Alternate estimators for H

The $m(q,\Delta)$ estimator for fBM

Let $X_t = \sigma B^H_t$, and set $SS^{(q)}_n := \frac{1}{n}\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^q$. Then

$$\mathbb{E}(SS^{(q)}_n) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^q) \;=\; \Delta^{qH}\frac{1}{n}\sum_{i=1}^n \mathbb{E}(|X_i - X_{i-1}|^q) \;=\; \sigma^q \mathbb{E}(|Z|^q)\Delta^{qH} \;=\; \sigma^q K_q \Delta^{qH} \quad (1)$$

where $\Delta = \frac{1}{n}$, $K_q = \mathbb{E}(|Z|^q) = \frac{2^{q/2}}{\sqrt{\pi}}\Gamma(\frac{q+1}{2})$, (for $q > -1$) and $Z \sim N(0,1)$. This naturally leads to the estimates $(\hat{H}_n, \hat{\sigma}_n)$ for (H,σ) defined by

$$SS^{(q)}_n \;=\; \hat{\sigma}^q_n K_q \Delta^{q\hat{H}_n}$$

if we have computed $SS^{(q)}_n$ for at least two Δ-values. Taking logs we see that

$$\log SS^{(q)}_n \;=\; q\log\hat{\sigma}_n + \log K_q \;+\; q\hat{H}_n \log\Delta$$

so we can perform **linear regression** on $\log SS^{(q)}_n$ vs $\log\Delta$ for a range of Δ-values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal $q\hat{H}_n$ (q is chosen by you, e.g. $q = 1, 2, 2.5, 3$ etc), and the **intercept** at $\log\Delta = 0$ is $q\log\hat{\sigma}_n + \log K_q$, from which we can compute $\hat{\sigma}_n$ since K_q has an explicit formula above. This is the $m(q,\Delta)$ estimator discussed in [GJR18]. One can then also compute the R^2**-statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of $\hat{H}_n$ and $\hat{\sigma}_n$.

To approximate the effect of using **realized variance** with m subwindows to estimate V_t (as in Part 2) , we can use the Central Limit Theorem approximation from FM02:

$$V_{i\Delta} \;=\; V_0 e^{\sigma B^H_{i\Delta}}(1 + \sqrt{\frac{2}{m}}\,\varepsilon_i)$$

where the ε_i's are i.i.d. $N(0,1)$ (and independent of B^H), so (using that $\log(1+x) = x + O(x^2)$), we see that $\log V_{i\Delta} = \log V_0 + \sigma B^H_{i\Delta} + \sqrt{\frac{2}{m}}\varepsilon_i + O(\frac{1}{m}) = \log V_0 + X_{i\Delta} + \sqrt{\frac{2}{m}}\varepsilon_i + O(\frac{1}{m})$.

For convenience we now define $\tilde{X}_{i\Delta} = X_{i\Delta} + \sqrt{\frac{2}{m}}\,\varepsilon_i$. Then $\tilde{X}_{i\Delta} - \tilde{X}_{(i-1)\Delta} = X_{i\Delta} - X_{(i-1)\Delta} + \sqrt{\frac{2}{m}}(\varepsilon_i - \varepsilon_{(i-1)}) \sim N(0, \sigma^2\Delta^{2H} + \frac{4}{m})$, and setting $\widetilde{SS}^{(2)}_n := \frac{1}{n}\sum_{i=1}^n |\tilde{X}_{i\Delta} - \tilde{X}_{(i-1)\Delta}|^2$, adding the effect of ε into the computation above we find that

$$\mathbb{E}(\widetilde{SS}^{(2)}_n) \;=\; \sigma^2\Delta^{2H} + \frac{4}{m}$$

(since $K_q = 1$ for $q = 2$) so we now regress $\log(\widetilde{SS}^{(2)}_n - \frac{4}{m})$ vs $\log\Delta$, which provides a smart adjustment to $\hat{H}_n$ (this adjustment is made in
`https://colab.research.google.com/drive/1jJGf4bVWETJqWRIMZ6STjsJ9jINgEaPd#scrollTo=e9fn6ABHNf52`

