MGS 2012: FUN Lecture 2
Purely Functional Data Structures

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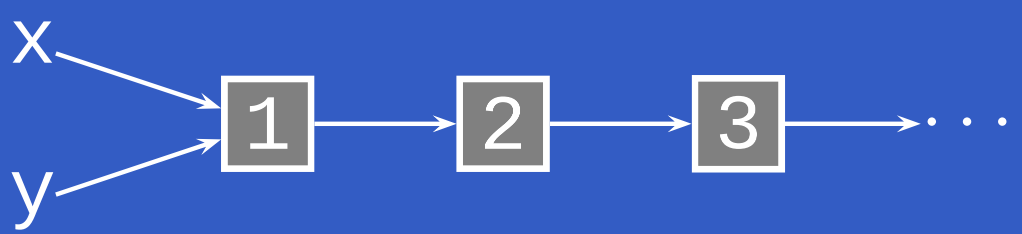
Purely Functional Data structures (1)

Why is there a need to consider purely functional data structures?

- The standard implementations of many data structures assume imperative update. To what extent truly necessary?
- Purely functional data structures are ***persistent***, while imperative ones are ***ephemeral***:
  - Persistence is a useful property in its own right.
  - Can't expect added benefits for free.

Purely Functional Data structures (2)

Linked list:



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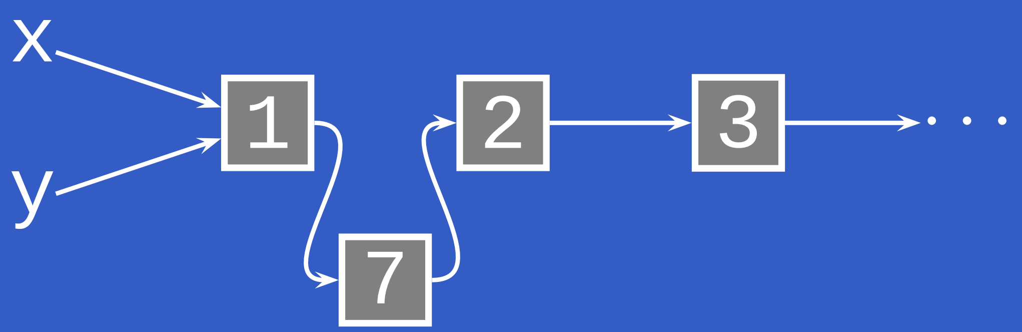
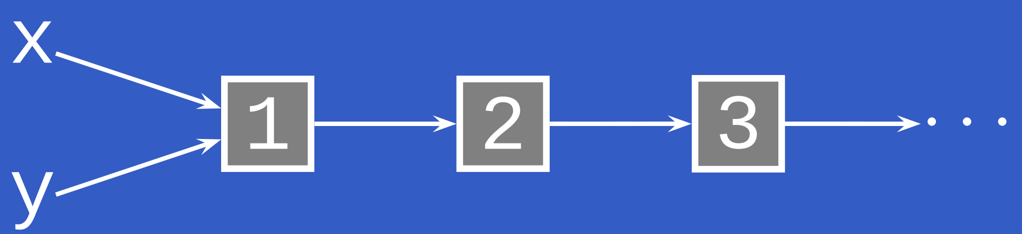


Purely Functional Data structures (3)

Linked list:



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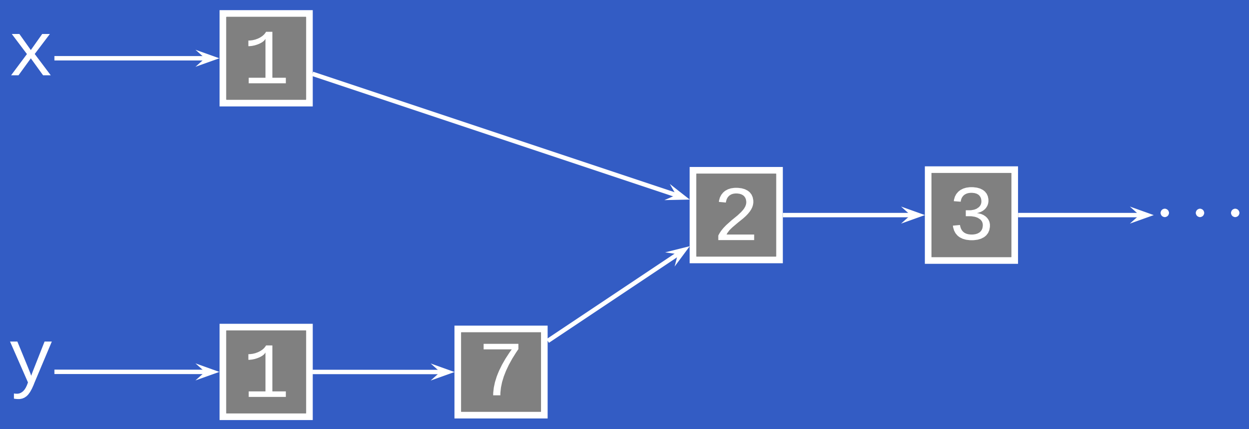

Purely Functional Data structures (4)

This lecture draws from:

> Chris Okasaki. *Purely Functional Data Structures*. Cambridge University Press, 1998.

