Fixpoint Monads

Reconciling Domain and Metric Theoretic Fixpoints

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Abstract

We show that least fixpoints of endomaps on pointed domains and unique fixpoints of contractive endomaps on complete metric spaces arise from one and the same categorical structure, namely that of a *fixpoint monad* corresponding to a pointed endofunctor. This specializes to the existing notion of a fixpoint object. As an application we give a criterion for algebraic compactness.

1 Introduction

Traditionally, denotational models of computation have been based on some form of partial order (*domain*) [Plo81], with recursion modelled using Tarski's least fixpoint theorem. An alternative approach, developed especially for modelling concurrent computations, is the metric-theoretic one [BR92], where recursion is modelled using Banach's unique fixpoint theorem. Various attempts have already been made to reconcile these two different approaches (see, eg, [Smy88, Wag94]). Here we address one particular question, namely whether it is possible to provide a uniform framework for understanding both Tarski's and Banach's fixpoint theorems, despite their apparent heterogeneous nature.

A motivation for our investigation is [Esc99], where a metric model of PCF is given which is very close to the traditional domain-theoretic model [Sco93]. Here we give a conceptual explanation of some constructions performed in that work. A crucial observation is that ε-contractive maps $X \to Y$ between metric spaces are in bijection with maps of type $Id_\varepsilon X \to Y$, where Id_ε is the endofunctor scaling down the distance between the points of a space by a factor $\varepsilon \in [0, 1)$. Contractive *endomaps* thus appear as algebras $Id_\varepsilon X \to X$ of the endofunctor Id_ε.

We use this correspondence as a basis for an axiomatization of the notion of contractive endomap and Banach's fixpoint theorem, following closely the domain-theoretic axiomatization which led to the notion of *fixpoint object* [CP92]. However, we do not use the notion of lifting monad as our starting point. There are at least two reasons for that. Firstly, the the obvious candidate for a metric counterpart of lifting is some form of down-scaler endofunctor, but

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down-scalers cannot be given the structure of a monad. The problem is that there is no map from $Id_\varepsilon X$ to X which is non-distance increasing, hence, in particular, no multiplication. Secondly, while the initial algebra of the *endofunctor part* of the lifting monad is a fixpoint object, providing the basic structure to compute least fixpoints of endomaps on pointed domains, the initial algebra of any metric down-scaler endofunctor is the trivial empty metric space.

We overcome the second obstacle by noticing that although the initial algebra, i.e. the free algebra over the initial object, is trivial, all other free algebras of Id_ε can be used to obtain the fixpoints of ε-contractive maps given by Banach's theorem in essentially the same way as for the fixpoints of endomaps on cppos from the fixpoint object. We proceed then by axiomatizing the relevant structure involved in the above remarks, obtaining the new notions of σ-contractive map and of fixpoint monad relative to arbitrary pointed endofunctors $\langle S, \sigma \rangle$.

The theorems we prove are routine generalizations of those for fixpoint objects, once one realizes various unexploited generalities in the latter. For instance, one can start from a pointed endofunctor rather than from a monad in order to define the fixpoint object. This is crucial, because the down-scaler Id_ε comes with a natural point, namely the evident 'almost identity' map $id^\varepsilon : Id \to Id_\varepsilon$, hence the structure of the lifting monad relevant for fixpoints is shared by the down-scalers. In fact, we generalize the motivating theorem for fixpoint objects [Sim92, Thm 4.3] by considering fixpoints of (possibly weakly) σ-contractive maps. This not only accounts for contractive maps in the metric sense, but also for all endomaps on cppos, the least element of a cppo giving algebraic structure which makes any endomap into an algebra of the lifting endofunctor.

A further sign of the soundness of our generalization is that the theorems relating fixpoint objects and final coalgebras [Fre91] also generalize to fixpoint monads. Indeed, we easily prove that the existence of a certain family of final coalgebras gives rise to a fixpoint monad. In the converse direction, we use the notion of *locally contractive* endofunctors on monoidal closed categories and the fact that they are algebraically compact.

