# Rough Bergomi revisited - exact and minimal-variance hedging for VIX and European options, and exact calibration to multiple smiles

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**Abstract**

For a generalized rough Bergomi-type model, we formally show how to replicate a VIX option with dynamic trading in a VIX future, and a European option, using the Clark-Ocone formula from Malliavin calculus. We also compute the minimal variance hedge for a European call when we can only dynamically hedge with the underlying, which is relevant in practice since dynamic trading with a VIX future will incur a larger bid-offer spread. This builds on the work of Keller-Ressel[KR22] who derives asymptotic approximations for the $\Delta$-hedge for the SABR and rough SABR models from [FG22], and as a by-product we construct a rough volatility model with an exact fit to target laws $\mu_1, \mu_2, ...$ at multiple maturities $T_1 < T_2 < ..$ (with $\mu_1, \mu_2, ..$ in convex order) where the squared volatility process $v$ can be characterized explicitly using the Clark-Ocone formula. We also provide numerical simulations, and explain how to adapt the well known Renault-Touzi[RT96] conditioning trick to reduce the sample variance of Monte Carlo estimates for the European call hedge at each time instant.[1]

## 1 Introduction

The Rough Bergomi (rBergomi) model introduced in [BFG16] has been a popular, tractable and much cited rough volatility model. For the rBergomi model (and the original Rough Fractional Stochastic Volatility (RFSV) model driven by fBM in [GJR18]), the log of the instantaneous variance process $V$ is Gaussian so the VIX index is approximately log-normally distributed and hence produces VIX smiles which are almost flat, not the concave upward-sloping smiles that we typically see in practice. A two-factor (also known as a skewed) Rough Bergomi model with a linear combination of exponential terms with two different $\nu$-values as discussed in [Guy21] (see also [JMP21] for extensions/variations e.g. using two different $H$-values and [AL24]) can often fit a single-maturity short-maturity VIX smile very well but if we do this it typically struggles to achieve sufficient at-the-money skew for options on the SPX itself with the same maturity (see [Guy21] and we have also seen this phenomenon first hand in testing).

Statistical evidence for rough volatility in the literature has typically involved assuming an RFSV model where $\log V$ is a multiple of fBM and testing the monofractal scaling relationship $m(q,\Delta) = c_q \Delta^{qH}$ with log-log plots using realized variance (see e.g. [GJR18] for definition of $m(q,\Delta)$) (which is essentially a GMM approach) but without any analysis of $p$-values/significance tests. Testing the residuals implied by the data using the inverse Cholesky/MLE approach with goodness-of-fit tests outlined in [F23b] shows that (at least for the SPX this decade), daily realized variance with 1min sampling intervals does not follow a standard RFSV model (see code on author's website). A two-factor RFSV model can achieve a reasonable fit to daily SPX returns post-covid (from e.g. Jan 22-July 24, see code link in [F24b]), but if we only have access to SPX prices at 1min increments one has to take this with a pinch of salt since even if we are just trying to estimate the volatility of a standard Brownian motion $W$, we know that

$$\sqrt{m}\,(\sum_{i=0}^{m-1}(W_{(i+1)T/m}-W_{iT/m})^2-T) \quad \sim \quad \sqrt{m}\,(\sum_{i=0}^{m-1}\Delta t Z_i^2 - T) \quad = \quad \sqrt{m}\,T(\frac{1}{m}\sum_{i=0}^{m-1}Z_i^2-1) \;\stackrel{w}{\to}\; N(0,2T)$$

where $\Delta t = T/m$ and $Z_1, Z_2, ...$ is a sequence of i.i.d. standard Normals, and we are using the Central Limit Theorem and that $\mathbb{E}(Z_i^2)=1$ and $\mathbb{E}(Z_i^4)=3$ so $\mathrm{Var}(Z_i^2)=2$. Hence the Relative Standard Error (RSE) here in estimating the $V$ is $\approx 1.65\sqrt{\frac{2}{m}}$, and for a 6.5hr trading day, 1min intervals correspond to $m=390$, so the RSE even when just estimating unknown constant volatility is 11.8% (see also Theorem 2.1 in [FTR19] and Example 3.1 in [BCPV23] and [AJ14] for more involved results on this phenomenon in a stochastic volatility setting but where the $\frac{2}{m}$ variance answer is the same, for essentially the same reason outlined here). In estimating $V$ we essentially incur additional white noise with variance $\frac{2}{m}$ which can lead to underestimating $H$, see also [CD22] for further discussion). One could use Garman-Klass or Rogers-Satchell et al. type estimators which incorporate additional information about the high and low prices to derive minimal variance estimators for volatility, but these results

[1] We thank Alan Lewis as always for many stimulating discussions.

typically assume the true volatility is also constant so they can appeal to classical results on the joint distribution of the min, max and terminal level or the range of Brownian motion.

