A diffusion-type process with a given joint law for the terminal level and supremum at an independent exponential time

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Abstract

We construct a weak solution to the stochastic functional differential equation $X_t = x_0 + \int_0^t \sigma(X_s, M_s)dW_s$, where $M_t = \sup_{0\leq s\leq t} X_s$. Using excursion theory, we then solve explicitly the following problem: for a natural class of joint density functions $\mu(y,b)$, we specify $\sigma(.,.)$, so that X is a martingale, and the terminal level and supremum of X, when stopped at an independent exponential time ξ_λ, is distributed according to μ. We can view $(X_{t\wedge\xi_\lambda})$ as an alternate solution to the problem of finding a continuous local martingale with a given joint law for the maximum and the drawdown, which was originally solved by Rogers[Rog93] using excursion theory. This complements the recent work of Carr[Carr09] and Cox,Hobson&Obłój[CHO09], who consider a standard one-dimensional diffusion evaluated at an independent exponential time[1].

1 Introduction

Carr[Carr09] and Cox,Hobson&Obłój[CHO09] consider the following problem: suppose μ is a given probability distribution on $\mathbb{R}$. Find a time-homogeneous martingale diffusion process (X_t), such that X_T is distributed according to μ, where T is an independent exponential random variable. More specifically, for μ sufficiently regular, find a function $\sigma : \mathbb{R} \mapsto \mathbb{R}^+$ such that

$$X_T = x_0 + \int_0^T \sigma(X_t)dW_t \sim \mu \tag{1}$$

where (W_t) is a Brownian motion. This problem arises naturally in [Carr09], who proposes modelling a stock price process under a risk-neutral measure as a time-homogeneous martingale diffusion, time-changed by an independent gamma subordinator: $S_t = X_{\Gamma_t}$. The gamma clock is normalized so that $T = \Gamma_{t^*}$ has an exponential distribution, where t^* is the maturity of options whose prices are given or observed. Carr takes the Laplace transform of the Dupire forward PDE for call options at a fixed time, and solves for the volatility in terms of the call prices at the exponential time. [Carr09] solves for σ explicitly, but does not discuss existence and uniqueness. Cox,Hobson&Obłój[CHO09] added some rigour to the work of Carr, solving this problem in a more general setting when the target distribution may not have a density, using a generalized diffusion process.

If X is a general uniformly integrable martingale and M is its running supremum process, the martingale property of X imposes certain constraints on the law of (X_∞, M_∞), see e.g. [BD63], [DubGil78], [Rog93], [AY79]. If $\bar{\mu}$ denotes the law of X_∞, then [BD63] show that

$$\bar{\mu} \prec \mathbb{P}(M_\infty \in .) \prec \bar{\mu}^*$$

where $\prec$ denotes the usual stochastic ordering of probabilities, and $\bar{\mu}^*$ denotes the Hardy transform of $\bar{\mu}$. When $\bar{\mu}$ has no atoms, $\bar{\mu}^*$ is the law of $b_{\bar{\mu}}(Z)$, where Z has the law $\bar{\mu}$ and $b_{\bar{\mu}}$ is the *barycentre* function of $\bar{\mu}$. [Rog93] characterizes the possible joint laws of the variables X_∞, M_∞. For a continuous local martingale X starting at zero, [Rog93] shows that

$$\mathbb{P}(M_\infty > b)db \geq \mathbb{E}((b - X_\infty)1_{M_\infty \in db})\,. \tag{2}$$

If X is also uniformly integrable, the inequality (2) becomes an equality, and we use this equality in subsection 4.4. Moreover, for a given joint law ν for the maximum and the drawdown: $(M_\infty, M_\infty - X_\infty)$, (2) is also a sufficient condition for the existence of a continuous local martingale with the prescribed joint law ν. To establish existence, [Rog93] provides a delicate construction which involves generating an independent random variable for each downward excursion of standard Brownian motion from a new maximum, and then stopping the Brownian motion when the size of the excursion (i.e. the drawdown) exceeds the independent random variable.

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Brunick&Shreve[BS10] consider a multi-dimensional Itō process, and then construct a weak solution to a so-called *diffusion-type* equation (a one-dimensional SDE with path-dependent coefficients, using the terminology on page 225 in Lipster&Shiryaev[LS01]) that mimics the marginals of the Itō process at each fixed time. They also show how to match the joint distribution at each fixed time of various statistics of the Itō process, including the running maximum and running average of one component of the Itō process. This extends the earlier work of Gyöngy[Gyö86]. Remarkably, [BS10] also prove a similar result *on path space*: for any Itō process with integrable drift and covariance, its law on path space can be mimicked by a weak solution to an SDE with path-dependent coefficients, i.e. a diffusion-type process which depends just on its own history; the diffusion coefficient at a given time for the mimicking process is the just the expectation of the diffusion coefficient of the original Itō process at the same time, conditioned on the entire path up to that time.

In this article, we first review the solution in [Carr09], and we discuss qualitative features of the associated Local Variance Gamma model. We then construct a weak solution X to the one-dimensional stochastic functional differential equation $X_t = x_0 + \int_0^t \sigma(X_s, M_s)dW_s$, where $M_s = m_0 \vee \sup_{0\leq s\leq t} X_s$, $m_0 \geq x_0$. (X_t) is constructed as a time-changed Brownian motion $X_t = B_{A_t}$, by modifying the proof in Engelbert&Schmidt[ES84] for a regular one-dimensional diffusion, and X is a diffusion-type process. In addition, we show that $(X_{t\wedge\xi_\lambda})$ is a uniformly integrable martingale, where ξ_λ is an independent exponential random variable. We then show how to approximate this process with arbitrary accuracy with a process whose diffusion coefficient is piecewise constant as a function of the maximum. Using excursion theory for regular diffusions, we then compute an expression for the joint density of the terminal level and the supremum at an independent exponential time, and we show that the joint density satisfies a forward Kolmogorov equation. Integrating twice, we obtain a forward PDE for up-and-out put options, and we then use this to back out a volatility function so that X has a given joint density for the terminal level and the supremum at ξ_λ.