2 Terminology and Notation

Metric Spaces In this paper by a *metric space* we intend a *non-empty* set X together with a *bounded* metric $d : X \times X \to [0, \infty)$. Without loss of generality, we assume the bound to be 1, hence $d : X \times X \to [0, 1]$. Such metric spaces form a category with morphisms $\langle X, d \rangle \to \langle X', d' \rangle$ given by non-distance increasing functions $f : X \to X'$, ie $d'(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for all x_1 and x_2 in X. A special case of non-distance increasing functions are the ε-contractive ones, where $\varepsilon \in [0, 1)$ and $d'(f(x_1), f(x_2)) \leq \varepsilon \cdot d(x_1, x_2)$ for all x_1 and x_2 in X. (See, eg, [Dug66] for more on metric spaces.)

Algebras A pointed endofunctor $\langle S, \sigma \rangle$ consists of an endofunctor on a category $\mathcal{C}$ and a natural transformation $\sigma : Id \Rightarrow S$. (The latter is called the point of S, but will shall call points also the maps $1 \to X$ from a terminal object.) In this paper we consider algebras of endofunctors S, of pointed endofunctors $\langle S, \sigma \rangle$, and of monads $\langle S, \sigma, \mu \rangle$ on a category $\mathcal{C}$. All three consist of maps $f : SX \to X$, but this map has to satisfy the equations $id_X = f \circ \sigma_X$ and $f \circ Sf = f \circ \mu_X$ if S is a monad, only the first equation if S is a pointed endofunctor, and no equation at all if S is just an endofunctor. When we say that f is an algebra of an endofunctor S we mean that f has to satisfy no equation at all, even if S is a pointed endofunctor or a monad.

The definition of algebra homomorphism $h : \langle X, f\rangle \to \langle Y, g\rangle$ is independent from the equations for the algebra structure; that is, in all three cases it consists of a morphism $h : X \to Y$ in the underlying category such that

$$\begin{tikzcd} SX \arrow[r, "Sh"] \arrow[d, "f"'] & SY \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes. In the sequel, S-*Alg* will denote the category of algebras of the endofunctor S. The symbol T will be reserved for a monad, and T-*Alg* for the corresponding category.

3 σ-Contractive Maps

Definition 3.1 Let $\langle S, \sigma\rangle$ be a pointed endofunctor. We say that a map $f : X \to Y$ is *weakly σ-contracting* if it factors through $\sigma_X : X \to SX$; if there is a *unique* $\overline{f} : SX \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[dr, "f"'] & SX \arrow[d, dashed, "\overline{f}"] \\ & Y \end{tikzcd}$$

commutes then we say that f is *σ-contracting* and we call such $\overline{f}$ the σ-factor of f. □

Example 3.2 Lifting with eta in cpos. A map is weakly contractive iff its image has a lower bound. In this case a factor maps bottom to a lower bound. Hence, strong contractivity hardly ever occurs.

Example 3.3 The functor is down-scaler for a given fixed ε, the point is the "identity". Contractive maps are contractive maps in the metric sense, with contractivity factor ε. Since the "identity" is epi, weak and strong contractivity coincide, as the following proposition shows.

Proposition 3.4

1. If the pushout of $\sigma_X : SX \to Y$ along f exists and it is the identity on Y then f is σ-contracting.

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "f"'] & SX \arrow[d, "\overline{f}"] \\ Y \arrow[r, "id_Y"'] & Y \end{tikzcd}$$

2. If σ is pointwise epi and f is weakly σ-contracting, then the pushout of $\sigma_X : SX \to Y$ along f exists and it is the identity on Y, hence f is σ-contracting.

