The inconsistency of a Brouwerian continuity principle with the Curry–Howard interpretation

Martín Escardó and Chuangjie Xu

University of Birmingham, UK

Abstract. If all functions $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ are continuous then $0 = 1$. This is a theorem of intensional (and hence of extensional) intuitionistic dependent-type theories, with existence in the formulation of continuity expressed as a Σ type via the Curry-Howard interpretation. With an intuitionistic notion of anonymous existence, defined as the propositional truncation of Σ, it is consistent that all such functions are continuous. A model is Johnstone's topological topos. On the other hand, any of these two intuitionistic conceptions of existence give the same, consistent, notion of uniform continuity for functions $(\mathbb{N} \to 2) \to \mathbb{N}$, again valid in the topological topos. It is open whether the consistency of (uniform) continuity extends to homotopy type theory. The theorems of type theory informally proved here are also formally proved in Agda, but the development presented here is self-contained and doesn't show Agda code.

Keywords: Dependent types, intensional Martin-Löf type theory, Curry-Howard interpretation, constructive mathematics, Brouwerian continuity axioms, anonymous existence, propositional truncation, function extensionality, homotopy type theory, topos theory.

1 Introduction

We show that a continuity principle that holds in Brouwerian intuitionistic mathematics becomes false when we move to its Curry–Howard interpretation. We formulate and prove this in an intensional version of intuitionistic type theory (Section 2). Another Brouwerian (uniform) continuity principle, however, is logically equivalent to its Curry–Howard interpretation (Section 4). In order to be able to formulate and prove this logical equivalence, we need a type theory in which both a formula and its Curry–Howard interpretation can be expressed (Section 3). For example, toposes admit both $\forall, \exists$ (via the subobject classifier) and Π, Σ (via their local cartesian closedness) and hence qualify. We adopt the HoTT-book [16] approach of working with propositional truncation $\| - \|$ to express $\exists (x : X).A(x)$ as the propositional truncation of $\Sigma(x : X).A(x)$. This is related to related to NuPrl's bracket types [13], Maietti's mono-types [12], and Awodey–Bauer bracket types in extensional type theory [1]. Here by a proposition we mean a type whose elements are all equal in the sense of the identity type, as in the HoTT book. In a topos with identity types understood as equalizers,

the propositions are the truth values (subterminal objects), and the propositional truncation of an object X is its support, namely the image of the unique map $X \to 1$ to the terminal object, which is the truth value of the inhabitedness of X, without necessarily revealing an inhabitant, and the equation

$$(\exists(x : X).A(x)) = \|\Sigma(x : X).A(x)\|$$

holds. In HoTT, this is taken as the definition of $\exists$, with truncation taken as a primitive notion. But we don't (need to) work with the homotopical understanding of type theory or the univalence axiom here.

1.1 The continuity of all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$

In Brouwerian intuitionistic mathematics, all functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ on the Baire space $\mathbb{N}^{\mathbb{N}} = (\mathbb{N} \to \mathbb{N})$ are continuous [2,17]. This means that, for any sequence $\alpha : \mathbb{N}^{\mathbb{N}}$ of natural numbers, the value $f\alpha$ of the function depends only on a finite prefix of the argument $\alpha : \mathbb{N}^{\mathbb{N}}$. If we write $\alpha =_n \beta$ to mean that the sequences α and β agree at their first n positions, a precise formulation of this continuity principle is

$$\forall(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}). \ \forall(\alpha : \mathbb{N}^{\mathbb{N}}). \ \exists(n : \mathbb{N}). \ \forall(\beta : \mathbb{N}^{\mathbb{N}}). \ \alpha =_n \beta \to f\alpha = f\beta.$$

It is well known that this statement cannot be proved in higher-type Heyting arithmetic (HA^{ω}), but that is consistent and validated by the model of Kleene–Kreisel continuous functionals, and also realizable with Kleene's second combinatory algebra K_2 [2].

We show that, in intensional Martin-Löf type theory, the Curry–Howard interpretation of the above continuity principle is false: It is a theorem of intensional MLTT, even without universes, that

$$\left(\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta\right) \to 0 = 1.$$

We prove this by adapting Kreisel's well-known argument that, e.g. in HA^{ω}, extensionality, choice and continuity are together impossible [11]. The difference here is that

1. We work in *intensional* type theory.

2. Choice for the Σ interpretation of existence is a theorem of type theory.

So what is left to understand is that extensionality is not needed in Kreisel's argument when it is rendered in type theory (Section 2).