The quadratic rough Heston model ([GJR20]) has an explicit formula for sampling the VIX (cf. chapter 6.2 in [Rom22]), obtained via the solution to a linear VIE in terms of the resolvent of the fractional kernel of the $Z$ process. and using the Gamma kernel $K(t)=e^{-\lambda t}t^{\alpha-1}$, the model often has an uncanny ability to fit close-to-1-month and 2 month SPX and 1month VIX smiles simultaneously very well with only 5 parameters $(\alpha,a,c,\lambda,\theta)$, setting $b=0$ W.L.O.G. and also fitting $Z_0$), but calibrated $H=\alpha-\frac{1}{2}$ values can often bounce around a lot from (e.g. 0.132 in Jan 2023 but .0624 in June 2024, see numerical results in [F24] and code on the author's website).

Exact hedging for call options under the rough Heston model is formulated in [ER18], where the call option price satisfies an PPDE in terms of a Fréchet derivative with respect to the entire forward variance cuve $\xi_t(u)$ (see also chapter 5 in [VZ19]) which evolves as $d\xi_t(u)=\frac{1}{\lambda}f^{\alpha,\lambda}(u-t)\nu\sqrt{V_t}dB_t$ where $f^{\alpha,\lambda}(.)$ is the *Mittag-Leffler* function, but it is rather difficult to implement this in practice since the call payoff also has to be re-written as a Fourier integral involving complex exponential contracts, each of which can then in turn be replicated with dynamic trading in the underlying and a continuum of forward variance contracts (see the final paragraph in subsection 1.1 below for a brief discussion on the practical difficulties in using the latter).

A standard GARCH(1,1) model with i.i.d. (skewed) $t$-distributed residuals typically fits daily historical returns very well across a wide range of assets and time windows when we apply standard goodness of fit tests (Kolmogorov-Smirnov, Shapiro-Wilks etc) to the residuals implied by daily returns using maximum likelihood estimates for the model parameters (see e.g. [F23] and [NPP14] and code on the author's website), and one can lower the sample variance of the MLEs and still get good $p$-values by using additional intraday data with (non) equidistant time intervals chosen so that the realized variance has been historically approximately constant in each interval (again see code on author's website).

Section 2.2 in Keller-Ressel[KR22] gives a concise background on mean-variance hedging so we do not repeat this here, and derives asymptotic approximations for the mean variance hedge using the original SABR formula for the SABR model with $\beta=1$ and the recent rough SABR formula from [FG22] for the rough Bergomi case (see also section 4.1 in [Schw95] for the original formulation of the discrete-time variance optimal hedge). For the QGARCH(1,1) model discussed above, we can also use deep learning to approximate the mean-variance hedge by exploiting the Markov nature of the model, essentially just adding an extra dimension to existing code which uses deep hedging to approximate the classical Black-Scholes hedging strategy. One can also attempt to price options with transaction costs using deep hedging with exponential indifference pricing but for this we need to keep track of the agent's risky wealth as additional state variable (see e.g. [BGTW19],[BMPW22] and related articles on this theme).

Related to the calibration problem discussed in the abstract, [CL21] shows how to calibrate a (one-dimensional) Bass[Bass83] martingale to given marginals at two different maturities; fitting a single maturity is elementary, but jointly fitting to two maturities requires an iterative fixed point scheme of the form $F^{n+1}=\mathcal{A}F^n$ for some non-linear integral operator $\mathcal{A}$ (see Theorem 2.1 in [CL21]) where $\mathcal{A}$ is a map from the space of distribution functions on $\mathbb{R}$ to itself. The aforementioned fixed point scheme just requires numerically computing two Gaussian convolution integrals, inverting a cdf and then iterating the procedure, for which [AMP23] establish existence and uniqueness (and linear convergence) results, and (in our experience) the scheme converges very quickly in practice.

