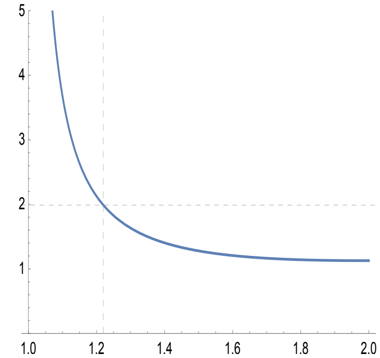

Figure 1: Plot of $F(\alpha,t)$ in q3 as a function of α (for $t=1$ fixed).

Homework 5

1. Plot the value of a European call option as a function of the initial stock price at $t=0$ for two different maturities T_1 and T_2 (assume that $0<T_1<T_2$ and the interest rate $r=0$). Derive the asymptotic behaviour of the call price as $S_0\to\infty$.

Solution. The unique no-arbitrage call option price $C(S_0,0)$ at $t=0$ satisfies

$$\frac{C(S_0,0)}{S_0} \;=\; \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}((S_T-K)^+) \;=\; \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}((S_0e^{\sigma W_T-\frac{1}{2}\sigma^2T}-K)^+) \;=\; \mathbb{E}^{\mathbb{Q}}((e^{\sigma W_T-\frac{1}{2}\sigma^2T}-\frac{K}{S_0})^+)\,.$$

This tends to 1 as $S_0\to\infty$ by the dominated convergence theorem, since the expression inside the expectation is less than or equal to $\mathcal{E}=e^{\sigma W_T-\frac{1}{2}\sigma^2T}$ (and tends monotonically to $\mathcal{E}$ as $S_0\to\infty$ a.s. since $\frac{K}{S_0}\to 0$), and $\mathbb{E}^{\mathbb{Q}}(\mathcal{E})=1<\infty$. Hence we have shown that $C(S_0,0)\sim S_0$ as $S_0\to\infty$.

2. Consider a portfolio of buying $\frac{1}{\Delta}$ call options with strike K, and selling $\frac{1}{\Delta}$ call options of strike $K+\Delta$ for $\Delta>0$. Plot the terminal payoff function $f(S)$ of this portfolio as a function of S. What does this payoff tend to as $\Delta\to 0$?

Solution. The payoff function $f(S)=0$ for $S\le K$, $f(S)=\frac{1}{\Delta}(S-K-(S-K-\Delta))=1$ for $S\ge K+\Delta$, and $f(S)=\frac{1}{\Delta}(S-K)$ for $S\in[K,K+\Delta]$, and in particular we see that $f(.)$ is continuous at $S=K$ and $K+\Delta$ (This is known as a **call spread** strategy.). From a picture, we see that this tends to a digital call payoff $1_{S_T\ge K}$ as $\Delta\to 0$.

3. A **symmetric α-stable process** X with parameters $\alpha\in(0,2]$, $\sigma>0$ is a generalization of Brownian motion, which has independent stationary increments like Brownian motion but now $\mathbb{E}(e^{iu(X_t-X_s)}|X_s)=e^{-(t-s)\sigma^\alpha|u|^\alpha}$ for $u\in\mathbb{R}$ and $0\le s\le t$, so X is only a (multiple of) BM if $\alpha=2$, but for $\alpha<2$ the increments of X are not normally distributed. If $\bar{X}_t=\max_{0\le s\le t}X_s$ and $\underline{X}_t=\min_{0\le s\le t}X_s$, it can be shown that

$$\mathbb{E}(\bar{X}_t-\underline{X}_t) \;=\; F(\alpha,t) \;:=\; \frac{2\alpha\Gamma(1-\frac{1}{\alpha})}{\pi}t^{\frac{1}{\alpha}}$$

for $\alpha\in(1,2]$. Use this identity to define a statistical estimator $\hat{\alpha}$ for α (assuming $\sigma=1$) from observed values for $\bar{X}_t$ and $\underline{X}_t$. Is $\hat{\alpha}$ biased? (you may use that $\frac{\partial^2}{\partial\alpha^2}F(\alpha,t)>0$).

