail in Section 7.3. This observation relies on Proposition 7.14 below and we prefer to present examples of dcpos with small compact bases first (Section 7.2).

Lemma 7.13. *A map $\beta : B \to D$ is a small compact basis for a dcpo D if and only if β is a small basis for D and $\beta(b)$ is compact for every $b : B$.*

Proof. If $\beta(b)$ is compact for every $b : B$, then $\beta(b) \sqsubseteq x$ if and only if $\beta(b) \ll x$ for every $b : B$ and $x : D$, so that $\twoheaddownarrow_\beta x \simeq \downarrow_\beta x$ for every $x : D$. In particular, $\twoheaddownarrow_\beta x$ approximates x if and only if $\downarrow_\beta x$ does, which completes the proof. $\square$

Proposition 7.14. *A small compact basis contains every compact element. That is, if $\beta : B \to D$ is a small compact basis for a dcpo D and $x : D$ is compact, then there exists $b : B$ such that $\beta(b) = x$.*

Proof. Suppose we have a compact element $x : D$. By compactness of x and the fact that $x = \downarrow_\beta x$, there exists $b : B$ with $\beta(b) \ll x$ such that $x \sqsubseteq \beta(b)$. But then $\beta(b) = x$ by antisymmetry. $\square$

Now that we have the theory of small bases we turn to examples illustrating small bases in practice. Our examples will involve small *compact* bases and an example of a dcpo with a small basis that is not compact will have to wait until Section 8.2 when we have developed the ideal completion.

Example 7.15. The map $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$ defined by $0 \mapsto \mathbf{0}_{\mathcal{U}}$ and $1 \mapsto \mathbf{1}_{\mathcal{U}}$ is a small compact basis for $\Omega_{\mathcal{U}}$. In particular, $\Omega_{\mathcal{U}}$ is algebraic.

The basis $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$ defined above has the special property that it is *dense* in the sense of [14, `TypeTopology.Density`]: its image has empty complement, i.e. the type $\Sigma_{P:\Omega_{\mathcal{U}}} \neg(\Sigma_{b:\mathbf{2}}\, \beta(b) = P)$ is empty.

Proof of Example 7.15. By Example 4.5, every element in the image of β is compact. Moreover, $\Omega_{\mathcal{U}}$ is locally small, so we only need to prove that for every $P : \Omega_{\mathcal{U}}$ the family $\downarrow_\beta P$ is directed with supremum P. The domain of the family is inhabited, because $\beta(0)$ is the least element. Semidirectedness also follows easily, since $\mathbf{2}$ has only two elements for which we have $\beta(0) \sqsubseteq \beta(1)$. Finally, the supremum of $\downarrow_\beta P$ is obviously below P. Conversely, if P holds, then $\bigsqcup \downarrow_\beta P = \mathbf{1} = P$. The final claim follows from Lemma 7.12. ☐

Example 7.16. For a set $X : \mathcal{U}$, the map $\beta : (\mathbf{1} + X) \to \mathcal{L}_{\mathcal{U}}(X)$ given by $\mathrm{inl}(\star) \mapsto \bot$ and $\mathrm{inr}(x) \mapsto \eta(x)$ is a small compact basis for $\mathcal{L}_{\mathcal{U}}(X)$. In particular, $\mathcal{L}_{\mathcal{U}}(X)$ is algebraic.

Similar to Example 7.15, the basis $\beta : (\mathbf{1} + X) \to \mathcal{L}_{\mathcal{U}}(X)$ defined above is also dense.

Proof of Example 7.16. By Example 4.6, every element in the image of β is compact. Moreover, the lifting is locally small, so we only need to prove that for every partial element l, the family $\downarrow_\beta l$ is directed with supremum l. The domain of the family is inhabited, because $\beta(\mathrm{inl}(\star))$ is the least element. Semidirectedness also follows easily: First of all, $\beta(\mathrm{inl}(\star))$ is the least element. Secondly, if we have $x, x' : X$ such that $\beta(\mathrm{inr}(x)), \beta(\mathrm{inr}(x')) \sqsubseteq l$, then because $\beta(\mathrm{inr}(x)) \equiv \eta(x)$ is defined, we must have $\beta(\mathrm{inr}(x)) = l = \beta(\mathrm{inr}(x'))$ by definition of the order. Finally, the supremum of $\downarrow_\beta l$ is obviously a partial element below l. Conversely, if l is defined, then $l = \eta(x)$ for some $x : X$, and hence, $l = \eta(x) \sqsubseteq \bigsqcup \downarrow_\beta l$. The final claim follows from Lemma 7.12. ☐

Example 7.17. For a set $X : \mathcal{U}$, the map $\beta : \mathrm{List}(X) \to \mathcal{P}_{\mathcal{U}}(X)$ from Definition 4.11 (whose image is the type of Kuratowski finite subsets of X) is a small compact basis for $\mathcal{P}_{\mathcal{U}}(X)$. In particular, $\mathcal{P}_{\mathcal{U}}(X)$ is algebraic.

Proof of Example 7.17. By Lemma 4.12 and Example 4.13, all elements in the image of β are compact. Moreover, $\mathcal{P}_{\mathcal{U}}(X)$ is locally small, so we only need to prove that for every $A : \mathcal{P}(X)$ the family $\downarrow_\beta A$ is directed with supremum A, but this was also proven in Example 4.13. The final claim follows from Lemma 7.12. ☐

At this point the reader may ask whether we have any examples of dcpos which can be equipped with algebraicity data but that do not have a small compact basis. The following example shows that this can happen in our predicative setting:

Example 7.18. The lifting $\mathcal{L}_{\mathcal{V}}(P)$ of a proposition $P : \mathcal{U}$ can be given algebraicity data, but has a small compact basis if and only if P is $\mathcal{V}$-small. Thus, requiring that $\mathcal{L}_{\mathcal{V}}(P)$ has a small basis for every proposition $P : \mathcal{U}$ is equivalent to the propositional resizing principle that every proposition in $\mathcal{U}$ is equivalent to one in $\mathcal{V}$.

Proof of Example 7.18. Note that $\mathcal{L}_{\mathcal{V}}(P)$ is simply the type of propositions in $\mathcal{V}$ that imply P. It has algebraicity data, because given such a proposition Q, the family

$$\begin{aligned} Q + \mathbf{1}_{\mathcal{V}} &\to \mathcal{L}_{\mathcal{V}}(P) \\ \mathrm{inl}(q) &\mapsto \mathbf{1}_{\mathcal{V}} \\ \mathrm{inr}(\star) &\mapsto \mathbf{0}_{\mathcal{V}} \end{aligned}$$

is directed, has supremum Q and consists of compact elements. But if $\mathcal{L}_{\mathcal{V}}(P)$ had a small compact basis $\beta : B \to \mathcal{L}_{\mathcal{V}}(P)$, then we would have $P \simeq \exists_{b:B}(\beta(b) \simeq \mathbf{1}_{\mathcal{V}})$ and the latter is $\mathcal{V}$-small. Conversely, if P is equivalent to $P_0 : \mathcal{V}$, then $\mathcal{L}_{\mathcal{V}}(P)$ is isomorphic to $\mathcal{L}_{\mathcal{V}}(P_0)$, which has a small compact basis by Example 7.16. □

Example 7.19. In classical mathematics, it is known [23, Proposition 2.6] that every well-ordered set C with a top element $\top$ is an algebraic lattice, and every compact element of it is equal to the least element or of the form $c + 1$ for some $c \in C \setminus \{\top\}$. The ordinals in univalent foundations, as introduced in [73, Section 10.3] and further developed by the second author in [15], give a constructive example of a large (even impredicatively) algebraic sup-lattice without a small basis. In [32, Theorem 5.8] we showed that the large poset of small ordinals has small suprema, so indeed the ordinals form a sup-lattice. (This result needs small set quotients.) Moreover, they cannot have a small basis, as otherwise we could take the supremum of all ordinals in the basis which would yield a greatest ordinal which does not exist (as a consequence of [73, Lemma 10.3.21]). It remains to show that the sup-lattice of ordinals is algebraic. This follows from the following two facts.

(i) Every successor ordinal, i.e. one of the form $\alpha + \mathbf{1}$, is compact.

(ii) Every ordinal α is the supremum of the family $x : \alpha \mapsto \alpha \downarrow x + \mathbf{1}$, where $\alpha \downarrow x$ denotes the ordinal of elements of α that are strictly less than x.

While the family in (ii) is not necessarily directed, this does not pose a problem, since we can take its directification (see Definition 9.10 later) by considering finite joins of elements in the family which are necessarily compact again by Lemma 4.7.

For proving these facts, we recall from [15, `Ordinals.OrdinalOfOrdinals`] that the order $\preceq$ on ordinals can be characterised as follows:

$$\alpha \preceq \beta \quad \iff \quad \forall_{x:\alpha} \exists_{y:\beta}\, \alpha \downarrow x = \beta \downarrow y.$$

We now prove (i): Suppose that $\alpha + \mathbf{1} \preceq \bigsqcup_{i:I} \beta_i$. Since $\alpha = (\alpha + \mathbf{1}) \downarrow \mathrm{inr}(\star)$, there exists $s : \bigsqcup_{i:I} \beta_i$ with $\alpha = \bigsqcup_{i:I} \beta_i \downarrow s$. By [33, Lemma 15] there exist $j : I$ and $b : \beta_j$ with $\bigsqcup_{i:I} \beta_i \downarrow s = \beta_j \downarrow b$. Hence, $\alpha = \beta_j \downarrow b$, but then it follows that $\alpha + \mathbf{1} \preceq \beta_j$.

