

# Searchable types in HoTT/UF

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The problem addressed here

$$\begin{aligned} \mathbb{Z} &= \{0, 1\} \\ &\simeq 1 + 1 \neq \mathbb{Z} \end{aligned}$$

Given a set  $X$  and  $p: X \rightarrow \mathbb{Z}$ .

- either exhibit  $x \in X$  such that  $p(x) = 0$  ( $\Rightarrow$  root of  $p$ )
- or else determine that  $P$  has no root.

For which sets  $X$  can this be done?

- In terms of computation, this is a exhaustive search problem.
- In terms of logic, this is a choice problem.
- In terms of topology, this turns out to be a compactness problem.

Can we exhaustively search an infinite set mechanically?

Can we prove non-trivial instances of choice?

# our type theory

Martin-Löf Type Theory

MLTT  $\mathbb{O}, \mathbb{1}, \mathbb{N}, +, \times, \Sigma, \Pi, \text{Id}, \mathcal{M}, W$

+

univalence (So in particular we have functional and propositional extensionality)

+  
set quotients ( $\Leftrightarrow$  propositional truncations + set replacement)

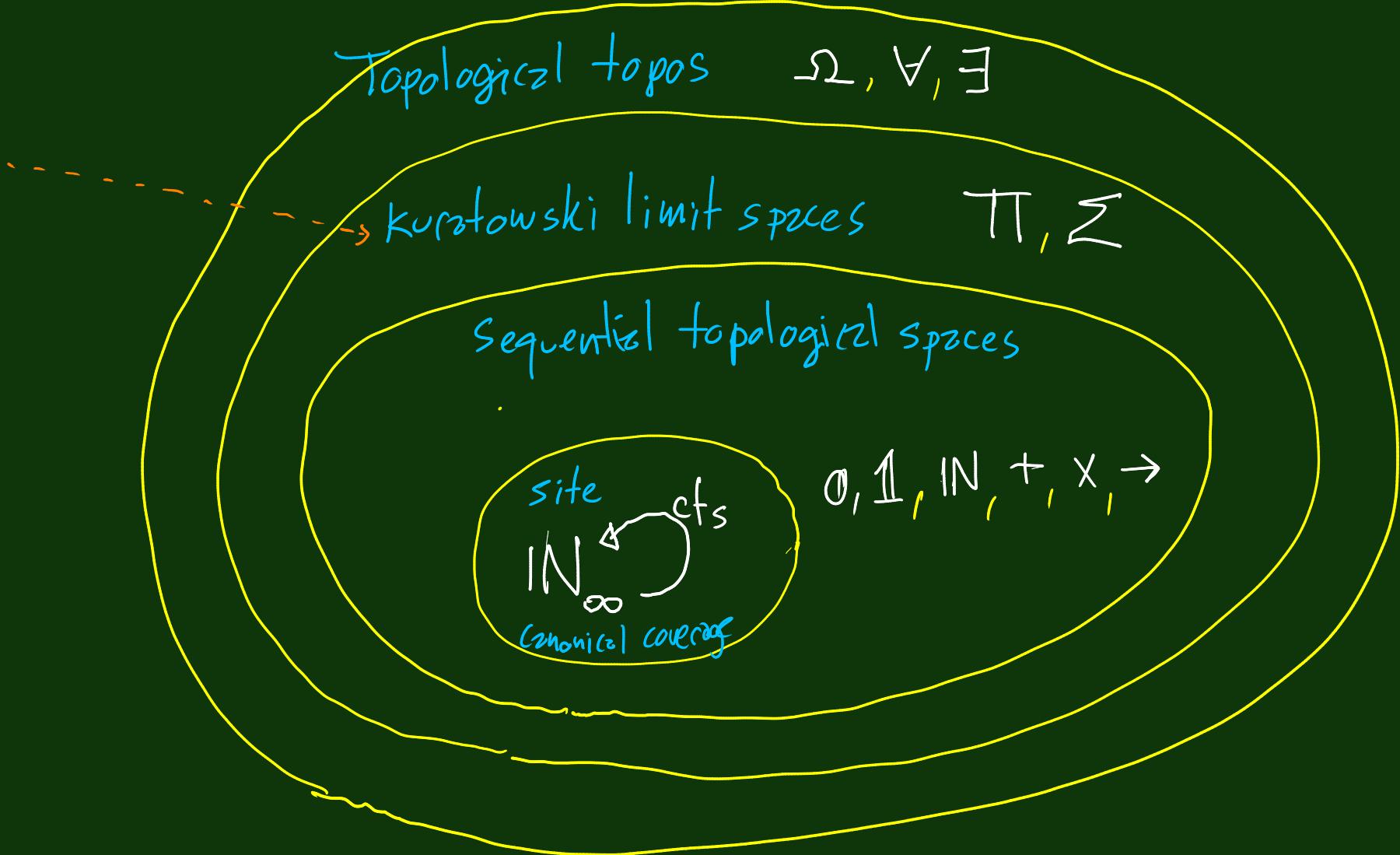
Many Models

- Types are sets.
- Types are spaces.
- Types are "sets with computational structure" (realizability).
- Types are the objects of a topos.
- Types are homotopy types.

We reason constructively, so:  
Our results hold in all models.

[One particular model plays a guiding role]

concrete  
sheaves



Johnstone  
1979

Examples of MLTT definable objects in that topos

- $\mathbb{N}$  and  $\mathbb{Z} \stackrel{\text{def}}{=} \mathbb{1} + \mathbb{1}$  get the discrete topology.
  - $\mathbb{N} \rightarrow \mathbb{2}$  is the Cantor Space, and  $\mathbb{N} \rightarrow \mathbb{N}$  is the Baire space.