The above two versions of the notion of continuity can be usefully compared by considering the interpretations of HA^{ω} and MLTT in Johnstone's *topological topos* [8]. The point of this topos is that it fully embeds a large cartesian closed category of continuous maps of topological spaces, the sequential topological spaces, and the larger locally cartesian closed category of Kuratowski limit spaces [14]. As discussed above, any topos has $\exists, \forall, \Sigma, \Pi$ and hence models both intuitionistic predicate logic and dependent type theory. We have that

1. The formula

$$\forall(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}). \; \forall(\alpha : \mathbb{N}^{\mathbb{N}}). \; \exists(n : \mathbb{N}). \; \forall(\beta : \mathbb{N}^{\mathbb{N}}). \; \alpha =_n \beta \to f\alpha = f\beta$$

   is true in the topological topos.

   The informal reading of this is "all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous".

2. There is a function

$$(\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta) \to 0 = 1$$

   in the topological topos, or indeed in any topos whatsoever, by our version of Kreisel's argument.

   The informal reading of this is "not all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous".

But there is no contradiction in the formal versions of the above statements: they simultaneously hold in the same world, the topological topos. From a hypothetical inhabitant of

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\Sigma(n : \mathbb{N}).\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\alpha =_n \beta \to f\alpha = f\beta$$

we get a modulus-of-continuity functional

$$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N},$$

by projection (rather than by choice in the topos-logic sense), which gives a modulus of continuity $n = M(f, \alpha)$ of the function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ at the point $\alpha : \mathbb{N}^{\mathbb{N}}$. Kreisel's argument derives a contradiction from the existence of M. What this shows, then, is that although every function *is* continuous, there is no continuous way of finding a modulus of continuity of a given function f at a given point α. There is no continuous M. Perhaps the difference between the seemingly contradictory statements becomes clearer if we formulate them type theoretically with(out) propositional truncation: In the topological topos, the object

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}). \; \Pi(\alpha : \mathbb{N}^{\mathbb{N}}). \; \|\Sigma(n : \mathbb{N}). \; \Pi(\beta : \mathbb{N}^{\mathbb{N}}). \; \alpha =_n \beta \to f\alpha = f\beta\|$$

is inhabited, but

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}). \; \Pi(\alpha : \mathbb{N}^{\mathbb{N}}). \; \Sigma(n : \mathbb{N}). \; \Pi(\beta : \mathbb{N}^{\mathbb{N}}). \; \alpha =_n \beta \to f\alpha = f\beta$$

is not.

1.2 The uniform continuity of all functions $2^{\mathbb{N}} \to \mathbb{N}$

The above situation changes radically when we move from the Baire space to the Cantor space, and from continuous functions to uniformly continuous functions.

Another Brouwerian continuity principle is that all functions from the Cantor space $2^{\mathbb{N}} = (\mathbb{N} \to 2)$ to the natural numbers are uniformly continuous:

$$\forall (f \colon 2^{\mathbb{N}} \to \mathbb{N}). \ \exists (n \colon \mathbb{N}). \ \forall (\alpha, \beta \colon 2^{\mathbb{N}}). \ \alpha =_n \beta \to f\alpha = f\beta.$$

Again this is not provable in HA^{ω} but consistent and validated by the model of continuous functionals, by realizability over K_2, and by the topological topos. We have also constructively developed a model analogous to the topological topos in [18].

By the above discussion, the above principle is equivalent to

$$\Pi (f \colon 2^{\mathbb{N}} \to \mathbb{N}). \ \| \Sigma (n \colon \mathbb{N}). \ \Pi (\alpha, \beta \colon 2^{\mathbb{N}}). \ \alpha =_n \beta \to f\alpha = f\beta \|.$$

We show that this, in turn, is logically equivalent to its untruncated version

$$\Pi (f \colon 2^{\mathbb{N}} \to \mathbb{N}). \ \Sigma (n \colon \mathbb{N}). \ \Pi (\alpha, \beta \colon 2^{\mathbb{N}}). \ \alpha =_n \beta \to f\alpha = f\beta.$$

In particular, it follows that this object is inhabited (by a global point) in the topological topos. Each inhabitant gives, by projection, a "fan functional" $(2^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$ that continuously assigns a modulus of uniform continuity to its argument. There is a canonical one, which assigns the least modulus of uniform continuity.