To see that (ii) is true, we first notice that $\alpha \downarrow x + \mathbf{1} \preceq \alpha$ holds (using the characterisation of the partial order) so that α is an upper bound for the family. Now suppose that β is another upper bound. We need to show that $\alpha \preceq \beta$. So let $x : \alpha$ be arbitrary. Since β is an upper bound of the family, there is $b : \beta$ with $\alpha \downarrow x = (\alpha \downarrow x + \mathbf{1}) \downarrow \mathrm{inr}\, \star = \beta \downarrow b$, so we are done.

7.3. The canonical basis of compact elements

So far, our development of algebraic dcpos (with small compact bases) has resulted from specialising the treatment of continuous dcpos with small bases. In this section we take a closer look at the algebraic case.

Classically, the subset K of compact elements of an algebraic dcpo D forms a basis for D. In our predicative context, we only consider small bases, and a priori there is no reason for K to be a small type. However, if D comes equipped with a small compact basis, then set replacement implies that K is in fact small.

We recall the *set replacement* principle from [32, Definition 3.27]: it asserts that the image of a map $f : X \to Y$ is $\mathcal{U} \sqcup \mathcal{V}$-small if X is $\mathcal{U}$-small and Y is a locally $\mathcal{V}$-small set, where local smallness refers to smallness of the identity types. We also recall [32, Theorem 3.29] that set replacement is logically equivalent to the existence of small set quotients.

If we additionally assume univalence, then the relevant smallness condition is a property [32, Section 2.3], which means that having an unspecified small compact basis is sufficient.

In particular, with set replacement and univalence, we can show:

$$\begin{gathered}\|D \text{ has a specified small compact basis}\| \to \\ D \text{ has a specified small compact basis}\end{gathered}$$

Using the terminology of [39, Definition 3.9], we may thus say that the type expressing that D has a specified small compact basis has split support. This observation is due to Ayberk Tosun (private communication) who also formalised the result for spectrality in the context of locale theory in predicative univalent foundations [4, Theorem 4.17].

Lemma 7.20. *Every element x of an algebraic dcpo is the directed supremum of all compact elements below x.*

Proof. Writing D for the algebraic dcpo, and letting $x : D$ be arbitrary, we have to show that the inclusion family

$$\iota_x : (\Sigma_{c:D}(c \text{ is compact}) \times (c \sqsubseteq x)) \to D$$

is directed with supremum x. Since this is a proposition, we may assume to be given algebraicity data for D. Thus, we have a directed family $\kappa_x : I_x \to D$ of compact elements with supremum x. By directedness, I_x is inhabited, so we see that the domain of ι_x is inhabited too. For semidirectedness, assume we have compact elements $c_1, c_2 : D$ below x. Since x is the directed supremum of κ_x, there exist elements $i_1, i_2 : I_x$ with $c_1 \sqsubseteq \kappa_x(i_1)$ and $c_2 \sqsubseteq \kappa_x(i_2)$. By directedness of κ_x, there then exists $i : I_x$ such that c_1 and c_2 are both below $\kappa_x(i)$. But $\kappa_x(i)$ is a compact element below x so we are done. Finally, we show that ι_x has supremum x. Clearly, x is an upper bound for ι_x. Now suppose that y is any other upper bound. It then suffices to show that $\kappa_x(i) \sqsubseteq y$ for all $i : I_x$. But each $\kappa_x(i)$ is a compact element below x, so this holds since y is an upper bound for ι_x. □

Lemma 7.21. *Assuming set replacement, if a $\mathcal{V}$-dcpo is equipped with a small compact basis, then the subtype of compact elements is $\mathcal{V}$-small. If we additionally assume univalence, then having an unspecified small compact basis suffices.*

Proof. Let $\beta : B \to D$ be the small compact basis of the $\mathcal{V}$-dcpo D. Notice that β factors through the subtype K of compact elements of D. Moreover, by Proposition 7.14 the map $\beta : B \to K$ is surjective. Hence, K is equivalent to the image of $\beta : B \to D$. Now an application of set replacement finishes the proof, since B is small and K is locally small because it is a subtype of D which is locally small by antisymmetry, Proposition 7.5 and Lemma 7.13.

Assuming univalence, being small is a property [32, Proposition 2.8], so that the result follows from the above and the universal property of the truncation. □

Proposition 7.22. *Assuming univalence and set replacement, the types expressing that a dcpo has a specified, resp. unspecified, small compact basis are logically equivalent.*

Proof. In one direction, we simply apply the propositional truncation. In the other direction, we apply Lemmas 7.20 and 7.21 to see that

$$K_s \simeq K \hookrightarrow D$$

is a small compact basis for the $\mathcal{V}$-dcpo D, where K denotes the subtype of compact elements and K_s is the $\mathcal{V}$-small copy of K. □

We note that the above cannot be promoted to an equivalence of types, because the type of specified small compact bases is not a proposition. This may seem puzzling because there is a unique basis—*as a subset*—which consists of compact elements. If we had asked for the map $\beta : B \to D$ in the definition of a small compact basis to be an embedding, then (ignoring size issues for a moment) the resulting type is a proposition: it has a unique element in case the dcpo is algebraic, given by the *subset* of compact elements. (This is true because any basis must contain all compact elements.)

We illustrate why we do not impose this requirement by revisiting Example 7.17. This example showed that the canonical map from lists into the powerset of a set X is a small compact basis for the algebraic dcpo $\mathcal{P}(X)$. This map is not an embedding, as any permutation of a list will give rise to the same subset. If we insisted on having an embedding, we would instead have to use the inclusion of the Kuratowski finite subsets $\mathcal{K}(X)$ into $\mathcal{P}(X)$. However, $\mathcal{K}(X)$ is not a small type without additional assumptions, such as HITs or more specifically, set replacement (as $\mathcal{K}(X)$ is precisely the image of the inclusion of lists into the powerset).

Returning to the main line of thought, we conclude that, in the presence of set replacement and univalence, if there is some unspecified small compact basis, then the subset of compact elements is small.

8. The round ideal completion

We have seen that in continuous dcpos, the basis essentially "generates" the whole dcpo, because the basis suffices to approximate any of its elements. It is natural to ask whether one can start from a more abstract notion of basis and "complete" it to a continuous dcpo. Following Abramsky and Jung [1, Section 2.2.6], but keeping track of size, this is exactly what we do here using the notion of an *abstract basis* and the *round ideal completion*.

Definition 8.1 (Abstract basis)**.** An **abstract $\mathcal{V}$-basis** is a type $B : \mathcal{V}$ with a binary relation $\prec\, : B \to B \to \mathcal{V}$ that is proposition-valued, transitive and satisfies **nullary** and **binary interpolation**:

(i) for every $a : B$, there exists $b : B$ with $b \prec a$, and

(ii) for every $a_1, a_2 \prec b$, there exists $a : B$ with $a_1, a_2 \prec a \prec b$.

Definition 8.2 (Ideal, (round) ideal completion, $\mathcal{V}$-Idl$(B, \prec)$)**.**

(i) A subset $I : \mathcal{P}_{\mathcal{V}}(B)$ of an abstract $\mathcal{V}$-basis $(B, \prec)$ is a **$\mathcal{V}$-ideal** if it is a directed lower set with respect to $\prec$. That it is a lower set means: if $b \in I$ and $a \prec b$, then $a \in I$ too.

(ii) We write $\mathcal{V}$-Idl$(B, \prec)$ for the type of $\mathcal{V}$-ideals of an abstract $\mathcal{V}$-basis $(B, \prec)$ and call $\mathcal{V}$-Idl$(B, \prec)$ the **(round) ideal completion** of $(B, \prec)$.

For the remainder of this section, we will fix an abstract $\mathcal{V}$-basis $(B, \prec)$ and consider its $\mathcal{V}$-ideals. The name round ideal completion is justified by Lemma 8.4 below.

Definition 8.3 (Union of ideals, $\bigcup \mathcal{I}$)**.** Given a family $\mathcal{I} : S \to \mathcal{V}$-Idl$(B, \prec)$ of ideals, indexed by $S : \mathcal{V}$, we write

$$\bigcup \mathcal{I} :\equiv \{b \in B \mid \exists_{s:S}(b \in \mathcal{I}_s)\}.$$

The following lemma is proved just like in the classical case [1, Section 2.2.6].

Lemma 8.4.

(i) If $\mathcal{I} : S \to \mathcal{V}\text{-}\mathrm{Idl}(B, \prec)$ is directed, then $\bigcup \mathcal{I}$ is an ideal.

(ii) The round ideal completion is a $\mathcal{V}$-dcpo when ordered by subset inclusion. Paying attention to the universe levels, the ideals form a large but locally small $\mathcal{V}$-dcpo because $\mathcal{V}\text{-}\mathrm{Idl}(B, \prec) : \mathcal{V}\text{-}\mathrm{DCPO}_{\mathcal{V}^+,\mathcal{V}}$.

(iii) The ideals of an abstract basis are round*: for every element a of an ideal I, there exists $b \in I$ such that $a \prec b$.*

Roundness makes up for the fact that we have not required an abstract basis to be reflexive. If it is, then (Section 8.1) the ideal completion is algebraic.