The $m(q,\Delta)$ estimator for the RL process

If $X_t = \sigma Z^H_t$ where $Z^H_t = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ is an RL process, then for $q = 2$ and $H \in (0, \frac{1}{2})$ one can show that

$$\mathbb{E}(SS^{(2)}_n) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^2) \;=\; \sigma^q\Delta^{2H}\frac{1}{n}\sum_{i=1}^n \mathbb{E}(|Z^H_i - Z^H_{i-1}|^2) \;\sim\; \sigma^2 c_H \Delta^{2H}$$

as $n \to \infty$ for some constant $c_H > 0$,

$$\log(SS^{(2)}_n) \;=\; 2\log\hat{\sigma}_n \;+\; \log c_{\hat{H}} \;+\; 2\hat{H}_n \log\Delta$$

so we can still perform linear regression as before, but now the intercept is $2\log\hat{\sigma}_n \;+\; \log c_{\hat{H}}$, which leads to an adjusted estimate for $\hat{\sigma}_n$ (I will try to compute c_H explicitly, otherwise c_H has to be estimated numerically with Monte Carlo).

The Han-Schied [HS21] estimator

Let $X_t = \sigma B^H_t$ and let

$$\theta_{m,k} \quad = \quad 2^{\frac{m}{2}}(2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}) \quad = \quad -2^{\frac{m}{2}}(X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{2k}{2^{m+1}}})$$

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for $R(s,t) = \mathbb{E}(B^H_s B^H_t)$, one can check that

$$\mathbb{E}(\theta^2_{m,k}) \quad = \quad \sigma^2 2^{m-2H(1+m)}(4-4^H)\,. \tag{2}$$

Then setting $s_n^2 = \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\theta^2_{m,k}$, we see that $\mathbb{E}(s_n^2) = \sum_{m=0}^{n-1} 2^m \mathbb{E}(\theta^2_{m,k})$ (since (2) does not depend on k) which simplifies to

$$\mathbb{E}(s_n^2) \quad = \quad \sigma^2(4^{n(1-H)} - 1) \quad \sim \quad \sigma^2 4^{n(1-H)} \quad = \quad \sigma^2 2^{2n(1-H)} \tag{3}$$

as $n \to \infty$, which suggests an estimator $\hat{H}_n$ defined by $s_n = \hat{\sigma}_n\, 2^{n(1-\hat{H}_n)}$ which (assuming $\hat{\sigma}_n = O(1)$ as $n \to \infty$) we can re-arrange as

$$\hat{H}_n \quad = \quad 1 - \frac{1}{n}\log_2(\frac{s_n}{\hat{\sigma}_n}) \quad = \quad 1 - \frac{1}{n}\log_2 s_n \,+\, O(\frac{1}{n})$$

where $\log_2$ denotes the base-2 logarithm, so (ignoring the $O(\frac{1}{n})$ remainder term), we recover the Han-Schied[HS21] estimator $\hat{H}_n = 1 - \frac{1}{n}\log_2 s_n$. Then

$$\begin{aligned}
\mathbb{E}(\hat{H}_n) \quad = \quad 1 - \frac{1}{n}\mathbb{E}(\log_2(s_n)) \quad = \quad 1 - \frac{1}{2n}\mathbb{E}(\log_2(s_n^2)) \quad \geq \quad 1 - \frac{1}{2n}\log_2\mathbb{E}(s_n^2) \quad &= \quad 1 - \frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)} - 1)) \\
&\geq \quad 1 - \frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)})) \\
&= \quad 1 - \frac{1}{2n}\log_2(\sigma^2) \,-\, \frac{1}{2n}\log_2(4^{n(1-H)})) \\
&= \quad H - \frac{1}{2n}\log_2(\sigma^2)
\end{aligned}$$

and the final line is $> H$ if $\sigma < 1$, so $\mathbb{E}(\hat{H}_n) > H$ if $\sigma < 1$. See also discussion on optimal scaling factors in [HS21].

For more a general process X, (under certain conditions) [HS21] show that $\hat{H}_n \to R$ as $n \to \infty$, where $R = 1/q$, where q is the critical p-value at which $\lim_{n\to\infty}\sum_{i=1}^n |X_{i\Delta} - X^H_{(i-1)\Delta}|^p$ switches from being zero (for $p > q$) to $+\infty$ (for $p < q$) (see Eq 2.2 in [HS21]).