Proof. 1) If $g : SX \to Y$ is such that $g \circ \sigma_X = f$, then, by the universal property of pushouts, there is a unique $g' : Y \to Y$ such that $g' \circ id_Y = id_Y$ and $g' \circ g = \overline{f}$, which implies that $g = \overline{f}$.

2) Let $\overline{f}$ be a σ-factor for f. Take $h : SX \to Z$ and $k : Y \to Z$ such that $k \circ f = h \circ \sigma_X$. Then $k \circ \overline{f} \circ \sigma_X = h \circ \sigma_X$, hence, since σ_X is epi, $k \circ \overline{f} = h$, which means that k is the desired unique mediating morphism from Y to Z. □

Note that if $f : X \to Y$ is (weakly) σ-contractive so are the composites $f \circ g$ and $h \circ f$, for all maps $g : X' \to X$ and $h : Y \to Y'$. In particular, since any map $X \to 1$ into a terminal object is trivially contracting, constant maps $X \to 1 \to Y$ are contractive.

If $\langle S, \sigma, \mu \rangle$ is a monad and X is an algebra for the *monad* S, then every map $f : X \to Y$ is trivially weakly σ-contractive, because the algebra structure cancels the unit σ_X of the monad S. This is the case when X is a cppo and S is the lifting monad (although σ is not pointwise epi).

We now consider the σ-contractive maps for a pointed endofunctor $\langle S, \sigma \rangle$ on a category $\mathcal{C}$ as the *objects* of a category. The arrows $f \to g$ are pairs of maps h and k in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "k"'] & Y' \end{tikzcd}$$

commutes. Then, clearly, such a category is isomorphic to the comma category $(S \downarrow Id)$:

$$\begin{tikzcd} SX \arrow[r, "Sh"] \arrow[d, "\overline{f}"'] & SX' \arrow[d, "\overline{g}"] \\ Y \arrow[r, "k"'] & Y' \end{tikzcd}$$

This restricts to an isomorphism between the subcategory of σ-contractive *endomaps* and the category S-*Alg* of algebras $SX \to X$ of the endofunctor S.

4 Fixpoint Monads

If the (evident) forgetful functor $U : S\text{-}Alg \to \mathcal{C}$ has a left adjoint, then we denote the monad generated by this adjunction by $T = \langle T, \eta, \mu \rangle$, and the free S-algebra over an object X by

$$\delta_X : STX \to TX$$

Note that δ_X is natural in X. We call T the *monad generated by* S.

Definition 4.1 Let T be a monad on a category $\mathcal{C}$ with a terminal object 1. Let 1 also denote the constantly 1 endofunctor on $\mathcal{C}$. Then T is a *fixpoint monad* for a pointed endofunctor (S, σ) if it is generated by the endofunctor S and the parallel natural transformations $\delta \circ \sigma_T$ and id_T have an equalizer $\infty : 1 \to T$. □

Note that ∞ is then a fixpoint for $\delta \circ \sigma_T$

$$\begin{tikzcd} 1 \arrow[d, "\infty"'] \arrow[drr, "\infty"] & & \\ T \arrow[r, "\sigma_T"'] & ST \arrow[r, "\delta"'] & T \end{tikzcd} \tag{1}$$

and that the naturality of ∞_X is a consequence of the universal property of equalizers.

Example 4.2 The monad T generated by the endofunctor Id_ε is the monad studied in [Esc99]. Its action on a (non-empty, 1-bounded) complete metric space $X = \langle X, d \rangle$ is the set

$$TX = (X \times \mathbb{N}) \cup \{\infty\}$$

endowed with the metric $\overline{d} : TX \times TX \to [0,1]$ defined by

$$\overline{d}(\infty, \infty) = 0, \qquad \overline{d}(x^{(n)}, \infty) = \overline{d}(\infty, x^{(n)}) = \varepsilon^n$$

