INDUCTION AND RECURSION ON THE REAL LINE

MARTÍN HÖTZEL ESCARDÓ

Department of Computing, Imperial College
180 Queen's Gate, London SW7 2BZ, United Kingdom
E-mail: mhe@doc.ic.ac.uk

ABSTRACT

We characterize the real line by properties similar to the so-called *Peano axioms* for natural numbers. These properties include an induction principle and a corresponding recursion scheme. The recursion scheme allows us to define functions such as addition, multiplication, exponential, logarithm, sine, arc sine, etc. from simpler ones. In order to obtain such a characterization, we introduce a notion of infinitely iterated composition of morphisms in categories, and we state a fixed point theorem and an infinite composition theorem for uniform spaces.

1 Introduction

We characterize the real line by properties similar to the so-called *Peano axioms* for natural numbers [10, 11, 19]. These properties include an induction principle and a corresponding recursion scheme. The recursion scheme allows us to define functions such as addition, multiplication, exponential, logarithm, sine, arc sine, etc. from simpler ones.

1.1 Programme

We begin by characterizing the unit interval $\mathbb{I} = [0, 1]$. For that purpose we introduce "successor functions" $S_0, S_1 : \mathbb{I} \to \mathbb{I}$ in such a way that every point of the unit interval arises as an "infinitely iterated composition" of the successor functions. If Y is a space for which infinite compositions of functions $t_0, t_1 : Y \to Y$ exist and generate a subspace of Y, we can define a function $h : \mathbb{I} \to Y$ by the rule

> *In order to compute $h(x)$, replace S_0 by t_0, and S_1 by t_1 in the construction of x, obtaining a construction of a point $h(x) \in Y$.*

This function satisfies the recursive equations

$$\begin{aligned} h(S_0(x)) &= t_0(h(x)) \\ h(S_1(x)) &= t_1(h(x)). \end{aligned}$$

The notion of infinitely iterated composition is defined for sequences of morphisms $\{f_n : X_{n+1} \to X_n | i \in \omega\}$ in any category. When the infinite composition exists, it is a morphism $X_\infty \to X_0$, denoted by $\bigodot_{n=0}^{\infty} f_n$. In the cases we consider, X_∞ is usually a terminal object, so that $\bigodot_{n=0}^{\infty} f_n$ is a global element of X_0. The main property enjoyed by infinite composition is

$$\bigodot_{n=i}^{\infty} f_n \;=\; f_i \circ \bigodot_{n=i+1}^{\infty} f_n. \tag{1}$$

The infinite composition of a constant sequence of morphisms f behaves as a "fixed morphism" of f. Indeed, denoting $\bigodot_{n=0}^{\infty} f$ by f^∞, we have that

$$f^\infty \;=\; f \circ f^\infty,$$

by virtue of equation 1. Therefore, the domains and codomains of composable sequences of morphisms of concrete categories must be mathematical structures with "good fixed point properties".

An appealing possibility is to use metric spaces as objects and contractions as morphisms. But in order to define metric spaces we need real numbers, and the real number system is what we are going to axiomatize.

We use uniform spaces to avoid this circularity [3]. We generalize the notion of contraction to uniform spaces and state appropriate fixed point and infinite composition theorems for contractions and sequences of contractions in uniform spaces. *A posteriori*, the uniform spaces we consider turn out to be induced by metric spaces. *Therefore, the reader who is not acquainted with uniform spaces but has some experience with metric spaces might prefer to read "metric space" whenever it says "uniform space", since the language used to talk about uniform spaces is essentially the same as the one used to talk about metric spaces.*

In order to characterize the real line, we consider the successor functions acting on the whole real line and postulate the existence of "predecessor functions" P_0 and P_1, which are the inverses of S_0 and S_1. Every real number arises as a finite application of the predecessor functions to a point of the unit interval.

The real line is both "inwards" and "outwards" infinite, in the sense that we can find points arbitrarily near to each other and points arbitrarily far from each other. Its inwards infinite character is accounted for by the successor functions, and its outwards infinite character is accounted for by the predecessor functions. Its completeness property is accounted for by the formation of infinite compositions.

There is a connection between the algebraic structure of real numbers given by the successor functions and their usual algebraic structure given by addition. To begin with, it is possible recursively define addition from the successor functions. Conversely, the successor functions can be expressed in terms of addition.

Moreover, there is a correspondence between recursive definitions and topological group homomorphisms, which shows that the presentations of the real line as a structure $(\mathbb{R}, S_0, S_1)$ and as a structure $(\mathbb{R}, +, 0)$ are essentially the same.

1.2 Contents

This work is organized as follows. Section 2 introduces induction and recursion for natural and integer numbers, and section 3 relates definition by recursion on integer numbers to group homomorphisms. Section 4 states a fixed point theorem for uniform spaces after a quick introduction to them. Section 5 defines the notion of infinite composition of morphisms in categories and states an existence for theorem sequences of contractions on a complete uniform space. Section 6 gives the axiomatic presentation of the real line sketched above, and section 7 relates definition by recursion on real numbers to topological group homomorphisms. Section 8 shows how to derive iterative programs from recursive definitions. Section 9 contains some concluding remarks and discusses some possible future work.

The purpose of sections 2 and 3 is to motivate sections 6 and 7 respectively, which are the main sections of this work. The definitions and propositions contained in sections 2 and 3 are analogous to the definitions and propositions contained in sections 6 and 7 respectively, and they occur in the same order. In this analogy, the unit interval corresponds to the natural numbers, and the real line corresponds to the integer numbers. Sections 4 and 5 lay down the technical foundations needed for the main sections. As far as the author knows, the material contained in them, including the definition of infinitely iterated composition of morphisms in general categories, is original. Most of the contents of sections 2 and 3 is standard, but the author has never seen the extension of recursion to integer numbers, and the correspondence between recursive definitions on integer numbers and group homomorphisms presented in these sections. This is a consequence of a correspondence between recursive definitions on natural numbers and monoid homomorphisms, which is omitted from this paper due to lack of space. But the correspondence should be obvious after reading section 3.

2 Induction and recursion on integer numbers

2.1 Unary systems

A ***unary system*** consists of a set X together with a function $s : X \to X$ and an element $a \in X$ [19]. The function s is said to be the ***successor function*** of the unary system, and the element a is said to be its ***zero element***. An element in the image of s is said to be a ***successor***.

A ***homomorphism*** from a unary system (X, s, a) to a unary system (Y, t, b) is a function $h : X \to Y$ such that $h(a) = b$ and $h \circ s = t \circ h$. Unary systems and homomorphisms form a category under ordinary function composition.

Examples 2.1

1. Let $\mathbb{N}$ be the monoid of natural numbers under addition and define $S(x) = x+1$. Then $(\mathbb{N}, S, 0)$ is a unary system, which we call the ***natural unary system***.

