

A simplified unified propagator model for signed order flow, concave price impact and rough volatility

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Abstract

In this note, we simplify the setup in [MORS26] by exogenously modelling signed order flow as an integrated hyper-rough Heston process with Holder exponent in $(\frac{1}{2}, 1)$, which obviates the need for Hawkes processes, and lengthy scaling limit arguments with C -tightness etc.

0.1 Positive order flow and the inversion formula

$$\Delta_t = \int_0^t f(t-s)(B_{F_s^2} - B_{F_s^1})ds$$

so

$$\mathbb{E}(\Delta_t^2) = \int_0^t f(t-s)\mathbb{E}(\Delta_s)ds$$

Similar to Theorem 3.1 in [MORS26]¹, we consider a hyper-rough Heston process $(F_t^+)_t \geq 0$ for cumulative positive order flow which satisfies

$$F_t^+ = g(t) + \int_0^t f(t-s)B_{F_s^+}ds$$

where B is a standard Brownian motion with $B_0 = 0$ and $f(t) = e^{-\theta t}t^{H-\frac{1}{2}}$ is the usual Gamma kernel with $H \in (-\frac{1}{2}, \frac{1}{2})$, $\theta > 0$ so $f \in L^1$ and we assume $g \in C^1$. F_t^+ is a.s. $(2\alpha - \varepsilon) \wedge 1$ -Hölder continuous where $\alpha = H + \frac{1}{2}$ (see e.g. Theorem 3.1 in [JM20] for details)² and F^+ is an increasing process which is non-differentiable if $H \leq 0$.

Lemma 0.1 $\mathbb{E}(F_t^+) < \infty$.

Proof. See Appendix A. ■

Corollary 0.2 Let $S = F_s^+$ and $T = F_t^+$ for $0 \leq s \leq t$, then $\mathbb{E}(B_{u \wedge T}^2) = \mathbb{E}(u \wedge T) \leq \mathbb{E}(T) < \infty$, so $(B_{u \wedge T})$ is bounded in L^2 (and hence U.I.) and $B_{u \wedge T} \rightarrow B_T$ a.s., so (from e.g. Theorem 13.7 and Lemma 5.10 in [Will91]) $0 = \lim_{u \rightarrow \infty} \mathbb{E}(B_{u \wedge T}) = \mathbb{E}(B_T)$, and from the OST for U.I. martingales applied to $X_u = B_{u \wedge T}$ (e.g. Theorem 5.20 in [Eth18]), $\mathbb{E}(B_T | \mathcal{F}_S^B) = B_S$, so $B_{F_t^+}$ is an \mathcal{G}_t^+ -martingale, where $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$ (see also [AJ21] for a weak formulation on this).

Add proof of strong existence/uniqueness

If h is a function with $h * f \equiv 1$, then we have the inversion formula:

$$\begin{aligned} \int_0^t h(t-s)(F_s^+ - g(s))ds &= \int_0^t h(t-s) \int_0^s f(s-u)B_{F_u^+}duds = \int_0^t \int_u^t h(t-s)f(s-u)dsB_{F_u^+}du \\ &= \int_0^t \int_0^{t-u} h(t-s-u)f(s)dsB_{F_u^+}du \\ &= \int_0^t (h * f)(t-u)B_{F_u^+}du = \int_0^t B_{F_u^+}du \end{aligned}$$

(where Fubini is justified since $F_t^+ < \infty$ a.s. for finite t and B is a.s. continuous), and hence

$$\frac{d}{dt} \int_0^t h(t-s)(F_s^+ - g(s))ds = B_{F_t^+} \tag{1}$$

Lebesgue a.e. Taking Laplace transforms, the condition $h * f \equiv 1$ becomes $\hat{f}(\lambda)\hat{h}(\lambda) = \frac{1}{\lambda}$, from which we find that

$$h(t) = \frac{\theta^\alpha}{\Gamma(\alpha)} \left(1 - \frac{\Gamma(-\alpha, t\theta)}{\Gamma(-\alpha)}\right)$$

where $\Gamma(a, z) = \int_z^\infty s^{a-1}e^{-s}ds$ is the incomplete Gamma function, and in particular $h(t) = O(t^{-H-\frac{1}{2}})$ as $t \rightarrow 0$.

