Compactly generated Hausdorff locales

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Abstract

The main theorem is that two natural notions of compactly generated locale coincide in the Hausdorff case. One goes back to work of Kelley on exponential laws for maps of topological Hausdorff spaces which are continuous on compact subspaces. The other goes back to work of Lawson on the duality of continuous posets and of Hofmann and Mislove on local compactness and continuous lattices; it was formulated by Hofmann and Lawson, who already considered part of the relationship to the first in the case of topological spaces. Some partial results in the absence of the Hausdorff separation axiom are also reported.

Key words: Compactly generated locale, continuity on compact subspaces, Hausdorff locale, k-space, Lawson duality, Hofmann–Mislove theorem, local compactness, continuous lattice.

MSC: 54D50, 06D22, 54C08, 06B35

1 Introduction

We define the *Kelleyification* of a locale X, denoted by KX, to be the colimit of the diagram whose vertices are the compact sublocales of X and whose arrows are the embeddings between the vertices in the lattice of sublocales of X. Because a continuous map defined on KX is essentially the same thing as a consistent family of continuous maps defined on the compact sublocales of X, the role of KX is to articulate the notion of a not necessarily continuous map on X which is continuous on the compact sublocales of X. We thus define a *compactly continuous map* on X with values on a locale Y to be a continuous map $KX \to Y$.

Because the embeddings of the compact sublocales of X into X form a cocone, there is a unique map $\varepsilon_X : KX \to X$ through which every inclusion of a compact sublocale of X factors via the leg of this compact sublocale into the colimit object KX — see Section 4 below for details. According to the above idea, composition with this canonical map shows that every continuous map on X 'is' compactly continuous. Thus, we can say that X is compactly generated if the converse holds: every compactly continuous map on X is genuinely continuous.

1.1 DEFINITION We say that a locale X is *compactly generated in the sense of Kelley* if the canonical map $\varepsilon_X : KX \to X$ is a homeomorphism (isomorphism in the category of locales).

The localic version of the Hofmann–Mislove theorem [8] states that the first Lawson dual (preframe of Scott open filters) of the topology of a locale is isomorphic to the collection of compact fitted sublocales under reverse inclusion [13, 23]. With this in mind, the following requires that the topology of the locale be the Lawson dual of the preframe of compact fitted sublocales. For details, see Section 2 below.

1.2 DEFINITION We say that a locale is *compactly generated in the sense of Hofmann and Lawson* if its topology is canonically isomorphic to its second Lawson dual.

Hofmann and Lawson [7] proved that if a Hausdorff topological space is compactly generated in the sense of Kelley then it is also compactly generated in their sense. We prove this and the converse for locales.

1.3 THEOREM *For Hausdorff locales, the above two notions of compact generation coincide.*

But we can say more.

1.4 THEOREM *The second Lawson dual of the topology of a Hausdorff locale is isomorphic to the topology of the Kelleyification of the locale.*

In particular, because double Lawson dualization is functorial, the assignment $X \mapsto KX$ is the object part of a functor from Hausdorff locales that makes the family of maps $\varepsilon_X : KX \to X$ into a natural transformation — but this observation is not developed in this paper.

Open problems. Of course, one would like to know that the Kelleyification operation is idempotent on Hausdorff locales. For this, we need KX and X to have the same compact sublocales if X is Hausdorff. By 4.3 below, we know that every compact sublocale of X is a sublocale of KX. Because direct images of compact sublocales are compact, the canonical map $\varepsilon : KX \to X$ gives a compact sublocale $\varepsilon[Q]$ of X for each compact sublocale Q of KX, which, by the previous observation, is again a compact sublocale of KX. What we don't know at the time of writing is whether the compact sublocales Q and $\varepsilon[Q]$ are the same. This difficulty doesn't arise in the topological case because there KX has the same set of points as X with a finer topology and $\varepsilon : KX \to X$ is just the identity on points, so that a compact subset Q of KX and its image $\varepsilon[Q]$ are trivially the same — see e.g. Mac Lane [17, Section VII-8] or Borceux [2, Section 7.2].

To get a coreflection, one needs that, moreover, the Kelleyification be also Hausdorff. Alternatively, one could work with a weakening of the Hausdorff separation axiom. One which has proved useful for the purposes of the technical development of this paper is that every compact sublocale be closed and Hausdorff, but it is not clear whether this weakening fits the bill. (Of course, once one has such a coreflection, the above notion of compactly continuous map coincides with that of morphism of the co-Kleisli category of the comonad induced by the coreflection.)

Regarding cartesian closedness, if each compact sublocale of X is closed and Hausdorff then it is locally compact and hence exponentiable in the category of locales by Hyland [9]. Thus, it is natural to attempt to construct the exponential Y^X of two compactly generated Hausdorff(-like) locales by taking the Kelleyification of the limit of the exponentials Y^Q for $Q \leq X$ compact. By Johnstone [12], if each Q were an open locale (as a locale on its own, not as a sublocale of X), then Y^Q would be Hausdorff if Y is, but there is no reason to expect compact sublocales to be open. But, even if Y^Q proves to be Hausdorff(-like), it is not clear how much of the separation property survives the limit process.

Outline of technical development. As a first step, we show that the topology of KX, for X Hausdorff, is isomorphic to the frame of nuclei j on $(\mathcal{O}\,X)^\wedge$ with $j^{-1}(1)$ Scott open, where the preframe $(\mathcal{O}\,X)^\wedge$ is the Lawson dual of the topology $\mathcal{O}\,X$ of the locale X. This is the content of Theorem 5.1. Section 4 contains the calculation of the colimit KX that leads to Theorem 5.1. As a second step, we show that such nuclei are in bijection with the second Lawson dual $(\mathcal{O}\,X)^{\wedge\wedge}$. This is formulated as Theorem 5.8 and amounts to Theorem 1.4 above. Chasing the isomorphisms, Theorem 1.3 follows as a corollary. All of this relies on representing $(\mathcal{O}\,X)^\wedge$ as a subpreframe of the frame $\mathcal{O}\,X$. This is the content of Section 3 and in particular of 3.11. The constructions of the second step mimic those of the localic Hofmann–Mislove isomorphism (as recalled in Section 2). We show in Section 7 that the second step and the Hofmann–Mislove theorem are in fact both instances of a more general phenomenon, which is formulated as Theorem 7.7. This also shows that the Hausdorff separation axiom is not necessary to perform the second step. However, the proof for this special case is interesting on its own right and hence has been included in Section 5.

Foundations. We develop our work in the familiar language of, say, ZF set theory. However, we emphasize that logical principles that are not necessarily valid in toposes other than that of sets are not invoked [14]. In particular, excluded-middle and choice principles are not used. However, notice that the powerset axiom *is* used, so that, for example, our results are not necessarily expressible or valid in formal topology [19] as developed in intuitionistic type theory [18].

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2 Locales

In this section we fix terminology and notation and recall known facts. Our main reference to locale theory is Johnstone [11] (see also [14, Chapter C]).

Frames, locales and continuous maps. A *frame* is a complete lattice in which finite meets distribute over arbitrary joins, and a homomorphism of frames is a map that preserves these operations. The category of frame homomorphisms is denoted by $\mathbf{Frm}$. A *locale* X is specified by a frame $\mathcal{O}\,X$, called the *topology* of X. The elements of the topology are called *opens* and are ranged over by the letters u, v, w. The minimum and maximum opens are denoted by 0 and 1. A *continuous map* $f\colon X \to Y$ of locales is specified by a frame homomorphism $f^*\colon \mathcal{O}\,Y \to \mathcal{O}\,X$. As a map of posets, this has a right adjoint, which is denoted by $f_*\colon \mathcal{O}\,X \to \mathcal{O}\,Y$. The category of continuous maps of locales is denoted by $\mathbf{Loc}$. Formally, $\mathbf{Loc}$ is defined as the opposite of the category $\mathbf{Frm}$ of homomorphisms of frames [10].

Sublocales and nuclei. A continuous map $j\colon X \to Y$ of locales is a *sublocale embedding* if $j^*\colon \mathcal{O}\,Y \to \mathcal{O}\,X$ is a surjection, which is equivalent to saying that the composite $j^* j_*$ is the identity of $\mathcal{O}\,X$.