2. There is a single-element unary system, which is of course unique up to isomorphism. We call such a unary system a ***trivial unary system***.

■

Recall that a subset of a set X is closed under a function $f : X \to X$ if it contains $f(x)$ whenever it contains x.

Proposition 2.1 *A unary system X is initial iff it enjoys the following properties:*

1. *its successor function is injective;*

2. *every element is a zero or a successor;*

3. *no element is both a zero and a successor;*

4. *(induction) if A is subset of X which contains the zero element and is closed under the successor function, then $A = X$.*

Corollary 2.2 *The natural unary system is initial.*

Therefore, the properties given in proposition 2.1 uniquely characterize the natural unary system up to isomorphism, since any two initial objects are isomorphic.

If (Y, t, b) is a unary system, the equations

$$\begin{aligned} h(0) &= b \\ h(S(x)) &= t(h(x)) \end{aligned}$$

are said to be a ***simple recursive definition*** of the unique homomorphism $h : \mathbb{N} \to Y$. Note that $n = S^n(0)$, so that

$$h(S^n(0)) = t^n(b).$$

Therefore, in order to compute $h(n)$ it suffices to replace S by t and 0 by b in the construction of n, obtaining a construction of an element $h(n) \in Y$.

2.2 Unary systems with predecessor

A ***unary system with predecessor*** is a unary system whose successor function is a bijection. Its inverse is said to be its ***predecessor function***.

Example 2.2 Let $\mathbb{Z}$ be the group of integer numbers and define $S(x) = x + 1$. Then $(\mathbb{Z}, S, 0)$ is a unary system with predecessor, which we call the ***integer unary system***. Its predecessor function is denoted by P. ■

Proposition 2.3 *A unary system X is initial in the category of unary systems with predecessor iff it enjoys the following properties:*

1. *the zero element is distinct from its successor;*
2. *(induction) if A is a non-empty subset of X closed under the successor and predecessor functions, then $A = X$.*

Corollary 2.4 *The integer unary system is initial in the category of unary systems with predecessor.*

Therefore, the properties given in proposition 2.3 uniquely characterize the integer unary system up to isomorphism.

If (Y, t, b) is a unary system with predecessor, the equations

$$\begin{aligned} h(0) &= b \\ h(S(x)) &= t(h(x)) \end{aligned}$$

are said to be a ***simple recursive definition*** of the unique homomorphism $h : \mathbb{Z} \to Y$. Note that the equation

$$h(P(x)) = q(h(x),$$

where q is the predecessor function of Y, is implied by the above equations. If $f : A \to A$ is a bijection and n is a negative integer number, f^n is defined to be $(f^{-1})^{-n}$, so that $n = S^n(0)$ for all integer numbers n. Therefore

$$h(S^n(0)) = t^n(b).$$

2.3 Derived predecessorless unary systems

Given a unary system (X, s, a) with predecessor, the unary system $(\bar{X}, \bar{s}, a)$ such that $\bar{X}$ is the least set which contains a and is closed for the successor function, and such that $\bar{s}$ is the restriction of s to $\bar{X} \to \bar{X}$, is said to be its ***derived predecessorless unary system***.

Example 2.3 The predecessorless unary system derived from the integer unary system is the natural unary system. ■

Proposition 2.5 *If a unary system with predecessor is initial, so is its derived predecessorless unary system. Moreover, if Y and Y' are predecessorless unary systems derived from initial unary systems X and X', then the unique homomorphism from X to X' restricts to a homomorphism from Y to Y'.*

Proposition 2.6 *Given an initial unary system Y, there is an essentially unique initial unary system X with predecessor such that Y is derived from X. Moreover, if Y' is a unary system derived from a unary system X' with predecessor, then the unique homomorphism from Y to Y' can be extended to a homomorphism from X to X' in a unique way.*

2.4 Definition by cases for unary systems

Proposition 2.7 *Let (X, s, a) be an initial unary system, Y be a set, b be an element of Y, and $g : X \to Y$ be a function. Then there is a unique function $h : X \to Y$ such that*

$$\begin{aligned} h(a) &= b \\ h(s(x)) &= g(x). \end{aligned}$$

Equivalently, the following diagram is a co-product diagram in the category of sets and functions:

$$X \xrightarrow[s]{} X \xleftarrow[a]{} 1$$

The above equations are said to be a ***definition by cases*** of h.

3 Recursive definitions on integer numbers and group homomorphisms

3.1 Monadic groups

A ***monadic group*** is a group $(G, +, 0)$ together with an element $1 \in G$, called its ***unit***. For any monadic group $(G, +, 0, 1)$, the function $s(x) = x + 1$ is said to be its ***successor function***.

A ***homomorphism of monadic groups*** is a homomorphism of groups which preserves the unit. It is easy to check that monadic group homomorphisms preserve the successor function.

Proposition 3.1 *If $(G, +, 0, 1)$ is a monadic group with successor function s, then $(G, s, 0)$ is a unary system with predecessor.*

Proposition 3.2 *Let $(G, +, 0, 1)$ be a monadic group and s be its successor function Then $(G, s, 0)$ is an initial unary system with predecessor iff $0 \neq 1$, and the least set which contains 0 and is closed under the successor and predecessor functions is G.*

A unary system G with predecessor obtained by the above construction is said to be ***derived from the monadic group*** G.

Proposition 3.3 *Let $(G, s, 0)$ be an initial unary system with predecessor derived from a monadic group G, and $(H, t, 0)$ be a unary system with predecessor derived from a monadic group H. Then a function $G \to H$ is a homomorphism of monadic groups iff it is a homomorphism of unary systems.*

By the ***monadic additive integers*** we mean the group of integer numbers under addition with unit 1.

Corollary 3.4 *The monadic additive integers are initial in the category of monadic groups.*

3.2 Recursive definition of the group structure

Proposition 3.5 *Let (X, s, a) be an initial unary system with predecessor, Y and Z be sets, and $c : Y \to Z$ and $u : X \times Y \times Z \to Z$ be functions such that $(Z, z \mapsto u(x, y, z), c(y))$ is a unary system with predecessor for all $x \in X$ and $y \in Y$. Then there is a unique function $h : X \times Y \to Z$ such that*

$$\begin{aligned} h(a, y) &= c(y) \\ h(s(x), y) &= u(x, y, h(x, y)) \end{aligned}$$

The above equations are said to be a ***primitive recursive definition*** of h.