Technical conditions aside, the diffusion-type process in this article and the continuous martingale M in section 3 in [Rog93] can both match the same pre-specified joint density for the maximum and the terminal level; the difference is that the process in [Rog93] stops when the *size* of an excursion from the running maximum exceeds a certain random length, but our X process stops when the *duration* of an excursion from the maximum exceeds an independent exponential random time i.e. the excursion is "marked" (this follows from the lack of memory property of exponential random variables). The advantage of the approach taken here versus that in [BS10] is that we do not need a Itō process to begin with whose joint law(s) then have to mimicked; rather we just specify a single joint law for the terminal level and supremum, and then provide sufficient conditions to be able match that law. In principle, this means we have a time-changed diffusion model, whose volatility can be chosen so as to be perfectly consistent with the observed prices of all up-and-out put options at all strikes and barrier levels at a single maturity (which includes all European put options as a special case when the barrier is at infinity).

In section 6, we solve the related problem of constructing a diffusion process with a given law for the maximum at a fixed or an exponential time, using a nice result in Carraro,ElKaroui&Obłój[CElO09].

2 Review of standard identities for a one-dimensional diffusion

Let X be a regular[2] diffusion process on $\mathbb{R}$ with infinitesimal generator

$$\mathcal{G} = \frac{1}{2}\sigma(x)^2\frac{\partial^2}{\partial x^2}$$

and assume that $\sigma(x)$ is continuous and strictly positive and that $\pm\infty$ are natural boundaries for X. We can construct a regular one-dimensional diffusion process from a Brownian motion (B_t) via a stochastic time-change [3]. Let $\mathbb{P}_x$ denote the probability measure associated with $X_0 = x$ and $p(t,x,y)$ be the transition density with respect to the speed measure density $m(x) = \frac{1}{\sigma(x)^2}$ given by

$$\mathbb{P}_x(X_t \in dy) = p(t,x,y)m(y)dy\,. \tag{3}$$

The transition density $p(t,x,y)$ is symmetric, that is $p(t,x,y) = p(t,y,x)$, and the *Green's function* (or *resolvent kernel*) of X is given by

$$R_\lambda(x,y) = \int_0^\infty e^{-\lambda t}p(t;x,y)dt = w_\lambda^{-1}\,\phi_\lambda(x\vee y)\,\psi_\lambda(x\wedge y)\,, \tag{4}$$

[2] i.e. $\mathbb{P}_x(H_y < \infty) > 0$ for all $x, y \in \mathbb{R}$ where $H_y = \inf\{t : X_t = y\}$.

[3] see the result by Engelbert&Schmidt in Theorem 5.5.4 in Karatzas&Shreve[KS91], and Cox,Hobson&Obłój[CHO09].

for $\lambda > 0$, where $\psi_\lambda(x)$ and $\phi_\lambda(x)$ are the unique (up to a multiplicative constant) increasing and decreasing non-negative solutions of the ordinary differential equation[4]

$$\mathcal{G}f = \lambda f \qquad (\lambda > 0) \tag{5}$$

and $w_\lambda = \frac{1}{2}[\psi'_\lambda(y)\phi_\lambda(y) - \psi_\lambda(y)\phi'_\lambda(y)]$ is the *Wronskian*, which depends only on λ. From (5) we see that

$$\frac{1}{2}\frac{\partial^2 \psi_\lambda}{\partial x^2} = \lambda m(x)\psi_\lambda\,. \tag{6}$$

From page 18 in Borodin&Salminen[BorSal02], we have the following left tail behaviour for $\psi_\lambda(.)$

$$\lim_{x\to-\infty}\psi_\lambda(x) = 0 \quad , \quad \lim_{x\to-\infty}\frac{\partial\psi_\lambda}{\partial x} = 0\,. \tag{7}$$

2.1 First hitting times

Let $H_y = \inf\{t : X_t = y\}$ be the first passage time to y [5]. The distribution of H_y has a $\mathbb{P}_x$-density:

$$\mathbb{P}_x(H_y \in dt) = f_{xy}(t)dt\,.$$

Let $M_t = \sup_{0\le s\le t} X_s$ and ξ_λ be an independent exponential random variable. Recall the following formula for the Laplace transform of H_y

$$\mathbb{E}_x(e^{-\lambda H_y}) = \mathbb{P}_x(\xi_\lambda > H_y) = \frac{R_\lambda(x,y)}{R_\lambda(y,y)} = \int_0^\infty e^{-\lambda t} f_{xy}(t)dt = \begin{cases} \psi_\lambda(x)/\psi_\lambda(y) & (x \le y)\,, \\ \phi_\lambda(x)/\phi_\lambda(y) & (x \ge y) \end{cases} \tag{8}$$

(see [BorSal02]). Conditioning on $\{H_b < \xi_\lambda\}$ and using the strong Markov property, it can be shown that

$$\mathbb{P}_x(X_{\xi_\lambda} \in dy, M_{\xi_\lambda} \in db) = \frac{\psi_\lambda(x)\psi_\lambda(y)}{\psi_\lambda(b)^2}2\lambda m(y)dydb \qquad (y \le b)$$

(see section II.19 in [BorSal02], and Csáki,Foldes&Salminen[CFM87] for the proof). Both endpoints are natural boundaries, so the $\mathbb{P}_x$-density of X_{ξ_λ} integrates to 1, and the conditional density $\mathbb{P}_x(X_{\xi_\lambda} \in dy \,|\, M_{\xi_\lambda} = b)$ is given by

$$\mathbb{P}_x(X_{\xi_\lambda} \in dy \,|\, M_{\xi_\lambda} = b) = -\frac{\frac{\psi_\lambda(x)\psi_\lambda(y)}{\psi_\lambda(b)^2}2\lambda m(y)}{\frac{\partial}{\partial b}[\frac{\psi_\lambda(x)}{\psi_\lambda(b)}]}dy = \frac{2\lambda\psi_\lambda(y)\,m(y)}{\psi'_\lambda(b)}dy \qquad (y \le b)\,. \tag{9}$$

Thus we see that

$$\int_{-\infty}^b \frac{2\lambda\psi_\lambda(y)\,m(y)}{\psi'_\lambda(b)}dy = 1\,. \tag{10}$$

3 A weak solution to the SFDE

Let $(B_t, \mathbb{P}_{x_0})$ be a standard Brownian motion defined on some $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ with $B_0 = x_0 \in \mathbb{R}$, $\bar{B}_t = \sup_{0\le s\le t} B_s$, and assume that $\mathcal{F}_t$ satisfies the usual conditions[6]. We let $\mathbb{E}_{x_0} = \mathbb{E}^{\mathbb{P}_{x_0}}$ and we assume $\mathcal{F}_0$ is rich enough to support random variables independent of B. Let $\sigma(.,.)$ be continuous with $0 < \sigma(x,m) \le \sigma_{\max} < \infty$, and consider the a.s. strictly increasing process

$$T_t = \int_0^t \frac{1}{\sigma(B_s, \bar{B}_s \vee m_0)^2}ds = \int_0^t \tilde{m}(B_s, \bar{B}_s \vee m_0)ds\,,$$

for $m_0 \ge x_0$, where

$$\tilde{m}(x,m) = \frac{1}{\sigma(x,m)^2}$$

and $0 < \tilde{m}_{\min} = 1/\sigma^2_{\max} \le \tilde{m}(x,m)$. Now let $A_t = \inf\{s \ge 0 : T_s > t\}$ be the "inverse" of T_t for $t > 0$, and set

$$X_t = B_{A_t}\,. \tag{11}$$

From here on, we let $\mathbb{P}_{x,m}$ denote $\mathbb{P}_{x_0}$ when $x_0 = x$ and $m_0 = m$.

