

# A simplified unified propagator model for signed order flow, concave price impact and rough volatility

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## Abstract

In this note, we simplify the setup in [MORS26] by exogenously modelling signed order flow as the difference of two integrated hyper-rough Heston process with Hölder exponent in  $(\frac{1}{2}, 1)$ , which endogenously determines the price process, the propagator mechanism and the long-run permanent price impact. This obviates the need to use nearly unstable heavy-tailed Hawkes processes, and lengthy macroscopic scaling limit arguments with  $C$ -tightness.

## 1.1 Persistent order flow

Similar to Theorem 3.1 in [MORS26]<sup>1</sup>, we consider a weak solution the Stochastic Volterra Equation

$$F_t^+ = g(t) + \int_0^t f(t-s)B_{F_s^+}ds \quad (1)$$

for a standard Brownian motion  $B$  with  $B_0 = 0$ , where  $f(t) = e^{-\theta t}t^{H-\frac{1}{2}}$  with  $H \in (-\frac{1}{2}, \frac{1}{2})$ ,  $\theta > 0$  so  $f \in L^1$  and  $g(t) = V_0 t$  for some  $V_0 > 0$  (note  $\theta$  replaces the ergodicity parameter  $\lambda$  in Theorem 3.1 in [MORS26]) (weak existence and uniqueness for (1) is established in section 2 of [AJ21]).  $F_t^+$  is a.s.  $(2\alpha - \varepsilon) \wedge 1$ -Hölder continuous where  $\alpha = H + \frac{1}{2}$  (see e.g. Theorem 3.1 in [JM20] for details)<sup>2</sup> and  $F^+$  is an increasing process which is non-differentiable if  $H \leq 0$ .

**Lemma 1.1**  $\mathbb{E}(F_t^+) < \infty$ .

**Proof.** See Appendix A. ■

**Corollary 1.2**  $B_{F_t^+}$  is an  $\mathcal{G}_t^+$ -martingale, where  $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$ .

**Proof.** Let  $S = F_s^+$  and  $T = F_t^+$  for  $0 \leq s \leq t$ , then  $T$  and  $S$  are  $\mathcal{F}^B$ -stopping times with  $0 \leq S \leq T$ . To see this, consider two Brownian paths  $\omega, \omega'$  with  $\omega(r) = \omega(r')$ , then  $F_t^+(\omega) \leq u \Rightarrow F_t^+(\omega') \leq u$ . By the contrapositive,  $F_t^+(\omega') > u \Rightarrow F_t^+(\omega) \geq u$ , so  $F_t^+(\omega) > u \Rightarrow F_t^+(\omega') \geq u$ , hence  $1_{F_t^+(\omega) \leq u} = 1_{F_t^+(\omega') \leq u}$ , so  $\{F_t^+(\omega) \leq u\} \in \mathcal{F}_u^B$ .

Then  $\mathbb{E}(B_{u \wedge T}^2) = \mathbb{E}(u \wedge T) \leq \mathbb{E}(T) = \mathbb{E}(F_t^+) < \infty$  (from the previous lemma), so  $(B_{u \wedge T})$  is bounded in  $L^2$  (and hence  $M_u = B_{u \wedge T}$  is an  $\mathcal{F}_u^B$ -martingale with  $\mathbb{E}(M_u^2) = \mathbb{E}(u \wedge T) \leq \mathbb{E}(T) < \infty$  and hence is UI) so from the OST for U.I. martingales applied to  $M$  (e.g. Theorem 5.20 in [Eth18]),  $\mathbb{E}(B_T | \mathcal{F}_S^B) = B_S$ , and the martingale property follows. ■

## 1.2 The inversion formula

If  $h$  is a function with  $h * f \equiv 1$ , then we have the inversion formula:

$$\begin{aligned} \int_0^t h(t-s)(F_s^+ - g(s))ds &= \int_0^t h(t-s) \int_0^s f(s-u)B_{F_u^+}duds = \int_0^t \int_u^t h(t-s)f(s-u)dsB_{F_u^+}du \\ &= \int_0^t \int_0^{t-u} h(t-s-u)f(s)dsB_{F_u^+}du \\ &= \int_0^t (h * f)(t-u)B_{F_u^+}du = \int_0^t B_{F_u^+}du \end{aligned}$$

