VIX options under Markov stochastic volatility models

We now consider a typical **Markov stochastic volatility model** for a log stock price process $X_t = \log S_t$:

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t}(\rho dW_t + \bar{\rho} dB_t) \\ dV_t &= \kappa(\theta - V_t)dt + \alpha(V_t)dW_t \end{aligned}$$

where W and B are two independent Brownian motions, with $\kappa \geq 0$, $\alpha(.) > 0$ and $V_0 = v_0 > 0$. The theoretical value of the **VIX index** at time t is

$$\mathrm{VIX}_t = \sqrt{-\frac{2}{\Delta}\mathbb{E}(\log\frac{S_{t+\Delta}}{S_t}|\mathcal{F}_t)} \tag{1}$$

where S_t is the **S&P 500** index value at time t, and in practice $\Delta = 30$ days ($\frac{1}{12}$ of a year). The VIX index is an important barometer of market volatility/stress. Since

$$\log S_{t+\Delta} = \log S_t - \frac{1}{2}\int_t^{t+\Delta} V_s ds + \int_t^{t+\Delta} \sqrt{V_s}(\rho dW_s^2 + \bar{\rho} dW_s^1)$$

and the conditional expectation of the final term is zero at time t since the stochastic integral is a martingale, we can re-write (1) as

$$\mathrm{VIX}_t^2 = \frac{1}{\Delta}\mathbb{E}(\int_t^{t+\Delta} V_u du|\mathcal{F}_t) = \frac{1}{\Delta}\int_t^{t+\Delta}\mathbb{E}(V_u|\mathcal{F}_t)du = \frac{1}{\Delta}\int_t^{t+\Delta}\mathbb{E}(V_u|V_t)du \tag{2}$$

where the final equality follows because V is a Markov process (note this is not true for the rough Heston model below). From this we see that we can interpret VIX_t^2 as a rolling **variance swap rate** at time t over the next 1 month time period.

A VIX call option is a European call option on VIX_T for some maturity T, so the payoff of a VIX call option is $\max(\mathrm{VIX}_T - K, 0)$ at time T. Integrating the SDE for V_t above and setting $\alpha(v) = \sigma\sqrt{v}$, we see that

$$\begin{aligned} V_t &= V_0 + \int_0^t \kappa(\theta - V_u)du + \int_0^t \sigma\sqrt{V_u}dW_u^2 \\ \Rightarrow \quad \mathbb{E}(V_t) &= V_0 + \mathbb{E}(\int_0^t \kappa(\theta - V_u)du) = V_0 + \int_0^t \kappa(\theta - \mathbb{E}(V_u))du \end{aligned}$$

since the second term in the previous equation is a stochastic integral and thus has zero expectation. Differentiating we see that $f(t) = \mathbb{E}(V_t)$ satisfies the ordinary differential equation:

$$\frac{df}{dt} = \kappa(\theta - f(t))$$

with initial condition $f(0) = V_0$, which has solution

$$f(t) = \theta + e^{-\kappa t}(V_0 - \theta)\,. \tag{3}$$

For (2), we need to be able to compute $\mathbb{E}(V_u|V_t)$. But since $V_u|V_t = v \sim V_{u-t}|V_0 = v$, from (3) we see that

$$\mathbb{E}(V_u|V_t) = \theta + e^{-\kappa(u-t)}(V_t - \theta)$$

for $u \geq t$ i.e. we just replace t with $u - t$ and V_0 with V_t in $f(t)$, so setting $t = T$ in (2) we see that

$$\mathrm{VIX}_T^2 = \frac{1}{\Delta}\int_T^{T+\Delta}(\theta + e^{-\kappa(u-T)}(V_T - \theta))du\,.$$

We can compute the integral here explicitly since V_T does not depend on u, and we obtain

$$\mathrm{VIX}_T^2 = \frac{1 - e^{-\kappa\Delta}}{\kappa\Delta}V_T + \frac{\theta}{\kappa\Delta}(e^{-\kappa\Delta} + \kappa\Delta - 1) = aV_T + b$$

for $\kappa > 0$. If $\kappa = 0$, V is a martingale, so $\mathbb{E}(V_t|V_u)$ in (2) is just V_u, so in this case $\mathrm{VIX}_t^2 = V_t$.

