King's College London

UNIVERSITY OF LONDON

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Candidate No: .... **Desk No:** ....

MSc EXAMINATION

7CCMFM02 RISK NEUTRAL VALUATION MOCK QUESTIONS

JANUARY 2026

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS. FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO ALL FOUR QUESTIONS.

WITHIN A GIVEN QUESTION, THE RELATIVE WEIGHTS OF THE DIFFERENT PARTS ARE INDICATED BY A PERCENTAGE FIGURE.

NO CALCULATORS ARE PERMITTED.

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TURN OVER WHEN INSTRUCTED

1. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Let

$$X_t \;=\; \int_0^t \frac{T-t}{T-u} dW_u \,.$$

Write down the distribution of X_t for $t \in [0, T)$. **[30%]**

Solution. For any process of the form $X_t = \int_0^t \phi(s) dW_s$ with $\int_0^t \phi(s)^2 ds < \infty$, we have that $X_t \sim N(0, \int_0^t \phi(s)^2 ds)$ (see Ito's Lemma chapter), so

$$X_t \;=\; N(0, \int_0^t (\frac{T-t}{T-u})^2 du)$$

b. Let $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$, and let

$$S(x) \;=\; \int_c^x e^{-2\int_c^\xi \frac{\mu(\zeta)}{\sigma(\zeta)^2} d\zeta} d\xi \,.$$

Using Ito's lemma, derive the SDE satisfied by $Y_t = S(X_t)$. Hint: use the fundamental theorem of calculus. **[40%]**

Solution. From the fundamental theorem of calculus, we see that

$$S'(x) \;=\; e^{-2\int_c^x \frac{\mu(\zeta)}{\sigma(\zeta)^2} d\zeta} \quad , \quad S''(x) \;=\; -2\frac{\mu(x)}{\sigma(x)^2} S'(x) \,.$$

Then from Ito's lemma

$$\begin{aligned}
dY_t &= S'(X_t)(\mu(X_t)dt + \sigma(X_t)dW_t) \;+\; \frac{1}{2} S''(X_t)\sigma(X_t)^2 dt \\
&= S'(X_t)\sigma(X_t)dW_t \\
&= S'(S^{-1}(Y_t))\sigma(S^{-1}(Y_t))dW_t \;=\; \tilde{\sigma}(Y_t)dW_t
\end{aligned}$$

where $\tilde{\sigma}(y) = S'(S^{-1}(y))\sigma(S^{-1}(y))$ (note S is invertible since $S'(x) > 0$).

c. State whether the following statements are true or false:

- Brownian motion is twice differentiable in x and once in t
- Brownian motion can be strictly increasing on a non-zero time interval $[0, \delta]$
- Brownian motion is differentiable and continuous in t
- Brownian motion has stationary increments
- The stock price process for the Black-Scholes model can hit zero in finite time

[50%]

Solution.

FFFTF (4th option true from 2nd and 3rd properties of Brownian motion: $W_t - W_s \sim N(0, t-s)$ for $s \leq t$ and $W_t - W_s$ independent of $(W_u)_{0 \leq u \leq s}$. Fifth option false because $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ where W is a Brownian motion under $\mathbb{P}$, but S_t can only be zero if $W_t = -\infty$, but W_t is continuous, and (from a standard result in analysis) any continuous function is bounded on a finite closed interval.

2. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Let $X_t = \sigma W_t$ and $\bar{X}_t = \max_{0\leq s\leq t} X_s$ (you may use that $\bar{X}_t$ is a non-decreasing process and hence has zero quadratic variation unlike Brownian motion). By applying the Ito product rule to $\bar{X}_t(\bar{X}_t - X_t)$, compute $\mathbb{E}(\bar{X}_t(\bar{X}_t - X_t))$

Solution.

$$d(\bar{X}_t(\bar{X}_t - X_t)) \;=\; d\bar{X}_t(\bar{X}_t - X_t) + \bar{X}_t(d\bar{X}_t - dX_t)\,.$$