  - $\mathbb{N}_\infty \stackrel{\text{def}}{=} \sum_{\alpha: \mathbb{N} \rightarrow \mathbb{2}} \prod_{i: \mathbb{N}} \alpha_i \geq \alpha_{i+1}$  is the one-point compactification of  $\mathbb{N}$ .
  - $\sum_{x: \mathbb{N}_\infty} ((x = \infty) \rightarrow \mathbb{2})$  looks like this
- $$\begin{array}{ccccccc} \vdots & \frac{1}{\cdot} & \frac{2}{\cdot} & \frac{3}{\cdot} & \frac{4}{\cdot} & \dots & \frac{\infty}{\cdot} \\ & \downarrow & & & & & \end{array} \quad \begin{array}{c} \vdots \\ \infty \end{array}$$
- $$\begin{array}{c} \underline{n} \stackrel{\text{def}}{=} 1^n 0^w \\ \infty \stackrel{\text{def}}{=} 1^w \end{array} \quad \begin{array}{c} \mathbb{N} \hookrightarrow \mathbb{N}_\infty \\ n \mapsto \underline{n} \end{array}$$
- This is compact  $T_1$  but not Hausdorff.
- We have  $\{0, 1, \dots, \infty\} \cap \{0, 1, \dots, \infty\} = \mathbb{N}$

$\xrightarrow{\text{compact}}$        $\xrightarrow{\text{not compact}}$

Mathematical expression of the problem in our system

We can pick  
a root of p  
if it has any.

$$\text{TP}: X \rightarrow 2, (\sum_{x:X} p x = 0) + (\prod_{x:X} p x = 1)$$

$$\models \sum_{x:X} p x = 0$$

- Stranger than excluded middle.
- We are making a choice.

We have  $\sum$  rather than  $\exists$ .

| We ask which types  $X$  satisfy this choice principle.  
| Definition. We call such types compact.

All types are compact  $\Leftrightarrow$  global choice holds

Global choice: We can choose a point of every non-empty type.

$$\prod_{X:\mathcal{U}} \underbrace{\exists X}_{X \text{ is non-empty}} \rightarrow X$$

E.g. Voevodsky's model of simplicial sets

- Stronger than choice, which is consistent with univalence.
- Contradicts univalence.
- But there are plenty of compact types in HoTT/UF.
- The ones we can construct are all equipped with well-orders.

