The real P&L in Black-Scholes and Dupire Delta hedging

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Abstract: We derive the real Profit&Loss(P&L) that accrues in what option traders do every day: continuously Delta (Δ)-hedge a call, by substituting the option's *running* implied volatility (generally stochastic) into the Black-Scholes(BS) Δ-formula. The result provides formal justification for a heuristic rule-of-thumb that a trader's P&L on continuously Δ-hedged positions is his vega times times the daily change in implied vol. For *Digital* options, we show that if we Δ-hedge them at the running implied vol of the associated vanilla, then we have hidden exposure to the volatility skew dynamics. We go on to derive the P&L incurred when we periodically re-calibrate a local volatility model (LVM) to the smile, and Δ-hedge under the premise that the most recently fitted LVM *is* the correct model. As asides, we demonstrate how a Investment Bank can trade implied volatility *directly* (in the home currency of the bank), using At-The-Money(ATM) FX forward starting options, and derive a new forward equation for Vanillas which contains localized vol-of-vol and vol skew terms, which extends the results of Dupire

*This article was completed during an internship as a Quantitative Analyst in FX Derivatives at HSBC Bank plc, 8 Canada Square, London E14 5HQ

1. Summary of relevant literature

Schonbucher[22] considered the problem of exogenously specifying Markovian dynamics for the BS implied volatility of *one* call option. Under this framework, the call option value is just a function of three variables (the Stock price S_t, calendar time t, and the BS implied volatility $\hat{\sigma}_t$), so we can easily apply Itō's lemma to obtain an SDE for the call option value. Using Itō, he derived a no-arbitrage condition for the drift of $\hat{\sigma}_t$, which ensures that the discounted call value evolves as a martingale. The condition that the call have zero drift also gives rise to an interesting pde linking the call option's vanna ($\frac{\partial \text{Vega}}{\partial S_t}$) and volga ($\frac{\partial \text{Vega}}{\partial \hat{\sigma}_t}$) to the risk neutral drift of $\hat{\sigma}_t$, and the correlation between S_t and $\hat{\sigma}_t$.

Brace,Goldys,vanderHoek&Womersley[4] extend this idea to exogenously imposing dynamics for the evolution of the *entire smile.* The drift of $\hat{\sigma}_t$, and the stock's *instantaneous* volatility process σ_t, and thus the S_t process itself, are *endogenously determined* by imposing that the discounted call price evolve as a martingale, and the feedback condition that the implied volatility squared times the time to expiration (the *dimensionless implied variance*) tend to zero as we approach maturity. σ_t is equal to the just maturing ATM implied vol, as we might expect. This forces the call to converge to its intrinsic value at maturity. If the endogenously determined S_t process is a bona fide martingale, as opposed to just a local martingale, then arbitrage is precluded. Unfortunately, the largest class of vol-of-vol (the lognormal volatility of $\hat{\sigma}_t$) processes for which this is the case is not known. They go on to show that their general stochastic implied volatility framework encompasses all the standard existing stochastic volatility models (for which the discounted stock price is *a-priori* a true martingale) as special cases. They subsequently demonstrate that while the initial smile determines the marginal stock price distribution from Dupire's results, the vol-of-vol process determines the *joint* stock price distribution, to which the price of forward starting options are very sensitive. In Appendix 2, we exploit the fact that the $\frac{\partial \text{vega}}{\partial \text{vol}}$ of an ATM vanilla is negligible, to show how to more or less replicate a linear contract which pays us ATM EUR/USD implied vol at some time in the future (in EUR), by buying an ATM forward starting option.

In section 1 of this paper, we use Schonbucher's methodology to derive the P&L that a trader makes when he buys a call, and continuously Δ-hedges it with the future, by plugging the running (stochastic) $\hat{\sigma}_t$ into the BS Δ-formula. We go on to show that if we continuously Δ-hedge a Digital at the running implied vol of the associated Vanilla, we incure a hidden Delta *and* volatility skew risk.