The one-dimensional Bass martingale is also the solution to the martingale optimization problem $\inf_{X\in\mathcal{M}^c:X_t=X_0+\int_0^t\sigma_s dW_s:X_T\sim\mu}\mathbb{E}(\int_0^T(\sigma_t-1)^2dt)$, (where $\mathcal{M}^c$ is the space of continuous martingales) which is clearly also the solution to $\sup_{X\in\mathcal{M}^c:X_t=X_0+\int_0^t\sigma_s dW_s:X_T\sim\mu}\mathbb{E}(\int_0^T\sigma_t dt)$ (see e.g. introduction of [BST23] and section 1.3 in [BBHK20]); hence the Bass martingale is a *stretched Brownian motion*[2], which (formally at least) can also be dualized as $\sup_{f\in C_b(\mathbb{R})}(-\int_{\mathbb{R}}fd\mu+\inf_{\sigma\in\mathcal{A}}(\mathbb{E}(f(X_T)+\int_0^T(\sigma_t-1)^2dt))$ (for a suitable space of adapted processes $\mathcal{A}$) in the spirit of [GLOW22],[GLW22], which leads to a HJB equation for the inner inf.

## 1.1 Outline of article

In section 2 of this article, we show how to perfectly replicate a VIX option with dynamic trading in a VIX future using the Clark-Ocone formula, which essentially gives explicit expressions for the integrand in the martingale representation theorem in terms of Malliavin derivatives. In section 3 we extend this analysis to synthesize a European option with dynamic trading in the underlying and a VIX future, and we also compute the minimal variance hedge for a European call when we can only dynamically hedge with the underlying, and we provide numerical simulations. Our Malliavin approach is numerically intensive as it requires computing a conditional expectation with Monte Carlo at each time instant, but obviates the need for e.g. deep learning with RNNs (see e.g. [BGTW19],[BMPW22],[HTZ21]) or use of Gâteaux derivatives/functional Itô calculus, PPDEs and infinite-dimensional hedging portfolios (cf. [ER18],[HTZ21],[VZ19],[FHT22]). The latter approach is somewhat impractical because for this approach we need a continuum of forward variance contracts $\xi_t(u)$, each of which has to be replicated

[2]See [BST23], [BBST23] and [BBHK20] for more on this, and extension to higher dimensions and randomized $X_0$

with an "infinitesimal calendar spread" of log contracts which in turn has to be replicated with an infinite number of European options, each of which will have a much larger bid-offer spread than the underlying in practice. These results are easily extended to mixed/two-factor rough Bergomi models which give better fits to VIX smiles in practice.

## 2 Hedging VIX options

Let $W$ denote a standard Brownian motion and $\mathcal{F}_t^W$, and consider a generalized Rough Bergomi model for a log stock price process $X_t = \log S_t$ for which the squared spot volatility process $V$ satisfies

$$V_t = \xi_0(t) e^{Z_t - \frac{1}{2}\mathrm{Var}(Z_t)} \tag{1}$$

under a risk-neutral measure $\mathbb{Q}$, where $Z_t = \int_0^t \kappa(t-s)dW_s$ for some $\kappa \in L^2([0,T])$, so $\mathrm{Var}(Z_t) = \int_0^t \kappa(t-s)^2 ds = \int_0^t \kappa(s)^2 ds$. Note the usual standard rough Bergomi model assumes $c = 1$. We can easily extend the results in this paper to the case when $V_t = \xi_0(t) e^{Z_t - \frac{1}{2}\mathrm{Var}(Z_t)}$. A popular choice is the Gamma kernel: $\kappa(t) = t^{H-\frac{1}{2}} e^{-\theta t}$ for $H \in (0, \frac{1}{2}]$ and $\theta \geq 0$, where the roughness and ergodicity of $Z$ are controlled by $H$ and $\theta$ respectively. Then we can easily verify that $\xi_t(u) := \mathbb{E}(V_u | \mathcal{F}_t^W)$ satisfies

$$\xi_t(u) = \xi_0(u)\, e^{\int_0^t \kappa(u-r)dW_r \,-\, \frac{1}{2}\int_0^t \kappa(u-r)^2 dr}$$

and

$$d\xi_t(u) = \kappa(u-t)\xi_t(u)dW_t \tag{2}$$

so $\xi_t(u)$ is a driftless time-inhomogenous Geometric Brownian motion for each $u$ and $\xi_t(u)$ is an $\mathcal{F}_t^W$-martingale.