We now recall the fundamental existence result for driftless one-dimensional SDEs in Engelbert&Schmidt[ES84], whose proof is also given in Theorem 5.5.4 in [KS91] and also discussed in [ES85]:

[4] This is the absolutely continuous case discussed on page 18 in [BorSal02], see also Davydov&Linestky[DavLin01]
[5] By convention, if X never reaches y, $H_y = \infty$.
[6] $\mathcal{F}_t$ is right continuous and $\mathcal{F}_0$ contains all $\mathcal{F}$ sets of measure zero.

Theorem 3.1 *There exists a non-trivial weak solution to the one-dimensional stochastic differential equation*

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s$$

if and only if the function $m(x) = \sigma(x)^{-2}$ is locally integrable.

The following proposition is a variant of this theorem for a simple one-dimensional stochastic functional differential equation (SFDE):

Proposition 3.2 *The process (X_t) in (11) is a non-exploding weak solution to the one-dimensional SFDE*

$$X_t = x_0 + \int_0^t \sigma(X_s, M_s) dW_s \tag{12}$$

for some Brownian motion W, where $M_t = m_0 \vee \sup_{0 \le s \le t} X_s$.

Proof. See Appendix A. ∎

Remark 3.1 Given the bound on σ, it should also be possible to establish *uniqueness in law* for X as well. However, this will not be needed in this article, so we defer the details for future work.

We refer the reader to Mao[Mao97] and Mohammed[Moh84] for existence and uniqueness results for general SFDEs.

3.1 The martingale property and uniform integrability

Proposition 3.3 *(X_t) is a square integrable $\mathcal{F}_{A_t}$-martingale, and $(X_{t\wedge\xi_\lambda})$ is a uniformly integrable $\mathcal{F}_{A_t}$-martingale.*

Proof. First, note that

$$\mathbb{E}(X_t^2) = x_0^2 + \mathbb{E}(\int_0^t \sigma(X_s, M_s)^2 ds) \le x_0^2 + \sigma_{\max}^2 t\,, \tag{13}$$

so X_t is a square integrable. From the independence of ξ_λ, we have

$$\mathbb{E}(X_{t\wedge\xi_\lambda}^2) \le x_0^2 + \sigma_{\max}^2 \mathbb{E}(t \wedge \xi_\lambda) \le x_0^2 + \frac{\sigma_{\max}^2}{\lambda}$$

for all $t > 0$, so the family $(X_{t\wedge\xi_\lambda})$ is U.I. (see e.g. Lemma 13.3 in [Will91]). The martingale properties follow easily. ∎

From here on, we make the following additional assumption on σ:

Assumption 3.4 *$\sigma(x, m)$ is Lipschitz in m:*

$$|\sigma(x, m_1) - \sigma(x, m)| \le K|m_1 - m| \tag{14}$$

for some $K > 0$.

From Assumption 3.4, we see that for $-R \le x \le m \le R$, we have

$$\begin{aligned} |\tilde{m}(x, m_1) - \tilde{m}(x, m)| = |\frac{1}{\sigma^2(x, m_1)} - \frac{1}{\sigma^2(x, m)}| \quad &= \quad |\frac{(\sigma(x, m) - \sigma(x, m_1))(\sigma(x, m) + \sigma(x, m_1))}{\sigma^2(x, m_1)\sigma^2(x, m)}| \\ &\le \quad K_1(R)\,|m_1 - m| \end{aligned} \tag{15}$$

for some $K_1(R)$; thus $\tilde{m}(x, m)$ is Lipschitz in m for $-R \le x \le m \le R$.

3.2 Almost sure convergence for an approximating sequence of diffusion processes

Proposition 3.5 *Let $\tilde{m}_n(x,m)=\tilde{m}(x,\frac{1}{n}[mn])$ for $n\geq 1$, so that $\tilde{m}_n(x,m)$ is piecewise constant in m, and let*

$$X_t^n = B_{A_t^n} \tag{16}$$

where A_t^n is the strictly increasing continuous inverse of

$$T_t^n = \int_0^{\tau_{m_0}\wedge t} \tilde{m}(B_s,m_0)ds + \int_{\tau_{m_0}\wedge t}^{t} \tilde{m}_n(B_s,\bar{B}_s)ds \tag{17}$$

for $t\geq 0$, where $\tau_b=\inf\{s:B_s=b\}$. Let ξ_λ denote an $\mathcal{F}_0$-measurable exponential random variable with parameter λ, independent of X. Then $(X_t^n,M_t^n)\to(X_t,M_t)$ a.s., and $(X_{\xi_\lambda}^n,M_{\xi_\lambda}^n)\to(X_{\xi_\lambda},M_{\xi_\lambda})$ a.s.

Proof. We first assume that $m_0=x_0$ for simplicity. By definition of the inverse process we have

$$t = \int_0^{A_t^n}\tilde{m}_n(B_s,\bar{B}_s)ds = \int_0^{A_t}\tilde{m}(B_s,\bar{B}_s)ds \geq A_t\tilde{m}_{\min}\,. \tag{18}$$

We first assume that $A_t\leq A_t^n$. $A_t<\infty$ a.s., hence $\sup_{0\leq s\leq A_t}|B_s|<\infty$ a.s. Then, using the local Lipschitz property of $\tilde{m}$ in (15), and (18), we see that

$$\begin{aligned}
0 &= \int_0^{A_t}\tilde{m}(B_s,\bar{B}_s)ds - \int_0^{A_t^n}\tilde{m}_n(B_s,\bar{B}_s)ds \\
&= \int_0^{A_t}[\tilde{m}(B_s,\bar{B}_s)-\tilde{m}_n(B_s,\bar{B}_s)]ds - \int_{A_t}^{A_t^n}\tilde{m}_n(B_s,\bar{B}_s)ds \\
&\leq \frac{K_1(\omega)A_t}{n} - \tilde{m}_{\min}(A_t^n-A_t) \\
&\leq \frac{K_1(\omega)t}{\tilde{m}_{\min}n} - \tilde{m}_{\min}(A_t^n-A_t)
\end{aligned} \tag{19}$$

for some $K_1(\omega)$, i.e. depending on the sample path. Rearranging, we find that

$$|A_t^n - A_t| \leq \frac{K_1(\omega)t}{\tilde{m}_{\min}^2 n}\,,$$

By a similar argument, we obtain the same inequality for the case $A_t^n\leq A_t$.

