Alternate estimators for H

The $m(q,\Delta)$ estimator for fBM

Let $X_t = \sigma B_t^H$, and set $SS_n^{(q)} := \frac{1}{n}\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^q$. Then

$$\mathbb{E}(SS_n^{(q)}) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^q) \;=\; \Delta^{qH}\frac{1}{n}\sum_{i=1}^n \mathbb{E}(|X_i - X_{i-1}|^q) \;=\; \sigma^q\mathbb{E}(|Z|^q)\Delta^{qH} \;=\; \sigma^q K_q \Delta^{qH} \quad (1)$$

where $\Delta = \frac{1}{n}$, $K_q = \mathbb{E}(|Z|^q) = \frac{2^{q/2}}{\sqrt{\pi}}\Gamma(\frac{q+1}{2})$, (for $q > -1$) and $Z \sim N(0,1)$. This naturally leads to the estimates $(\hat{H}_n, \hat{\sigma}_n)$ for (H,σ) defined by

$$SS_n^{(q)} \;=\; \hat{\sigma}_n^q\, K_q \Delta^{q\hat{H}_n}$$

if we have computed $SS_n^{(q)}$ for at least two Δ-values. Taking logs we see that

$$\log SS_n^{(q)} \;=\; q\log\hat{\sigma}_n + \log K_q \;+\; q\hat{H}_n \log\Delta$$

so we can perform **linear regression** on $\log SS_n^{(q)}$ vs $\log\Delta$ for a range of Δ-values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal $q\hat{H}_n$ (q is chosen by you, e.g. $q = 1, 2, 2.5, 3$ etc), and the **intercept** at $\log\Delta = 0$ is $q\log\hat{\sigma}_n + \log K_q$, from which we can compute $\hat{\sigma}_n$ since K_q has an explicit formula above. This is the $m(q,\Delta)$ estimator discussed in [GJR18]. One can then also compute the R^2**-statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of $\hat{H}_n$ and $\hat{\sigma}_n$.

To approximate the effect of using **realized variance** with m subwindows to estimate V_t (as in Part 2) , we can use the Central Limit Theorem approximation from FM02:

$$V_{i\Delta} \;=\; V_0 e^{\sigma B_{i\Delta}^H}(1 + \sqrt{\frac{2}{m}}\,\varepsilon_i)$$

where the ε_i's are i.i.d. $N(0,1)$ (and independent of B^H), so (using that $\log(1+x) = x + O(x^2)$), we see that $\log V_{i\Delta} = \log V_0 + \sigma B_{i\Delta}^H + \sqrt{\frac{2}{m}}\varepsilon_i + O(\frac{1}{m}) = \log V_0 + X_{i\Delta} + \sqrt{\frac{2}{m}}\varepsilon_i + O(\frac{1}{m})$.

For convenience we now define $\tilde{X}_{i\Delta} = X_{i\Delta} + \sqrt{\frac{2}{m}}\,\varepsilon_i$. Then $\tilde{X}_{i\Delta} - \tilde{X}_{(i-1)\Delta} = X_{i\Delta} - X_{(i-1)\Delta} + \sqrt{\frac{2}{m}}(\varepsilon_i - \varepsilon_{(i-1)}) \sim N(0, \sigma^2\Delta^{2H} + \frac{4}{m})$, and setting $\widetilde{SS}_n^{(2)} := \frac{1}{n}\sum_{i=1}^n |\tilde{X}_{i\Delta} - \tilde{X}_{(i-1)\Delta}|^2$, adding the effect of ε into the computation above we find that

$$\mathbb{E}(\widetilde{SS}_n^{(2)}) \;=\; \sigma^2\Delta^{2H} + \frac{4}{m}$$

(since $K_q = 1$ for $q = 2$) so we now regress $\log(\widetilde{SS}_n^{(2)} - \frac{4}{m})$ vs $\log\Delta$, which provides a smart adjustment to $\hat{H}_n$ (this adjustment is made in
`https://colab.research.google.com/drive/1jJGf4bVWETJqWRIMZ6STjsJ9jINgEaPd#scrollTo=e9fn6ABHNf52`