The VIX index is a well known estimator of future volatility, which is quoted in the market. Theoretically the value of the VIX index time is $t \leq T$ is given by $\mathrm{VIX}_t = \sqrt{\frac{1}{\Delta}\int_t^{t+\Delta} \xi_t(u)du}$ for some $\Delta > 0$. Then we can consider an option on the VIX which pays

$$F = f(\mathrm{VIX}_T)$$

at time $T$ (and we assume interest rates are zero for simplicity). For the specific case of a VIX call option, $f(x) = (x-K)^+$ and $f'(x) = 1_{x>k}$, even though $f'$ is not Lipshitz, we can compute Malliavin derivatives using a suitable approximation procedure (see e.g. end of page 333 in Nualart[Nua06]).

From the Clark-Ocone formula, we have

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_t^W F | \mathcal{F}_t^W) dW_t \tag{3}$$

where $D_t^W F$ is the Malliavin derivative of $F$ with respect to $W$.

Recall that we compute $D_t^W F$ by perturbing $W$ by a function $H(t)$, such that $\int_0^T h(t)^2 dt < \infty$ where $h(t) = H'(t)$ and $h \in L^2([0,T])$. We denote the perturbed value of $F$ by $F(W + \varepsilon H)$. Then $D_t^W F$ is the (in general) random function such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(W + \varepsilon H) - F(W)) = \int_0^T D_t^W F \cdot h(t) dt$$

if such a function exists. For our $F = f(\mathrm{VIX}_T)$ payoff here, formally using the chain rule, we see that

$$\begin{aligned} D_t^W F = f'(\mathrm{VIX}_T) D_t^W \mathrm{VIX}_T = f'(\mathrm{VIX}_T) D_t^W \sqrt{\mathrm{VIX}_T^2} &= f'(\mathrm{VIX}_T)\frac{1}{2}(\mathrm{VIX}_T^2)^{-\frac{1}{2}} D_t^W(\mathrm{VIX}_T^2) \\ &= \frac{f'(\mathrm{VIX}_T)}{2\mathrm{VIX}_T} \cdot D_t^W \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)du \\ &= \frac{f'(\mathrm{VIX}_T)}{2\mathrm{VIX}_T} \cdot \frac{1}{\Delta}\int_T^{T+\Delta} D_t^W \xi_T(u)du\,. \end{aligned} \tag{4}$$

Using that $\xi_t(u) = \xi_0(u)\, e^{\int_0^t \kappa(u-r)dW_r \,-\, \frac{1}{2}\int_0^t \kappa(u-r)^2 dr}$ we see that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\xi_T(u)(W + \varepsilon H) - \xi_T(u)(W)) = \xi_T(u) \cdot \int_0^T \kappa(u-r)h(r)dr\,.$$

Then we can just read off $D_r^W \xi_T(u)$ as whatever function is in front of $h(r)$; in this case $D_t^W \xi_T(u) = \xi_T(u)\kappa(u-t)$ hence

$$D_t^W F = \frac{f'(\mathrm{VIX}_T)}{2\mathrm{VIX}_T} \cdot \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)\kappa(u-t)du \tag{5}$$

