FM04 Module Class Test 2026

1. **Hitting time for Brownian motion with drift**

   Let $X_t = \gamma t + W_t$ for $\gamma \in \mathbb{R}$ where W is standard Brownian motion, and let $H_b = \inf\{t : X_t = b\}$ for $b > 0$. Using that $\mathbb{E}(e^{-qH_b}) = e^{b(\gamma-\sqrt{\gamma^2+2q})}$ for $q > 0$, which of the following statements is correct:

   - $\lim_{q\to 0} \mathbb{E}(e^{-qH_b}) = 1$ for all $\gamma \in \mathbb{R}$ by the bounded convergence theorem
   - H_b is finite a.s. for all $\gamma \in \mathbb{R}$
   - $\lim_{q\to 0} \mathbb{E}(e^{-qH_b}) = \mathbb{E}(1_{H_b<\infty}) = e^{b(\gamma-\sqrt{\gamma^2})}$ for all $\gamma \in \mathbb{R}$ (see q4, Hwk4) ✓
   - If X hits $a > 0$ in finite time then the process $(H_b)_{b\geq 0}$ has a jump to infinity at some $b \leq a$

2. **CGMY Moments**

   Let X be a one-sided CGMY process with $\log \mathbb{E}(e^{pX_t}) = t\int_0^\infty (e^{px} - 1 - px1_{x\leq 1})\nu(x)dx$ for $p \in \mathbb{R}$, where $\nu(x) = \frac{Ce^{-Mx}}{x^{1+Y}}$ with $C, M > 0$ and $Y \in (1, 2)$. Then

   - $\mathbb{E}(e^{MX_t}) = \infty$ (see q1, Hwk4)
   - $\mathbb{E}(e^{MX_t})$ is finite ✓
   - $\lim_{p\nearrow M} \mathbb{E}(e^{pX_t}) = \infty$
   - $\mathbb{E}(X_t) < 0$

3. **Jump time**

   Let X be a Lévy process with Lévy density $\nu(x)$, and let $\nu(A) = \int_A \nu(x)dx$. Then the expected length of time for the first positive jump of size ≥ 1 is

   - $\nu([1, \infty))$
   - $\frac{1}{\nu([1,\infty))}$ (see q2, Hwk4) ✓
   - $\nu([1, \infty))t$
   - $\frac{1}{\nu((-\infty,-1]\cup[1,\infty))t}$

4. **Hitting time process for arithmetic Brownian motion**

   Let W be a standard Brownian motion and $X_t = W_t + \gamma t$ for $\gamma \geq 0$. Then $\mathbb{E}(e^{-qH_b}) = e^{b(\gamma-\sqrt{\gamma^2+2q})} = e^{b\int_0^\infty (e^{-qx}-1)\frac{e^{-Mx}}{\sqrt{2\pi x^3}}dx}$ for $b, q \geq 0$, where $H_b = \inf\{t : X_t = b\}$ and $M = \frac{1}{2}\gamma^2$. The process $(H_b)_{b\geq 0}$

   - has positive-only jumps but is not an increasing process
   - is a one-sided CGMY process with $Y = \frac{1}{2}$ $(\nu(x) = \frac{Ce^{-Mx}}{x^{1+Y}}1_{x>0})$ ✓
   - is the quadratic variation of X
   - is a CGMY process with $Y = \frac{3}{2}$

5. **fBM fourth moment**

   Let B^H be a fractional Brownian motion with Hurst exponent $H \in (0,1)$. Then

   - $\mathbb{E}((B_t^H)^4) = 3t^{2H}$
   - $\mathbb{E}((B_t^H)^4) = 3t^{4H}$ $B_t^H \sim N(0, t^{2H}) \sim t^H Z$ where $\mathbb{E}(Z^4) = 3$, so $\mathbb{E}((B_t^H)^4) = \mathbb{E}((t^H Z)^4))$ where $Z \sim N(0,1)$ ✓
   - $\mathbb{E}((B_t^H)^4) = 2t^H$
   - $\mathbb{E}((B_t^H)^4) = t^{4H}$