Finally, we recall that $X_t=B_{A_t}$ and B is continuous a.s, thus $X_t-X_t^n=B_{A_t}-B_{A_t^n}\to 0$ so $X_{\xi_\lambda}-X_{\xi_\lambda}^n = B_{A_{\xi_\lambda}}-B_{A_{\xi_\lambda}^n}\to 0$ a.s., because ξ_λ is also finite a.s. By the continuity of B, we also have $M_t^n\to M_t$ and $M_{\xi_\lambda}^n\to M_{\xi_\lambda}$. For $m_0>x_0$, $X_t=X_t^n$ for $0\leq t\leq H_{m_0}$, and for $t>H_{m_0}$ we just apply a similar argument. ■

4 Excursions from the maximum

Recall the regular one-dimensional diffusion process X on $\mathbb{R}$ introduced in section 2. We know that $0<m(x)<\infty$ for all $x\in\mathbb{R}$ so X is recurrent, that is $\mathbb{P}_x(L_\infty^a=\infty)=1$ for all $a\in\mathbb{R}$, where L_t^a is the local time at a, defined by

$$L_t^a = l_{A_t}^a$$

where l_t^a is the local time at a for the Brownian motion B used for the time-change $X_t=B_{A_t}$. X also satisfies the occupation time formula $\int_0^t f(X_s)ds=\int_{-\infty}^{\infty}f(x)L_t^x\,m(dx)$ (see Theorem 49.1 in [RW87] and [PY03]).

We can consider the excursions of X below its past maximum. These excursions, when indexed by the level at which they begin, can be regarded as a Poisson point process $\Xi=\{\Xi_s:s\geq 0\}$ in $\mathbb{R}_+\times U$, where $U=\{$continuous $f:\mathbb{R}_+\mapsto\mathbb{R}_+\}$ such that for some $\zeta>0$, $f^{-1}(0,\infty)=(0,\zeta)$. The independent increments property of the first passage process $(H_b)_{b\geq x_0}$ implies that (Ξ_s) is Poissonian in nature, with non-homogeneous intensity. We might expect that the excursion measure for excursions below the maximum when the maximum is at level b be governed by twice the Itō excursion law corresponding to excursions below the *fixed* level b (when indexing by local time not the maximum), and this is proved in [Fitz85] (see also section 2.5 in [PY03]) [7].

[7] Note that X is recurrent so X cannot have a terminal excursion of infinite lifetime below a terminal maximum as for the transient case e.g. driftless Geometric Brownian motion (see [PY03] for more on this case)

The excursion measure $\mathbf{n}(b,.)$ at the level b can be described in various ways, in particular the Itō-McKean and the Williams characterizations (see [SVY07] for details). If A is a Borel subset of U, then the number of points of $(\Xi_s)_{s\geq x_0}$ in $[x_0, b]\times A$ is a Poisson variable with mean

$$\int_{x_0}^{b} \mathbf{n}(s,A)ds\,.$$

In particular, the *excursion entrance law* (i.e. the rate of excursions with lifetime longer than t which fall in dy at time t when the current maximum is x) is given by

$$\mathbf{n}(\epsilon_t \in dy) = 2m(dy)f_{yx}(t)\,.$$

If we now consider an independent Poisson marking process on X with rate λ on the calendar time-scale (see [Rog89] and section VI.49 in [RW87] for details), then the process of λ-marked excursions for $(H_b)_{b\geq x_0}$ is also an inhomogeneous Poisson process, and the excursion measure of marked excursions falling in dy at the first mark is given by the Laplace transform of the entrance law:

$$\lambda n_\lambda(dy) = 2\int_0^\infty \lambda e^{-\lambda t} f_{yb}(t)m(dy)dt = \frac{2\lambda\psi_\lambda(y)}{\psi_\lambda(b)}m(dy)\,, \tag{20}$$

where we have used (8) to obtain the last equality (see Proposition 4 in [PY03], Theorem 1 in [SVY07] and [Fitz85]). The rate of all marked excursions $\lambda n_\lambda(1)$ is then obtained by integrating over y and using the identity in (10):

$$\lambda n_\lambda(1) = \int_{-\infty}^{b} \frac{2\lambda\psi_\lambda(y)}{\psi_\lambda(b)}m(dy) = \frac{\psi'_\lambda(b)}{\psi_\lambda(b)}\,. \tag{21}$$

4.1 A family of Sturm-Liouville equations

Definition 4.1 *Let $\psi^{(m)}_\lambda(x)$ denote the family of unique positive increasing solutions to the Sturm-Liouville equation*

$$\frac{\partial^2\psi^{(m)}_\lambda}{\partial x^2} = 2\lambda\tilde{m}(x,m)\psi^{(m)}_\lambda \qquad\qquad (\lambda>0,\, m\geq x)\,, \tag{22}$$

subject to $\psi^{(m)}_\lambda(0)=1$.

Remark 4.1 $\psi^{(m)}_\lambda(x)$ is just the same $\psi_\lambda(x)$ that appears in section 2 for a regular diffusion $dX_t = \sigma(X_t,m)dW_t$, with diffusion coefficient equal to $\sigma(.,m)$ for m *fixed*, and the multiplicative constant chosen so that $\psi_\lambda(0)=1$. In probabilistic terms, $\psi_\lambda(x)$ is given as follows:

$$\psi^{(m)}_\lambda(x) \;=\; \begin{cases} \mathbb{E}_x(e^{-\lambda H_0}) & (x\leq 0)\,, \\ 1/\mathbb{E}_0(e^{-\lambda H_x}) & (x\geq 0) \end{cases}$$

(see section V.50 in [RW87] for details). Appendix 8 in [RY99] gives an explicit construction of $\psi_\lambda(x)$ by transforming to a Riccati equation.