(where Fubini is justified since  $F_t^+ < \infty$  a.s. for finite  $t$  and  $B$  is a.s. continuous), and hence

$$B_{F_t^+} = \frac{d}{dt} \int_0^t h(t-s)(F_s^+ - g(s))ds = \frac{d}{dt} \int_0^t h(t-s)F_s^+ds - \frac{d}{dt} \int_0^t h(t-s)g(s)ds \quad (2)$$

<sup>1</sup>see also section 5.4 in [FGS21]

<sup>2</sup>we can simulate  $F^+$  using the Monte Carlo scheme in [AA25] using Normal Inverse Gaussian variates

Lebesgue a.e., where the final equality holds since  $\int_0^t h(t-s)(F_s^+ - g(s))ds$  and  $\int_0^t h(t-s)g(s)ds$  are both absolutely continuous, and hence so is their sum. Moreover, setting  $H(t) = \int_0^t h(s)ds$ , integrating by parts we see that

$$\int_0^t H(t-s)dF_s^+ = F_s^+ H(t-s) \Big|_{s=0}^{s=t} + \int_0^t h(t-s)F_s^+ ds = \int_0^t h(t-s)F_s^+ ds.$$

Taking Laplace transforms, the condition  $h * f \equiv 1$  becomes  $\hat{f}(\lambda)\hat{h}(\lambda) = \frac{1}{\lambda}$ , from which we find that

$$h(t) = \frac{\theta^\alpha}{\Gamma(\alpha)} \left(1 - \frac{\Gamma(-\alpha, t\theta)}{\Gamma(-\alpha)}\right)$$

where  $\Gamma(a, z) = \int_z^\infty s^{a-1}e^{-s}ds$  is the incomplete Gamma function, and  $h(t) = O(t^{-H-\frac{1}{2}})$  as  $t \rightarrow 0$ .

### 1.3 Signed order flow

Again following [MORS26], we now assume the *signed* order flow for an asset is

$$V_t = F_t^+ - F_t^-$$

where  $F_t^- = g(t) + \int_0^t f(t-s)W_{F_s^-}ds$  and  $W$  is another Brownian motion independent of  $B$  (so  $F^-$  is an i.i.d. copy of  $F^+$ ). Then we see that

$$\frac{d}{dt} \int_0^t h(t-s)V_s ds = \frac{d}{dt} \int_0^t H(t-s)dV_s = \int_0^t h(t-s)dV_s = B_{F_t^+} - W_{F_t^-} \quad (3)$$

since the  $g$  terms cancel (and the second equality follows from the Stieltjes-Leibniz rule using the known Hölder continuity of  $F^\pm$  and that  $|h(u)| \leq Cu^{-\alpha}$  so  $\int_0^t |h(t-s)||dV_s| < \infty$ ), and for  $0 \leq t \leq u$  we have

$$\mathbb{E}(F_u^+ | \mathcal{F}_{F_t^+}^B) = g(u) + \mathbb{E}\left(\int_0^u f(u-s)B_{F_s^+}ds | \mathcal{G}_t^+\right) = g(u) + \int_0^t f(u-s)B_{F_s^+}ds + B_{F_t^+} \int_t^u f(u-s)ds$$

(and similarly for  $\mathbb{E}(F_u^- | \mathcal{F}_{F_t^-}^W)$ ), where  $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$ , since  $(B_{F_t^+})_{t \geq 0}$  is a  $\mathcal{G}_t^+$ -martingale by Corollary 1.2.

