Background, advice and ideas for Project 3

- The theoretical definition of the $\mathrm{VIX}_t^2 = -\frac{2}{\Delta}\mathbb{E}^{\mathbb{Q}}(\log(e^{-(r-q)\Delta}\frac{S_{t+\Delta}}{S_t})|\mathcal{F}_t)$, where $\Delta = 30$ days and $\mathcal{F}_t$ and $\mathbb{Q}$ are the market filtration and risk-neutral measure being used to price European options on the SPX, and r and q are the interest rate and dividend yield on the SPX (assumed constant here for simplicity). In practice this formula is approximated using prices of a finite set of European options (which comes from the Breeden-Litzenberger formula applied to the log payoff). The VIX is not a tradeable contract in itself, but you can trade futures on the VIX (see chapter 1 of FM14 notes on KEATS).

- You can try adapting your code to maximize exp utility using options on more than 1 asset, e.g. European options paying $(X-K)^+$, $(Y-K)^+$ and/or $(X-KY)^+$ (for a range of strike values K) as in Project 3, where X and Y are e.g. two exchange rates at some future maturity date (option price data for this is given in Table 1 in
`https://martinforde.github.io/FXCrossSmiles-updated.pdf`) or see data at
`https://www.investing.com/currencies/eur-usd-options`, and use 2d Gauss-Legendre quadrature to integrate over the joint density of (X,Y) or Monte Carlo to replace the 1d integration scheme in your Part 2, and either choose or fit a market model using a time series of data from e.g. Yahoo finance which (ideally) ends on the same as date as the option prices are quoted, e.g. a GARCH model (see code and documentation on my website). Or you can e.g. look at SPX and VIX options together and price options using Monte-Carlo rather than numerical integration (can use
`https://github.com/jgatheral/QuadraticRoughHeston/blob/main/qrheston_simulation.ipynb` on github for this).

  Just switching from exponential to power utility in Part 3 is not going to impress the markers.

- Can discuss background concepts for general non-linear optimizations problems: the **Karush–Kuhn–Tucker (KKT)** conditions, **complementary slackness**, primal/dual feasibility, **KKT/Lagrange multipliers**, **saddlepoints**, **sensitivities**, subgradients etc. (see e.g.
`https://en.wikipedia.org/wiki/Karush-Kuhn-Tucker_conditions`

- **Very instructive toy practice question**: Consider an agent who can buy x units of a single stock whose initial price is p and/or leave some of their money in cash time zero, and assume interest rates are zero for simplicity, with a total wealth constraint of w, with no further trading after time zero. Assume the terminal stock price is $\sim \mathrm{Exp}(1)$, and the trader wishes to maximize their expected exponential utility i.e. $-\mathbb{E}(e^{-\lambda(xS+w-xp)})$ over $x \in (-\infty, w/p]$, with risk-aversion parameter λ. Compute the optimal amount of stock to buy, and comment on the answer. Note that the utility function $U(w) = -e^{-\lambda w}$ is concave and increasing, capturing the economic intuition that more money makes us happier but the marginal utility derived from an additional pound decreases as our wealth increases.

  Solution. The problem is to maximize

$$\mathbb{E}(U(xS+w-xp)) \;=\; \mathbb{E}(-e^{-\lambda(xS+w-xp)}) \;=\; \int_0^\infty -e^{-\lambda(xS+w-xp)}e^{-S}dS \;=\; -\frac{e^{-w\lambda+px\lambda}}{1+x\lambda}1_{\{\lambda x+1>0\}} - \infty\cdot 1_{\{\lambda x+1\leq 0\}}$$

  over $x \in (-\infty, w/p]$. Differentiating this expression wrt x and setting the answer to zero, we find that the optimal stock holding is

$$x^* \;=\; \frac{1-p}{p\lambda} \tag{1}$$

  if $\frac{1-p}{p\lambda} < w/p$ and $\lambda x + 1 > 0$ so the feasible set is $x \in (-\frac{1}{\lambda}, \frac{w}{p})$; otherwise $x^* = w/p$ (i.e. we hit the wealth constraint). Note we have a minus sign in front of the expectation because we need $U(x)$ to be **concave**.

