

A simplified unified propagator model for signed/unsigned order flow, concave price impact and rough volatility

Abstract

We simplify the setup in [MORS26] by exogenously modelling the signed order flow as an fBM-type process with $H > \frac{1}{2}$, which avoids the need for Hawkes processes, Mittag-Leffler functions and lengthy scaling limit arguments.

0.1 Positive order flow and the inversion formula

Similar to Theorem 3.1 in [MORS26]¹, we consider a hyper-rough Heston process $(F_t^+)_{t \geq 0}$ for cumulative positive order flow which satisfies

$$F_t^+ = g(t) + \int_0^t f(t-s)B_{F_s^+}ds$$

where B is a standard Brownian motion with $B_0 = 0$ and $f(t) = e^{-\theta t}t^{H-\frac{1}{2}}$ is the usual Gamma kernel with $H \in (-\frac{1}{2}, \frac{1}{2})$, $\theta > 0$ so $f \in L^1$ and we assume $g \in C^1$. F_t^+ is a.s. $(2\alpha - \varepsilon) \wedge 1$ -Hölder continuous where $\alpha = H + \frac{1}{2}$ (see e.g. Theorem 3.1 in [JM20] for details)² and F^+ is an increasing process which is non-differentiable if $H \leq 0$. Then if $h * f \equiv 1$, we have the inversion formula:

$$\begin{aligned} \int_0^t h(t-s)(F_s^+ - g(s))ds &= \int_0^t h(t-s) \int_0^s f(s-u)B_{F_u^+}duds = \int_0^t \int_u^t h(t-s)f(s-u)dsB_{F_u^+}du \\ &= \int_0^t \int_0^{t-u} h(t-s-u)f(s)dsB_{F_u^+}du \\ &= \int_0^t (h * f)(t-u)B_{F_u^+}du = \int_0^t B_{F_u^+}du \end{aligned}$$

where Fubini is justified since $F_t^+ < \infty$ a.s. for finite t and B is a.s. continuous, and thus

$$\frac{d}{dt} \int_0^t h(t-s)(F_s^+ - g(s))ds = B_{F_t^+} \quad (1)$$

Lebesgue a.e. Taking Laplace transforms, the condition $h * f \equiv 1$ becomes $\hat{f}(\lambda)\hat{h}(\lambda) = \frac{1}{\lambda}$, from which we find that

$$h(t) = \frac{\theta^\alpha}{\Gamma(\alpha)} \left(1 - \frac{\Gamma(-\alpha, t\theta)}{\Gamma(-\alpha)}\right)$$

where $\Gamma(a, z) = \int_z^\infty s^{a-1}e^{-s}ds$ is the incomplete Gamma function, and in particular $h(t) = O(t^{-H-\frac{1}{2}})$ as $t \rightarrow 0$.

0.2 Persistent signed order flow

Again following [MORS26], we now assume the *signed* order flow for an asset is

$$V_t = F_t^+ - F_t^-$$

where $F_t^- = g(t) + \int_0^t f(t-s)W_{F_s^-}ds$ and W is another Brownian motion independent of B (so F^- is an i.i.d. copy of F^+). Then we see that

$$\frac{d}{dt} \int_0^t h(t-s)V_sds = B_{F_t^+} - W_{F_t^-}$$

since the g terms cancel, and for $0 \leq t \leq u$ we have

$$\mathbb{E}(F_u^+ | \mathcal{F}_{F_t^+}^B) = g(u) + \mathbb{E}(\int_0^u f(u-s)B_{F_s^+}ds | \mathcal{G}_t^+) = g(u) + \int_0^t f(u-s)B_{F_s^+}ds + B_{F_t^+} \int_t^u f(u-s)ds$$

(and similarly for $\mathbb{E}(F_u^- | \mathcal{F}_{F_t^-}^W)$), where $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$, assuming $(B_{F_t^+})_{t \geq 0}$ is a \mathcal{G}_t^+ -martingale (see e.g. Section 7 in [AJ21] for more on this).