Carr&Madan[7] derived the P&L that accrues when a trader sells an European style contingent claim for a *constant* BS implied volatility of $\hat{\sigma}$, and dynamically BS Δ-hedges it with the future, by always plugging the *constant* value $\hat{\sigma}$ into the

BS Δ-formula. Their expression for the BS Δ-hedging error (which they obtained using only Itō's lemma and the BS pde) is as follows:

$$I_T = \int_0^T \frac{1}{2} F_t^2 \, e^{r(T-t)} \, \frac{\partial^2 C}{\partial F^2}(F_t, t; \hat{\sigma}) \, [\, \hat{\sigma}^2 \, - \, \sigma_t^2 \,] \, dt \tag{1.1}$$

where $\frac{\partial^2 C}{\partial F^2}(F_t, t; \hat{\sigma})$ is the gamma of the contingent claim (wrt F_t, not S_t) under the BS model, with implied volatility equal to $\hat{\sigma}$.

The purpose of section 2 is to extend Carr&Madan's result to the case when Δ-hedging under the premise that some LVM holds true, where the local volatility surface is backed out from the market prices of vanillas at time 0 using the Dupire forward pde, but where we also re-compute the market implied local volatility surface periodically (i.e. at known discrete times), and always Δ-hedge under the most recently calibrated LVM.

In Appendix 1, we derive a new forward equation for a European call, by deriving a forward pde for the infinitesimal calendar spread of calls, which holds under any stochastic volatility framework without jumps, and extends the beautiful result of Dupire[13]. This, and similar equations become useful in backing out information from the smile about the vol-of-vol, skew, and/or mean reversion term structure when we make simplifying assumptions on the volatility process

2. Delta-hedging at the call's running implied volatility

Let $F_t = S_t \, e^{(r-q)(T'-t)}$ denote the future on S_t (for delivery at T'), and $\hat{\sigma}_t$ be the Black-Scholes implied volatility of a call expiring at T'. We assume that investors can trade in continuous time, that the interest rate r and dividend yield q are constant, and that F_t is a continuous martingale. Let the instantaneous volatility σ_t of F_t follow some arbitrary, unspecified stochastic process, and let F be governed by the following SDE (under the *physical* measure P):

$$\frac{dF}{F} = \mu \, dt \, + \, \sigma_t \, dW \tag{2.2}$$

where dW is standard Brownian motion, and we assume that $\hat{\sigma}_t$ also follows some unspecified, but sufficiently well behaved process, with (possibly stochastic) drift α under $\mathbb{P}$, which satisfies the boundary and feedback conditions discussed in

Brace,Goldys,vanderHoek&Womersley[4]. Now suppose that under some risk-neutral measure $\mathbb{P}^*$, $\hat{\sigma}_t$ has drift α^*. It is well known that F has to evolve as a martingale under a risk-neutral measure. The forward call value $\bar{C} = e^{r(T'-t)} C$ is just a function of t, F_t and $\hat{\sigma}_t$, so we can apply Itō's lemma to obtain:

$$
\begin{aligned}
d\bar{C} &= \bar{C}_t \, dt + \bar{C}_F \, dF + \frac{1}{2} \bar{C}_{FF} \, dF^2 \\
&+ \bar{C}_{\hat{\sigma}} \, d\hat{\sigma} + \frac{1}{2} \bar{C}_{\hat{\sigma}\hat{\sigma}} \, d\hat{\sigma}^2 + \bar{C}_{F\hat{\sigma}} \, dF \, d\hat{\sigma} \\
&= [\, \bar{C}_t + \frac{1}{2} \bar{C}_{FF} \, F^2 \, \sigma_t^2 + \bar{C}_{\hat{\sigma}} \, \alpha^* \,] \, dt + [\, \frac{1}{2} \bar{C}_{\hat{\sigma}\hat{\sigma}} \, d\hat{\sigma}^2 + \bar{C}_{F\hat{\sigma}} \, F \, \sigma_t \, dW \, d\hat{\sigma} \,] \\
&+ \bar{C}_F \, F \, \sigma_t \, dW + \bar{C}_{\hat{\sigma}} \, [\, d\hat{\sigma} - \mathbb{E}^{\mathbb{P}}(d\hat{\sigma}) \,] \\
&+ \bar{C}_F \, F \, \mu \, dt + \bar{C}_{\hat{\sigma}} \, [\, \mathbb{E}^{\mathbb{P}}(d\hat{\sigma}) - \alpha^* \,]
\end{aligned}
\tag{2.3}
$$