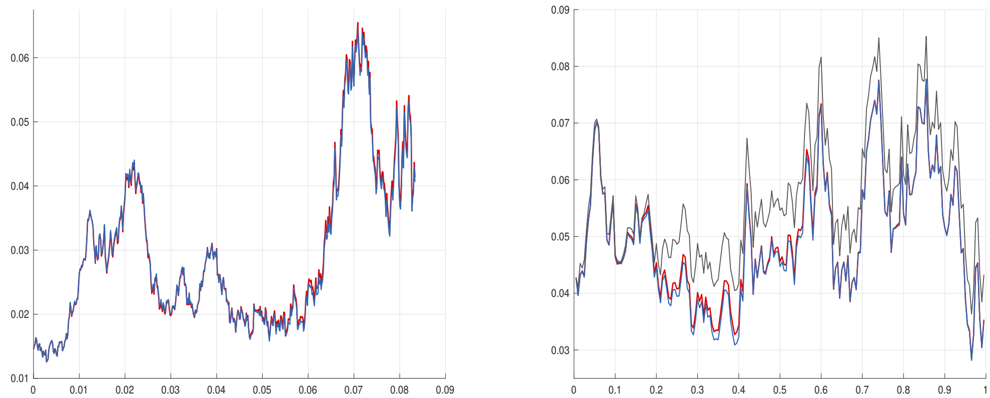


Figure 1: On the left here we have plotted the price of a VIX call option (blue) and the wealth process for the Clark-Ocone replication strategy (red) as a function of time for a standard rough Bergomi model with $\kappa(t) = \nu t^{H-\frac{1}{2}}$, $c = 1$, $\xi_0(u) \equiv V_0 = .04$, $H = 0.1$, $\nu = 1$, $T = \frac{1}{12}$ and strike $K = .2$. We used $N = 500$ time steps for the single "outer" Monte Carlo path and $N_2 = 200$ time steps with $M = 250000$ paths and antithetic sampling for the nested Monte Carlo at each time point to compute $\mathbb{E}_t(D_t^W F)$ in Eq (5) (code available on request). We used 20 Gauss-Legendre quadrature points to compute the VIX, and we see that both paths are almost indistinguishable. On the right we have plotted the price of a European call option (blue) and the wealth process for the Clark-Ocone replication strategy (red) and the wealth process for the minimal-variance hedging strategy (grey), for the standard RFSV model with $V_0 = .01$, $H = 0.2$, $\nu = 0.5$, $c = 1$, $\rho = -0.2$, $T = 1$ and $K = 1$ with $N = 200$, $N_2 = 500$ and $M = 800000$ using the Cholesky decomposition to sample the fBM exactly, and we have used (first and second) moment matching to approximate the $\kappa$ integrals in (12) (again with 20 point Gauss-Legendre quadrature). To obtain a plot like this one should use *common random numbers* (i.e. the same random seed) for each inner Monte Carlo run (as one should when estimating Greeks with finite differences, see e.g. [Hau] for details on why this is beneficial), which significantly lowers the sample variance in estimating the change in the call price over each time step. Of course theoretically the red and blue lines should be identical but in practice there is numerical error due to the finite number of steps and paths, which is typically larger when $H$ is smaller, $\nu$ is larger and/or $\rho$ is closer to $-1$.

and recall that $\mathrm{VIX}_T^2 = \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)du$. There are two integrals in this expression, which can can be computed using Gauss-Legendre quadrature; specifically we need to jointly sample $\xi_T(u)$ at the $n$-point Gaussian-Legendre quadrature abscissae values $(u_i^n)_{i=1}^n$ values for the interval $[T, T+\Delta]$ and $\log \xi_T(u)$ are jointly Gaussian, so in principle we can use the Cholesky decomposition for this although in practice this often fails because if we set $Z_t^u := \int_0^t (u-s)^{H-\frac{1}{2}} dW_s$, the covariance matrix for $Z_t^{(u_i^n)}$ (with $t$ fixed) is very close to singular beyond a certain $n$-value, so we can either use basic short time step Monte Carlo instead or a hybrid of the Cholesky method and a basic Euler-type discretization of $Z_t^u$ with second moment matching for the latter (or the truncated Cholesky method in Algorithm 3.7 in [JMP21]).

If $f(x) = x$, then a VIX call is just a VIX future, so theoretically we can replicate a VIX option using a VIX future, by holding

$$\frac{\mathbb{E}(\frac{f'(\mathrm{VIX}_T)}{2\mathrm{VIX}_T} \cdot \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)\kappa(u-t)du|\mathcal{F}_t^W)}{\mathbb{E}(\frac{1}{2\mathrm{VIX}_T} \cdot \frac{1}{\Delta}\int_T^{T+\Delta} \xi_T(u)\kappa(u-t)du|\mathcal{F}_t^W)}$$

VIX futures at each time instant $t$. This may be desirable in practice since the bid-offer spread on VIX futures (in percentage terms) may be lower than for VIX options or a variance swap synthetically replicated with a finite number of Europeans. As of 22 Dec 2023, the bid-ask spread on VIX futures was \$0.05 with the VIX index itself at 13.50, and the spread on close-to-the-money VIX options was \$0.03 to \$0.04.[3]

# 3 Hedging European options

Now consider a generalized Rough Bergomi model for a log stock price process $X_t$:

$$\begin{aligned} X_t &= X_0 - \frac{1}{2}\int_0^t V_s ds + \int_0^t \sqrt{V_s}(\rho dW_s + \bar{\rho} dB_s) \\ V_t &= V_0 e^{\int_0^t \kappa(t-s)dW_s - \frac{1}{2}\int_0^t \kappa(t-s)^2 ds} \end{aligned} \tag{6}$$

[3]Data obtained from CBOE data services and Charles Schwab.

where $\bar{\rho} = \sqrt{1-\rho^2}$ and $B$ is another Brownian motion independent of $W$, and we assume $\rho \in [-1,0]$ which ensures that $S$ is a true $\mathcal{F}_t^{W,B}$-martingale (see Gassiat[Gass19] for details). Then