In order to establish the above logical equivalence, we prove the following general principle for "exiting truncations": If A is a family of types indexed by natural numbers such that

1. $A(n)$ is a proposition for every $n \colon \mathbb{N}$, and
2. $A(n)$ implies that $A(m)$ is decidable for every $m < n$,

then
$$\| \Sigma (n \colon \mathbb{N}).A(n) \| \to \Sigma (n \colon \mathbb{N}).A(n).$$

From anonymous existence one gets explicit existence in this case.

2 Continuity of functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$

We reason informally, but rigorously, in type theory, where, as above, we use the equality sign to denote identity types, unless otherwise indicated. A formal proof, written in Agda [3,4,15], is available at [6], but the development here is self-contained and doesn't show Agda code.

The following says that the Curry–Howard interpretation of "all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous" is false.

Theorem 1. *If*

$$\Pi (f \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\Pi (\alpha \colon \mathbb{N}^{\mathbb{N}}).\Sigma (n \colon \mathbb{N}).\Pi (\beta \colon \mathbb{N}^{\mathbb{N}}). \ \alpha =_n \beta \to f\alpha = f\beta$$

then $0 = 1$.

We take the conclusion to be $0 = 1$ rather than the empty type because we are not assuming a universe for the sake of generality. The argument below gives $0 = 1$, and, as is well known, to get to the empty type from $0 = 1$ a universe is needed.

Proof. Let 0^ω denote the infinite sequence of zeros, that is, $\lambda i.0$, and let $0^n k^\omega$ denote the sequence of n many zeros followed by infinitely many k's. Then

$$(0^n k^\omega) =_n 0^\omega \quad \text{and} \quad (0^n k^\omega)(n) = k.$$

Assume $\Pi(f\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}).\Pi(\alpha\colon \mathbb{N}^\mathbb{N}).\Sigma(n\colon \mathbb{N}).\Pi(\beta\colon \mathbb{N}^\mathbb{N}).\alpha =_n \beta \to f\alpha = f\beta$. By projection, with $\alpha = 0^\omega$, this gives a modulus-of-continuity function

$$M\colon (\mathbb{N}^\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$$

such that

$$\Pi(f\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}).\Pi(\beta\colon \mathbb{N}^\mathbb{N}).0^\omega =_{Mf} \beta \to f(0^\omega) = f\beta. \tag{1}$$

We use M to define a function $f\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}$ such that $M(f)$ cannot be a modulus of continuity of f and hence get a contradiction. Let

$$m = M(\lambda\alpha.0),$$

and define $f\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}$ by

$$f\beta = M(\lambda\alpha.\beta(\alpha m)).$$

The crucial observation is that, by simply expanding the definitions, we have the judgemental equalities

$$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha m)) = M(\lambda\alpha.0) = m,$$

because $0^\omega(\alpha m) = 0$.

By the defining property (1) of M, and the crucial observation,

$$\Pi(\beta\colon \mathbb{N}^\mathbb{N}).0^\omega =_{Mf} \beta \to m = f\beta. \tag{2}$$

Using this, each of the two cases $Mf = 0$ and $Mf > 0$ implies $0 = 1$.

Assume $Mf = 0$. By (2),

$$\Pi(\beta\colon \mathbb{N}^\mathbb{N}).m = f\beta.$$

This with $\beta = \lambda i.i$ gives

$$m = f(\lambda i.i) = M(\lambda\alpha.\alpha m),$$

by expanding the definition of f. By the defining property (1) of M, this means that

$$\Pi(\alpha\colon \mathbb{N}^\mathbb{N}).0^\omega =_m \alpha \to 0 = \alpha m.$$

But this gives $0 = 1$ if we choose the sequence $\alpha = 0^m 1^\omega$, because $0^m 1^\omega(m) = 1$.

