

## Alternate estimators for $H$

### The $m(q, \Delta)$ estimator from [GJR18]

For the first task, let  $SS_n^{(q)} := \frac{1}{n} \sum_{i=1}^n |B_{i\Delta}^H - B_{(i-1)\Delta}^H|^q$ . Then

$$\mathbb{E}(SS_n^{(q)}) = \frac{1}{n} \mathbb{E}\left(\sum_{i=1}^n |B_{i\Delta}^H - B_{(i-1)\Delta}^H|^q\right) = \Delta^{qH} \frac{1}{n} \sum_{i=1}^n \mathbb{E}(|B_i^H - B_{i-1}^H|^q) = \mathbb{E}(|Z|^q) \Delta^{qH} = K_q \Delta^{qH} \quad (1)$$

where  $K_q = \mathbb{E}(|Z|^q) = \frac{2^{q/2}}{\sqrt{\pi}} \Gamma(\frac{q+1}{2})$ , with  $q > -1$  and  $Z \sim N(0, 1)$ . From this we can then derive the estimator for the first task. Alternatively, if let  $X = \sigma B_t^H$  and now define  $SS_n^{(q)} = \frac{1}{n} \sum_{i=1}^n |X_{i\Delta}^H - X_{(i-1)\Delta}^H|^q$ , and assume  $H$  and  $\sigma$  are unknown, then (1) changes to

$$\mathbb{E}(SS_n^{(q)}) = \sigma^q K_q \Delta^{qH}$$

which leads to the estimates  $(\hat{H}_n, \hat{\sigma}_n)$  for  $(H, \sigma)$  defined by

$$SS_n^{(q)} = \hat{\sigma}_n^q K_q \Delta^{q\hat{H}_n}$$

if we have computed  $SS_n^{(q)}$  for at least two  $\Delta$ -values. Taking logs we see that

$$\log SS_n^{(q)} = q \log \hat{\sigma}_n + \log K_q + q \hat{H}_n \log \Delta$$

so we can perform **linear regression** on  $\log SS_n^{(q)}$  vs  $\log \Delta$  for a range of  $\Delta$ -values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal  $q \hat{H}_n$  ( $q$  is chosen by you, e.g.  $q = 1, 2, 2.5, 3$  etc), and the **intercept** at  $\log \Delta = 0$  is  $q \log \hat{\sigma}_n + \log K_q$ , from which we can compute  $\hat{\sigma}_n$  since  $K_q$  has an explicit formula above. This is the  $m(q, \Delta)$  estimator discussed in [GJR18]. One can then also compute the  **$R^2$ -statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of  $\hat{H}_n$  and  $\hat{\sigma}_n$ .

### The Han-Schied [HS21] estimator

Let  $X_t = \sigma B_t^H$  and let

$$\theta_{m,k} = 2^{\frac{m}{2}} (2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}) = -2^{\frac{m}{2}} (X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{2k}{2^{m+1}}})$$

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for  $R(s, t) = \mathbb{E}(B_s^H B_t^H)$ , one can check that

$$\mathbb{E}(\theta_{m,k}^2) = \sigma^2 2^{m-2H(1+m)} (4 - 4^H). \quad (2)$$

Then setting  $s_n^2 = \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} \theta_{m,k}^2$ , we see that  $\mathbb{E}(s_n^2) = \sum_{m=0}^{n-1} 2^m \mathbb{E}(\theta_{m,k}^2)$  (since (2) does not depend on  $k$ ) which simplifies to

$$\mathbb{E}(s_n^2) = \sigma^2 (4^{n(1-H)} - 1) \sim \sigma^2 4^{n(1-H)} = \sigma^2 2^{2n(1-H)} \quad (3)$$

as  $n \rightarrow \infty$ , which suggests an estimator  $\hat{H}_n$  defined by  $s_n = \hat{\sigma}_n 2^{n(1-\hat{H}_n)}$  which (assuming  $\hat{\sigma}_n = O(1)$  as  $n \rightarrow \infty$ ) we can re-arrange as

$$\hat{H}_n = 1 - \frac{1}{n} \log_2 \left( \frac{s_n}{\hat{\sigma}_n} \right) = 1 - \frac{1}{n} \log_2 s_n + O\left(\frac{1}{n}\right)$$

where  $\log_2$  denotes the base-2 logarithm, so (ignoring the  $O(\frac{1}{n})$  remainder term), we recover the Han-Schied[HS21] estimator  $\hat{H}_n = 1 - \frac{1}{n} \log_2 s_n$ . Then

$$\begin{aligned} \mathbb{E}(\hat{H}_n) &= 1 - \frac{1}{n} \mathbb{E}(\log_2(s_n)) = 1 - \frac{1}{2n} \mathbb{E}(\log_2(s_n^2)) \geq 1 - \frac{1}{2n} \log_2 \mathbb{E}(s_n^2) = 1 - \frac{1}{2n} \log_2 (\sigma^2 (4^{n(1-H)} - 1)) \\ &\geq 1 - \frac{1}{2n} \log_2 (\sigma^2 (4^{n(1-H)})) = 1 - \frac{1}{2n} \log_2 (\sigma^2) - \frac{1}{2n} \log_2 (4^{n(1-H)}) \\ &= H - \frac{1}{2n} \log_2 (\sigma^2) \end{aligned}$$

and the final line is  $> H$  if  $\sigma < 1$ , so  $\mathbb{E}(\hat{H}_n) > H$  if  $\sigma < 1$ . See also discussion on optimal scaling factors in [HS21].