A *nucleus* of a locale is a finite-meet-preserving closure operator on the topology of the locale. Under the pointwise ordering, nuclei form a frame (and hence the topology of another locale). Meets are given pointwise, but joins are harder to calculate, as a pointwise join may fail to be idempotent. Thus, a non-empty join is above the pointwise join. But notice that, by virtue of the frame distributivity law, non-empty pointwise joins are inflationary and preserve finite meets.

The sublocales of any locale are in order-reversing bijection with its nuclei. A nucleus j of a locale X induces a sublocale X_j whose topology consists of the fixed points of j. The embedding $X_j \to X$ is given by the continuous map whose defining frame homomorphism sends $u \in \mathcal{O}\,X$ to $j(u) \in \mathcal{O}\,X_j$. Conversely, any sublocale embedding $j\colon X' \to X$ induces the nucleus $j = j_* j^*$ on $\mathcal{O}\,X$ which makes X' homeomorphic to X_j.

Open and closed sublocales. Every open u can be regarded as a sublocale via the *open nucleus*

$$u^\circ(v) = (u \Rightarrow v),$$

where the operator $\Rightarrow$ is *Heyting implication*, that is, $(u \Rightarrow v)$ is the largest $w \in \mathcal{O}\,X$ with $w \wedge u \leq v$. This nucleus has a boolean complement, referred to as a *closed nucleus*, given by

$$u^\Box(v) = u \vee v.$$

Notice that the topology of the sublocale induced by this nucleus is $\uparrow u$, the principal filter generated by u in the topology of the original locale.

The *closure* of a sublocale S of a locale X is the meet of the closed sublocales F of X with $S \leq F$, denoted by $\bar{S}$.

Fitted sublocales. A sublocale is called *fitted* if it is the meet of its neighbourhoods. We approach this notion via nuclei. For any nucleus j, the set $j^{-1}(1)$ is a filter because j preserves finite meets. Remembering that the frame of nuclei is dually isomorphic to the coframe of sublocales, the following says that $j^{-1}(1)$ is the open-neighbourhood filter of the sublocale induced by j. For later use, we formulate this in more generality than is needed for the purposes of the present discussion. Notice that the notion of a nucleus makes sense for any meet-semilattice, and that the notion of an open nucleus makes sense for any meet-semilattice with Heyting implication.

2.1 *For any nucleus j on a meet-semilattice L with Heyting implication and any $u \in L$, the inequality $u^\circ \leq j$ holds if and only if $j(u) = 1$.*

PROOF If $u^\circ \leq j$ then $1 = (u \Rightarrow u) = u^\circ(u) \leq j(u)$. Conversely, assume that $j(u) = 1$ and let $w \leq u^\circ(v) = (u \Rightarrow v)$. Then $w \wedge u \leq v$ and hence $j(w) \wedge j(u) \leq j(v)$. Since $j(u) = 1$ and $w \leq j(w)$, we have that $w \leq j(v)$. Considering $w = u^\circ(v)$, we conclude that $u^\circ(v) \leq j(v)$, as required. □

In particular, $\bigvee\{u^\circ \mid j(u) = 1\} \le j$ for any nucleus j. The nucleus j is *fitted* if this is actually an equality.

Compact locales and sublocales. A locale X is *compact* if it satisfies the Heine-Borel property, that is, every open cover of X has a finite subcover. Equivalently, X is compact if every directed open cover of the maximum open 1 has 1 as a member. A sublocale (as a locale on its own) is compact if and only if its filter of open neighbourhoods (in its host locale) is Scott open (that is, inaccessible by directed joins of opens). If the sublocale is induced by a nucleus j, this is equivalent to saying that the filter $j^{-1}(1)$ is Scott open. It is natural to restrict attention to fitted sublocales in compactness considerations, because a sublocale has the same neighbourhoods as the fitted sublocale that arises by taking the meet of its neighbourhoods, and hence, from the point of view of the Heine-Borel property, they are indistinguishable.

Preframes and Lawson duality. A *preframe* is a poset with finite meets and directed joins in which the former distribute over the latter, and a homomorphism of preframes is a map that preserves these operations. The *Lawson dual* of a preframe L is the set of Scott open filters of L, which is also a preframe, denoted by $L^\wedge$, with finite meets and directed joins given by finite intersections and directed unions [16]. For any preframe L there is a *canonical map* $e\colon L \to L^{\wedge\wedge}$ defined by

$$e(u) = \{\phi \in L^\wedge \mid u \in \phi\},$$

which is a preframe homomorphism. For any preframe homomorphism $h\colon L \to M$ there is a preframe homomorphism $h^\wedge\colon M^\wedge \to L^\wedge$ that sends $\gamma \in M^\wedge$ to the set of $u \in L$ with $h(u) \in \gamma$. This construction exhibits Lawson dualization as a contravariant endofunctor of the category of homomorphisms of preframes, making the canonical map $e\colon L \to L^{\wedge\wedge}$ into a natural transformation.

The Hofmann–Mislove theorem. In its localic form, this theorem says that the map that sends a sublocale Q to the set of opens u with $Q \le u$ is an order-revering bijection from the set of compact fitted sublocales to the Lawson dual of the topology. Its inverse takes a filter to its meet in the lattice of sublocales. In terms of nuclei, the bijection sends j to $j^{-1}(1)$ and its inverse sends ϕ to $\bigvee\{u^\circ \mid u \in \phi\}$.

The dual Hofmann–Mislove theorem. For the purposes of this paragraph only, we say that an ideal of compact fitted sublocales, with respect to the sublocale ordering, is *co-Scott open* if it is inaccessible by meets of filtered sets of compact fitted sublocales. The following can be interpreted as saying that a locale is compactly generated in the sense of Hofmann and Lawson if and only if the "dual Hofmann–Mislove theorem" holds.

2.2 *A locale X is compactly generated in the sense of Hofmann and Lawson if and only if the map that sends a sublocale u to the set of compact fitted sublocales Q with $Q \le u$ is a bijection from open sublocales to co-Scott open ideals of compact fitted sublocales.*

PROOF Write $\mathcal{Q}\,X$ to denote the set of compact sublocales of X under *reverse* inclusion. Then co-Scott open ideals of compact fitted sublocales are the same thing as Scott open filters of $\mathcal{Q}\,X$, that is, elements of $(\mathcal{Q}\,X)^{\wedge}$. The Hofmann–Mislove theorem gives an isomorphism $\mathcal{Q}\,X \to (\mathcal{O}\,X)^{\wedge}$ and hence, by contravariance of Lawson duality, we get an isomorphism $(\mathcal{O}\,X)^{\wedge\wedge} \to (\mathcal{Q}\,X)^{\wedge}$. By composition with the canonical map $\mathcal{O}\,X \to (\mathcal{O}\,X)^{\wedge\wedge}$, we get a map $\mathcal{O}\,X \to (\mathcal{Q}\,X)^{\wedge}$, which is an isomorphism if and only if the canonical map is. An easy calculation shows that this map sends an open u to the set of compact fitted sublocales Q with $Q \leq u$. $\square$

Regular locales. We shall approach the Hausdorff separation axiom via the stronger condition of regularity. An open u is said to be *well inside* an open v, written $u \lessdot v$, if $\bar{u} \leq v$, where the sublocale $\bar{u}$ is the closure of the open sublocale u as discussed above. Without mentioning sublocales, this amounts to saying that there is an open w with $u \wedge w = 0$ and $v \vee w = 1$. This is in turn equivalent to saying that $v \vee \neg u = 1$, where $\neg u$ is the *Heyting complement* of u, that is, the largest open w disjoint from u. A locale is called *regular* if every open is a join of opens well inside it.

Hausdorff locales. A locale is called *Hausdorff* if its diagonal is closed. We don't use this definition directly in this paper. In the technical development that follows, we use the following facts: (1) Hausdorff locales are closed under the formation of sublocales, (2) compact sublocales of Hausdorff locales are closed [22], (3) regular locales are Hausdorff, (4) compact Hausdorff locales are regular.