## Ordinals

$X$  equipped with a proposition-valued relation  $\lessdot$  satisfying

1.  $\lessdot$  is transitive

2. If two points have the same predecessors then they are equal.

3.  $\lessdot$  satisfies transfinite induction

$$\left( \prod_{x:X} \left( \prod_{y:X} (y \lessdot x \rightarrow P_y) \rightarrow P_x \right) \right)$$
$$\rightarrow \prod_{x:X} P_x$$

•  $X$  is automatically a set by (2) (its identity types are propositions)

• Trichotomy  $x < y \vee x = y \vee x > y$  is equivalent to excluded middle.

• But there are lots of well-ordered types that are trichotomous

The large type of all small ordinals

Univalence implies that this type

1. Is a (large) ordinal,
2. Has suprema of arbitrary small families.

## Functions $p:X \rightarrow \mathbb{Z}$

They classify complemented subtypes of  $X$ .

$$X \simeq \left( \sum_{x:X} p x = 0 \right) + \left( \sum_{x:X} p x = 1 \right).$$

$\begin{array}{ccc} Y & \longrightarrow & 1 \\ \downarrow & & \downarrow 0 \\ X & \xrightarrow{p} & 2 \end{array}$

complemented

In models:

- Topological topos. They classify open subspaces.
- Realizability toposes. They classify decidable subobjects.

complemented c.e. subobjects with c.e. complement.

## Totally separated types

Recall

Definition. A type  $X$  is called compact if

$$\prod p:X \rightarrow \mathbb{Z}, (\sum_{x:X}, px=0) + (\prod_{x:X}, px=1).$$

This definition is not good unless there are plenty of maps  $X \rightarrow \mathbb{Z}$ .

Definition. A type  $X$  is called totally separated if

$$\prod_{x,y:X}, (\prod p:X \rightarrow \mathbb{Z}, px=py) \rightarrow x=y.$$

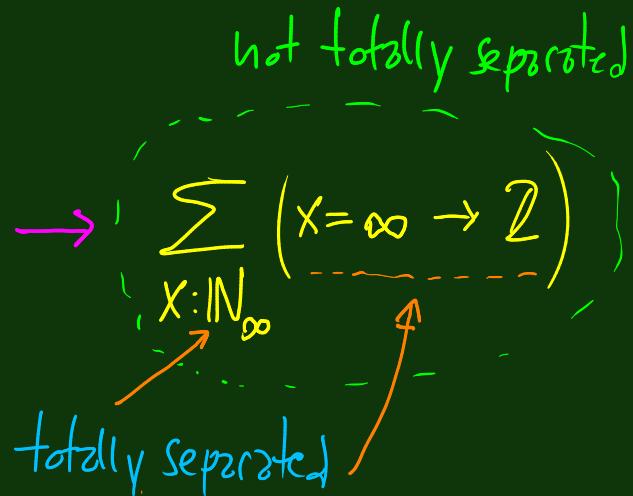
In the topological topos. The clopens separate the points.  
(Topological notion with the same name.)

## Some facts

1. Totally separated types are sets (their identity types are propositions)
2. They form an exponential ideal (more generally a "T1-ideal") and are closed under  $+$ ,  $\times$ , retracts and include  $0, 1, 2, \mathbb{N}, \mathbb{N}_\infty$  and all discrete types (those with decidable equality).
3. They are not closed under  $\Sigma$  in general.

**Example.** In the topological topos, the type  $\sum_{X:\mathbb{N}_\infty} (x = \infty \rightarrow 2)$  is not totally separated.

(Compact totally separated spaces are Hausdorff. Also known as Stone spaces.)



4. Define the simple types to be the smallest collection of types including  $\emptyset, \mathbb{1}, \mathbb{N}$  and closed under  $\times, +, \rightarrow$ .

The simple types are all totally separated (by  $(2)$  above).

