Homework 5

1. Let $(Z_t)_{t\in\mathbb{R}}$ be a Gaussian process with autocovariance

$$r(\tau) \;=\; \mathbb{E}(Z_t Z_{t+\tau}) \;=\; (1+|\tau|)^{-\gamma}$$

for $\gamma > 0$. For two correlated $N(0,1)$ random variables X, Y with $\mathbb{E}(XY) = \rho$ it can be shown that $\mathbb{E}(\mathrm{sgn}(X)\mathrm{sgn}(Y)) = \frac{2}{\pi}\arcsin\rho$. Let $\varepsilon_t = \mathrm{sgn}(Z_t)$. Compute the asymptotic behaviour of $\mathbb{E}(\varepsilon_t\varepsilon_{t+\tau})$ as $\tau \to \infty$.

Solution.

$$\mathbb{E}(\varepsilon_t\varepsilon_{t+\tau}) \;=\; \frac{2}{\pi}\arcsin\mathbb{E}(Z_t Z_{t+\tau})\,.$$

But $\arcsin x = x + O(x^3)$ as $x \to 0$, so

$$\mathbb{E}(\varepsilon_t\varepsilon_{t+\tau}) \;\sim\; \frac{2}{\pi}\mathbb{E}(Z_t Z_{t+\tau}) \;=\; \frac{2}{\pi}(1+|\tau|)^{-\gamma} \;\sim\; \frac{2}{\pi}|\tau|^{-\gamma}\,.$$

2. Let X denote a Lévy process and assume X_t has a density for simplicity. Then there exist two non-negative, non-decreasing Lévy processes (L^{-1}, H) for which

$$\mathbb{E}(e^{-qL_t^{-1} - zH_t}) \;=\; e^{-\kappa(q,z)t} \tag{1}$$

for $q, z > 0$, where

$$\kappa(q,z) \;=\; \exp\Big(\int_0^\infty \int_0^\infty (e^{-s} - e^{-qs-zx})\frac{1}{s}\mathbb{P}(X_s \in dx)ds\Big)\,.$$

Take the limit as $q \to 0$ in (1) with $z = 0$. What happens if X is Brownian motion?

Solution.

$$\lim_{q\to 0}\mathbb{E}(e^{-qL_t^{-1}}) \;=\; \mathbb{E}(1_{L_t^{-1}<\infty}) \;=\; \mathbb{P}(L_t^{-1} < \infty) \;=\; e^{-t\kappa(0,0)}$$

where

$$\kappa(0,0) \;=\; \exp\Big(\int_0^\infty \int_0^\infty (e^{-s} - 1)\frac{1}{s}\mathbb{P}(X_s \in dx)ds\Big) \;=\; \exp\Big(-\int_0^\infty \frac{1}{s}(1-e^{-s})\mathbb{P}(X_s > 0)ds\Big)\,.$$

If X is Brownian motion then $\mathbb{P}(X_s > 0) = \frac{1}{2}$ is independent of s, so the integral simplifies to $\frac{1}{2}\int_0^\infty \frac{1}{s}(1-e^{-s})ds = \infty$, so $\kappa(0,0) = 0$, and $\mathbb{P}(L_t^{-1} < \infty) = 1$.