We finish this section with a class of examples of compactly generated locales in the sense of Hofmann and Lawson. A locale is called *locally compact* if its topology is a continuous lattice in the sense of Dana Scott [20, 6, 11]. It is known that, under countable choice, all locally compact locales are compactly generated in this sense, where choice is used in order to establish existence of enough Scott open filters of opens. For a stably locally compact locale (a locally compact locale such that for every open $u \ll 1$, the set $\Uparrow u$ is a filter), choice is not needed, and locally compact Hausdorff locales are stably locally compact.

2.3 *Locally compact Hausdorff locales are compactly generated.*

3 The preframe of cocompact opens

Because the compact sublocales of a Hausdorff locale are closed, we can represent them by their open complements. More generally, the compact closed sublocales of any locale are in order-reversing bijection with a subpreframe of the topology of the locale. Although the propositions of this section are straightforward, they form the base for the development that follows.

3.1 DEFINITION We say that an open is *cocompact* if its boolean complement in the lattice of sublocales is compact. The poset of cocompact opens of a locale X is denoted by $\mathcal{C}\,X$ and cocompact opens are ranged over by the letters c, d, e.

Because the topology of the closed sublocale whose complement is the open u is the frame $\uparrow u$, the first and second assertions of the following proposition are equivalent.

Since a sublocale X_j of a locale X induced by a nucleus j is compact if and only if its open-neighbourhood filter $j^{-1}(1)$ is Scott open and since the complement of an open u is the sublocale induced by the closed nucleus $u^{\square} = (v \mapsto u \vee v)$, the equivalence of the first and the third follow (but it is also easy to prove this directly).

3.2 *The following are equivalent for any open $c \in \mathcal{O}\, X$ of a locale X.*

1. *c is cocompact.*
2. *1 is a compact element of the frame $\uparrow c = \{u \in \mathcal{O}\, X \mid c \leq u\}$.*
3. *The filter $\{u \in \mathcal{O}\, X \mid c \vee u = 1\}$ is Scott open.*

In particular, the maximum open 1 is always cocompact. But there is no reason why any other cocompact open should exist, even if the locale is Hausdorff. For example, the smallest dense sublocale of the localic real line is Hausdorff but has no points and hence no compact sublocales, as every non-null compact locale has (classically) at least one point [11]. The following is an immediate consequence of 3.2(2), because the relation $c \leq u$ implies $\uparrow u \subseteq \uparrow c$.

3.3 *If c is cocompact, u is open and $c \leq u$ then u is cocompact.*

3.4 COROLLARY *For any locale, the cocompact opens are closed under the formation of non-empty joins and Heyting implication in the topology.*

PROOF Closure under non-empty joins is immediate and closure under Heyting implication follows from the fact that the latter is an inflationary map in its second argument. □

Notice that cocompact opens are closed under the formation of the empty join (the minimum open 0), if and only if the locale is compact. By 3.3, this is equivalent to saying that all opens are cocompact and amounts to the familiar fact that in a compact locale all closed sublocales are compact.

The following is an immediate consequence of the fact that the collections of compact and of closed sublocales are each closed under the formation of finite joins, but it is easy to prove it directly using 3.2(3).

3.5 *Cocompact opens are closed under the formation of finite meets.*

PROOF We have already observed that they are closed under the formation of the empty meet (the maximum open 1). Let c and d be cocompact and let $\mathcal{U}$ be a directed collection of opens with $1 = \bigvee \mathcal{U} \vee (c \wedge d) = (\bigvee \mathcal{U} \vee c) \wedge (\bigvee \mathcal{U} \vee d)$. Then $1 = \bigvee \mathcal{U} \vee c$ and $1 = \bigvee \mathcal{U} \vee d$, and hence there are $u, v \in \mathcal{U}$ such that already $1 = u \vee c$ and $1 = v \vee d$. By directedness, there is $w \in \mathcal{U}$ with $u \leq w$ and $v \leq w$, and hence with $1 = w \vee c$ and $1 = w \vee d$ and therefore $1 = (w \vee c) \wedge (w \vee d) = w \vee (c \wedge d)$, which shows that $c \wedge d$ is cocompact. □

By a *Heyting preframe* we mean a preframe with Heyting implication.

3.6 COROLLARY *For any locale, the cocompact opens form a sub Heyting preframe of the topology.*

In fact, it is almost a subframe, as it may only lack closure under the formation of the empty join.

The material developed in the remainder of this section can be postponed until Section 5. We begin with a simple observation.

If a filtered collection of compact closed sublocales has null meet, then some member of the collection is already null. Formulated in terms of cocompact opens, this amounts to saying that if $\bigvee D = 1$ for a directed collection D of cocompact opens, then already $1 \in D$. In other words:

3.7 *For any locale, the top open* 1 *is compact in the preframe of cocompact opens.*

PROOF Let D be a directed set of cocompact opens with $\bigvee D = 1$ and choose any $d \in D$. Because d is cocompact, 1 is compact in $\uparrow d$ and hence $1 \in D \cap \uparrow d$ because this set has the same join as D by directedness. $\square$

We now discuss the way in which the Hausdorff separation axiom will be approached in Sections 5 and 6 below. As we have seen in Section 2, Hausdorff locales are closed under the formation of sublocales, and compact sublocales of Hausdorff locales are closed. Hence every compact sublocale of a Hausdorff locale is closed and Hausdorff. It is precisely this property of Hausdorff locales that is relevant for the arguments of those sections.

3.8 DEFINITION We say that a locale is *weakly Hausdorff* if every compact sublocale is closed and Hausdorff.

No doubt, there are many sensible ways in which the Hausdorff separation axiom can be weakened. This is the one which is relevant for the purposes of Sections 5 and 6 below.

We exploit the fact that, in the presence of compactness, Hausdorffness and regularity coincide. If Q is a compact sublocale of a weakly Hausdorff locale X, then its complement is a cocompact open, say c, and the topology of Q is the frame $\uparrow c \subseteq \mathcal{O}\,X$. The Heyting complement of $d \in \uparrow c$ calculated in the frame $\uparrow c$ is $(d \Rightarrow c)$ calculated in $\mathcal{O}\,X$, which coincides with that computed in $\mathcal{C}\,X$ by the above development.

3.9 DEFINITION We write $e \lessdot_c d$ to denote the *well-inside* relation of the frame $\uparrow c$, that is, the condition $d \vee (e \Rightarrow c) = 1$. We emphasize that when we assert that $e \lessdot_c d$ we have to make sure that d and e belong to $\uparrow c$ because this is a relation on $\uparrow c$ by definition.

Thus, saying that the compact sublocale Q is regular amounts to saying that every $d \in \uparrow c$ is the join of the cocompact opens $e \lessdot_c d$. In this case, in order to conclude that $d \leq d'$ for d and d' in $\uparrow c$, it is enough to show that $e \lessdot_c d$ implies $e \leq d'$ for every $e \in \uparrow c$. We use this and the following related method of proof for establishing inequalities of cocompact opens of a weakly Hausdorff locale.

3.10 *Assume that the frame* $\uparrow c$ *is regular and let* $d, d' \in \uparrow c$*. If* $d \vee e = 1$ *implies* $d' \vee e = 1$ *for every* $e \in \uparrow c$ *then* $d \leq d'$*.*

PROOF If $e' \lessdot_c d$ then $d \vee (e' \Rightarrow c) = 1$ and hence $d' \vee (e' \Rightarrow c) = 1$ considering $e = (e' \Rightarrow c) \in \uparrow c$, that is, $e' \lessdot_c d'$, which implies $e' \leq d'$. The result then follows from regularity. □

A simple application of this is the following lemma. Recall that $(\mathcal{O}\, X)^\wedge$ denotes the Lawson dual of $\mathcal{O}\, X$ as defined in Section 2.

3.11 *For a weakly Hausdorff locale* X*, the map* $\alpha\colon\ \mathcal{C}\, X \to (\mathcal{O}\, X)^\wedge$ *defined by*

$$\alpha(c) = \{u \in \mathcal{O}\, X \mid c \vee u = 1\}$$

is an isomorphism.

