Addendum to the paper
Infinite sets that satisfy the principle of omniscience in all varieties of constructive mathematics

Martín Escardó
University of Birmingham, UK

Draft version of October 7, 2011

Abstract

We apply the omniscience of $\mathbb{N}_\infty$ to derive *Ishihara's Tricks*. We also offer a constructive version of *Grilliot's Trick*, and argue that it is not related to the omniscience of $\mathbb{N}_\infty$. Finally, we investigate the nature of the maps $\mathbb{N}_\infty \to 2$ in the absence of continuity axioms, using the omniscience of $\mathbb{N}_\infty$ as a tool.

1 Preliminaries

1.1 Terminology and notation

We assume the terminology and notation of [3], with a few important departures and clarifications explained below.

Operations, functions, maps and mappings. By an *operation* we mean a function that is not necessarily extensional, and by a *function* or *map* or *mapping* we mean an extensional operation. This is standard terminology in Bishop's mathematics. But we still work within spartan constructive mathematics.

Subsets and extensionality. Subsets are taken to be extensional by definition, with respect to the equality of their ambient sets.

Apartness and strong extensionality. We remark that we use the symbol $\neq$ for negation of equality in [3], rather than for apartness as often done by Bishop mathematics practitioners. We work with the standard apartness on $2^{\mathbb{N}}$ and its restriction to $\mathbb{N}_\infty$ [3], and with the negation-of-equality apartness on discrete sets, such as 2 and $\mathbb{N}$. If X and Y are equipped with apartness relations $\sharp$, a function $f\colon X \to Y$ is is called *strongly extensional* if it reflects apartness, in the sense that $f(x) \sharp f(x') \implies x \sharp x'$. Whenever we speak of strongly extensional functions, we implicitly assume that their source and target come equipped with (standard or given) apartness relations.

1.2 Basic arithmetic on $\mathbb{N}_\infty$

For $x \in \mathbb{N}_\infty$, define $x + 1 \in \mathbb{N}_\infty$ by cases as

$$(x+1)_0 = 1, \qquad (x+1)_{n+1} = x_n.$$

Then, by induction for the first equation and by cases for the second one,

$$\underline{n} + 1 = \underline{n+1}, \qquad \infty + 1 = \infty.$$

Also
$$x + 1 = x \implies x = \infty, \text{ and } x + 1 = \infty \implies x = \infty.$$
We order $\mathbb{N}_\infty$ lexicographically, or equivalently pointwise [3, Section 6]:
$$x \leq y \iff x_i \leq y_i \text{ forall } i.$$
Hence the minimum and maximum functions $\min, \max \colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ are given pointwise,
$$\min(x, y)(i) = \min(x_i, y_i), \qquad \max(x, y)(i) = \max(x_i, y_i),$$
and
$$x \leq y \iff \min(x, y) = x \iff \max(x, y) = y.$$
Notice that
$$\min(\underline{m}, \underline{n}) = \underline{\min(m, n)}, \qquad \min(\underline{m}, \infty) = \min(\infty, \underline{m}) = \underline{m},$$
$$\max(\underline{m}, \underline{n}) = \underline{\max(m, n)}, \qquad \max(\underline{m}, \infty) = \max(\infty, \underline{m}) = \infty.$$
Define also
$$\min \colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$$
by
$$\min(x, n) = \sum_{i<n} x_i = \sup\{i \in [0, n] \mid x_i = 1\},$$
where of course the sup of the empty set is 0, and define $\min \colon \mathbb{N} \times \mathbb{N}_\infty \to \mathbb{N}$ by $\min(n, x) = \min(x, n)$. Then
$$\min(\underline{m}, n) = \min(m, n), \qquad \min(\infty, n) = n, \qquad \underline{\min(x, n)} = \min(x, \underline{n}).$$
Later we'll often write $\underline{n}$ as simply n by an abuse of notation, which is justified by the arthmetical properties discussed above.