Or we can jointly estimate H and σ by performing linear regression since

$$\log s_n \quad = \quad \log\hat{\sigma}_n \,+\, n(1-\hat{H}_n)\log 2 \tag{4}$$

but we now have to compute $\log s_n$ for a range of different n-values to get a line of best fit, for which the slope is $(1-\hat{H}_n)\log 2$ and the intercept is $\log\hat{\sigma}_n$.

You can then draw histograms of $\hat{H}_n$ if you simulate M fBM paths and compute the sample variance for $\hat{H}_n$ (or a confidence interval), or compute $\hat{H}_n$ for real data, e.g. using the SPX data file or data from yahoo finance.

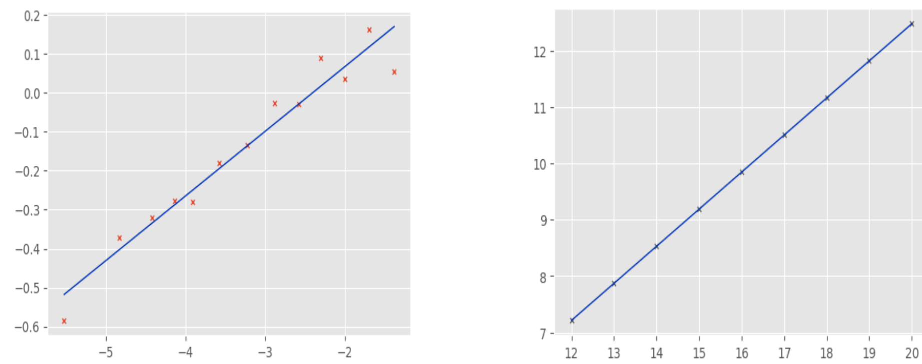

Figure 1: On the left, we see estimates of H for the SPX using the $m(q,\Delta)$ method for the SPX from 3rdJan22-15thJul24 for $q=2$ for which $\hat{H}=0.0830$ and $\hat{\sigma}=1.221$ (see similar plots in [GJR18]. On the right we see the linear regression in (4) for the Han-Schied method ($\log s_n$ vs n) for a true fBM path with 2^{20} time points, for which $\hat{H}=0.0508$, and $\hat{\sigma}=1.010$.

The rough Heston model

The driftless rough Heston model satisfies

$$V_t \;=\; V_0+\nu\int_0^t (t-u)^{H-\frac{1}{2}}\sqrt{V_u}\,dW_u\,.$$

Then $\mathbb{E}(V_t)=V_0$, and V has covariance function:

$$\begin{aligned}\mathbb{E}((V_s-V_0)(V_t-V_0)) &= \nu^2\mathbb{E}\Big(\int_0^s (s-u)^{H-\frac{1}{2}}\sqrt{V_u}\,dW_u\cdot\int_0^t (t-r)^{H-\frac{1}{2}}\sqrt{V_r}\,dW_r\Big)\\ &= \nu^2\int_0^s (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}\mathbb{E}(V_u)\,du\\ &= V_0\nu^2\int_0^s (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}\,du \;=\; V_0\nu^2\bar{R}(s,t)\end{aligned}$$

for $0\le s\le t$, where $\bar{R}(s,t)$ is the covariance function for the Riemann-Liouville (RL) process $Z_t=\int_0^t(t-u)^{H-\frac{1}{2}}dW_u$ used for the rough Bergomi model (note Z is a Gaussian process but V is not), but the explicit formula for $\bar{R}(s,t)$ is more complicated than the $R(s,t)$ formula for fBM.

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H \;=\; c_H\Big(\int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}})\,dW_s+\int_0^t (t-s)^{H-\frac{1}{2}}\,dW_s\Big) \;=\; c_H(A_t+Z_t)$$

for $t\ge 0$, in terms of an RL process Z (and note that A_t is known at time zero for all $t\ge 0$), and $c_H=\big(\frac{2H\,\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\,\Gamma(2-2H)}\big)^{\frac{1}{2}}$. Note also that A_t and Z_t are independent.

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