Now assume $Mf > 0$ instead. For any $\beta \colon \mathbb{N}^\mathbb{N}$, by the continuity of $\lambda\alpha.\beta(\alpha m)$, by the definition of f, and by the defining property (1) of M, we have that

$$\Pi(\alpha \colon \mathbb{N}^\mathbb{N}).0^\omega =_{f\beta} \alpha \to \beta 0 = \beta(\alpha m).$$

Considering $\beta = 0^{Mf} 1^\omega$, this gives

$$\Pi(\alpha \colon \mathbb{N}^\mathbb{N}).0^\omega =_m \alpha \to \beta 0 = \beta(\alpha m),$$

because $f\beta = m$ as $0^\omega =_{Mf} \beta$ and $f(0^\omega) = m$. Considering $\alpha = 0^m (Mf)^\omega$, this in turn gives $0 = \beta 0 = \beta(\alpha m) = \beta(Mf) = 1$. □

Remark 1 (Thomas Streicher, personal communication). The conversion

$$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha m)) = M(\lambda\alpha.0) = m$$

in the above proof relies on the ξ-rule (reduction under λ), which is not available in a system based on the combinators S and K rather than the λ-calculus. Usually HA^ω is taken in combinatory form, in which case one needs some form of extensionality to conclude that $f(0^\omega) = m$, and this explains how we avoid the extensionality hypothesis in Kreisel's original argument. But notice that the ξ-rule holds in categorical models.

Therefore the argument of the above proof shows that:

Theorem 2. *In HA^ω, the ξ-rule, the axiom of choice, and the continuity of all functions $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ are together impossible.*

Another observation, offered to us independently by Thorsten Altenkirch, Thierry Coquand and Per Martin-Löf (personal communication), is that the continuity of a function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ implies that it is extensional in the sense that it maps pointwise equal arguments to equal values, and so the continuity axiom has some amount of extensionality built into it.

The above formulation and proof of Theorem 1 assumes natural numbers, identity types, Π and Σ types, and no universes. But it uses only the identity type of the natural numbers. If we assume a universe U, this identity type doesn't need to be assumed, because it can be defined by induction. We first define a U-valued equality relation, where in the left-hand side $\mathbb{0}$ is the empty type and $\mathbb{1}$ is the unit type with element $\star$,

$$(0 = 0) = \mathbb{1}, \quad (m + 1 = 0) = (0 = n + 1) = \mathbb{0}, \quad (m + 1 = n + 1) = (m = n).$$

Then we define refl $: \Pi(n : N).n = n$ by induction as

$$\mathrm{refl}(0) = \star, \qquad \mathrm{refl}(n + 1) = \mathrm{refl}(n),$$

and $J : \Pi(A : \Pi(m, n).m = n \to U).(\Pi n.A\,n\,n\,(\mathrm{refl}(n))) \to \Pi(m, n, p).A\,m\,n\,p$
by

$$J\,A\,r\,0\,0\,\star = r\,0,$$
$$J\,A\,r\,(m+1)\,0\,p = \mathbb{O}\text{-rec}(p),$$
$$J\,A\,r\,0\,(n+1)\,p = \mathbb{O}\text{-rec}(p),$$
$$J\,A\,r\,(m+1)\,(n+1)\,p = J(\lambda mn.A(m+1)(m+1))(\lambda n.r(n+1))\,m\,n\,p.$$

where $\mathbb{O}\text{-rec} : \Pi\{X : U\}.\mathbb{O} \to X$ is the recursion combinator of the empty type. The usual computation rule, or judgemental equality, for J when it is given as primitive doesn't hold here, but the above J is enough to define transport (substitution) and hence symmetry, transitivity and application (congruence), which are enough to carry out the above proof formally (and we have checked this in Agda [6]). Hence the theorem and its proof can be expressed in a type theory without a primitive equality type. All is needed to formulate and prove Theorem 1 is a type theory with $\mathbb{O}, \mathbb{1}, \mathbb{N}, \Pi, \Sigma, U$.

3 Propositional truncation and existential quantification

We recall the notion of propositional truncation from the HoTT book and use it to define the quantifiers $\exists, \forall$ (in a slightly different way from that in the HoTT book) so that they satisfy the Lawvere's adjointness conditions that correspond to their intuitionistic introduction and elimination rules.

A difference is that, instead of adding propositional truncations for all types to our type theory, we define what a propositional truncation for a given type is. For some types, the propositional truncation is already definable, including the types needed in our discussion of uniform continuity in Section 4.