1.3 Apartness on $\mathbb{N}_\infty$

The standard apartness on $2^{\mathbb{N}}$ and $\mathbb{N}_\infty$ are defined in [3, Section 6]. We will use the following fact for $x \in \mathbb{N}_\infty$:
$$x \mathrel{\sharp} x + 1 \implies \exists n \in \mathbb{N}(x = \underline{n}).$$
This is seen as follows. By definition of the standard apartness relation, x and $x+1$ differ at some index, and so one of them (and hence the other) is apart from ∞. So it remains to show that
$$x \mathrel{\sharp} \infty \implies \exists n \in \mathbb{N}(x = \underline{n}).$$
The hypothesis simply means that $x_i = 0$ for some i, and hence the conclusion follows from [3, Lemma 3.2]. Of course, which is very important for the results developed below, one cannot derive the same conclusion from the weaker hypothesis that $x \neq \infty$. Notice also that
$$\underline{m} \mathrel{\sharp} \underline{n} \iff m \neq n,$$
so that the embedding $(n \mapsto \underline{n}) \colon \mathbb{N} \to \mathbb{N}_\infty$ is strongly extensional. Notice that
$$\max(x, y) \mathrel{\sharp} \infty \implies x \mathrel{\sharp} \infty.$$
The following is related to the fact that the condition $x = \underline{n}$ is decidable for any $x \in \mathbb{N}_\infty$ and $n \in \mathbb{N}$.

Lemma 1.1 *A function $p\colon \mathbb{N}_\infty \to 2$ is strongly extensional iff $p(x) \neq p(\infty)$ implies $x \sharp \infty$ for every $x \in \mathbb{N}_\infty$*

Proof ($\Leftarrow$) Assume that $p(x) \neq p(y)$. Then either $p(x) \neq p(\infty)$ or $p(y) \neq p(\infty)$ by co-transitivity. Assume the first case without loss of generality. Then $x \sharp \infty$ by the hypothesis, and $x \sharp y$ or $y \sharp \infty$ again by co-transitivity. In the first case we are done, and hence assume the second. Then there are $m, n \in \mathbb{N}$ with $x = \underline{m}$ and $y = \underline{n}$. If we had $m = n$, the extensionality of p would give $p(x) = p(\underline{m}) = p(\underline{n}) = p(y)$, and hence we must have $m \neq n$, which shows that $x \sharp y$, as required. $\square$

1.4 Continuity and discontinuity

We say that a map $p\colon \mathbb{N}_\infty \to 2$ is continuous if

$$\exists n \in \mathbb{N}\, \forall m \geq n (p(m) = p(\infty)),$$

where we write m to mean $\underline{m}$. Of course there are continuous maps:

Proposition 1.2 *If a sequence $\alpha\colon \mathbb{N} \to 2$ is eventually constant, then it extends to a strongly extensional, continuous map $p\colon \mathbb{N}_\infty \to 2$.*

Proof The hypothesis is that $\exists n \in \mathbb{N}\, \forall m > n(\alpha_m = \alpha_n)$. Let $p(x) = \alpha_{\min(x,n)}$, where $\min\colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$ is defined in Section 1.2. Then $p(m) = \alpha_m$ if $m < n$, and $p(m) = \alpha_n$ if $m \geq n$, so that $p(m) = \alpha_m$ for every $m \in \mathbb{N}$. Also $p(\infty) = \alpha_n$, which shows that p is continuous. If $p(x) \neq p(\infty)$, that is, $\alpha_{\min(x,n)} \neq \alpha_n$, then $\min(x, n) \neq n$ and so there is $i < n$ with $x_i = 0$, and hence $x \sharp \infty$. Therefore p is strongly extensional by Lemma 1.1. $\square$

We say that $p\colon \mathbb{N}_\infty \to 2$ is *discontinuous* if

$$\forall n \in \mathbb{N}\, \exists m \geq n(p(m) \neq p(\infty)).$$

This is a strengthening and positive formulation of the negation of continuity. However, Corollary 2.4 below shows that, in the strongly extensional case, the two notions agree, with an application of the omniscience of $\mathbb{N}_\infty$.