6. **Expected maximum at exponential time**

   For the symmetric α-stable process X with $\alpha \in (0,2)$, we have that $\mathbb{E}(e^{-\beta \bar{X}_{e_q}}) = e^{-\frac{1}{\pi}\int_0^\infty \frac{\beta}{u^2+\beta^2}\log(1+\frac{u^\alpha}{q})\,du}$ for $\beta \geq 0$, where $\bar{X}_t = \max_{0\leq s\leq t} X_s$ and $e_q \sim \text{Exp}(q)$ is independent of X. Which of the following statements is correct:

   - $\mathbb{E}(\bar{X}_{e_q}) = \frac{1}{\pi}\int_0^\infty \frac{1}{u^2}\log(1+\frac{u^\alpha}{q})\,du$ which is finite for all $\alpha \in (0,2)$
   - $\mathbb{E}(\bar{X}_{e_q}) = \infty$
   - $\mathbb{E}(\bar{X}_{e_q}) = 0$
   - $\mathbb{E}(\bar{X}_{e_q}) = \frac{1}{\pi}\int_0^\infty \frac{1}{u^2}\log(1+\frac{u^\alpha}{q})\,du$, which is $+\infty$ for $\alpha \in (0,1]$ (Compute $-\frac{d}{d\beta}\mathbb{E}(e^{-\beta \bar{X}_{e_q}})|_{\beta=0}$, and use that $\log(1+x) = x + O(x^2)$) ✓

7. **Hitting time for Black-Scholes model**

   Let $S_t = S_0 e^{\mu t + \sigma W_t}$ with $\mu \in \mathbb{R}, \sigma, S_0 > 0$, where W is standard Brownian motion, and let $H_b = \inf\{t : S_t = b\}$ and $\bar{S}_t = \max_{0\leq u\leq t} S_u$.

   - The flat periods of $(\bar{S}_t)_{t\geq 0}$ correspond to the jumps of $(H_b)_{b\geq 0}$ (rotate the graph of $\bar{S}_t$)) ✓
   - The flat periods of $(\log S_t)_{t\geq 0}$ correspond to the jumps of $(H_b)_{b\geq 0}$
   - The flat periods of $(H_b)_{b\geq 0}$ correspond to the jumps of $(\bar{S}_t)_{t\geq 0}$
   - $(S_t)_{t\geq 0}$ is a Lévy process

8. **Approximating a Lévy process**

   We can approximate a Lévy process with positive-only jumps with a compound Poisson process with

   - rate $\lambda_\epsilon = \int_\epsilon^\infty \nu(x)dx$ and jump size density $\mu_\epsilon(x) = \frac{\nu(x)1_{x\geq\epsilon}}{\lambda_\epsilon}$ (see Lecture 2 and Hwk 4) ✓
   - rate $\lambda_\epsilon = \int_\epsilon^\infty \nu(x)dx$ and jump size density $\mu_\epsilon(x) = \frac{\nu(x)}{\lambda_\epsilon}$
   - rate $\lambda_\epsilon = \int_0^\infty \nu(x)dx$ and jump size density $\mu_\epsilon(x) = \frac{\nu(x)}{\lambda_\epsilon}$
   - rate $\lambda_\epsilon = \int_\epsilon^\infty \nu(x)dx$ and jump size density $\mu_\epsilon(x) = \nu(x)1_{x\geq\epsilon}$

9. **Spectrally negative Lévy process**

   Consider a Lévy process X with negative-only jumps and a non-zero Brownian component, and let $e_q \sim \mathrm{Exp}(q)$ be independent of X. Which of the following statements is true:

   - $\bar{X}_{e_q}$ is Exponentially distributed (see q5, Hwk4) ✓
   - $\underline{X}_{e_q}$ is Exponentially distributed but $\bar{X}_{e_q}$ is not
   - $\mathbb{E}(e^{iuX_{e_q}}) = \mathbb{E}(e^{iu\bar{X}_{e_q}})\,\mathbb{E}(e^{-iu\underline{X}_{e_q}})$
   - X will hit $-\infty$ in finite time a.s.