

We can re-write their u_t equation in operator form:

$$u_t = a_t + \int_0^t B(t, s) u_s ds = a + (Bu)(t)$$

so (formally at least) we can invert using the Neumann series:

$$u = (I - B)^{-1}(a) = (I + B + B^2 + \dots)a$$

(see page 37 in their paper). Define the resolvent R (of the second kind) of B via

$$I = (I - B)(I + R) = I - B + R - BR$$

so

$$R = B + BR.$$

Applying both sides here to an arbitrary function $f \in C([0, T])$ and assuming $R = R^B$ is also of Volterra form $(Rf)(t) = \int_0^t r(t, u) f(u) du$, we find that

$$\begin{aligned} \int_0^t r(t, u) f(u) du &= \int_0^t B(t, u) f(u) du + \int_0^t B(t, s) \int_0^s r(s, u) f(u) du ds \\ \Rightarrow \int_0^t r(t, u) f(u) du &= \int_0^t B(t, u) f(u) du + \int_0^t \int_u^t r(s, u) B(t, s) ds f(u) du \\ &\Rightarrow \int_0^t (r(t, u) - B(t, u) - \int_u^t r(s, u) B(t, s) ds) f(u) du = 0. \end{aligned}$$

This has to hold for all f , so

$$r(t, s) = B(t, s) + \int_s^t B(t, v) r(v, s) dv$$

for Leb a.e. $s \in [0, t]$. Note also the similar (but simpler) form of the optimal solution in Theorem 2.2 in my paper with Ben+Leandro where $u_t = \bar{u}(t) + \int_0^t k(v, t) dW_v$.