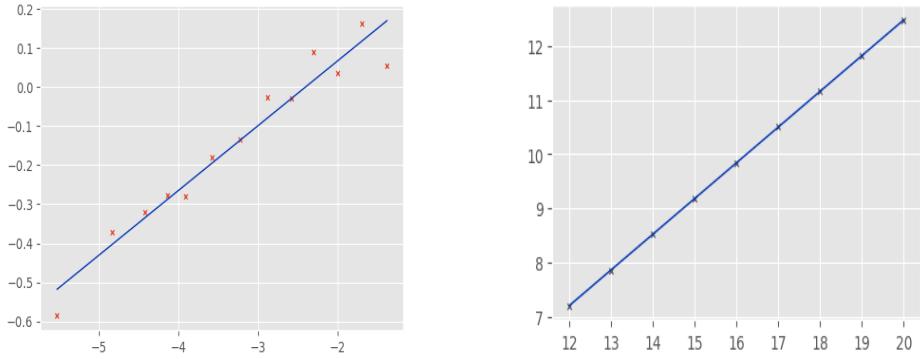


Figure 1: On the left, we see estimates of  $H$  for the SPX using the  $m(q, \Delta)$  method for the SPX from 3rdJan22-15thJul24 for  $q = 2$  for which  $\hat{H} = 0.0830$  and  $\hat{\sigma} = 1.221$  (see similar plots in [GJR18]). On the right we see the linear regression in (4) for the Han-Schied method ( $\log s_n$  vs  $n$ ) for a true fBM path with  $2^{20}$  time points, for which  $\hat{H} = 0.0508$ , and  $\hat{\sigma} = 1.010$ .

For more a general process  $X$ , (under certain conditions) [HS21] show that  $\hat{H}_n \rightarrow R$  as  $n \rightarrow \infty$ , where  $R = 1/q$ , where  $q$  is the critical  $p$ -value at which  $\lim_{n \rightarrow \infty} \sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}^H|^p$  switches from being zero (for  $p > q$ ) to  $+\infty$  (for  $p < q$ ) (see Eq 2.2 in [HS21]).

Or we can jointly estimate  $H$  and  $\sigma$  by performing linear regression since

$$\log s_n = \log \hat{\sigma}_n + n(1 - \hat{H}_n) \log 2 \quad (4)$$

but we now have to compute  $\log s_n$  for a range of different  $n$ -values to get a line of best fit, for which the slope is  $(1 - \hat{H}_n) \log 2$  and the intercept is  $\log \hat{\sigma}_n$ .

You can then draw histograms of  $\hat{H}_n$  if you simulate  $M$  fBM paths and compute the sample variance for  $\hat{H}_n$  (or a confidence interval), or compute  $\hat{H}_n$  for real data, e.g. using the SPX data file or data from yahoo finance.

## Discussion on the rough Heston model, and the Riemann-Liouville process

The driftless rough Heston model satisfies

$$V_t = V_0 + \nu \int_0^t (t-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u.$$

Then  $\mathbb{E}(V_t) = V_0$ , and  $V$  has covariance function:

$$\begin{aligned} \mathbb{E}((V_s - V_0)(V_t - V_0)) &= \nu^2 \mathbb{E}\left(\int_0^s (s-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \cdot \int_0^t (t-r)^{H-\frac{1}{2}} \sqrt{V_r} dW_r\right) \\ &= \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \mathbb{E}(V_u) du \\ &= \nu^2 V_0 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du = V_0^2 \nu^2 \bar{R}(s, t) \end{aligned}$$

for  $0 \leq s \leq t$ , where  $\bar{R}(s, t)$  is the covariance function for the Riemann-Liouville (RL) process  $Z_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u$  used for the rough Bergomi model, (note  $Z$  is a Gaussian process but  $V$  is not), but the explicit formula for  $\bar{R}(s, t)$  is more complicated than the  $R(s, t)$  formula for fBM.

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H = c_H \left( \int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}) dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \right) = c_H (A_t + Z_t)$$

for  $t \in \mathbb{R}$ , in terms of the RL process  $Z$  above (and note that  $A_t$  is known at time zero for all  $t \geq 0$ ), and  $c_H = (\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)})^{\frac{1}{2}}$ . Note also that  $A_t$  and  $Z_t$  are independent.

## References

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