1.5 The generic convergent sequence

This justifies the terminology adopted in [3]:

Lemma 1.3 *Any Cauchy sequence $x\colon \mathbb{N} \to X$ in a complete metric space X extends to a strongly extensional map $x\colon \mathbb{N}_\infty \to X$ with $x_\infty = \lim_n x_n$.*

Proof For any $\alpha \in \mathbb{N}_\infty$ define a sequence $y = y^\alpha \in X^{\mathbb{N}}$ by induction as $y_0 = x_0$, $y_{n+1} = x_{n+1}$ if $\alpha_n = 1$ and $y_{n+1} = y_n$ if $\alpha_n = 0$. This is a Cauchy sequence, and by the completeness of X we can define $x_\alpha = \lim_n y_n^\alpha$, and the stated requirements are easily verified. $\square$

Moreover, such an extension is continuous, and any strongly extensional continuous function $x\colon \mathbb{N}_\infty \to X$ arises in this way. But we don't need these two additional facts.

1.6 Choice and decidability

The following is folklore — see e.g. [4]:

Lemma 1.4 (Using choice) *For any two subsets P, Q of a set X with $P \cup Q = X$, there are disjoint sets $P' \subseteq P$ and $Q' \subseteq Q$ with $P' \cup Q' = X$.*

Proof By the hypothesis, $\forall x \in X \exists y \in 2(y = 0 \implies x \in P \wedge y = 1 \implies x \in Q)$, and by choice, $\exists p\colon X \to 2(\forall x \in X(p(x) = 0 \implies x \in P \wedge p(x) = 1 \implies x \in Q)$. To conclude, let $P' = p^{-1}(0)$ and $Q' = p^{-1}(1)$. $\square$

1.7 Taboos

Recall the following definitions (limited principle of omniscience, weak limited principle of omniscience, and Markov's principle) where α ranges over $2^{\mathbb{N}}$:

$$\begin{aligned} \text{LPO} &\iff \exists n \in \mathbb{N}(\alpha_n = 0) \vee \forall n \in \mathbb{N}(\alpha_n = 1), \\ \text{WLPO} &\iff \forall n \in \mathbb{N}(\alpha_n = 1) \vee \neg\forall n \in \mathbb{N}(\alpha_n = 1), \\ \text{MP} &\iff \neg\forall n \in \mathbb{N}(\alpha_n = 1) \implies \exists n \in \mathbb{N}(\alpha_n = 0). \end{aligned}$$

Clearly LPO $\iff$ WLPO $\wedge$ MP. The first two are taboos in all varieties of constructive mathematics, and the third is considered dubious in some but not all varieties. Notice that a sequence $\alpha \in 2^{\mathbb{N}}$ satisfies one of the above conditions if and only if the sequence $r(\alpha) \in \mathbb{N}_\infty$ satisfies the same condition, where $r\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is the retraction defined in [3, Proposition 3.1]. Hence these principles can be formulated in terms of the sets $\mathbb{N}_\infty$ and $\mathbb{N}$ as follows:

$$\begin{aligned} \text{LPO} &\iff \mathbb{N}_\infty = \underline{\mathbb{N}} \cup \{\infty\} &&\iff \forall x \in \mathbb{N}_\infty(x \sharp \infty \vee x = \infty), \\ \text{WLPO} &\iff \mathbb{N}_\infty = \mathbb{N}_\infty \setminus \{\infty\} \cup \{\infty\} &&\iff \forall x \in \mathbb{N}_\infty(x \neq \infty \vee x = \infty), \\ \text{MP} &\iff \mathbb{N}_\infty \setminus \{\infty\} = \underline{\mathbb{N}}, &&\iff \forall x \in \mathbb{N}_\infty(x \neq \infty \implies x \sharp \infty). \end{aligned}$$

The first one was already remarked immediately after [3, Lemma 3.4]. We also claimed next to this that the denumerability of $\mathbb{N}_\infty$ implies LPO. But one has to be careful here. If by denumerability we mean that there is a bijection, then this is clear. But if we mean that there is a surjection, the situation is subtler. We will discuss this in a future version of these notes. For the moment we just report without proof that the existence of a surjection does definitely lead to a taboo, which is not so easily formulated.

2 Ishihara's tricks from the omniscience of $\mathbb{N}_\infty$

We claim that the following lemma is the essence of *Ishihara's First Trick* [7, 2]. The point of the lemma is that the quantifications are over $\mathbb{N}$ rather than $\mathbb{N}_\infty$. As above, by an abuse of notation, we will often write n to mean $\underline{n}$.