$$\overline{d}(x^{(n)}, y^{(n)}) = \varepsilon^n \cdot d(x, y), \qquad \overline{d}(x^{(n)}, y^{(m)}) = \varepsilon^{\min(n,m)} \qquad \text{for } n \neq m.$$

where $x^{(n)}$ stands for (x, n). We regard the points of TX as "abstract computations" of elements of X. The free algebra structure $\delta_X : STX \to TX$ is the *delay operator*

$$\delta_X(x^{(n)}) = x^{(n+1)}, \qquad \delta_X(\infty) = \infty.$$

Finally, the unit $\eta_X : X \to TX$ and the multiplication $\mu_X = \varphi_{\delta_X} : T^2X \to TX$ are given as follows:

$$\eta_X(x) = x^{(0)}, \qquad \mu_X\left(\left(x^{(n)}\right)^{(m)}\right) = x^{(m+n)}.$$

Example 4.3 The monad T generated by the lifting endofunctor L is defined as follows. Its action on a cpo X is the set

$$TX = (X \times \mathbb{N}) \cup (\mathbb{N} \cup \{\infty\})$$

ordered by:

$$x^{(n)} \sqsubseteq y^{(n)} \iff x \sqsubseteq y \qquad n \sqsubseteq x^{(m)} \iff n \leq m$$

$$n \sqsubseteq m \iff n \leq m \qquad n \sqsubseteq \infty$$

where, again, $x^{(n)}$ stands for (x, n). The free algebra structure is defined in a similar way as in the metric case, and for the extra points of $\mathbb{N}$ it is the successor. So is the unit [??what about multiplication??]

Proposition 4.4 If an endofunctor S generates a monad T then the canonical comparison functor

$$K : S\text{-}Alg \to T\text{-}Alg$$

between the algebras of the *endofunctor* S and the algebras of the *monad* T is an isomorphism. $\square$

Before giving concrete examples, we give the general construction of this folklore isomorphism. The functor K maps an S-algebra $\mathfrak{f} : SX \to X$ to the T-algebra $\varphi_{\mathfrak{f}} : TX \to X$, obtained by transposing the identity $id_X : X \to X = U\langle X, \mathfrak{f}\rangle$ across the adjunction from $\mathcal{C}$ to S-Alg:

$$\begin{tikzcd} STX \arrow[r, "S\varphi_{\mathfrak{f}}"] \arrow[d, "\delta_X"'] & SX \arrow[d, "\mathfrak{f}"] \\ TX \arrow[r, "\varphi_{\mathfrak{f}}"'] & X \end{tikzcd} \tag{2}$$

Example 4.5 Given a map $f : LX \to X$, the structure map sends n to $f^n(f(\bot))$, and a point $x^{(n)}$ to $f^n(x)$.

Example 4.6 Given an ε-contractive map f the structure map sends ∞ to the fixpoint of f and a point $x^{(n)}$ to $f^n(x)$.

In other words, $\varphi_{\mathfrak{f}}$ is the counit of the adjunction applied to the algebra $\langle X, \mathfrak{f}\rangle$. On morphisms, the comparison functor behaves as the identity. Therefore:

Remark 4.7 $\varphi_{\mathfrak{f}}$ is natural in $\mathfrak{f}$, in the sense that for every commuting square

$$\begin{tikzcd} SX \arrow[r, "Sh"] \arrow[d, "\mathfrak{f}"'] & SY \arrow[d, "\mathfrak{g}"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

we have $\varphi_{\mathfrak{g}} \circ Th = h \circ \varphi_{\mathfrak{f}}$:

$$\begin{tikzcd} TX \arrow[r, "Sh"] \arrow[d, "\varphi_{\mathfrak{f}}"'] & STY \arrow[d, "\varphi_{\mathfrak{g}}"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

$\square$

The inverse of K acts on a T-algebra $TX \to X$ by precomposing it with $SX \stackrel{S\eta_X}{\to} STX \stackrel{\delta_X}{\to} TX$.