5. In the topological topos, a subtype of a simple type is compact in the above type-theoretic sense iff it is compact in the topological sense.

In this case the inclusion is a section and hence the subtype is itself totally separated.

A so-called constructive taboo.

- The set  $\mathbb{N}$  of natural numbers **fails** to be compact
- The compactness of  $\mathbb{N}$  amounts to Bishop's LPO  
(Limited Principle of Omniscience).

- More precisely, LPO is independent of MLTT
- False in realizability models (not computable)  
in topological models (not continuous)
  - True in the model of classical sets (by choice)

Probably the simplest infinite example

$$\mathbb{N}_\infty := \sum \alpha = 2^{\mathbb{N}}, \prod i : \mathbb{N}, \alpha_i \geq \alpha_{i+1}$$

That is, the type of decreasing binary sequences.

$$\underline{n} := 1^n 0^\omega$$

$$\infty := 1^\omega$$

Theorem of HoTT/UF

The type  $\mathbb{N}_\infty$  is compact.

(JSL '2013)

↳ Done in a weaker system  
(Gödel's system T)

We have an injection  $\mathbb{N} \rightarrow \mathbb{N}_\infty$

$$n \mapsto \underline{n}$$

## Proof sketch (with the difficult part omitted)

- Given  $p: \mathbb{N}_\infty \rightarrow \mathbb{Z}$ , (not assumed be continuous)  
define  $\beta_n = \min(p_0, p_1, \dots, p_n)$  Formulas for the infimum of the set of roots.
- This is clearly decreasing.
- Now we check whether  $p\beta=0$  or  $p\beta=1$ .
  - If  $p\beta=0$  then we've found a root.
  - If  $p\beta=1$  then  $p\alpha=1$  for all  $\alpha: \mathbb{N}_\infty$  and so there is no root. (This is easy classically and less so constructively.)

In the pub  $\mathbb{N}_\infty$  there is a person  $\beta: \mathbb{N}_\infty$  such that if  $\beta$  drinks, then everybody drinks.

Some consequences | (decision procedures)

(1) For every  $p : \mathbb{N}_\infty \rightarrow 2$  either  $\prod_{n:\mathbb{N}} p_n = 1$  or  $\neg \prod_{n:\mathbb{N}} p_n = 1$   
 (JSL'2013)

Quantification over the natural numbers ! Not over  $\mathbb{N}_\infty$ .

(2) Given  $f : \mathbb{N}_\infty \rightarrow \mathbb{N}$ , we can decide whether it is not continuous.

(3) There is some discontinuous  $f : \mathbb{N}_\infty \rightarrow \mathbb{N}$  iff WLPO holds

(Bishop's principle of Weak Limited omniscience,  $\prod p : \mathbb{N} \rightarrow 2, (\prod n. p_n = 1) + \neg (\prod n. p_n = 1)$   
 which is also independent of MLTT)

(MSCS'2015)

## Some applications of the compactness of $\mathbb{N}_\infty$

1. Pierre Predic & Chsd E. Brown. Arxiv '2019  
Cantor-Bernstein implies excluded middle  
arxiv 1904.09193  
(Also implemented in Coq-)

2. Dag Normann & William Tait. Springer '2017  
On the Computability of the Fan Functional  
(They use the system T compactness of  $\mathbb{N}_\infty$   
to fill a gap in an unpublished but widely  
circulated 1958 manuscript by Tait.)

[ Compact sets in our type theory ]

- (1)  $\emptyset, \perp$  and  $\mathbb{N}_\infty$  are compact . Baby Tychonoff.
- (2) If  $X$  and  $Y$  are compact then so are  $X+Y$  and  $\overbrace{X \times Y}$ .
- (3) If  $X$  is a compact set and  $A$  is a family of compact sets indexed by  $X$ , then its disjoint union  $\sum_{x:X} A_x$  is a compact set.
- (4) If furthermore
  - (a) we have a function that picks an element of  $A_x$  for any given  $x:X$ , and
  - (b) the set  $X$  has at most one element,
 then the cartesian product  $\prod_{x:X} A_x$  is compact . Micro-Tychonoff.

