Formula sheet

Note: any characteristic function for a Lévy process will be given to you in a class test/exam question

- Let $X_t = \mu t + \sigma W_t$, where W is Brownian motion. Then

$$F(b|x) \quad = \quad \mathbb{P}(\bar{X}_t \le b \,|\, X_t = x) \quad = \quad 1 - e^{-\frac{2b(b-x)}{\sigma^2 t}} \tag{1}$$

for $b \ge \max(x, 0)$, where $\bar{X}_t = \max_{0\le s\le t} X_s$. This allows us to sample $\bar{X}_t$ given X_t as $F^{-1}(U|X_t)$, where

$$F^{-1}(u|x) \quad = \quad \frac{1}{2}(x + \sqrt{x^2 - 2t\sigma^2 \log(1-u)})$$

with $U \sim U[0,1]$ and $F^{-1}(\,.\,|x) : (0,1) \to [\max(x,0), \infty]$ is the inverse of the function on the right in (1) (viewed as a function of b), which is very useful for Monte Carlo pricing of **barrier options**. Note $F(b|x)$ (and hence $F^{-1}(u|x)$) do not depend on μ, but obviously the distribution of X_t depends on μ.

- Let (Y_i) be a sequence of i.i.d. random variables with density $\mu(x)$ and $(N_t)_{t\ge 0}$ be a **Poisson process** for which $\mathbb{E}(e^{pN_t}) = e^{\lambda t(e^p - 1)}$ (and recall that $N_t \in \mathbb{N}$), and assume N_t is independent of (Y_i), and let $X_t = \sum_{i=1}^{N_t} Y_i$. X is known as a **compound Poisson process**, and

$$\mathbb{E}(e^{iuX_t - q[X,X]_t}) \quad = \quad e^{\lambda t \int_{-\infty}^{\infty}(e^{iux - qx^2} - 1)\mu(x)dx} \tag{2}$$

for $u \in \mathbb{R}$, $q \ge 0$.

- Let X be a random variable with $\int_{-\infty}^{\infty} |\mathbb{E}(e^{iuX})| du < \infty$. Then X has a density $f_X(x)$, which can be computed using the **inverse Fourier transform**:

$$f_X(x) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\mathbb{E}(e^{iuX})du\,.$$

Similarly, for the joint density $f_{X,Y}(x,y)$ of two random variables X, Y is given by the double IFT:

$$f_{X,Y}(x,y) \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iux - ivy}\,\mathbb{E}(e^{iuX + ivY})dudv$$

so e.g. we can apply this to (2) with $[X,X]_t = Y$ and $q = iu$.

- General definition of a Lévy process is here: `https://en.wikipedia.org/wiki/L%C3%A9vy_process` Brownian motion, Poisson processes, compound Poisson processes and α-stable processes are all examples of Levy processes.

- A **symmetric** α**-stable process** X with parameters $\alpha \in (0,2]$, $\sigma > 0$ is a generalization of Brownian motion, which has **independent stationary increments** like Brownian motion but $\mathbb{E}(e^{iuX_t}) = e^{-t\sigma^\alpha |u|^\alpha}$ for $u \in \mathbb{R}$, so $\mathbb{E}(X_t^2) = \infty$ if $\alpha < 2$. For the special case $\alpha = 2$, $X_t = \sqrt{2}\,\sigma W_t$ where W is a standard Brownian motion.

- A general Lévy process with no Brownian component has the **Lévy-Khintchine** representation

$$\mathbb{E}(e^{iuX_t}) \quad = \quad \exp(t(i\gamma u - \frac{1}{2}\sigma^2 u + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux 1_{|x|\le 1})\nu(x)dx)) \tag{3}$$

where the Lévy density $\nu(x) \ge 0$ must satisfy $\int_{-\infty}^{\infty} \nu(x)\min(1,x^2)dx < \infty$. If the iux integral here is finite, it can be absorbed into a modified γ term to get

$$\mathbb{E}(e^{iuX_t}) \quad = \quad \exp(t(i\tilde{\gamma}u + \int_{-\infty}^{\infty}(e^{iux} - 1)\nu(x)dx))$$

where $\tilde{\gamma} = \gamma - \int_{-\infty}^{\infty} x 1_{|x|\le 1})\nu(x)dx)$, which agrees with the form for the compound Poisson case when $\int_{-\infty}^{\infty}\nu(x)dx < \infty$. A compound Poisson process is a Lévy process with $\nu(x) = \lambda\mu(x)$ where $\mu(x)$ is the jump density as above.