¹see also section 5.4 in [FGS21]

²we can simulate F^+ using the Monte Carlo scheme in [AA25] using Normal Inverse Gaussian variates

0.3 Price dynamics and the propagator model

Making the usual assumption that the asset price $P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u | \mathcal{G}_t)$ where $F_t = F_t^+ - F_t^-$ and $\mathcal{G}_t = \sigma(\mathcal{G}_t^+, \mathcal{G}_t^-)$ with $\mathcal{G}_t^- = \mathcal{F}_{F_t^-}^W$ (for some $\kappa > 0$)³, we find that $f(u-s) \rightarrow 0$ as $u \rightarrow \infty$ since $\theta > 0$, but $\lim_{u \rightarrow \infty} \int_t^u f(u-s)ds = c_{H,\theta}$ where $c_{H,\theta} = \theta^{-\frac{1}{2}-H}\Gamma(\frac{1}{2} + H)$, so

$$P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u^+ - F_u^- | \sigma(\mathcal{F}_{F_t^+}^B, \mathcal{F}_{F_t^-}^W)) = \kappa c_{H,\theta} (B_{F_t^+} - W_{F_t^-})$$

i.e. P is the difference of two i.i.d. hyper-rough Heston models (each with correlation $\rho = 1$ since each process only has one driving Brownian motion), and

$$P_t = \kappa c_{H,\theta} \frac{d}{dt} \int_0^t h(t-s) V_s ds$$

which is a weak formulation of the usual Propagator model, and the *unsigned* order flow is $U_t = F_t^+ + F_t^-$.

0.4 Power-law price impact

In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 up to time t_0 is given by

$$MI(t) = \frac{d}{dt} \int_0^t h(t-s) s ds = \frac{\theta^{-\alpha_-} (t\theta\Gamma(-\alpha) - t\theta\Gamma(-\alpha, t\theta) - \Gamma(\alpha_-, 0) + \Gamma(\alpha_-, t\theta))}{\Gamma(-\alpha)\Gamma(\alpha)}$$

for $0 \leq t < t_0$ (where $\alpha_- = \frac{1}{2} - H$) for $0 \leq t \leq t_0$ which is $O(t^{\frac{1}{2}-H})$ as $t \rightarrow 0$ and globally concave in t for $H \in (-\frac{1}{2}, \frac{1}{2})$, consistent with empirical evidence (note the usual square-root impact law corresponds to $H = 0$ here), and we can show that the asymptotic (i.e. permanent) impact as $t \rightarrow \infty$ (given no trading after t_0) is

$$MI(\infty) = \frac{t_0 \theta^\alpha}{\Gamma(\frac{1}{2} + H)}$$

(this is the asymptotic grey line in the plot below). For $H = \frac{1}{2}$, $h(t) = \theta$ which corresponds to pure permanent price impact; conversely $\lim_{\theta \rightarrow 0} h(t) = \text{const.} \times t^{-\frac{1}{2}-H}$. Recall from above that F_t^+ has Hölder regularity $(2\alpha - \varepsilon) \wedge 1$, and empirical evidence in [MORS26] suggests that $2\alpha \in (\frac{1}{2}, 1)$ which corresponds to $H \in (-\frac{1}{4}, 0)$.

0.5 Extensions

If we consider a mixed model where $f(t) = e^{-\theta t} t^{H-\frac{1}{2}} + e^{-\theta_2 t} t^{H_2-\frac{1}{2}}$ with $H \in (-\frac{1}{4}, 0)$ and $H_2 \in (0, \frac{1}{2})$, then we find that

$$h(t) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} (-c)^n \sum_{k=0}^{\infty} \frac{(\beta n)_k}{k!} \delta^k \theta^{-(\mu_n+k)} \frac{\gamma(\mu_n+k, \theta t)}{\Gamma(\mu_n+k)}$$

where $\beta = \frac{1}{2} + H_2$, $c = \Gamma(\beta)/\Gamma(\alpha)$, $\Delta = \theta - \theta_2$, $\mu_n = \beta n - \alpha(n+1)$, $\gamma(a, z) = \int_0^z s^{a-1} e^{-s} ds$ is the lower incomplete Gamma function and $(a)_k$ denotes the Pochhammer symbol. We can then use two (possibly different) functions f functions of this form for F^\pm , and the presence of H_2 here means the price dynamics will now also have a non hyper-rough component, which is arguably more realistic.

³note we are assuming $P_0 = 0$ without loss of generality since we can easily add a P_0 term

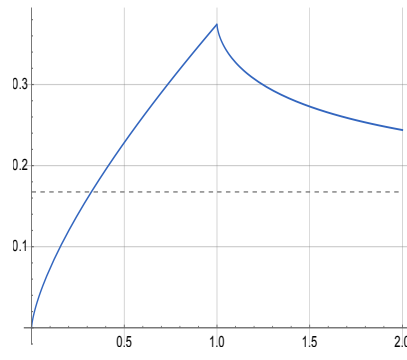


Figure 1: Concave price impact of an exogenous metaorder executed at constant trading speed 1 over $[0, 1]$ with $H = -0.2$, $\theta = 0.1$ and $\kappa = 1$. The blue line asymptotes to the constant level $MI(\infty)$ (grey line) as $t \rightarrow \infty$ which represents the asymptotic permanent price impact.

References

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