Solution. We just solve $\bar{X}_t-\underline{X}_t=F(\hat{\alpha},t)$ for $\hat{\alpha}$ to get $\hat{\alpha}$ (see plot of $F(\alpha,t)$ above for $t=1$). For the 2nd part, we see that

$$F(\alpha,t) \;=\; \mathbb{E}(\bar{X}_t-\underline{X}_t) \;=\; \mathbb{E}(F(\hat{\alpha},t)) \;\ge\; F(\mathbb{E}(\hat{\alpha}),t)$$

where the final inequality follows from **Jensen's inequality** from last lecture. Assuming the $\ge$ is actually a $>$ here, we can apply F^{-1} to both sides and (since F^{-1} is decreasing), we see that $\alpha<\mathbb{E}(\hat{\alpha})$, so $\hat{\alpha}$ is biased.

4. Let $M_t^{(n)} = \max_{0 \le k \le n} W_{kt/n}$ denote the **discretely sampled** maximum of W. Is $M_t^{(n)}$ more or less than $M_t = \max_{0 \le s \le t} W_s$? It is known that

$$\mathbb{E}(M_t^{(n)}) \quad = \quad \sqrt{\frac{t}{2\pi n}}\, H_n^{(\frac{1}{2})}$$

where $H_n^{(\frac{1}{2})} = \sum_{k=1}^{n} k^{-\frac{1}{2}}$ is known as the nth Harmonic number of order $\frac{1}{2}$. Using that $H_n^{\frac{1}{2}} \sim 2\sqrt{n}$ as $n \to \infty$, show this formula is consistent with the formula $\mathbb{E}(R_t) = 2\sqrt{\frac{2t}{\pi}}$ in Hwk4, q5, where R_t is the range of W.

Solution. $M_t^{(n)} \le M_t$. Using the given asymptotic relation, we see that

$$\mathbb{E}(M_t^{(n)}) \quad \sim \quad \sqrt{\frac{t}{2\pi n}}\, 2\sqrt{n} \quad = \quad \sqrt{\frac{2t}{\pi}}$$

and we see this is half $\mathbb{E}(R_t)$ as expected, since $\mathbb{E}(R_t) = \mathbb{E}(M_t) - \mathbb{E}(m_t) = 2\mathbb{E}(M_t)$ since $m_t \sim -M_t$, where $m_t = \min_{0 \le s \le t} W_s$.

5. Consider the famous **GARCH(1,1)** discrete-time stochastic volatility model where

$$\begin{aligned} r_t &= \sqrt{V_t}\,\varepsilon_t \\ V_t &= \omega + \alpha r_{t-1}^2 + \beta V_{t-1} \end{aligned}$$

for $t \in \mathbb{Z}$ and $\omega, \alpha, \beta > 0$, where $r_t = (S_t - S_{t-1})/S_{t-1}$ is the return on a stock price process S at time t, and ε_t is an i.i.d. sequence of random variables with $\mathbb{E}(\varepsilon_t) = 0$ and $\mathrm{Var}(\varepsilon_t) = 1$. Express V_t in terms of the ε_t sequence only, and compute $\mathbb{E}(V_t)$ and give a condition on the model parameters for the formula to make sense.

Solution. We first note that

$$V_t \quad = \quad \omega + \alpha r_{t-1}^2 + \beta V_{t-1} \quad = \quad \omega + \alpha V_{t-1}\varepsilon_{t-1}^2 + \beta V_{t-1} \quad = \quad \omega + A_t V_{t-1} \tag{1}$$

where $A_t = \alpha\varepsilon_{t-1}^2 + \beta$. "Unrolling" this expression, we see that

$$\begin{aligned} V_t \quad = \quad \omega + A_t(\omega + A_{t-1}V_{t-2}) \quad &= \quad \omega + A_t(\omega + A_{t-1}(\omega + A_{t-2}V_{t-3})) \\ &= \quad \omega + A_t\omega + A_tA_{t-1}\omega + A_tA_{t-1}A_{t-2}V_{t-3} \\ &= \quad \omega(1 + A_t + A_tA_{t-1} + A_tA_{t-1}A_{t-2} + \ldots)\,. \end{aligned}$$

We also note that $\mathbb{E}(A_t) = \alpha + \beta$ for all t, so

$$\mathbb{E}(V_t) \quad = \quad \omega(1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots) \quad = \quad \frac{\omega}{1 - \alpha - \beta}$$

(using the formula for the sum of a **Geometric series**), so the model only makes sense when $\alpha + \beta < 1$, which is known as the **stationarity condition** which ensures that V_t has the same distribution for all t (see notes on my webpage for more details on the model, many students have looked at this model in their projects and we often use the student t-distribution for the ε_t's).