Does arbitrary Tychonoff hold ?

Is the Cantor type  $\mathbb{N} \rightarrow 2$  <sup>(probably)</sup> compact in our type theory ?

- The compactness of  $\mathbb{N} \rightarrow 2$  is independent.

| No. |

- True in the topological topos ( notions of compactness coincide )
- False in Hyland's effective topos ( Kleene tree to blame )  
( realizability topos over Kleene's  $K_1$  )
- True in the Kleene-Vesley topos  
( realizability over Kleene's  $K_2$  )

Perhaps amazingly, these two toposes have the same simple types.  
( more precisely, the full subcategories on the objects that arise as the interpretation of the simple types are equivalent. )

## Building more compact sets

- The compact sets that we have constructed so far are all well-ordered.

- (1)  $\emptyset$
- (2)  $\mathbb{N}_\infty$
- (3)  $X+Y$
- $X \times Y$
- (3)  $\sum_{x:X} L_x$

lexicographic order

Problem with  $\Sigma$

- But we can't get very high ordinals with just the above.

- This is what we address next

after we address this

Ordinal sums

$\alpha : \text{Ord}$

$\beta : \langle \alpha \rangle \rightarrow \text{Ord}$

$$(x, y) < (x', y') \stackrel{\text{def}}{=} (x < x') + \sum_{p: x = x'} \text{transport } p \text{ } y = y'$$

---

E.g.  $\alpha \stackrel{\text{def}}{=} \omega$

:

$\beta_2$

|

$\beta_1$

|

$\beta_0$

|

## Problem with ordinal sums

The lexicographic order is not extensional in general  
Can derive excluded middle ↗

If is extensional if

This is very lucky because our  
compact ordinals do have top.

- 1) The given orders have top elements - then so does the sum
- or 2) The given orders are trichotomous - then so is the sum
- or 3) The given orders are cotransitive - but I have no reason to believe  
that so would be the sum
- or 4) Excluded middle holds (by (2))

