Locating the Scott topology

(Note)

Martín Escardó
School of Computer Science, University of Birmingham, Birmingham B15 2TT, England
`m.escardo@cs.bham.ac.uk` `http://www.dcs.st-and.ac.uk/~mhe`

Reinhold Heckmann
AbsInt Angewandte Informatik GmbH, Stuhlsatzenhausweg 69, D-66123 Saarbrücken, Germany
`heckmann@absint.com` `http://www.cs.uni-sb.de/~heckmann`

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In the *Compendium of Continuous Lattices* [1], it is shown that the way-below relation of a meet-continuous lattice is the intersection of the approximating auxiliary relations, and hence that a complete lattice is continuous if and only if it is meet-continuous and it has a smallest approximating auxiliary relation. We define approximating topologies, and we show that the Scott topology of a meet-continuous lattice is the intersection of the approximating topologies, concluding that a meet-continuous lattice is continuous if and only if it has a smallest approximating topology.

An ***auxiliary relation*** on a complete lattice is a binary relation $\prec$ that satisfies the following conditions:

1. $x \prec y$ implies $x \leq y$,
2. $x' \leq x \prec y \leq y$ implies $x' \prec y'$,
3. $x \prec y$ and $x' \prec y$ together imply $x \vee x' \prec y$.
4. $\perp \prec y$,

1 *The way-below relation of a complete lattice is an auxiliary relation.*

An auxiliary relation $\prec$ on a complete lattice is ***approximating*** if and only if every element x of the lattice is the join of the elements $x' \prec x$.

2 *A complete lattice is continuous if and only if its way-below relation is approximating.*

A complete lattice is ***meet-continuous*** if binary meets distribute over directed joins.

3 *Continuous lattices are meet-continuous.*

4 *The way-below relation of a meet-continuous lattice is the intersection of the approximating relations.*

5 Corollary *A complete lattice is continuous if and only if it is meet continuous and it has a smallest approximating relation.*

For proofs of the above facts, see pages 43–45 of the *Compendium*. We consider a variation of this theme. We first recall more known facts. We say that a topology on a preordered set is ***compatible*** if its specialization order coincides with the preorder. The ***Alexandroff topology*** of a preordered set has as open sets the upper sets, and the ***upper topology*** is the smallest topology for which the principal ideals are closed.

6 *A topology on a preordered set is compatible if and only if it is smaller than the Alexandroff topology and larger than the upper topology.*

Since the Scott topology is between the two, one has the following immediate consequence.

7 *The Scott topology is compatible.*

8 Definition For a compatible topology T on a complete lattice L, define a binary relation $\prec_T$ on L by

$$x \prec_T y \text{ iff there is } U \in T \text{ with } y \in U \text{ and } x \leq z \text{ for all } z \in U.$$

That is, $x \prec_T y$ if and only if y belongs to the T-interior of $\uparrow x$.

9 Lemma *For any compatible topology T on a complete lattice, $\prec_T$ is an auxiliary relation.*

An element $x \in L$ is ***T-approximated*** if it is the join of the elements $u \prec_T x$, and T is ***approximating*** if every element of L is T-approximated. That is, the topology T is approximating if and only if the induced auxiliary relation $\prec_T$ is approximating. Notice that $x \prec_T x$ if and only if $\uparrow x$ is open. Hence the relation $\prec_T$ is reflexive if and only if T is the Alexandroff topology, in which case $x \prec_T y$ if and only if $x \leq y$. Therefore the Alexandroff topology is the largest approximating topology, but in a rather uninteresting way.

Although the relations $x \prec_{\text{Scott}} y$ and $x \ll y$ don't coincide in arbitrary complete lattices, they do in continuous lattices. In more detail, we have the following.

10 *For elements x and y of a complete lattice,*

1. *$x \prec_{\text{Scott}} y$ implies $x \ll y$,*
2. *$x \ll y$ implies $x \prec_{\text{Scott}} y$ provided the Scott topology is approximating or the lattice is continuous.*

The following corollary is the original formulation of continuity by Dana Scott [2].