The $m(q,\Delta)$ estimator for the RL process

If $X_t = \sigma Z_t^H$ where $Z_t^H = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dW_s$ is an RL process, then for $q = 2$ and $H \in (0, \frac{1}{2})$ one can show that

$$\mathbb{E}(SS_n^{(2)}) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^2) \;=\; \sigma^q\Delta^{2H}\frac{1}{n}\sum_{i=1}^n \mathbb{E}(|Z_i^H - Z_{i-1}^H|^2) \;\sim\; \sigma^2 c_H \Delta^{2H}$$

as $n \to \infty$ where $c_H = -\frac{4H\Gamma(\frac{1}{2}+H)\Gamma(-2H)}{\Gamma(\frac{1}{2}-H)} > 0$. Hence for the regression we now consider

$$\log(SS_n^{(2)}) \;=\; 2\log\hat{\sigma}_n \;+\; \log c_{\hat{H}} \;+\; 2\hat{H}_n\log\Delta$$

so now the intercept is $2\log\hat{\sigma}_n \;+\; \log c_{\hat{H}}$, which leads to an adjusted estimate for $\hat{\sigma}_n$

The Han-Schied [HS21] estimator

Let $X_t = \sigma B_t^H$ and let

$$\theta_{m,k} \;=\; 2^{\frac{m}{2}}(2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}) \;=\; -2^{\frac{m}{2}}(X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{2k}{2^{m+1}}})$$

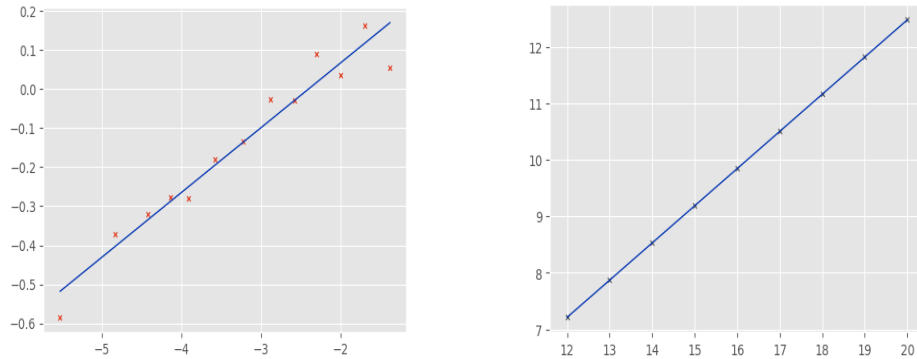

Figure 1: On the left, we see estimates of H for the SPX using the $m(q,\Delta)$ method for the SPX from 3rdJan22-15thJul24 for $q=2$ for which $\hat{H}=0.0830$ and $\hat{\sigma}=1.221$ (see similar plots in [GJR18]. On the right we see the linear regression in (4) for the Han-Schied method ($\log s_n$ vs n) for a true fBM path with 2^{20} time points, for which $\hat{H}=0.0508$, and $\hat{\sigma}=1.010$.

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for $R(s,t)=\mathbb{E}(B_s^H B_t^H)$, one can check that

$$\mathbb{E}(\theta_{m,k}^2) \quad = \quad \sigma^2 2^{m-2H(1+m)}(4-4^H)\,. \tag{2}$$

Then setting $s_n^2=\sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\theta_{m,k}^2$, we see that $\mathbb{E}(s_n^2)=\sum_{m=0}^{n-1}2^m\mathbb{E}(\theta_{m,k}^2)$ (since (2) does not depend on k) which simplifies to

$$\mathbb{E}(s_n^2) \quad = \quad \sigma^2(4^{n(1-H)}-1) \quad \sim \quad \sigma^2 4^{n(1-H)} \quad = \quad \sigma^2 2^{2n(1-H)} \tag{3}$$

as $n\to\infty$, which suggests an estimator $\hat{H}_n$ defined by $s_n=\hat{\sigma}_n\,2^{n(1-\hat{H}_n)}$ which (assuming $\hat{\sigma}_n=O(1)$ as $n\to\infty$) we can re-arrange as

$$\hat{H}_n \quad = \quad 1-\frac{1}{n}\log_2(\frac{s_n}{\hat{\sigma}_n}) \quad = \quad 1-\frac{1}{n}\log_2 s_n \;+\; O(\frac{1}{n})$$

where $\log_2$ denotes the base-2 logarithm, so (ignoring the $O(\frac{1}{n})$ remainder term), we recover the Han-Schied[HS21] estimator $\hat{H}_n=1-\frac{1}{n}\log_2 s_n$. Then