  Note that the **fair price** i.e. the **expected payout** of the stock is $\mathbb{E}(S) = 1$, so (1) says that we **buy stock when the stock is underpriced**, and **sell when the stock is overpriced**, and $|x^*|$ is smaller when λ is larger i.e. when the agent is more **risk-averse**.

  The same issue crops up in Part 2: you should check that the fair price of your optimal portfolio under the market model (i.e. the expected payout) is $\geq$ the market price of this portfolio.

  You can try and come up with variant of this example in your Part 1, and possibly include indifference pricing for e.g. an additional call option. See `https://arxiv.org/abs/2403.00139` for the case when we have European options at all strikes.

  In the project S is replaced by ξ and $\log\xi$ is Normally distributed, and there are call options in the problem; in particular note that $\mathbb{E}(e^{-\lambda x(\xi_T-K)^+}) = \infty$ if $x < 0$, because we have the exponential of the exponential of a Gaussian here. **This same right tail issue arises in Part 2 and you should think about how to avoid this**.

- **General case with d assets and no liquidity constraints or bid-ask spreads**. Consider a financial market defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with d assets with **random payoffs** $(S_1, ..., S_d)$ at time T (which are **linearly independent**) with market prices π_i at $t = 0$ (these assets can include **European call/put options**). Let $Y_i = S_i - \pi_i$, and assume a financial agent can only trade at time zero.

  Derive the **first-order optimality condition** for an agent to maximize their **expected utility** $\mathbb{E}(U(b \cdot Y))$ over $b \in \mathbb{R}^d$, where $U : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $U''(x) < 0$ and b_i is the position in the i'th asset and we assume that $\mathbb{E}(U(b \cdot Y)) < \infty$ for all $b \in \mathbb{R}^d$.

  Solution. As in first year calculus, we compute derivatives wrt each b_i and then set the answer to zero:

$$\frac{\partial}{\partial b_i}\mathbb{E}(U(b \cdot Y)) \quad = \quad \mathbb{E}(Y_i U'(b \cdot Y)) \quad = \quad 0$$

  for $i = 1, ..., d$, i.e. we have d equations for the d unknowns $b_1^*, ..., b_d^*$ for the optimal portfolio allocation b^*. Note we can re-write this as

$$\mathbb{E}^{\mathbb{Q}}(Y_i) \quad = \quad 0 \tag{2}$$

  where we define a **new probability measure** as $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(\frac{U'(b^* \cdot Y)}{\mathbb{E}^{\mathbb{P}}(U'(b^* \cdot Y))} 1_A)$ for events $A \in \mathcal{F}$, and (as a sanity check) we note that $\mathbb{Q}(\Omega) = 1$.

  Under the moment condition stated in the question, it turns out that a unique solution b^* exists if the **no-arbitrage** condition is satisfied: if $\mathbb{P}(b \cdot Y > 0) > 0$ then $\mathbb{P}(b \cdot Y < 0) > 0$. In this case, from (2), we see that under $\mathbb{Q}$, all contracts are priced according to the market, i.e. $\mathbb{E}^{\mathbb{Q}}(S_i) = \pi_i$. $\mathbb{Q}$ is known as a **risk-neutral measure**. We can solve this maximization problem numerically using MOSEK.

  In the project we use the **exponential utility function** $U(x) = -e^{-\lambda x}$, in which case (for $\lambda = 1$) we are computing $\max_b(-\log \mathbb{E}(e^{-b \cdot Y})) = \max_b(-\log \mathbb{E}(e^{b \cdot Y})) = -\min_b \log \mathbb{E}(e^{b \cdot Y})$, i.e. minus the **minimum of the log mgf** of Y.