

Convex order condition

Let $(X_t)_{t \geq 0}$ denote a martingale. Then for $0 \leq T_1 \leq T_2$:

$$\begin{aligned}\mathbb{E}((X_{T_2} - K)^+) &= \mathbb{E}(\mathbb{E}((X_{T_2} - K)^+ | X_{T_1})) \quad (\text{from the \textbf{tower property}}) \\ &\geq \mathbb{E}(\mathbb{E}(X_{T_2} - K | X_{T_1})^+) \\ &\quad (\text{from the \textbf{conditional Jensen inequality} applied to the convex function } f(x) = (x - K)^+) \\ &= \mathbb{E}((X_{T_1} - K)^+).\end{aligned}$$

Hence we see that call option prices with maturity T_2 are \geq call option prices with maturity T_1 . This is known as the **convex ordering** condition, which we can write as $\mu_{T_1} \preceq \mu_{T_2}$, where μ_t denotes the density of X_t .

Bass martingale with random initial starting distribution

Let B^α denote a Brownian motion with $B_0^\alpha \sim \alpha$ (i.e. a random initial starting point with density $\alpha(x)$, and assume the process $B_{(\cdot)}^\alpha - B_0^\alpha$ is independent of B_0^α). Then the density of B_t^α is

$$\int_{-\infty}^{\infty} R_t(y - x)\alpha(x)dx = (R_t * \alpha)(y).$$

Moreover

$$\mathbb{E}(F(B_1^\alpha) | B_t^\alpha = x) = \int_{-\infty}^{\infty} R_{1-t}(y - x)F(y)dy = (R_{1-t} * F)(x).$$

Now let $M_t = \mathbb{E}(F(B_1^\alpha) | B_t^\alpha)$ for $t \in (0, 1]$. We wish to choose F and α so that $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$, for two given distributions μ_0 and μ_1 (both with zero expectations), with μ_0, μ_1 in convex order.

Then

$$M_t = (R_{1-t} * F)(B_t^\alpha). \quad (1)$$

Let μ be a probability density. The **push-forward** $F_\# \mu$ of μ by F is the distribution of $F(X)$ if $X \sim \mu$, so

$$\mathbb{P}(F(X) \leq x) = \mathbb{P}(X \leq F^{-1}(x)) = \int_{-\infty}^{F^{-1}(x)} \mu(y)dy. \quad (2)$$

Thus if μ_t denotes the density of M_t , (1) implies that

$$\mu_t = (R_{1-t} * F)_\#(R_t * \alpha)$$

since the distribution of B_t is $R_t * \alpha$. In particular

$$\begin{aligned}\mu_0 &= (R_1 * F)_\#(\alpha) \\ \mu_1 &= F_\#(R_1 * \alpha)\end{aligned} \quad (3)$$

since $R_0 * f = f$ for any f . This suggests an **alternating iterative scheme**:

$$\mu_0 = (R_1 * F^n)_\#(\alpha^{n+1}), \quad (4)$$

$$\mu_1 = F_\#^{n+1}(R_1 * \alpha^{n+1}) \quad (5)$$

to solve for (α, F) , with $F^0(x) = x$ as the initial guess. For any probability distribution μ , let G_μ denote the cdf associated with μ . Then to compute α^{n+1} , we use (2) applied to (3):

$$\begin{aligned}G_{\mu_0}(x) &= \int_{-\infty}^{(R_1 * F^n)^{-1}(x)} \alpha^{n+1}(y)dy \\ \Rightarrow G_{\mu_0}((R_1 * F^n)(x)) &= \int_{-\infty}^x \alpha^{n+1}(y)dy = G_{\alpha^{n+1}}(x)\end{aligned}$$

and we can differentiate the final line wrt x to get the density $\alpha^{n+1}(x)$ using the fundamental theorem of calculus. Using this α^{n+1} , the solution to the second equation in (5) is then $F^{n+1}(x) = G_{\mu_1}^{-1}(G_{R_1 * \alpha^{n+1}}(x))$, and we then repeat this iterative process.

Deriving the Conze-Henry-Labordere fixed point equation for the distribution function of α

Sampling: draw from α , and then set $X_0 = G_{\mu_0}^{-1} \circ G_\alpha(B_0) = (R_1 * F)(B_0)$

Recall that for two probability densities ν_1 and ν_2 , for $h = G_{\nu_1}^{-1}(G_{\nu_2})$, $\nu_1 = h_\# \nu_2$. Applying this to (3) (assuming F is strictly increasing) we see that

$$\begin{aligned} R_1 * F &= G_{\mu_0}^{-1} \circ G_\alpha \\ &= G_{\mu_1}^{-1} \circ G_{R_1 * \alpha} = G_{\mu_1}^{-1} \circ (R_1 * G_\alpha) \end{aligned} \tag{6}$$

using that $G_{R_1 * \alpha} = R_1 * G_\alpha$. To check this identity, we take derivatives of the right hand side wrt x to get

$$\frac{d}{dx} \int_{-\infty}^{\infty} R_1(y) G_\alpha(x-y) dy = \int_{-\infty}^{\infty} R_1(y) G'_\alpha(x-y) dy = \int_{-\infty}^{\infty} R_1(y) \alpha(x-y) dy = (R_1 * \alpha)(x).$$

Then using (6) and then (7), we see that

$$G_\alpha = G_{\mu_0} \circ (R_1 * F) = G_{\mu_0} \circ (R_1 * (G_{\mu_1}^{-1} \circ (R_1 * G_\alpha))) = \Phi(G_\alpha) \tag{7}$$

where Φ is shorthand for the all operators successively being applied to G_α on the right hand side.

This is conceptually similar to a simple non-linear 1d equation of the form $x = g(x)$ in first year Numerical analysis, which we can solve using the fixed point method $x_{n+1} = g(x_n)$ if $|g'(x)| < 1$. We can use the same method here except now the scalar x_n is replaced by a function G_α^n , so the iterative scheme becomes

$$G_\alpha^{n+1} = \Phi(G_\alpha^n) \tag{8}$$

which (under suitable conditions) converges to a function $G_\alpha^\infty(\cdot)$, which is the desired cdf for α so as to make $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$. Note once we have G_α can compute the distribution of B_1^α and hence the required function F to make $F(B_1^\alpha) \sim \mu_1$.

Short piece of Python code to implement (8) available at

<https://colab.research.google.com/drive/1neZR8aRTJnJ4PspoHCMt6kroGJEP87HL?usp=sharing>