Hawkes processes

(updated 11th Aug 7.12am). Consider a time-inhomogenous Poisson process $(N_t)_{t\geq 0} \in \mathbb{N}$ whose intensity is itself a random process λ_t which evolves as

$$\lambda_t \;=\; \mu + \int_{[0,t]} \phi(t-s)dN_s$$

i.e. λ depends on the history of N itself, where μ is a positive constant and ϕ a positive function. For this reason we say that N is **self-exciting**, and we can view this as a special type of **Stochastic Volterra Equation** (SVE) with no Brownian motion. The meaning of the λ_t is that $\lim_{h\to 0}\frac{1}{h}\mathbb{P}(N_{t+h}-N_t>0|\mathcal{F}_t^{\lambda,N})=\lambda_t$, and note we can re-write λ as

$$\lambda_t \;=\; \lambda_0 + \sum_{0\leq s_i\leq t}\phi(t-s_i)$$

where $s_1, s_2, ...$ are the random **jump times** of N, which we can also take as the definition of λ.

If we let $M_t = N_t - \int_0^t \lambda_u du$, then M is a martingale and we can re-write the λ equation as

$$\lambda_t \;=\; \mu + \int_{[0,t]}\phi(t-s)(dM_t+\lambda_t dt) \;=\; \mu + (\phi * dM)_t + (\phi*\lambda)_t\,.$$

For any $f\in L^1(0,\infty)$, we can easily check that $\|\phi * f\| = \|\phi\|\,\|f\|$, where all norms will refer to the $L^1(0,\infty)$ norm, so the operator $\phi*$ is a contraction on $L^1(0,\infty)$ if $\|\phi\|=\int_0^\infty\phi(u)du<1$, so the inverse $(I-\phi*)^{-1}=I+(\phi*)+(\phi*)^2+...$ is well defined. We can re-write the eq for λ_t as

$$(I-\phi*)\lambda \;=\; \mu + \phi * dM\,.$$

To make further sense of $(I-\phi*)^{-1}$, we look for a function ψ such that

$$(I-\phi*)^{-1}f \;=\; (I+\psi*)f$$

for any test function $f\in L^1(0,\infty)$, so

$$\begin{aligned} f \;&=\; (I-\phi*)(I+\psi*)f \;=\; (I-\phi*)(f+\psi*f) \\ &=\; f-\phi*f+\psi*f-\phi*\psi*f \end{aligned}$$

which we can re-write in operator form (i.e. without the f) as $\phi*\psi=\psi-\phi$. ψ is known as the **resolvent** of ϕ, note definition here is opposite way round to chap 3 in FM14. Applying this to our Hawkes process i.e. setting $f(t)=\lambda_t$, we see that

$$\begin{aligned} \lambda \;&=\; (I-\phi*)^{-1}(\mu+\phi*dM) \;=\; (1+\psi)*\mu + (I+\psi)*(\phi*dM) \\ &=\; \mu+\psi*\mu+(\phi*+(\phi*)^2+...)*dM \\ &=\; \mu+\psi*\mu+\psi*dM \end{aligned}$$

where $\psi=\sum_{k=1}^\infty(\phi*)^k$, which is shorthand for

$$\lambda_t \;=\; \mu+\mu\int_{[0,t]}\psi(t-s)ds + \int_{[0,t]}\psi(t-s)dM_s \tag{1}$$

The propagator model - concave price impact from Hawkes order flow

Consider two independent Hawkes processes $N_t^{\pm}$ with associated intensities $\lambda_t^{\pm}$ which evolve as with

$$\lambda_t^{\pm} \;=\; \mu+\mu\int_0^t\psi(t-s)ds+\int_0^t\psi(t-s)dM_s^{\pm}$$

where $dM_t^{\pm}=dN_t^{\pm}-\lambda_t^{\pm}dt$. Then

$$\begin{aligned} \mathbb{E}_t(N_u^+) \;&=\; g(t)+\mathbb{E}_t\Big(\int_0^u\int_0^s\psi(s-v)dM_v^+ds\Big) \;=\; g(t)+\mathbb{E}_t\Big(\int_0^u\int_v^u\psi(s-v)dsdM_v^+\Big) \\ &=\; g(t)+\int_0^t\int_v^u\psi(s-v)dsdM_v^+ \end{aligned}$$

for some function $g(t)$, and note that $\int_v^\infty \psi(s-v)ds = \int_0^\infty \psi(s)ds = \|\psi\|_1$. Then if we assume the current price $P_t = \kappa \lim_{u\to\infty} \mathbb{E}(N_u^+ - N_u^- | \mathcal{F}_t)$ for some constant $\kappa > 0$, then

$$\frac{1}{\kappa}P_t \;=\; \lim_{u\to\infty} \mathbb{E}(N_u^+ - N_u^- | \mathcal{F}_t) \;=\; \int_0^t \sigma(dM_v^+ - dM_v^-)$$

where $\sigma = \|\psi\|$. Then

$$\begin{aligned}\frac{1}{\sigma}P_t \;&=\; N_t^+ - N_t^- - \int_0^t \lambda_s^+ ds + \int_0^t \lambda_s^- ds \;=\; N_t^+ - N_t^- - \int_0^t \int_u^t \phi(s-u)ds(dN_u^+ - dN_u^-)\\ &=\; \int_0^t (1 - \int_0^{t-u} \phi(s)ds)(dN_u^+ - dN_u^-)\end{aligned}$$