We will look at some examples of how ***numerical representations*** can be used to derive purely functional data structures.

Numerical Representations (1)

Strong analogy between lists and the usual representation of natural numbers:

```
data List a =
      Nil
    | Cons a (List a)

tail (Cons _ xs) = xs

append Nil           ys = ys
append (Cons x xs) ys =
    Cons x (append xs ys)
```

```
data Nat =
      Zero
    | Succ Nat

pred (Succ n) = n

plus Zero n     = n
plus (Succ m) n =
    Succ (plus m n)
```

Numerical Representations (2)

This analogy can be taken further for designing ***container*** structures because:

- inserting an element resembles incrementing a number
- combining two containers resembles adding two numbers

etc.

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etc.

Thus, representations of natural numbers with certain properties ***induce*** container types with similar properties. Called ***Numerical Representations***.

Random Access Lists

We will consider ***Random Access Lists*** in the following. Signature:

```
data RList a

empty   :: RList a
isEmpty :: RList a -> Bool
cons    :: a -> RList a -> RList a
head    :: RList a -> a
tail    :: RList a -> RList a
lookup  :: Int -> RList a -> a
update  :: Int -> a -> RList a
           -> RList a
```

Positional Number Systems (1)

- A number is written as a ***sequence*** of ***digits*** $b_0 b_1 \ldots b_{m-1}$, where $b_i \in D_i$ for a fixed family of digit sets given by the positional system.
- b_0 is the ***least significant*** digit, b_{m-1} the ***most significant*** digit (note the ordering).
- Each digit b_i has a ***weight*** w_i. Thus:

$$\mathrm{value}(b_0 b_1 \ldots b_{m-1}) = \sum_{0}^{m-1} b_i w_i$$

where the fixed sequence of weights w_i is given by the positional system.

Positional Number Systems (2)

- A number is written written in ***base*** B if $w_i = B^i$ and $D_i = \{0, \ldots, B-1\}$.
- The sequence w_i is usually, but not necessarily, increasing.
- A number system is ***redundant*** if there is more than one way to represent some numbers (disallowing trailing zeroes).
- A representation of a positional number system can be ***dense***, meaning including zeroes, or ***sparse***, eliding zeroes.

Exercise 1: Positional Number Systems

Suppose $w_i = 2^i$ and $D_i = \{0, 1, 2\}$. Give three different ways to represent 17.

Exercise 1: Solution

- 10001, since $\mathrm{value}(10001) = 1 \cdot 2^0 + 1 \cdot 2^4$
- 1002, since $\mathrm{value}(1002) = 1 \cdot 2^0 + 2 \cdot 2^3$
- 1021, since $\mathrm{value}(1021) = 1 \cdot 2^0 + 2 \cdot 2^2 + 1 \cdot 2^3$
- 1211, since
  $\mathrm{value}(1211) = 1 \cdot 2^0 + 2 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$

From Positional System to Container

Given a positional system, a numerical representation may be derived as follows:

- for a container of size n, consider a representation $b_0 b_1 \ldots b_{m-1}$ of n,
- represent the collection of n elements by a ***sequence of trees*** of size w_i such that there are b_i trees of that size.

For example, given the positional system of exercise 1, a container of size 17 might be represented by 1 tree of size 1, 2 trees of size 2, 1 tree of size 4, and 1 tree of size 8.

What Kind of Trees?

The kind of tree should be chosen depending on needed sizes and properties. Two possibilities:

- ***Complete Binary Leaf Trees***

```
data Tree a = Leaf a
            | Node (Tree a) (Tree a)
```