Lemma 2.1 *If a map $p\colon \mathbb{N}_\infty \to 2$ is strongly extensional, then*

$$\exists n \in \mathbb{N}(p(n) \neq p(\infty)) \vee \forall n \in \mathbb{N}(p(n) = p(\infty)).$$

Proof Because the condition $p(x) = p(\infty)$ is decidable for any $x \in X$, the omniscience of $\mathbb{N}_\infty$ gives $\exists x \in \mathbb{N}_\infty(p(x) \neq p(\infty))$ or $\forall x \in \mathbb{N}_\infty(p(x) = p(x_\infty))$. If the first case holds, then $x \sharp \infty$ by the strong extensionality of p, and hence $x = n$ for some $n \in \mathbb{N}$, so that $p(n) = p(\infty)$ by the extensionality of p. If the second case holds, then in particular $\forall n \in \mathbb{N}(p(n) = p(\infty))$ by considering $x = n$. $\square$

The following theorem is almost literally the general version [2, Proposition 1] of *Ishihara's First Trick* from [7], with a restriction and a generalization:

1. The disjointness condition is not assumed in [2].

   This assumption is more restrictive, but allows us to avoid the axiom of choice. Although the axiom of (countable) choice is not explicitly mentioned in [2], it is tacitly applied. We confine the application of choice to Lemma 1.4.

2. A complete metric on X and a Cauchy sequence $x_n \in X$ are assumed in [2].

   Without the metric assumption, and replacing Cauchy sequences by strongly extensional maps $x\colon \mathbb{N}_\infty \to X$, we are more general, in view of Lemma 1.3.

Theorem 2.2 *If P, Q are disjoint subsets a set X with $P \cup Q = X$ and $x\colon \mathbb{N}_\infty \to X$ is a strongly extensional map with*

$$\forall y \in X(y \sharp x_\infty \vee y \notin Q),$$

then

$$\forall n \in \mathbb{N}(x_n \in P) \vee \exists n \in \mathbb{N}(x_n \in Q).$$

Proof Define $q\colon X \to 2$ by $q(x) = 0 \iff x \in Q$. Then $q(x_\infty) = 1$ and so the hypothesis amounts to $y \sharp x_\infty \vee q(y) = q(x_\infty)$, which is equivalent to the implication $q(y) \neq q(x_\infty) \implies y \sharp x_\infty$. Hence the map $p = q \circ x$ is strongly extensional by Lemma 1.1, and the result follows from Lemma 2.1 applied to p. □

Assuming choice as in [2], we recover [2, Proposition 1], using projective covers if necessary, as in [9], and Lemmas 1.3 and 1.4.

We now derive a form of *Ishihara's Second Trick* [2, Proposition 2] from two nested applications of Lemma 2.1 (using the idea of proof of [3, Theorem 4.1(3-4)]).

Lemma 2.3 *If $p\colon \mathbb{N}_\infty \to 2$ is strongly extensional, then*

$$\exists n \in \mathbb{N}\, \forall m \geq n(p(m) = p(\infty)) \vee \forall n \in \mathbb{N}\, \exists m \geq n(p(m) \neq p(\infty)).$$

With the terminology of Section 1.4, this amounts to saying that p is either continuous or discontinuous.

Corollary 2.4 *If a strongly extensional map $p\colon \mathbb{N}_\infty \to 2$ fails to be continuous, then it is discontinuous in the positive sense defined in Section 1.4.*

Proof of Lemma 2.3. Define $q\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to 2$ by $q(x, y) = p(\max(x, y))$. Because p and $\max\colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$ are strongly extensional so is q. By Lemma 2.1 applied to the function $(y \mapsto q(x, y))\colon \mathbb{N}_\infty \to 2$, there is $r\colon \mathbb{N}_\infty \to 2$ such that

$$\begin{aligned} r(x) = 1 &\iff \exists m \in \mathbb{N}(q(x, m) \neq q(x, \infty)), \\ r(x) = 0 &\iff \forall m \in \mathbb{N}(q(x, m) = q(x, \infty)). \end{aligned}$$