Note this is just $F(V_T)$ for some **affine** function $F(v) = av + b$ for some a, b, so we can easily now estimate $\mathbb{E}(\max(\mathrm{VIX}_T - K, 0))$ (i.e. the VIX call price with strike K) using Monte Carlo as $\mathbb{E}(\max(\sqrt{F(V_T)} - K, 0))$. You do not need to simulate S itself to price this VIX option so we only need simulate the Brownian motion which drives V. One can also price the **VIX future** which pays VIX_T at time T, i.e. just set $K = 0$.

We can also compute VIX implied volatility, by replacing S_0 by $\sqrt{\mathbb{E}(\mathrm{VIX}_T)}$ in the Black-Scholes formula. End-of-day SPX and VIX option price data can be bought very cheaply from **CBOE datashop**.

Remark 0.1 Note that $b \geq 0$, so $\mathrm{VIX}_T^2 \geq b$, which means that for $K \leq \sqrt{b}$, $\mathbb{E}(\max(\mathrm{VIX}_T - K, 0)) = \mathbb{E}(\mathrm{VIX}_T) - K$, i.e. the VIX option is just worth its **intrinsic value**, and hence has zero implied volatility. So for $b > 0$, the VIX implied volatility tends to zero as $K \searrow \sqrt{b}$.

Can use `scipy.optimize.minimize` to perform minimization in Python, or `fmins` in MATLAB. We typically calibrate stoc vol models by minimizing the sum of squares difference between model and market implied volatilities

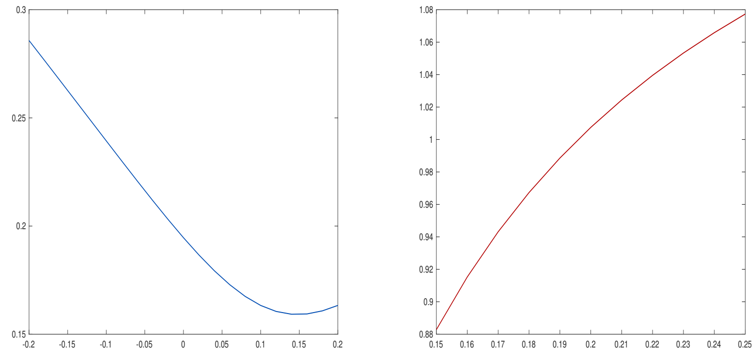

Figure 1: Here we have plotted SPX (left) and VIX (right) implied volatility smiles for $V_0 = .04$, $\kappa = 4$, $\theta = .045$, $\rho = -.65$, and $\alpha(v) = \nu v^p$ with $p = 1$, $\nu = 3$ and $T = 30$ days, using the Matlab code in KEATs. The x-axis for the SPX smile is log-moneyness $x = \log \frac{K}{S_0}$, and the x-axis for the VIX smile is the VIX option strike K. The VIX future price is $\mathbb{E}(\mathrm{VIX}_T) = .1958$.

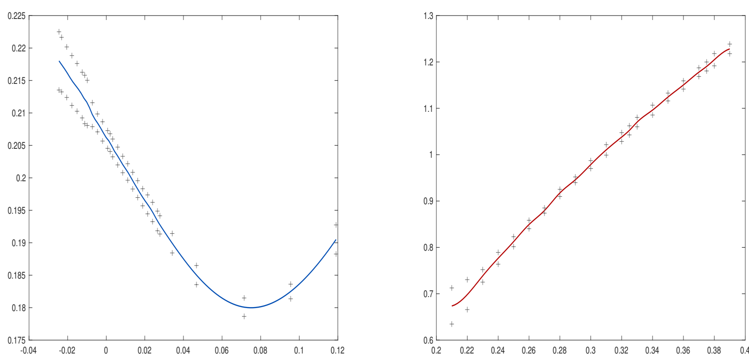

Figure 2: Here we have plotted the real bid and ask SPX (left) and VIX (right) implied volatility smiles also for $T = 30$ days on 19th Dec 2022 and a cubic spline through the midpoint implied vols, using the CBOE datashop data in the Excel sheets on KEATs. Here the $T = 30$ day SPX future price is 3817.29 and the VIX future price is .2438, with 1 month US treasury rate $r = .0395$.