  Sizes: $2^n, n \geq 0$

- ***Complete Binary Trees***

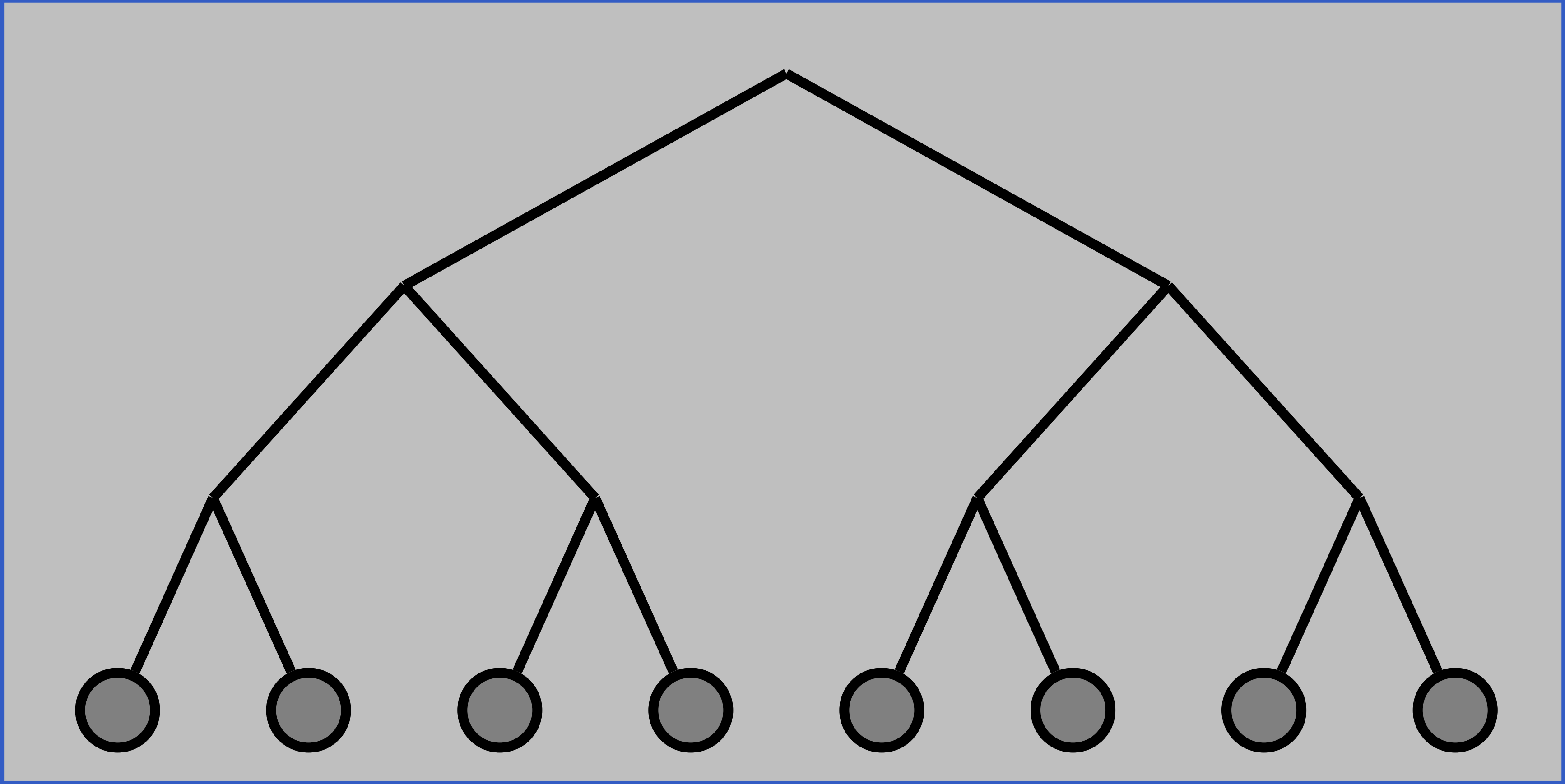
```
data Tree a = Leaf a
            | Node (Tree a) a (Tree a)
```

  Sizes: $2^{n+1} - 1, n \geq 0$

(Balance has to be ensured separately.)