Proposition 3.6 *Let $(G, s, 0)$ be an initial unary system with predecessor derived from a monadic group $(G, +, 0, 1)$, and let p be its predecessor function. Then the equations*

$$\begin{aligned} 0 + y &= y \\ s(x) + y &= s(x + y) \end{aligned}$$

are a primitive recursive definition of addition, and the equations

$$\begin{aligned} -0 &= 0 \\ -s(x) &= p(-x) \end{aligned}$$

are a simple recursive definition of the opposite function.

4 A fixed point theorem for uniform spaces

4.1 Uniform spaces

Recall that a filter of sets is a collection $\mathcal{F}$ of subsets of a set X such that $B \supseteq A \in \mathcal{F}$ implies $B \in \mathcal{F}$, and $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$; and recall that a filter of (binary) relations on a set X is a filter of subsets of $X \times X$.

A ***uniform space*** is a set X together with a filter $\mathcal{U}$ of reflexive relations on X such that $U^{-1} \in \mathcal{U}$ for every $U \in \mathcal{U}$, and such that there is a $V \in \mathcal{U}$ with $V \circ V \subseteq U$ for every $U \in \mathcal{U}$ [3]. It is ***separating*** if $\bigcap \mathcal{U} = \{(x, x) | x \in X\}$. The set $\mathcal{U}$ is said to be a ***uniformity*** on X. The members of $\mathcal{U}$ are said to be the ***entourages*** of X. Two points x and y are said to be U***-close*** if xUy.

A ***base for a uniform space*** X is a family of entourages of X such that every entourage of X contains a member of the family.

A ***metric*** on a set X ***induces a uniformity*** on X, specified by the base $\{U_r | r \in \mathbb{R}^+\}$, where xU_ry if $d(x, y) < r$. Indeed, the notion of uniformity is an abstraction of the concept of metric. For a uniformity $\mathcal{U}$ derived from a metric d, the fact that every $U \in \mathcal{U}$ is reflexive corresponds to the fact that $d(x, x) = 0$, the fact that $U^{-1} \in \mathcal{U}$ for every $U \in \mathcal{U}$ corresponds to the fact that $d(x, y) = d(y, x)$, and the fact that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$ corresponds to the triangular inequality $d(x, y) + d(y, z) \geq d(x, z)$. Such a uniformity $\mathcal{U}$ is always separating, due to the fact that $d(x, y) = 0$ implies $x = y$.

A function f between uniform spaces is ***uniformly continuous*** if for every entourage V of the codomain of f there is an entourage U of its domain such that xUy implies $f(x)Vf(y)$. Uniform spaces and uniformly continuous functions form a category under ordinary function composition.

A ***uniformity*** on a set X ***induces a topology*** on X, whose open sets are the subsets O of X such that for every $x \in O$ there exists an entourage U

of X for which $U(x) \subseteq X$. Here $U(x)$ denotes the set $\{y|xUy\}$. A uniformly continuous function is continuous for the induced topologies, but the converse is not true in general.

If a metric d induces a uniformity $\mathcal{U}$, and the uniformity $\mathcal{U}$ induces a topology τ, then the topology τ is the one induced by the metric d, usually specified by a base of open spheres.

From our point of view, the main advantage that uniform spaces have over metric spaces is that they give us a notion of closeness independent of the concept of real number.

4.2 Completeness

Recall that a ***net*** is a set indexed by a directed set. A net $\{x_n|n \in D\}$ in a uniform space X ***converges to a point*** y if for every entourage U of X there is an n in D such that $x_i U y$ for all $i \geq n$. Such a point y is said to be a ***limit of the net***. If the space is separating, every net has at most one limit. A ***Cauchy net*** on X is a net $\{x_n|n \in D\}$ such that for every entourage U of X there is an n in D such that $x_i U x_j$ for all $i, j \geq n$. Every convergent net is a Cauchy net. We say that a uniform space is ***complete*** if every Cauchy net x_n has a limit. If the space is separating the limit is unique and is denoted by $\lim_n x_n$. If x_n is a Cauchy net and f is uniformly continuous, then $f(x_n)$ is also a Cauchy net.

4.3 Contractions and fixed points

Recall that a non-expansive map on a metric space X is a function $f : X \to X$ such that $d(f(x), f(y)) \leq d(x, y)$, and that a contraction is a function $f : X \to X$ for which there is a positive real number $r < 1$ such that $d(f(x), f(y)) < rd(x, y)$. Such an r is said to be a contraction factor for f. Every contraction is non-expansive, and non-expansive maps are uniformly continuous. We generalize the notion of contraction to uniform spaces as follows, based on the observation that if f is a contraction then $d(f^i(x), f^j(y)) < r^n d(x, y)$ for all $i, j \geq n$.

Let X be a uniform space. A function $f : X \to X$ is ***non-expansive*** if for every entourage V there is an entourage $U \subseteq V$ such that xUy implies $f(x)Uf(y)$. It is immediate that every non-expansive map is uniformly continuous. A ***contraction*** on X is a non-expansive map $f : X \to X$ such that for every entourage U and all points x and y there is a natural number n with the property that $f^i(x)Uf^j(y)$ for all $i, j \geq n$. A map which associates such natural numbers to entourages and pairs of points is said to be a ***contraction modulus*** for f. It is easy to check that a contraction in a metric space is also a contraction in the induced uniform space.

Proposition 4.1 *Every contraction on a complete separating uniform space has a unique fixed point.*

Proof. (Existence) Let f be such a contraction and x_0 be any point of the space. Then $x_n = f^n(x_0)$ is a Cauchy sequence. In fact, if U is an entourage, there exists an n such that $f^i(x_0)Uf^j(x_0)$ for all $i, j \geq n$. This means that $x_i U x_j$ for all $i, j \geq n$. Now, $f(\lim_n x_n) = \lim_n f(x_n) = \lim_n x_{n+1} = lim_n x_n$, since f is uniformly continuous and the space is complete. Therefore $\lim_n x_n$ is a fixed point of f. (Unicity) Let a and b be fixed points of a contraction f. As for every entourage U and all x and y there is an n such that $f^n(x)Uf^n(y)$, by taking $x = a$ and $y = b$ we conclude that aUb for every entourage U. Therefore $a = b$, since the space is separating. ■

We denote the unique fixed point of a contraction f on a complete separating uniform space by f^∞.

4.4 A note on semi-uniform spaces

We make no use of the fact that for every entourage U of a uniform space there is an entourage V such that $V \circ V \subseteq U$. Spaces which do not necessarily satisfy this property are said to be ***semi-uniform spaces***. We make no use of the fact that U^{-1} is an entourage for every entourage U either. Spaces which do not necessarily satisfy this property are said to be ***quasi-uniform spaces*** [17]. Also, spaces which do not necessarily satisfy some of the properties referred above are said to be ***quasi-semi-uniform spaces***. Thus, a quasi-semi-uniform space is just a set together with a filter of reflexive relations on it. But it is not clear what are the appropriate notions of Cauchy net, limit of a net, completeness, etc. for such spaces, and our fixed point theorem depend on them. M. B. Smyth is currently developing such notions for semi-uniform spaces.

It seems relevant to work in the more general category of semi-uniform spaces. Smyth was able to obtain the unit interval as an inverse limit of finite semi-uniform spaces [18] (see also [16]), and this inverse limit construction appears to be strongly related to the principles of induction and recursion on the unit interval stated in section 6.

5 An infinite composition theorem for uniform spaces

5.1 Infinite composition of morphisms in categories

Consider any category, and let $\{f_n | n \in \omega\}$ be a sequence of morphisms such that $\mathrm{dom}(f_n) = \mathrm{cod}(f_{n+1})$. If we put $X_n = \mathrm{cod}(f_n)$, this means that $f_n : X_{n+1} \to X_n$. An ***infinite composition*** of the sequence f_n is a morphism $f_0^\infty : X_\infty \to X_0$

for which there are morphisms $f_i^\infty : X_\infty \to X_i$, $i > 0$, such that the diagram

$$\begin{tikzcd}
 & & & & X_\infty \arrow[dlll, "f_{i+1}^\infty"'] \arrow[dll, "f_i^\infty"'] \arrow[d, "f_2^\infty"] \arrow[dr, "f_1^\infty"] \arrow[drr, "f_0^\infty"] & & \\
\ldots & X_{i+1} \arrow[r, "f_i"] & X_i & \ldots & X_2 \arrow[r, "f_1"] & X_1 \arrow[r, "f_0"] & X_0
\end{tikzcd}$$

is a limit of the diagram

$$\begin{tikzcd}
\ldots & X_{i+1} \arrow[r, "f_i"] & X_i & \ldots & X_2 \arrow[r, "f_1"] & X_1 \arrow[r, "f_0"] & X_0
\end{tikzcd}$$

In this case f_i^∞ is an infinite composition of the sequence $\{f_{n+i} | n \in \omega\}$. If we write $\bigodot_{n=i}^\infty f_n = f_i^\infty$, we have that

$$\bigodot_{n=i}^{\infty} f_n = f_i \circ \bigodot_{n=i+1}^{\infty} f_n = f_i \circ f_{i+1} \circ \cdots \circ f_{i+j} \circ \bigodot_{n=i+j+1}^{\infty} f_n \tag{2}$$

$$\bigodot_{n=i+1}^{\infty} f_n = \bigodot_{n=i}^{\infty} f_{n+1} \tag{3}$$

5.2 Infinite composition of contractions in uniform spaces

Proposition 5.1 *Let X be a complete separating uniform space and let $\{f_n : X \to X | n \in \omega\}$ be a sequence of contractions with a bound for their contraction moduli. Then f_n has an infinite composition, which is a global element of X (in the category of uniform spaces and uniformly continuous functions).*

We identify this global element of X and the corresponding point of X. In most cases we consider, the underlying set of the sequence of contractions f_i is finite. Under this circumstance, there is always a bound for the contraction moduli.

If f_n is a constant sequence of contractions f, then its infinite composition is essentially the unique fixed point of f, that is, $\bigodot_{i=0}^\infty f = f^\infty$. Therefore this is a generalization of the fixed point theorem.

6 Induction and recursion on the real line

6.1 Binary systems

See examples 6.1 and notes 6.2 below for a motivation of the following definition.

A ***binary system*** consists of a complete separating uniform space X together with contractions $s_0, s_1 : X \to X$ such that $s_0(s_1^\infty) = s_1(s_0^\infty)$. The contractions s_0 and s_1 are said to be the ***left*** and ***right successor functions*** of the binary system. The points s_0^∞ and s_1^∞ are said to be its ***end-points***, and the point $s_0(s_1^\infty) = s_1(s_0^\infty)$ is said to be its ***centre***. A point in the image of s_0 is said to be a ***left successor***, and a point in the image of s_1 is said to be a ***right successor***. A ***successor*** is a left successor or a right successor.

A ***homomorphism*** from a binary system (X, s_0, s_1) to a binary system (Y, t_0, t_1) is a uniformly continuous function $h : X \to Y$ such that $h \circ s_i = t_i \circ h$ for $i = 0, 1$. Binary systems and homomorphisms form a category under ordinary function composition.

Examples 6.1

1. Let $\mathbb{I} = [0, 1]$ be the unit interval, together with the uniformity induced by the metric $d(x, y) = |x - y|$. Now define $S_0(x) = x/2$ and $S_1(x) = (x + 1)/2$. Then $(\mathbb{I}, S_0, S_1)$ is a binary system, which we call the ***unit binary system***. Its end-points are 0 and 1, and its centre is 1/2. The left successors are the points in $[0, 1/2]$ and the right successors are the points in $[1/2, 1]$.

   A ***binary sequence*** is a sequence of elements of the set $\mathbf{2} = \{0, 1\}$. A ***binary expansion of a point*** $x \in \mathbb{I}$ is any binary sequence $\{a_i | i \in \omega\}$ such that $x = \sum_{i=0}^{\infty} a_i 2^{-(i+1)}$. It is easy to prove that a_i is a binary expansion of x iff $x = \bigodot_{i=0}^{\infty} S_{a_i}$, so that infinite compositions of the successor functions yield all points of the unit interval. The fact that $S_0(S_1^\infty) = S_1(S_0^\infty) = 1/2$ is then related to the fact that the binary sequences 01^ω and 10^ω are both binary expansions of 1/2.

2. There is a single-point binary system, which is of course unique up to isomorphism. We call such a binary system a ***trivial binary system***.

■

A subset of a complete separating uniform space X is said to be ***inductive*** if it contains the limits of Cauchy nets on it. Equivalently, it is inductive if it is topologically closed for the topology induced by the uniformity on X.

Proposition 6.1 *A binary system X is initial iff it enjoys the following properties:*

1. *its end-points are distinct;*
2. *its successor functions are injective;*
3. *every point is a successor;*
4. *the only point which is both a left successor and a right successor is its centre;*

5. *(induction) if A is a non-empty inductive subset of X closed under the successor functions, then $A = X$.*

Corollary 6.2 *The unit binary system is initial.*

Therefore, the properties given in proposition 6.1 uniquely characterize the unit binary system up to isomorphism.

If (Y, t_0, t_1) is a binary system, the equations

$$\begin{aligned} h(S_0(x)) &= t_0(h(x)) \\ h(S_1(x)) &= t_1(h(x)) \end{aligned}$$

are said to be a ***simple recursive definition*** of the unique homomorphism $h : \mathbb{I} \to Y$. This homomorphism is explicitly obtained by

$h(x) = \bigodot_{i=0}^{\infty} t_{a_i}$, where $\{a_i | i \in \omega\}$ is any binary expansion of x.

This is proved by induction on $\mathbb{I}$. It follows that

$h(\bigodot_{i=0}^{\infty} S_{a_i}) = \bigodot_{i=0}^{\infty} t_{a_i}$.

Notes 6.2

1. If no point were both a left successor and a right successor, then the complements of the images of s_0 and s_1 would form a disconnection of X. But the unit interval is a connected space. Hence the condition $s_0(s_1^{\infty}) = s_1(s_0^{\infty})$ is a way of enforcing connectedness. Therefore, it can be called a ***connectedness axiom***.

2. If no connectedness axiom had been imposed, the initial system would be Cantor space, which is totally disconnected.

3. This is not the only possible connectedness axiom. For instance, $s_0 \circ s_1 \circ s_1 = s_1 \circ s_0 \circ s_0$ is another possibility. This condition is related to golden ratio expansions of real numbers [7]. Hence we might speak of a ***golden system*** when this condition is imposed.

4. Yet another possibility is to work with three contractions s_{-1}, s_0 and s_{+1} subject to the equations $s_0 \circ s_{-1} = s_{-1} \circ s_1$ and $s_0 \circ s_1 = s_1 \circ s_{-1}$. This condition is related to signed digit binary expansions of real numbers (see [7] again). In this case we might speak of a ***signed binary system***. Homomorphisms are defined in the obvious way. We call binary systems as defined above ***unsigned binary systems*** to distinguish them from the signed ones.

5. It is not hard to prove that if (X, s_{-1}, s_0, s_{+1}) is a signed binary system, then (X, s_{-1}, s_{+1}) is an unsigned binary system, which is initial if the signed one is. Conversely, if (X, s_{-1}, s_{+1}) is an initial unsigned binary system, then the equations $s_0(s_{-1}(x)) = s_{-1}(s_1(x))$ and $s_0(s_1(x)) = s_1(s_{-1}(x))$ happen to be a *definition by cases* (see section 6.7) of a contraction s_0, so that they yield a signed binary system. Moreover, homomorphisms of unsigned binary systems preserve this contraction. Therefore, there is no essential difference between signed and unsigned binary systems.

6. There are other representations of real numbers which are not based on x-ary expansions for any x. For example, there is the ***Stern-Brocot representation*** of points of the compact space $[0, \infty]$ [8]. Surprisingly enough, this representation is also related to binary systems. The so-called *Stern-Brocot algorithm* uses two functions $r_0, r_1 : [0, \infty] \to [0, \infty]$ to generate all rational numbers, without repetition, from the number 1. These functions are defined by $r_0(x) = x/(x + 1)$ and $r_1(x) = x + 1$ with the convention that $r_0(\infty) = 1$ and $r_1(\infty) = \infty$. It turns out that $([0, \infty], r_0, r_1)$ is an initial binary system with centre 1, and left and right end-points 0 and ∞. The left successors are the points in $[0, 1]$ and the right successors are the points in $[1, \infty]$. It can be called the ***rational binary system***.

7. The main advantage of using binary systems instead of systems with other connectedness axioms is that binary systems are closely related to the ordinary algebraic structure of the real line (see section 7).

■

6.2 Binary systems with predecessors

A ***binary system with predecessors*** is a binary system whose successor functions are isomorphisms of uniform spaces. Its inverses are said to be its ***left*** and ***right predecessor functions***.

Example 6.3 Let $\mathbb{R}$ be the ***uniform real line***, that is, the set of real numbers together with the uniformity induced by the metric $d(x, y) = |x - y|$, and define $S_0(x) = x/2$, $S_1(x) = (x + 1)/2$. Then $(\mathbb{R}, S_0, S_1)$ is a binary system with predecessors, which we call the ***real binary system***. Its predecessor functions are denoted by P_0 and P_1. ■

Proposition 6.3 *A binary system X is initial in the category of binary systems with predecessors iff it enjoys the following properties:*

1. *its end-points are distinct;*

2. *(induction) if A is a non-empty inductive subset of X closed under the successor and predecessor functions, then $A = X$.*

Corollary 6.4 *The real binary system is initial in the category of binary systems with predecessors.*

Therefore, the properties given in proposition 6.3 uniquely characterize the real binary system up to isomorphism.

If (Y, t_0, t_1) is a binary system with predecessors, the equations

$$\begin{aligned} h(S_0(x)) &= t_0(h(x)) \\ h(S_1(x)) &= t_1(h(x)) \end{aligned}$$

are said to be a ***simple recursive definition*** of the unique homomorphism $h : \mathbb{R} \to Y$. Note that the equations

$$\begin{aligned} h(P_0(x)) &= q_0(h(x)) \\ h(P_1(x)) &= q_1(h(x)), \end{aligned}$$

where q_0, q_1 are the predecessors of Y, are implied by the above equations.

For the purposes of this work it is convenient to adopt the following definition. A ***binary expansion of a real number*** x consists of a natural number n, a number $k \in \mathbf{2}$, and a binary sequence a_i such that

$$x = k(1 - 2^n) + 2^n \sum_{i=0}^{\infty} a_i 2^{-(i+1)}.$$

It is not hard to show that (n, k, a_i) is a binary expansion of x iff

$$x = P_k^n \left(\bigodot_{i=0}^{\infty} S_{a_i} \right).$$

Then an explicit definition of h is given by

$h(x) = q_k^n \left(\bigodot_{i=0}^{\infty} t_{a_i}\right)$, where (n, k, a_i) is any binary expansion of x.

Therefore

$$h\left(P_k^n \left(\bigodot_{i=0}^{\infty} S_{a_i}\right)\right) = q_k^n \left(\bigodot_{i=0}^{\infty} t_{a_i}\right).$$

6.3 Derived predecessorless binary systems

Given a binary system (X, s_0, s_1) with predecessors, the binary system $(\bar{X}, \bar{s_0}, \bar{s_1})$ such that $\bar{X}$ is the closure of the least set which contains the centre and is closed for the successor functions, and such that $\bar{t_0}, \bar{t_1}$ are the restrictions of s_0, s_1 to $\bar{X} \to \bar{X}$, is said to be its ***derived predecessorless binary system***.

Example 6.4 The predecessorless binary system derived from the real binary system is the unit binary system. ■

Proposition 6.5 *If a binary system with predecessors is initial, so is its derived predecessorless binary system. Moreover, if Y and Y' are predecessorless binary systems derived from initial binary systems X and X', then the unique homomorphism from X to X' restricts to a homomorphism from Y to Y'.*

Proposition 6.6 *Given an initial binary system Y, there is an essentially unique initial binary system X with predecessors such that Y is derived from X. Moreover, if Y' is a binary system derived from a binary system X' with predecessors, then the unique homomorphism from Y to Y' can be extended to a homomorphism from X to X' in a unique way.*

6.4 Definition by cases for binary systems

Proposition 6.7 *Let (X, s_0, s_1) be an initial binary system, and Y be a topological space. If $g_0, g_1 : X \to Y$ are continuous functions such that $g_0(s_1^\infty) = g_1(s_0^\infty)$, then there is a unique continuous function $h : X \to Y$ such that*

$$\begin{aligned} h(s_0(x)) &= g_0(x) \\ h(s_1(x)) &= g_1(x). \end{aligned}$$

Equivalently, the following diagram is a pushout in the category of topological spaces and continuous functions:

$$\begin{tikzcd}
X & X \arrow[l, "s_0"'] \\
X \arrow[u, "s_1"] & 1 \arrow[u, "s_1^\infty"'] \arrow[l, "s_0^\infty"]
\end{tikzcd}$$

The above equations are said to be a ***definition by cases*** of h. Note that the implied diagonal arrow of the pushout diagram is the centre of the binary system. We omit a definition by cases scheme for the real line.

An explicit definition of h in the case $X = \mathbb{I}$ is given by

$$h(x) = \begin{cases} g_0(2x) & \text{if } x \leq 1/2 \\ g_1(2x-1) & \text{if } x \geq 1/2 \end{cases}$$

Equality of real numbers is not decidable in general [1, 2, 14]. Therefore, in order to compute h, we need to start the computations of $g_0(2x)$, $g(2x-1)$, and the comparison x versus $1/2$ simultaneously. If $x \neq 1/2$, as soon as we discover which of the cases $x < 1/2$ or $x > 1/2$ hold, we can abort the computation of $g_0(2x)$ or the computation of $g_1(2x-1)$. If $x = 1/2$, we have that $g_0(2x) = g_1(2x-1)$. Thus the definition by cases scheme resembles the so-called *parallel conditional* [13]. A similar phenomenon occurs for functions defined by simple recursion on the unit interval or the real line.

7 Recursive definitions on real numbers and topological group homomorphisms

7.1 Dyadic topological groups

A ***dyadic group*** is a monadic group $(G, +, 0, 1)$ such that G is Abelian and for every element y there is a unique x such that $y = 2x$, denoted by $y/2$. See section 3.1 for the definition of monadic group.

Let $(G, +, 0, 1)$ be any dyadic group. By definition, the function $p_0(x) = 2x$ is a bijection, which has $s_0(x) = x/2$ as its inverse. Moreover, it preserves the neutral element and addition, so that it is an isomorphism. It is immediate that these functions have 0 as a unique fixed point. Now define $p_1(x) = 2x - 1 = p_0(x) - 1$. Then p_1 is also a bijection, and it and its inverse $s_1(x) = (x+1)/2 = s_0(x+1)$ have 1 as a unique fixed point. Denote the unique fixed points of s_0 and s_1 by s_0^∞ and s_1^∞. Then $s_0(s_1^\infty) = s_0(1) = s_0(0+1) = s_1(0) = s_1(s_0^\infty)$.

We call s_0, s_1, p_0, p_1 the ***successor*** and ***predecessor functions*** of G. Note that

$$x + 1 = p_0(s_1(x)) \quad (4)$$

$$x - 1 = p_1(s_0(x)) \quad (5)$$

A ***homomorphism of dyadic groups*** is a homomorphism of monadic groups. It is easy to check that dyadic group homomorphisms preserve the successor and predecessor functions.

Recall that a topological group is a group $(G, +, 0)$ together with a topology on G such that the addition and opposite functions are continuous with respect to the topology; and that the topological group homomorphisms are the group homomorphisms which are continuous.

The ***induced uniformity*** on an Abelian topological group G is specified by the base $\{(x, y) | \text{there is a neighbourhood } O \text{ of } 0 \text{ such that } x - y \in O\}$ [3]. The induced uniformity induces the original topology. Every homomorphism of Abelian topological groups is uniformly continuous for the induced uniformity.

An Abelian topological group is said to be ***complete*** if it is complete as a uniform space under the induced uniformity.

Example 7.1 The group of real numbers under addition is dyadic with unit 1. It is topological for the usual topology of the real line. The induced uniformity on it is the uniformity induced by the metric $d(x, y) = x - y$. Therefore it is complete. ■

Lemma 7.1 *Let G be a dyadic topological group, and s_0, s_1, p_0, p_1 be its successor and predecessor functions. Then s_0 and s_1 are contractions and p_0 and p_1 are uniformly continuous for the induced uniformity.*

Corollary 7.2 *If G is T_0, then (G, s_0, s_1) is binary system with predecessors.*

Proposition 7.3 *Let $(G, +, 0, 1)$ be a dyadic T_0 topological group, and s_0, s_1 be its successor functions. Then (G, s_0, s_1) is an initial binary system with predecessors iff G is complete, $0 \neq 1$, and the least set which contains 0 and is closed under the successor and predecessor functions is dense in G.*

A binary system G with predecessors obtained by the above construction is said to be ***derived from the dyadic topological group*** G.

Proposition 7.4 *Let (G, s_0, s_1) be an initial binary system with predecessors derived from a dyadic topological group G, and (H, t_0, t_1) be a predecessorless binary with predecessors derived from a dyadic topological group H. Then a uniformly continuous function $G \to H$ is a homomorphism of dyadic topological groups iff it is a homomorphism of binary systems.*

By the ***dyadic additive real line*** we mean the topological group of real numbers under addition with unit 1.

Corollary 7.5 *The dyadic additive real line is initial in the category of complete dyadic topological groups.*

7.2 Exponential and logarithmic functions

Proposition 7.4 allows us to obtain recursive definitions of the exponential, logarithmic, and trigonometric functions [6]. Here we present the recursive definitions of the exponential and logarithm functions.

Examples 7.2

1. The exponential function $\exp(x) = e^x$ is the unique homomorphism from the topological group of real numbers under addition to the topological group of positive real numbers under multiplication such that $\exp(1) = e$ [4]. These groups are dyadic with units 1 and e respectively. The binary system with predecessors derived from the former is the real binary system, and the binary system with predecessors derived from the latter is $(\mathbb{R}^+, T_0, T_1)$ where $T_0(x) = \sqrt{x}$ and $T_1(x) = \sqrt{ex}$. Therefore $\exp : \mathbb{R} \to \mathbb{R}^+$ has the recursive definition

$$\begin{aligned}
\exp(S_0(x)) &= T_0(\exp(x)) \\
\exp(S_1(x)) &= T_1(\exp(x))
\end{aligned}$$

2. Similarly, the natural logarithm function $\ln : \mathbb{R}^+ \to \mathbb{R}$ has the following recursive definition:

$$\begin{array}{rcl} \ln(T_0(x)) & = & S_0(\ln(x)) \\ \ln(T_1(x)) & = & S_1(\ln(x)) \end{array}$$

In examples 8.1 we derive iterative programs from these recursive definitions. ■

7.3 Recursive definition of the group structure

Proposition 7.6 *Let (X, s_0, s_1) and (Y, t_0, t_1) be initial binary systems with predecessors, Z be a complete uniform space, and $u_{i,j} : Z \to Z$ for $i, j \in \mathbf{2}$ be contractions such that $(Z, u_{i,0}, u_{i,1})$ and $(Z, u_{0,j}, u_{1,j})$ are binary systems with predecessors for $i, j \in \mathbf{2}$, and such that $(Z, u_{i,j}, u_{1-i,1-j})$ are binary systems with predecessors for $i, j \in \mathbf{2}$, all with the same centre. Then there is a unique uniformly continuous function $h : X \times Y \to Z$ such that*

$$\begin{array}{rcl} h(s_0(x), t_0(x)) & = & u_{0,0}(h(x, y)) \\ h(s_0(x), t_1(x)) & = & u_{0,1}(h(x, y)) \\ h(s_1(x), t_0(x)) & = & u_{1,0}(h(x, y)) \\ h(s_1(x), t_1(x)) & = & u_{1,1}(h(x, y)) \end{array}$$

The above equations are said to be a ***double recursive definition*** of h.

Proposition 7.7 *Let (G, s_0, s_1) be an initial binary system with predecessors derived from a dyadic topological group $(G, +, 0, 1)$, and let p_0, p_1 be its predecessor functions. Then the equations*

$$\begin{array}{rcl} s_0(x) + s_0(y) & = & s_0(x + y) \\ s_0(x) + s_1(y) & = & s_1(x + y) \\ s_1(x) + s_0(y) & = & s_1(x + y) \\ s_1(x) + s_1(y) & = & s_2(x + y) \quad \textit{where } s_2 = s_1 \circ p_0 \circ s_1 \end{array}$$

are a double recursive definition of addition, and the equations

$$\begin{array}{rcl} -s_0(x) & = & s_0(-x) \\ -s_1(x) & = & s_{-1}(-x) \quad \textit{where } s_{-1} = s_0 \circ p_1 \circ s_0 \end{array}$$

are a simple recursive definition of the opposite function.

7.4 Recursive definition of multiplication

Proposition 7.8 *Let (X, s_0, s_1) be an initial binary system with predecessors, Y be a uniform space, Z be a complete uniform space, and $t_0, t_1 : X \times Y \times Z \to Z$ be uniformly continuous functions such that $(Z, z \mapsto t_0(x, y, z), z \mapsto t_1(x, y, z))$ is a binary system with predecessors for all $x \in X$ and $y \in Y$, and such that all such binary systems have a bound for the contraction moduli of the successor functions. Then there is a unique continuous function $h : X \times Y \to Z$ such that*

$$\begin{aligned} h(s_0(x), y) &= t_0(x, y, h(x, y)) \\ h(s_1(x), y) &= t_1(x, y, h(x, y)) \end{aligned}$$

The above equations are said to be a ***primitive recursive definition*** of h.

A ***multiplication operation*** for a dyadic group $(G, +, 0, 1)$ is any operation $(x, y) \mapsto xy$ on G such that $0y = 0$ and $(x + 1)y = xy + y$.

Proposition 7.9 *Let (G, s_0, s_1) be an initial binary system with predecessors derived from a dyadic topological group $(G, +, 0, 1)$. Then G admits a unique multiplication operation, which has the following primitive recursive definition:*

$$\begin{aligned} s_0(x) \cdot y &= s_0(xy) \\ s_1(x) \cdot y &= s_0(xy + y) \end{aligned}$$

8 Imperative programs derived from recursive definitions

In this section we consider an ideal imperative programming language. The imperative language that we consider is an extension of the language introduced by Dijkstra [5] (see also [9]). We deliberately omit a definition of the extension, since it will be obvious from context.

Since we are interested in total correctness, we adopt the following definition. If S is a (possibly non-deterministic) program and P and Q are properties of states, then $\{P\}S\{Q\}$ means that (1) all computations of S starting with a state that satisfies P terminate, and (2) the final state of such computations satisfies the property Q. Nothing is asserted about computations that start in states which do not satisfy P.

In order to measure the convergence of the programs presented in this section, we associate metrics to topological groups. A metric d on an Abelian group $(G, +, 0)$ is ***invariant*** if $d(x+z, y+z) = d(x, y)$. From [12] we know that every T_0 topological group is T_2, and from [4] we know that any T_2 Abelian topological group with a countable base has an invariant metric d which induces its uniformity. Denote $d(0, x)$ by $|x|$. Then $d(x, y) = |x - y|$. The mapping $x \mapsto |x|$ enjoys the following properties:

1. $|-x| = x$
2. $|x+y| \leq |x| + |y|$
3. $|x| = 0$ iff $x = 0$

The results in the following section hold for any choice of such a metric d, and depend only on the laws displayed above.

8.1 Bounded iteration

Proposition 8.1 *Let (G, s_0, s_1) be an initial binary system with predecessors derived from a dyadic topological group $(G, +, 0, 1)$, (H, t_0, t_1) be a predecessor-less binary with predecessors derived from a dyadic topological group $(H, \cdot, 1, e)$ with a countable base, and $h : G \to H$ be the unique homomorphism from G to H. Then*

$\{ x \in \bar{G}$ and $n \in \omega \}$
$y := 1$;
$r := e$;
`do` n `times`
  `if` $x \in s_0(\bar{G}) \rightarrow x := s_0^{-1}(x)$;
  $\Box\ x \in s_1(\bar{G}) \rightarrow x := s_1^{-1}(x)$; $y := ry$;
  `fi`
  $r := t_0(r)$;
`od`
$\{ y \in \bar{H}$ and $|h(x) - y| < |e^{-n}| \}$

Corollary 8.2 *Let P be the program displayed above. Then for every entourage V of H there is a natural number n such that*

$$\{x \in \bar{G}\}P\{y \in \bar{H} \text{ and } h(x)Vy\}.$$

Examples 8.1 (Cf. examples 7.2)

1. (Exponential) If G is the group of real numbers under addition with unit 1 and H is the group of positive real numbers under multiplication with unit e, then the application of the proposition gives us the following algorithm for computing $\exp : [0, 1] \to [1, e]$