¹see also section 5.4 in [FGS21]

²we can simulate F^+ using the Monte Carlo scheme in [AA25] using Normal Inverse Gaussian variates

0.2 Persistent signed order flow

Again following [MORS26], we now assume the *signed* order flow for an asset is

$$V_t = F_t^+ - F_t^-$$

where $F_t^- = g(t) + \int_0^t f(t-s)W_{F_s^-} ds$ and W is another Brownian motion independent of B (so F^- is an i.i.d. copy of F^+). Then we see that

$$\frac{d}{dt} \int_0^t h(t-s)V_s ds = B_{F_t^+} - W_{F_t^-}$$

since the g terms cancel, and for $0 \leq t \leq u$ we have

$$\mathbb{E}(F_u^+ | \mathcal{F}_{F_t^+}^B) = g(u) + \mathbb{E}\left(\int_0^u f(u-s)B_{F_s^+} ds | \mathcal{G}_t^+\right) = g(u) + \int_0^t f(u-s)B_{F_s^+} ds + B_{F_t^+} \int_t^u f(u-s) ds$$

(and similarly for $\mathbb{E}(F_u^- | \mathcal{F}_{F_t^-}^W)$, where $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$, since $(B_{F_t^+})_{t \geq 0}$ is a \mathcal{G}_t^+ -martingale by Corollary 0.2).

0.3 Price dynamics and the propagator model

Making the usual assumption that the asset price $P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u | \mathcal{G}_t)$ where $F_t = F_t^+ - F_t^-$ and $\mathcal{G}_t = \sigma(\mathcal{G}_t^+, \mathcal{G}_t^-)$ with $\mathcal{G}_t^- = \mathcal{F}_{F_t^-}^W$ (for some $\kappa > 0$)³, we find that $f(u-s) \rightarrow 0$ as $u \rightarrow \infty$ since $\theta > 0$, but $\lim_{u \rightarrow \infty} \int_t^u f(u-s) ds = c_{H,\theta}$ where $c_{H,\theta} = \theta^{-\frac{1}{2}-H} \Gamma(\frac{1}{2}+H)$, so

$$P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u^+ - F_u^- | \mathcal{G}_t) = \kappa c_{H,\theta} (B_{F_t^+} - W_{F_t^-})$$

i.e. P is the difference of two i.i.d. hyper-rough Heston models (each with correlation $\rho = 1$ since each process only has one driving Brownian motion), and

$$P_t = \kappa c_{H,\theta} \frac{d}{dt} \int_0^t h(t-s)V_s ds$$

which is a weak formulation of the usual Propagator model, and the *unsigned* order flow is $U_t = F_t^+ + F_t^-$.

0.4 Power-law price impact

In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 up to time t_0 is given by

$$MI(t) = \frac{d}{dt} \int_0^t h(t-s) s ds = \frac{\theta^{-\alpha_-} (t\theta\Gamma(-\alpha) - t\theta\Gamma(-\alpha, t\theta) - \Gamma(\alpha_-, 0) + \Gamma(\alpha_-, t\theta))}{\Gamma(-\alpha)\Gamma(\alpha)}$$

for $0 \leq t < t_0$ (where $\alpha_- = \frac{1}{2} - H$) for $0 \leq t \leq t_0$ which is $O(t^{\frac{1}{2}-H})$ as $t \rightarrow 0$ and globally concave in t for $H \in (-\frac{1}{2}, \frac{1}{2})$, consistent with empirical evidence (note the usual square-root impact law corresponds to $H = 0$ here), and we can show that the asymptotic (i.e. permanent) impact as $t \rightarrow \infty$ (given no trading after t_0) is

$$MI(\infty) = \frac{t_0 \theta^\alpha}{\Gamma(\frac{1}{2} + H)}$$

(this is the asymptotic grey line in the plot below). For $H = \frac{1}{2}$, $h(t) = \theta$ which corresponds to pure permanent price impact; conversely $\lim_{\theta \rightarrow 0} h(t) = \text{const.} \times t^{-\frac{1}{2}-H}$. Recall from above that F_t^+ has Hölder regularity $(2\alpha - \varepsilon) \wedge 1$, and empirical evidence in [MORS26] and below suggests that $2\alpha \in (\frac{1}{2}, 1)$ which corresponds to $H \in (-\frac{1}{4}, 0)$.