so

$$P_t \;=\; \int_0^t \zeta(t-u)(dN_u^+ - dN_u^-) \tag{2}$$

where $\zeta(t) = \kappa\sigma(1 - \int_0^t \phi(s)ds) \searrow \kappa\sigma(1 - \|\phi\|)$ as $t \to \infty$, so

$$\zeta'(t) \;=\; -\kappa\sigma\phi(t) \;=\; -\zeta(0)\phi(t)$$

and since $\zeta(\infty) > 0$ under our standing assumption that $\|\phi\| < 1$, we interpret $\zeta(\infty)$ as the (non-transient) permanent price impact component of ζ. Hence from the Volterra form in (2), we see that P is a **propagator model** (e.g. like transient price impact) but also retains the martingale property.

Impact of a metaorder executed at constant rate before and after completion

Consider the additional contribution from an additional agent who buys at a fixed rate v for duration τ. Then impact the cumulative impact time t is

$$P_t \;=\; \int_0^t \zeta(t-u)(dN_u^+ - dN_u^-) \,+\, v\int_0^{t\wedge\tau} \zeta(t-s)ds$$

whose expectation is $MI(t) = v\int_0^{t\wedge\tau}\zeta(t-s)ds$, which we call the **market impact function**, see plot below where we see **concave price impact** up to τ, and then decay thereafter (which is broadly consistent with empirical findings where $MI(t)$ is often found to be $const. \times t^{\frac{1}{2}}$ for $t \le \tau$ (the so-called **square root impact law**.)

Example ϕ and ψ functions

One can easily check that $\|\phi\| = \frac{\|\psi\|}{1+\|\psi\|}$ so $\|\psi\| = \frac{\|\phi\|}{1-\|\phi\|}$. A common choice for ϕ is

$$\phi(t) \;=\; \nu t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$$

where $E_{\alpha,\alpha}$ is the **Mittag-Leffler** function (which is **heavy-tailed** since $\phi(t) \sim \frac{const.}{t^{1+\alpha}}$ as $t \to \infty$), and $\int_0^\infty \phi(t)dt = \frac{\nu}{\lambda}$ so we choose $\nu < \lambda$. For this choice of ϕ, the resolvent is

$$\psi(t) \;=\; \nu t^{\alpha-1} E_{\alpha,\alpha}(-(\lambda-\nu)t^\alpha)$$

and as $\nu \nearrow \lambda$, $\|\phi\| \nearrow 1$ and $\|\psi\| \nearrow \infty$ (see also table below).

	$\psi(t)$	$\phi(t)$
Constant	c	$c\,e^{-ct}$
Fractional	$\frac{c\,t^{\alpha-1}}{\Gamma(\alpha)}$	$c\,t^{\alpha-1}E_{\alpha,\alpha}(-c\,t^\alpha)$
Exponential	$c\,e^{-\lambda t}$	$c\,e^{-(\lambda+c)\,t}$
Gamma	$\frac{c\,e^{-\lambda t}t^{\alpha-1}}{\Gamma(\alpha)}$	$c\,e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-c\,t^\alpha)$

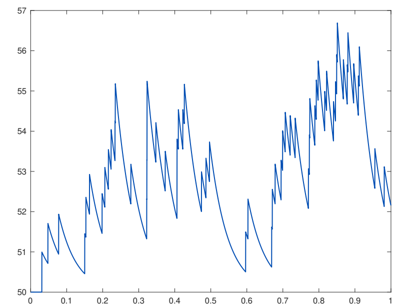

Figure 1: Here we we have simulated the intensity process of the form $\lambda_t = \lambda_0 + \int_0^t k(t-s)dN_s$ for $k(t) = e^{-10t}$ and $\lambda_0 = 50$.

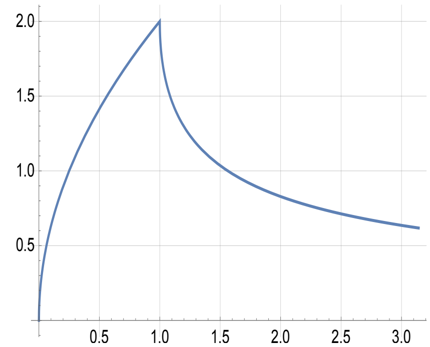

Figure 2: Here we see a typical concave market impact function $MI(t) = \int_0^{t\wedge\tau} \zeta(t-s)ds$ with $\tau = 1$.

Interpretation of a Hawkes process in terms of population dynamics

Let us define a population model: At time zero, there are no individuals. Some individuals (migrants) arrive as a uniform Poisson process with intensity μ. If a migrant arrives at time s, the birth dates of its children form a Poisson process of intensity $\phi(t-s)$ at time t, with $\int_0^\infty \phi(t)dt \quad < \quad 1$. In the same way, if a child is born at s', the birth dates of its children form a Poisson process of intensity $\phi(\cdot - s')$. Let N_t be the number of individuals who were born or migrated until time t. Then N is a Poisson-type process with intensity

$$\lambda_t \quad = \quad \mu + \int_0^t \phi(t-s)\, dN_s \tag{3}$$

i.e. N is a Hawkes process. This captures the notion of the process being self-exciting.

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