Then $r(\infty) = 0$ because otherwise $\exists m \in \mathbb{N}(q(\infty, m) \neq q(\infty, \infty))$, which would amount to $p(\infty) \neq p(\infty)$. Now assume that $r(x) \neq r(\infty)$, that is, $r(x) = 1$. Then $q(x, m) \neq q(x, \infty)$ for some $m \in \mathbb{N}$, which amounts to $p(\max(x, m)) \neq p(\infty)$, and the strong extensionality of p gives $\max(x, m) \sharp \infty$ and so $x \sharp \infty$. Hence r is strongly extensional by Lemma 1.1. By Lemma 2.1 applied to r, and expanding the definitions of q and r, using the fact that $r(\infty) = 0$, we conclude that

$$\begin{aligned} &\exists n \in \mathbb{N}\, \forall m \in \mathbb{N}(p(\max(n, m)) = p(\max(n, \infty))) \\ \vee \quad &\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}(p(\max(n, m)) \neq p(\max(n, \infty))), \end{aligned}$$

which is equivalent to the desired conclusion. □

Lemma 2.3 amounts to a form of *Ishihara's Second Trick*, with the same modifications discussed in the paragraph preceding Theorem 2.2, and, moreover, with a weaker hypothesis than that of [2, Proposition 2]:

Theorem 2.5 *If P, Q are disjoint subsets a set X with $P \cup Q = X$ and $x\colon \mathbb{N}_\infty \to X$ is a strongly extensional map with*

$$\forall y \in X(y \sharp x_\infty \vee y \notin Q),$$

then

$$\forall n \in \mathbb{N}\, \exists m \geq n(x_m \in P) \vee \exists n \in \mathbb{N}\, \forall m \geq n(x_m \in Q).$$

Proof Literally the same as that of Theorem 2.2, but using Lemma 2.3 rather than 2.1 in the final step. □

We again use Lemmas 1.3 and 1.4 to get [2, Proposition 2] (with the weaker hypothesis) as a corollary.

3 Grilliot's Trick from a constructive perspective

In the context of higher-type recursion theory developed within classical mathematics, Grilliot [5, Lemma 1] showed that one can effectively define a functional $E : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ such that

$$E(h) = 0 \iff \exists n \in \mathbb{N}(h(n) = 0)$$

from any effectively discontinuous $F\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$. This is known as *Grilliot's Trick* in the research community, and sometimes in print, as in e.g. [6]. Here F is called effectively discontinuous if there is a sequence $g_i\colon \mathbb{N} \to \mathbb{N}$ with limit f, with both g_i and f recursive in F, such that $F(f) \neq \lim_i F(g_i)$.

We reproduce Grilliot's argument, for comparison with a constructive counter-part given below. By taking a subsequence, we may assume that $F(f) \neq F(g_i)$ for every i. Again by taking a subsequence, we may assume that $g_i(j) = f(j)$ for all $j \leq i$. Notice that, in both cases, one needs unbounded search to find the next element of the subsequence. If one now defines $J\colon (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$ by

$$J(h)(j) = g_i(j), \text{where } i = \inf\{n \in [0, j] \mid h(n) = 0\},$$

and where the infimum of the empty set is of course the largest element j of the integer interval $[0, j]$, then

$$J(h) = \begin{cases} f & \text{if } \forall n \in \mathbb{N}(h(n) \neq 0), \\ g_i & \text{if } h(i) = 0 \text{ and } \forall n < i(h(n) \neq 0). \end{cases}$$

Therefore E can be defined by

$$E(h) = 0 \iff F(J(h)) \neq F(f).$$

Although the definitions are explicit, their correcteness proofs rely on classical logic (Hartley [6] addresses this to some extent).

We offer Theorem 3.2 below as a constructive counter-part of Grilliot's Trick, where we take discontinuity, as defined in Section 1.4 and investigated in Section 2, as a constructive replacement of the notion of effective discontinuity. With this stronger (classically equivalent) notion, we avoid unbounded search, but we haven't managed to get away without countable choice. Another difference is our consideration of maps $p\colon \mathbb{N}_\infty \to 2$ rather than $F\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$, but this is inessential.

Lemma 3.1 *Any monotone increasing, inflationary map $g\colon \mathbb{N} \to \mathbb{N}$ extends to a strongly extensional map $G\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ with $G(\infty) = \infty$.*

The conditions on g are $m \leq n \implies g(m) \leq g(n)$ and $n \leq g(n)$ respectively.