Proposition 4.1 $\psi^{(m)}_\lambda(x)$ *is continuous in m.*

Proof. See Appendix B. ■

4.2 The joint density of $(X_{\xi_\lambda}, M_{\xi_\lambda})$

We now return to the X process in (11), and recall that $H_b = \inf\{t : X_t = b\}$ denotes the first hitting time of X to $b\geq x_0$.

Definition 4.2 *Let*

$$\Psi^{(m)}_\lambda(x,b) \;=\; \mathbb{E}_{x,m}(e^{-\lambda H_b})$$

denote the Laplace transform of the first hitting time to b from below.

Proposition 4.2 *We have the following expression for the joint density of* $(X_{\xi_\lambda}, M_{\xi_\lambda})$

$$\mathbb{P}_{x,m}(X_{\xi_\lambda} \in dy, M_{\xi_\lambda} \in db) \,=\, \hat{p}(y,b)dydb \,=\, \Psi^{(m)}_\lambda(x,b)\,\frac{2\lambda\psi^{(b)}_\lambda(y)}{\psi^{(b)}_\lambda(b)}\tilde{m}(y,b)dydb \qquad\qquad (b>m)\,. \tag{23}$$

Proof. We again recall the approximating process (X^n_t) in (16) and set $x_0 = x, m_0 = m$ as before. We know that $(X^n_{\xi_\lambda}, M^n_{\xi_\lambda}) \to (X_{\xi_\lambda}, M_{\xi_\lambda})$ a.s.,

$$\mathbb{E}_{x,m}(1_{M_{\xi_\lambda}\in[b,b+\delta],X_{\xi_\lambda}\in[y,y+\delta]}) = \lim_{n\to\infty}\mathbb{E}_{x,m}(1_{M^n_{\xi_\lambda}\in[b,b+\delta],X^n_{\xi_\lambda}\in[y-\delta,y+\delta]})\,. \tag{24}$$

for $\delta > 0$. As before, (X^n_t) is just a standard one-dimensional diffusion process for $t \in [H_{\frac{k}{n}}, H_{\frac{k+1}{n}}]$ for some Brownian motion W^n for all $k \in \mathbb{N}$. Hence, using the excursion theory description and the identity (20) for the rate of marked excursions falling in dy at the mark, we have

$$\begin{aligned}
\mathbb{E}_{x,m}(1_{M_{\xi_\lambda}\in[b,b+\delta],X_{\xi_\lambda}\in[y,y+\delta]}) \;&=\; \lim_{n\to\infty}\mathbb{E}_{x,m}(1_{M^n_{\xi_\lambda}\in[b,b+\delta],\,X^n_{\xi_\lambda}\in[y,y+\delta]})\\
&=\; \lim_{n\to\infty}\mathbb{P}_{x,m}(M^n_{\xi_\lambda}\geq b)\,[1-\exp(-\int_b^{b+\delta}\int_y^{y+\delta}\frac{2\lambda\psi^{(\frac{1}{n}[nu])}_\lambda(y)}{\psi^{(\frac{1}{n}[nu])}_\lambda(u)}\tilde{m}_n(y,u)dydu)]\\
&=\; \mathbb{P}_{x,m}(M_{\xi_\lambda}\geq b)\,[1-\exp(-\int_b^{b+\delta}\int_y^{y+\delta}\frac{2\lambda\psi^{(u)}_\lambda(y)}{\psi^{(u)}_\lambda(u)}\tilde{m}(y,u)dydu)]\\
&=\; \Psi^{(m)}_\lambda(x,b)\,[1-\exp(-\int_b^{b+\delta}\int_y^{y+\delta}\frac{2\lambda\psi^{(u)}_\lambda(y)}{\psi^{(u)}_\lambda(u)}\tilde{m}(y,u)dydu)]
\end{aligned} \tag{25}$$

for δ sufficiently small, where we have used the continuity of $\psi^{(m)}_\lambda(x)$ in m. Letting δ tend to zero, we see that

$$\hat{p}(y,b) = \lim_{\delta\to 0}\frac{1}{\delta^2}\mathbb{E}_{x,m}(1_{M_{\xi_\lambda}\in[b,b+\delta],\,X_{\xi_\lambda}\in[y,y+\delta]}) \;=\; \Psi^{(m)}_\lambda(x,b)\,\frac{2\lambda\psi^{(b)}_\lambda(y)}{\psi^{(b)}_\lambda(b)}\tilde{m}(y,b)\,.$$

■

From here on we set $x_0 = m_0$ so $M_t = \sup_{0\leq s\leq t}X_s$, i.e. there is no previous historical maximum at $t = 0$.

Corollary 4.3 $\hat{p}(y,b)$ *satisfies the forward Kolmogorov equation*

$$\frac{\partial^2}{\partial y^2}[\frac{1}{2}\sigma(y,b)^2\hat{p}] \;=\; \lambda\hat{p} \qquad\qquad (y<b). \tag{26}$$

Proof. As a function of y, $\hat{p}(y,b)$ is just a multiple of $\psi^{(b)}_\lambda(y)\tilde{m}(y,b)$. The result then just follows from the Sturm-Liouville equation (6). ■

Remark 4.2 From (7), we know that $\lim_{x\to-\infty}\psi_\lambda(x) = 0$ and $\lim_{x\to-\infty}\frac{\partial\psi_\lambda}{\partial x} = 0$. Applying this to (23), we see that

$$\lim_{y\to-\infty}\sigma(y,b)^2\hat{p}(y,b) = 0 \quad , \quad \lim_{y\to-\infty}\frac{\partial}{\partial y}[\sigma(y,b)^2\hat{p}(y,b)] = 0\,. \tag{27}$$

4.3 The forward equation for an up-and-out put option

Proposition 4.4 *Let* $\hat{V}(K,b) = \int_{-\infty}^{\infty}(K-y)^+\hat{p}(y,b)dy = \frac{1}{db}\mathbb{E}_{x_0,x_0}((K-X_{\xi_\lambda})^+1_{M_{\xi_\lambda}\in db})$. *Then* $\hat{V}$ *satisfies the forward equation*

$$\frac{1}{2}\sigma(K,b)^2\frac{\partial^2\hat{V}}{\partial K^2} = \lambda\hat{V} \qquad\qquad (K\leq b)\,. \tag{28}$$

Remark 4.3 Note that $\hat{V} = \frac{\partial\hat{P}}{\partial b}$, where

$$\hat{P}(K,b) = \mathbb{E}_{x_0,x_0}((K-X_{\xi_\lambda})^+1_{M_{\xi_\lambda}<b})$$

is the price of an up-and-out put option on X_{ξ_λ}. Thus we can obtain the joint density $\hat{p}(y,b)$ from observed up-and-out put prices, if $\hat{P}\in C^{2,1}$.