Theorem 4.8 If T is a fixpoint monad for a pointed endofunctor (S, σ), then, for every S-algebra structure $\mathfrak{f} : SX \to X$ the point

$$\mathbf{Y}_{\mathfrak{f}} \stackrel{\text{def}}{=} \varphi_{\mathfrak{f}} \circ \infty_X : 1 \to X$$

has the following two properties:

1. (σ-*fixpoint*) $\mathbf{Y}_{\mathfrak{f}}$ is a σ-fixpoint for $\mathfrak{f}$, ie $\mathbf{Y}_{\mathfrak{f}} = \mathfrak{f} \circ \sigma_X \circ \mathbf{Y}_{\mathfrak{f}}$:

$$\begin{tikzcd} 1 \arrow[d, "\mathbf{Y}_{\mathfrak{f}}"'] \arrow[drr, "\mathbf{Y}_{\mathfrak{f}}"] & & \\ X \arrow[r, "\sigma_X"'] & SX \arrow[r, "\mathfrak{f}"'] & X \end{tikzcd}$$

2. (*uniformity*) $\mathbf{Y}_{\mathfrak{f}}$ is natural in $\mathfrak{f}$ in the sense that for all S-algebra homomorphisms $h : \langle X, \mathfrak{f}\rangle \to \langle Y, \mathfrak{g}\rangle$

$$\begin{tikzcd} SX \arrow[r, "Sh"] \arrow[d, "\mathfrak{f}"'] & SY \arrow[d, "\mathfrak{g}"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

we have $\mathbf{Y}_{\mathfrak{g}} = h \circ \mathbf{Y}_{\mathfrak{f}}$:

$$\begin{tikzcd} 1 \arrow[d, "\mathbf{Y}_{\mathfrak{f}}"'] \arrow[dr, "\mathbf{Y}_{\mathfrak{g}}"] & \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

Moreover:

3. (*uniqueness*) $\mathbf{Y}$ is the unique operator verifying the two properties above.

Proof. 1) Just note that the following diagram commutes because of diagrams (1) and (2) and of the naturality of σ

$$\begin{tikzcd} 1 \arrow[r, "\infty_X"] \arrow[ddr, "\infty_X"'] & TX \arrow[r, "\varphi_{\mathfrak{f}}"] \arrow[d, "\sigma_{TX}"] & X \arrow[d, "\sigma_X"] \\ & STX \arrow[r, "S\varphi_{\mathfrak{f}}"] \arrow[d, "\delta_X"] & SX \arrow[d, "\mathfrak{f}"] \\ & TX \arrow[r, "\varphi_{\mathfrak{f}}"'] & X \end{tikzcd}$$

2) By naturality of ∞ and φ:

$$\mathbf{Y}_{\mathfrak{g}} = \varphi_{\mathfrak{g}} \circ \infty_Y = \varphi_{\mathfrak{g}} \circ Th \circ \infty_X = h \circ \varphi_{\mathfrak{f}} \circ \infty_X = h \circ \mathbf{Y}_{\mathfrak{f}}$$

3) By the equalizing property of ∞, we have that any other σ-fixpoint operator $\overline{\mathbf{Y}}$ has the property that $\overline{\mathbf{Y}}_{\delta_X} = \infty_X$. Then, if $\overline{\mathbf{Y}}$ satisfies the second property,

$$\overline{\mathbf{Y}}_{\mathfrak{f}} = \varphi_{\mathfrak{f}} \circ \overline{\mathbf{Y}}_{\delta_X} = \varphi_{\mathfrak{f}} \circ \infty_X = \mathbf{Y}_{\mathfrak{f}}$$

because, by definition, $\varphi_{\mathfrak{f}}$ is an S-algebra homomorphism between δ_X and $\mathfrak{f}$. $\square$

(Cf [Sim92, Thm 4.3] and [Mul92, Thm 3.12].)

Example 4.9 Cpos.