Proof Let $G(x)(n) = 1 \iff n < g(\min(x, n+1))$, where the function $\min\colon \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{N}$ is defined in Section 1.2. Then $G(x) \in \mathbb{N}_\infty$, because this amounts to $n + 1 < g(\min(x, n+2)) \implies n < g(\min(x, n+1))$. We have $G(\infty) = \infty$ because $G(\infty)(n) = 1$ for any n. The map G extends g, i.e. $G(\underline{k}) = \underline{g(k)}$, if and only if $G(\underline{k})(n) = 1 \iff \underline{g(k)}(n) = 1$, that is, $n < g(\min(k, n+1)) \iff n < g(k)$, which is seen as follows. ($\Rightarrow$): By monotonicity, $g(\min(k, n+1)) \leq g(k)$, and hence transitivity gives $n < g(k)$ from the hypothesis. ($\Leftarrow$): By inflationarity, $n < g(n+1)$ and hence the hypothesis gives $n < \min(g(k), g(n+1)) = g(\min(k, n+1)$, because g is monotone. Finally, to prove strong extensionality, assume $G(x) \sharp G(y)$, that is, $G(x)(n) \neq G(y)(n)$ for some n. Without loss of generality, assume that $G(x)(n) = 0$ and $G(y)(n) = 1$, that is, $g(\min(x, n+1)) \leq n < g(\min(y, n+1)$. Then $x \leq n < y$, and hence $x_n = 0$ and $y_n = 1$, which shows that $x \sharp y$, as required. $\square$

The extension G is also continuous, for any version of continuity we can think of, but we don't need to know this. Our proof of the following is essentially the same as that of Grilliot's Trick, but we argue constructively, as discussed above:

Theorem 3.2 *Assuming countable choice:*

1. *If there is a discontinuous map $p\colon \mathbb{N}_\infty \to 2$, then WLPO holds.*

2. *If there is a discontinuous strongly extensional map $p\colon \mathbb{N}_\infty \to 2$, LPO holds.*

Proof Applying choice to the discontinuity hypothesis $\forall n \exists m \geq n(p(n) \neq p(\infty)$, we get a map $g\colon \mathbb{N} \to \mathbb{N}$ with $g(n) \geq n$ and $p(g(n)) \neq p(\infty)$. We can ensure, with search bounded by $g(n)$ after choice is applied, that $g(n)$ is the least $m \geq n$ with $p(m) \neq p(\infty)$, so that g is monotone increasing and hence extends a strongly extensional map $G\colon \mathbb{N}_\infty \to \mathbb{N}_\infty$ with $G(\infty) = \infty$ by Lemma 3.1. To prove WLPO we show that $x = \infty$ or $x \neq \infty$ for any $x \in \mathbb{N}_\infty$, and to prove LPO we show that $x = \infty$ or $x \sharp \infty$. We reduce these tests to the decidable condition $p(G(x)) = p(\infty)$:

(*i*) $p(G(x)) = p(\infty)$: Then $p(G(x)) \neq p(G(n))$ for any $n \in \mathbb{N}$ because $p(G(n)) = p(g(n)) \neq p(\infty)$, and hence $x \neq n$ by the extensionality of the map $p \circ G$. Therefore $x = \infty$ by [3, Lemma 3.3].

(*ii*) $p(G(x)) \neq p(\infty) = p(G(\infty))$: Then $x \neq \infty$ by the extensionality of $p \circ G$, and $x \sharp \infty$ if p, and hence $p \circ G$, is strongly extensional. □

The essence of the above proof is this: From a map p with modulus of discontinuity g, construct a function p' such that $p'(n) \neq p'(\infty)$ for every $n \in \mathbb{N}$, so that for any given $x \in \mathbb{N}_\infty$ we can test whether $x = \infty$ holds by checking whether the decidable condition $p'(x) = p'(\infty)$ holds. This is the function $p \circ G$. The role of composition with G is to elimitate the values of p at arguments in $\mathbb{N}$ that agree with $p(\infty)$.