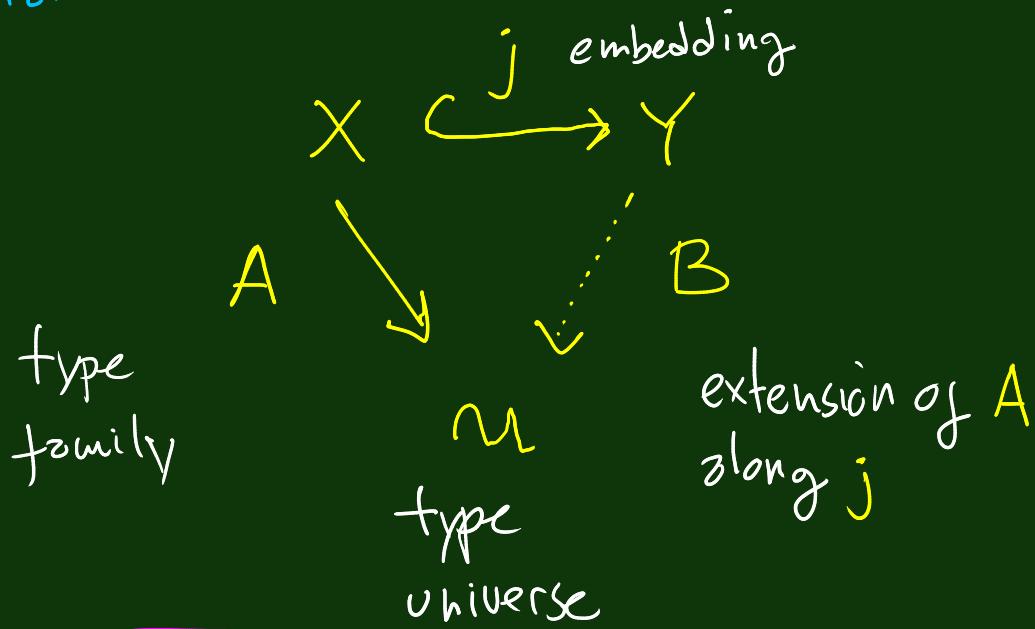
## Supremum of ordinals

- Does every family  $\alpha : I \rightarrow \text{Ord}$  have a supremum constructively?
- As far as I know, this hadn't been answered before.  
Left open by Forsberg, Kraus & Xu (MFCS'21).
- Two answers, by Tom de Jong & myself independently.  
Both implemented in Agda by Tom.
- Mine is as follows. Define  $\sum_{i:I} \langle \alpha_i \rangle \rightarrow \text{Ord}$   
 $(i, x) \mapsto \alpha_i \downarrow x$

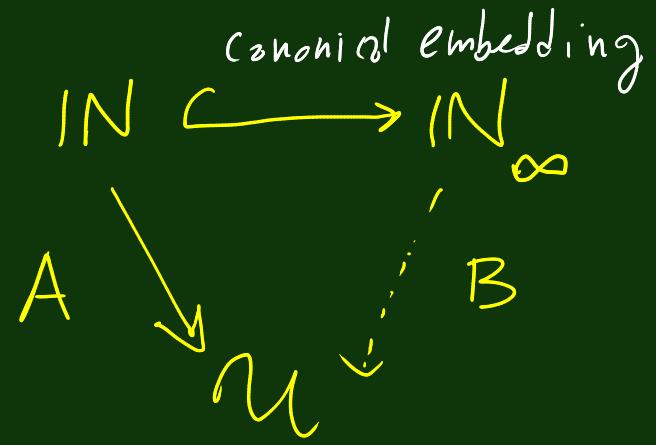
Then the supremum is just the image of this function.  
Although  $\text{Ord}$  is large, the image is small.

# Extending families of compact sets

General situation:



Interested in:



Want: If  $\prod_x A_x$  compact  
for every  $x: X$ , then  $\prod_y B_y$  compact.  
for every  $y: Y$ .

Because then: By (3), if  $Y$  is also  
compact, then  $\sum_{y: Y} B_y$  compact too.

$$j^*(y) = \sum_{x:x} j_x = y$$

Family extension problem

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & Y \\ A & \downarrow & B \\ M & \curvearrowright & \end{array}$$

( MSCS'2021. "Injective types in univalent mathematics")

This set has at most one element.  
(because  $j$  is an embedding)

Smallest solution (left kan extension):  $B_y := \sum_{(x_1): j^{-1}(y)} A_x$

Largest solution (right kan extension):  $B_y := \prod_{(x_1): j^{-1}(y)} A_x$

→ If this works for the wish of the previous board. ] why? By Micro-Tychonoff

## Summary of the previous reasoning

$$X \xrightarrow{j \text{ given}} Y$$

given

$$A \downarrow \begin{matrix} \vdots \\ M \end{matrix} \quad B_y := \overline{\bigcap_{x: j^{-1}(y)} A_x}$$

Special case  
of interest:

$$\mathbb{N} \hookrightarrow \mathbb{N}_\infty$$

$$A \downarrow \begin{matrix} \vdots \\ M \end{matrix} \quad B$$

**Theorem** If the set  $A_x$  is compact for every  $x: X$ , then the set  $B_y$  is compact for every  $y: Y$ .

**Corollary** If additionally  $Y$  is compact, then so is  $\sum_{y: Y} B_y$ .

In the special case of interest we have  $B(\infty) \simeq 1$

But this is not enough

The above says that type universes  $\mathcal{U}$  are injective

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ A \downarrow & \curvearrowleft \bar{A}(y) = \overline{\pi} & A x \\ \mathcal{U} & \curvearrowright & (x, -) : j^{-1}(y) \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ \alpha \downarrow & \curvearrowleft \bar{\alpha} & \\ \text{ord}_{\mathcal{U}} & & \end{array}$$

We need to order  $\alpha(y)$ .