PROOF Let $\phi \in (\mathcal{O}\, X)^\wedge$. By the Hofmann–Mislove theorem and the fact that every compact sublocale of X is closed, there is a least one $c \in \mathcal{C}\, X$ with $\alpha(c) = \phi$ (and a unique such c whose complement in the lattice of sublocales is fitted, but we don't use this knowledge). Hence the map is a surjection. In order to show that it is an injection, let $c, c' \in \mathcal{C}\, X$ with $\alpha(c) = \alpha(c')$. Then, for every $u \in \mathcal{O}\, X$, we have that $c \vee u = 1$ if and only if $c' \vee u = 1$. Because X is weakly Hausdorff, the frame $\uparrow(c \wedge c')$ is regular, and hence we conclude that $c = c'$ by 3.10, as required. □

Notice that this shows that every compact sublocale of a weakly Hausdorff locale is fitted.

4 The colimit construction

Let X be a locale and $C \subseteq \mathcal{O}\, X$ be an upper set regarded as a poset under the inherited order. For the sake of motivation, the reader should keep in mind that, in our applications, X will be Hausdorff and C will be the Heyting preframe $\mathcal{C}\, X$ of cocompact opens of X discussed in the previous section. But, technically speaking, the precise nature of X and C turn out to be irrelevant for the purposes of this section.

For each $c \in C$ let X_c be the complement of c in the lattice of sublocales, i.e.

$$\mathcal{O}\, X_c = \uparrow c.$$

If $d \geq c$ then we have a sublocale embedding $i_{dc}\colon X_d \to X_c$ given by

$$i^*_{dc}(u) = d \vee u,$$

and it is clear that if $e \geq d \geq c$ then

$$\begin{tikzcd} X_e \arrow[rr, "i_{ed}"] \arrow[dr, "i_{ec}"'] & & X_d \arrow[dl, "i_{dc}"] \\ & X_c, & \end{tikzcd}$$

and that $i_{cc}\colon X_c \to X_c$ is the identity. In other words, this construction produces a functor $F : C^{\mathrm{op}} \to \mathbf{Loc}$, given by $F(c) = X_c$ on objects and by $F(d \geq c) = (X_d \xrightarrow{i_{dc}} X_c)$ on arrows. Let $i_c\colon X_c \to KX$, for $c \in C$, be the colimiting cone of this functor. When X is Hausdorff and $C = \mathcal{C}\,X$, the locale KX is the Kelleyification of X as defined in the introduction.

Because colimits in $\mathbf{Loc}$ are limits in $\mathbf{Frm}$, which can be calculated as in the category of sets, the topology of the locale KX can be taken as

$$\begin{aligned} \mathcal{O}\,KX &= \{j \in \prod_{c\in C} \uparrow c \mid \forall d \geq c,\ j(d) = i^*_{dc}(j(c))\} \\ &= \{j\colon C \to C \mid \forall d \geq c,\ j(d) = d \vee j(c)\}, \end{aligned}$$

where the first equation holds by the nature of limits in the category of sets and the second because $j \in \prod_{c\in C} \uparrow c$ if and only if $c \leq j(c)$ for all $c \in C$ and because this is entailed by the condition that $j(d) = d \vee j(c)$ for all $d \geq c$ as the choice $c = d$ shows. The maps $i_c\colon X_c \to KX$, for $c \in C$, are then given by

$$i^*_c(j) = j(c),$$

as these are the projections in the category of sets. If $f_c\colon X_c \to Y$, for $c \in C$, are continuous maps that constitute another cocone, that is, such that for $d \geq c$,

$$\begin{tikzcd} X_d \arrow[rr, "i_{dc}"] \arrow[dr, "f_d"'] & & X_c \arrow[dl, "f_c"] \\ & Y, & \end{tikzcd}$$

then the unique continuous map $f\colon KX \to Y$ such that

$$\begin{tikzcd} X_c \arrow[rr, "i_c"] \arrow[dr, "f_c"'] & & KX \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

for every $c \in C$ is given by

$$f^*(v) = (c \mapsto f^*_c(v)),$$

again by the nature of limits in the category of sets.

Now considering $Y = X$ and $f_c = \varepsilon_c$ in the above construction, where the continuous map $\varepsilon_c\colon X_c \to X$ is the closed inclusion

$$\varepsilon^*_c(u) = c \vee u,$$

we obtain a cocone and hence a unique map $\varepsilon\colon KX \to X$ with

$$\begin{tikzcd} X_c \arrow[rr, "i_c"] \arrow[dr, "\varepsilon_c"'] & & KX \arrow[dl, "\varepsilon"] \\ & X & \end{tikzcd}$$

for every $c \in C$, given by

$$\varepsilon^*(u) = (c \mapsto c \vee u).$$

We now have a closer look at the topology of KX. As observed in Section 2, the notion of a nucleus makes sense in any meet-semilattice.

4.1 *If C is closed under the formation of finite meets, then any $j \in \mathcal{O}KX$ is a nucleus on C.*

PROOF We have already seen that j is inflationary. We know that $j(d) = j(c) \vee d$ whenever $d \geq c$. This with the choice $d = j(c)$ shows that $j(j(c)) = j(c) \vee j(c) = j(c)$ and hence that j is idempotent. For given d and d', the choice $c = d \wedge d'$, which implies $d \geq c$ and $d' \geq c$, gives $j(d) = j(d \wedge d') \vee d$ and $j(d') = j(d \wedge d') \vee d'$. It follows that $j(d) \wedge j(d') = (j(d \wedge d') \vee d) \wedge (j(d \wedge d') \vee d') = j(d \wedge d') \vee (d \wedge d') = j(d \wedge d')$ using distributivity and again the fact that j is inflationary, which shows that j preserves finite meets. $\square$

4.2 *The following are equivalent for any function $j : C \to C$.*

1. $j \in \mathcal{O}KX$.
2. $j(d) = j(c) \vee d$ *for all* $d \geq c$ *in* C.
3. $j(c \vee u) = j(c) \vee u$ *for all* $c \in C$ *and* $u \in \mathcal{O}X$.
4. $j(c \vee e) = j(c) \vee e$ *for all* $c, e \in C$.

PROOF $(1 \Leftrightarrow 2)$: This has already been established. $(2 \Rightarrow 3)$: Choosing $d = c \vee u$, as we may because C is an upper set, we have $c \leq d$ and hence (2) gives $j(c \vee u) = j(c) \vee c \vee u = j(c) \vee u$ as $c \leq j(c)$ by assumption. $(3 \Rightarrow 4)$: Immediate. $(4 \Rightarrow 2)$: For $d \geq c$ we have that $c \vee d = d$ and hence using (4) with $e = d$ we get $j(d) = j(c) \vee d$, as required. $\square$

We finish this section by considering the right adjoints of the defining frame homomorphisms of the maps involved in the colimit. This material can be postponed until Section 6 below.

Those for the cocone maps $i_c : X_c \to KX$ are given by

$$(i_c)_*(d) = (e \mapsto d \vee e)$$

for all $d \in \uparrow c$, because with this definition we have that

$$i_c^*((i_c)_*(d)) = i_c^*(e \mapsto d \vee e) = d \vee c = d,$$

and that

$$(i_c)_*(i_c^*(j)) = (i_c)_*(j(c)) = (e \mapsto j(c) \vee e) = (e \mapsto j(c \vee e)) \geq j,$$

and because (right) adjoints are unique.

4.3 *For every $c \in C$, the cocone map $i_c : X_c \to KX$ is a sublocale embedding.*

PROOF This follows from the first chain of equations. $\square$

Now let $f_c\colon X_c \to Y$, for $c \in C$, constitute a cocone and $f\colon KX \to Y$ be the unique continuous map with $f \circ i_c = f_c$ for all $c \in C$. By the adjoint-monotone-map theorem and the fact that $f^*(v) = (c \mapsto f_c^*(v))$, we have that

$$\begin{aligned}
f_*(j) &= \text{largest } v \in \mathcal{O}\,Y \text{ s.t. } f^*(v) \le j \\
&= \text{largest } v \in \mathcal{O}\,Y \text{ s.t. } \forall c \in C,\ f_c^*(v) \le j(c) \\
&= \text{largest } v \in \mathcal{O}\,Y \text{ s.t. } \forall c \in C,\ v \le (f_c)_*(j(c)) \\
&= \text{largest } v \in \mathcal{O}\,Y \text{ s.t. } v \text{ is a lower bound of } \{(f_c)_*(j(c)) \mid c \in C\} \\
&= \bigwedge_{c \in C} (f_c)_*(j(c)).
\end{aligned}$$

We specialize this to a particular case of interest.