Notice that this argument (due to Grilliot) *doesn't* use the omniscience of $\mathbb{N}_\infty$, and *doesn't* have it as a corollary. The argument shows that the hypothetical existence of a discontinuous function entails a taboo, LPO, the omniscience of $\mathbb{N}$ (rather than that of $\mathbb{N}_\infty$), under the assumption of strong extensionality, or just WLPO without the assumption. The omniscience of $\mathbb{N}_\infty$, which is a fact rather than a taboo, is proved with an essentially different argument. However, using the omniscience of $\mathbb{N}_\infty$ as in Corollary 2.4 to get discontinuity from non-continuity, we have (relying on countable choice):

Corollary 3.3 *If some strongly extensional map $p\colon \mathbb{N}_\infty \to 2$ fails to be continuous, then LPO must hold.*

By *Kreisel's Tricks* we mean the ideas attributed to Kreisel in Exercise 1 of [1, page 581]). They sketch how to (1) perform search over $\mathbb{N}_\infty$ (in system T assuming classical logic to prove correctness) and (2) use this to effectively decide whether or not a function has a particular kind of discontinuity (namely $p(n) = 0$ for all n and $p(\infty) = 1$, cf. Lemma 2.1). This exercise was the starting point of this work, as stated in [3, Section 9]. In summary, our analysis and constructive reworking of Kreisel's and Grilliot's ideas allow the following reading of the various tricks:

1. Grilliot's Trick = The existence of a discontinuous function implies a taboo.

2. Kreisel's Tricks = Omniscience of $\mathbb{N}_\infty$ & a special case of decidable continuity.

3. Ishihara's Trick = Non-continuous functions are discontinuous.

The paper [3] is a constructive reworking of Kreisel's Trick (and more), this section is a constructive reworking of Grilliot's Trick, and the previous section is a derivation of Ishihara's Trick from the constructive version of Kreisel's Trick. The following section develops additional information.

4 The nature of the maps $\mathbb{N}_\infty \to 2$

We now briefly investigate the nature of the maps $\mathbb{N}_\infty \to 2$, still in the absence of continuity axioms. In particular, we investigate the collection of sequences $\alpha\colon \mathbb{N} \to 2$ that arise as restrictions of functions $p\colon \mathbb{N}_\infty \to 2$. Equivalently, we investigate the sequences $\alpha \in 2^{\mathbb{N}}$ that can be extended to functions $p \in 2^{\mathbb{N}_\infty}$. It turns out that, even in the absence of continuity axioms, they are fairly restricted in character. Eventually constant sequences constructively satisfy the LPO condition, and hence the WLPO condition too. Although without continuity axioms one cannot prove that only the eventually constant sequences $\alpha \in 2^{\mathbb{N}}$ can be extended to maps $p \in 2^{\mathbb{N}_\infty}$, we can show that only those that satisfy the WLPO condition can be extended (Theorem 4.2). We begin with a lemma that mixes quantification over $\mathbb{N}_\infty$ and $\mathbb{N}$.

Lemma 4.1 *For any map $p\colon \mathbb{N}_\infty \to 2$,*

$$\exists x \in \mathbb{N}_\infty (x \neq \infty \wedge p(x) = 0) \vee \forall n \in \mathbb{N}(p(n) = 1).$$

Proof Because the condition $p(y) = p(y+1)$ is decidable, the omniscience of $\mathbb{N}_\infty$ tells us that one of the following two cases holds:

$$(1) \quad \exists y \in \mathbb{N}_\infty (p(y) \neq p(y+1)) \quad \vee \quad (2) \quad \forall y \in \mathbb{N}_\infty (p(y) = p(y+1)).$$

(1) Then $y \neq \infty$, for if we had $y = \infty$, we would also have $y = y+1$ and hence $p(y) = p(y+1)$ by extensionality. Hence we also have $y+1 \neq \infty$, for if we had $y+1 = \infty$ we would have $y = \infty$. Because one of $p(y)$ and $p(y+1)$ must be zero, we conclude that $\exists x \in \mathbb{N}_\infty (x \neq \infty \wedge p(x) = 0)$.

