Hawkes processes

Consider a time-inhomogenous Poisson process $(N_t)_{t\geq 0} \in \mathbb{N}$ whose intensity is itself a random process λ_t which evolves as

$$\lambda_t = \mu + \int_{[0,t]} \phi(t-s)dN_s$$

i.e. λ depends on the history of N itself, where μ is a positive constant and ϕ a positive function. For this reason we say that N is **self-exciting**, and this is a special type of **Stochastic Volterra equation** with no Brownian motion. The meaning of the λ_t is that $\lim_{h\to 0} \frac{1}{h}\mathbb{P}(N_{t+h} - N_t > 0|\mathcal{F}_t^{\lambda,N}) = \lambda_t$, and note we can re-write λ as

$$\lambda_t = \lambda_0 + \sum_{0\leq s_i \leq t} \phi(t-s_i)$$

where $s_1, s_2, ...$ are the random **jump times** of N, which we can also take as the definition of λ.

If we let $M_t = N_t - \int_0^t \lambda_u du$, then M is a martingale and we can re-write the λ equation as

$$\lambda_t = \mu + \int_{[0,t]} \phi(t-s)(dM_t + \lambda_t dt) = \mu + (\phi * dM)_t + (\phi * \lambda)_t \,.$$

Note that

$$\|\phi * \lambda\|_\infty \leq \|\lambda\|_\infty \int_{[0,t]} \phi(t-s)ds = \|\lambda\|_\infty \int_{[0,t]} \phi(u)du < \|\lambda\|_\infty \int_0^\infty \phi(u)du < \|\lambda\|_\infty$$

if $\|\phi\| = \int_0^\infty \phi(u)du < 1$.

So $\phi*$ is a contraction on $C_b[0,\infty)$ under the sup norm, so its inverse is well defined, and also on $C_b[0,T]$. We can re-write this in operator notation as

$$(I - \phi*)\lambda = \mu + \phi * dM \,.$$

To make sense of $(I-\phi*)^{-1}$, we look for a function ψ such that

$$(I-\phi*)^{-1}f = (I+\psi*)f \,.$$

for any test function f, so

$$\begin{aligned} f &= (I-\phi*)(I+\psi*)f = (I-\phi*)(f+\psi * f) \\ &= f - \phi * f + \psi * f - \phi * \psi * f \end{aligned}$$

which we can re-write in operator form (i.e. without the f) as $\phi * \psi = \psi - \phi,$. ψ is known as the **resolvent of ϕ, note definition here is opposite way round to chap 3 in FM14**. Applying this to our Hawkes process i.e. setting $f(t) = \lambda_t$, we see that

$$\begin{aligned} \lambda &= (I-\phi*)^{-1}(\mu + \phi * dM) = (1+\psi) * \mu + (I+\psi) * (\phi * dM) \\ &= \mu + \psi * \mu + (\phi * + (\phi*)^2 + ...) * dM \\ &= \mu + \psi * \mu + \psi * dM \end{aligned}$$

where $\psi = \sum_{k=1}^\infty (\phi*)^k$, which is shorthand for

$$\lambda_t = \mu + \mu \int_{[0,t]} \psi(t-s)ds + \int_{[0,t]} \psi(t-s)dM_s \tag{1}$$

The propagator model - concave price impact from Hawkes order flow

Consider two independent Hawkes processes $N_t^\pm$ with associated intensities $\lambda_t^\pm$ which evolve as with

$$\lambda_t^\pm = \mu + \mu \int_0^t \psi(t-s)ds + \int_0^t \psi(t-s)dM_s^\pm$$

where $dM_t^\pm = dN_t^\pm - \lambda_t^\pm dt$. Then

$$\begin{aligned} \mathbb{E}_t(N_u^+) &= g(t) + \mathbb{E}_t(\int_0^u \int_0^s \psi(s-v)dM_v^+ ds) = g(t) + \mathbb{E}_t(\int_0^u \int_v^u \psi(s-v)ds dM_v^+) \\ &= g(t) + \int_0^t \int_v^u \psi(s-v)ds dM_v^+ \end{aligned}$$

for some function $g(t)$, and note that $\int_v^\infty \psi(s-v)ds = \int_0^\infty \psi(s)ds = \|\psi\|_1$. Then if we assume the current price $P_t = \kappa \lim_{u\to\infty} \mathbb{E}(N_u^+ - N_u^- | \mathcal{F}_t)$ for some constant $\kappa > 0$, then

$$\frac{1}{\kappa} P_t \quad = \quad \lim_{u\to\infty} \mathbb{E}(N_u^+ - N_u^- | \mathcal{F}_t) \quad = \quad \int_0^t \sigma(dM_v^+ - dM_v^-)$$

where $\sigma = \|\psi\|$. Then

$$\begin{aligned} \frac{1}{\sigma} P_t &= N_t^+ - N_t^- - \int_0^t \lambda_s^+ ds + \int_0^t \lambda_s^- ds = N_t^+ - N_t^- - \int_0^t \int_u^t \phi(s-u)ds(dN_u^+ - dN_u^-) \\ &= \int_0^t (1 - \int_0^{t-u} \phi(s)ds)(dN_u^+ - dN_u^-) \end{aligned}$$

so

$$P_t \quad = \quad \int_0^t \zeta(t-u)(dN_u^+ - dN_u^-) \tag{2}$$

where $\zeta(t) = \kappa\sigma(1 - \int_0^t \phi(s)ds)$, so

$$\zeta'(t) \quad = \quad -\kappa\sigma\phi(t) \quad = \quad -\zeta(t)\phi(t)\,.$$

Hence from the Volterra form in (2), we see that P is a **propagator model** (e.g. like transient price impact) but also retains the martingale property.