- A general **KoBoL**/CGMY-type Lévy process has $\nu(x) = \frac{c_+ e^{-Mx}}{x^{1+Y_+}} 1_{x>0} + \frac{c_- e^{-G|x|}}{|x|^{1+Y_-}} 1_{x<0}$ for $Y_+, Y_- \in (0,2)$, for which

$$\mathbb{E}(e^{iuX_t}) \quad = \quad \exp(t(i\gamma u \,+\, c_+\Gamma(-Y_+)((M - iu)^{Y_+} - M^{Y_+}) \,+\, c_-\Gamma(-Y_-)((G + iu)^{Y_-} - G^{Y_-}))) \tag{4}$$

for $Y_\pm \ne 1$ and $u \in \mathbb{R}$, and we take the *principal branch* for the power functions (or if we set $u = -ip$ so $iu = p$ with $p \in \mathbb{R}$, then the right hand side is valid for $p \in [-G, M]$ and $\mathbb{E}(e^{pX_t}) < \infty$ in this range). An α-stable process has $G = M = 0$ and $\alpha = Y_+ = Y_-$; a symmetric α-stable process has $c_+ = c_-$ and $\gamma = 0$, and an α-stable process with positive-only jumps has $c_- = 0$, so in this case $\mathbb{E}(e^{iuX_t}) = e^{iu\gamma t + c_+\Gamma(-Y_+)(-iu)^{Y_+}t}$, and this process is non-decreasing if $Y_+ \in (0,1)$ and $\gamma \ge 0$ (note we usually set $\gamma = 0$ for this case).

- Let X be a Lévy process and e_q an Exp(q) random variable independent of X. Then $\bar{X}_{e_q}$ and $\bar{X}_{e_q} - X_{e_q}$ are independent, and $\bar{X}_{e_q} - X_{e_q} \sim -\underline{X}_{e_q}$, and if X has a density $\rho_t(x)$, then

$$\Phi_q^+(z) \quad := \quad \mathbb{E}(e^{iz\bar{X}_{e_q}}) \quad = \quad \exp\Big(\int_0^\infty t^{-1}e^{-qt}\int_0^\infty (e^{izx}-1)\rho_t(x)dxdt\Big) \qquad (z \in \mathbb{R}) .$$

- From this we find in Hwk 2 that

$$\mathbb{E}(\bar{X}_t) \quad = \quad \int_0^t \frac{\mathbb{E}(X_s^+)}{s} ds .$$

- A zero-mean Gaussian process B_t^H is called standard **fractional Brownian motion** (fBM) with **Hurst exponent** $H \in (0,1)$ if

$$R_H(s,t) \quad = \quad \mathbb{E}(B_t^H B_s^H) - \mathbb{E}(B_t^H)\mathbb{E}(B_s^H) \quad = \quad \mathbb{E}(B_t^H B_s^H) \quad = \quad \frac{1}{2}(t^{2H} + s^{2H} - (t-s)^{2H})$$

  for $0 \le s \le t$. When $H \in (0, \frac{1}{2})$, B^H is **rougher** than standard BM, and when $H \in (\frac{1}{2}, 1)$, B^H is **smoother** than standard BM; more specifically B^H is $H - \varepsilon$ **Hölder continuous** which means that $|B_t^H - B_s^H| \le c_1(\omega)|t-s|^{H-\varepsilon}$ a.s. for any $\varepsilon \in (0, H)$ where $c_1(\omega)$ is a (in general random) constant.