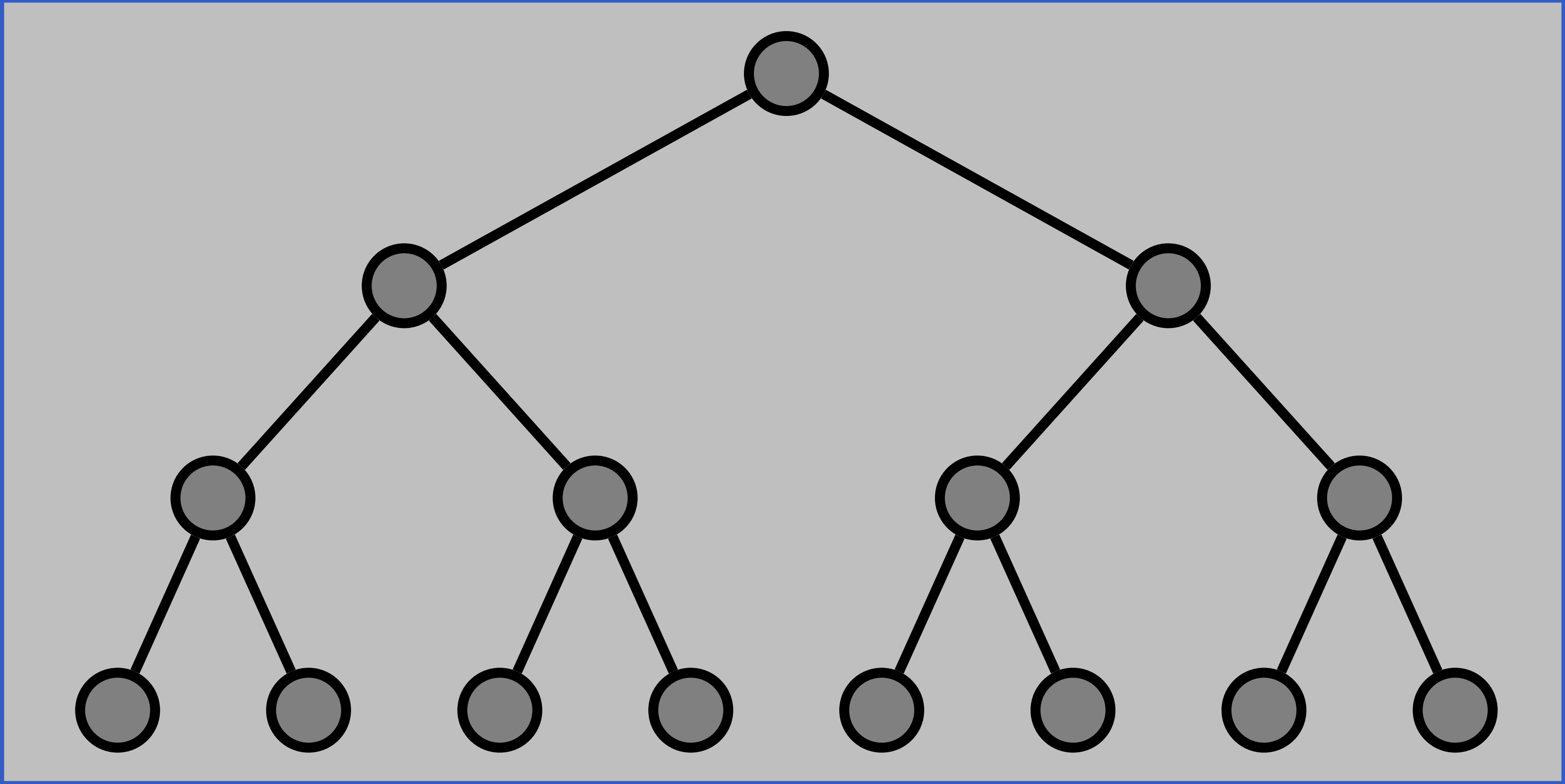
Example: Complete Binary Leaf Tree

Size $2^3 = 8$:



Example: Complete Binary Tree

Size $2^4 - 1 = 15$:



Binary Random Access Lists (1)

Binary Random Access Lists are induced by

- the usual binary representation, i.e. $w_i = 2^i$, $D_i = \{0, 1\}$
- complete binary leaf trees

Thus:

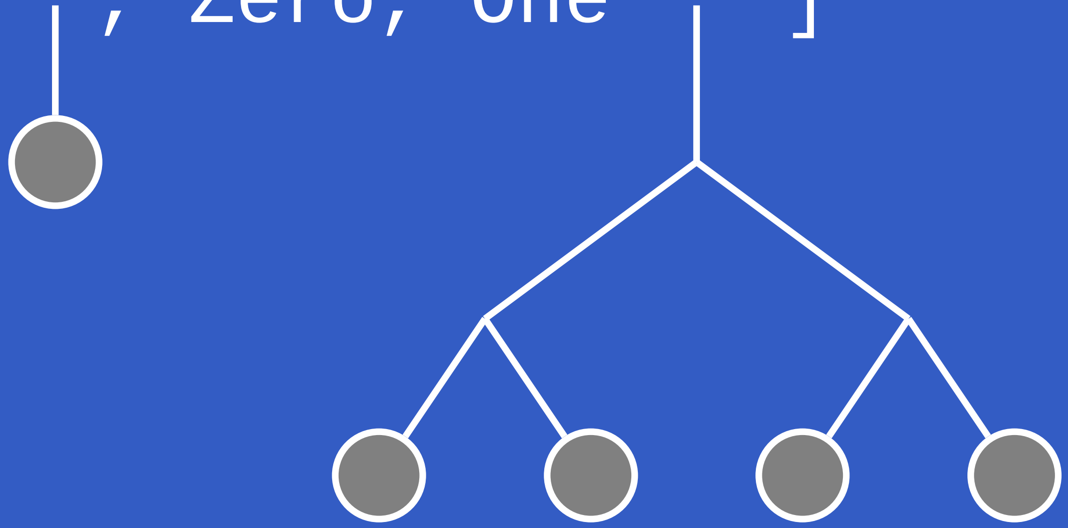
```
data Tree a = Leaf a
            | Node Int (Tree a) (Tree a)
data Digit a = Zero | One (Tree a)
type RList a = [Digit a]
```

The `Int` field keeps track of tree size for speed.

