Proof of the searchability of $\mathbb{N}_\infty$

Martín Hötzel Escardó

Principle of omniscience

$$\Pi(p\colon X \to 2), (\Sigma(x : X), p(x) = 0) + (\Pi(x : X), p(x) = 1).$$

Can be proved for X finite (not for X subfinite in general).

For $X = \mathbb{N}$ this is LPO, so can't be proved.

For $X = 2^{\mathbb{N}}$ can be proved from Brouwerian assumptions.
(Continuity, fan theorem. We don't do this here.)

Drinker paradox

In every pub there is a person a such that if a drinks then everybody drinks.

$$\Pi(p\colon X \to 2), \Sigma(a : X), p(a) = 1 \implies \Pi(x : X), p(x) = 1.$$

For X inhabited, this is equivalent to the omniscience of X.

Selection of roots of 2-valued functions

A selection function for a set X is a functional $\varepsilon\colon (X \to 2) \to X$ such that for all $p\colon X \to 2$,

$$\boxed{p(\varepsilon(p)) = 1 \implies \Pi(x : X), p(x) = 1.}$$

Equivalently, the function p has a root if and only if $\varepsilon(p)$ is a root.

$$\boxed{p(\varepsilon(p)) = 0 \Longleftarrow \Sigma(x : X), p(x) = 0.}$$

Searchable sets

We say that a type is searchable if it has a selection function.

The generic convergent sequence

$$\mathbb{N}_\infty = \Sigma(x : 2^{\mathbb{N}}), \Pi(i : \mathbb{N}), x_i \geq x_{i+1}.$$

Also known as the one-point compactification of the natural numbers.
(And it is the final co-algebra of the functor $X \mapsto 1 + X$.)

The univalent set $\mathbb{N}_\infty$ has elements $\underline{n} = 1^n 0^\omega$ and $\infty = 1^\omega$.

Saying that every element of $\mathbb{N}_\infty$ is of one of the forms $\underline{n}$ or ∞ amounts to LPO.

Lemma. $\Pi(x : \mathbb{N}_\infty), (\Pi(n : \mathbb{N}), x \neq \underline{n}) \implies x = \infty.$

Proof. For any i, if we had $x_i = 0$, then we would have $x = \underline{n}$ for some $n < i$, and so we must have $x_i = 1$.

We don't have LPO, but we have the following

Lemma (Density). For all $p\colon \mathbb{N}_\infty \to 2$, if

1. $p(\underline{n}) = 1$ for every $n : \mathbb{N}$, and

2. $p(\infty) = 1$,

then

3. $p(x) = 1$ for every $x : \mathbb{N}_\infty$.

Proof. If we had $p(x) \neq 1$, then we would have $x \neq \underline{n}$ for every $n : \mathbb{N}$, and hence $x \neq \infty$, by the previous lemma, which contradicts the hypothesis.

$\mathbb{N}_\infty$ is searchable and hence omniscient

Proof. Given $p\colon \mathbb{N}_\infty \to 2$, let

$$\varepsilon(p) = \lambda i. \min_{n \le i} p(\underline{n}).$$

Clearly $\varepsilon(p) : \mathbb{N}_\infty$ (it is clearly a decreasing sequence). Also

$$\begin{array}{ll} (0) & \Pi(n : \mathbb{N}), \varepsilon(p) = \underline{n} \implies p(\underline{n}) = 0, \\ (1) & \varepsilon(p) = \infty \implies \Pi(n : \mathbb{N}), p(\underline{n}) = 1. \end{array}$$

We need to show that $p(\varepsilon(p)) = 1 \implies \Pi(x : \mathbb{N}_\infty), p(x) = 1$.

Claim 0. $\boxed{p(\varepsilon(p)) = 1 \implies \Pi(n \in \mathbb{N}), \varepsilon(p) \neq \underline{n}.}$

Proof. We know that $\Pi(n : \mathbb{N}), \varepsilon(p) = \underline{n} \implies p(\underline{n}) = 0$.

But, for any $n : \mathbb{N}$, if we have $\varepsilon(p) = \underline{n}$, the hypothesis of the claim gives $p(\underline{n}) = 1$.

Claim 1. $\boxed{p(\varepsilon(p)) = 1 \implies \varepsilon(p) = \infty.}$

Proof. This follows from Claim 0 and the previous lemma that

$$\Pi(x : \mathbb{N}_\infty), (\Pi(n : \mathbb{N}), x \neq \underline{n}) \implies x = \infty.$$

Claim 2. $\boxed{p(\varepsilon(p)) = 1 \implies \Pi(n : \mathbb{N}), p(\underline{n}) = 1.}$

Proof. This follows from the previous fact $\varepsilon(p) = \infty \implies \Pi(n : \mathbb{N}), p(\underline{n}) = 1$.

Claim 1. $\boxed{p(\varepsilon(p)) = 1 \implies \varepsilon(p) = \infty.}$

Claim 2. $\boxed{p(\varepsilon(p)) = 1 \implies \Pi(n : \mathbb{N}), p(\underline{n}) = 1.}$

Claim 3. $\boxed{p(\varepsilon(p)) = 1 \implies p(\infty) = 1.}$

Proof. This follows from Claim 1 and function extensionality.

Claim 4. $\boxed{p(\varepsilon(p)) = 1 \implies \Pi(x : \mathbb{N}_\infty), p(x) = 1.}$

Proof. This follows from Claims 2 and 3 and the density Lemma.

Q.E.D.

Addendum

$\varepsilon(p)$ is the infimum of the set of roots of p.

(The infimum of the empty set is the top element ∞.)

So it is the least root if p has a some root.

Consequences

WLPO is also undecided

$$\Pi(p : \mathbb{N} \to 2), (\Pi(n : \mathbb{N}), p(n) = 1) + \neg\Pi(x : \mathbb{N}), p(n) = 1$$

But we have:

Theorem. $\Pi(p : \mathbb{N}_\infty \to 2), (\Pi(n : \mathbb{N}), p(\underline{n}) = 1) + \neg\Pi(n : \mathbb{N}), p(\underline{n}) = 1.$

The point is that now we quantify over $\mathbb{N}$,
although the function p is defined on $\mathbb{N}_\infty$.

More consequences

1. Every function $f : \mathbb{N}_\infty \to \mathbb{N}$ is constant or not.

2. Any two functions $f, g : \mathbb{N}_\infty \to \mathbb{N}$ are equal or not.

3. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ has a minimum value, and it is possible to find a point at which the minimum value is attained.

4. For any function $f : \mathbb{N}_\infty \to \mathbb{N}$ we can find a point $x : \mathbb{N}_\infty$ such that if f has a maximum value, the maximum value is x.

5. Any function $f : \mathbb{N}_\infty \to \mathbb{N}$ is not continuous, or not-not continuous.

6. There is a non-continuous function $f : \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds.