

Taking "algebraically" seriously
in the definition of
algebraically inductive type

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12th ASSUME seminar, 5th June 2025
Nottingham, UK

Abstract

Theorem. In a 1-topos, the following two categories are isomorphic, with an isomorphism that is the identity on objects:

1. Pullback-natural, associative, algebraically injective objects.
2. Algebras of the partial-map classifier (aka lifting) monad.

- Partial results towards the ∞ -topos situation.
- We work in HoTT/UF.

I. Algebraic injectives (MScs'2021)

Def. Algebraic injective structure on a type D consists of

1. An extension operation, for any types X and Y ,

$$(-) \mid (-) : (X \rightarrow D) \times (X \hookrightarrow Y) \rightarrow (Y \rightarrow D).$$

fibers are propositions.

2. For each map $f : X \rightarrow D$ and embedding $j : X \hookrightarrow Y$,

a choice of an identification $(f \mid j) \circ j = f$, as illustrated by

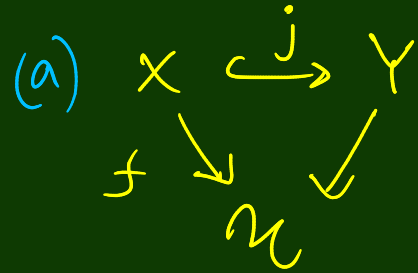
$$\begin{array}{ccc} X & \xhookrightarrow{j} & Y \\ f \searrow & & \swarrow f \mid j \\ & D & \end{array}$$

Some examples

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(They need univalence.)

1. $D := \mathcal{U}$



$$(f|j)(y) := \prod_{(x, -) : \text{fiber } j \ y} f(x) \cdot$$

(Right Kan extension.)

(b)

Use \sum instead.

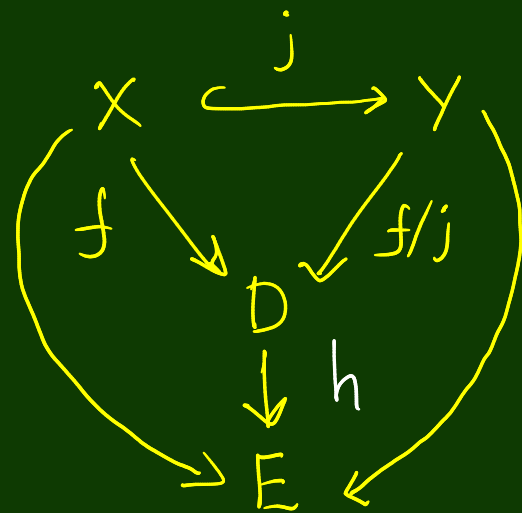
(Left Kan extension.)

2. The type Ω of propositions. Use \forall or \exists .

3. Universes of n -types.

4. Algebras of the lifting monad. We'll come back to this.

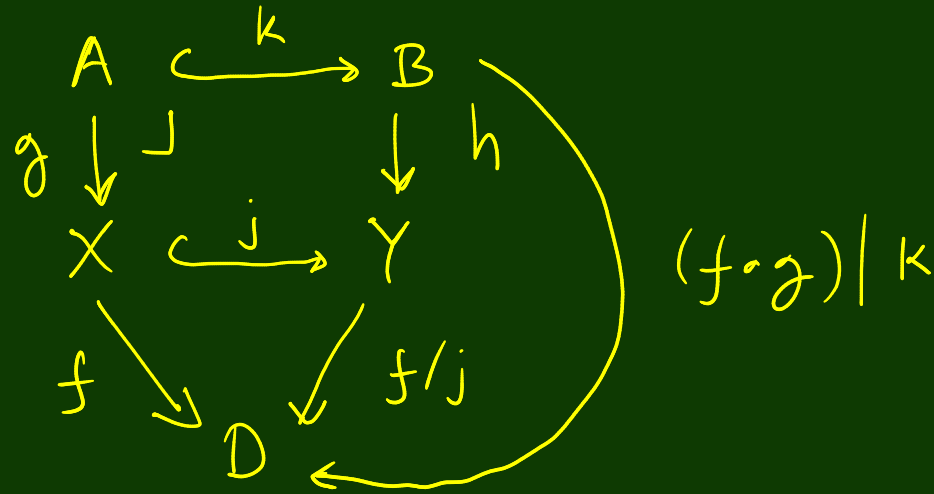
Homomorphisms of algebraic injectives



$$h \circ (f|j) = (h \circ f) | j$$

Pullback naturality

Previous examples are all pullback natural.



$$(f|j) \circ h = (f \circ g)|k$$

It is essential that the square is a pullback.

Consider the non-pullback square

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & 1 \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\quad} & 1 \end{array}$$

for a counter-example.

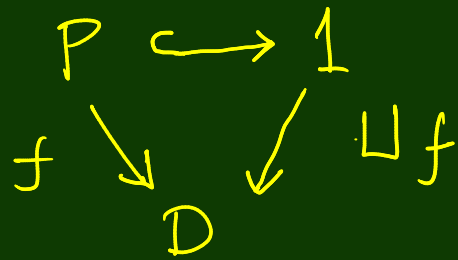
Associativity

$$\begin{array}{ccccc}
 X & \xhookrightarrow{j} & Y & \xhookrightarrow{k} & Z \\
 & \searrow f & \downarrow f|_j & \swarrow & \\
 & & D & &
 \end{array}
 \quad f|(k \circ j) = (f|_j)|_k$$

Examples (Mscs' 2024) $D := \mathcal{U}$ with extension given by Π or Σ .

II. Algebraic floppy structure

MSCS'2021



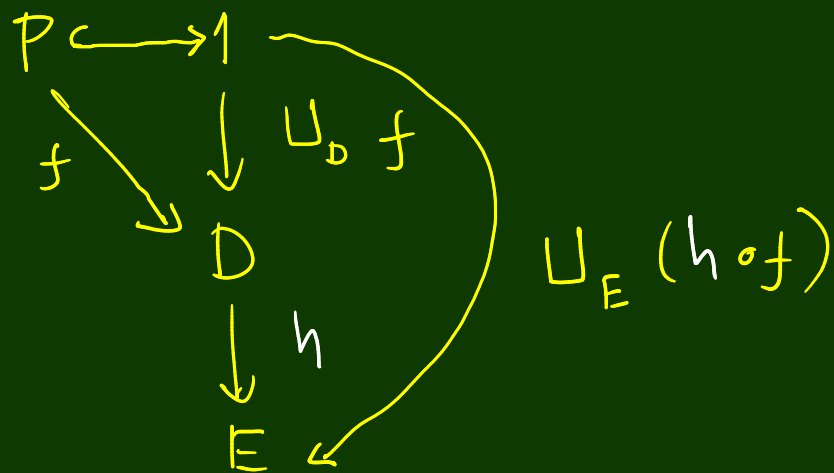
Every partial element
of D can be extended
to a total element.

N.B. T.F.A.E.

1. The map $P \rightarrow 1$ is an embedding.
2. The type P is a proposition.

Trivial fact. Algebraic inductive structure is, in particular,
an algebraic floppy structure.

Homomorphisms



$$h \circ U_D f = U_E(h \circ f)$$

construction | Algebraic fibby \rightarrow algebraic injective | MSCS'2021

$$\begin{array}{ccc}
 X & \xrightarrow{j} & Y \\
 f \searrow & & \swarrow \\
 & D &
 \end{array}
 \quad (f|j)(\gamma) := \bigsqcup_{(x,-) \in \underbrace{\text{fiber } j}_Y} f x$$

Fact (new observation)

This algebraic injective structure is pullback natural.
(Not only fiber-natural)

$$\begin{array}{ccc}
 P_Y & \longrightarrow & 1 \\
 p_{r_1} \downarrow & & \downarrow \bigsqcup (f \circ p_{r_1}) \\
 X & \xrightarrow{f} & D
 \end{array}$$

Fiberwise extension.

$$\begin{array}{ccc}
 P_Y & \xrightarrow{c} & 1 \\
 p_{r_1} \downarrow & \lrcorner & \downarrow y \\
 X & \xrightarrow{j} & Y \\
 f \searrow & & \swarrow f|j \\
 & D &
 \end{array}
 \quad \bigsqcup (f \circ p_{r_1})$$

Get this by fibbiness.

III. The lifting monad

M.H.E & C. Kuopp CSL'2017 &
TypeTopology 2018

$$\mathcal{L}X := \sum_{P:\Omega} (P \xrightarrow{\varphi} X)$$

The type of partial elements of X .

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ X & \xrightarrow{f} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ \mathcal{L}X & \xrightarrow{\mathcal{L}f} & \mathcal{L}Y \end{array}$$

$$(\underline{1}, \lambda \cdot x) \quad (P, \varphi) \longmapsto (P, f \circ \varphi)$$

$$\text{is-def(ined)} : \mathcal{L}X \rightarrow \Omega := p_{c1}$$

$$\text{value} : (\ell : \mathcal{L}X) \rightarrow \text{is-def } \ell \rightarrow X := p_{c2}$$

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathcal{L}Y \\ \mathcal{L}X & \xrightarrow{g^\#} & \mathcal{L}Y \end{array}$$

$$(P, \varphi) \longmapsto$$

is a prop,
as required.

$$\left(\left(\sum_{P:\Omega} \text{is-def}(f(\varphi P)) \right), \right. \\ \left. \lambda(p, d) . \text{value}(f(\varphi P)) d \right)$$

Monad algebras

1. Structure map

$$\sqcup : \mathcal{Z}A \rightarrow A$$

Extend a partial element to a total element!
So \mathcal{Z} -algebras give algebraic fluffy structure.
(MSCS'2021)

2. Unit law

$$\sqcup (\lambda(-:1). a) = a \text{ for every } a:A.$$

Extension "property", as for fluffy types.
(extension data!)

3. Associativity law

$$\bigsqcup_{p:P} \bigsqcup_{q:Q_p} f(p,q) = \bigsqcup_{r: \sum_{p:P} Q_p} f r$$

$$P : \Omega$$

$$Q : P \rightarrow \Omega$$

$$f : \sum_{p:P} Q_p \rightarrow A$$

Previously not counted for
when discussing injectivity

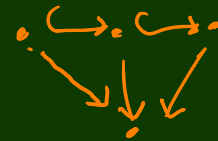
IV. Putting I-III together

So \mathcal{Z} -algebra structure = associative algebraic flabby structure.

Lemma Let \sqcup be the algebraic flabby structure induced by
a given algebraic injective structure \sqcap that is pullback natural.

Then \sqcup is associative iff \sqcap is associative

$$\sqcup \sqcup = \sqcup$$



Ongoing work. Replace "iff" by " \simeq ".
(For sets we have this.)

Lemma Let $|$ be the algebraic injective structure induced by
a given algebraic fibbing structure \sqcup .
Then $|$ is always pullback natural. (We've already discussed this.)

Lemma. The round trip $\sqcup \mapsto | \mapsto \sqcup'$
is always the identity on both extension operators and extension data.

Lemma The round trip $| \mapsto \sqcup \mapsto |'$
is the identity on extension operators
iff $|$ is pullback natural.

But what about
extension data?
Ongoing.

Theorem. Let D be any type.

1. Then

$$\begin{aligned} & \text{pullback-natural, associative injective structure on } D \\ & \iff \text{associative algebraic flabby structure on } D \\ & = \text{ } \mathcal{L}\text{-algebra structure on } D. \end{aligned}$$

2. If D is a set, then " \iff " in (1) becomes an equivalence " \simeq ".

- What is missing to always have β type equivalence?
 - check that the pullback-naturality data is unchanged by round trips.
 - check that the associativity data is unchanged by round trips.

(ongoing work, perhaps not difficult.)

- But there is still something else missing.

V. Is \mathcal{Z} really a monad?

Nobody knows what a monad on types is in HoTT/UF.

People do know what monads on ∞ -toposes are, though.

But we don't know how to say that in the language of HoTT/UF.

The problem is how to specify coherence data for the monad laws.

Speculative ideas

This is the part of the talk in which I make a fool of myself.

Try to define monad on types as follows.

1. Data

(a) A function $T : \text{Type} \rightarrow \text{Type}$

(May well change type levels.)

(b) A function $\eta : X \rightarrow TX$

(parametric in X .)

(c) A function $(f \mapsto f^\#) : (X \rightarrow TY) \rightarrow (TX \rightarrow TY)$ (parametric in X, Y)

Then we can define

$$\mu : TTX \rightarrow TX := (\text{id}_{TX})^\#$$

$$T : (X \rightarrow Y) \rightarrow (TX \rightarrow TY) := f \mapsto (\eta_Y \circ f)^\#$$

2. No equations. Instead ...

2. Declare TX to be a free algebra w.r.t. TY 's.

This is given by a universal property, and so is property rather than data.

$$\begin{array}{ccccc}
 X & \xrightarrow{\eta_x} & TX & \xleftarrow{\mu_x} & TTX \\
 \searrow \forall f & & \downarrow \exists! h & & \downarrow TTh \\
 & & TY & \xleftarrow{\mu_y} & TTY
 \end{array}$$

The square says that h is an algebra homomorphism.

In HoTT/UF ,

$$(f: X \rightarrow TY) \rightarrow \text{is-contr} \left(\sum_{h: TX \rightarrow TY} \left(h \circ \eta_x \sim f \right) \times \left(h \circ \mu_x \sim \mu_y \circ TTh \right) \right)$$

2.

$$\begin{array}{ccccc}
 X & \xrightarrow{\eta_x} & TX & \xleftarrow{\mu_x} & TTX \\
 \searrow \forall f & & \downarrow \exists! h & & \downarrow TTh \\
 & & Ty & \xleftarrow{\mu_y} & TTy
 \end{array}$$

In HoTT/UF ,

$$(f: X \rightarrow Ty) \rightarrow \text{is-contr} \left(\sum_{h: TX \rightarrow Ty} \left(h \circ \eta_x \sim f \right) \times \left(h \circ \mu_x \sim \mu_y \circ TTy \right) \right)$$

Likely problem.

h should be $f^\#$, but how could we conclude that?