Binary Random Access Lists (2)

Example: Binary Random Access List of size 5:

```
[ One   , Zero, One    ]
```



Binary Random Access Lists (3)

The increment function on dense binary numbers:

```
inc [] = [One]
inc (Zero : ds) = One : ds
inc (One  : ds) = Zero : inc ds  -- Carry
```

Binary Random Access Lists (4)

Inserting an element first in a binary random access list is analogous to `inc`:

```
cons :: a -> RList a -> RList a
cons x ts = consTree (Leaf x) ts

consTree :: Tree a -> RList a -> RList a
consTree t []               = [One t]
consTree t (Zero : ts)      = (One t : ts)
consTree t (One t' : ts) =
    Zero : consTree (link t t') ts
```

Binary Random Access Lists (5)

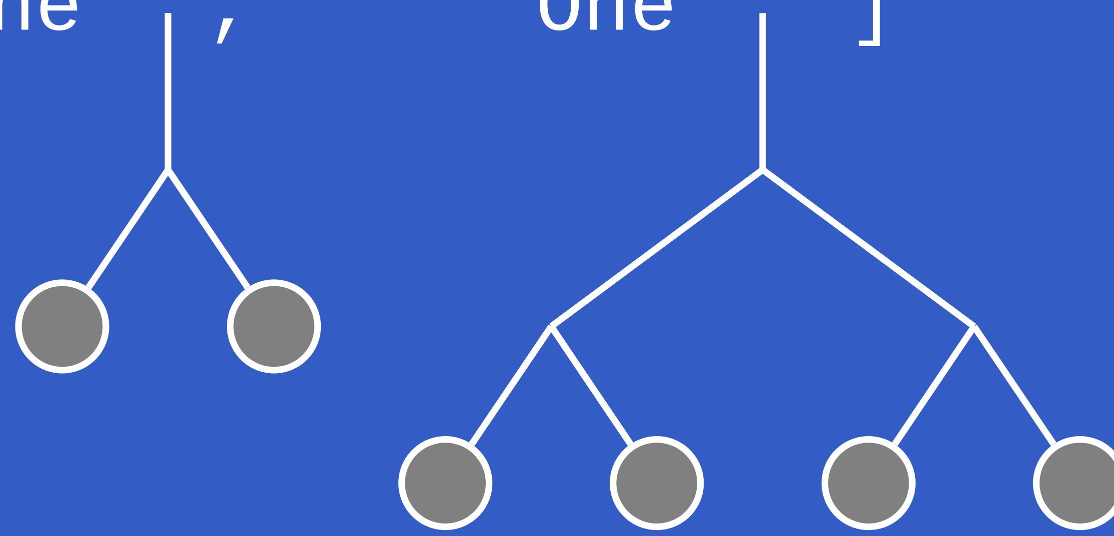
The utility function `link` joins two equally sized trees:

```
-- t1 and t2 are assumed to be the same size
link t1 t2 = Node (2 * size t1) t1 t2
```

Binary Random Access Lists (6)

Example: Result of consing element onto list of size 5:

```
[ Zero, One  ,        One   ]
```



Exercise 2: `unconsTree`

The decrement function on dense binary numbers:

```
dec [One] = []
dec (One  : ds) = Zero : ds
dec (Zero : ds) = One : dec ds  -- Borrow
```

Define `unconsTree` following the above pattern:

```
unconsTree :: RList a -> (Tree a, RList a)
```

And then `head` and `tail`:

```
head :: RList a -> a
tail :: RList a -> RList a
```

Exercise 2: Solution (1)

```
unconsTree :: RList a -> (Tree a, RList a)
unconsTree [One t]        = (t, [])
unconsTree (One t : ts) = (t, Zero : ts)
unconsTree (Zero  : ts) = (t1, One t2 : ts’)
    where
        (Node _ t1 t2, ts’) = unconsTree ts
```

Note: partial operation.

Exercise 2: Solution (2)

```
head :: RList a -> a
head ts = x
    where
        (Leaf x, _) = unconsTree ts


tail :: RList a -> RList a
tail ts = ts'
    where
        (_, ts') = unconsTree ts
```

Binary Random Access Lists (7)

Lookup is done in two stages: first find the right tree, then lookup in that tree:

```
lookup :: Int -> RList a -> a
lookup i (Zero : ts) = lookup i ts
lookup i (One t : ts)
    | i < s        = lookupTree i t
    | otherwise    = lookup (i - s) ts
    where
        s = size t
```

Note: partial operation.

Binary Random Access Lists (8)

```
lookupTree :: Int -> Tree a -> a
lookupTree _ (Leaf x) = x
lookupTree i (Node w t1 t2)
    | i < w `div` 2 =
        lookupTree i t1
    | otherwise =
        lookupTree (i - w `div` 2) t2
```

The operation `update` has exactly the same structure.