4.4 *The right adjoint of the defining frame homomorphism of the canonical map $\varepsilon\colon KX \to X$ is given by*

$$\varepsilon_*(j) = \bigwedge_{c \in C} j(c).$$

PROOF $\varepsilon_*(j) = \bigwedge_c (\varepsilon_c)_*(j(c)) = \bigwedge_c c \vee j(c) = \bigwedge_c j(c)$. $\square$

4.5 COROLLARY *The canonical map $\varepsilon_X\colon KX \to X$ is a homeomorphism if and only if the following two conditions hold.*

1. *Every $j \in \mathcal{O}\,KX$ is of the form $j(c) = u \vee c$ for a unique $u \in \mathcal{O}\,X$.*

2. *Every $u \in \mathcal{O}\,X$ is a meet of members of C.*

PROOF Because adjoints are unique and inverses are adjoints, the frame homomorphism $\varepsilon^*\colon \mathcal{O}\,X \to \mathcal{O}\,KX$ is an isomorphism if and only if $\varepsilon^*(\varepsilon_*(j)) = j$ and $\varepsilon_*(\varepsilon^*(u)) = u$. $\square$

5 The second Lawson dual

In this section we consider the colimit construction KX of the previous section relative to the choice $C = \mathcal{C}\,X$, the Heyting preframe of cocompact opens of X. According to the programme set in the introduction, our goal is to show that $\mathcal{O}\,KX$ and $(\mathcal{O}\,X)^{\wedge\wedge}$ are isomorphic if X is Hausdorff, where $(\mathcal{O}\,X)^{\wedge\wedge}$ is the second Lawson dual of $\mathcal{O}\,X$ as defined in Section 2. Because we know that $(\mathcal{O}\,X)^{\wedge\wedge} \cong (\mathcal{C}\,X)^{\wedge}$ by 3.11, it suffices to show that $\mathcal{O}\,KX$ and $(\mathcal{C}\,X)^{\wedge}$ are isomorphic. Our construction is strikingly similar to that of the Hofmann–Mislove isomorphism. This similarity is the subject of Section 7 below.

As remarked in the paragraph preceding 2.1, the notions of nuclei and open nuclei make sense in any Heyting preframe, and for any nucleus j on a preframe, the set

$$\nabla j = j^{-1}(1)$$

is a filter.

5.1 THEOREM *Let X be a locale and $j\colon \mathcal{C}X \to \mathcal{C}X$ be a nucleus. If X is weakly Hausdorff then the following conditions are equivalent.*

1. $j \in \mathcal{O}KX$.
2. j *is Scott continuous.*
3. $\nabla j \in (\mathcal{C}X)^\wedge$.

(Under no assumptions on X, the implications $1 \Rightarrow 2 \Rightarrow 3$ hold.)

PROOF $(1 \Rightarrow 2)$: We use 4.2. Let $D \subseteq \mathcal{C}X$ be directed and choose any $c \in D$. Then, using 4.2(4) twice and the fact that the binary-join operation preserves directed joins in any of its arguments, $j(\bigvee D) = j(c \vee \bigvee D) = j(c) \vee \bigvee D = \bigvee_{d \in D} j(c) \vee d = \bigvee_{d \in D} j(c \vee d) = \bigvee_{e \in D} j(e)$, where the last equation holds by directedness of D, because if $d \in D$ then there is some $e \in D$ above d and c and hence above $c \vee d$.

$(2 \Rightarrow 3)$: The set $\{1\}$ is Scott open in $\mathcal{C}X$ by 3.7.

$(3 \Rightarrow 1)$: Assume that X is weakly Hausdorff. It suffices to show that $j(d) \leq d \vee j(c)$ for $d \geq c$ because the other inequality holds as j is inflationary and monotone. We use 3.10. Both $j(d)$ and $d \vee j(c)$ belong to $\uparrow c$ because j is monotone and inflationary and $d \geq c$. Let $e \in \uparrow c$ with $1 = e \vee j(d)$. We need to conclude that $1 = e \vee d \vee j(c)$. Since $e \vee j(d) \leq j(e) \vee j(d) \leq j(e \vee d)$, Scott openness of ∇j and regularity of $\uparrow c$ show that there is some $b \lessdot_c e \vee d$ such that already $1 = j(b)$. By 2.1, the second condition gives $(b \Rightarrow c) \leq j(c)$, and this and the first give $1 = e \vee d \vee (b \Rightarrow c) \leq e \vee d \vee j(c)$, as required. $\square$

(Notice that if $j \in \mathcal{O}KX$ then j preserves binary joins by virtue of 4.2 and hence all non-empty joins by the above if X is weakly Hausdorff — but this fact doesn't seem to be directly relevant to our considerations.)

For a filter $\phi \subseteq \mathcal{C}X$, let $\Delta\phi\colon \mathcal{C}X \to \mathcal{C}X$ be the map given by

$$\Delta\phi(c) = \bigvee\{d \Rightarrow c \mid d \in \phi\}.$$

That is, $\Delta\phi$ is the pointwise join of the open nuclei $d^\circ\colon \mathcal{C}X \to \mathcal{C}X$ for $d \in \phi$. Notice that this is the join of a directed set because the map $d \mapsto d^\circ$ transforms meets into joins and ϕ is a filter.

We shall soon discuss whether $\Delta\phi$ is a nucleus (and hence the join of the nuclei $c^\circ\colon \mathcal{C}X \to \mathcal{C}X$ for $c \in \phi$). In the following proposition we allow ourselves to write $\nabla\Delta\phi$ for the inverse image of 1 along $\Delta\phi$ in the absence of this knowledge. Its proof reuses an argument applied by Johnstone in order to prove the localic Hofmann–Mislove Theorem [13].

5.2 *If X is any locale and $\phi \in (\mathcal{C}X)^\wedge$ then $\nabla\Delta\phi = \phi$.*

PROOF Let $c \in \nabla\Delta\phi$, that is, $\Delta\phi(c) = 1$. Because $1 \in \phi$ and ϕ is Scott open, we conclude by directedness of the defining join of $\Delta\phi$ that $(d \Rightarrow c) \in \phi$ for some $d \in \phi$. Hence $c \geq (d \Rightarrow c) \wedge d$ is in ϕ too because ϕ is a filter, which shows that $\nabla\Delta\phi \subseteq \phi$. Conversely, let $c \in \phi$. Then $(c \Rightarrow c) = 1 \in \phi$ and hence $\Delta\phi(c) = 1$, that is, $c \in \nabla\Delta\phi$, which shows that $\phi \subseteq \nabla\Delta\phi$. $\square$

5.3 *If X is weakly Hausdorff and j is any nucleus on $\mathcal{C}\,X$ then $\Delta\nabla j = j$.*

PROOF $\Delta\nabla j(c) = \bigvee\{d \Rightarrow c \mid j(d) = 1\} \leq j(c)$ by 2.1. In order to prove the inequality in the other direction, let $e \lessdot_c j(c)$. Then $1 = j(c) \vee (e \Rightarrow c) \leq j(c) \vee j(e \Rightarrow c) \leq j(c \vee (e \Rightarrow c)) = j(e \Rightarrow c)$. Considering $d = (e \Rightarrow c)$ we conclude that $e \leq ((e \Rightarrow c) \Rightarrow c) = (d \Rightarrow c) \leq \Delta\nabla j(c)$. The result then follows by regularity of the frame $\uparrow c$. □

It remains to show that $\Delta\phi$ is a nucleus.

5.4 *For any locale X and any filter $\phi \subseteq \mathcal{C}\,X$, the map $\Delta\phi$ is inflationary and preserves finite meets.*

PROOF By virtue of the preframe distributivity law, endomaps that are inflationary and preserve finite meets are closed under the formation of pointwise directed joins, and the map $\Delta\phi$ is the pointwise directed join of the open nuclei d° for $d \in \phi$. □

In order to prove idempotence, we need to assume that the locale is weakly Hausdorff. We use the following lemma, which reuses ideas of [4, Lemma 5.1].