### 1.4 Price dynamics and the propagator model

Making the usual assumption that the asset price  $P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u | \mathcal{G}_t)$  where  $F_t = F_t^+ - F_t^-$  and  $\mathcal{G}_t = \sigma(\mathcal{G}_t^+, \mathcal{G}_t^-)$  with  $\mathcal{G}_t^- = \mathcal{F}_{F_t^-}^W$  (for some  $\kappa > 0$ )<sup>3</sup>, we find that  $f(u-s) \rightarrow 0$  as  $u \rightarrow \infty$  since  $\theta > 0$ , but  $\lim_{u \rightarrow \infty} \int_t^u f(u-s)ds = c_{H,\theta}$  where  $c_{H,\theta} = \theta^{-\frac{1}{2}-H}\Gamma(\frac{1}{2} + H)$ , so

$$P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u^+ - F_u^- | \mathcal{G}_t) = \kappa c_{H,\theta} (B_{F_t^+} - W_{F_t^-})$$

i.e.  $P$  is the difference of two i.i.d. hyper-rough Heston models (each with correlation  $\rho = 1$  since each process only has one driving Brownian motion), and we recover the Propagator equation

$$P_t = \kappa c_{H,\theta} \int_0^t h(t-s)dV_s$$

and the *unsigned* order flow is  $U_t = F_t^+ + F_t^-$ .

### 1.5 Power-law price impact

In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 up to time  $t_0$  is given by

$$MI(t) = \frac{d}{dt} \int_0^t h(t-s)sds = \frac{\theta^{-\alpha}(-t\theta\Gamma(-\alpha) - t\theta\Gamma(-\alpha, t\theta) - \Gamma(\alpha_-, 0) + \Gamma(\alpha_-, t\theta))}{\Gamma(-\alpha)\Gamma(\alpha)}$$

for  $0 \leq t < t_0$  (where  $\alpha_- = \frac{1}{2} - H$ ) for  $0 \leq t \leq t_0$  which is  $O(t^{\frac{1}{2}-H})$  as  $t \rightarrow 0$  (for  $H \neq \frac{1}{2}$ ) and globally concave in  $t$  for  $H \in (-\frac{1}{2}, \frac{1}{2})$ , consistent with empirical evidence (note the usual square-root impact law corresponds to  $H = 0$  here), and we can show that the asymptotic (i.e. permanent) impact as  $t \rightarrow \infty$  (given no trading after  $t_0$ ) is

$$MI(\infty) = \frac{t_0\theta^\alpha}{\Gamma(\frac{1}{2} + H)}$$

<sup>3</sup>note we are assuming  $P_0 = 0$  without loss of generality to ease notation, but we can easily add a positive  $P_0$  term

(this is the asymptotic grey line in the plot below).

For the special case  $H = \frac{1}{2}$  that we have previously excluded,  $h(t) = \theta$ , which corresponds to linear permanent price impact; conversely  $\lim_{\theta \rightarrow 0} h(t) = \text{const.} \times t^{-\frac{1}{2}-H}$ . Recall from above that  $F_t^+$  has Hölder regularity  $(2\alpha - \varepsilon) \wedge 1$ , and empirical evidence in [MORS26] and below suggests that  $2\alpha \in (\frac{1}{2}, 1)$  which corresponds to  $H \in (-\frac{1}{4}, 0)$ .

## 1.6 Extensions

If we augment  $f$  to a function of the form  $f(t) = e^{-\theta_1 t} t^{H-\frac{1}{2}} + e^{-\theta_2 t} t^{H_2-\frac{1}{2}}$ , then we find that

$$h(t) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} (-c)^n \sum_{k=0}^{\infty} \frac{(\beta n)_k}{k!} \delta^k \theta^{-(\mu_n+k)} \frac{\gamma(\mu_n+k, \theta t)}{\Gamma(\mu_n+k)}$$

where  $\beta = \frac{1}{2} + H_2$ ,  $c = \Gamma(\beta)/\Gamma(\alpha)$ ,  $\delta = \theta - \theta_2$ ,  $\mu_n = \beta n - \alpha(n+1)$ ,  $\gamma(a, z) = \int_0^z s^{a-1} e^{-s} ds$  is the lower incomplete Gamma function and  $(a)_k$  denotes the Pochhammer symbol. We can then use two (possibly different)  $f$  functions to drive  $F^\pm$ , and if  $H_2 \in (0, \frac{1}{2})$ ,  $P_t$  will have a conventional (i.e. non-hyper) rough component, which may be more realistic. We can also assume  $W$  and  $B$  correlated which gives greater flexibility in modelling skew for  $P_t$ .