But $\bar{C} = \mathbb{E}^{\mathbb{P}^*}((S_{T'} - K)^+ \,|\, \mathcal{F}_t)$ is a martingale under $\mathbb{P}^*$, and the 1st line in (2.3) is exactly the drift of $\bar{C}$ under $\mathbb{P}^*$, so the 1st line vanishes. The terms in the 2nd line are entirely stochastic (i.e. have no drift), and the 3rd line gives the drift of $\bar{C}$ under $\mathbb{P}$, so we can integrate (2.3) over $[0, T]$ $(T \leq T')$ to give:

$$
\bar{C}_T - \bar{C}_0 = C_T \, e^{r(T'-T)} - C_0 \, e^{rT'} = \int_0^T \bar{C}_F \, dF + \int_0^T \bar{C}_{\hat{\sigma}} \, (d\hat{\sigma} - \alpha^* \, dt)
\tag{2.4}
$$

Dividing through by $e^{r(T'-T)}$, and re-arranging, we get:

$$
\begin{aligned}
& C_T - C_0 \, e^{rT} - e^{-r(T'-T)} \int_0^T \bar{C}_F \, dF \\
&= C_T - C_0 \, e^{rT} - \int_0^T e^{r(T-t)} \, C_F \, dF \\
&= \int_0^T e^{r(T-t)} \, C_{\hat{\sigma}} \, [\, d\hat{\sigma} - \alpha^* \, dt \,]
\end{aligned}
\tag{2.5}
$$

The lhs is the P&L at time T on the strategy of borrowing C_0 at time 0 to fund a long position in the call (where the loan is paid back at T), and a dynamic

position in $-C_F$ futures. Here we have used the fact that the future marks-to-market continuously, and any profit(loss) generated by the futures trading is either lent(borrowed) in the money market, with loans being paid back at T. Here we have the advantage over Carr&Madan[7]'s *fixed* implied vol Δ-hedge, because we can explicitly compute the Δ-hedging P&L, even when we sell the option prior to expiration, because we are dealing with a *market* model of stochastic volatility.

Now $C_F = \frac{\partial C}{\partial F}(F_t, t; \hat{\sigma}_t)$ is exactly the value we obtain when we plug the running implied volatility $\hat{\sigma}_t$ into the forward BS Δ-formula. Thus the rhs *is* the P&L when we Δ-hedge at the running implied volatility. Note that even if we sell the option at a higher implied volatility than where we bought at, and Δ-hedge, we are *not* guaranteed a profit (as some traders erroneously believe), because the rhs P&L is not merely $\int_0^T d\hat{\sigma}_t$, and we could have high vega on a day when implied went down significantly. Vega will be higher when then is *longer* to maturity, and when the option is close to being ATM.