5.5 *Let X be a weakly Hausdorff locale and $\phi \in (\mathcal{C}\,X)^\wedge$.*

1. *If $d \in \phi$ and $e \wedge d \leq c$ then $e \leq \Delta\phi(c)$.*
2. *If $c \leq c'$ and $e \lessdot_c \Delta\phi(c')$ then $e \wedge d \lessdot_c c'$ for some $d \in \phi$.*

PROOF (1): By definition of Heyting implication.

(2): Firstly, notice that because $\Delta\phi$ is inflationary and monotone, $\Delta\phi(c')$ belongs to $\uparrow c$, as required for the relation $\lessdot_c$. By the assumption, $1 = \Delta\phi(c') \vee (e \Rightarrow c)$. Hence, by directedness of the defining join of $\Delta\phi(c')$ and cocompactness of $(e \Rightarrow c)$, there is some $d' \in \phi$ with $1 = (d' \Rightarrow c') \vee (e \Rightarrow c)$. It follows that $e \lessdot_c (d' \Rightarrow c')$. Because $(d' \vee c' \Rightarrow c') = (d' \Rightarrow c')$ and because ϕ, being a filter, is upper closed, we may assume that $d' \geq c'$ and hence that $d' \in \uparrow c$. Then, because ϕ is Scott open and $\uparrow c$ is a regular frame, there is some $d \lessdot_c d'$ in ϕ. Finally, because the well-inside relation is multiplicative [11, Lemma III-1.1(iv)], we conclude that $e \wedge d \lessdot_c (d' \Rightarrow c') \wedge d' \leq c'$, as required. □

5.6 *If X is weakly Hausdorff and $\phi \in (\mathcal{C}\,X)^\wedge$ then $\Delta\phi$ is idempotent and hence a nucleus.*

PROOF Let $e \lessdot_c j(j(c))$. By two successive applications of 5.5(2), we first conclude that $e \wedge d \lessdot_c j(c)$ for some $d \in \phi$ and then that $e \wedge d \wedge d' \leq c$ for some $d' \in \phi$. Since $d \wedge d' \in \phi$ as ϕ is a filter, we conclude by 5.5(1) that $e \leq j(c)$. By regularity of $\uparrow c$, we conclude that $j(j(c)) \leq j(c)$, as required. □

5.7 COROLLARY *If X is weakly Hausdorff and $\phi \in (\mathcal{C}\,X)^\wedge$ then $\Delta\phi \in \mathcal{O}\,KX$.*

PROOF Let $j = \Delta\phi$. Then j is a nucleus with $\nabla j = \nabla\Delta\phi = \phi \in (\mathcal{C}\,X)^\wedge$. Hence the result follows from 5.1. □

5.8 THEOREM *If X is a weakly Hausdorff locale then the assignment $j \mapsto \nabla j$ is an isomorphism from $\mathcal{O}\,KX$ to $(\mathcal{C}\,X)^{\wedge}$, with inverse given by $\phi \mapsto \Delta\phi$.*

We have seen in 3.11 that the map $\alpha\colon\ \mathcal{C}\,X \to (\mathcal{O}\,X)^{\wedge}$ defined by

$$\alpha(c) = \{u \in \mathcal{O}\,X \mid c \vee u = 1\}$$

is an isomorphism for any weakly Hausdorff locale X. Dualizing $\alpha\colon\ \mathcal{C}\,X \to (\mathcal{O}\,X)^{\wedge}$, we get an isomorphism $\alpha^{\wedge}\colon\ (\mathcal{O}\,X)^{\wedge\wedge} \to (\mathcal{C}\,X)^{\wedge}$.

5.9 COROLLARY *For any weakly Hausdorff locale, the map $\Delta \circ \alpha^{\wedge}\colon (\mathcal{O}\,X)^{\wedge\wedge} \to \mathcal{O}\,KX$ is an isomorphism.*

This concludes the proof of Theorem 1.4. Recall from Section 2 that the canonical map $e\colon\ \mathcal{O}\,X \to (\mathcal{O}\,X)^{\wedge\wedge}$ is defined by $e(u) = \{\phi \in (\mathcal{O}\,X)^{\wedge} \mid u \in \phi\}$.

5.10 COROLLARY *A weakly Hausdorff locale X is compactly generated in the sense of Hofmann and Lawson if and only if the map $\beta\colon\ \mathcal{O}\,X \to (\mathcal{C}\,X)^{\wedge}$ defined by*

$$\beta(u) = \alpha^{\wedge} \circ e(u) = \{c \in \mathcal{C}\,X \mid c \vee u = 1\}$$

is an isomorphism.

PROOF Because $\alpha^{\wedge}$ is an isomorphism, the canonical map is an isomorphism if and only if β is an isomorphism. □

Now observe that the diagram

$$\begin{array}{ccc} \mathcal{O}\,X & \xrightarrow{\varepsilon^*} & \mathcal{O}\,KX \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle \nabla} \\ (\mathcal{O}\,X)^{\wedge\wedge} & \xrightarrow[\alpha^{\wedge}]{} & (\mathcal{C}\,X)^{\wedge} \end{array}$$

commutes if X is weakly Hausdorff.

5.11 COROLLARY *For any weakly Hausdorff locale X, the map $\varepsilon_X\colon KX \to X$ is a homeomorphism if and only if the map $e_X\colon\ \mathcal{O}\,X \to (\mathcal{O}\,X)^{\wedge\wedge}$ is an isomorphism.*

PROOF $\alpha^{\wedge}$ and ∇ are isomorphisms if X is weakly Hausdorff. □

This concludes the proof of Theorem 1.3.

6 Locales with enough compact sublocales

We say that a locale X has *enough compacts* if $u \leq v$ holds in $\mathcal{O}\,X$ whenever $Q \leq u$ implies $Q \leq v$ for every compact sublocale Q of X. By 2.2, this is equivalent to saying that the canonical map $e_X\colon\ \mathcal{O}\,X \to (\mathcal{O}\,X)^{\wedge\wedge}$ is an injection. Hence compactly generated weakly Hausdorff locales have enough compacts. By 2.3, locally compact Hausdorff locales are compactly generated. Using classical logic, a locale has enough compacts if and only if it has enough points, because points are compact sublocales and, conversely, because a non-null compact locale has at least one point [11]. However, this is not the case in general, because the localic real line fails to have enough points in some toposes but is locally compact Hausdorff in all of them.

The following characterization is an immediate corollary of the results of the preceding section and of 4.4.

6.1 *The following conditions are equivalent for any weakly Hausdorff locale X.*

1. *X has enough compacts.*
2. *ε^* is an injection.*
3. *If $u \leq v \vee c$ for all $c \in \mathcal{C}\,X$ then $u \leq v$.*
4. *$\varepsilon_* \circ \varepsilon^* = \mathrm{id}_{\mathcal{O}\,X}$.*
5. *$v = \bigwedge\{c \in \mathcal{C}\,X \mid v \leq c\}$ for any $v \in \mathcal{O}\,X$.*
6. *If $v \leq c$ implies $u \leq c$ for all $c \in \mathcal{C}\,X$ then $u \leq v$.*
7. *$\nabla \circ \varepsilon^*$ is an injection.*
8. *If $u \vee c = 1$ implies $v \vee c = 1$ for all $c \in \mathcal{C}\,X$ then $u \leq v$.*

PROOF By the diagram preceding 5.11, the canonical map $e\colon\ \mathcal{O}\,X \to (\mathcal{O}\,X)^{\wedge\wedge}$ is an injection if and only if $\varepsilon^*\colon\ \mathcal{O}\,X \to \mathcal{O}\,KX$ is an injection, and ∇ is an isomorphism if X is weakly Hausdorff. The result then follows by a routine unfolding of definitions. □

For the following, it is appropriate to formulate both weak and strong versions.