3.1 Propositional truncation

We adopt the terminology of the HoTT book, which clashes with the terminology of the Curry–Howard interpretation of (syntactical) propositions as types. For us, a *proposition* is a subsingleton, or a type whose elements are all equal, in the sense of the identity type, here written "=" again as in the HoTT book:

$$\mathrm{isProp}\,X = \Pi(x, y\colon X).x = y.$$

Perhaps a better terminology, compatible with that of topos theory, would be *truth value*, in order to avoid the clash. But we will stick to the terminology *proposition*, and occasionally use *truth value* synonymously, for emphasis.

A propositional truncation of a type X, if it exists, is a proposition $\|X\|$ together with a map $|-| : X \to \|X\|$ such that for any proposition P and $f : X \to P$ we can find $\bar{f} : \|X\| \to P$. Because P is a proposition, this map $\bar{f}$ is automatically unique up to pointwise equality, and we have $\bar{f}|x| = f(x)$, and hence a propositional truncation is a reflection in the categorical sense, giving a

universal map of X into a proposition. This can also be understood as a recursion principle, or elimination rule:

$$\text{isProp}\, P \to (X \to P) \to \|X\| \to P,$$

for any types P and X. The induction principle, in this case, can be derived from the recursion principle, but in practice it is seldom needed.

In HoTT, propositional truncations for all types are given as higher-inductive types, with the judgemental equality $\bar{f}|x| = f(x)$. From the existence of the truncation of the type $\mathbb{1} + \mathbb{1}$ with this judgemental equality, one can prove function extensionality (any two pointwise equal functions are equal) [10]. The assumption that $\|X\| \to X$ for every type X gives a constructive taboo (and also contradicts univalence) [9].

However, for some types X, not only can a propositional truncation $\|X\|$ be constructed in MLTT, but also there is a map $\|X\| \to X$:

1. If $P = \mathbb{0}$ or $P = \mathbb{1}$, or more generally if P is any proposition, we can take $\|P\| = P$, of course. In particular, if $X \to \mathbb{0}$, we can take $\|X\| = X$; but also we can take $\|X\| = \mathbb{0}$, even though we can't say $X = \mathbb{0}$ without univalence.

2. If we have an inhabitant of X then we can take $\|X\| = \mathbb{1}$. The map $\|X\| \to X$ simply picks the given inhabitant.

3. More generally, if X is logically equivalent to a proposition P, then we can take $\|X\| = P$, and we make profitable use of this simple fact.

4. If X is any type and $g : X \to X$ is a constant map in the sense that any two of its values are equal, we can take $\|X\|$ to be the type $\Sigma(x : X).g(x) = x$ of fixed points of g, together with the function $X \to \|X\|$ that maps x to $(g(x), p)$, where p is an inhabitant of the type $g(g(x)) = g(x)$ coming from the constancy witness [9]. In this case the first projection gives a map $\|X\| \to X$. Given a map $f : X \to P$ into a proposition, we let $\bar{f} : \|X\| \to P$ be the first projection followed by $f : X \to P$ (and we don't use the fact that P is a proposition).

5. For any $f : \mathbb{N} \to \mathbb{N}$, the type $\Sigma(n : \mathbb{N}).f(n) = 0$, which may well be empty, has a constant endomap that sends (n, p) to (n', p'), where we take the least $n' \le n$ with $p' : f(n') = 0$, using the decidability of equality of $\mathbb{N}$ and bounded search. Hence not only $\|\Sigma(n : \mathbb{N}).f(n) = 0\|$ exists, but also $\|\Sigma(n : \mathbb{N}).f(n) = 0\| \to \Sigma(n : \mathbb{N}).f(n) = 0$.

