Homework 3

1. (Estimating volatility). Let $X_t = \mu t + \sigma W_t$. Show that $\hat{\sigma}_n^2 = \sum_{i=0}^{n-1}(X_{(i+1)/n} - X_{i/n})^2$ is a consistent estimator for σ^2 (i.e. that $\hat{\sigma}_n^2 \to \sigma^2$ as $n \to \infty$ in some sense). Is $\hat{\sigma}_n^2$ an unbiased estimator?

Solution.

$$\begin{aligned}
\hat{\sigma}_n^2 \;&=\; \sum_{i=0}^{n-1}(X_{(i+1)/n} - X_{i/n})^2 \;=\; \sum_{i=0}^{n-1}(\frac{\mu}{n} + \sigma(W_{(i+1)/n} - W_{i/n}))^2 \;\sim\; \sum_{i=1}^{n}(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}Z_i)^2 \\
&= \frac{\mu^2}{n} + \frac{2\mu\sigma}{n}\sum_{i=1}^{n}\frac{Z_i}{\sqrt{n}} + \frac{\sigma^2}{n}\sum_{i=1}^{n}Z_i^2 \\
&= \frac{\mu^2}{n} + \frac{2\mu\sigma}{\sqrt{n}}\cdot\frac{1}{n}\sum_{i=1}^{n}Z_i + \frac{\sigma^2}{n}\sum_{i=1}^{n}Z_i^2
\end{aligned}$$

where the Z_i's are i.i.d. $N(0,1)$, and we have used that $W_{(i+1)/n} - W_{i/n} \sim \frac{1}{\sqrt{n}}Z_i$ (from the third property of Brownian motion). The $\frac{\mu^2}{n}$ term in the final line trivially tends to zero, and the second term also tends to zero a.s. because $\frac{1}{n}\sum_{i=1}^{n} Z_i$ tends to $\mathbb{E}(Z_i) = 0$ from the SLLN. Hence $\hat{\sigma}_n^2$ tends to the constant σ^2 in distribution by applying the SLLN to the final term (which also implies convergence in probability), so $\hat{\sigma}_n^2$ is a consistent estimator for σ^2.

Note this applies to the log stock price $X_t = \log S_t$ for the Black-Scholes model if we just replace μ here with $\mu - \frac{1}{2}\sigma^2$, and the final limit does not depend on μ.

For the second part, for n finite, we see that $\mathbb{E}(\hat{\sigma}_n^2) = \frac{\mu^2}{n} + \sigma^2$; hence $\hat{\sigma}_n^2$ is only unbiased when $\mu = 0$.

2. Using the expression for $\mathbb{P}(S_T > K)$ in the Black-Scholes chapter, what can we deduce about convergence of S_t as $t \to \infty$ when $\mu = 0$.

Solution. For $\mu = 0$

$$\mathbb{P}(S_T > K) \;=\; \Phi^c(\frac{\log\frac{K}{S_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) \;\to\; 0$$

as $T \to \infty$, because $\frac{\log\frac{K}{S_0}+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \sim \frac{1}{2}\sigma\sqrt{T} \to +\infty$ as $T \to \infty$. Hence $\mathbb{P}(S_T > K) = \mathbb{P}(|S_T - 0| > K) \to 0$ for any $K > 0$, so $S_t \to 0$ **in probability** under $\mathbb{P}$ as $T \to \infty$.

3. (Quadratic co-variation of two correlated Brownian motions). Let W be a Brownian motion, and let $B_t = \rho W_t + \bar{\rho}\tilde{W}_t$ where $\bar{\rho} = \sqrt{1-\rho^2}$ and $\tilde{W}_t$ is another BM independent of W. Then it can be shown that B is also a Brownian motion and $\mathbb{E}(W_t B_t) = \rho t$. Compute

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}(W_{(i+1)/n} - W_{i/n})(B_{(i+1)/n} - B_{i/n})\,.$$

Solution. The sum here has the same distribution as

$$\sum_{i=0}^{n-1}\sqrt{\Delta t}Z_i\cdot\sqrt{\Delta t}(\rho Z_i + \bar{\rho}\tilde{Z}_i) \;=\; \frac{1}{n}\sum_{i=0}^{n-1}Z_i(\rho Z_i + \bar{\rho}\tilde{Z}_i) \;\to\; \rho$$

where $\Delta t = \frac{1}{n}$, and Z_i and $\tilde{Z}_i$ are two independent sequences of i.i.d. standard Normals. The convergence then follows from the SLLN.