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(X(W,B+\varepsilon H) - X(W,B)) \quad = \quad \bar{\rho} \int_0^T \sqrt{V_t}\, h(t) dt$$

so we can read off that

$$D_t^B X_T \quad = \quad \bar{\rho}\sqrt{V_t}\,. \tag{7}$$

Similarly

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(V_t(W+\varepsilon H) - V_t(W)) \quad = \quad V_t \int_0^t \kappa(t-s) h(s) ds$$

so $D_r^W V_t = \kappa(t-r) 1_{r \leq t} V_t$ and $D_r^W \sqrt{V_t} = \frac{1}{2} V_t^{-\frac{1}{2}} \kappa(t-r) 1_{r\leq t} V_t = \frac{1}{2}\kappa(t-r) 1_{r \leq t}\sqrt{V_t}$. Thus

$$\begin{aligned} D_r^W X_T &= -\frac{1}{2}\int_0^T D_r^W V_s ds \,+\, \int_0^T D_r^W(\rho\sqrt{V_s} dW_s + \bar{\rho}\sqrt{V_s} dB_s) \\ &= -\frac{1}{2}\int_0^T \kappa(s-r) 1_{r\leq s} V_s ds \,+\, \frac{1}{2}\bar{\rho}\int_0^T \kappa(s-r) 1_{r \leq s}\sqrt{V_s} dB_s \,+\, \frac{1}{2}\rho\int_0^T \kappa(s-r) 1_{r\leq s}\sqrt{V_s} dW_s \,+\, \rho\sqrt{V_r} \end{aligned}$$

so

$$D_t^W X_T \quad = \quad -\frac{1}{2}\int_t^T \kappa(s-t) V_s ds \,+\, \frac{1}{2}\bar{\rho}\int_t^T \kappa(s-t)\sqrt{V_s} dB_s \,+\, \frac{1}{2}\rho\int_t^T \kappa(s-t)\sqrt{V_s} dW_s \,+\, \rho\sqrt{V_t}\,. \tag{8}$$

**Remark 3.1** For the case of a European call option, $f(x) = (e^x - K)^+$ which has the issue that $\mathbb{E}(f(X_T)^2) = \infty$ if $\rho > \frac{1}{\sqrt{2}}$ (see main result in [Gass19]), so $f$ is not Malliavin differentiable. However, using the put-call parity $C - P = S - K$, we can re-write $f$ as $f(X_T) = S_T - K + (K - S_T)^+$, and the $S_T - K$ part is trivial to hedge, and clearly $\mathbb{E}((K - S_T)^2) < \infty$, so this is not an issue.

Then from the two-dimensional Clark-Ocone formula, we have

$$F \quad = \quad \mathbb{E}(F) \,+\, \int_0^T \phi_t dB_t \,+\, \int_0^T \psi_t dW_t \tag{9}$$

where $F = f(X_T)$, $\phi_t = \mathbb{E}(D_t^B F|\mathcal{F}_t^{W,B})$ and $\psi_t = \mathbb{E}(D_t^W F|\mathcal{F}_t^{W,B})$, and hence

$$C_t \quad := \quad \mathbb{E}(F|\mathcal{F}_t^{W,B}) \quad = \quad \mathbb{E}(F) \,+\, \int_0^t \phi_s dB_s \,+\, \int_0^t \psi_s dW_s\,. \tag{10}$$

From the chain rule, we know that

$$D_t^B F \quad = \quad f'(X_T) D_t^B X_T \quad , \quad D_t^W F \quad = \quad f'(X_T) D_t^W X_T \tag{11}$$

and we have explicit expressions for $D_t^B X_T$ and $D_t^B X_T$ in (7) and (8) above. Then using that

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}(\rho dW_t + \bar{\rho} dB_t) \\ dC_t &= \psi_t dW_t \,+\, \phi_t dB_t \end{aligned}$$

we see that

$$d\langle C,S\rangle_t \quad = \quad S_t\sqrt{V_t}(\rho\psi_t \,+\, \bar{\rho}\phi_t) dt$$

so the minimal variance stock holding at time $t$ is

$$\theta_t \quad = \quad \frac{d\langle C,S\rangle_t}{d\langle S_t\rangle} \quad = \quad \frac{\rho\psi_t \,+\, \bar{\rho}\phi_t}{S_t\sqrt{V_t}}$$

(see also section 10.4 in [CT04] for general background on mean variance hedging and application to exponential Lévy models).