The familiar fixpoint operator of domain theory is recovered by considering maps

$$SX \xrightarrow{x} X \xrightarrow{f} X$$

where x is an algebra for the *pointed* endofunctor $\langle S, \sigma\rangle$, hence $x \circ \sigma_x = id_X$. Then we have that for every endomap $f : X \to X$ on a (S,σ)-algebra $\langle X, x\rangle$ the point

$$\mathbf{Y}^x_f \stackrel{\text{def}}{=} \varphi_{f\circ x} \circ \infty_X : 1 \to X$$

is a fixpoint for f:

$$\mathbf{Y}_f^x = f \circ x \circ \sigma_X \circ \mathbf{Y}_f^x = f \circ \mathbf{Y}_f^x$$

Moreover, $\mathbf{Y}$ is natural in $\langle f, x\rangle$ in the sense that for all endomaps $f : X \to X$ and $g : Y \to Y$ on (S, σ)-algebras $\langle X, x\rangle$ and $\langle Y, v\rangle$ and all homomorphisms $h : X \to Y$

$$\begin{tikzcd}
SX \arrow[r, "Sh"] \arrow[d, "x"'] & SY \arrow[d, "y"] \\
X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X \arrow[r, "h"'] & Y
\end{tikzcd}$$

we have $\mathbf{Y}_g^y = h \circ \mathbf{Y}_f^x$. $\mathbf{Y}$ is the unique operator verifying the two properties above. Note that we do not need to assume that S is a monad, although when we take S to be the lifting monad on cpos then its (monad) algebras are property rather than structure, hence the superscript x can be dropped. The fixpoint operator is then the one given by the least fixpoint theorem for endomaps on cppos.

Example 4.10 We can define a fixpoint operator $\mathbf{Y}$ for σ-contractive endomaps $f : X \to X$:

$$\mathbf{Y}_f \stackrel{\text{def}}{=} \varphi_{\overline{f}} \circ \infty_X : 1 \to X$$

Clearly:

$$\mathbf{Y}_f = f \circ \mathbf{Y}_f$$

and $\mathbf{Y}_f$ is natural in f in the sense that for all σ-contractive endomaps $f : X \to X$ and $g : Y \to Y$ and all commuting squares

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X \arrow[r, "h"'] & Y
\end{tikzcd}$$

we have $\mathbf{Y}_g = h \circ \mathbf{Y}_f$. Moreover, $\mathbf{Y}$ is the unique operator verifying the two properties above. It corresponds to the fixpoint operator given by Banach's fixpoint theorem.

Remark 4.11 If T is a fixpoint monad for a pointed endofunctor on a category $\mathcal{C}$ with an initial object 0, then all equalizers ∞_X are determined by ∞_0, because $\infty_X = \mathbf{Y}_{\delta_X}$ hence, by uniformity, ∞_X is the composition of $\infty_0 = \mathbf{Y}_{\delta_0}$ with the unique homomorphism from the initial algebra δ_0 to δ_X. This explains why in categories with initial objects one can focus on the *fixpoint object* $T0$ rather than on the whole monad T.

In particular, for cppos, $T0$ is given by the ordered natural numbers together with a top infinity point. In the metric case, when $\langle S, \sigma\rangle$ is $(Id_\varepsilon, id^\varepsilon)$, there is no algebra $x : Id_\varepsilon X \to X$ such that $x \circ id_X^\varepsilon = id_X$. Thus, instead of arbitrary endomaps between algebras of the *pointed* endofunctor $(Id_\varepsilon, id^\varepsilon)$, one considers ε-contractive endomaps. Since id^ε is pointwise epi, there is an isomorphism $f \mapsto \overline{f}$ between the category of ε-contractive endomaps and the category of Id_ε-algebras.

5 Fixpoints and Final Coalgebras

Let us consider the case when the category $\mathcal{C}$ has binary coproducts and the endofunctor S *freely generates a monad* T, in the sense that $TX \cong X + STX$ is an initial algebra for the endofunctor $(X + S)$ for every X in $\mathcal{C}$.

