

## Background, advice and ideas for Project 3

- The theoretical definition of the  $VIX_t^2 = -\frac{2}{\Delta} \mathbb{E}^{\mathbb{Q}}(\log(e^{-(r-q)\Delta} \frac{S_{t+\Delta}}{S_t}) | \mathcal{F}_t)$ , where  $\Delta = 30$  days and  $\mathcal{F}_t$  and  $\mathbb{Q}$  are the market filtration and risk-neutral measure being used to price European options on the SPX, and  $r$  and  $q$  are the interest rate and dividend yield on the SPX (assumed constant here for simplicity). In practice this formula is approximated using prices of a finite set of European options (which comes from the Breeden-Litzenberger formula applied to the log payoff). The VIX is not a tradeable contract in itself, but you can trade futures on the VIX (see chapter 1 of FM14 notes on KEATS).
- You can try adapting your code to maximize exp utility using options on more than 1 asset, e.g. European options paying  $(X - K)^+$ ,  $(Y - K)^+$  and/or  $(X - KY)^+$  (for a range of strike values  $K$ ) as in Project 3, where  $X$  and  $Y$  are e.g. two exchange rates at some future maturity date (option price data for this is given in Table 1 in <https://martinforde.github.io/FXCrossSmiles-updated.pdf>) or see data at <https://www.investing.com/currencies/eur-usd-options>, and use 2d Gauss-Legendre quadrature to integrate over the joint density of  $(X, Y)$  or Monte Carlo to replace the 1d integration scheme in your Part 2, and either choose or fit a market model using a time series of data from e.g. Yahoo finance which (ideally) ends on the same as date as the option prices are quoted, e.g. a GARCH model (see code and documentation on my website). Or you can e.g. look at SPX and VIX options together and price options using Monte-Carlo rather than numerical integration (can use [https://github.com/jgatheral/QuadraticRoughHeston/blob/main/qrho\\_simulation.ipynb](https://github.com/jgatheral/QuadraticRoughHeston/blob/main/qrho_simulation.ipynb) on github for this).

**Just switching from exponential to power utility in Part 3 is not going to impress the markers.**

- Can discuss background concepts for general non-linear optimizations problems: the **Karush–Kuhn–Tucker (KKT)** conditions, **complementary slackness**, primal/dual feasibility, **KKT/Lagrange multipliers**, **saddlepoints**, **sensitivities**, subgradients etc. (see e.g. [https://en.wikipedia.org/wiki/Karush-Kuhn-Tucker\\_conditions](https://en.wikipedia.org/wiki/Karush-Kuhn-Tucker_conditions))
- **Very instructive toy practice question:** Consider an agent who can buy  $x$  units of a single stock whose initial price is  $p$  and/or leave some of their money in cash time zero, and assume interest rates are zero for simplicity, with a total wealth constraint of  $w$ , with no further trading after time zero. Assume the terminal stock price is  $\sim \text{Exp}(1)$ , and the trader wishes to maximize their expected exponential utility i.e.  $-\mathbb{E}(e^{-\lambda(xS+w-xp)})$  over  $x \in (-\infty, w/p]$ , with risk-aversion parameter  $\lambda$ . Compute the optimal amount of stock to buy, and comment on the answer. Note that the utility function  $U(w) = -e^{-\lambda w}$  is concave and increasing, capturing the economic intuition that more money makes us happier but the marginal utility derived from an additional pound decreases as our wealth increases.

**Solution.** The problem is to maximize

$$\mathbb{E}(U(xS + w - xp)) = \mathbb{E}(-e^{-\lambda(xS+w-xp)}) = \int_0^\infty -e^{-\lambda(xS+w-xp)} e^{-S} dS = -\frac{e^{-w\lambda+px\lambda}}{1+x\lambda} 1_{\{\lambda x+1>0\}} - \infty \cdot 1_{\{\lambda x+1\leq 0\}}$$

over  $x \in (-\infty, w/p]$ . Differentiating this expression wrt  $x$  and setting the answer to zero, we find that the optimal stock holding is

$$x^* = \frac{1-p}{p\lambda} \quad (1)$$

if  $\frac{1-p}{p\lambda} < w/p$  and  $\lambda x + 1 > 0$  so the feasible set is  $x \in (-\frac{1}{\lambda}, \frac{w}{p})$ ; otherwise  $x^* = w/p$  (i.e. we hit the wealth constraint). Note we have a minus sign in front of the expectation because we need  $U(x)$  to be **concave**.