3.2 Quantification

For a universe U, let Prop be the type of propositions in U:

$$\text{Prop} = \Sigma(X : U).\,\text{isProp}\, X.$$

If we assume that all types in U come with designated propositional truncations, then we have a reflection

$$r : U \to \mathrm{Prop}$$

that sends $X : U$ to the pair $(\|X\|, p)$ with $p : \mathrm{isProp}\,\|X\|$ coming from the assumption. In the other direction, we have an embedding

$$s : \mathrm{Prop} \to U,$$

given by the projection. For $X : U$ we have

$$s(r(X)) = \|X\|.$$

(We also have that s is a section of r if propositional univalence holds.) For a fixed type $X : U$, the type constructors Π and Σ can be regarded as having type

$$\Pi, \Sigma : (X \to U) \to U.$$

We define

$$\forall, \exists : (X \to \mathrm{Prop}) \to \mathrm{Prop},$$

by, for any $A : X \to \mathrm{Prop}$,

$$\exists(A) = r(\Sigma(s \circ A)),$$
$$\forall(A) = r(\Pi(s \circ A)),$$

which we also write more verbosely as

$$(\exists(x : X).A(x)) = r(\Sigma(x : X).s(A(x))),$$
$$(\forall(x : X).A(x)) = r(\Pi(x : X).s(A(x))).$$

This is essentially the same as the definition in the HoTT book, except that we give different types to $\exists, \forall$. With the type given in the book, $\forall$ gets confused with Π, because, with function extensionality, a product of propositions is a proposition, and so there is no need to distinguish $\forall$ from Π in the book.

The point of choosing the above types is that now it is easy to justify that these quantifiers do satisfy the intuitionistic rules for quantification. It is enough to show that they satisfy Lawvere's adjointness conditions. For $P, Q : \mathrm{Prop}$, define

$$(P \leq Q) = (s(P) \to s(Q)).$$

This is a pre-order (and a partial order if propositional univalence holds). Now endow the function type $(X \to \mathrm{Prop})$ with the pointwise order (using Π to define it). Then the quantifiers $\exists, \forall : (X \to \mathrm{Prop}) \to \mathrm{Prop}$ are the left and right adjoints to the exponential transpose $\mathrm{Prop} \to (X \to \mathrm{Prop})$ of the projection

$$\mathrm{Prop} \times X \to \mathrm{Prop},$$

using the universal property of truncation. The exponential transpose maps P to $\lambda x.P$. Hence the adjointness condition for the existential quantifier amounts to

$$\exists(A) \leq P \iff A \leq \lambda x.P.$$

Expanding the definitions, this amounts to

$$\|\Sigma(x:X).s(A(x))\| \to s(P) \iff \Pi(x:X).s(A(x)) \to s(P),$$
$$\iff (\Sigma(x:X).s(A(x))) \to s(P).$$

So we need to check that

$$\|\Sigma(x:X).s(A(x))\| \to s(P) \iff (\Sigma(x:X).s(A(x))) \to s(P)$$

holds, but this is the case by the defining property of propositional truncation. For the sake of completeness, we also check the adjointness condition

$$P \leq \forall(A) \iff \lambda x.P \leq A.$$

For this we need function extensionality (which follows from the assumption of truncations supporting the judgemental equality discussed above). By definition, this amounts to

$$s(P) \to \|\Pi(x:X).s(A(x))\| \iff \Pi(x:X).s(P) \to s(A(x)).$$

But $\Pi(x:X).s(P) \to s(A(x)) \iff s(P) \to \Pi(x:X).s(A(x))$, and so the above is equivalent to

$$s(P) \to \|\Pi(x:X).s(A(x))\| \iff s(P) \to \Pi(x:X).s(A(x)).$$

But this again holds by the defining property of truncation, because, by function extensionality, a product of propositions is a proposition, and each $s(A(x))$ is a proposition. This explains why $\forall$ is identified with Π in the HoTT book.

Having established that the quantifiers $\exists, \forall : (X \to \text{Prop}) \to \text{Prop}$ defined from Σ and Π with truncation (via the reflection $r : U \to \text{Prop}$) do satisfy the adjointness conditions corresponding to the introduction and elimination rules of intuitionistic logic, in practice we prefer to use the notation of the HoTT book, with $\exists(x:X).A(x)$ defined as $\|\Sigma(x:X).A(x)\|$ for $A:X \to U$, or even avoid $\exists$ altogether, and just use truncation explicitly, as in the next section.

4 Uniform continuity of functions $2^{\mathbb{N}} \to \mathbb{N}$

We now compare the untruncated formulation of the uniform continuity principle

$$\Pi(f\colon 2^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(n\colon\mathbb{N}).\ \Pi(\alpha,\beta\colon 2^{\mathbb{N}}).\ \alpha =_m \beta \to f\alpha = f\beta$$

with its truncated version

$$\Pi(f\colon 2^{\mathbb{N}} \to \mathbb{N}).\ \|\Sigma(n\colon\mathbb{N}).\ \Pi(\alpha,\beta\colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta\|.$$

A formal counter-part in Agda of this section is available at [6].