Proof. Integrating (26) with respect to y, and using (27) we have

$$\frac{\partial}{\partial y}[\frac{1}{2}\sigma(y,b)^2\hat{p}]\,|_{y=K} = \lambda\int_{-\infty}^{K}\hat{p}(y,b)dy \qquad\qquad (K \le b).$$

Integrating again, and using (27) and the Breeden-Litzenberger formula, we obtain

$$\frac{1}{2}\sigma(K,b)^2\hat{p}(K) = \lambda\int_{-\infty}^{\infty}(K-y)^+\hat{p}(y,b)dy \;=\; \lambda\hat{V}(K,b) \;=\; \lambda\frac{1}{db}\mathbb{E}_{x_0,x_0}((K-X_{\xi_\lambda})^+1_{M_{\xi_\lambda}\in db}) \qquad\qquad (y \le b).$$

Using that $\hat{p}(K,b) = \frac{\partial^2\hat{V}}{\partial K^2}$, we see that $\hat{V}$ satisfies the forward equation

$$\frac{1}{2}\sigma(K,b)^2\frac{\partial^2\hat{V}}{\partial K^2} = \lambda\hat{V} \qquad\qquad (K \le b)\,.$$

■

4.4 Rogers' condition

Let $\nu(b,u)$ denote the joint density of the maximum and the drawdown $(M_{\xi_\lambda}, M_{\xi_\lambda} - X_{\xi_\lambda})$ corresponding to $\hat{p}(y,b)$. From Proposition 3.3, we know that $(X_{t\wedge\xi_\lambda})$ is a continuous uniformly integrable $\mathcal{F}_{A_t}$-martingale. Thus, from Theorem 3.1 in Rogers[Rog93], the joint density ν must satisfy

$$\int_b^{\infty}\int_0^{\infty}\nu(m,u)dudm = \int_0^{\infty}u\nu(b,u)du\,, \tag{29}$$

which has the interpretation

$$\mathbb{P}(M_{\xi_\lambda} > b) \;=\; \frac{1}{db}\mathbb{E}((M_{\xi_\lambda} - X_{\xi_\lambda})1_{M_{\xi_\lambda}\in db}) \;=\; \frac{1}{db}\mathbb{E}((b - X_{\xi_\lambda})1_{M_{\xi_\lambda}\in db})\,. \tag{30}$$

5 Backing out $\sigma(x,m)$ from a given joint density for the maximum and the terminal level

Assumption 5.1 *Let μ be a probability measure on $R = \{(y,b)\in\mathbb{R}^2 : y \le b,\, x_0 \le b < \infty\}$. Assume that μ has a bounded, continuous and strictly positive density $\mu(y,b)$ on R, with $\int_R(b+|y|)\mu(y,b)dydb < \infty$, $\int_R y\mu(y,b)dydb = x_0$, and assume that the ν density associated with μ satisfies Rogers' condition in* (29).

Proposition 5.2 *Let Assumption 5.1 hold, and define*

$$\tilde{m}(K,b) = \frac{\mu(K,b)}{2\lambda\int_{-\infty}^{\infty}(K-y)^+\mu(y,b)dy} \qquad\qquad (K \le b,\ \lambda > 0) \tag{31}$$

and assume that $0 < \sigma(K,b) = \tilde{m}(K,b)^{-\frac{1}{2}} \le \sigma_{\max} < \infty$ with σ continuous, and assume $\tilde{m}$ satisfies the Lipschitz condition in Assumption 3.4. Let (X_t) denote the process in (11) *with $\tilde{m}(x,m)$ given by* (31)*, and ξ_λ be an $\mathcal{F}_0$-measurable $\mathbb{P}_{x_0,x_0}$-exponential random variable with rate λ, independent of X. Then $(X_{\xi_\lambda}, M_{\xi_\lambda}) \sim \mu$ under $\mathbb{P}_{x_0,x_0}$.*

Proof. The proof just follows from (28). ■

5.1 The problem case $\mu(x_0, x_0) = 0$ - infinite volatility at time zero

Letting $b \searrow x_0$, Rogers' condition in (30) tends to the following

$$1 = \frac{1}{db}\mathbb{E}((x_0 - X_{\xi_\lambda})1_{M_{\xi_\lambda} \in db})|_{b=x_0}\,, \tag{32}$$

and note that we have an indicator function on the event $\{M_{\xi_\lambda} = x_0\}$ i.e. that a new maximum is not attained on $[0, \xi_\lambda]$. Combining (32) with (31), we see that

$$\tilde{m}(x_0, x_0) = \frac{\mu(x_0, x_0)}{2\lambda \int_{-\infty}^{\infty}(x_0 - y)^+ \mu(y, x_0)dy} = \frac{\mu(x_0, x_0)}{2\lambda \int_{-\infty}^{x_0}(x_0 - y)\mu(y, x_0)dy} = \frac{\mu(x_0, x_0)}{2\lambda}\,, \tag{33}$$

so $\sigma(x_0, x_0) = \infty$ if and only if $\mu(x_0, x_0) = 0$.

We wish to preclude the case where $\sigma(x_0, x_0) = \infty$. However, for many processes of interest, e.g. Brownian motion at a fixed time, Brownian motion at an independent Gamma time (not exponential), a mixture of Brownian motions at a fixed time (see section 7), the joint density $\mu(y, b)$ is zero at the starting point (x_0, x_0), but nowhere else; hence the calibrated $\sigma(x_0, x_0) = \infty$ in Proposition 5.2. In this case, we replace $\mu(y, b)$ by the truncated density $\mu_\delta(y, b)$, defined as follows

$$\mu_\delta(y, b) \quad = \quad \begin{cases} \epsilon & (x_0 - \delta \le y \le x_0 + \delta,\ x_0 \le b \le x_0 + \delta)\,, \\ \mu(y, b) & \text{(otherwise)} \end{cases} \tag{34}$$

for $\delta > 0$, where $\epsilon = \frac{1}{2\delta^2}\int_{x_0}^{x_0+\delta}\int_{x_0-\delta}^{m}\mu(y, m)dydm$ is a constant, constructed so as to make $\mu_\delta(y, b)$ integrate to 1 (draw a picture!). We can then apply Proposition 5.2 to $\mu_\delta(y, b)$ for δ arbitrarily small; in this sense we can approximate the original joint density μ to arbitrary precision.