4. (Estimating volatility). Let $X_t = \mu t + \sigma W_t$ and let $\bar{X}_t = \max_{0\le s\le t} X_s$ and $\underline{X}_t = \min_{0\le s\le t} X_s$. Using that

$$\mathbb{E}^{\mathbb{P}}(\bar{X}_t(\bar{X}_t - X_t) + \underline{X}_t(\underline{X}_t - X_t)) \;=\; \sigma^2 t \tag{1}$$

derive an **unbiased estimate** for σ^2 from n daily observations of $X = \log S$ using the daily returns $r_i := X_{i\Delta t} - X_{(i-1)\Delta t}$, daily (relative) highs $H_i = \max_{s\in[(i-1)\Delta t, i\Delta t]}(X_s - X_{(i-1)\Delta t})$, and daily (relative) lows $L_i = \min_{s\in[(i-1)\Delta t, i\Delta t]}(X_s - X_{(i-1)\Delta t})$ for $i \in \mathbb{N}$, where $\Delta t = 1$ day.

Solution. X has i.i.d. increments and the r_i's are the increments of X with time increment 1 (since $\Delta t = 1$) so we see that $r_i \sim X_1$ for all i.

Moreover, for each i, the *process* $X_s - X_{(i-1)\Delta t}$ for $s \in [(i-1)\Delta t, i\Delta t]$ is independent (and distributed the same) as the process $X_s - X_{(j-1)\Delta t}$ for $s \in [(j-1)\Delta t, j\Delta t]$ for $j \neq i$, so (in particular) the $H_i(H_i - r_i)$'s are i.i.d. and so are the $L_i(L_i - r_i)$ (this doesn't mean that $H_i(H_i - r_i)$ and $L_i(L_i - r_i)$ are independent of each other, but we don't require that).

Hence from this i.i.d. property, we see that

$$\mathbb{E}^{\mathbb{P}}\Big(\frac{1}{n}\sum_{i=1}^{n}(H_i(H_i - r_i) + L_i(L_i - r_i))\Big) \quad = \quad \mathbb{E}^{\mathbb{P}}(H_i(H_i - r_i) + L_i(L_i - r_i)) \quad = \quad \sigma^2 \Delta t$$

so $\hat{\sigma}^2 := \frac{1}{n\Delta t}\sum_{i=1}^{n}(H_i(H_i - r_i) + L_i(L_i - r_i))$ has expectation σ^2, and hence is an unbiased estimate for σ^2, which is robust to unknown μ.

5. (Double barrier computation). Let $X_t = \gamma t + W_t$, $M_t := \max_{0 \le s \le t} X_s$ and $m_t := \min_{0 \le s \le t} X_s$. Using that

$$\mathbb{P}(X_t \in dx, M_t < b, m_t > a) \quad = \quad -\frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\lambda_n t} e^{\gamma x - \frac{1}{2}\gamma^2 t} \sin\Big(\frac{n\pi(x-a)}{b-a}\Big)\sin\Big(\frac{n\pi a}{b-a}\Big)dx$$

for $a < 0 < b$ where $\lambda_n = \frac{n^2\pi^2}{2(b-a)^2}$, explain how you would use this to compute the cdf of the **two-sided maximum** $R_t := \max_{0 \le s \le t}|X_s|$, and the density of $\tau = \min\{t : |X_t| \ge r\}$. Is $\tau = \min\{t : |X_t| = r\}$?

Solution.

$$\mathbb{P}(R_t < r) \quad = \quad \mathbb{P}(M_t < r, m_t > -r)\,.$$

We compute this by setting $b = r$ and $a = -r$, and then integrating each term of the series from $x = -r$ to r to compute the right hand side (we assume we can interchange integral and series without proof).

For the second part, the events $\{R_t < r\}$ and $\{\tau > t\}$ are equivalent; hence $\mathbb{P}(R_t < r) = \mathbb{P}(\tau > t)$, so the density of τ is

$$\frac{d}{dt}\mathbb{P}(\tau \le t) \quad = \quad \frac{d}{dt}(1 - \mathbb{P}(\tau > t)) \quad = \quad -\frac{d}{dt}\mathbb{P}(\tau > t) \quad = \quad -\frac{d}{dt}\mathbb{P}(R_t < r)$$

and we explained how to compute $\mathbb{P}(R_t < r)$ in the first part of the solution. For the final part, yes because W_t is continuous as a function of t and hence X cannot jump over a barrier.