(2) If $p(0) = 0$, then the example $x = 0$ shows that $\exists x \in \mathbb{N}_\infty (x \neq \infty \wedge p(x) = 0)$. Otherwise we conclude that $\forall n \in \mathbb{N}(p(n) = 1)$ by induction on n. □

We now quantify over $\mathbb{N}$ only:

Theorem 4.2 *For any map $p\colon \mathbb{N}_\infty \to 2$,*

$$\forall n \in \mathbb{N}(p(n) = 1) \vee \neg\forall n \in \mathbb{N}(p(n) = 1).$$

Proof If the second disjunct of the lemma holds, there is nothing to prove, and hence assume the first. By [3, Lemma 3.3], the condition $x \neq \infty$ implies $\neg\forall n \in \mathbb{N}(x \neq n)$. Therefore we conclude that $\neg\forall n \in \mathbb{N}(p(n) = 1)$ holds, for if we had $\forall n \in \mathbb{N}(p(n) = 1)$ we would have $\forall n \in \mathbb{N}(x \neq n)$ by extensionality and the fact that $p(x) = 0$, which would be a contradiction. □

If a sequence $\alpha \in 2^{\mathbb{N}}$ can be extended to a strongly extensional map $p \in 2^{\mathbb{N}}$, then it must satisfy the LPO condition:

Theorem 4.3 *For any strongly extensional map $p\colon \mathbb{N}_\infty \to 2$,*

$$\exists n \in \mathbb{N}(p(n) = 0) \vee \forall n \in \mathbb{N}(p(n) = 1).$$

Proof We again consider the cases (1) and (2) of Lemma 4.1.

(1) By the strong extensionality of p, we have that $y \sharp y+1$ and hence $y = n$ for some $n \in \mathbb{N}$ by Section 1.2. Because one of $p(y)$ and $p(y+1)$ must be zero, we conclude that $\exists n \in \mathbb{N}(p(n) = 0)$.

(2) The argument is literally the same as that of case (2) of Lemma 4.1. □

Notice that this is similar to, but not quite the same as, Lemma 2.1. With the same kind of argument as in Lemma 2.3, using two nested applications of Theorem 4.3, we conclude:

Corollary 4.4 *For any strongly extensional map $p\colon \mathbb{N}_\infty \to 2$,*

$$\exists n \in \mathbb{N}\, \forall m \geq n(p(m) = 1) \vee \forall n \in \mathbb{N}\, \exists m \geq n(p(m) = 0).$$

References

[1] H.P. Barendregt. *The Lambda-Calculus: its Syntax and Semantics.* North-Holland, 1984.

[2] D. Bridges, D. van Dalen, and H. Ishihara. Ishihara's proof technique in constructive analysis. *Indag. Math. (N.S.)*, 14(2):163–168, 2003.

[3] M. H. Escardó. Infinite sets that satisfy the principle of omniscience in all varieties of constructive mathematics. University of Birmingham, UK, `http://www.cs.bham.ac.uk/~mhe/papers/index.html`, July 2011.

[4] M. H. Escardó and P. Oliva. Searchable sets, Dubuc–Penon compactness, omniscience principles, and the Drinker Paradox. In F. Ferreira, H. Guerra, E. Mayordomo, and J. Rasga, editors, *Computability in Europe 2010, Abstract and Handout Booklet*, pages 168–177. Centre fo Applied Mathematics and Information Technology, Department of Mathematics, University of Azores, 2010.

[5] T. J. Grilliot. On effectively discontinuous type-2 objects. *J. Symbolic Logic*, 36:245–248, 1971.

[6] J. P. Hartley. Effective discontinuity and a characterisation of the superjump. *J. Symbolic Logic*, 50(2):349–358, 1985.

[7] H. Ishihara. Continuity and nondiscontinuity in constructive mathematics. *J. Symbolic Logic*, 56(4):1349–1354, 1991.

[8] H. Ishihara. Informal constructive reverse mathematics. Technical Report CDMTCS-229, Centre for Discrete Mathematics and Theoretical Computer Science, Auckland University, January 2004. `http://www.cs.auckland.ac.nz/CDMTCS/researchreports/229hajime.pdf`.

[9] E. Palmgren. Bishop's set theory. TYPES summer school, `http://www2.math.uu.se/~palmgren/ErikP_BISH.ps`, 2005.