Binary Random Access Lists (9)

Time complexity:

- `cons`, `head`, `tail`, perform $O(1)$ work per digit, thus $O(\log n)$ worst case.
- `lookup` and `update` take $O(\log n)$ to find the right tree, and then $O(\log n)$ to find the right element in that tree, so $O(\log n)$ worst case overall.

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Time complexity for `cons`, `head`, `tail` disappointing: can we do better?

Skew Binary Numbers (1)

Skew Binary Numbers:

- $w_i = 2^{i+1} - 1$ (rather than 2^i)
- $D_i = \{0, 1, 2\}$

Representation is redundant. But we obtain a ***canonical form*** if we insist that only the least significant non-zero digit may be 2.

Note: The weights correspond to the sizes of ***complete*** binary trees.

Skew Binary Numbers (2)

Theorem: Every natural number n has a unique skew binary canonical form.
Proof ***sketch***. By induction on n.

- Base case: the case for 0 is direct.

Skew Binary Numbers (3)

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Skew Binary Numbers (3)

- Inductive case. Assume n has a unique skew binary representation $b_0 b_1 \ldots b_{m-1}$
  - If the least significant non-zero digit is smaller than 2, then $n+1$ has a unique skew binary representation obtained by adding 1 to the least significant digit b_0.
  - If the least significant non-zero digit b_i is 2, then note that $1 + 2(2^{i+1} - 1) = 2^{i+2} - 1$. Thus $n+1$ has a unique skew binary representation obtained by setting b_i to 0 and adding 1 to b_{i+1}.

Exercise 3a: Skew Binary Numbers

- Give the canonical skew binary representation for 31, 30, 29, and 28.

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- Give the canonical skew binary representation for 31, 30, 29, and 28.
- Solution: 00001, 0002, 0021, 0211

Exercise 3b: Skew Binary Numbers

Assume a ***sparse*** skew binary representation of the natural numbers

```
type Nat = [Int]
```

where the integers represent the ***weight*** of each ***non-zero*** digit. Assume further that the integers are stored in increasing order, except that the first two may be equal indicating that the smallest non-zero digit is 2. E.g. `28 = [3,3,7,15]`.

Implement a function `inc` to increment a natural number. (E.g. `inc [3,3,7,15] = [7,7,15]`)

Exercise 3b: Solution

```
inc :: Nat -> Nat
inc (w1 : w2 : ws)
    | w1 == w2 = w1 * 2 + 1 : ws
inc ws            = 1 : ws
```

Note! No carry propagation!

E.g.:

```
   inc [1,3,7,15]  = [1,1,3,7,15]
   inc [1,1,3,7,5] = [3,3,7,15]
   inc [3,3,7,15]  = [7,7,15]
```

Skew Binary Random Access Lists (1)

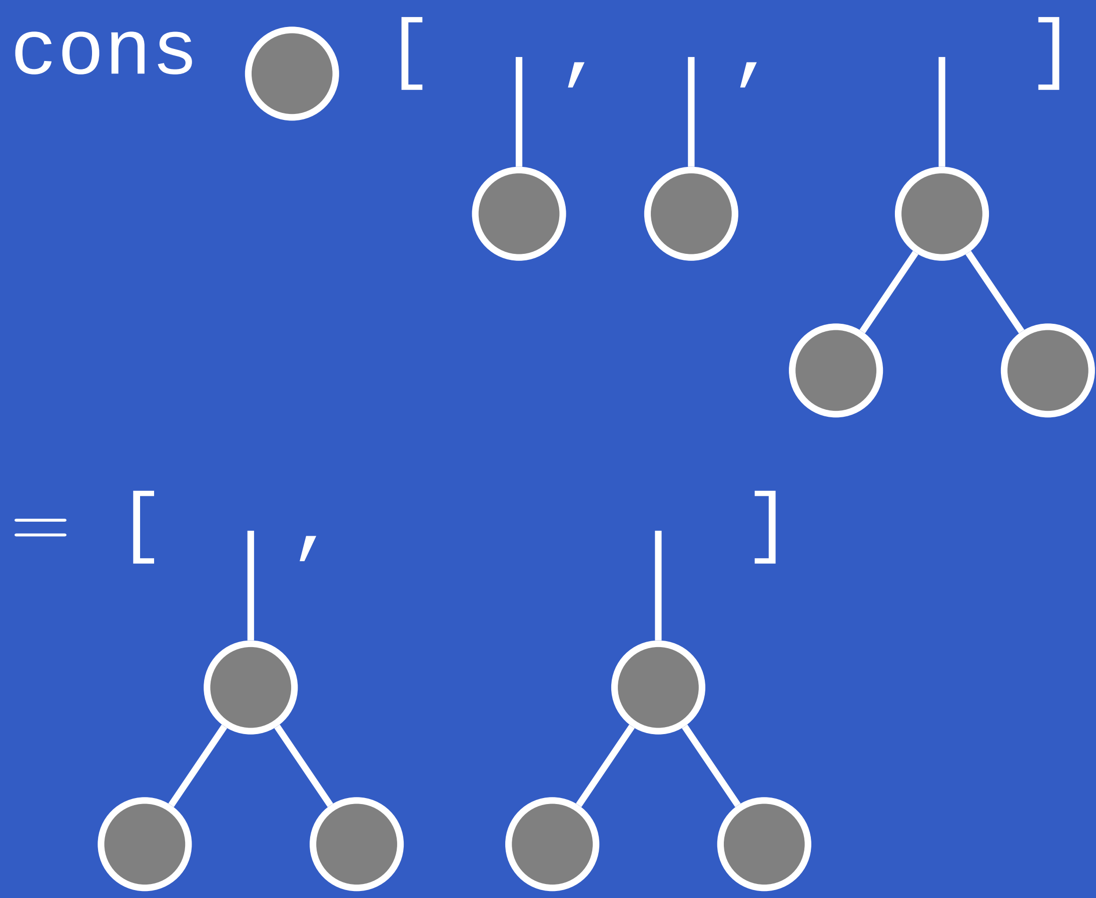
```
data Tree a = Leaf a | Node (Tree a) a (Tree a)
type RList a = [(Int, Tree a)]

empty :: RList a
empty = []

cons :: a -> RList a -> RList a
cons x ((w1, t1) : (w2, t2) : wts) | w1 == w2 =
    (w1 * 2 + 1, Node t1 x t2) : wts
cons x wts = ((1, Leaf x) : wts)
```