We define, for  $\alpha, \beta : (\lambda y)$ ,  
 $\alpha < \beta = \overline{\pi} \alpha \sigma < \beta \sigma$   
 $\sigma : f^{-1}(y)$

We also need:

Theorem. The type of (topped) ordinals also is injective.

Proof That of the injectivity of  $\geq$  universe for + additional construction of the order + checking it works.  $\square$

Next

- Compact ordinals induced by Brouwer ordinal expressions.
- A generalization to a universe is Tarski of compact ordinals.

## Brouwer ordinal codes

They seem to be due to Hilbert!

A type  $B$  inductively defined by Constructors

$Z : B$

"Zero"

$S : B \rightarrow B$

"Successor"

$L : (\mathbb{N} \rightarrow B) \rightarrow B$

"Limit"

# Four interpretations of Brouwer codes as ordinals

0)

$$[\![z]\!] = \emptyset$$

$$[\![Sb]\!] = [\![b]\!] + 1$$

$$[\![Lb]\!] = \sup_i [\![b_i]\!]$$

standard interpretation

1)

$$[\![z]\!] = \emptyset$$

$$[\![Sb]\!] = [\![b]\!] + 1$$

$$[\![Lb]\!] = \sum_i [\![b_i]\!]$$

trichotomous interpretation

2)

$$[\![z]\!] = 1 \leftarrow !$$

$$[\![Sb]\!] = [\![b]\!] + 1$$

$$[\![Lb]\!] = \sup_i \overline{[\![b_i]\!]}$$

compact interpretation

3)

$$[\![z]\!] = 1 \leftarrow !$$

$$[\![Sb]\!] = [\![b]\!] + 1$$

$$[\![Lb]\!] = \sum_i \overline{[\![b_i]\!]}$$

overline means extension to  $\mathbb{N}_\infty$   
by infectivity

compact totally separated  
interpretation

Assuming excluded middle

which admittedly is not very useful for what we're investigating

standard

$$[\![ b ]\!]_{\sup} \leq [\![ b ]\!]_{\sum}$$

$\wedge$

compact

$$[\![ b ]\!]_{\sup} \leq [\![ b ]\!]_{\sum}$$

$\wedge$

trichotomous

In the next page we see what happens constructively here.  
compact totally separated

why do we need excluded middle?

Because  $(-) + 1$  is monotone  $\Leftrightarrow$  excluded middle holds.

## Theorems

The ordinal

$$[\![ b ]\!]_{\Sigma}$$

- is discrete, in fact trichotomous
- is  $\geq$  retract of  $\mathbb{N}$
- So countable
- Not compact unless LPO holds

The ordinal

$$[\![ b ]\!]_{\Xi}$$

- is Compact
- is a retract of  $\mathbb{N} \rightarrow 2$
- so totally separated
- is not countable unless LPO holds
- is not discrete unless LPO holds

Even better:  
Every decidable  
subset is either  
empty or has  
at least one element.

- There is an order-preserving-reflecting embedding

$$[\![ e ]\!]_1 \hookrightarrow [\![ e ]\!]^1$$

In models:

The embedding doesn't have a computable/continuous inverse.

whose image has empty complement.

- LPO  $\Rightarrow$  this embedding is a bijection  $\Rightarrow$  WLPO.

| Illustration | The ordinal  $\omega+1$ .



- Discrete
- compact iff LPO
- countable

Every decreasing sequence  
is of one of the forms

$1^n 0^\omega$  and  $1^\omega$ .

- compact
  - discrete iff WLPO
  - countable iff LPO
  - bijection iff LPO,
  - but its image has empty complement.
- There is no decreasing sequence other than  $1^\omega_0$  and  $1^\omega$ .

Universes is to Tarski of compact ordinals

We define  $E : \mathcal{U}_o$

$\Delta : E \rightarrow \text{Ord}^{\text{Top}}$  trichotomous

by induction-excursion. Then we define, by recursion,

$K : E \rightarrow \text{Ord}^{\text{Top}}$  compact  
(totally separated?)