## 1.7 Empirical results

Below we have tabulated estimates of  $2\alpha$  (i.e. the Hölder exponent of  $(F_t)_{t \geq 0}$ ) using signed order flow for major US tech stocks using the model-independent  $\hat{H}_{L,K}^\pi(X)$  estimator in Eq 7 in Cont&Das[CD24] which is consistent and allows for irregularly spaced data (with their  $L/K = \lceil \sqrt{N} \rceil$  where  $N$  is the total sample size). For this we have used the free data at <https://data.lobsterdata.com/info/DataSamples.php>:<sup>4</sup>

	GOOG	AAPL	MSFT	AMZN	INTC
$N$	49284	118326	410881	57360	404496
$2\hat{\alpha}$	0.692	0.720	0.847	0.735	0.890
$\hat{H}$	-0.154	-0.140	-.0767	-0.133	-0.0553

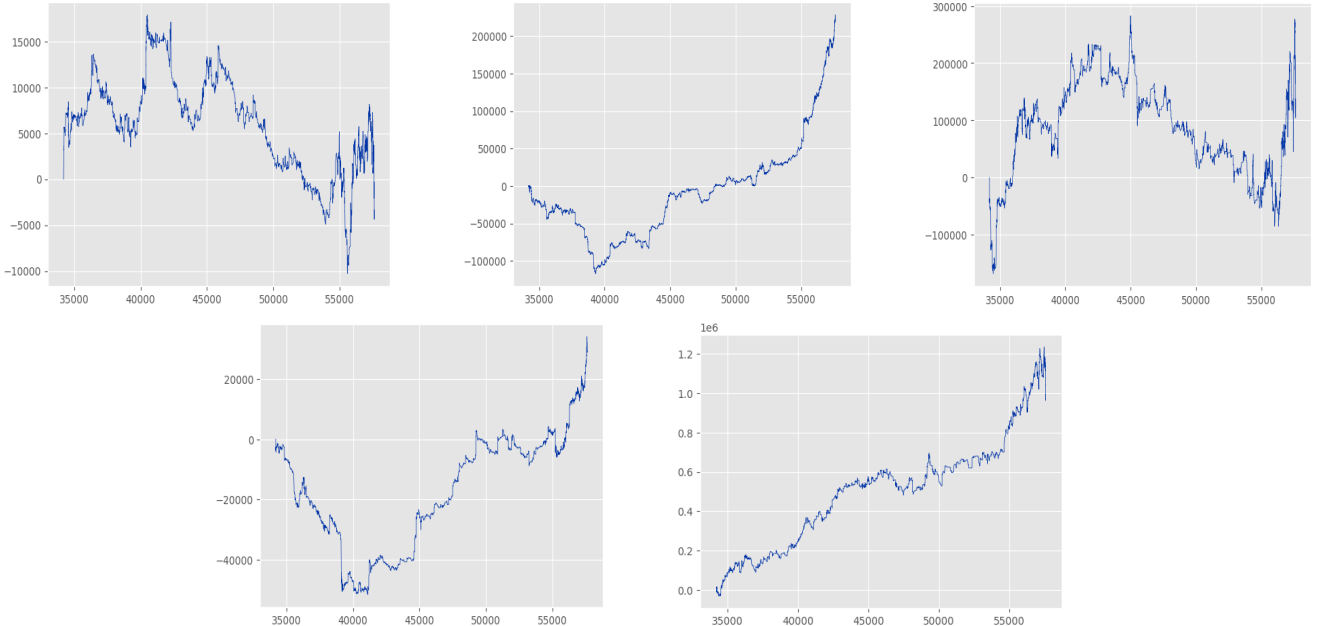


Figure 1: Signed order flow for GOOG, AAPL, MSFT, AMZN and INTC

<sup>4</sup>only using columns 4 and 5 from the lobsterdata.csv files which correspond to actual realized trades.

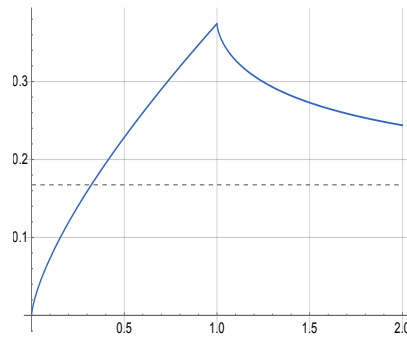


Figure 2: Concave price impact of an exogenous metaorder executed at constant trading speed 1 over  $[0, 1]$  with  $H = -0.2$ ,  $\theta = 0.1$  and  $\kappa = 1$ . The blue line asymptotes to the constant level  $MI(\infty)$  (grey line) as  $t \rightarrow \infty$  which represents the asymptotic permanent price impact.