0.5 Extensions

If we augment f to a function of the form $f(t) = e^{-\theta_1 t^{H-\frac{1}{2}}} + e^{-\theta_2 t^{H_2-\frac{1}{2}}}$ with $H_2 \in (0, \frac{1}{2})$, $\theta_2 > 0$, then we find that

$$h(t) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} (-c)^n \sum_{k=0}^{\infty} \frac{(\beta n)_k}{k!} \delta^k \theta^{-(\mu_n+k)} \frac{\gamma(\mu_n+k, \theta t)}{\Gamma(\mu_n+k)}$$

where $\beta = \frac{1}{2} + H_2$, $c = \Gamma(\beta)/\Gamma(\alpha)$, $\delta = \theta - \theta_2$, $\mu_n = \beta n - \alpha(n+1)$, $\gamma(a, z) = \int_0^z s^{a-1} e^{-s} ds$ is the lower incomplete Gamma function and $(a)_k$ denotes the Pochammer symbol. We can then use two (possibly different) f functions to drive F^\pm , and the presence of the positive H_2 parameter means the price dynamics will now have a (non-hyper) rough component, which is arguably more realistic. We can also assume W and B correlated which gives greater flexibility in capturing skew for the price process P_t .

³note we are assuming $P_0 = 0$ without loss of generality to ease notation, but we can easily add a positive P_0 term

0.6 Numerics

Below we have tabulated estimates of 2α (i.e. the Hölder exponent of F_t) using signed order flow for major US tech stocks using the model-independent $\hat{H}_{L,K}^\pi(X)$ estimator in Eq 7 in Cont&Das[CD24] which is consistent and allows for irregularly spaced data (with their $L/K = [\sqrt{N}]$ where N is the total data size). For this we have used the free data at <https://data.lobsterdata.com/info/DataSamples.php>: ⁴

	GOOG	AAPL	MSFT	AMZN	INTC
N	49284	118326	410881	57360	404496
$2\hat{\alpha}$	0.692	0.720	0.847	0.735	0.890
\hat{H}	-0.154	-0.140	-0.0767	-0.133	-0.0553