We work in a type theory with $\mathbb{0}, \mathbb{1}, \mathbb{2}, \mathbb{N}, \Pi, \Sigma, \mathrm{Id}$. This time, identity types for types other than $\mathbb{N}$ are needed, but universes are not. But we need more:

1. In principle, we would have to assume the presence of truncations, for example as defined in the HoTT book and explained in the previous section, from which function extensionality follows [10].

2. However, in turns out that function extensionality alone suffices, because it implies the existence of the propositional truncation mentioned above, and hence we can omit propositional truncations from our type theory. (But it doesn't seem to be possible to remove the assumption of function extensionality in the theorem proved here).

Hence we don't assume propositional truncations in our type theory.

Theorem 3. *Assuming function extensionality, for every $f : 2^{\mathbb{N}} \to \mathbb{N}$ the type*

$$\Sigma(n \colon \mathbb{N}).\ \Pi(\alpha, \beta \colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta$$

has a propositional truncation, and the proposition

$$\Pi(f \colon 2^{\mathbb{N}} \to \mathbb{N}).\ \|\Sigma(n \colon \mathbb{N}).\ \Pi(\alpha, \beta \colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta\|$$

is logically equivalent to the type

$$\Pi(f \colon 2^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(n \colon \mathbb{N}).\ \Pi(\alpha, \beta \colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta.$$

Lemma 1. *Function extensionality implies that, for any $f : 2^{\mathbb{N}} \to \mathbb{N}$, the type family*

$$A(n) = \Pi(\alpha, \beta \colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta$$

satisfies the following conditions:

1. *$A(n)$ is a proposition for every $n \colon \mathbb{N}$, and*
2. *$A(n)$ implies that $A(m)$ decidable for every $m < n$,*

Proof. By Hedberg's Theorem [7], equality of natural numbers is a proposition. Hence so is $A(n)$, because, by function extensionality, a product of a family of propositions is a proposition. To conclude that for all n, if $A(n)$ holds then $A(m)$ is decidable for all $m < n$, it is enough to show that for all n, if $A(n+1)$ holds then $A(n)$ is decidable. For every n, the type

$$B(n) = \Pi(s \colon 2^n).\ f(s0^{\omega}) = f(s1^{\omega}),$$

is decidable, because $\mathbb{N}$ has decidable equality and finite products of decidable types are also decidable. Now let $n : \mathbb{N}$ and assume $A(n+1)$. To show that $A(n)$ is decidable, it is enough to show that $A(n)$ is logically equivalent to $B(n)$, because then $B(n) \to A(n)$ and $\neg B(n) \to \neg A(n)$ and hence we can decide $A(n)$ by reduction to deciding $B(n)$.

The implication $A(n) \to B(n)$ holds without considering the assumption $A(n+1)$. To see this, assume $A(n)$ and let $s : 2^n$. Taking $\alpha = s0^\omega$ and $\beta = s1^\omega$, we conclude from $A(n)$ that $f(s0^\omega) = f(s1^\omega)$, which is the conclusion of $B(n)$.

Now assume $A(n+1)$ and $B(n)$. To establish $A(n)$, let $\alpha, \beta : 2^{\mathbb{N}}$ with $\alpha =_n \beta$. We need to conclude that $f(\alpha) = f(\beta)$. By the decidability of equality of 2, either $\alpha(n) = \beta(n)$ or not. If $\alpha(n) = \beta(n)$, then $\alpha =_{n+1} \beta$, and hence $f(\alpha) = f(\beta)$ by the assumption $A(n+1)$. If $\alpha_n \neq \beta_n$, we can assume w.l.o.g. that $\alpha_n = 0$ and $\beta_n = 1$. Now take $s = \alpha_0 \alpha_1 \ldots \alpha_{n-1} (= \beta_0 \beta_1 \ldots \beta_{n-1})$. Then $\alpha =_{n+1} s0^\omega$ and $s1^\omega =_{n+1} \beta$, which together with $A(n+1)$ imply $f(\alpha) = f(s0^\omega)$ and $f(s1^\omega) = f(\beta)$. But $f(s0^\omega) = f(s1^\omega)$ by $B(n)$, and hence $f(\alpha) = f(\beta)$ by transitivity. $\square$

