

Taking "algebraically" seriously  
in the definition of  
algebraically inductive type

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# Abstract

**Theorem.** In a 1-topos, the following two categories are isomorphic, with an isomorphism that is the identity on objects:

1. Pullback-natural, associative, algebraically injective objects.
2. Algebras of the partial-map classifier (aka lifting) monad.

- Partial results towards the  $\infty$ -topos situation.
- We work in HoTT/UF.

# I. Algebraic injectives (MScs'2021)

Def. Algebraic injective structure on a type  $D$  consists of

1. An extension operation, for any types  $X$  and  $Y$ ,

$$(-) \mid (-) : (X \rightarrow D) \times (X \hookrightarrow Y) \rightarrow (Y \rightarrow D).$$

fibers are propositions.

2. For each map  $f : X \rightarrow D$  and embedding  $j : X \hookrightarrow Y$ ,

a choice of an identification  $(f \mid j) \circ j = f$ , as illustrated by

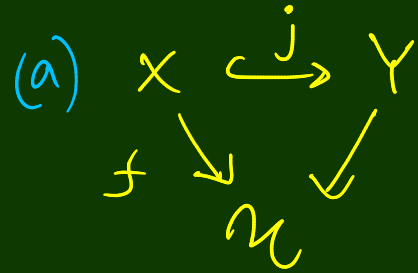
$$\begin{array}{ccc} X & \xhookrightarrow{j} & Y \\ f \searrow & & \swarrow f \mid j \\ & D & \end{array}$$

## Some examples

MSCS'2021

(They need univalence.)

1.  $D := \mathcal{U}$



$$(f|j)(y) := \prod_{(x, -) : \text{fiber } j \ y} f(x) \cdot$$

(Right Kan extension.)

(b)

Use  $\sum$  instead.

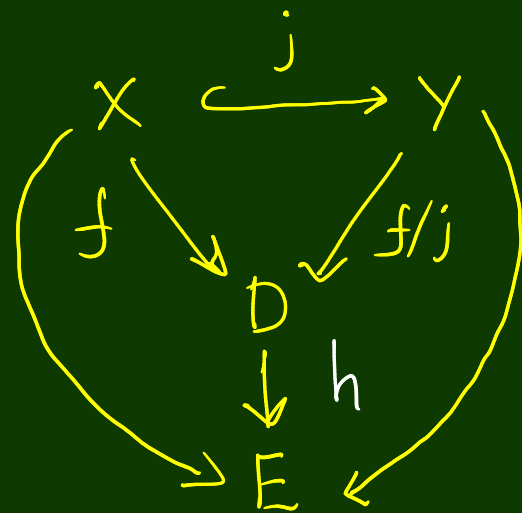
(Left Kan extension.)

2. The type  $\Omega$  of propositions. Use  $\forall$  or  $\exists$ .

3. Universes of  $n$ -types.

4. Algebras of the lifting monad. We'll come back to this.

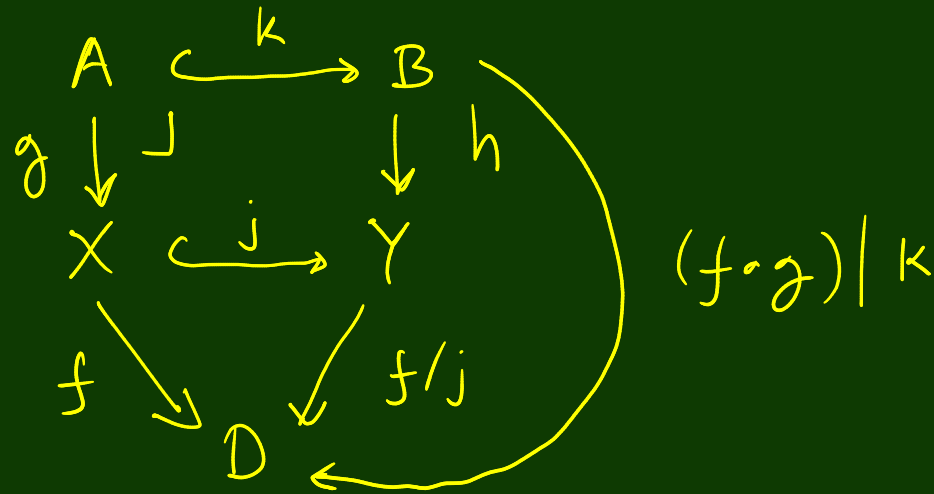
# Homomorphisms of algebraic injectives



$$h \circ (f|j) = (h \circ f) | j$$

# Pullback naturality

Previous examples are all pullback natural.