Proposition 5.1 If, for every X in $\mathcal{C}$, $TX \cong X + STX$ is a final $(X + S)$-coalgebra then T is a fixpoint monad for $\langle S, \sigma \rangle$ for every point $\sigma : Id \Rightarrow S$ of S.

We need the following technical lemma.

Lemma 5.2 Let G and H be two endofunctors on a category $\mathcal{C}$ connected by a natural transformation $\tau : H \Rightarrow GH$. Let $a : A \to GHA$ be a final GH-coalgebra with inverse $a' : GHA \to A$. Then, the algebra $a' \circ \tau_A : HA \to A$ is final with respect to homomorphisms from H-coalgebras, ie for every H-coalgebra $y : Y \to HY$, there exists a unique $h : Y \to A$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "y"'] & A \\ HY \arrow[r, "Hh"'] & HA \arrow[u, "a' \circ \tau_A"'] \end{tikzcd}$$

commutes.

Proof. Simply turn $y : Y \to HY$ into a GH-coalgebra by composing it with $\tau_Y : HY \to GHY$ and then use finality. $\square$

Proof of Proposition 5.1. Define ∞_X using the finality property of δ_X given by the above lemma for $\tau = inr : S \Rightarrow X + S$:

$$\begin{tikzcd} 1 \arrow[r, dashed, "\infty_X"] \arrow[d, "\sigma_1"'] & TX \\ S1 \arrow[r, "S(\infty_X)"'] & STX \arrow[u, "\delta_X"'] \end{tikzcd}$$

Then, by the naturality of σ:

$$\delta_X \circ \sigma_{TX} \circ \infty_X = \delta_X \circ S(\infty_X) \circ \sigma_1 = \infty_X$$

Further, that ∞ is an equalizer can be proved as follows. Consider a map $f : A \to TX$ such that

$$f = \delta_X \circ \sigma_{TX} \circ f \tag{3}$$

The claim is that f factors through $\infty_X : 1 \to TX$. By the finality property of δ_X and the definition of ∞_X, it suffices to show that the following two diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma_A"'] & TX \\ SA \arrow[r, "Sf"'] & STX \arrow[u, "\delta_X"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "!"] \arrow[d, "\sigma_A"'] & 1 \arrow[d, "\sigma_1"] \\ SA \arrow[r, "S!"'] & S1 \end{tikzcd}$$

But the commutativity of the second is trivial and the commutativity of the first follows from the naturality of σ and the assumption (3). $\square$

Algebraic Compactness

In the rest of this section we assume familiarity with the results in [Koc72]. Thus recall that a monoidal pointed endofunctor $\langle S, \sigma \rangle$ on a monoidal category $\mathcal{C} = \langle \mathcal{C}, \otimes, I \rangle$ is equipped with a natural transformation

$$\mathfrak{m}_{X,Y} : X \otimes Y \to S(X \otimes Y)$$

which gives rise to a tensorial strength $\mathfrak{t}_{X,Y} : X \otimes SY \to S(X \otimes Y)$

$$\begin{tikzcd} X \otimes SY \arrow[r, "\sigma \otimes id"] \arrow[dr, "\mathfrak{t}"'] & SX \otimes SY \arrow[d, "\mathfrak{m}"] \\ & S(X \otimes Y) \end{tikzcd}$$

Also recall that in a (*monoidal*) *closed category*, ie a category where $\_ \otimes X$ has a right adjoint $[X, \_]$, there is a 1-1 correspondence between tensorial strengths and ordinary strengths; the latter internalize the action of endofunctors on morphisms:

$$\mathfrak{s}_{X,Y} : [X, Y] \to [SX, SY]$$

(see [Koc72, Thm 1.3]).

