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Another look at countable products  
of separable sets

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Theory Lab Lunch



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(Another look at my LIES'2007 paper.)

(1)

searchable subset of a type X

$$A \subseteq X$$

$$\varepsilon = (X \rightarrow B) \rightarrow X$$

$$p \mapsto a \in A \text{ s.t.}$$

$$\text{if } \exists a \in A. p(a) \text{ then } p(a_0)$$

$$SX \stackrel{\text{def}}{=} (X \rightarrow B) \rightarrow X$$

Exhaustible subset

$$QX = (X \rightarrow B) \rightarrow B$$

$$\exists = (X \rightarrow B) \rightarrow B$$

$$p \mapsto \text{truth value } \exists a \in A. p(a)$$

Proposition

searchable  $\Rightarrow$  exhaustible

Proof

$$SX \longrightarrow QX$$

$$\varepsilon \mapsto \lambda p. p(\varepsilon p)$$

□

Proposition

computable images of exhaustible sets are exhaustible.  
searchable searchble

Proof (1) let  $f: X \rightarrow Y$ . Then  $\exists y \in f(A). q(y) \Leftrightarrow \exists x \in A. q(f(x))$

$$(X \rightarrow Y) \longrightarrow (QX \longrightarrow QY)$$

$$f \mapsto \lambda \varepsilon. \lambda p. \exists (x \mapsto f(x)). p(\varepsilon p)$$

(2) To find  $y \in f(A)$  s.t.  $q(y)$ , find  $x \in A$  s.t.  $q(f(x))$   
and apply  $f$  to it.

$$(X \rightarrow Y) \longrightarrow (SX \longrightarrow SY)$$

$$f \mapsto \lambda \varepsilon. \lambda q. f(\varepsilon(\lambda x. p(f(x))))$$

□



(This part from Barbados notes 2004.)

(2)

Proposition Singletons (consisting of a single element) are exhaustive & searchable.

Proof (1)  $X \longrightarrow QX$   
 $x \longmapsto \lambda p. p(x)$

(2)  $X \longrightarrow SX$   
 $x \longmapsto \lambda p. x$   $\square$

Proposition A union of exhaustive sets is exhaustive.  
 searchable searchable

Proof (1)  $x \in \bigcup \mathcal{A} \iff \exists A \in \mathcal{A} \text{ s.t. } x \in A$

$Q \bigcup \mathcal{A} \longrightarrow Q \bigcup \mathcal{A}$   
 $\exists \longmapsto \lambda p. \exists (\lambda \epsilon. \epsilon(p))$

(2) To find  $x \in \bigcup \mathcal{A}$ , find  $A \in \mathcal{A}$  s.t.  $\exists a \in A$  s.t.  $p(x)$ . & then find

$S \bigcup \mathcal{A} \longrightarrow S \bigcup \mathcal{A}$   
 $\epsilon \longmapsto \lambda p. \epsilon (\lambda \epsilon. p(\epsilon p)) (p)$

$\square$

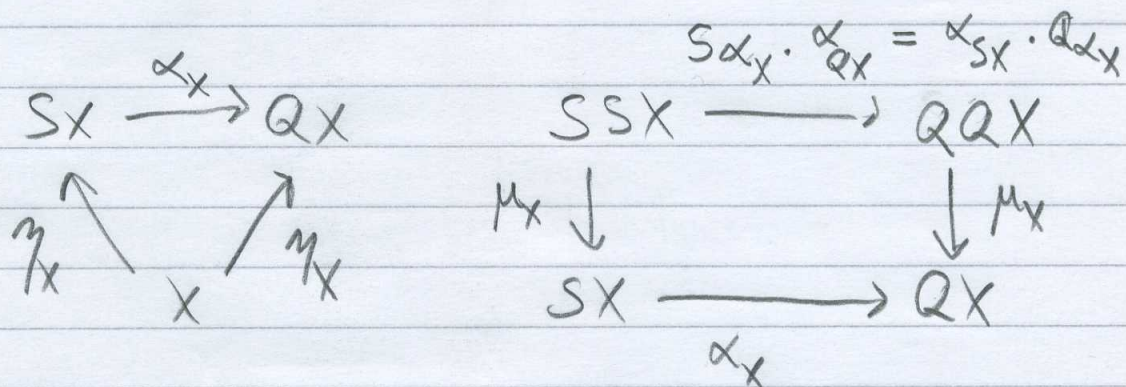
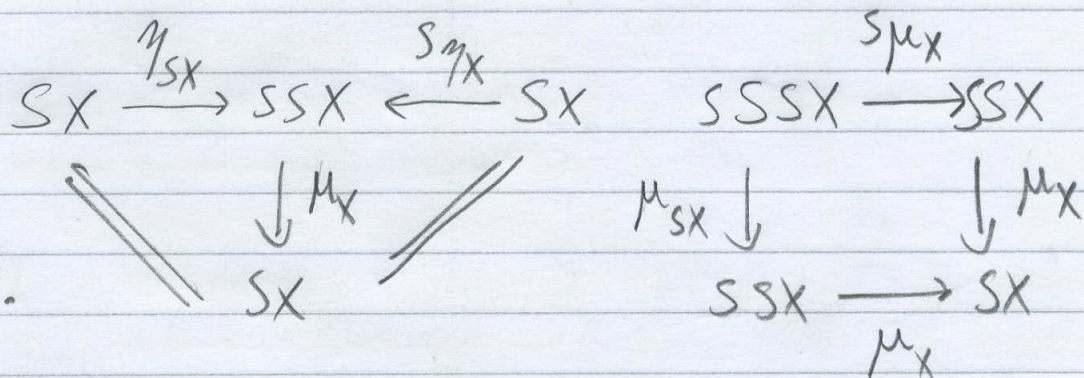


③

We have two monads and a monad morphism.