$$(f|j) \circ h = (f \circ g)|k$$

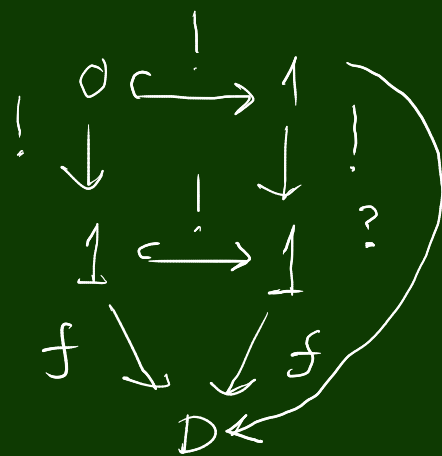
It is essential that the square is a pullback.

Consider the non-pullback square

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\quad} & 1 \end{array}$$

for a counter-example.

The counter-example in detail



$$x := 1$$

$$(f \circ !)|!(*) = \bigsqcup_{x:=0} (\dots) = !1$$

To get a counter-example, just choose  $f x = (\perp \rightarrow \emptyset)$

So this counter-example works for the injective induced by any algebraic fixity structure.



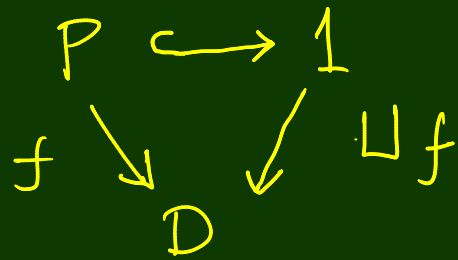
# Associativity

$$\begin{array}{ccccc}
 X & \xhookrightarrow{j} & Y & \xhookrightarrow{k} & Z \\
 & \searrow f & \downarrow f|_j & \swarrow & \\
 & & D & & 
 \end{array}
 \quad f|(k \circ j) = (f|_j)|_k$$

Examples (Mscs' 2024)  $D := \mathcal{U}$  with extension given by  $\Pi$  or  $\Sigma$ .

## II. Algebraic floppy structure

MSCS'2021



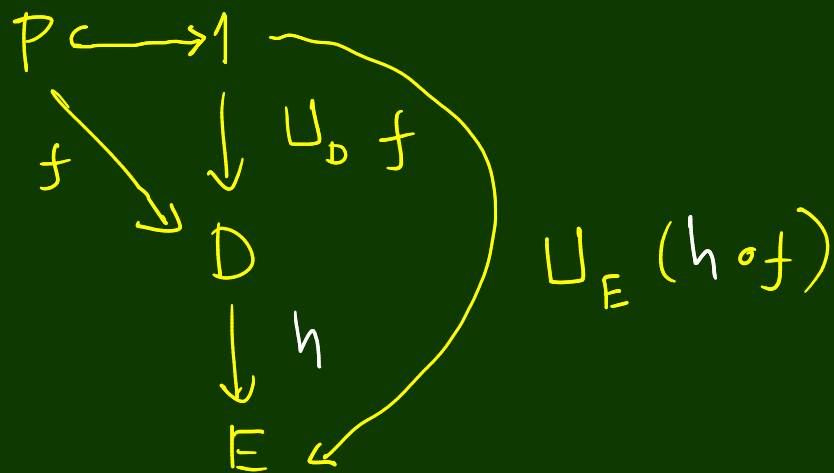
Every partial element  
of  $D$  can be extended  
to a total element

N.B. T.F.A.E.

1. The map  $P \rightarrow 1$  is an embedding.
2. The type  $P$  is a proposition.

Trivial fact. Algebraic inductive structure is, in particular,  
an algebraic floppy structure.

# Homomorphisms



$$h \circ U_D f = U_E(h \circ f)$$

# construction | Algebraic fibby $\rightarrow$ algebraic injective | MSCS'2021

$$\begin{array}{ccc}
 X & \xrightarrow{j} & Y \\
 f \searrow & & \swarrow \\
 & D & 
 \end{array}
 \quad (f|j)(\gamma) := \bigsqcup_{(x,-) \in \underbrace{\text{fiber } j}_Y} f x$$

Fact (new observation)

This algebraic injective structure is pullback natural.  
(Not only fiber-natural)

$$\begin{array}{ccc}
 P_Y & \longrightarrow & 1 \\
 p_{r_1} \downarrow & & \downarrow \bigsqcup (f \circ p_{r_1}) \\
 X & \xrightarrow{f} & D
 \end{array}$$

Fiberwise extension.

$$\begin{array}{ccc}
 P_Y & \xrightarrow{c} & 1 \\
 p_{r_1} \downarrow & \lrcorner & \downarrow y \\
 X & \xrightarrow{j} & Y \\
 f \searrow & & \swarrow f|j \\
 & D & 
 \end{array}
 \quad \bigsqcup (f \circ p_{r_1})$$

Get this by flabbiness.