Definition 8.5 (Principal ideal, $\downarrow b$)**.** The **principal ideal** of an element $b : B$ is defined as the subset $\downarrow b :\equiv \{a \in B \mid a \prec b\}$. Observe that the principal ideal is indeed an ideal: it is a lower set by transitivity of $\prec$, and inhabited and semidirected precisely by nullary and binary interpolation, respectively.

Lemma 8.6. *The assignment $b \mapsto \downarrow b$ is monotone, i.e. if $a \prec b$, then $\downarrow a \subseteq \downarrow b$.*

Proof. By transitivity of $\prec$. □

Lemma 8.7. *Every ideal is the directed supremum of its principal ideals. That is, for an ideal I, the family $(\Sigma_{b:B}(b \in I)) \xrightarrow{b \mapsto \downarrow b} \mathcal{V}\text{-}\mathrm{Idl}(B, \prec)$ is directed and has supremum I.*

Proof. Since ideals are lower sets, we have $\downarrow b \subseteq I$ for every $b \in I$. Hence, the union $\bigcup_{b \in I} \downarrow b$ is a subset of I. Conversely, if $a \in I$, then by roundness of I there exists $a' \in I$ with $a \prec a'$, so that $a \in \bigcup_{b \in I} \downarrow b$. So it remains to show that the family is directed. Notice that it is inhabited, because I is an ideal. Now suppose that $b_1, b_2 \in I$. Since I is directed, there exists $b \in I$ such that $b_1, b_2 \prec b$. But this implies $\downarrow b_1, \downarrow b_2 \subseteq \downarrow b$ by Lemma 8.6, so the family is semidirected, as desired. □

Lemma 8.8. *The following are equivalent for every two ideals I and J:*

(i) $I \ll J$;

(ii) there exists $b \in J$ such that $I \subseteq \downarrow b$;

(iii) there exist $a \prec b$ such that $I \subseteq \downarrow a \subseteq \downarrow b \subseteq J$.

In particular, if b is an element of an ideal I, then $\downarrow b \ll I$.

Proof. We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). So suppose that $I \ll J$. Then J is the directed supremum of its principal ideals by Lemma 8.7. Hence, there exists $b \in J$ such that $I \subseteq \downarrow b$ already, which is exactly (ii). Now suppose that we have $a \in J$ with $I \subseteq \downarrow a$. By roundness of J, there exists $b \in J$ with $a \prec b$. But then $I \subseteq \downarrow a \subseteq \downarrow b \subseteq J$ by Lemma 8.6 and the fact that J is a lower set, establishing (iii). Now suppose that condition (iii) holds and that J is a subset of some directed join of ideals $\mathcal{J}$ indexed by a type $S : \mathcal{V}$. Since $a \in {\downarrow} b \subseteq J$, there exists $s : S$ such that $a \in \mathcal{J}_s$. In particular, ${\downarrow} a \subseteq \mathcal{J}_s$ because ideals are lower sets. Hence, if $a' \in I \subseteq {\downarrow} a$, then $a' \in \mathcal{J}_s$, so $I \subseteq \mathcal{J}_s$, which proves that $I \ll J$.

Finally, if b is an element of an ideal I, then ${\downarrow} b \ll I$, because (ii) implies (i) and ${\downarrow} b \subseteq {\downarrow} b$ obviously holds. $\square$

Theorem 8.9. *The principal ideals* ${\downarrow}(-) : B \to \mathcal{V}\text{-Idl}(B, \prec)$ *yield a small basis for* $\mathcal{V}\text{-Idl}(B, \prec)$*. In particular,* $\mathcal{V}\text{-Idl}(B, \prec)$ *is continuous.*

Proof. First of all, note that the way-below relation on $\mathcal{V}\text{-Idl}(B, \prec)$ is small-valued because of Lemma 8.8. So it remains to show that for every ideal I, the family $(\Sigma_{b:B}({\downarrow} b \ll I)) \xrightarrow{b \mapsto {\downarrow} b} \mathcal{V}\text{-Idl}(B, \prec)$ is directed with supremum I. That the domain of this family is inhabited follows from Lemma 8.8 and the fact that I is inhabited. For semidirectedness, suppose we have $b_1, b_2 : B$ with ${\downarrow} b_1, {\downarrow} b_2 \ll I$. By Lemma 8.8 there exist $c_1, c_2 \in I$ such that ${\downarrow} b_1 \subseteq {\downarrow} c_1$ and ${\downarrow} b_2 \subseteq {\downarrow} c_2$. Since I is directed, there exists $b \in I$ with $c_1, c_2 \prec b$. But now ${\downarrow} b_1 \subseteq {\downarrow} c_1 \subseteq {\downarrow} b \ll I$ by Lemmas 8.6 and 8.8 and similarly, ${\downarrow} b_2 \subseteq {\downarrow} b \ll I$. Hence, the family is semidirected, as we wished to show. Finally, we show that I is the supremum of the family. If $b \in I$, then, since I is round, there exists $c \in I$ with $b \prec c$. Moreover, ${\downarrow} c \ll I$ by Lemma 8.8. Hence, b is included in the join of the family. Conversely, if we have $b : B$ with ${\downarrow} b \ll I$, then ${\downarrow} b \subseteq I$, so I is also an upper bound for the family. $\square$

8.1. The round ideal completion of a reflexive abstract basis

If the relation of an abstract basis is reflexive, then we obtain an algebraic dcpo, as we show now.

Lemma 8.10. *If* $\prec : B \to B \to \mathcal{V}$ *is proposition-valued, transitive and reflexive, then* $(B, \prec)$ *is an abstract basis.*

Proof. The interpolation properties for $\prec$ are easily proved when it is reflexive. $\square$

Lemma 8.11. *If an element* $b : B$ *is reflexive, i.e.* $b \prec b$ *holds, then* $b \in I$ *if and only if* ${\downarrow} b \subseteq I$ *for every ideal* I*.*

Proof. The left-to-right implication holds because I is a lower set and the converse holds because $b \in {\downarrow} b$ as b is assumed to be reflexive. $\square$

Lemma 8.12. *If* $b : B$ *is reflexive, then its principal ideal* ${\downarrow} b$ *is compact.*

Proof. Suppose that we have $b : B$ such that $b \prec b$ holds and that ${\downarrow} b \subseteq \bigcup \mathcal{I}$ for some directed family $\mathcal{I}$ of ideals. By Lemma 8.11, we have $b \in \bigcup \mathcal{I}$, which means that there exists s in the domain of $\mathcal{I}$ such that $b \in \mathcal{I}_s$. Using Lemma 8.11 once more, we see that ${\downarrow} b \subseteq \mathcal{I}_s$, proving that ${\downarrow} b$ is compact. $\square$

Theorem 8.13. *If $\prec$ is reflexive, then a small* compact *basis for $\mathcal{V}\text{-Idl}(B, \prec)$ is given by the principal ideals $\downarrow(-) : B \to \mathcal{V}\text{-Idl}(B, \prec)$. In particular, $\mathcal{V}\text{-Idl}(B, \prec)$ is algebraic.*

Proof. This follows from Theorem 8.9 and Lemmas 7.13 and 8.12. $\square$

Theorem 8.14. *If $f : B \to D$ is a monotone map to a $\mathcal{V}$-dcpo D, then the map $\bar{f} : \mathcal{V}\text{-Idl}(B, \prec) \to D$ defined by taking an ideal I to the supremum of the directed family $f \circ \mathrm{pr}_1 : (\Sigma_{b:B}(b \in I)) \to D$ is Scott continuous. Moreover, if $\prec$ is reflexive, then $\bar{f}$ is the unique Scott continuous map making the diagram*

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr, "\downarrow(-)"'] & & D \\
& \mathcal{V}\text{-Idl}(B, \prec) \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

commute.

Proof. Note that $f \circ \mathrm{pr}_1 : (\Sigma_{b:B}(b \in I)) \to D$ is indeed a directed family, because I is a directed subset of B and f is monotone. For Scott continuity of $\bar{f}$, assume that we have a directed family $\mathcal{I}$ of ideals indexed by $S : \mathcal{V}$. We first show that $\bar{f}(\bigcup \mathcal{I})$ is an upper bound of $\bar{f} \circ \mathcal{I}$. So let $s : S$ be arbitrary and note that $\bar{f}(\bigcup \mathcal{I}) \sqsupseteq \bar{f}(\mathcal{I}_s)$ as soon as $\bar{f}(\bigcup \mathcal{I}) \sqsupseteq f(b)$ for every $b \in \mathcal{I}_s$. But for such b we have $b \in \bigcup \mathcal{I}$, so this holds. Now suppose that y is an upper bound of $\bar{f} \circ \mathcal{I}$. To show that $\bar{f}(\bigcup \mathcal{I}) \sqsubseteq y$, it is enough to prove that $f(b) \sqsubseteq y$ for every $b \in \mathcal{I}$. But for such b, there exists $s : S$ such that $b \in \mathcal{I}_s$ and hence, $f(b) \sqsubseteq \bar{f}(\mathcal{I}_s) \sqsubseteq y$.