Also, note that $\bar{C}_t + \frac{1}{2} F^2 \hat{\sigma}^2 \bar{C}_{FF} = 0$, because this is just the BS pde (expressed in forward terms), and $\bar{C}$ is just our familiar forward BS valuation function. Using this, and the fact that the drift term of $\bar{C}$ under $\mathbb{P}^*$ (i.e. the top line in (2.3)) has to be zero, we obtain

$$[\frac{1}{2} \bar{C}_{FF} F^2 (\sigma_t^2 - \hat{\sigma}_t^2) + \bar{C}_{\hat{\sigma}} \alpha^*] dt + \frac{1}{2} \bar{C}_{\hat{\sigma}\hat{\sigma}} d\hat{\sigma}^2 + \bar{C}_{F\hat{\sigma}} F \sigma_t dW d\hat{\sigma} = 0$$

$$(2.6)$$

dividing by dt, and re-writing this in option trader slang, we obtain:

$$\frac{1}{2} \text{Gamma} \cdot \text{F}^2(\sigma_t^2 - \text{vol}^2) + \text{Vega} \cdot \text{voldrift}$$

$$+ \quad \frac{1}{2} \text{Volga} \cdot \text{volofvol}^2 + \text{Vanna} \cdot \text{F} \sigma_t \cdot \text{corr}(\text{Stock}, \text{vol}) = 0 \qquad (2.7)$$

where vol $= \hat{\sigma}$, Gamma $= \bar{C}_{FF}$, Volga $= \bar{C}_{\hat{\sigma}\hat{\sigma}} = \frac{\partial \text{Vega}}{\partial \text{vol}}$ is the vol convexity, and Vanna $= \bar{C}_{F\hat{\sigma}} = \frac{\partial \text{Vega}}{\partial \text{F}} = \frac{\partial \text{Delta}}{\partial \text{vol}}$, if Delta $= \bar{C}_F$.

3. Delta-hedging Digitals

It is well known that the (forward) value of Digital call option is:

$$\bar{D} = \frac{\partial}{\partial K}\bar{C}(S, K, t, \hat{\sigma}(K)) \quad = \quad \frac{\partial \bar{C}}{\partial K} + \frac{\partial \bar{C}}{\partial \hat{\sigma}} \cdot \frac{\partial \hat{\sigma}}{\partial K}$$

$$= \quad \bar{D}^{\mathrm{BS}}(F_t, \hat{\sigma}, t) + \mathrm{vega} \cdot \mathrm{volskew} \qquad (3.8)$$

where $\bar{D}^{BS}(F_t, \hat{\sigma}, t)$ is the forward value function of a Digital under BS, vega $= \frac{\partial \bar{\mathrm{C}}}{\partial \hat{\sigma}}$ is the Vanilla call's vega, and volskew $= \frac{\partial \hat{\sigma}}{\partial \mathrm{K}}$ is the slope of the implied vol surface at K. $\bar{D}$ is now a function of *four* variables (the fourth of which is volskew). Proceeding as before, we obtain the SDE:

$$\begin{aligned} d\bar{D} \quad &= \quad \bar{D}_F \, dF \; + \; \bar{D}_{\hat{\sigma}} \, (d\hat{\sigma} \; - \; \mathbb{E}^{\mathbb{P}^*}(d\hat{\sigma})) \\ &+ \quad \bar{C}_{\hat{\sigma}} \, (d\mathrm{volskew} \; - \; \mathbb{E}^{\mathbb{P}^*}(d\mathrm{volskew})) \end{aligned} \qquad (3.9)$$

But $\bar{D}_F = \frac{\partial \bar{D}^{BS}}{\partial F} + \frac{\partial^2 \bar{C}}{\partial F \partial \sigma} \cdot \frac{\partial \sigma}{\partial K} + \frac{\partial \bar{C}}{\partial \sigma} \cdot \frac{\partial^2 \sigma}{\partial K \partial F} = \mathrm{DIG}_{\Delta}^{\mathrm{BS}} + \mathrm{Vanna} \cdot \mathrm{volskew} \, + \, \mathrm{vega} \cdot \frac{\partial \mathrm{volskew}}{\partial \mathrm{F}}$, where $\frac{\partial \bar{D}^{BS}}{\partial F} = \mathrm{DIG}_{\Delta}^{\mathrm{BS}}$ is the number we obtain when we plug the running implied vol $\hat{\sigma}$ of the K-strike Vanilla into the BS (forward) Δ-formula for a Digital. So if we Δ-hedge the Digital in this way, in addition to the implied vol risk, we are exposed to the $[\,\mathrm{Vanna} \cdot \mathrm{volskew} \, + \, \mathrm{vega}\,]\,\mathrm{dF}$ term *and* to vega (dvolskew $-$ $\mathbb{E}^{\mathbb{P}^*}$(volskew)). Ideally, we could eliminate this underlying and vol skew exposure by plugging the vol implied by the market price of the digital at any time into the BS Digital Δ-formula, but in practice Digitals are not sufficiently liquid for this to be a feasible game plan.