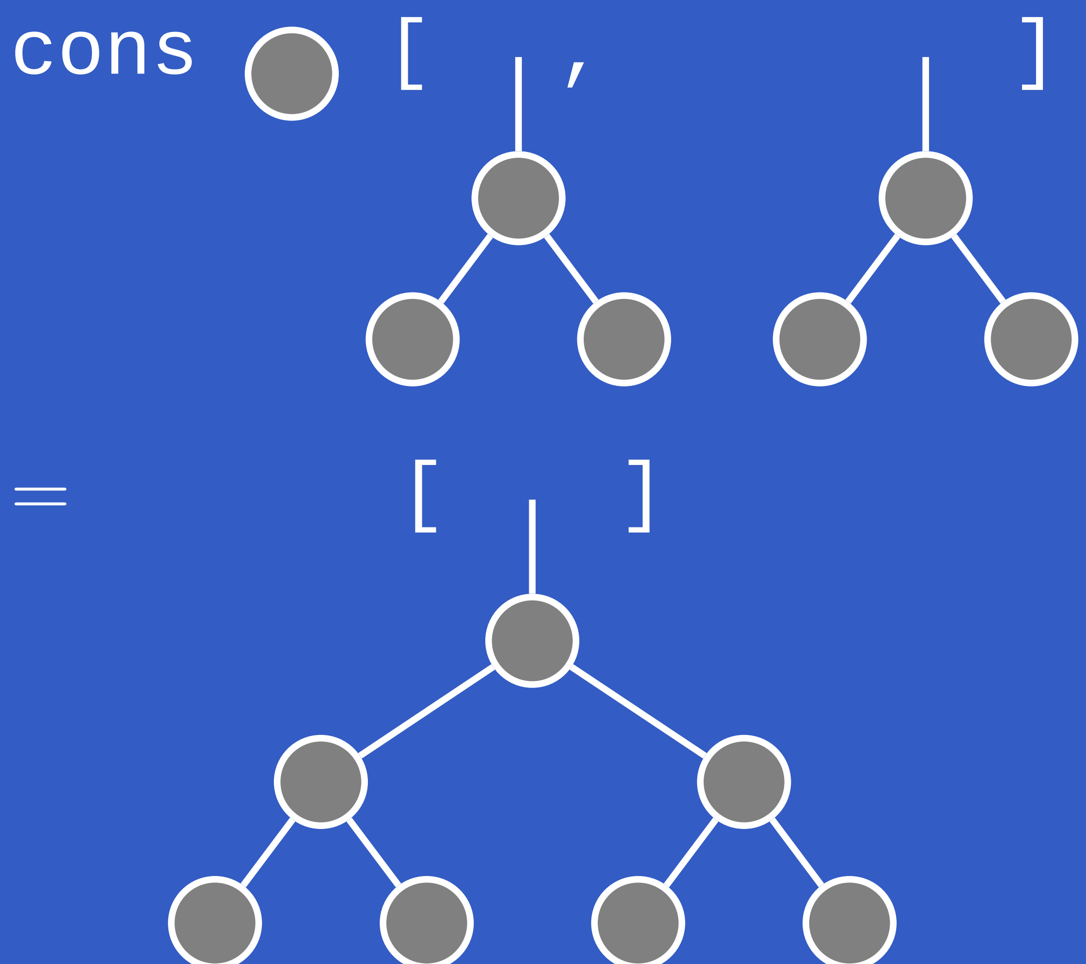
Skew Binary Random Access Lists (2)