Finally, if $\prec$ is reflexive, then we prove that $\bar{f}(\downarrow b) = f(b)$ for every $b : B$ by antisymmetry. Since $\prec$ is assumed to be reflexive, we have $b \in \downarrow b$ and therefore, $f(b) \sqsubseteq \bar{f}(\downarrow b)$. Conversely, for every $c \prec b$ we have $f(c) \sqsubseteq f(b)$ by monotonicity of f and hence, $\bar{f}(\downarrow b) \sqsubseteq f(b)$, as desired. Uniqueness is proved easily, because if $g : \mathcal{V}\text{-Idl}(B, \prec) \to D$ is Scott continuous with $g(\downarrow b) = f(b)$, then for an arbitrary ideal I we have $g(I) = g(\bigcup_{b \in I} \downarrow b) = \bigsqcup_{b \in I} g(\downarrow b) = \bigsqcup_{b \in I} f(b) \equiv \bar{f}(I)$ by Lemma 8.7 and Scott continuity of g. $\square$

8.2. Example: the ideal completion of the dyadics rationals

We describe an example of a continuous dcpo, built using the ideal completion, that is not algebraic. In fact, this dcpo has no compact elements at all.

We inductively define a type and an order representing dyadic rationals $m/2^n$ in the interval $(-1, 1)$ for integers m, n. The intuition for the upcoming definitions is the following: Start with the point 0 in the middle of the interval. Then consider the two functions, respectively standing for *left* and *right*,

$$\begin{aligned}
l, r &: (-1, 1) \to (-1, 1) \\
l(x) &:\equiv (x - 1)/2 \\
r(x) &:\equiv (x + 1)/2
\end{aligned}$$

that generate the dyadic rationals. Observe that $l(x) < 0 < r(x)$ for every $x : (-1, 1)$. Accordingly, we inductively define the following types.

Definition 8.15 (Dyadics, $\mathbb{D}$, $\prec$)**.** The type of **dyadics** $\mathbb{D} : \mathcal{U}_0$ is the inductive type with these three constructors

$$\mathrm{m} : \mathbb{D} \quad \mathrm{l} : \mathbb{D} \to \mathbb{D} \quad \mathrm{r} : \mathbb{D} \to \mathbb{D}.$$

We also inductively define $\prec : \mathbb{D} \to \mathbb{D} \to \mathcal{U}_0$ as

$$\begin{array}{lll} \mathrm{m} \prec \mathrm{m} \;\;\;\coloneqq \mathbf{0} & \mathrm{l}(x) \prec \mathrm{m} \;\;\;\coloneqq \mathbf{1} & \mathrm{r}(x) \prec \mathrm{m} \;\;\;\coloneqq \mathbf{0} \\ \mathrm{m} \prec \mathrm{l}(y) \coloneqq \mathbf{0} & \mathrm{l}(x) \prec \mathrm{l}(y) \coloneqq x \prec y & \mathrm{r}(x) \prec \mathrm{l}(y) \coloneqq \mathbf{0} \\ \mathrm{m} \prec \mathrm{r}(y) \coloneqq \mathbf{1} & \mathrm{l}(x) \prec \mathrm{r}(y) \coloneqq \mathbf{1} & \mathrm{r}(x) \prec \mathrm{r}(y) \coloneqq x \prec y. \end{array}$$

Lemma 8.16. *The type of dyadics is a set with decidable equality.*

Proof. Sethood follows from having decidable equality by Hedberg's Theorem. To see that $\mathbb{D}$ has decidable equality, one can use a standard inductive proof. ☐

Definition 8.17 (Trichotomy, density, having no endpoints)**.** We say that a binary relation $<$ on a type X is

- **trichotomous** if exactly one of $x < y$, $x = y$ or $y < x$ holds.
- **dense** if for every $x, y : X$, there exists some $z : X$ such that $x < z < y$.
- **without endpoints** if for every $x : X$, there exist some $y, z : X$ with $y < x < z$.

Lemma 8.18. *The relation $\prec$ on the dyadics is proposition-valued, transitive, irreflexive, trichotomous, dense and without endpoints.*

Proof. That $\prec$ is proposition-valued, transitive, irreflexive and trichotomous is all proven by a straightforward induction on the definition on $\mathbb{D}$. That it has no endpoints is witnessed by the fact that for every $x : \mathbb{D}$, we have

$$\mathrm{l}\,x \prec x \prec \mathrm{r}\,x \tag{†}$$

which is proven by induction on $\mathbb{D}$ as well. We spell out the inductive proof that it is dense, making use of (†). Suppose that $x \prec y$. Looking at the definition of the order, we see that we need to consider five cases.

- If $x = \mathrm{m}$ and $y = \mathrm{r}\,y'$, then we have $x \prec \mathrm{r}(\mathrm{l}(y')) \prec y$.
- If $x = \mathrm{l}(x')$ and $y = \mathrm{m}$, then we have $x \prec \mathrm{l}(\mathrm{r}(x')) \prec y$.
- If $x = \mathrm{l}(x')$ and $y = \mathrm{r}\,y'$, then we have $x \prec \mathrm{m} \prec y$.
- If $x = \mathrm{r}(x')$ and $y = \mathrm{r}\,y'$, then we have $x' \prec y'$ and therefore, by induction hypothesis, there exists $z' : \mathbb{D}$ such that $x' \prec z' \prec y'$. Hence, $x \prec \mathrm{r}(z') \prec y$.
- If $x = \mathrm{l}(x')$ and $y = \mathrm{l}(y')$, then $x' \prec y'$ and so, by induction hypothesis, there exists $z' : \mathbb{D}$ such that $x' \prec z' \prec y'$. Thus, $x \prec \mathrm{l}(z') \prec y$. ☐

Proposition 8.19. *The pair* $(\mathbb{D}, \prec)$ *is an abstract* $\mathcal{U}_0$*-basis.*

Proof. By Lemma 8.18 the relation $\prec$ is proposition-valued and transitive. Moreover, that it has no endpoints implies unary interpolation. For binary interpolation, suppose that we have $x \prec z$ and $y \prec z$. Then by trichotomy there are three cases.

- If $x = y$, then using density and our assumption that $x \prec z$, there exists $d : \mathbb{D}$ with $y = x \prec d \prec z$, as desired.
- If $x \prec y$, then using density and our assumption that $y \prec z$, there exists $d : \mathbb{D}$ with $y \prec d \prec z$, but then also $x \prec d$ since $x \prec y$, so we are done.
- If $x \prec y$, then the proof is similar to that of the second case. □

Proposition 8.20. *The ideal completion* $\mathcal{U}_0\text{-Idl}(\mathbb{D}, \prec) : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_0}$ *is continuous with small basis* $\downarrow(-) : \mathbb{D} \to \mathcal{U}_0\text{-Idl}(\mathbb{D}, \prec)$*. Moreover, it cannot be algebraic, because none of its elements are compact.*

Proof. The first claim follows from Theorem 8.9. Now suppose for a contradiction that we have a compact ideal I. By Lemma 8.8, there exists $x \in I$ with $I \subseteq {\downarrow} x$. But this implies $x \prec x$, which is impossible as $\prec$ is irreflexive. □

8.3. Ideal completions of small bases

Given a $\mathcal{V}$-dcpo D with a small basis $\beta : B \to D$, we show that there are two natural ways of turning B into an abstract basis. Either define $b \prec c$ by $\beta(b) \ll \beta(c)$, or by $\beta(b) \sqsubseteq \beta(c)$. Taking their $\mathcal{V}$-ideal completions we show that the former yields a continuous dcpo isomorphic to D, while the latter yields an algebraic dcpo (with a small compact basis) in which D can be embedded. The latter fact will find application in Section 9.2, while the former gives us a presentation theorem: every dcpo with a small basis is isomorphic to a dcpo of ideals. In particular, if $D : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{T}}$ has a small basis, then it is isomorphic to a dcpo with simpler universe parameters, namely $\mathcal{V}\text{-Idl}(B, \ll_\beta) : \mathcal{V}\text{-DCPO}_{\mathcal{V}^+,\mathcal{V}}$. Of course a similar result holds for dcpos with a small compact basis. In studying these variations, it is helpful to first develop some machinery that all of them have in common.

Fix a $\mathcal{V}$-dcpo D with a small basis $\beta : B \to D$. In what follows we conflate the family $\twoheaddownarrow_\beta x : (\Sigma_{b:B}(\beta(b) \ll x)) \xrightarrow{\beta \circ \mathrm{pr}_1} D$ with its associated subset $\{b \in B \mid \beta(b) \ll x\}$, formally given by the map $B \to \Omega_{\mathcal{V}}$ defined as $b \mapsto \exists_{b:B}(\beta(b) \ll x)$.

Lemma 8.21. *The assignment* $x : D \mapsto \twoheaddownarrow_\beta x : \mathcal{P}(B)$ *is Scott continuous.*

Proof. Note that $\twoheaddownarrow_\beta(-)$ is monotone: if $x \sqsubseteq y$ and $b : B$ is such that $\beta(b) \ll x$, then also $\beta(b) \ll y$. So it suffices to prove that $\twoheaddownarrow_\beta(\bigsqcup \alpha) \subseteq \bigcup_{i:I} \twoheaddownarrow_\beta \alpha_i$. So suppose that $b : B$ is such that $\beta(b) \ll \bigsqcup \alpha$. By Lemma 7.6, there exists $c : B$ with $\beta(b) \ll \beta(c) \ll \bigsqcup \alpha$. Hence, there exists $i : I$ such that $\beta(b) \ll \beta(c) \sqsubseteq \alpha_i$ already, and therefore, $b \in \bigcup_{j:J} \twoheaddownarrow_\beta \alpha_j$, as desired. □

By virtue of the fact that β is a small basis for D, we know that taking the directed supremum of $\downarrow_\beta x$ equals x for every $x : D$. In other words, $\downarrow_\beta(-)$ is a section of $\bigsqcup(-)$. The following lemma gives conditions for the other composite to be an inflation or a deflation.