$$\begin{aligned}
\mathbb{E}(\hat{H}_n) \quad = \quad 1-\frac{1}{n}\mathbb{E}(\log_2(s_n)) \quad = \quad 1-\frac{1}{2n}\mathbb{E}(\log_2(s_n^2)) \quad \geq \quad 1-\frac{1}{2n}\log_2\mathbb{E}(s_n^2) \quad &= \quad 1-\frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)}-1)) \\
&\geq \quad 1-\frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)})) \\
&= \quad 1-\frac{1}{2n}\log_2(\sigma^2) \;-\; \frac{1}{2n}\log_2(4^{n(1-H)})) \\
&= \quad H-\frac{1}{2n}\log_2(\sigma^2)
\end{aligned}$$

and the final line is $>H$ if $\sigma<1$, so $\mathbb{E}(\hat{H}_n)>H$ if $\sigma<1$. See also discussion on optimal scaling factors in [HS21].

For more a general process X, (under certain conditions) [HS21] show that $\hat{H}_n\to R$ as $n\to\infty$, where $R=1/q$, where q is the critical p-value at which $\lim_{n\to\infty}\sum_{i=1}^n|X_{i\Delta}-X_{(i-1)\Delta}^H|^p$ switches from being zero (for $p>q$) to $+\infty$ (for $p<q$) (see Eq 2.2 in [HS21]).

Or we can jointly estimate H and σ by performing linear regression since

$$\log s_n \quad = \quad \log\hat{\sigma}_n \;+\; n(1-\hat{H}_n)\log 2 \tag{4}$$

but we now have to compute $\log s_n$ for a range of different n-values to get a line of best fit, for which the slope is $(1-\hat{H}_n)\log 2$ and the intercept is $\log\hat{\sigma}_n$.

You can then draw histograms of $\hat{H}_n$ if you simulate M fBM paths and compute the sample variance for $\hat{H}_n$ (or a confidence interval), or compute $\hat{H}_n$ for real data, e.g. using the SPX data file or data from yahoo finance.

The rough Heston model

The driftless rough Heston model satisfies

$$V_t \;=\; V_0 \,+\, \nu \int_0^t (t-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \,.$$

Then $\mathbb{E}(V_t) = V_0$, and V has covariance function:

$$\begin{aligned}
\mathbb{E}((V_s - V_0)(V_t - V_0)) &= \nu^2 \mathbb{E}(\int_0^s (s-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \cdot \int_0^t (t-r)^{H-\frac{1}{2}} \sqrt{V_r} dW_r) \\
&= \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \mathbb{E}(V_u) du \\
&= V_0 \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du \;=\; V_0 \nu^2 \bar{R}(s,t)
\end{aligned}$$

for $0 \le s \le t$, where $\bar{R}(s,t)$ is the covariance function for the Riemann-Liouville (RL) process $Z_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u$ used for the rough Bergomi model (note Z is a Gaussian process but V is not), but the explicit formula for $\bar{R}(s,t)$ is more complicated than the $R(s,t)$ formula for fBM.

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H \;=\; c_H(\int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}) dW_s \,+\, \int_0^t (t-s)^{H-\frac{1}{2}} dW_s) \;=\; c_H(A_t + Z_t)$$

for $t \ge 0$, in terms of an RL process Z (and note that A_t is known at time zero for all $t \ge 0$), and $c_H = (\frac{2H\,\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\,\Gamma(2-2H)})^{\frac{1}{2}}$. Note also that A_t and Z_t are independent.

References

[BFG16] Bayer, C., P.K.Friz, and J.Gatheral, "Pricing under rough volatility", *Quantitative Finance*,16(6), pp. 887–904, 2016.

[Cha14] Chang, Y-C., "Efficiently Implementing the Maximum Likelihood Estimator for Hurst Exponent", *Mathematical Problems in Engineering*, vol. 2014

[CD22] Cont, R. and P.Das, "Rough Volatility: Fact or Artefact?", preprint, 2022

[F23] Forde, M., "Statistical issues and calibration problems under rough and Markov volatility", `https://martinforde.github.io/Talk.pdf`

[GJR18] Gatheral, J., T.Jaisson and M.Rosenbaum, "Volatility is rough", *Quantitative Finance*, 18(6), 2018

[HS21] Han, X. and A.Schied, "The Hurst roughness exponent and its model-free estimation", preprint, 2021

[HS24] Han, X. and A.Schied, "Estimating the roughness exponent of stochastic volatility from discrete observations of the integrated variance", preprint, 2024