6 The special case $dX_t = \sigma(M_t)dW_t$ - a diffusion process with a given law for the maximum at a fixed time

This interesting special case is studied in Carraro,ElKaroui&Obłój[CElO09]. For any locally bounded positive Borel function u, associate the primitive function $U(x) = x_0 + \int_a^x u(s)ds$, and let $\bar{W}_t = \sup_{0\le s\le t} W_s$. The Azéma-Yor process associated with U is

$$X_t^U = U(\bar{W}_t) - u(\bar{W}_t)(\bar{W}_t - W_t) = x_0 + \int_0^t u(\bar{W}_s)\,dW_s\,,$$

because $(\bar{W}_t - W_t)d\bar{W}_t = 0$. Note that $\bar{X}_t^U = U(\bar{W}_t)$, so

$$dX_t^U = \sigma(\bar{X}_t^U)dW_t \tag{35}$$

where $\sigma = u(U^{-1})$. From this, we see that we can choose U (and thus σ) so that $\bar{X}_t^U$ has a given law, because we know the law of $\bar{W}_t$ is just a one-sided Gaussian distribution. This is tantamount to fitting all the single-barrier no-touch option prices at a single-maturity i.e. options that pay $1_{M_t \le b}$ at t for all $b \ge x_0$. Alternatively, we can choose σ so that X_{ξ_λ} has a given law, where ξ_λ is an $\mathcal{F}_0$-measurable exponential random variable with parameter λ, independent of X

We can also use a regular one-dimensional diffusion $dX_t = \sigma(X_t)dW_t$ to fit the law of M_{ξ_λ} using the standard identities from section 2:

$$\mathbb{E}_{x_0}(e^{-\lambda H_b}) = \mathbb{P}_{x_0}(M_{\xi_\lambda} > b) = \frac{\psi_\lambda(x_0)}{\psi_\lambda(b)}\,, \quad \frac{1}{2}\sigma(x)^2\psi_\lambda''(x) = \lambda\psi_\lambda\,,$$

and then back out $\sigma(x)$ from a pre-specified complementary cumulative distribution function for the maximum $\bar{F}(b) = \frac{\psi_\lambda(x_0)}{\psi_\lambda(b)}$ for all $b \ge x_0$, using the formula

$$\sigma(b)^2 = \frac{2\lambda\psi_\lambda(b)}{\psi_\lambda''(b)}$$

(note that we can set $\psi_\lambda(x_0) = 1$ without loss of generality).

7 Examples

7.1 Fitting to a mixture of Local Variance Gamma model densities

For standard Brownian motion evaluated at an independent exponential time, using the reflection principle we have

$$\hat{p}(y,b) = 2\lambda e^{-\sqrt{2\lambda}\,(2b-y)} \qquad (y \le b). \tag{36}$$

We now wish to a find a $\sigma(x,m)$ such that $(X_{\xi_\lambda}, M_{\xi_\lambda})$ has the following pdf

$$\hat{p}(y,b) = \frac{1}{2}\left[2e^{-\sqrt{2}(2b-y)} + 4e^{-2(2b-y)}\right],$$

for $\lambda = 1$ and $X_0 = 0$. This pdf is a mixture density of (36) for $\lambda = 1, 2$. Using Eq (31), we find that

$$\frac{2}{3} \le \sigma(K,b)^2 = \frac{1 + e^{-\sqrt{2}(\sqrt{2}-1)(2b-K)}}{1 + 2e^{-\sqrt{2}(\sqrt{2}-1)(2b-K)}} \le 1 \qquad (K \le b). \tag{37}$$

The marginal density of X_{ξ_λ} is symmetric around zero, and we see that $\sigma(X_t, M_t)$ is a bounded, strictly increasing function of $M_t + M_t - X_t$ i.e. the reflection of X_t around M_t.

7.2 Fitting to a symmetric cdf using the reflection principle

Motivated by the reflection principle for Brownian motion, consider calibrating to a joint cdf of the following form

$$\mathbb{P}(M > b, X < y) = \bar{F}(2b - y) \tag{38}$$

for $y < b$, where $\bar{F}$ is a complementary c.d.f. with a *symmetric* density f centred around zero, with $f'(x) < 0$ for $x > 0$, $f'(x) > 0$ for $x < 0$, and f not differentiable at $x = 0$. Then $\sigma(K,b)$ takes the particularly simple form

$$\sigma(K,b)^2 = -\frac{2\bar{F}(2b-K)}{f'(2b-K)} \ge 0, \tag{39}$$

and we see that $\sigma(.,.)$ depends only on $z = 2K - b$, so we write $\sigma(K,b) \equiv \sigma(z)$. We also have to preclude the possibility that $\sigma(z)$ explodes as $z \to \infty$. To this end, we also impose that

$$\limsup_{z\to\infty} \sigma(z)^2 = \limsup_{z\to\infty} -\frac{2\bar{F}(z)}{f'(z)} < \infty. \tag{40}$$

If we try and fit to the joint density of standard Brownian motion evaluated at fixed time 1

$$\hat{p}(y,b) = 2(2b-y)\frac{\sqrt{2}}{\sqrt{\pi}}\, e^{-\frac{1}{2}(2b-y)^2} \qquad (y \le b), \tag{41}$$

we are in the problem case $\mu(x_0, x_0) = 0$ discussed in subsection 5.1. If we perform the aforementioned δ-truncation procedure in (34), we find that the calibrated $\sigma(z)$ function tends to

$$\sigma(z)^2 = \frac{2\Phi^c(z)}{z n(z)} \tag{42}$$

as $\delta \to 0$, where $n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi^c(x) = \int_x^\infty n(z)dz$. Note that $\sigma(z) \to \infty$ as $z \to 0$ and $\sigma(z) \to 0$ as $z \to \infty$, so we see that condition (40) is satisfied.

7.3 Fitting to a Variance Gamma model-type density

To reproduce a Variance Gamma model type density, we can consider Brownian motion evaluated at the sum of two independent Exponential rvs i.e. a Gamma random variable. In this case we have

$$\hat{p}(y,b) = (2b-y)\sqrt{2}e^{-\sqrt{2}(2b-y)} \qquad (y \le b). \tag{43}$$

and

$$\sigma(K,b)^2 = 1 + \frac{\sqrt{2}}{2b-K}, \tag{44}$$

so $\mu(0,0) = 0$ and $\sigma(0,0) = \infty$, so again we have to perform the δ-truncation.