### 3.0.1 The RFSV model and lack of Malliavin differentiability at $t = 0$

We can easily extend the analysis to the case when $Z_t = \int_0^t \kappa(s,t)dW_s$ and $V_t = e^{\nu Z_t - \frac{1}{2}c\mathbb{E}(Z_t^2)}$ for some $c \in \mathbb{R}$ and $\kappa(.,t) \in L^2([0,t])$ for all $t$ in $[0,T]$, e.g. the case when $Z$ is standard fBM (for which the explicit formula for $\kappa$ is given in e.g. Eq 3 in [FZ17] in terms of the incomplete $\beta$ function, and the associated stochastic volatility model when $c = 1$ is the RFSV model from [GR18], which is also used in [CD22]). In this case (8) becomes

$$D_t^W X_T \quad = \quad -\frac{1}{2}\int_t^T \kappa(t,s)V_s ds \,+\, \frac{1}{2}\bar{\rho}\int_t^T \kappa(t,s)\sqrt{V_s}dB_s \,+\, \frac{1}{2}\rho\int_t^T \kappa(t,s)\sqrt{V_s}dW_s \,+\, \rho\sqrt{V_t} \tag{12}$$

as long as the integrals and stochastic integrals here are well defined. Surprisingly, this is not the case a $t = 0$ when $Z$ is standard fBM, since $\kappa(t,s) \to +\infty$ as $t \searrow 0$ for $s > 0$ and $\lim_{t\to 0}\int_t^T \kappa(t,s)^q ds = \infty$ for $q \in \{1,2\}$ so $D_t^W X_T$ is undefined at $t = 0$, but we can still use (12) if we only start hedging after time zero (see second plot in Figure 1 for a numerical simulation).

### 3.0.2 The minimal-variance hedge for VIX options

If we wish to compute the minimal variance hedge for a VIX option (i.e. dynamically hedging with $S$ alone), then this only really makes sense if $\rho \neq 0$; in this case clearly $D_t^B F = 0$, so the minimal variance hedge is $\theta_t = \frac{\rho\psi_t}{S_t\sqrt{V_t}}$ where $D^W F$ for this problem is the expression in (4), but this strategy does not typically work very well unless $\rho$ is close to $-1$.

## 3.1 Variance reduction for computing the hedge amount using Monte Carlo

A European call option corresponds to $f(x) = (e^x - e^k)^+$, and from the tower property we can reduce the sample variance of the numerical estimation of $\mathbb{E}(D_t^W F|\mathcal{F}_t)$ with Monte Carlo by conditioning on $B$ (similar to the classic Renault-Touzi[RT96] conditioning trick) as

$$\mathbb{E}(D_t^W F|\mathcal{F}_t^{W,B}) \quad = \quad \mathbb{E}(\mathbb{E}(D_t^W F|\mathcal{F}_T^W)|\mathcal{F}_t^{W,B})) \quad = \quad \mathbb{E}(\mathbb{E}(f'(X_T)D_t^W X_T|\mathcal{F}_T^W)|\mathcal{F}_t^{W,B})$$

and we can use that $X_T$ and $D_t^W X_T$ are bivariate Normal conditioned on $\mathcal{F}_T^W$ to compute the inner conditional expectation explicitly in terms of the Erf function in e.g. Mathematica (we omit the details here for the sake of brevity). As usual this trick is more effective when $|\rho|$ is smaller, and we gain no benefit when $|\rho| = 1$. We can also use antithetic sampling, even if $|\rho| = 1$.

# 4 Exact calibration to single or multiple smiles - a rough Bergomi Bass model

If $\phi$ is chosen so $F = g(X_T)$ has a given target law $\mu$ on $(0,\infty)$ with a strictly positive density[4] with $\int_0^\infty x\mu(x)dx = S_0$, then setting $S_t = \mathbb{E}(F|\mathcal{F}_t^{W,B})$ in (10) yields a martingale price process $(S_t)_{t\in[0,T]}$ with $S_T \sim \mu$. We further assume that we can write $g$ as $g(x) = e^x + f(x)$ so $S_T = e^{X_T} + f(X_T)$ where $f$ is bounded and differentiable, and $\mathbb{E}(f(X_T)^2) < \infty$ (see Remark 3.1) and $S_t = e^{X_t} + \mathbb{E}(f(X_T)|\mathcal{F}_t^{W,B})$.