11 *A complete lattice is continuous if and only if its Scott topology is approximating.*

We now characterize the Scott topology among all approximating topologies, obtaining an alternative characterization of continuity for complete lattices.

12 Theorem *The Scott topology of a meet-continuous lattice is the intersection of the approximating topologies.*

Proof In order to see that any approximating topology T contains the Scott topology, let x be a member of a Scott open set U. By definition of approximation, there is a $u \prec_T x$ in U, which means that there is a T-open neighbourhood V of x with $u \leq v$ for all $v \in V$. Since U is Alexandroff open, V is contained in U. Therefore U, being a union of T-open sets, is T-open. Conversely, given an ideal I, we say that an Alexandroff open set U is I-open if either I intersects U or $\bigvee I \notin U$. Then the I-open sets form a topology, and, by construction, the intersection of such topologies is the Scott topology. To see that the I-topology is approximating, we consider two cases. (1) $x \leq \bigvee I$. For $i \in I$ we have that $i \prec_T i$. Since I is an ideal, $x \wedge i \in I$ and hence $x \wedge i \prec_T x$ for any $i \in I$. By meet-continuity, the join of such meets is x, and hence x is I-approximated. (2) $x \not\leq \bigvee I$. Then the principal ideal generated by x is I-open and hence $x \prec_T x$, which shows that x is T-approximated. □

The above proof relies on the principle of excluded middle—we don't know how to avoid it, and maybe it is unavoidable.

13 Corollary *A complete lattice is continuous if and only if it is meet-continuous and it has a smallest approximating topology.*

This easily generalizes to posets with binary meets and directed joins in which the former distribute over the latter—we omit the routine details, but notice that the notions of auxiliary relation and approximating auxiliary relation have to be slightly reformulated as least elements and binary joins are not available in the general case.

The assumption of meet-continuity in the above theorem and corollary cannot be removed, as the following example shows.

14 Example Consider two disjoint copies $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}' = \{0', 1', 2', \dots\}$ of the natural numbers under their natural order. To make this into a complete lattice, add bottom and top elements $\perp$ and $\top$. This is not meet-continuous, because $1 \wedge n' = \perp$ and $1 \wedge \top = \top$. If T is an approximating topology, then all open sets are upper sets. On the other hand, for all $x \neq \top$, the condition $x \prec_T x$ must hold, and thus $\uparrow x$ is open. Then $\{\top\} = \uparrow 1 \cap \uparrow 1'$ is open as well, and so all upper sets are open, so that T is the Alexandroff topology. Therefore the Scott topopology is not the intersection of the approximating topologies in this example.

We now consider an alternative version of approximation, first considered by Scott for a particular topology.

15 Lemma *A compatible topology T on a complete lattice is approximating if and only if every element x is the* ***limit inferior*** *of its T-neighbourhoods, in the sense that*

$$x = \bigvee\{\bigwedge U \mid x \in U \in T\}.$$

Proof Assume that T is approximating. It is enough to show that $x \leq \bigvee\{\bigwedge U \mid x \in U \in T\}$, because the other inequality always holds. To establish the inequality it is enough to show that $y \leq \bigvee\{\bigwedge U \mid x \in U \in T\}$ for all $y \prec_T x$, because x is T-approximated. By definition of $y \prec_T x$, there is a $U \in T$ with $x \in U$ and $y \leq u$ for $u \in U$. Hence $y \leq \bigwedge U$, and we are done. Conversely, suppose that the equality holds. It is clear that the join is directed, and that if $x \in U$ we have that $\bigwedge U \prec_T x$. Hence x is T-approximated. $\square$

This gives rise to a characterization of continuity for complete lattices that doesn't refer to the way-below relation or the Scott topology.

16 Theorem *A complete lattice is continuous if and only if it is meet-continuous and has a smallest compatible topology for which every element is the limit inferior of its neighbourhoods.*

Concluding remark We have seen that, in a meet-continuous lattice, the way-below relation is the intersection of the approximating auxiliary relations, and the Scott topology is the intersection of the approximating topologies. How do these two facts relate? We haven't looked at this question yet.

References

[1] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices.* Springer-Verlag, 1980.

[2] D.S. Scott. Continuous lattices. In F.W. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136, 1972.