Lemma 8.22. *Let $I : \mathcal{P}_{\mathcal{V}}(B)$ be a subset of B such that its associated family $\bar{I} : (\Sigma_{b:B}(b \in I)) \xrightarrow{\beta \circ \mathrm{pr}_1} D$ is directed.*

(i) If the conjunction of $\beta(b) \sqsubseteq \beta(c)$ and $c \in I$ implies $b \in I$, then $\downarrow_\beta \bigsqcup \bar{I} \subseteq I$.

(ii) If for every $b \in I$ there exists $c \in I$ such that $\beta(b) \ll \beta(c)$, then $I \subseteq \downarrow_\beta \bigsqcup \bar{I}$.

In particular, if both conditions hold, then $I = \downarrow_\beta \bigsqcup \bar{I}$.

Note that the first condition says that I is a lower set with respect to the order of D, while the second says that I is round with respect to the way-below relation.

Proof. (i) Suppose that I is a lower set and let $b : B$ be such that $\beta(b) \ll \bigsqcup \bar{I}$. Then there exists $c \in I$ with $\beta(b) \sqsubseteq \beta(c)$, which implies $b \in I$ as desired, because I is assumed to be a lower set. (ii) Assume that I is round and let $b \in I$ be arbitrary. By roundness of I, there exists $c \in I$ such that $\beta(b) \ll \beta(c)$. But then $\beta(b) \ll \beta(c) \sqsubseteq \bigsqcup \bar{I}$, so that $b \in \downarrow_\beta \bigsqcup \bar{I}$, as we wished to show. ☐

Lemma 8.23. *Suppose that we have $\prec : B \to B \to \mathcal{V}$ and let $x : D$ be arbitrary.*

(i) If $b \prec c$ implies $\beta(b) \sqsubseteq \beta(c)$ for every $b, c : B$, then $\downarrow_\beta x$ is a lower set w.r.t. $\prec$.

(ii) If $\beta(b) \ll \beta(c)$ implies $b \prec c$ for every $b, c : B$, then $\downarrow_\beta x$ is semidirected w.r.t. $\prec$.

Proof. (i) This is immediate, because $\downarrow \beta x$ is a lower set with respect to the order relation on D. (ii) Suppose that the condition holds and that we have $b_1, b_2 : B$ such that $\beta(b_1), \beta(b_2) \ll x$. Using binary interpolation in the basis, there exist $c_1, c_2 : B$ with $\beta(b_1) \ll \beta(c_1) \ll x$ and $\beta(b_2) \ll \beta(c_2) \ll x$. Hence, $c_1, c_2 \in \downarrow_{\beta x}$ and moreover, by assumption we have $b_1 \prec c_1$ and $b_2 \prec c_2$, as desired. ☐

8.3.1. Ideal completion with respect to the way-below relation

Lemma 8.24. *If $\beta : B \to D$ is a small basis for a $\mathcal{V}$-dcpo D, then $(B, \ll_\beta)$ is an abstract $\mathcal{V}$-basis where $b \ll_\beta c$ is defined as $\beta(b) \ll \beta(c)$.*

Remark 8.25. The definition of an abstract $\mathcal{V}$-basis requires the relation on it to be $\mathcal{V}$-valued. Hence, for the lemma to make sense we appeal to the fact that β is a *small* basis which tells us that we can substitute $\beta(b) \ll \beta(c)$ by an equivalent type in $\mathcal{V}$.

Proof of Lemma 8.24. The way-below relation is proposition-valued and transitive. Moreover, $\ll_\beta$ satisfies nullary and binary interpolation precisely because we have nullary and binary interpolation in the basis for the way-below relation by Lemma 7.6. $\square$

The following theorem is a presentation result for dcpos with a small basis: every such dcpo can be presented as the round ideal completion of its small basis.

Theorem 8.26. *The map* $\downarrow_\beta(-) : D \to \mathcal{V}\text{-Idl}(B, \ll_\beta)$ *is an isomorphism of* $\mathcal{V}$*-dcpos.*

Proof. First of all, we should check that the map is well-defined, i.e. that $\downarrow_\beta x$ is an $(B, \ll_\beta)$-ideal. It is an inhabited subset by nullary interpolation in the basis and a semidirected lower set because the criteria of Lemma 8.23 are satisfied when taking $\prec$ to be $\ll_\beta$. Secondly, the map $\downarrow_\beta(-)$ is Scott continuous by Lemma 8.21.

Now notice that the map $\beta : (B, \ll_\beta) \to D$ is monotone and that the Scott continuous map it induces by Theorem 8.14 is exactly the map $\bigsqcup$: $\mathcal{V}\text{-Idl}(B, \ll_\beta) \to D$ that takes an ideal I to the supremum of its associated directed family $\beta \circ \mathrm{pr}_1 : (\Sigma_{b:B}(b \in I)) \to D$.

Since β is a basis for D, we know that $\bigsqcup \downarrow_\beta x = x$ for every $x : D$. So it only remains to show that $\downarrow_\beta \circ \bigsqcup$ is the identity on $\mathcal{V}\text{-Idl}(B, \ll_\beta)$, for which we will use Lemma 8.22. So suppose that $I : \mathcal{V}\text{-Idl}(B, \ll_\beta)$ is arbitrary. Then we only need to prove that

(i) the conjunction of $\beta(b) \sqsubseteq \beta(c)$ and $c \in I$ implies $b \in I$ for every $b, c : B$;

(ii) for every $b \in I$, there exists $c \in I$ such that $\beta(b) \ll \beta(c)$.

Note that (ii) is just saying that I is a round ideal w.r.t. $\ll_\beta$, so this holds. For (i), suppose that $\beta(b) \sqsubseteq \beta(c)$ and $c \in I$. By roundness of I, there exists $c' \in I$ such that $c \ll_\beta c'$. But then $\beta(b) \sqsubseteq \beta(c) \ll \beta(c')$, so that $b \ll_\beta c'$ which implies that $b \in I$, because ideals are lower sets. $\square$

8.3.2. Ideal completion with respect to the order relation

Lemma 8.27. *If* $\beta : B \to D$ *is a small basis for a* $\mathcal{V}$*-dcpo* D*, then* $(B, \sqsubseteq_\beta)$ *is an abstract* $\mathcal{V}$*-basis where* $b \sqsubseteq_\beta c$ *is defined as* $\beta(b) \sqsubseteq \beta(c)$*.*

Proof. The relation $\sqsubseteq_\beta$ is reflexive, so this follows from Lemma 8.10. $\square$

Remark 8.28. The definition of an abstract $\mathcal{V}$-basis requires the relation on it to be $\mathcal{V}$-valued. Hence, for the lemma to make sense we appeal to Proposition 7.5 to know that D is locally small which tells us that we can substitute $\beta(b) \sqsubseteq \beta(c)$ by an equivalent type in $\mathcal{V}$.

Theorem 8.29. *The map* $\downarrow_\beta(-) : D \to \mathcal{V}\text{-Idl}(B, \sqsubseteq_\beta)$ *is the embedding in an embedding-projection pair. In particular,* D *is a retract of the algebraic dcpo* $\mathcal{V}\text{-Idl}(B, \sqsubseteq_\beta)$ *that has a small compact basis. Moreover, if* β *is a small* compact *basis, then the map is an isomorphism.*

Proof. First of all, we should check that the map is well-defined, i.e. that $\downarrow_\beta x$ is an $(B, \sqsubseteq_\beta)$-ideal. It is an inhabited subset by nullary interpolation in the basis and a semidirected lower set because the criteria of Lemma 8.23 are satisfied when taking $\prec$ to be $\sqsubseteq_\beta$. Secondly, the map $\downarrow_\beta(-)$ is Scott continuous by Lemma 8.21.

Now notice that the map $\beta : (B, \sqsubseteq_\beta) \to D$ is monotone and that the continuous map it induces by Theorem 8.14 is exactly the map $\bigsqcup : \mathcal{V}\text{-Idl}(B, \sqsubseteq_\beta) \to D$ that takes an ideal I to the least upper bound of its associated directed family $\beta \circ \mathrm{pr}_1 : (\Sigma_{b:B}(b \in I)) \to D$.

Since β is a basis for D, we know that $\bigsqcup \downarrow_\beta x = x$ for every $x : D$. So it only remains to show that $\downarrow_\beta \circ \bigsqcup$ is a deflation, for which we will use Lemma 8.22. So suppose that $I : \mathcal{V}\text{-Idl}(B, \sqsubseteq_\beta)$ is arbitrary. Then we only need to prove that the conjunction of $\beta(b) \sqsubseteq \beta(c)$ and $c \in I$ implies $b \in I$, but this holds, because I is a lower set with respect to $\sqsubseteq_\beta$.

Finally, assume that β is a small compact basis. We show that $\downarrow_\beta \circ \bigsqcup$ is also inflationary in this case. So let I be an arbitrary ideal. By Lemma 8.22 it is enough to show that for every $b \in I$, there exists $c \in I$ such that $\beta(b) \ll \beta(c)$. But by assumption, $\beta(b)$ is compact, so we can simply take c to be b. ☐

Combining Theorem 7.8, Theorems 8.9 and 8.26, and Theorems 8.13 and 8.29, we obtain the following result:

Corollary 8.30.