8 Conclusion

We have constructed a weak solution to the stochastic functional differential equation $X_t = x_0 + \int_0^t \sigma(X_s, M_s)dW_s$ via a time-changed Brownian motion. Using excursion theory for excursions away from the maximum for a regular diffusion process, we were able to specify $\sigma(.,.)$, so that X is a martingale, and the terminal level and supremum of X, when stopped at an independent exponential time ξ_λ, has a pre-specified joint density μ. For the problem case when $\mu(x_0, x_0) = 0$, the initial volatility is infinite, so instead we just fit the joint law to arbitrary accuracy by truncating the density around (x_0, x_0). Possible future directions for this line of research include: (i) providing a cleaner solution for the problem case $\mu(x_0, x_0) = 0$ which does not involves truncation and instead deals rigorously with the peculiar issue of infinite initial volatility and (ii) relaxing the conditions on the two-dimensional speed measure $\tilde{m}$, e.g. impose that $\tilde{m}$ is just continuous instead of Lipschitz continuous, and removing the upper bound on σ.

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A Proof of Proposition 3.2

We first establish that X is non-exploding. $\sigma(x,m)>0$, so $\tilde{m}(x,m)<\infty$, and (B_t) is continuous a.s., hence $T_t<\infty$ a.s. for $0<t<\infty$. Also, from the lower bound on $\tilde{m}$, we see that $T_\infty=\lim_{s\to\infty}T_s=\infty$ a.s. Consequently, A_t is strictly increasing and continuous a.s., with $A_t<\infty$ for $0\le t<\infty$ and $A_\infty=\lim_{t\to\infty}A_t=\infty$ a.s.. Thus $|X_t|=|B_{A_t}|<\infty$ a.s., so X is non-exploding.

From here on, we proceed as in the proof of Theorem 5.5.4 in [KS91]. From Problem 3.4.5(v) in [KS91], we can verify that each A_t is a stopping time for $\{\mathcal{F}_s\}$. Set $\mathcal{G}_t=\mathcal{F}_{A_t}$. A is continuous, so $\{\mathcal{G}_s\}$ inherits the usual conditions from $\{\mathcal{F}_s\}$ (see Problem 1.2.23 in [KS91]). From the optional sampling theorem (Problem 1.3.23 in [KS91]) and the identity $A_{t\wedge T_s}=A_t\wedge s$ (Problem 3.4.5 (ii),(v) in [KS91]), we have for $0\le t_1\le t_2<\infty$ and $n\ge 1$

$$\mathbb{E}(X_{t_2\wedge A_n}|\mathcal{G}_{t_1})=\mathbb{E}(B_{A_{t_2}\wedge n}|\mathcal{F}_{A_{t_1}})=B_{A_{t_1}\wedge n}=X_{t_1\wedge T_n}\,,\quad \text{a.s.}$$

where $T_n:=\inf\{t:A_t\ge n\}$. Since $\lim_{n\to\infty}T_n=\infty$ a.s., we see that X is a continuous local martingale. Furthermore, $X^2_{t\wedge T_n}-A_{t\wedge T_n}=B^2_{A_t\wedge n}-(A_t\wedge n)$ is a continuous martingale for each $n\ge 1$, so the increasing process associated with X is given by

$$\langle X\rangle_t=A_t \qquad (0\le t<\infty,\text{ a.s.})\,. \tag{A-1}$$

We now compute the explicit form of A. The function $u\mapsto T_u(\omega)$ for $u\in\mathbb{R}$ is absolutely continuous, and the change of variable $v=T_u(\omega)$ is equivalent to $A_v=u$, so we have

$$A_t(\omega)=\int_0^{A_t(\omega)}\sigma^2(B_u(\omega),\bar{B}_u(\omega)\vee m_0)dT_u(\omega)=\int_0^t\sigma^2(X_v(\omega),M_v(\omega))dv\,. \tag{A-2}$$

(recall that $B_0=x_0$ and $\bar{B}_t=m_0\vee\sup_{0\le s\le t}B_s$). From (A-1) and (A-2) and the fact that $A_t<\infty$, we conclude that $\langle X\rangle$ is absolutely continuous. Theorem 3.4.2 in [KS91] asserts the existence of a Brownian motion $(\tilde{W}_t)$ and a measurable, adapted process (ρ_t) on a possibly extended probability space $\{\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}\}$ such that

$$X_t=x_0+\int_0^t\rho_v d\tilde{W}_v\,,\qquad \langle X\rangle_t=\int_0^t\rho_v^2dv;\quad 0\le t<\infty,\quad \tilde{\mathbb{P}}\text{ a.s.}$$

In particular $\tilde{\mathbb{P}}(\rho_t^2=\sigma^2(X_t,M_t)$ for Lebesgue a.e. $t\ge 0)=1$. We can set

$$W_t=\int_0^t\operatorname{sgn}(\rho_v)d\tilde{W}_v;\qquad 0\le t<\infty;$$

and W is itself a Brownian motion (Theorem 3.3.16 in [KS91]). Then

$$X_t=x_0+\int_0^t\rho_v d\tilde{W}_v=x_0+\int_0^t\sigma(X_v,M_v)dW_v\,.$$

Thus (X,W) is a (non-trivial) weak solution to (12).

B Proof of Proposition 4.1

Recall the family of Brownian motions $(B_t, \mathbb{P}_{x_0})$ defined at the start of section 3 with $B_0 = x_0$ under $\mathbb{P}_{x_0}$. For $x < 0$, by the usual time-change construction and Remark 4.1, we can re-write $\psi_\lambda^{(m)}(x)$ as

$$\psi_\lambda^{(m)}(x) = \mathbb{E}_x(\exp\{-\lambda \int_0^{\tau_0} \tilde{m}(B_s, m)ds\}) .$$

The continuity of $\psi_\lambda^{(m)}(x)$ in m then follows from the continuity of $\tilde{m}(.,.)$ in m and the bounded convergence theorem. Similarly for $x > 0$ we have

$$\psi_\lambda^{(m)}(x) = 1/\mathbb{E}_0(\exp\{-\lambda \int_0^{\tau_x} \tilde{m}(B_s, m)ds\}) ,$$

and continuity in m follows for the same reasons.