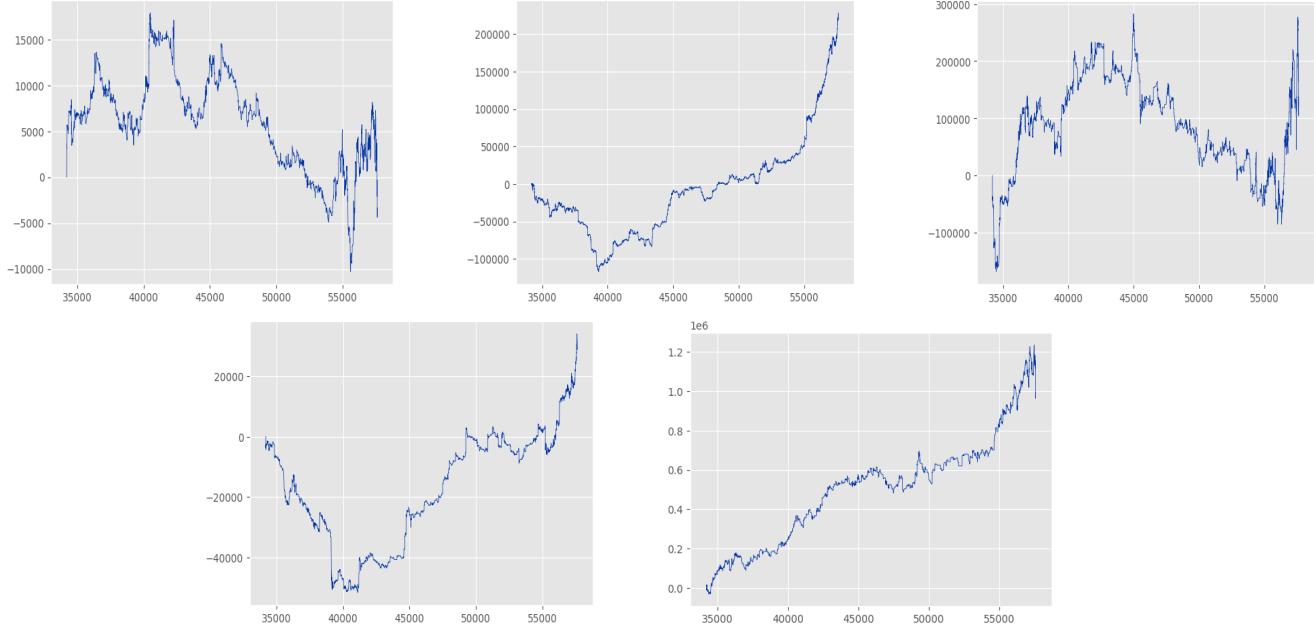


Figure 1: Signed order flow for GOOG, AAPL, MSFT, AMZN and INTC

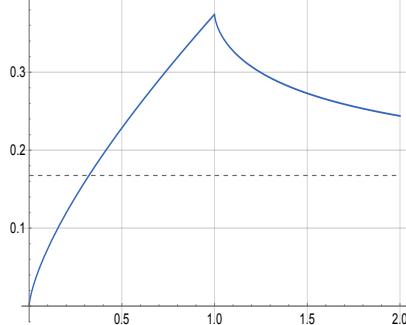


Figure 2: Concave price impact of an exogenous metaorder executed at constant trading speed 1 over $[0, 1]$ with $H = -0.2$, $\theta = 0.1$ and $\kappa = 1$. The blue line asymptotes to the constant level $MI(\infty)$ (grey line) as $t \rightarrow \infty$ which represents the asymptotic permanent price impact.

References

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⁴only using columns 4 and 5 from the lobsterdata.csv file which correspond to actual realized trades.

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A Appendix A

Proof. Set $G_t := \sup_{0 \leq r \leq t} |g(r)|$, $K_t := \int_0^t |f(r)| dr > 0$. For $n \geq 1$ define the level hitting time $\tau_n := \inf\{s \geq 0 : F_s^+ \geq n\}$.

Since F^+ is continuous and nondecreasing, on $\{\tau_n \leq t\}$ we have $F_{\tau_n}^+ = n$ and $F_r^+ \leq n$ for all $0 \leq r \leq \tau_n$. Fix $n \geq 1$ and work on the event $\{\tau_n \leq t\}$. For $0 \leq r \leq \tau_n$ we have $|B_{F_r^+}| \leq \sup_{0 \leq u \leq n} |B_u|$. Setting $s = \tau_n$ and taking absolute values we see that

$$\begin{aligned} n &= g(\tau_n) + \int_0^{\tau_n} f(\tau_n - r) B_{F_r^+} dr \leq |g(\tau_n)| + \int_0^{\tau_n} |f(\tau_n - r)| |B_{F_r^+}| dr \\ &\leq G_t + \left(\int_0^{\tau_n} |f(\tau_n - r)| dr \right) \sup_{0 \leq u \leq n} |B_u| \\ &= G_t + \left(\int_0^{\tau_n} |f(s)| ds \right) \sup_{0 \leq u \leq n} |B_u| \leq G_t + K_t \sup_{0 \leq u \leq n} |B_u|. \end{aligned}$$

Hence for $n > G_t$, on $\{\tau_n \leq t\}$ we have $\{\sup_{0 \leq u \leq n} |B_u| \geq \frac{n-G_t}{K_t}\}$. Since $F_t^+ \geq n$ iff $\tau_n \leq t$, we obtain

$$\mathbb{P}(F_t^+ \geq n) \leq \mathbb{P}\left(\sup_{0 \leq u \leq n} |B_u| \geq \frac{n-G_t}{K_t}\right), \quad n > G_t.$$

By the standard reflection principle bound $\mathbb{P}(\sup_{0 \leq u \leq n} |B_u| \geq a) \leq 4 \exp(-\frac{a^2}{2n})$, for $a > 0$, so for $n > G_t$,

$$\mathbb{P}(F_t^+ \geq n) \leq 4 \exp\left(-\frac{(n-G_t)^2}{2nK_t^2}\right).$$

In particular, if $n \geq 4G_t$ then $(n-G_t)^2/n \geq \frac{1}{2}n$, hence

$$\mathbb{P}(F_t^+ \geq n) \leq 4e^{-\frac{n}{4K_t^2}}, \quad n \geq 4G_t. \quad (\text{A-1})$$

Since $F_t^+ \geq 0$, we also know that $\mathbb{E}(F_t^+) = \int_0^\infty \mathbb{P}(F_t^+ \geq x) dx$. Now set $m := \lceil 4G_t \rceil$. Then

$$\mathbb{E}(F_t^+) \leq m + \int_m^\infty \mathbb{P}(F_t^+ \geq x) dx = m + \sum_{n=m}^\infty \int_n^{n+1} \mathbb{P}(F_t^+ \geq x) dx \leq m + \sum_{n=m}^\infty \mathbb{P}(F_t^+ \geq n). \quad (\text{A-2})$$

Using (A-1) for all $n \geq m$,

$$\sum_{n=m}^\infty \mathbb{P}(F_t^+ \geq n) \leq \sum_{n=m}^\infty 4 \exp\left(-\frac{n}{4K_t^2}\right) < \infty. \quad (\text{A-3})$$

Combining (A-2) and (A-3) yields $\mathbb{E}(F_t^+) < \infty$. ■