

Homework 2

1. Let X be a Lévy process, and e_q an $\text{Exp}(q)$ random variable independent of X , and let $\rho_t(x)$ denote the density of X_t . Using that

$$\Phi_q^+(z) := \mathbb{E}(e^{iz\bar{X}_{e_q}}) = \exp\left(\int_0^\infty t^{-1} e^{-qt} \int_0^\infty (e^{izx} - 1) \rho_t(x) dx dt\right) \quad (1)$$

compute $\mathbb{E}(\bar{X}_t)$ (this question requires no knowledge of Lecture 3).

Solution.

$$(\Phi_q^+)'(0) = i\mathbb{E}(\bar{X}_{e_q}) = i \int_0^\infty e^{-qt} \frac{\mathbb{E}(X_t^+)}{t} dt.$$

But we also know that $i\mathbb{E}(\bar{X}_{e_q}) = i \int_0^\infty q e^{-qt} \mathbb{E}(\bar{X}_t) dt$, so (dividing both expressions by iq) we see that

$$\int_0^\infty e^{-qt} \mathbb{E}(\bar{X}_t) dt = \frac{1}{q} \int_0^\infty e^{-qt} \frac{\mathbb{E}(X_t^+)}{t} dt.$$

But by a standard simple property of Laplace transforms, $\frac{1}{q}\mathcal{L}f = \mathcal{L}F$, where \mathcal{L} denotes the Laplace transform operator and $F(t) = \int_0^t f(s) ds$ (see e.g. https://en.wikipedia.org/wiki/Laplace_transform#Properties_and_theorems). Hence (comparing both sides) we see that

$$\mathbb{E}(\bar{X}_t) = \int_0^t \frac{\mathbb{E}(X_s^+)}{s} ds$$

(we have seen this formula before in FM02 without proof).

2. The general **Wiener-Hopf** formula for a Lévy process states that

$$\mathbb{E}(e^{izX_{e_q}}) = \mathbb{E}(e^{iz\bar{X}_{e_q}})\mathbb{E}(e^{iz\bar{X}_{e_q}})$$

for $z \in \mathbb{R}$. Using (1), show that \bar{X}_{e_q} has a **Lévy-Khintchine** representation of the form $\log \Phi_q^+(z) = \int_0^\infty (e^{izx} - 1)\nu(x)dx$ for some non-negative function $\nu(x)$ (recall we refer to $\nu(x)$ as the Lévy density).

Solution. Interchanging integrals in (1), we see that

$$\log \Phi_q^+(z) := \log \mathbb{E}(e^{iz\bar{X}_{e_q}}) = \int_0^\infty (e^{izx} - 1) \int_0^\infty t^{-1} e^{-qt} \rho_t(x) dt dx = \int_0^\infty (e^{izx} - 1) \nu(x) dx$$

where $\nu(x) = \int_0^\infty t^{-1} e^{-qt} \rho_t(x) dt$.

Remark 0.1 If X is standard Brownian motion, using the known density $\rho_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}x^2/t}$ of W_t , we can easily check that $\nu(x) = \frac{e^{-x\sqrt{2q}}}{x} 1_{x>0}$, and we have seen in the FM02 Mock that $\bar{X}_{e_q} \sim \text{Exp}(\sqrt{2q})$. A Lévy process with this $\nu(x)$ function is known as a **Gamma process**, which is essentially a **one-sided CGMY process** with $Y = 0$.

3. Let $X_t = \sigma B_t^H$ where B^H is fBM, and let $\theta_{m,k} = -2^{\frac{m}{2}} (X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} + X_{\frac{2k}{2^{m+1}}})$ and $s_n^2 = \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} \theta_{m,k}^2$. Using that

$$\mathbb{E}(s_n^2) = \sigma^2 (4^{n(1-H)} - 1) \sim \sigma^2 4^{n(1-H)} = \sigma^2 2^{2n(1-H)}$$

and $s_n^2 \sim \sigma^2 2^{2n(1-H)}$ as $n \rightarrow \infty$ (i.e. the ratio of both sides tends to 1), derive an estimator \hat{H}_n for H using a ratio of s_n terms at different resolutions. Is \hat{H} scale-invariant? You may assume that $\mathbb{E}(\theta_{m,k}^2)$ is independent of k .

Solution. $\frac{s_n^2}{2^{2n(1-H)}} \rightarrow \sigma^2$ as $n \rightarrow \infty$, so $\frac{s_{n-1}^2}{2^{2(n-1)(1-H)}} \rightarrow \sigma^2$ and the ratio of these two terms satisfies

$$\frac{s_n^2}{s_{n-1}^2} \cdot 2^{-2(1-H)} \rightarrow 1.$$

Setting $\frac{s_n^2}{s_{n-1}^2} = 2^{2(1-\hat{H}_n)}$, we see that $\frac{2^{2(1-\hat{H}_n)}}{2^{2(1-H)}} \rightarrow 1$, so $2(1-\hat{H}_n) - 2(1-H) = 2(H-\hat{H}_n) \rightarrow 0$, i.e. \hat{H}_n is a consistent estimator, and we can write \hat{H}_n explicitly as

$$\hat{H}_n = 1 - \frac{1}{2} \log_2 \frac{s_n^2}{s_{n-1}^2}.$$

\hat{H}_n is scale-invariant since if we multiply the path X by λ , the ratio $\frac{s_n^2}{s_{n-1}^2}$ remains un-changed because the λ terms cancel.