

## Convex order condition

Let  $(X_t)_{t \geq 0}$  denote a martingale. Then for  $0 \leq T_1 \leq T_2$ :

$$\begin{aligned} \mathbb{E}((X_{T_2} - K)^+) &= \mathbb{E}(\mathbb{E}((X_{T_2} - K)^+ | X_{T_1})) \quad (\text{from the } \mathbf{tower \ property}) \\ &\geq \mathbb{E}(\mathbb{E}(X_{T_2} - K | X_{T_1})^+) \\ &\quad (\text{from the } \mathbf{conditional \ Jensen \ inequality} \text{ applied to the convex function } f(x) = (x - K)^+) \\ &= \mathbb{E}((X_{T_1} - K)^+). \end{aligned}$$

Hence we see that call option prices with maturity  $T_2$  are  $\geq$  call option prices with maturity  $T_1$ . This is known as the **convex ordering** condition, which we can write as  $\mu_{T_1} \preceq \mu_{T_2}$ , where  $\mu_t$  denotes the density of  $X_t$ .

## Bass martingale with random initial starting distribution

Let  $B^\alpha$  denote a Brownian motion with  $B_0^\alpha \sim \alpha$  (i.e. a random initial starting point with density  $\alpha(x)$ ), and assume the process  $B_{(\cdot)}^\alpha - B_0^\alpha$  is independent of  $B_0^\alpha$ . Then the density of  $B_t^\alpha$  is

$$\int_{-\infty}^{\infty} R_t(y - x) \alpha(x) dx = (R_t * \alpha)(y).$$

Moreover

$$\mathbb{E}(F(B_1^\alpha) | B_t^\alpha = x) = \int_{-\infty}^{\infty} R_{1-t}(y - x) F(y) dy = (R_{1-t} * F)(x).$$

Now let  $M_t = \mathbb{E}(F(B_1^\alpha) | B_t^\alpha)$  for  $t \in (0, 1]$ . We wish to choose  $F$  and  $\alpha$  so that  $M_0 \sim \mu_0$  and  $M_1 \sim \mu_1$ , for two given distributions  $\mu_0$  and  $\mu_1$  (both with zero expectations), with  $\mu_0, \mu_1$  in convex order.

Then

$$M_t = (R_{1-t} * F)(B_t^\alpha). \quad (1)$$

Let  $\mu$  be a probability density. We define the **push-forward**  $F_{\#}\mu$  of  $\mu$  by  $F$  as the distribution of  $F(X)$  if  $X \sim \mu$ , so

$$\mathbb{P}(F(X) \leq x) = \mathbb{P}(X \leq F^{-1}(x)) = \int_{-\infty}^{F^{-1}(x)} \mu(y) dy. \quad (2)$$

Thus if  $\mu_t$  denotes the density of  $M_t$ , (1) implies that

$$\mu_t = (R_{1-t} * F)_{\#}(R_t * \alpha)$$

since the distribution of  $B_t$  is  $R_t * \alpha$ . In particular

$$\begin{aligned} \mu_0 &= (R_1 * F)_{\#}(\alpha) \\ \mu_1 &= F_{\#}(R_1 * \alpha) \end{aligned} \quad (3)$$

since  $R_0 * f = f$  for any  $f$ . This suggests an alternating iterative scheme:

$$\begin{aligned} \mu_0 &= (R_1 * F^n)_{\#}(\alpha^{n+1}) \\ \mu_1 &= F_{\#}^{n+1}(R_1 * \alpha^{n+1}) \end{aligned}$$

to solve for  $(\alpha, F)$ , with  $F^0(x) = x$  as the initial guess. For any probability distribution  $\mu$ , let  $G_\mu$  denote the cdf associated with  $\mu$ . Then to compute  $\alpha^1$ , we use (2):

$$G_{\mu_0}(x) = \int_{-\infty}^{(R_1 * F)^{-1}(x)} \alpha^{n+1}(y) dy \Rightarrow G_{\mu_0}((R_1 * F)(x)) = \int_{-\infty}^x \alpha^{n+1}(y) dy = G_{\alpha^{n+1}}(x)$$

and we can then differentiate this wrt  $x$  to get  $\alpha^{n+1}(x)$  using the fundamental theorem of calculus.

## Deriving the Conze-Labordere fixed point equation for the distribution function of $\alpha$

Recall that for two probability densities  $\nu_1$  and  $\nu_2$ , for  $h = G_{\nu_1}^{-1}(G_{\nu_2})$ ,  $\nu_1 = h_{\#}\nu_2$ . Applying this to (3) (assuming  $F$  is strictly increasing) we see that

$$\begin{aligned} R_1 * F &= G_{\mu_0}^{-1} \circ G_{\alpha} \\ F &= G_{\mu_1}^{-1} \circ G_{R_1 * \alpha} = G_{\mu_1}^{-1} \circ (R_1 * G_{\alpha}) \end{aligned} \tag{4}$$

using that  $G_{R_1 * \alpha} = R_1 * G_{\alpha}$ . To check this identity, we take derivatives of the right hand side wrt  $x$  to get

$$\frac{d}{dx} \int_{-\infty}^{\infty} R_1(y) G_{\alpha}(x-y) dy = \int_{-\infty}^{\infty} R_1(y) G'_{\alpha}(x-y) dy = \int_{-\infty}^{\infty} R_1(y) \alpha(x-y) dy = (R_1 * \alpha)(x).$$

Then using (4) and then (5), we see that

$$G_{\alpha} = G_{\mu_0} \circ (R_1 * F) = G_{\mu_0} \circ (R_1 * (G_{\mu_1}^{-1} \circ (R_1 * G_{\alpha}))) = \Phi(G_{\alpha})$$

where  $\Phi$  is shorthand for the all operators successively being applied to  $G_{\alpha}$  on the right hand side.

This is conceptually similar to a simple non-linear 1d equation of the form  $x = g(x)$  in first year Numerical analysis, which we can solve using the fixed point method  $x_{n+1} = g(x_n)$  if  $|g'(x)| < 1$ . We can use the same method here except now the scalar  $x_n$  is replaced by a function  $G_{\alpha}^n$ , so the iterative scheme becomes

$$G_{\alpha}^{n+1} = \Phi(G_{\alpha}^n)$$

which (under suitable conditions) converges to a function  $G_{\alpha}^{\infty}(\cdot)$ , which is the desired cdf for  $\alpha$  so as to make  $M_0 \sim \mu_0$  and  $M_1 \sim \mu_1$ . Note once we have  $G_{\alpha}$  can compute the distribution of  $B_1^{\alpha}$  and hence the required function  $F$  to make  $F(B_1^{\alpha}) \sim \mu_1$ .