6.2 *(Weakly) Hausdorff locales with enough compacts form a coreflective subcategory of the category of (weakly) Hausdorff locales.*

PROOF Let $\varepsilon_X[KX]$ be the image of $\varepsilon_X\colon KX \to X$. Then $\varepsilon[KX]$ is a sublocale of X with $\varepsilon[KX] = X_{\varepsilon_* \circ \varepsilon^*}$, and it is clear from the above that $\varepsilon[KX]$ has enough compacts. Being a sublocale of a (weakly) Hausdorff locale, it is itself (weakly) Hausdorff. We show that the inclusion $\varepsilon[KX] \to X$ is universal among maps from (weakly) Hausdorff locales with enough compacts into X. Let $f\colon A \to X$ be a continuous map on a (weakly) Hausdorff locale with enough compacts. Because embeddings are monomorphisms, there is at most one continuous map $\bar{f}\colon A \to \varepsilon[KX]$ coextending f along $\varepsilon[KX] \to X$. To show that there is at least one, it suffices to show that the image $f[A] = X_{f_* \circ f^*}$ of f is a sublocale of $\varepsilon[KX]$, because then the corestriction of f to its image followed by the the inclusion of $f[A]$ into $\varepsilon[KX]$ gives a coextension. Because sublocales are in order-reversing bijection with nuclei, this amounts to showing that $\varepsilon_* \circ \varepsilon^* \leq f_* \circ f^*$, which is equivalent to $f^* \circ \varepsilon_* \circ \varepsilon^* \leq f^*$. Because A has enough compacts, it suffices to show that $f^*(\varepsilon_*(\varepsilon^*(v))) \vee c = 1$ implies $f^*(v) \vee c = 1$ for all $v \in \mathcal{O}\,X$ and $c \in \mathcal{C}\,A$. By 3.11, for every $c \in \mathcal{C}\,A$ there is some $d \in \mathcal{C}\,X$ with $\alpha(d) = (f^*)^{\wedge}(\alpha(c))$, which amounts to saying that $w \vee d = 1$ if and only if $f^*(w) \vee c = 1$ for all $w \in \mathcal{O}\,X$. Considering $w = \varepsilon_*(\varepsilon^*(v))$ and $w = v$, our goal amounts to showing that $\varepsilon_*(\varepsilon^*(v)) \vee d = 1$ implies $v \vee d = 1$. Assume the premise. Then $1 = \varepsilon_*(\varepsilon^*(v)) \vee d \leq \varepsilon_*(\varepsilon^*(v)) \vee \varepsilon_*(\varepsilon^*(d)) \leq \varepsilon_*(\varepsilon^*(v \vee d)) = \bigwedge\{v \vee d \vee d' \mid d' \in \mathcal{C}\,A\} = v \vee d$ by 4.4, and hence $1 = v \vee d$, as required. □

7 The Hofmann–Mislove theorem

In Section 5 we exhibited a construction which, as we have already emphasized, mimics the construction of the localic Hofmann–Mislove isomorphism as given by Johnstone [13]. It is the purpose of this section to show that both constructions are instances of the same phenomenon. By 4.1 and 5.1, we know that for a Hausdorff locale X, the topology of KX is the frame of nuclei j on $\mathcal{C}\,X$ such that $j^{-1}(1)$ is Scott open. By 3.11, we know that the Heyting preframe $\mathcal{C}\,X$ is isomorphic to the first Lawson dual $(\mathcal{O}\,X)^\wedge$ of $\mathcal{O}\,X$. By 5.8, it follows that the nuclei j on $(\mathcal{O}\,X)^\wedge$ such that $j^{-1}(1)$ is Scott open are in order-preserving bijection with $(\mathcal{O}\,X)^{\wedge\wedge}$. Thus, this gives a Hofmann–Mislove theorem "one level up" for Hausdorff locales. It turns out that the Hausdorff separation axiom is not needed for that purpose. The only crucial uses of (weak) Hausdorffness in Section 5 occur in Theorem 5.1 (and in the corollaries that lead to the theorems formulated in the introduction, via the use of 3.11). In fact, in this section we show that, for any Heyting preframe, the nuclei j with $j^{-1}(1)$ Scott open are in bijection with the first Lawson dual of the preframe. This can be applied to the topology of a locale (yielding the Hofmann–Mislove theorem) or to its Lawson dual (yielding the Hofmann–Mislove theorem one level up), provided it is a Heyting preframe. This is the case, for example, if the compact fitted sublocales are closed under binary meets in the lattice of sublocales.

The frame of Scott continuous nuclei on a preframe. Because Scott continuous nuclei on a preframe feature in 5.1, we briefly digress to discuss Scott continuous nuclei on general preframes. By the argument given in [3, Lemma 3.1.8], they form a frame. Finite meets and directed joins can be calculated pointwise. The join of an arbitrary collection of Scott continuous nuclei J is given by $(\bigvee J)(c) = \bigvee\{p(c) \mid p \in J^*\}$, where J^* is the set of finite compositions of members of J. Notice that the members of J^* are inflationary and preserve finite meets and that the join on the right-hand side is that of a directed set. As shown below, the set of all nuclei is a also frame, with the set of Scott continuous nuclei as a subframe.

The frame of nuclei on a preframe. The remainder of this section generalizes the results of the paper [5] in two ways. The possibility of the first generalization was already mentioned there. The fact that the nuclei on a preframe form a frame was first established by Simmons [21]. We consider another proof that gives information which is used in 7.6 below. Our main tool is the following lemma, which slightly generalizes a theorem by Pataraia with the essentially the same proof [5]. By an *inductive poset* we mean a dcpo (directed complete poset) with a least element, and by an *inductive subset* of an inductive poset we mean a subset that has the least element as a member and is closed under the formation of directed joins.

7.1 *Any set F of monotone inflationary maps on an inductive poset D has a least common fixed point. Moreover, any inductive subset of D that is closed under f for each $f \in F$ has the least common fixed point as a member.*

We begin with the following simple observation.

7.2 *If p and k are inflationary maps on an arbitrary poset with p monotone and k idempotent, then $p \leq k$ if and only if $p \circ k = k$.*

PROOF The inequality $k \leq p \circ k$ always holds because p is inflationary. If $p \leq k$ then $p \circ k \leq k \circ k = k$ because k is idempotent. Conversely, if $p \circ k = k$ then $p \leq p \circ k = k$ because p is monotone and k is inflationary. □

A *prenucleus* is an inflationary map that preserves finite meets, but which is not necessarily idempotent. (The propositions below hold in fact for a slightly more general notion of prenucleus considered by Banaschewski [1]. However, since this is not needed for our purposes, we omit the routine details.) Prenuclei on a preframe are closed under composition and under the formation of pointwise directed joins.

7.3 *If P is a set of prenuclei on a preframe then there is a smallest nucleus k with $p \leq k$ for all $p \in P$. Moreover, if Q is an inductive set of prenuclei such that $p \in P$ and $q \in Q$ together imply $p \circ q \in Q$, then $k \in Q$.*

PROOF Because each $p \in P$ is monotone and inflationary, so is the endofunction $q \mapsto p \circ q$ of the dcpo of prenuclei. By 7.1, these functions have a least common fixed point k, which belongs to any inductive set Q of prenuclei such that $p \circ q \in Q$ whenever $p \in P$ and $q \in Q$. In order to show that k is idempotent, let Q be the set of prenuclei q with $q \circ k \leq k$, which is clearly inductive. If $p \in P$ and $q \in Q$ then $q \circ k \leq k$ and hence $p \circ q \circ k \leq p \circ k = k$, which shows that $p \circ q \in Q$. It follows that $k \in Q$ and hence that k is idempotent, as required. Thus, k is the least nucleus satisfying the family of fixed-point equations $p \circ k = k$ with $p \in P$. The result then follows from 7.2. □

Considering the case in which P is a set of nuclei, we conclude that every set of nuclei has a least upper bound. (Considering the case in which P is a singleton $\{p\}$, we conclude that there is a smallest nucleus $\bar{p} \geq p$. Using the induction principle, we conclude that $\bar{p}$ and p have the same fixed points and that $\bar{p}(c)$ is the least fixed point of p above c. For prenuclei on a frame, this was first proved by Banaschewski [1].)