4. Dynamic Dupire Δ-hedging

Suppose an investor buys a T-maturity European style contingent claim at time 0, with terminal payoff function $f(S_T) = f(F_T)$, (where T is also the delivery date of the future). Imagine he calibrates some local volatility function $\sigma(F, t)$ to the smile using the Dupire forward pde. Let $\bar{V} = \bar{V}(F_t, t \, ; \, \sigma(F, t))$ denote the (T-forward) value function of the contingent claim at time t under this particular LVM. In general, $\bar{V} = e^{r(T-t)} V$ will *not* match the observed (forward) price of the contingent claim at a later time t, because the LVM is likely to be mis-specified, but

it can be calibrated to the initial smile, so the no-arbitrage "pay- today"premium for the claim is $e^{-rT} \bar{V}_0$. Working directly under $\mathbb{P}^*$, and proceeding along similar lines to Carr&Madan[7] for their version of the BS Δ-hedge, we apply Itō's lemma to $\bar{V}$ over $[0, T]$ to obtain:

$$\bar{V}_T - \bar{V}_0 \;=\; \int_0^T \bar{V}_t \, dt + \int_0^T \bar{V}_F \, dF + \int_0^T \frac{1}{2} \bar{V}_{FF} \, F^2 \, \sigma_t^2 \, dt \qquad (4.10)$$

But $\bar{V}$ solves the backward valuation equation $\bar{V}_t = -\frac{1}{2} \sigma^2(F, t) F^2 \bar{V}_{FF}$, so we can re-write (4.10) as:

$$f(S_T) - V_0 \, e^{rT} - \int_0^T \bar{V}_F \, dF \;=\; \int_0^T \frac{1}{2} \bar{V}_{FF} \, F^2 \, [\sigma_t^2 - \sigma^2(F, t)] \, dt \tag{4.11}$$

The lhs is equal to the terminal value of the contingent claim, minus what we have to pay back the bank for the money we borrowed initially to fund it, plus the P&L on a dynamic position in $-V_F$ futures over $[0, T]$ (remember that the future marks-to-market continuously). Thus the hedge misses its target by the amount on the rhs, namely the integral of the difference between the realized and the calibrated local variance , weighted by half the Dupire model *Dollar Gamma*. It seems intuitively reasonable that if the estimated local volatility function behaves similarly to the actual instantaneous volatility process, then this hedge should out perform Carr&Madan's BS Δ-hedge. If we wished to gain further insight into the truth or otherwise of this statement, we could code up a finite difference scheme to compute the Gamma surface $\bar{V}_{FF}(F, t)$ for the BS model, and the LVM, and compare to see which is larger at various points in the domain.

We can easily extend this idea to compute the P&L when we Δ-hedge under a calibrated or estimated LVM, and periodically re-calibrate the model at times T_i, $i = 1, ..., n$ over time (where T_N is maturity), and Δ-hedge under the most recently fitted model. The net P&L that accrues is:

$$
\begin{aligned}
I_T \;=\;& \sum_{i=1}^{N-1} e^{r(T_N - T_i)} \left[V(F_{T_i}, T_i; \sigma_{T_{i-1}}(F, t)) - V(F_{T_i}, T_i; \sigma_{T_i}(F, t)) \right] \\
&+ \sum_{i=0}^{N-1} \int_{T_i}^{T_{i+1}} e^{r(T_N - T_{i+1})} \, e^{r(T_{i+1} - t)} \, \frac{F_t^2}{2} \, \frac{\partial^2 V}{\partial F^2}(F_t, t; \sigma_{T_i}(F, t)) \, [\sigma_{T_i}^2(F_t, t) - \sigma_t^2] \, dt
\end{aligned}
$$

$$(4.12)$$

where $V(F_t, t\,;\sigma_{T_i}(F, t))$ is the value function of the contingent claim at time t, under the Dupire model that was calibrated at time T_i.

5. Directions for future research

We disagree with Hagan et al. and the majority of market practitioners that Dupire Δ-hedging is unequivocally worse than Black-Scholes hedging, because we do *not* have to Δ-hedge according to the *market implied* local vol surface, but

rather we can choose any LVM we want as a proxy for hedging, one of which has to outperform Black-Scholes simply by virtue of its generality. The problem is finding a suitable LVM to hedge under, which could be done by performing some kind of statistical minimal entropy analysis. Some work has been done by the author and others in deriving backward and forward pdes for a trader's P&L distribution, when the model under which he Δ-(and/or vega) hedges is mispecified.

The h-grail in this realm would be to formulate a model where we can exogenously specify the dynamics of the entire smile, or a two dimensional continuum of something. Another direction is examining under what assumptions on the volatility process can we back out information about (or lock-in) future localized and/or non-localized vol-of-vol, vol-skew and/or vol mean reversion , or alternatively from the smile *plus* the market prices of barrier options.

Acknowledgment

Appendix A: The local vol-of-vol surface, and a new forward equation for Vanillas

Dupire[13] demonstrated that the terminal payoff of $\frac{2}{K^2}$ infinitesimal calendar spreads of Vanillas (under a general Markovian stochastic volatility framework without jumps) is $\sigma_T^2\,\delta(F_T - K)$. Let us now take things one step further and apply a product rule version of the generalized Ito lemma to this payoff:

$$
\begin{aligned}
d[\,\sigma_T^2\,\delta(F_T - K)\,] \;=\;& \sigma_T^2\,\delta'(F_T - K)\,dF_T \\
+\;& \frac{1}{2}\,\sigma_T^2\,\delta''(F_T - K)\,dF_T^2 \\
+\;& 2\sigma_T\,d\sigma_T\,\delta(F_T - K) \\
+\;& d\sigma_T^2\,\delta(F_T - K) \\
+\;& 2\sigma_T\,\delta'(F_T - K)\,d\sigma_T\,dF_T
\end{aligned}
\tag{A.1}
$$

Now assume σ_t follows some arbitrary stochastic volatility process $d\sigma_t = \alpha\,dt + u\,dW_2$, with $dW_1\,dW_2 = \rho_t\,dt$. Applying the Expectation operator, using its linearity propery, F_t's martingale property and the sifting property of Diracs (a lá Derman&Kani[10]), we get:

$$
\begin{aligned}
d\,\mathbb{E}(\sigma_T^2\,\delta(F_T - K) \;=\;& \frac{1}{2}\,\frac{\partial^2}{\partial K^2}[\,K^2\,\mathbb{E}(\,\sigma_T^4\,\delta(F_T - K)\,dT\,] \\
+\;& 2\,\mathbb{E}(\,\sigma_T\,\alpha_T\,\delta(F_T - K)\,) \\
+\;& \mathbb{E}(u_T^2\,\delta(F_T - K)\,dT) \\
+\;& 2\,\mathbb{E}(\,\sigma_T^2\,\delta'(F_T - K)\,\rho_T\,F_T\,u_T\,dT\,)
\end{aligned}
\tag{A.2}
$$

from which we arrive at the following pde:

$$
\begin{aligned}
\frac{2}{K^2}\,\frac{\partial^2 U}{\partial T^2} \;=\;& \frac{1}{2}\,\frac{\partial^2}{\partial K^2}[\,K^2\,\mathbb{E}(\,\sigma_T^4\,|\,F_T = K)\,\frac{\partial^2 U}{\partial K^2}\,] \\[2mm]
+\;& \mathbb{E}(u_T^2\,|F_T = K))\,\frac{\partial^2 U}{\partial K^2} \\[2mm]
-\;& 2\,\frac{\partial}{\partial K}[\,K\,\mathbb{E}(\,\sigma_T^2\,\rho_T\,u_T\,|\,F_T = K)\,\frac{\partial^2 U}{\partial K^2}] \\[2mm]
+\;& 2\,\mathbb{E}(\,\sigma_T\,\alpha_T\,|F_T = K)\,\frac{\partial^2 U}{\partial K^2})
\end{aligned}
\tag{A.3}
$$

By inspection, this forward pde contains vol-of-vol, and correlation (vol skew) terms, both localized in the Stock price. It should be possible to extend this result to Knock-Out and Reverse Knock-Out options, where the local volatilities in this case are the risk neutral expectation of future variance, conditoned on the stock ending up at a particular level, *and* steering clear of the barrier

By Dupire's result, we can lock in future instantaneous variance V_T, or a contract paying $S_T^n V_T$ (for integer n), by buying an appropriate amount of infinitesimal calendar spreads at all strikes. We can then apply a similar analysis to that above to these terminal payoffs to derive forward *ODEs* for such contracts. If we then make the assumption that we are under an extended Heston model, where the mean reversion parameter and the vol-of vol are deterministic time dependent functions, and we "hand-pick" the correlation and long term variance parameters as constants (or alternatively fix the vol-of-vol, and let the correlation be time dependent), we can back out the mean reversion and vol-of-vol (or correlation) term structure from the forward ODEs. We can accomplish something similar for a special case of the Dynamic SABR model with $\gamma(t) = 1$ (see Hagan[19] and Forde[16]). The author has also developed a modus operandi for trading skew using convex options.

Appendix B: How to trade ATM volatility directly, using FX forward starting options

Consider an FX ATM forward starting call option (or cliquet) of maturity T', where the strike is determined at a later time $T < T'$ as being equal to $F_T = S_T e^{(r-q)\tau}$, where $\tau = T' - T$. The value of this option at time T is $F_T \left[2\, \Phi(\frac{1}{2}\, \hat{\sigma}_T^{ATM}\, \sqrt{\tau}) - 1 \right]$ (by homogeneity of the BS formula), where $\hat{\sigma}_T^{ATM}$ is the ATM implied volatility (for maturity T') at T. But if the underlying currency (i.e. the currency we have the right to buy for K at T') matches the underlying currency of the bank, they may wish to sell the cliquet at T, and then convert the proceeds of the sale back into their underlying currency. Furthermore, for typical values of maturity and implied vol observed in practice, $\Phi(\frac{1}{2}\, \hat{\sigma}_T^{ATM}\, \sqrt{\tau})$ is to all intents and purposes a linear function of $\hat{\sigma}_T^{ATM}$ (i.e. $\frac{\partial \text{Vega}}{\partial \text{vol}}$ is negative, but negligible), so we effectively have a modus operandi for directly trading where the ATM volatility at a particular point in time in the future. Strictly speaking, we are using S_t as the numéraire here, and the no-arbitrage price for the cliquet at time $t < T$ (in terms of the underlying currency) is:

$$\mathbb{E}^{\mathbb{P}^*}(2\, \Phi(\frac{1}{2}\, \hat{\sigma}_T^{ATM}\, \sqrt{\tau}) - 1 \,|\, \mathcal{F}_t) \tag{B.1}$$

where $\mathbb{P}^*$ is the martingale measure associated with this choice of numéraire, under which S_t has dynamics $dS_t = \sigma_t^2 \, dt \, + \, \sigma_t \, dW_t$.

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