Then in general if $\kappa(\tau) \sim const. \times t^{\frac{1}{2}-H}$ (for $H \in (0,\frac{1}{2})$) as $\tau \to 0$, $(S_t)_{t\geq 0}$ is a rough volatility model which fits the target distribution $\mu$ at $T$ exactly, because in this case $\log V$ is a Gaussian process which is $H - \varepsilon$ Hölder continuous for $\varepsilon \in (0,H)$.

In particular, $D_t^B X_T = \bar{\rho}\sqrt{V_t}$ (Eq (7)) and $D_t^W X_T$ both include a $\sqrt{V_t}$ term (Eq (8)), and $D_t^B F = f'(X_T)D_t^B X_T$ (see Eq (11)). Recall that $\phi_t = \mathbb{E}(D_t^B F|\mathcal{F}_t^{W,B}) = \bar{\rho}\sqrt{V_t}\,\mathbb{E}(f(X_T)|\mathcal{F}_t^{W,B})$, and we can view the $\mathbb{E}(f'(X_T)|\mathcal{F}_t^{W,B})$ term here as a generalized local volatility component since we can further re-write this term as $\bar{\rho}\sqrt{V_t}\,\mathbb{E}(f(X_T)|\sigma(\mathcal{F}^W, X_t))$. We can also compute exact or minimal-variance hedge quantities for options on $S_T$ for this model using the same computations as Section 3, which are compound options on $X_T$.

If we wish to fit a rough Bergomi Bass model to two target densities $\mu$ and $\nu$ at maturities $T$ and $T_2$ (both with mean $S_0$, with $0 < T < T_2$ and $\mu$ and $\nu$ in convex order), we first require a coupling $\pi \in \mathrm{Cpl}(\mu,\nu)$ in the martingale transport $\mathrm{MT}(\mu,\nu)$ of $\mu$ and $\nu$, i.e. such that $\int y\pi_x(dy) = x$ where $\pi(dx,dy) = \pi_x(dy)\mu(dx)$, so $\mathbb{E}(S_{T_2}|S_T) = S_T$ (see e.g. section 2.2 of [BST23] for definitions and notation here) which gives us a conditional distribution function $F_{S_{T_2}|S_T}(s_2;s_1)$ for $S_{T_2}$ given $S_T$. A viable/sensible choice for $\pi$ could be to use the two-maturity Bass martingale discussed in [CL21] (see also [BBHK20]), or the Carr Local Variance Gamma model in [Carr09] and [CN17].

[4] we can do this by setting $f(x) = F_\mu^{-1}(F_{X_T}(x))$, where $F_\mu$ is the distribution function of $\mu$ and $F_{X_T}$ is the distribution function of $X_T$; then $\phi$ is strictly monotonically increasing because $F_\mu$ and $F_{X_T}$ are strictly monotonically increasing, since $\mu$ has a strictly positive density by assumption and $X_T$ has a strictly positive density when $|\rho| < 1$ because $X_T|V_{0\leq t\leq T}$ is conditionally Gaussian.

By adapting the approach used in [BG24], a martingale model consistent with the two marginals here then takes the form

$$\begin{aligned} S_t &= \mathbb{E}(g(X_T)|\mathcal{F}_t^{W,B}) \qquad (t \in [0,T]) \\ S_t &= \mathbb{E}(F^{-1}_{S_{T_2}|S_T}(U;S_T)|\mathcal{F}_{t-T}^{\tilde{W},\tilde{B}}) \qquad (t \in (T,T_2]) \end{aligned} \tag{13}$$

where

$$U \;=\; F_{\tilde{X}_{T_2-T}}(\tilde{X}_{T_2-T}) \;\sim\; U[0,1]$$

and $\tilde{X}$ is the log stock price for another rough Bergomi model of the form in (1) with $\tilde{X}_0 = 0$ (independent of $X$, also driven by two independent Brownians $\tilde{W}$ and $\tilde{B}$). $S$ is continuous on $[0,T_2]$, in particular at $t=T$ since $\mathbb{E}(S_{T_2}|S_T) = S_T$ since we have used a martingale coupling. Note that the instantaneous variance process $V$ for this model will not in general be continuous at $T$, but this is also the case for the standard two-maturity Bass martingale in [CL21] as in (13). and we can also apply the Clark-Ocone formula to $S$ for $t \in (T,T_2]$, and we can extend this construction to $n$-maturities using the same conditional sampling trick in (13).

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