*(i) A $\mathcal{V}$-dcpo has a small basis if and only if it is isomorphic to $\mathcal{V}$-*Idl$(B, \prec)$ *for an abstract basis $(B, \prec)$.*

(ii) A $\mathcal{V}$-dcpo has a small compact basis if and only if it is isomorphic to $\mathcal{V}$-Idl$(B, \prec)$ *for an abstract basis $(B, \prec)$ where $\prec$ is reflexive.*

(iii) A $\mathcal{V}$-dcpo has a small basis if and only if it is a retract of a $\mathcal{V}$-dcpo with a small compact basis.

Hence every continuous $\mathcal{V}$-dcpo with a small basis is a retract of some algebraic $\mathcal{V}$-dcpo.

In particular, every $\mathcal{V}$-dcpo with a small basis is isomorphic to one whose order takes values in $\mathcal{V}$ and whose carrier lives in $\mathcal{V}^+$.

9. Bilimits and exponentials

9.1. Structurally continuous and algebraic bilimits

We show that bilimits are closed under equipment with continuity/algebraicity data. For the reminder of this section, fix a directed diagram of $\mathcal{V}$-dcpos $(D_i)_{i:I}$ with embedding-projection pairs $(\varepsilon_{i,j}, \pi_{i,j})_{i \sqsubseteq j \text{ in } I}$ between them as in Section 3.6. We stress that, throughout this section, the word "embedding" is only used in the domain-theoretic sense, i.e. it is reserved for one half of an embedding-projection pair, rather than in the homotopy type theory sense of having subsingleton fibers.

But notice that domain-theoretic embeddings are homotopy embeddings, because they are sections, and sections of sets are always homotopy embeddings [65].

Now suppose that for every $i : I$, we have $\alpha_i : J_i \to D_i$ with each $J_i : \mathcal{V}$. Then we define $J_\infty :\equiv \Sigma_{i:I} J_i$ and $\alpha_\infty : J_\infty \to D_\infty$ by $(i,j) \mapsto \varepsilon_{i,\infty}(\alpha_i(j))$, where $\varepsilon_{i,\infty}$ is as in Definition 3.38.

Lemma 9.1. *If every α_i is directed and we have $\sigma : D_\infty$ such that α_i approximates σ_i, then α_∞ is directed and approximates σ.*

Proof. Observe that α_∞ is equal to the supremum, if it exists, of the directed families $(\varepsilon_{i,\infty} \circ \alpha_i)_{i:I}$ in the ind-completion of D_∞, cf. the proof of Lemma 5.2. Hence, for directedness of α_∞, it suffices to prove that the family $i \mapsto \varepsilon_{i,\infty} \circ \alpha_i$ is directed with respect to the exceeds-relation. The index type I is inhabited, because we are working with a directed diagram of dcpos. For semidirectedness, we will first prove that if $i \sqsubseteq i'$, then $\varepsilon_{i',\infty} \circ \alpha_{i'}$ exceeds $\varepsilon_{i,\infty} \circ \alpha_i$.

So suppose that $i \sqsubseteq i'$ and $j : J_i$. As α_i approximates σ_i, we have $\alpha_i(j) \ll \sigma_i$. Because $\varepsilon_{i,i'}$ is an embedding, it preserves the way-below relation (Lemma 4.15), so that we get $\varepsilon_{i,i'}(\alpha_i(j)) \ll \varepsilon_{i,i'}(\sigma_i) \sqsubseteq \sigma_{i'} = \bigsqcup \alpha_{i'}$. Hence, there exists $j' : J_{i'}$ with $\varepsilon_{i,i'}(\alpha_i(j)) \sqsubseteq \alpha_{i'}(j')$ which yields $\varepsilon_{i,\infty}(\alpha_i(j)) = \varepsilon_{i',\infty}(\varepsilon_{i,i'}(\alpha_i(j))) \sqsubseteq \varepsilon_{i',\infty}(\alpha_{i'}(j'))$, completing the proof that $\varepsilon_{i',\infty} \circ \alpha_{i'}$ exceeds $\varepsilon_{i,\infty} \circ \alpha_i$.

Now to prove that the family $i \mapsto \varepsilon_{i,\infty} \circ \alpha_i$ is semidirected with respect to the exceeds-relation, suppose we have $i_1, i_2 : I$. Since I is a directed preorder, there exists $i : I$ such that $i_1, i_2 \sqsubseteq i$. But then $\varepsilon_{i,\infty} \circ \alpha_i$ exceeds both $\varepsilon_{i_1,\infty} \circ \alpha_{i_1}$ and $\varepsilon_{i_2,\infty} \circ \alpha_{i_2}$ by the above.

Thus, α_∞ is directed. To see that its supremum is σ, observe that

$$\begin{aligned}
\sigma &= \textstyle\bigsqcup_{i:I} \varepsilon_{i,\infty}(\sigma_i) && \text{(by Lemma 3.41)} \\
&= \textstyle\bigsqcup_{i:I} \varepsilon_{i,\infty}(\bigsqcup \alpha_i) && \text{(since } \alpha_i \text{ approximates } \sigma_i\text{)} \\
&= \textstyle\bigsqcup_{i:I} \bigsqcup \varepsilon_{i,\infty} \circ \alpha_i && \text{(by Scott continuity of } \varepsilon_{i,\infty}\text{)} \\
&= \textstyle\bigsqcup_{(i,j):J_\infty} \alpha_\infty(i,j),
\end{aligned}$$

as desired.

Finally, we wish to show that $\alpha_\infty(i,j) \ll \sigma$ for every $(i,j) : J_\infty$. But $\varepsilon_{i,\infty}$ is an embedding and therefore preserves the way-below relation while $\alpha_i(j)$ approximates σ_i, so we get $\alpha_\infty(i,j) \equiv \varepsilon_{i,\infty}(\alpha_i(j)) \ll \varepsilon_{i,\infty}(\sigma_i) \sqsubseteq \sigma$ where the final inequality holds because $\varepsilon_{i,\infty} \circ \pi_{i,\infty}$ is a deflation. ∎

Lemma 9.2. *If $\alpha_i(j)$ is compact for every $i : I$ and $j : J_i$, then all the values of α_∞ are compact too.*

Proof. Let $(i,j) : J_\infty$ be arbitrary. Since $\varepsilon_{i,\infty}$ is an embedding it preserves compact elements, so $\alpha_\infty(i,j) \equiv \varepsilon_{i,\infty}(\alpha_i(j))$ is compact. ∎

Theorem 9.3. *If each D_i comes equipped with continuity (resp. algebraicity) data, then we can give continuity (resp. algebraicity) data for D_∞.*

Proof. Let $\sigma : D_\infty$ be arbitrary. By assumption on each D_i, we have a directed family $\alpha_i : J_i \to D_i$ approximating σ_i. Hence, by Lemma 9.1, the family α_∞ is directed and approximates σ, giving continuity data for D_∞. For the algebraic case, we apply the above and Lemma 9.2. $\square$

Note that we do not expect to be able to prove that D_∞ is continuous if each D_i is, because it would require an instance of the axiom of choice to get continuity data on each D_i, and without those we have nothing to operate on.

Theorem 9.4. *If each D_i has a small basis $\beta_i : B_i \to D_i$, then the map*

$$\begin{array}{rcl} \beta_\infty : (B_\infty :\equiv \Sigma_{i:I} B_i) & \to & D_\infty \\ (i,b) & \mapsto & \varepsilon_{i,\infty}(\beta_i(b)) \end{array}$$

is a small basis for D_∞. Furthermore, if each β_i is a small compact basis, then β_∞ is a small compact basis too.

Proof. First of all, we must show that the proposition $\beta_\infty(i,b) \ll \sigma$ is small for every $i : I$, $b : B_i$ and $\sigma : D_\infty$. This is the case as the way-below relation on D_∞ has small values. Indeed, by Proposition 6.13 and Theorem 9.3, it suffices to prove that D_∞ is locally small. But this holds by Proposition 3.42 as each D_i is locally small by Proposition 7.5.

It remains to prove that, for an arbitrary element $\sigma : D_\infty$, the family $\downarrow_{\beta_\infty} \sigma$ given by $\left(\Sigma_{(i,b):B_\infty} \beta_\infty(i,b) \ll \sigma\right) \xrightarrow{\beta_\infty \circ \mathrm{pr}_1} D_\infty$ is directed with supremum σ. Note that for every $i : I$ and $b : B_i$, we have that $\beta_i(b) \ll \sigma_i$ implies

$$\beta_\infty(i,b) \equiv \varepsilon_{i,\infty}(\beta_i(b)) \ll \varepsilon_{i,\infty}(\sigma_i) \sqsubseteq \sigma,$$

since Lemma 4.15 tells us that the embedding $\varepsilon_{i,\infty}$ preserves the way-below relation. Hence, the identity map induces a well-defined map

$$\iota : (\Sigma_{i:I} \Sigma_{b:B_i} \beta_i(b) \ll \sigma_i) \to \left(\Sigma_{(i,b):B_\infty} \beta_\infty(i,b) \ll \sigma\right).$$

Lemma 7.7 now tells us that we only need to show that $\downarrow_{\beta_\infty} \sigma \circ \iota$ is directed and has supremum σ. But if we write $\alpha_i : (\Sigma_{b:B_i} \beta_i(b) \ll \sigma_i) \to D_i$ for the map $b \mapsto \beta_i(b)$, then we see that $\downarrow_{\beta_\infty} \sigma \circ \iota$ is given by α_∞ defined from α_i as in the start of this section. But then α_∞ is indeed seen to be directed with supremum σ by virtue of Lemma 9.1 and the fact that α_i approximates σ_i.