Lemma 5.3 If $\langle S, \sigma \rangle$ is a monoidal pointed endofunctor on a closed category $\mathcal{C}$, then $\mathfrak{s}_{X,A} : [X, A] \to [SX, SA]$ is weakly σ-contractive for all X and A in $\mathcal{C}$.

Proof. By definition, $\mathfrak{s}_{X,A} : [X, A] \to [SX, SA]$ is obtained by transposing the map $S\varepsilon \circ \mathfrak{m} \circ (\sigma \otimes id)$, therefore it is the transposition of $S\varepsilon \circ \mathfrak{m}$ precomposed with σ, which gives the desired factorization. $\square$

Definition 5.4 Let H be a monoidal endofunctor on a closed category $\mathcal{C}$. We say that H is *locally σ-contractive* if its strength is σ-contractive.

Theorem 5.5 Let $\langle S, \sigma \rangle$ be a monoidal pointed endofunctor on a closed category $\mathcal{C} = (\mathcal{C}, \otimes, I)$ generating a fixpoint monad. Then every locally σ-contractive endofunctor H on $\mathcal{C}$ is algebraically compact, ie if $\alpha : HA \cong A$ is an initial H-algebra then its inverse $\alpha' : A \cong HA$ is a final H-coalgebra.

Proof. We have to prove that for every $g : X \to HX$ there exists a unique $h : X \to A$ making the diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "g"'] & A \arrow[d, bend right, "\alpha'"'] \\ HX \arrow[r, "Hh"'] & HA \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

commute. By monoidal closure, the mapping $h \mapsto \alpha \circ Hh \circ g$ corresponds to a map

$$\Phi_g : [X, A] \to [X, A]$$

which is σ-contracting because it factors through the (by hypothesis) σ-contractive map $\mathfrak{s}_{X,A} : [X,A] \to [HX,HA]$ as follows:

$$[X,A] \xrightarrow{\mathfrak{s}_{X,A}} [HX,HA] \xrightarrow{[g,\alpha]} [X,A]$$

By, again, monoidal closure, $\mathbf{Y}_{\Phi_g}$ is the name of a coalgebra map from A to TX. This proves the existence part of the theorem.

Next, uniqueness. Note that, by the initiality of A, the map $\Phi_{\alpha'}$ can have only one fixpoint, namely (the name of) the identity on $[A,A]$. Therefore, by uniformity, Φ_g is completely determined by a (necessarily unique) homomorphism from $\Phi_{\alpha'}$ to Φ_g. But, as the following diagram shows, such a homomorphism is given by $[h,A]$, for any coalgebra homomorphism $h : X \to A$.

$$\begin{tikzcd}
[A,A] \arrow[r, "\mathfrak{s}_{A,A}"] \arrow[d, "{[h,A]}"'] & [HA,HA] \arrow[r, "{[\alpha',\alpha]}"] \arrow[d, "{[Hh,HA]}"'] & [A,A] \arrow[d, "{[h,A]}"] \\
[X,A] \arrow[r, "\mathfrak{s}_{X,A}"'] & [HX,HA] \arrow[r, "{[g,\alpha]}"'] & [X,A]
\end{tikzcd}$$

The first square commutes by naturality and the second by the fact that h is a coalgebra homomorphism. $\square$

Remark 5.6 [Note that this theorem does not apply to cppo...Strictness..] But maybe the idea could be elaborated to cover this case.

By taking $H = X + S$ we have:

Corollary 5.7 If T is a fixpoint monad freely generated by a monoidal pointed endofunctor $\langle S, \sigma \rangle$ on a closed category $\mathcal{C}$, then, for every X in $\mathcal{C}$, $TX \cong X + STX$ is a final $(X+S)$-coalgebra.

Example 5.8 ???

Future work. The main question we would like to address next is whether it is possible to glue the various down-scalers together and obtain an endofunctor whose algebras are contractive maps, irrespective of their contraction factor.

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