Convex order condition

Let $(X_t)_{t\geq 0}$ denote a martingale. Then for $0 \leq T_1 \leq T_2$:

$$\begin{aligned}
\mathbb{E}((X_{T_2}-K)^+) &= \mathbb{E}(\mathbb{E}((X_{T_2}-K)^+|X_{T_1})) \quad \text{(from the } \textbf{tower property}\\
&\geq \mathbb{E}(\mathbb{E}(X_{T_2}-K|X_{T_1})^+)\\
&\quad \text{(from the } \textbf{conditional Jensen inequality} \text{ applied to the convex function } f(x)=(x-K)^+)\\
&= \mathbb{E}((X_{T_1}-K)^+)\,.
\end{aligned}$$

Hence we see that call option prices with maturity T_2 are $\geq$ call option prices with maturity T_1. This is known as the **convex ordering** condition, which we can write as $\mu_{T_1} \preceq \mu_{T_2}$, where μ_t denotes the density of X_t.

Bass martingale with random initial starting distribution

Let B^α denote a Brownian motion with $B_0^\alpha \sim \alpha$ (i.e. a random initial starting point with density α). Then the density of B_t^α is

$$\int_{-\infty}^{\infty} R_t(y-x)\alpha(x)dx \quad = \quad (R_t * \alpha)(y)\,.$$

Moreover

$$\mathbb{E}(F(B_1^\alpha)|B_t^\alpha = x) \quad = \quad \int_{-\infty}^{\infty} R_{1-t}(y-x)F(y)dy \quad = \quad (R_{1-t} * F)(x)\,.$$

Now let $M_t = \mathbb{E}(F(B_1^\alpha)|B_t^\alpha)$ for $t \in (0,1]$. We wish to choose α so that $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$, for two given distributions μ_0 and μ_1 (both with zero expectations), with μ_0, μ_1 in convex order.

Then

$$M_t \quad = \quad (R_{1-t} * F)(B_t^\alpha)\,. \tag{1}$$

Let μ be a probability density. We define the **push-forward** $F_{\#}\mu$ of μ by F as the distribution of $F(X)$ if $X \sim \mu$, so

$$\mathbb{P}(F(X) \leq x) \quad = \quad \mathbb{P}(X \leq F^{-1}(x)) \quad = \quad \int_{-\infty}^{F^{-1}(x)} \mu(y)dy$$

Thus if μ_t denotes the density of M_t, (1) implies that

$$\mu_t \quad = \quad (R_{1-t} * F)_{\#}(R_t * \alpha)$$

since the distribution of B_t is $R_t * \alpha$. In particular

$$\begin{aligned}
\mu_0 &= (R_1 * F)_{\#}(\alpha) \\
\mu_1 &= F_{\#}(R_1 * \alpha)
\end{aligned} \tag{2}$$

since $R_0 * f = f$ for any f.

Deriving the fixed point equation for the distribution function of α

Recall that for two probability densities ν_1 and ν_2, for $h = G_{\nu_1}^{-1}(G_{\nu_2})$, $\nu_1 = h_{\#}\nu_2$. Applying this to (2) (assuming F is strictly increasing) we see that

$$\begin{aligned}
R_1 * F &= G_{\mu_0}^{-1} \circ G_\alpha \\
F &= G_{\mu_1}^{-1} \circ G_{R_1 * \alpha} \quad = \quad G_{\mu_1}^{-1} \circ (R_1 * G_\alpha)
\end{aligned} \tag{3}$$

using that $G_{R_1*\alpha} = R_1 * G_\alpha$. To check this identity, we take derivatives of the right hand side wrt x to get

$$\frac{d}{dx}\int_{-\infty}^{\infty} R_1(y)G_\alpha(x-y)dy \quad = \quad \int_{-\infty}^{\infty} R_1(y)G'_\alpha(x-y)dy \quad = \quad \int_{-\infty}^{\infty} R_1(y)\alpha(x-y)dy \quad = \quad (R_1 * \alpha)(x)\,.$$

Then using (3) and then (4), we see that

$$G_\alpha \quad = \quad G_{\mu_0} \circ (R_1 * F) \quad = \quad G_{\mu_0} \circ (R_1 * (G_{\mu_1}^{-1} \circ (R_1 * G_\alpha))) \quad = \quad \Phi(G_\alpha)$$

where Φ is shorthand for the all operators successively being applied to G_α on the right hand side. This is conceptually similar to a simple non-linear 1d equation of the form $x = g(x)$ in first year Numerical analysis, which we can solve using the fixed point method $x_{n+1} = g(x_n)$ if $|g'(x)| < 1$. We can use the same method here except now the scalar x_n is replaced by a function G_α^n, so the iterative scheme becomes

$$G_\alpha^{n+1} = \Phi(G_\alpha^n)$$

which (under suitable conditions) converges to a function $G_\alpha^\infty(.)$, which is the desired cdf for α so as to make $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$. Note once we have G_α can compute the distribution of B_1^α and hence the required function F to make $F(B_1^\alpha)) \sim \mu_1$.