The problem seems to disappear for idempotent monads

Tentative definition. An idempotent monad on types consists of

1. A function $T: \text{Type} \rightarrow \text{Type}$.

A family of functions $\eta_X: X \rightarrow TX$.

data
property

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ \downarrow f & \searrow & \downarrow \exists! h \\ TY & & TY \end{array}$$

Here $\exists!$ is contractibility of Σ .

$\text{ext}: (f: X \rightarrow TY) \rightarrow \exists! h: TX \rightarrow TY, h \circ \eta_X \sim f$.

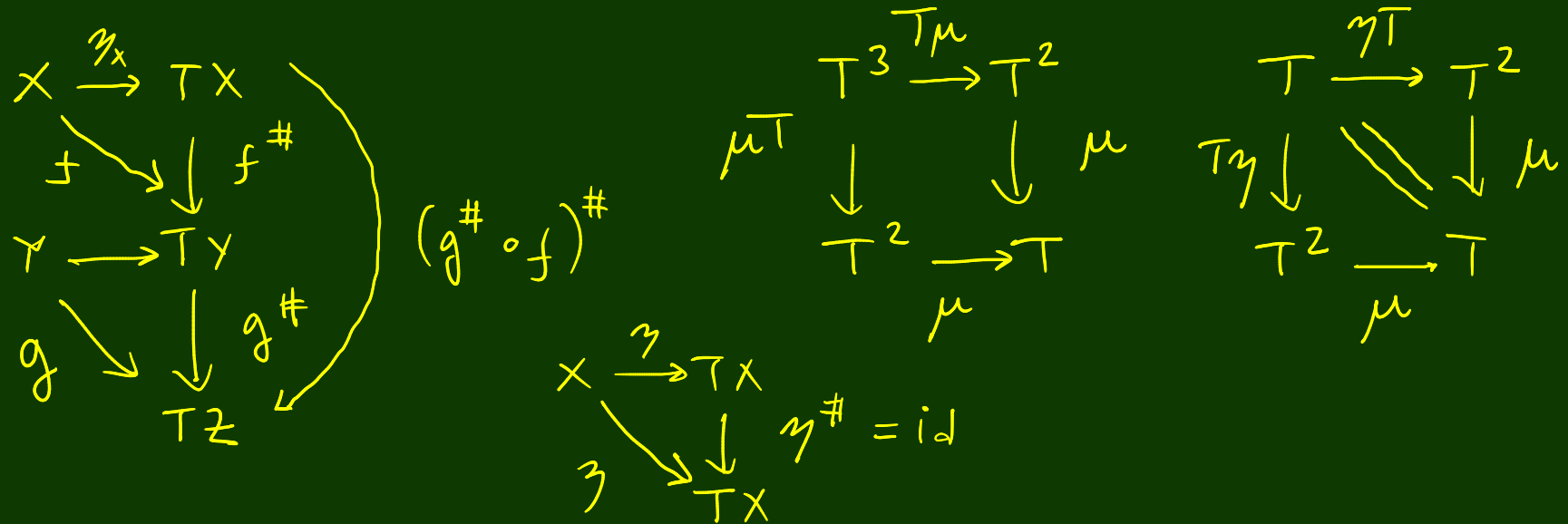
Idea: In an idempotent monad, every map of free algebras is a homomorphism

Then define $f^\# := \text{pr}_1 (\text{center of contraction } (\text{ext } f))$.
 $\nabla f := \text{pr}_2 (\text{center of contraction } (\text{ext } f))$.

That's the
complete
definition!

Then we can define $T : (X \rightarrow Y) \rightarrow TX \rightarrow TY$ and μ from $(-)^{\#}$ as before.

With **Eric Finster** we checked in my office whiteboard that the diagrams below commute.



Idempotency

The maps

$$TX \xrightarrow[\eta_{TX}]{T\eta_X} TTX$$

are equal.

The first one is

$$(\eta_{TX} \circ \eta_X)^\#$$

$$X \xrightarrow{\eta_X} TX \xrightarrow{\eta_{TX}} TTX.$$

To show that they are equal, it is enough to show that

$$T\eta_X \circ \eta_X = (\eta_{TX} \circ \eta_X)^\# \circ \eta_X.$$

But this holds by the definition of $(-)^\#$ and by \triangleleft .

Summary of \mathcal{V} and questions

1. This idea seems to work for *idempotent* monads on types.
2. We only give the data $T: \text{Type} \rightarrow \text{Type}$ and $\eta_X: X \rightarrow TX$.
3. Then we give a *property* of these *data*, formulated as a *universality condition*.
4. From this we automatically get the usual equations for idempotent monads.
5. The equations *should* be automatically *fully coherent*, as they are *deduced from property*.
6. Can we do the same for *arbitrary* monads on types, following the above outline?