Example: Consing onto list of size 5:



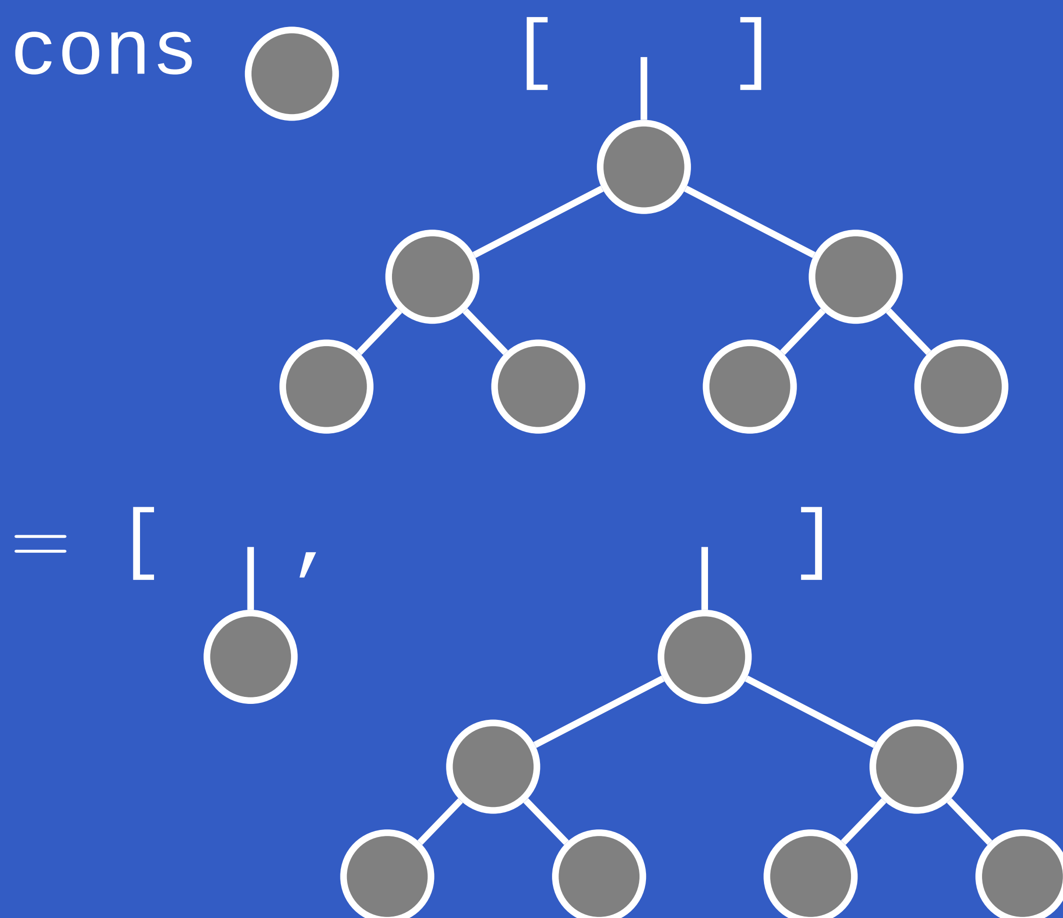
Skew Binary Random Access Lists (3)

Example: Consing onto list of size 6:



Skew Binary Random Access Lists (4)

Example: Consing onto list of size 7:



Skew Binary Random Access Lists (5)

```
head :: RList a -> a
head ((_, Leaf x)     : _) = x
head ((_, Node _ x _) : _) = x

tail :: RList a -> RList a
tail ((_, Leaf _): wts) = wts
tail ((w, Node t1 _ t2) : wts) =
    (w', t1) : (w', t2) : wts
    where
        w' = w `div` 2
```

Note: again, partial operations.

Skew Binary Random Access Lists (6)

```
lookup :: Int -> RList a -> a
lookup i ((w, t) : wts)
    | i < w       = lookupTree i w t
    | otherwise   = lookup (i - w) wts
lookupTree :: Int -> Int -> Tree a -> a
lookupTree _ _ (Leaf x) = x
lookupTree i w (Node t1 x t2)
    | i == 0    = x
    | i < w'    = lookupTree (i - 1) w' t1
    | otherwise = lookupTree (i - w' - 1) w' t2
    where
        w' = w `div` 2
```

Skew Binary Random Access Lists (7)

Time complexity:

- `cons`, `head`, `tail`: $O(1)$.
- `lookup` and `update` take $O(\log n)$ to find the right tree, and then $O(\log n)$ to find the right element in that tree, so $O(\log n)$ worst case overall.

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- `cons`, `head`, `tail`: $O(1)$.
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Okasaki:

> "Although there are better implementations of lists, and better implementations of (persistent) arrays, none are better at both."