A note on intersections of compact collections of open sets

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A topological space is *core-compact* iff each neighbourhood V of a point x contains a neighbourhood U of x with the property that every open cover of V has a finite subcover of U. This is equivalent to saying that its lattice of open sets is continuous in the sense of Dana Scott [2]. For example, locally compact Hausdorff spaces are core-compact. We prove the following.

> *Let $\mathcal{Q}$ be a collection of open sets of a core-compact space X. If $\mathcal{Q}$ is compact in the Scott topology of $\mathcal{O}\,X$, then $\bigcap \mathcal{Q}$ is open.*

We have never seen a formulation of this fact, but it ought to be known. Our proof is by abstract nonsense. As a preparation, we recall some basic definitions and facts. A topological space X is *exponential* iff the functor $X \times (-)$ has a right adjoint $(-)^X$. The exponential spaces are characterized precisely as the core-compact spaces [1]. If X is an exponential space and Y is an arbitrary space, the topology of Y^X is the Isbell topology, which is finer than the compact-open topology (but not always strictly finer). The *Sierpinski space*, denoted by $\mathbb{S}$, is the two-point lattice $\{0, 1\}$ under the Scott topology. Thus, $\{1\}$ is the only non-trivial open set and hence the continuous maps of a space X into $\mathbb{S}$ are the characteristic functions of opens of X. For any exponentiable space X, the function space $\mathbb{S}^X$ is homeomorphic to the lattice $\mathcal{O}\,X$ of open sets under the Scott topology. We are now ready to prove the proposition.

PROOF The evaluation map $(p, x) \mapsto p(x) : \mathbb{S}^X \times X \to \mathbb{S}$, being the transpose of the identity function $\mathbb{S}^X \to \mathbb{S}^X$, is continuous. Hence so are evaluation $X \times \mathbb{S}^X \to \mathbb{S}$ and its transpose $X \to \mathbb{S}^{\mathbb{S}^X}$. Identifying $\mathbb{S}^X$ with $\mathcal{O}\,X$ under the Scott topology, this gives a continuous map $N : X \to \mathbb{S}^{\mathcal{O}\,X}$ with $N(x)(U) = 1$ iff $x \in U$. Now, the set $\mathcal{V} = \{P \in \mathbb{S}^{\mathcal{O}\,X} \mid \mathcal{Q} \subseteq P^{-1}\{1\}\}$ is open (in the compact-open topology), and $x \in N^{-1}(\mathcal{V})$ iff $N(x) \in \mathcal{V}$ iff $N(x)(U) = 1$ for all $U \in \mathcal{Q}$ iff $x \in U$ for all $U \in \mathcal{Q}$ iff $x \in \bigcap \mathcal{Q}$. Therefore $\bigcap \mathcal{Q} = N^{-1}(\mathcal{V})$, which shows that it is open. $\square$

References

[1] B.J. Day and G.M. Kelly. On topological quotient maps preserved by pullbacks and products. *Mathematical Proceedings of the Cambridge Philosophical Society*, 67:553–558, 1970.

[2] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

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