Functoriality  
Naturality

&  
unit and  
associativity laws.



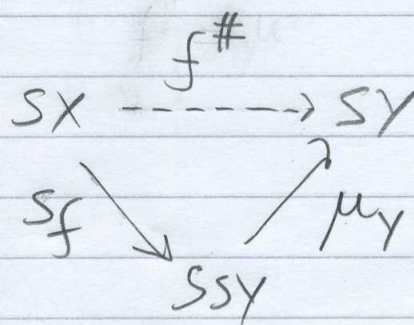
So what?

Want to derive other search algorithms from this.

These monads are automatically strong (we are in a cc & the functor, unit, multiplication are internalized)

Extension operator

$$f = X \rightarrow SY$$



This realizes the following  
proposition -



(4)

Proposition If  $\{B_x \subseteq Y \mid x \in X\}$  is a family of sets s.t.  $B_x$  is searchable uniformly in  $x \in X$  and  $A \subseteq X$  is searchable, then  $\bigcup_{x \in A} B_x$  is searchable.

Proof let  $f: X \rightarrow SY$  s.t.  $f(x) = \varepsilon_{B_x}$ .

Then  $\varepsilon_{\bigcup \{B_x \mid x \in A\}} = f^\#(\varepsilon_A)$ .  $\square$

In any ccc-monad the strength can be defined as

$$\begin{aligned} X \times TY &\longrightarrow T(X \times Y) \\ (x, \varepsilon) &\longmapsto \underbrace{(\lambda y. \eta(x, y))^\#}_{Y \longrightarrow T(X \times Y)}(\varepsilon) \end{aligned}$$

This redizes to

Proposition If  $x \in X$  and  $B \subseteq Y$  is searchable, then  $\{x\} \times B$  is searchable.

co-strength =

$$\begin{aligned} TX \times Y &\longrightarrow T(X \times Y) \\ (\varepsilon, y) &\longmapsto (\lambda x. \eta(x, y))^\#(\varepsilon) \end{aligned}$$



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Proposition If  $A \subseteq X$  and  $B \subseteq Y$  are searchable,  
then so is  $A \times B$ .

Proof This can be realized in two different ways because  
the monad is not commutative.

(i) Extend strength

$$TX \times TY \longrightarrow T(X \times Y)$$

$$(\varepsilon_A, \varepsilon_B) \longmapsto (\lambda x. \text{strength}(x, \varepsilon_B))^{\#}(\varepsilon_A)$$

(ii) Extend co-strength.  $\square$

Expanding all the definitions, (i) gives the same  
algorithm given in LICS'2007

$$TX \times TY \longrightarrow T(X \times Y)$$

$$(\varepsilon_A, \varepsilon_B) \longmapsto \begin{aligned} &\text{let } a_0 = \varepsilon_A(\lambda a. \exists_{b \in B} p(a, b)) \\ &\quad b_0 = \varepsilon_B(\lambda b. p(a_0, b)) \\ &\text{in } (a_0, b_0) \end{aligned}$$



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Curiosity (?)

$\Delta = (co) \text{ strength}$

$$TX \times TY \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} T(X \times Y)$$

$$\pi = \langle T\pi_X, T\pi_Y \rangle$$

In general,  $\pi \circ \Delta \neq \text{identity}$ . E.g.  $T = Q$

$$QX = (X \rightarrow B) \rightarrow B$$

But for  $T = S$ ,  $\pi \circ \Delta = \text{identity}$ .

Is that "important"?



LICS'2007 infinite product

$$X \times X^w \xrightleftharpoons[\text{cons}]{\langle \text{head}, \text{tail} \rangle} X^w$$

final coalgebra

$$\begin{array}{ccc}
 SX \times SX^w & \xrightarrow{\text{cons}'} & SX^w \\
 \text{strength} \searrow & & \nearrow S \text{ cons} \\
 & S(X \times X^w) & \\
 & \downarrow \text{this also has a retraction} & \\
 (SX)^w & \xrightarrow{\text{LICS functional } \Pi} & SX^w \\
 \uparrow \text{cons} & & \uparrow \text{cons}' \\
 SX \times (SX)^w & \xrightarrow{\text{Id} \times \Pi} & SX \times SX^w
 \end{array}$$

OR:

$$\Pi(\text{cons}(\varepsilon_0, \vec{\varepsilon})) = \text{cons}'(\varepsilon_0, \Pi(\vec{\varepsilon}))$$

It wasn't written like this in LICS'2007, but this is equivalent.