### III. The lifting monad

M.H.E & C. Kuopp CSL'2017 &  
TypeTopology 2018

$$\mathcal{L}X := \sum_{P:\Omega} (P \xrightarrow{\varphi} X)$$

The type of partial elements of  $X$ .

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ X & \xrightarrow{f} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ \mathcal{L}X & \xrightarrow{\mathcal{L}f} & \mathcal{L}Y \end{array}$$

$$(\underline{1}, \lambda \cdot x) \quad (P, \varphi) \mapsto (P, f \circ \varphi)$$

$$\text{is-def(ined)} : \mathcal{L}X \rightarrow \Omega := p_{c1}$$

$$\text{value} : (\ell : \mathcal{L}X) \rightarrow \text{is-def } \ell \rightarrow X := p_{c2}$$

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathcal{L}Y \\ \mathcal{L}X & \xrightarrow{g^\#} & \mathcal{L}Y \end{array}$$

$$(P, \varphi) \mapsto$$

is a prop,  
as required.

$$\left( \sum_{P:\Omega} \text{is-def}(g(\varphi P)) \right) \lambda(p, d) . \text{value}(g(\varphi P)) d$$

# Monad algebras

## 1. Structure map

$$\sqcup : \mathcal{Z}A \rightarrow A$$

Extend a partial element to a total element!  
So  $\mathcal{Z}$ -algebras give algebraic fluffy structure.  
(MSCS'2021)

## 2. Unit law

$$\sqcup (\lambda(-:1). a) = a \text{ for every } a:A.$$

Extension "property", as for fluffy types.  
(extension data !)

## 3. Associativity law

$$\sqcup_{p:P} \sqcup_{q:Q_p} f(p,q) = \sqcup_{r: \sum_{p:P} Q_p} f r$$

$$P : \Omega$$

$$Q : P \rightarrow \Omega$$

$$f : \sum_{p:P} Q_p \rightarrow A$$

Previously unaccounted for  
when discussing injectivity

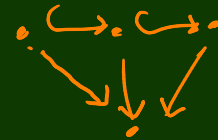
#### IV. Putting I-III together

So  $\mathcal{Z}$ -algebra structure = associative algebraic flabby structure.

Lemma Let  $\sqcup$  be the algebraic flabby structure induced by  
a given algebraic injective structure  $\sqcap$  that is pullback natural.

Then  $\sqcup$  is associative iff  $\sqcap$  is associative

$$\sqcup \sqcup = \sqcup$$



Ongoing work. Replace "iff" by " $\simeq$ ".  
(For sets we have this.)

Lemma Let  $|$  be the algebraic injective structure induced by a given algebraic fibbing structure  $\sqcup$ .  
Then  $|$  is always pullback natural. (We've already discussed this.)

Lemma The round trip  $\sqcup \mapsto | \mapsto \sqcup'$  is always the identity on both extension operators and extension data.

Lemma The round trip  $| \mapsto \sqcup \mapsto |'$  is the identity on extension operators iff  $|$  is pullback natural.

But what about extension data?  
Ongoing.



Theorem. Let  $D$  be any type.

1. Then

$$\begin{aligned} & \text{pullback-natural, associative injective structure on } D \\ & \iff \text{associative algebraic flabby structure on } D \\ & = \text{ } \mathcal{L}\text{-algebra structure on } D. \end{aligned}$$

2. If  $D$  is a set, then " $\iff$ " in (1) becomes an equivalence " $\simeq$ ".

- What is missing to always have 3 type equivalence?
  - check that the pullback-naturality data is unchanged by round trips.
  - check that the associativity data is unchanged by round trips.

(ongoing work, perhaps not difficult.)

- But there is still something else missing.

V. Is  $\mathcal{L}$  really a monad?

Nobody knows what a monad on types is in HoTT/UF.

People do know what monads on  $\infty$ -toposes are, though.

But we don't know how to say that in the language of HoTT/UF.

The problem is how to specify coherence data for the monad laws.