But if $d\bar{X}_t > 0$ [1], then $\bar{X}_t - X_t = 0$, so the $d\bar{X}_t(\bar{X}_t - X_t)$ term on the right is zero. Then integrating the remaining terms on both sides and using that the integral wrt dX_t has zero expectation, we see that

$$\begin{aligned}
\mathbb{E}(\bar{X}_t(\bar{X}_t - X_t)) \;=\; \mathbb{E}\Big(\int_0^t \bar{X}_s d\bar{X}_s\Big) \;&=\; \frac{1}{2}\mathbb{E}(\bar{X}_t^2) \;=\; \frac{1}{2}\mathbb{E}((\sigma\bar{W}_t)^2)\\
&=\; \frac{1}{2}\sigma^2\int_0^\infty b^2\frac{2}{\sqrt{2\pi t}}e^{-b^2/2t}db\\
&=\; \frac{1}{2}\sigma^2\int_{-\infty}^\infty b^2\frac{1}{\sqrt{2\pi t}}e^{-b^2/2t}db\\
&=\; \frac{1}{2}\sigma^2\mathbb{E}(W_t^2) \;=\; \frac{1}{2}\sigma^2 t
\end{aligned}$$

where we have used that the density of $\bar{W}_t$ is $\frac{2}{\sqrt{2\pi t}}e^{-\frac{b^2}{2t}}$ for $b \geq 0$ i.e. a one-sided Normal density (since $\mathbb{P}(\bar{W}_t > b) = 2\mathbb{P}(W_t > b)$).

b. Let $X_t = \gamma t + W_t$ with $\gamma \geq 0$. Then the first hitting time density for X to $b \geq 0$ is

$$f_{H_b}(t) \;=\; e^{\gamma b - \frac{1}{2}\gamma^2 t} f_{\tau_b}(t)$$

where $f_{\tau_b}(t) = \frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}$ is the first hitting time density for Brownian motion for a barrier $b > 0$. Write an expression for the cdf and density of $\bar{X}_t$.

Solution. Recall that $\{\bar{X}_t \geq b\} = \{H_b \leq t\}$, so

$$\mathbb{P}(\bar{X}_t \geq b) \;=\; \mathbb{P}(H_b \leq t) \;=\; \int_0^t f_{H_b}(s)ds$$

[1] more rigorously we mean that t is a *growth point* of M_t, i.e. where $M_{t+\delta} - M_t > 0$ for all $\delta > 0$

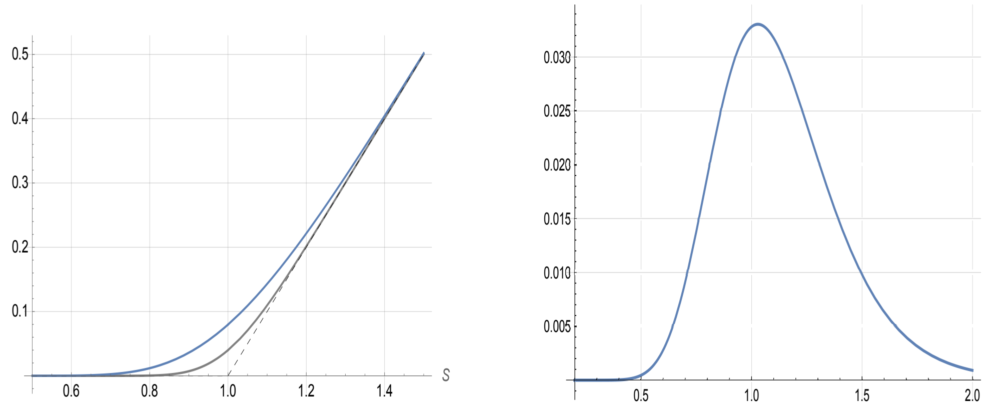

Figure 1: On the left we see the price of a call option as a function of S_0 for two different maturities. On the right we have plotted the value of calendar spread as a function of S_0 at $t = 0$ with $K = 1$, $\sigma = .2$, $T = 1$, $T_2 = 2$ and $r = 0$.

c. Plot the value of a standard calendar spread trade (i.e. buy a call option with maturity T_2 and sell a call with maturity $T < T_2$ with the same strike K) under the Black-Scholes model as a function of S_0 at $t = 0$ when $r = 0$. **[20%]**

Solution. We know the price of both calls tends to 0 as $S_0 \to 0$, and asymptotically behaves like $S_0 - K$ as $S_0 \to \infty$ (see first plot above), so the value of the overall strategy vanishes as $S_0 \to 0$ and as $S_0 \to \infty$, because the value is just the difference of the two calls.

The Delta of a call option tends to 0 as $S_0 \to 0$ and 1 as $S_0 \to \infty$ when $r = 0$, so the Delta of the overall strategy tends to zero as S_0 tends to 0 or ∞. Moreover, from the lecture notes we know that $\frac{\partial C}{\partial \tau}$ of a call is strictly positive, so the overall value of the strategy is positive (see 2nd plot above).