7.4 COROLLARY *The nuclei on a preframe form a complete lattice. Moreover, if J is a set of nuclei and Q is an inductive set of prenuclei such that $j \in J$ and $q \in Q$ together imply $j \circ q \in Q$, then $\bigvee J \in Q$.*

We refer to the above principle as *join induction*. It is clear that finite meets exist and can be calculated pointwise.

7.5 *The nuclei on a preframe form a frame.*

PROOF In remains to prove the frame distributivity law. We do this by join induction. Let k be a nucleus, J be a set of nuclei, and l denote $\bigvee\{k \wedge j \mid j \in J\}$. It is enough to show that $k \wedge \bigvee J \leq l$ because the other inequality holds in any complete lattice. By the preframe distributivity law and the fact that finite meets and directed joins of prenuclei are computed pointwise, the set Q of prenuclei q with $k \wedge q \leq l$ is inductive. If $j \in J$ then $k \wedge j \leq l$, and hence if $q \in Q$ then $(k \wedge j) \circ (k \wedge q) \leq l \circ l = l$. But $(k \wedge j) \circ (k \wedge q) = k \circ k \wedge k \circ q \wedge j \circ k \wedge j \circ q = k \wedge j \circ q$ because $k \leq k \circ q$ and $k \leq j \circ k$, which shows that $j \circ q \in Q$ and hence that $\bigvee J \in Q$. □

The Hofmann–Mislove–Johnstone theorem for Heyting preframes. We now generalize the Hofmann–Mislove theorem from frames to Heyting preframes with the same proof as given in [5], which is in turn essentially the same as that given in [13], except that the former uses the join-induction principle discussed above instead of transfinite induction.

As remarked above, in the presence of Heyting implication, the notions of open and fitted nuclei as defined for frames make sense for preframes. For easy comparison with Section 5, we use the letter C to denote an arbitrary Heyting preframe and the letters c, d, e to range over its members. For a nucleus j on C and a set $\phi \subseteq C$, define

$$\nabla j = j^{-1}(1), \qquad \Delta\phi = \bigvee\{d^\circ \mid d \in \phi\}.$$

Then ∇j is a filter and $\Delta\phi$ is a fitted nucleus, and it is clear that if $j \leq k$ then $\nabla j \subseteq \nabla k$ and that if $\phi \subseteq \gamma$ then $\Delta\phi \leq \Delta\gamma$. The only difficulty in the proof is that, in general, the defining join of $\Delta\phi$ cannot be computed pointwise (but we have seen an exception in Section 5, and another occurs in the paper [4]).

7.6 *Let j be a nucleus on a Heyting preframe C and ϕ be a subset of C.*

1. *$\Delta\nabla j \leq j$, and equality holds if j is fitted.*

2. *$\phi \subseteq \nabla\Delta\phi$, and equality holds if ϕ is a Scott open filter.*

PROOF (1): The stated inequality holds by 2.1. The assumption amounts to $j = \Delta\phi$ for some $\phi \subseteq C$. Hence the reverse inequality amounts to $\Delta\phi \leq \Delta\nabla\Delta\phi$, which follows from the inequality (2) and monotonicity of Δ.

(2): If $d \in \phi$ then $\Delta\phi(d) \geq d^\circ(d) = (d \Rightarrow d) = 1$ which amounts to $d \in \nabla\Delta\phi$. For the reverse inequality, we use the join-induction principle formulated in 7.4. Let Q be the set of prenuclei q with $q^{-1}(\phi) \subseteq \phi$, that is, such that $q(c) \in \phi$ implies $c \in \phi$. The least prenucleus, being the identity, belongs to Q. If $D \subseteq Q$ is a directed set with $(\bigvee D)(c) \in \phi$ then $q(c) \in \phi$ for some $q \in D$ because directed joins of prenuclei are computed pointwise and ϕ is Scott open, and $c \in \phi$ because $q \in Q$, which shows that $\bigvee D \in Q$. If $d^\circ \circ q(c) = (d \Rightarrow q(c)) \in \phi$ for $d \in \phi$ and $q \in Q$, then $q(c) \geq d \wedge (d \Rightarrow q(c))$ is in ϕ because ϕ is a filter, and hence so is c because $q \in Q$, from which it follows that $d^\circ \circ q \in Q$. By join-induction, the nucleus $\Delta\phi$ is in Q. Therefore $(\Delta\phi)^{-1}(1) \subseteq \phi$ because $1 \in \phi$. □

In analogy with the case of frames, it is natural to define a nucleus j on a preframe to be *compact* if the filter $j^{-1}(1)$ is Scott open, and we adopt this terminology in the formulation of the following theorem. However, we have seen that when the preframe is the Lawson dual of the topology of a Hausdorff locale, such nuclei are the *opens* of the Kelleyification of the locale. Essentially, what is going on is that Lawson dualization switches the roles of the notions of open and compact.

7.7 THEOREM *For any Heyting preframe, the assignment $j \mapsto \nabla j$ is a bijection from compact fitted nuclei to Scott open filters, with inverse given by $\phi \mapsto \Delta\phi$.*

References

[1] B. Banaschewski. Another look at the localic Tychonoff theorem. *Comment. Math. Univ. Carolin.*, 29(4):647–656, 1988.

[2] F. Borceux. *Handbook of Categorical Algebra. 2*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[3] M.H. Escardó. Properly injective spaces and function spaces. *Topology and its Applications*, 89(1–2):75–120, 1998.

[4] M.H. Escardó. The regular-locally-compact coreflection of stably locally compact locale. *Journal of Pure and Applied Algebra*, 157(1):41–55, 2001.

[5] M.H. Escardó. Joins in the frame of nuclei. *Applied Categorical Structures*, to appear.

[6] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

[7] K.H. Hofmann and J.D. Lawson. On the order theoretical foundation of a theory of quasicompactly generated spaces without separation axiom. In R.-E. Hoffmann, editor, *Continuous Lattices and Related Topics*, volume 27 of *Mathematik-Arbeitspapiere*, pages 143–160. University of Bremen, 1982.

[8] K.H. Hofmann and M. Mislove. Local compactness and continuous lattices. In B. Banaschewski and R.-E. Hoffmann, editors, *Continuous Lattices*, volume 871 of *Lecture Notes in Mathematics*, pages 209–248, 1981.

[9] M. Hyland. Function spaces in the category of locales. In *Continuous lattices,*, volume 871 of *Lect. Notes Math.*, pages 264–281, 1981. Proc. Conf., Bremen 1979.

[10] J.R. Isbell. Atomless parts of spaces. *Math. Scand.*, 31:5–32, 1972.

[11] P.T. Johnstone. *Stone Spaces*. Cambridge University Press, Cambridge, 1982.

[12] P.T. Johnstone. Open locales and exponentiation. In *Mathematical applications of category theory (Denver, Col., 1983)*, pages 84–116. Amer. Math. Soc., Providence, RI, 1984.

[13] P.T. Johnstone. Vietoris locales and localic semilattices. In *Continuous lattices and their applications (Bremen, 1982)*, pages 155–180. Dekker, New York, 1985.

[14] P.T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Number 43–44 in Oxford Logic Guides. Oxford Science Publications, 2002.

[15] J.L. Kelley. *General Topology*. D. van Nostrand, New York, 1955.

[16] J.D. Lawson. The duality of continuous posets. *Houston J. Math.*, 5(3):357–386, 1979.

[17] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1971.

[18] P. Martin-Löf. *Intuitionistic type theory*, volume 1 of *Studies in Proof Theory. Lecture Notes*. Bibliopolis, Naples, 1984. Notes by G. Sambin.

[19] G. Sambin. Some points in formal topology. *Theoretical Computer Science*, to appear.

[20] D.S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136, 1972.

[21] H. Simmons. Near-discreteness of modules and spaces as measured by Gabriel and Cantor. *J. Pure Appl. Algebra*, 56(2):119–162, 1989.

[22] J.J.C. Vermeulen. Some constructive results related to compactness and the (strong) Hausdorff property for locales. In *Category theory (Como, 1990)*, pages 401–409. Springer, Berlin, 1991.

[23] S.J. Vickers. Constructive points of powerlocales. *Math. Proc. Cambridge Philos. Soc.*, 122(2):207–222, 1997.

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