Finally, if every β_i is a small compact basis, then β_∞ is also a small compact basis because by Lemma 7.13 all we need to know is that $\beta_\infty(i,b) \equiv \varepsilon_{i,\infty}(\beta_i(b))$ is compact for every $i : I$ and $b : B_i$. But this follows from the fact that embeddings preserve compactness and that each $\beta_i(b)$ is compact. $\square$

9.2. Exponentials with small (compact) bases

Just as in the classical, impredicative setting, the exponential of two continuous dcpos need not be continuous [34]. However, with some work, we are able to show that E^D has a small basis provided that both D and E do and that E has all (not necessarily directed) $\mathcal{V}$-suprema, that is, E is a continuous lattice. We first establish this for small compact bases using *step functions* and then derive the result for compact bases using Theorem 8.29.

9.2.1. Single step functions

Suppose that we have a dcpo D and a pointed dcpo E. Classically [17, Exercise II-2.31], the single step function given by $d : D$ and $e : E$ is defined as

$$\begin{aligned} (\!|d \Rightarrow e|\!) &: D \to E \\ x &\mapsto \begin{cases} e & \text{if } d \sqsubseteq x; \\ \bot & \text{otherwise.} \end{cases} \end{aligned}$$

Constructively, we can't expect to make this case distinction in general, so we define single step functions using subsingleton suprema instead.

Definition 9.5 (Single step function, $(\!|d \Rightarrow e|\!)$)**.** The **single step function** given by two elements $d : D$ and $e : E$, where D is a locally small $\mathcal{V}$-dcpo and E is a pointed $\mathcal{V}$-dcpo, is the function $(\!|d \Rightarrow e|\!) : D \to E$ that maps a given $x : D$ to the supremum of the family indexed by the subsingleton $d \sqsubseteq x$ that is constantly e.

Note that we need the domain D to be locally small, because we need the type $d \sqsubseteq x$ to be a subsingleton in $\mathcal{V}$ to use the $\mathcal{V}$-directed-completeness of E. For the definition of $(\!|d \Rightarrow e|\!)$ to make sense, we need to know that the supremum of the constant family exists. This is the case by Lemma 3.17, which says that the supremum of a subsingleton-indexed family $\alpha : P \to E$ is given by the supremum of the directed family $\mathbf{1} + P \to E$ defined by $\mathrm{inl}(\star) \mapsto \bot$ and $\mathrm{inr}(p) \mapsto \alpha(p)$.

Lemma 9.6. *If $d : D$ is compact, then $(\!|d \Rightarrow e|\!)$ is Scott continuous for all $e : E$.*

Proof. Suppose that $d : D$ is compact and that $\alpha : I \to D$ is a directed family. We first show that $(\!|d \Rightarrow e|\!)(\bigsqcup \alpha)$ is an upper bound of $(\!|d \Rightarrow e|\!) \circ \alpha$. So let $i : I$ be arbitrary. Then we have to prove $\bigsqcup_{d \sqsubseteq \alpha_i} e \sqsubseteq \bigsqcup (\!|d \Rightarrow e|\!) \circ \alpha$. Since the supremum gives a lower bound of the upper bounds, it suffices to prove that $e \sqsubseteq \bigsqcup (\!|d \Rightarrow e|\!) \circ \alpha$ whenever $d \sqsubseteq \alpha_i$. But in this case we have $e = (\!|d \Rightarrow e|\!)(\alpha_i) \sqsubseteq \bigsqcup (\!|d \Rightarrow e|\!) \circ \alpha$, so we are done.

To see that $(\!|d \Rightarrow e|\!)(\bigsqcup \alpha)$ is a lower bound of the upper bounds, suppose that we are given $y : E$ such that y is an upper bound of $(\!|d \Rightarrow e|\!) \circ \alpha$. We are to prove that $\left(\bigsqcup_{d \sqsubseteq \bigsqcup \alpha} e\right) \sqsubseteq y$. Note that it suffices for $d \sqsubseteq \bigsqcup \alpha$ to imply $e \sqsubseteq y$. So assume that $d \sqsubseteq \bigsqcup \alpha$. By compactness of d there exists $i : I$ such that $d \sqsubseteq \alpha_i$ already. But then $e = (\!|d \Rightarrow e|\!)(\alpha_i) \sqsubseteq y$, as desired. ☐

Lemma 9.7. *A Scott continuous function $f : D \to E$ is above the single step function $(\!|d \Rightarrow e|\!)$ with $d : D$ compact if and only if $e \sqsubseteq f(d)$.*

Proof. Suppose that $(\!|d \Rightarrow e|\!) \sqsubseteq f$. Then $(\!|d \Rightarrow e|\!)(d) = e \sqsubseteq f(d)$, proving one implication. Now assume that $e \sqsubseteq f(d)$ and let $x : D$ be arbitrary. To prove that $(\!|d \Rightarrow e|\!)(x) \sqsubseteq f(x)$, it suffices that $e \sqsubseteq f(x)$ whenever $d \sqsubseteq x$. But if $d \sqsubseteq x$, then $e \sqsubseteq f(d) \sqsubseteq f(x)$ by monotonicity of f. ☐

Lemma 9.8. *If d and e are compact, then so is $(\!|d \Rightarrow e|\!)$ in the exponential E^D.*

Proof. Suppose that we have a directed family $\alpha : I \to E^D$ such that $(\!|d \Rightarrow e|\!) \sqsubseteq \bigsqcup \alpha$. Then we consider the directed family $\alpha^d : I \to E$ given by $i \mapsto \alpha_i(d)$. We claim that $e \sqsubseteq \bigsqcup \alpha^d$. Indeed, by Lemma 9.7 and our assumption that $(\!|d \Rightarrow e|\!) \sqsubseteq \bigsqcup \alpha$ we get $e \sqsubseteq (\bigsqcup \alpha)(d) = \bigsqcup \alpha^d$. Now by compactness of e, there exists $i : I$ such that $e \sqsubseteq \alpha^d(i) \equiv \alpha_i(d)$ already. But this implies $(\!|d \Rightarrow e|\!) \sqsubseteq \alpha_i$ by Lemma 9.7 again, finishing the proof. ☐

9.2.2. Exponentials with small compact bases

Fix $\mathcal{V}$-dcpos D and E with small compact bases $\beta_D : B_D \to D$ and $\beta_E : B_E \to D$ and moreover assume that E has suprema for all (not necessarily directed) families indexed by types in $\mathcal{V}$. We are going to construct a small compact basis on the exponential E^D.

Lemma 9.9. *If E is sup-complete, then every continuous function $f : D \to E$ is the supremum of the collection of single step functions $((\!|\beta_D(b) \Rightarrow \beta_E(c)|\!))_{b:B_D,c:B_E}$ that are below f.*

Proof. Note that f is an upper bound by definition, so it remains to prove that it is the least. Therefore suppose we are given an upper bound $g : D \to E$. We have to prove that $f(x) \sqsubseteq g(x)$ for every $x : D$, so let $x : D$ be arbitrary. Now $x = \bigsqcup \downarrow_{\beta_D} x$, because β_D is a small compact basis for D, so by Scott continuity of f and g, it suffices to prove that $f(\beta_D(b)) \sqsubseteq g(\beta_D(b))$ for every $b : B_D$. So let $b : B_D$ be arbitrary. Since β_E is a small compact basis for E, we have $f(\beta_D(b)) = \bigsqcup \downarrow_{\beta_E} f(\beta_D(b))$. So to prove $f(\beta_D(b)) \sqsubseteq g(\beta_D(b))$ it is enough to know that $\beta_E(c) \sqsubseteq g(\beta_D(b))$ for every $c : B_E$ with $\beta_E(c) \sqsubseteq f(\beta_D(b))$. But for such $c : B_E$ we have $(\!|\beta_D(b) \Rightarrow \beta_E(c)|\!) \sqsubseteq f$ and therefore $(\!|\beta_D(b) \Rightarrow \beta_E(c)|\!) \sqsubseteq g$ because f is an upper bound of such single step functions, and hence $\beta_E(c) \sqsubseteq g(\beta_D(b))$ by Lemma 9.7, as desired. ☐

Definition 9.10 (Directification)**.** In a poset P with finite joins, the **directification** of a family $\alpha : I \to P$ is the family $\bar{\alpha} : \mathrm{List}(I) \to P$ that maps a finite list to the join of its elements. It is clear that $\bar{\alpha}$ has the same supremum as α, and by concatenating lists, one sees that the directification yields a directed family, hence the name.

Lemma 9.11. *If each element of a family into a sup-complete dcpo is compact, then so are all elements of its directification.*

Proof. The supremum of the empty list is $\bot$ by Example 4.4, and hence the join of finitely many compact elements is compact by Lemma 4.7. ☐

Let us write $\sigma : B_D \times B_E \to E^D$ for the map that takes (b, c) to the single step function $(\!|\beta_D(b) \Rightarrow \beta_E(c)|\!)$ and

$$\beta : B :\equiv \mathrm{List}(B_D \times B_E) \to E^D$$

for its directification, which exists because E^D is $\mathcal{V}$-sup-complete as E is and suprema are calculated pointwise.