Lemma 2. *If a type X is logically equivalent to a proposition Q, then*

1. *X has the propositional truncation $\|X\| = Q$, and*
2. *$\|X\| \to X$.*

Proof. We have $X \to \|X\|$ because this is the assumption $X \to Q$. If $X \to P$ for some proposition P, then also $\|X\| \to P$, because this means $Q \to P$, which follows from the assumption $Q \to X$ and transitivity of implication. This shows that our definition of $\|X\|$ has the required property for truncations. And $\|X\| \to X$ is the assumption that $Q \to X$. $\square$

Lemma 3. *Function extensionality implies that for any family A of types indexed by natural numbers such that*

1. *$A(n)$ is a proposition for every $n : \mathbb{N}$, and*
2. *$A(n)$ implies that $A(m)$ decidable for every $m < n$,*

the type $\Sigma(n : \mathbb{N}).A(n)$ is logically equivalent to the proposition

$$P = \Sigma(k : \mathbb{N}).B(k)$$

where

$$B(k) = A(k) \times \Pi(i : \mathbb{N}).A(i) \to k \leq i.$$

Proof. By function extensionality, the product of a family of propositions is a proposition, and hence the type $\Pi(n : \mathbb{N}).A(n) \to k \leq n$ is a proposition, because the type $k \leq n$ is a proposition. Because the product of two propositions is a proposition, the type $B(k)$ is a proposition. But now if $B(k)$ and $B(k')$ then, by construction, $k = k'$. Hence any two inhabitants of P are equal, using the fact that $B(k)$ is a proposition, which means that P is indeed a proposition. By projection, $P \to \Sigma(n : \mathbb{N}).A(n)$. Conversely, if $(n, a) : \Sigma(n : \mathbb{N}).A(n)$ is given, then we can find, by the decidability of $A(m)$ for $m < n$, the minimal k such that there is $b : A(k)$, by search bounded by n, and this gives an element $(k, b, \mu) : P$ where $\mu : \Pi(i : \mathbb{N}).A(i) \to k \leq i$ is the minimality witness. This shows that $\Sigma(n : \mathbb{N}).A(n) \to P$ and concludes the proof. $\square$

Remark 2. Function extensionality in the above lemma can be avoided using the fact that the type of fixed points of a constant endomap is a proposition [9], where a map is constant if any two of its values are equal. Given $(n, a)\colon \Sigma(n\colon\mathbb{N}).A(n)$, we know that $A(m)$ is decidable for all $m < n$ and thus can find the minimal m such that $A(m)$, by search bounded by n, which gives an endomap of $\Sigma(n\colon\mathbb{N}).A(n)$. This map is constant, because any two minimal witnesses are equal, and because $A(n)$ is a proposition. Then we instead take P to be the type of fixed points of this constant map.

By Lemmas 1, 2, and 3, for any $f : 2^{\mathbb{N}} \to \mathbb{N}$, the truncation of the type

$$\mathrm{UC}(f) = \Sigma(n\colon \mathbb{N}).\ \Pi(\alpha, \beta\colon 2^{\mathbb{N}}).\ \alpha =_n \beta \to f\alpha = f\beta$$

exists and implies $\mathrm{UC}(f)$, which establishes Theorem 3.

Unfolding the above construction of the truncation, the truncated version of uniform continuity says that there is, using Σ to express existence, a minimal modulus of uniform continuity, making this use of Σ into a proposition, and, by function extensionality, the statement of uniform-continuity into a proposition too. Then the theorem says that this proposition is logically equivalent to the existence, using Σ again, of some modulus of uniform continuity. This statement is not a proposition, because any number bigger than a modulus of uniform continuity is itself a modulus of uniform continuity.

The situation here is analogous to that of quasi-inverses and equivalences in the sense of the HoTT book. The type expressing that a function has a quasi-inverse is not a proposition in general, but it is equivalent to the type expressing that the function is an equivalence, which is always a proposition. Hence being an equivalence is the propositional truncation of having a quasi-inverse.

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