We define  $E$ , using  $\Delta$ , inductively by the following constructors:

$$\begin{array}{c|c|c} \lceil \Pi \rceil : E & -\lceil + \rceil - : E \rightarrow E \rightarrow E & \lceil \sum \rceil : (e : E) \rightarrow (\langle \Delta e \rangle \rightarrow E) \rightarrow E \\ \lceil \omega + 1 \rceil : E & -\lceil \times \rceil - : E \rightarrow E \rightarrow E & \end{array}$$

$$E : \mathcal{U}_o$$

$$\Delta : E \rightarrow \text{Ord}^{\text{Top}}$$

$$K : E \rightarrow \text{Ord}^{\text{Top}}$$

$$i : (e : E) \rightarrow \langle \Delta e \rangle \rightarrow \langle Ke \rangle$$

$$l\text{-emb} : (e : E) \rightarrow \text{is-embedding } (\langle e \rangle)$$

$$\begin{array}{c|c|c} \lceil 1 \rceil : E & -\lceil + \rceil - : E \rightarrow E \rightarrow E & \lceil \sum \rceil : (e : E) \rightarrow (\langle \Delta e \rangle \rightarrow E) \rightarrow E \\ \lceil w+1 \rceil : E & -\lceil \times \rceil - : E \rightarrow E \rightarrow E & \end{array}$$

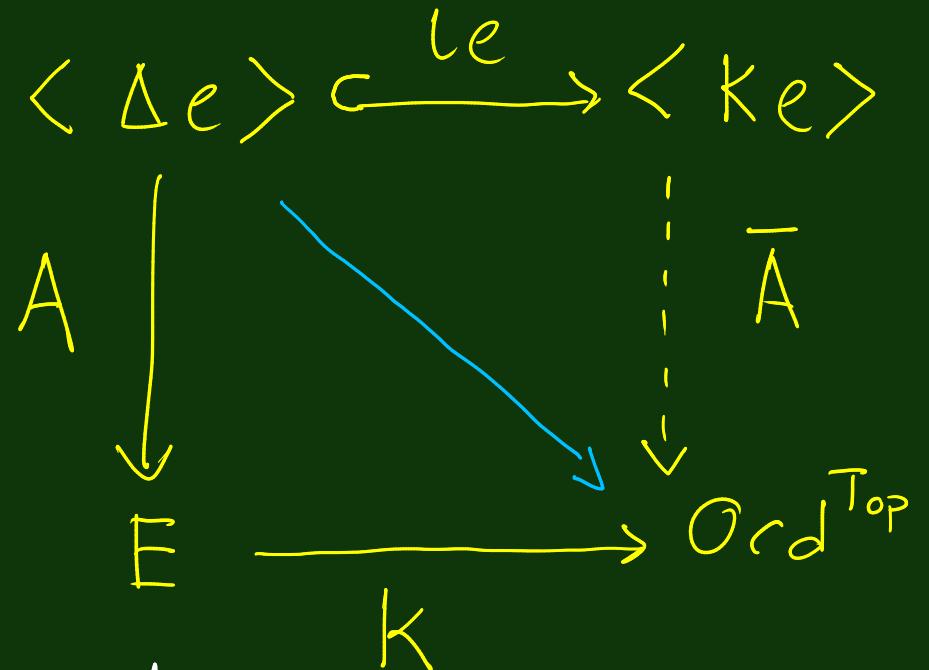
$\Delta \lceil \text{brocoli} \rceil = \text{brocoli}$  recursively.

$K$  is defined as  $\Delta$  except that rather than

$$\Delta(\lceil \sum e A \rceil) = \sum_{x : \langle \Delta e \rangle} \Delta(Ax)$$

We define

$$K(\lceil \sum e A \rceil) = \sum_{x : \langle Ke \rangle} K(\bar{A}x) \quad \leftarrow \text{What is } \bar{A} ?$$



We omit the definition  
of  $\iota$  and the proof that  
it is an embedding.

Answer: extension by  
infectivity. (We need right kan)

The End