Note that the **fair price** i.e. the **expected payout** of the stock is  $\mathbb{E}(S) = 1$ , so (1) says that we **buy stock when the stock is underpriced**, and **sell when the stock is overpriced**, and  $|x^*|$  is smaller when  $\lambda$  is larger i.e. when the agent is more **risk-averse**.

**Note the same issue is relevant for Part 2: you should check that the fair price of your optimal portfolio under the market model (i.e. the expected payout) is  $\geq$  the market price of this portfolio.**

**You can try and come up with variant of this example in your Part 1, and possibly include indifference pricing for e.g. an additional call option.** See <https://arxiv.org/abs/2403.00139> for the case when we have European options at all strikes.

In the project  $S$  is replaced by  $\xi$  and  $\log \xi$  is Normally distributed, and there are call options in the problem; in particular note that  $\mathbb{E}(e^{-\lambda x(\xi_T - K)^+}) = \infty$  if  $x < 0$ , because we have the exponential of the exponential of a Gaussian here. **This same right tail issue arises in Part 2 and you should think about how to avoid this.**

- **General case with  $d$  assets and no liquidity constraints or bid-ask spreads.** Consider a financial market defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $d$  assets with **random payoffs**  $(S_1, \dots, S_d)$  at time  $T$  (which are **linearly independent**) with market prices  $\pi_i$  at  $t = 0$  (these assets can include **European call/put options**). Let  $Y_i = S_i - \pi_i$ , and assume a financial agent can only trade at time zero.

Derive the **first-order optimality condition** for an agent to maximize their **expected utility**  $\mathbb{E}(U(b \cdot Y))$  over  $b \in \mathbb{R}^d$ , where  $U : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function with  $U''(x) < 0$  and  $b_i$  is the position in the  $i$ 'th asset and we assume that  $\mathbb{E}(U(b \cdot Y)) < \infty$  for all  $b \in \mathbb{R}^d$ .

**Solution.** As in first year calculus, we compute derivatives wrt each  $b_i$  and then set the answer to zero:

$$\frac{\partial}{\partial b_i} \mathbb{E}(U(b \cdot Y)) = \mathbb{E}(Y_i U'(b \cdot Y)) = 0$$

for  $i = 1, \dots, d$ , i.e. we have  $d$  equations for the  $d$  unknowns  $b_1^*, \dots, b_d^*$  for the optimal portfolio allocation  $b^*$ . Note we can re-write this as

$$\mathbb{E}^{\mathbb{Q}}(Y_i) = 0 \tag{2}$$

where we define a **new probability measure** as  $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left(\frac{U'(b^* \cdot Y)}{\mathbb{E}^{\mathbb{P}}(U'(b^* \cdot Y))} 1_A\right)$  for events  $A \in \mathcal{F}$ , and (as a sanity check) we note that  $\mathbb{Q}(\Omega) = 1$ .

Under the moment condition stated in the question, it turns out that a unique solution  $b^*$  exists if the **no-arbitrage** condition is satisfied: if  $\mathbb{P}(b \cdot Y > 0) > 0$  then  $\mathbb{P}(b \cdot Y < 0) > 0$ . In this case, from (2), we see that under  $\mathbb{Q}$ , all contracts are priced according to the market, i.e.  $\mathbb{E}^{\mathbb{Q}}(S_i) = \pi_i$ .  $\mathbb{Q}$  is known as a **risk-neutral measure**. We can solve this maximization problem numerically using MOSEK.

In the project we use the **exponential utility function**  $U(x) = -e^{-\lambda x}$ , in which case (for  $\lambda = 1$ ) we are computing  $\max_b (-\mathbb{E}(e^{-b \cdot Y})) = \max_b (-\mathbb{E}(e^{b \cdot Y})) = -\min_b \mathbb{E}(e^{b \cdot Y})$ , i.e. minus the **minimum of the mgf** of  $Y$ .