## References

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## A Appendix A

**Proof.** Set  $G_t := \sup_{0 \leq r \leq t} |g(r)|$ ,  $K_t := \int_0^t |f(r)| dr > 0$ . For  $n \geq 1$  define the level hitting time  $\tau_n := \inf\{s \geq 0 : F_s^+ \geq n\}$ .

Since  $F^+$  is continuous and nondecreasing, on  $\{\tau_n \leq t\}$  we have  $F_{\tau_n}^+ = n$  and  $F_r^+ \leq n$  for all  $0 \leq r \leq \tau_n$ . Fix  $n \geq 1$  and work on the event  $\{\tau_n \leq t\}$ . For  $0 \leq r \leq \tau_n$  we have  $|B_{F_r^+}| \leq \sup_{0 \leq u \leq n} |B_u|$ . Setting  $s = \tau_n$  and taking absolute values we see that

$$\begin{aligned}
 n &= g(\tau_n) + \int_0^{\tau_n} f(\tau_n - r) B_{F_r^+} dr \leq |g(\tau_n)| + \int_0^{\tau_n} |f(\tau_n - r)| |B_{F_r^+}| dr \\
 &\leq G_t + \left( \int_0^{\tau_n} |f(\tau_n - r)| dr \right) \sup_{0 \leq u \leq n} |B_u| \\
 &= G_t + \left( \int_0^{\tau_n} |f(s)| ds \right) \sup_{0 \leq u \leq n} |B_u| \leq G_t + K_t \sup_{0 \leq u \leq n} |B_u|.
 \end{aligned}$$

Hence for  $n > G_t$ , on  $\{\tau_n \leq t\}$  we have  $\{\sup_{0 \leq u \leq n} |B_u| \geq \frac{n - G_t}{K_t}\}$ . Since  $F_t^+ \geq n$  iff  $\tau_n \leq t$ , we obtain

$$\mathbb{P}(F_t^+ \geq n) \leq \mathbb{P}\left(\sup_{0 \leq u \leq n} |B_u| \geq \frac{n - G_t}{K_t}\right), \quad n > G_t.$$

By the standard reflection principle bound  $\mathbb{P}(\sup_{0 \leq u \leq n} |B_u| \geq a) \leq 4 \exp(-\frac{a^2}{2n})$ , for  $a > 0$ , so for  $n > G_t$ ,

$$\mathbb{P}(F_t^+ \geq n) \leq 4 \exp(-\frac{(n - G_t)^2}{2nK_t^2}).$$

In particular, if  $n \geq 4G_t$  then  $(n - G_t)^2/n \geq \frac{1}{2}n$ , hence

$$\mathbb{P}(F_t^+ \geq n) \leq 4e^{-\frac{n}{4K_t^2}}, \quad n \geq 4G_t. \quad (\text{A-1})$$

Since  $F_t^+ \geq 0$ , we also know that  $\mathbb{E}(F_t^+) = \int_0^\infty \mathbb{P}(F_t^+ \geq x) dx$ . Now set  $m := \lceil 4G_t \rceil$ . Then

$$\mathbb{E}(F_t^+) \leq m + \int_m^\infty \mathbb{P}(F_t^+ \geq x) dx = m + \sum_{n=m}^\infty \int_n^{n+1} \mathbb{P}(F_t^+ \geq x) dx \leq m + \sum_{n=m}^\infty \mathbb{P}(F_t^+ \geq n). \quad (\text{A-2})$$

Using (A-1) for all  $n \geq m$ ,

$$\sum_{n=m}^\infty \mathbb{P}(F_t^+ \geq n) \leq \sum_{n=m}^\infty 4 \exp(-\frac{n}{4K_t^2}) < \infty. \quad (\text{A-3})$$

Combining (A-2) and (A-3) yields  $\mathbb{E}(F_t^+) < \infty$ . ■