   $\{ x \in [0, 1]$ and $n \in \omega \}$
   $y := 1$;
   $r := e$;
   `do` n `times`
     `if` $x \leq 1/2 \rightarrow x := 2x$;
     $\Box\ x \geq 1/2 \rightarrow x := 2x - 1$; $y := y \times r$;
     `fi`
     $r := \sqrt{r}$;
   `od`
   $\{ y \in [1, e]$ and $|e^x - y| < e^{-n} \}$

2. (Logarithm) By interchanging the roles of G and H, and interchanging additive notation and multiplicative notation, we obtain:

```
{ y ∈ [1,e] and n ∈ ω }
x := 0;
r := 1;
do n times
      if y² ≤ e →y := y²;
      □ y² ≥ e →y := y²/e; x := x + r;
      fi
      r := r/2;
od
{ x ∈ [0,1] and |ln(y) − x| < 2⁻ⁿ }
```

By using similar techniques, it is possible to obtain simple iterative programs for computing sin and arcsin. ■

8.2 Unbounded iteration

Recall that an ordered group is a group $(G,+,0)$ together with a linear order $<$ such that $x>0$ and $y>0$ together imply $x+y>0$, and such that $x<y$ implies $x+z<y+z$. A ***dyadic ordered group*** is a dyadic group $(G,+,0,1)$ such that $(G,+,0)$ is an ordered group and $0 \leq 1$.

For any linear order, we denote the sets $\{x|x \leq a\}$, $\{x|a \leq x \leq a\}$, and $\{x|b \leq x\}$ by $(-\infty,a]$, $[a,b]$, and $[b,+\infty)$ respectively.

The following proposition allows us to extend examples 8.1 to the whole real line.

Proposition 8.3 *Let (G,s_0,s_1) be an initial binary system with predecessors derived from an ordered dyadic topological group $(G,+,0,1)$, (H,t_0,t_1) be a binary system with predecessors derived from a dyadic topological group $(H,\cdot,1,e)$ with a countable base, and $h: G \to H$ be the unique homomorphism from G to H. Then*

```
{ x ∈ G and n ∈ ω }
y := 1;
r := e;
do
    x ∈ (−∞,0]           →  x := s₁(x); y := y/r;
□  x ∈ [0,1/2] ∧ n > 0 →  x := s₀⁻¹(x); n := n − 1; r := t₀(r);
□  x ∈ [1/2,1] ∧ n > 0 →  x := s₁⁻¹(x); n := n − 1; r := t₀(r); y := ry;
□  x ∈ [1,+∞)            →  x := s₀(x); y := ry;
od
{ y ∈ H and |h(x) − y| < |e⁻ⁿ| }
```

9 Concluding remarks

It is possible to show that the recursion and the definition by cases schemes produce computable functions from given computable functions. Moreover, we can effectively derive procedures for computing such functions. We have shown that certain programs "compute" certain recursive functions (see section 8), but we have not shown that the ideal programming language used to define the programs has an effective operational semantics.

For several equivalent definitions of computable real numbers and functions see [7], where it is also shown that binary representation of real numbers is not suitable for real number computation. This seems to be in contradiction with the fact that the axiomatic presentation of the real line presented in this work is strongly related to binary expansions. This contradiction is only apparent. In fact, in order to show that the recursion and definition by cases schemes preserve computability, we do not use binary representation.

Although we have shown that it is possible to recursively define several real functions from simpler ones, we have not shown how the recursion and definition by cases schemes could be used to obtain a sensible collection of "primitive recursive real functions" from some base functions such as successor, predecessor, projections etc., similar to the collection of primitive recursive functions on natural numbers [15].

The characterization of the real numbers by the initial property of $(\mathbb{R}, S_0, S_1)$ can perhaps be used to define a notion of "real number object" in certain enriched categories, analogous to the definition of the notion of natural number object in cartesian closed categories [10].

There seems to be connections between the axiomatization of the unit interval presented here and its inverse limit construction given in [16]. Also, the pushout diagram shown in section 6.4 appears to be related to the domain equation $\mathbb{I} \cong \mathbb{I} +_1 \mathbb{I}$ presented in *loc. cit.*

Acknowledgements

This work has been supervised by Mike Smyth. I have benefited from discussions with him, A.C. Rocha Costa (by e-mail), Abbas Edalat, Iain Stewart and Paul Taylor. The commutative diagrams in this paper were drawn using Paul Taylor's package. The author has been sponsored by the Brazilian agency CNPq.

References

[1] M.J. Beeson. *Foundations of Constructive Mathematics.* Springer-Verlag, New York, 1985.

[2] H.J. Boehm and R. Cartwright. Exact real arithmetic: Formulating real numbers as functions. In Turner. D., editor, *Research Topics in Functional Programming*, pages 43–64. Addison-Wesley, 1990.

[3] N. Bourbaki. *General Topology*, volume 1. Addison-Wesley, London, 1966.

[4] N. Bourbaki. *General Topology*, volume 2. Addison-Wesley, London, 1966.

[5] E.W. Dijkstra. *A Discipline of Programming.* Prentice Hall, Englewood Cliffs, 1976.

[6] M.H. Escardó. Induction, recursion, and computation on the real line and some of its subspaces. Unpublished work in progress, 1994.

[7] P. Gianantonio. *A Functional Approach to Computability on Real Numbers.* PhD thesis, Università Degli Studi di Pisa, Dipartamento di Informatica, 1993.

[8] L.R. Graham, D.E. Knuth, and O. Patashnick. *Concrete Mathematics.* Addison-Wesley, New York, 1989.

[9] D. Gries. *The Science of Programming.* Springer, New York, 1981.

[10] J. Lambek and P.J. Scott. *Introduction to Higher Order Categorical Logic.* Cambridge University Press, Cambridge, 1986.

[11] S. MacLane and G. Birkhoff. *Algebra.* Macmillan, New York, 1st edition, 1967.

[12] G. McCarty. *Topology: an Introduction with Applications to Topological Groups.* McGraw-Hill, 1967.

[13] G. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5(1):223–255, 1977.

[14] H.G. Rice. Recursive real numbers. *Proc. Amer. Math. Soc.*, pages 784–791, 1954.

[15] H. Rogers. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.

[16] M.B. Smyth. Semi-metrics, closure spaces, and digital topology. To appear in *Theoretical Computer Science.*

[17] M.B. Smyth. Quasi-uniformities: Reconciling domains with metric spaces. In M. Main, A. Melton, M. Mislove, and D. Schimidt, editors, *Mathematical Foundations of Programming Languages*, pages 236–253, London, 1987. Springer-Verlag. LNCS 298.

[18] M.B. Smyth. The unit interval as an inverse limit of finite semi-uniform spaces. Verbal communication, July 1994.

[19] R. Stoll. *Set Theory and Logic.* W.H. Freeman and Company, San Fransisco, 1966.