Theorem 9.12. *The map β is a small compact basis for the exponential E^D, where E is assumed to be sup-complete.*

Proof. Firstly, every element in the image of β is compact by Lemmas 9.8 and 9.11. Secondly, for every $b : B$ and Scott continuous map $f : D \to E$, the type $\beta(b) \sqsubseteq f$ is small, because E^D is locally small by Proposition 7.9. Thirdly, for every such f, the family

$$(\Sigma_{b:B}(\beta(b) \sqsubseteq f)) \xrightarrow{\beta \circ \mathrm{pr}_1} E^D$$

is directed because β is the directification of σ. Finally, this family has supremum f by Lemma 9.9. ☐

9.2.3. Exponentials with small bases

We now present a variation of Theorem 9.12 but for (sup-complete) dcpos with small bases. In fact, we will prove it using Theorem 9.12 and the theory of retracts (Theorem 8.29 in particular).

Definition 9.13 (Closure under finite joins)**.** A small basis $\beta : B \to D$ for a sup-complete poset is **closed under finite joins** if we have $b_\perp : B$ with $\beta(b_\perp) = \perp$ and a map $\vee : B \to B \to B$ such that $\beta(b \vee c) = \beta(b) \vee \beta(c)$ for every $b, c : B$.

Lemma 9.14. *If D is a sup-complete dcpo with a small basis $\beta : B \to D$, then the directification of β is also a small basis for D. Moreover, by construction, it is closed under finite joins.*

Proof. Since β is a small basis for D, it follows by Proposition 7.5 that the way-below relation on D is small-valued. Hence, writing $\bar{\beta}$ for the directification of β, it remains to prove that $\downarrow_{\bar{\beta}} x$ is directed with supremum x for every $x : D$. But this follows easily from Lemma 7.7, because $\downarrow_{\beta} x$ is directed with supremum x and this family is equal to the composite

$$(\Sigma_{b:B}(\beta(b) \ll x)) \xhookrightarrow{b \mapsto [b]} \left(\Sigma_{l:\mathrm{List}(B)}\left(\bar{\beta}(l) \ll x\right)\right) \xrightarrow{\bar{\beta} \circ \mathrm{pr}_1} D,$$

where $[b]$ denotes the singleton list. ☐

Lemma 9.15. *If D is a $\mathcal{V}$-sup-complete poset with a small basis $\beta : B \to D$ closed under finite joins, then the ideal-completion $\mathcal{V}$-$\mathrm{Idl}(B, \sqsubseteq)$ is $\mathcal{V}$-sup-complete too.*

Proof. Since the $\mathcal{V}$-ideal completion is $\mathcal{V}$-directed complete, it suffices to show that $\mathcal{V}$-$\mathrm{Idl}(B, \sqsubseteq)$ has finite joins, because then we can turn an arbitrary small family into a directed one via Definition 9.10. As $\beta : B \to D$ is closed under finite joins, we have $b_\perp : B$ with $\beta(b_\perp) = \perp$ and we easily see that $\{b \in B \mid \beta(b) \sqsubseteq \beta(b_\perp)\}$ is the least element of $\mathcal{V}$-$\mathrm{Idl}(B, \sqsubseteq)$. Now suppose that we have two ideals $I, J : \mathcal{V}$-$\mathrm{Idl}(B, \sqsubseteq)$ and consider the subset K defined as

$$K :\equiv \{b \in B \mid \exists_{c \in I}\, \exists_{d \in J}\, (\beta(b) \sqsubseteq \beta(c \vee d))\}.$$

Notice that K is clearly a lower set. Also note that ideals are closed under finite joins as they are directed lower sets. Hence, $b_\perp \in I$ and $b_\perp \in J$, so that $b_\perp \in K$ and K is inhabited. For semidirectedness, assume $b_1, b_2 \in K$ so that there exist $c_1, c_2 \in I$ and $d_1, d_2 \in J$ with $\beta(b_1) \sqsubseteq \beta(c_1 \vee d_1)$ and $\beta(b_2) \sqsubseteq \beta(c_2 \vee d_2)$. Then $b_1 \vee b_2 \in K$, because $c_1 \vee c_2 \in I$ and $d_1 \vee d_2 \in J$ and $\beta(b_1 \vee b_2) \sqsubseteq \beta((c_1 \vee c_2) \vee (d_1 \vee d_2))$. Hence, K is a directed lower set. We claim that K is the join of I and J. First of all, I and J are both subsets of K, so it remains to prove that K is the least upper bound. To this end, suppose that we have an ideal L that includes I and J, and let $b \in K$ be arbitrary. Then there exist $c \in I$ and $d \in J$ with $\beta(b) \sqsubseteq \beta(c \vee d)$. But $I, J \subseteq L$ and ideals are closed under finite joins, so $c \vee d \in L$, which implies that $b \in L$ since L is a lower set. Therefore, K is the least upper bound of I and J, completing the proof. ☐

Theorem 9.16. *The exponential E^D of dcpos has a specified small basis if D and E have specified small bases and E is sup-complete.*

Proof. Suppose that $\beta_D : B_D \to D$ and $\beta_E : B_E \to E$ are small bases and that E is sup-complete. By Lemma 9.14 we can assume that $\beta_E : B_E \to E$ is closed under finite joins. We will write $\overline{D}$ and $\overline{E}$ for the respective ideal completions $\mathcal{V}\text{-Idl}\,(B_D, \sqsubseteq)$ and $\mathcal{V}\text{-Idl}(B_E, \sqsubseteq)$. Then Theorem 8.29 tells us that we have retracts

$$D \underset{r_D}{\overset{s_D}{\rightleftarrows}} \overline{D} \text{ and } E \underset{r_E}{\overset{s_E}{\rightleftarrows}} \overline{E}.$$

Composition yields a retract

$$E^D \underset{r}{\overset{s}{\rightleftarrows}} \overline{E}^{\overline{D}}$$

where $s(f) :\equiv s_E \circ f \circ r_D$ and $r(g) :\equiv r_E \circ g \circ s_D$. Now $\overline{D}$ and $\overline{E}$ have small compact basis by Theorem 8.13 and $\overline{E}$ is sup-complete by Lemma 9.15. Therefore, the exponential of $\overline{D}$ and $\overline{E}$ has a small basis by Theorem 9.12. Finally, Theorem 7.8 tells us that the retraction r yields a small basis on E^D, as desired. ☐

Note how, unlike Theorem 9.12, the above theorem does not give a nice description of the small basis for the exponential when we unfold the definitions. It may be possible to do so using function graphs, as is done in the classical setting of effective domain theory in [67, Section 4.1], and we leave this as an open question.

An application of the closure results of Theorems 9.4 and 9.12 is that Scott's D_∞ model of the untyped λ-calculus from Section 3.7 has a small compact basis.

Theorem 9.17. *Scott's D_∞ has a small compact basis and in particular is algebraic.*

Proof. By Example 7.16 the $\mathcal{U}_0$-dcpo D_0 has a small compact basis. Moreover, it is not just a $\mathcal{U}_0$-dcpo as it has suprema for all (not necessarily directed) families indexed by types in $\mathcal{U}_0$, as D_0 is isomorphic to $\Omega_{\mathcal{U}_0}$. Hence, by induction it follows that each D_n is $\mathcal{U}_0$-sup-complete. Therefore, by induction and Theorem 9.12 we get a small compact basis for each D_n. Thus, by Theorem 9.4, the bilimit D_∞ has a small basis too. ☐

10. Concluding remarks

Taking inspiration from work in category theory by Johnstone and Joyal [24], we gave predicatively adequate definitions of continuous and algebraic dcpos, and discussed issues related to the absence of the axiom of choice. We also presented predicative adaptations of the notions of a basis and the round ideal completion. The theory was accompanied by several examples: we described canonical small compact bases for the lifting and the powerset, and considered the round ideal completion of the dyadics. We also showed that Scott's D_∞ has a small compact basis and is thus algebraic in particular.

To prove that D_∞ had a small compact basis, we used that each D_n is a $\mathcal{U}_0$-sup-lattice, so that we could apply the results of Section 9.2. Example 7.16 tells us that $\mathcal{L}_{\mathcal{U}_0}(\mathbb{N})$ has a small compact basis too, but to prove that the $\mathcal{U}_0$-dcpos in the Scott model of PCF (see [27]) have small compact bases using the techniques of Section 9.2, we would need $\mathcal{L}_{\mathcal{U}_0}(\mathbb{N})$ to be a $\mathcal{U}_0$-sup-lattice, but it isn't. However, it is complete for *bounded* families indexed by types in $\mathcal{U}_0$ and we believe that is possible to generalise the results of Section 9.2 from sup-lattices to bounded complete posets. Classically, this is fairly straightforward, but from preliminary considerations it appears that constructively one needs to impose certain decidability criteria on the bases of the dcpos. For instance that the partial order is decidable when restricted to basis elements. Such decidability conditions were also studied in [28]. These conditions should be satisfied by the bases of the dcpos in the Scott model of PCF, but we leave a full treatment of bounded complete dcpos with bases satisfying such conditions for future investigations.

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