Impact of a metaorder executed at constant rate before and after completion

Consider the additional contribution from an additional agent who buys at a fixed rate v for duration τ. Then impact the cumulative impact time t is

$$P_t \quad = \quad \int_0^t \zeta(t-u)(dN_u^+ - dN_u^-) \; + \; v \int_0^{t\wedge\tau} \zeta(t-s)ds$$

whose expectation is $MI(t) = v\int_0^{t\wedge\tau} \zeta(t-s)ds$, which we call the **market impact function**, see plot below where we see **concave price impact** up to τ, and then decay thereafter (which is broadly consistent with empirical findings where $MI(t)$ is often found to be $const. \times t^{\frac{1}{2}}$ for $t \le \tau$ (the so-called **square root impact law**.)

Example ϕ and ψ functions

One can easily check that $\|\phi\| = \frac{\|\psi\|}{1+\|\psi\|}$ so $\|\psi\| = \frac{\|\phi\|}{1-\|\phi\|}$. A common choice for ϕ is

$$\phi(t) \quad = \quad \nu t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$$

where $E_{\alpha,\alpha}$ is the **Mittag-Leffler** function (which is **heavy-tailed** since $\phi(t) \sim \frac{const.}{t^{1+\alpha}}$ as $t\to\infty$), and $\int_0^\infty \phi(t)dt = \frac{\nu}{\lambda}$ so we choose $\nu < \lambda$. For this choice of ϕ, the resolvent is

$$\psi(t) \quad = \quad \nu t^{\alpha-1} E_{\alpha,\alpha}(-(\lambda-\nu)t^\alpha)$$

and as $\nu \nearrow \lambda$, $\|\phi\| \nearrow 1$ and $\|\psi\| \nearrow \infty$ (see also table below).

	$\psi(t)$	$\phi(t)$
Constant	c	$c\,e^{-ct}$
Fractional	$\frac{c\,t^{\alpha-1}}{\Gamma(\alpha)}$	$c\,t^{\alpha-1}E_{\alpha,\alpha}(-c\,t^\alpha)$
Exponential	$c\,e^{-\lambda t}$	$c\,e^{-(\lambda+c)\,t}$
Gamma	$\frac{c\,e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)}$	$c\,e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-c\,t^\alpha)$

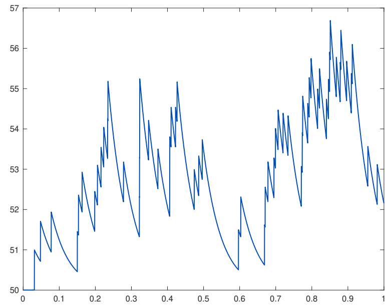

Figure 1: Here we we have simulated the intensity process of the form $\lambda_t = \lambda_0 + \int_0^t k(t-s)dN_s$ for $k(t) = e^{-10t}$ and $\lambda_0 = 50$.

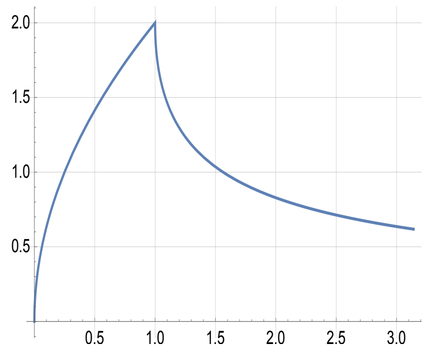

Figure 2: Here we see a typical concave impact function $\int_0^{t\wedge\tau} \zeta(t-s)ds$ with $\tau = 1$.

Interpretation of a Hawkes process in terms of population dynamics

Let us define a population model: At time zero, there are no individuals. Some individuals (migrants) arrive as a uniform Poisson process with intensity μ. If a migrant arrives at time s, the birth dates of its children form a Poisson process of intensity $\phi(t-s)$ at time t, with $\int_0^\infty \phi(t)dt \quad < \quad 1$. In the same way, if a child is born at s', the birth dates of its children form a Poisson process of intensity $\phi(\cdot - s')$. Let N_t be the number of individuals who were born or migrated until time t. Then N is a Poisson-type process with intensity

$$\lambda_t \quad = \quad \mu + \int_0^t \phi(t-s)\,dN_s \tag{3}$$

i.e. N is a Hawkes process. This captures the notion of the process being self-exciting.

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