Convex order condition

Let $(X_t)_{t\geq 0}$ denote a martingale. Then for $0 \leq T_1 \leq T_2$:

$$\begin{aligned}
\mathbb{E}((X_{T_2}-K)^+) &= \mathbb{E}(\mathbb{E}((X_{T_2}-K)^+|X_{T_1})) \quad \text{(from the \textbf{tower property}}\\
&\geq \mathbb{E}(\mathbb{E}(X_{T_2}-K|X_{T_1})^+)\\
&\quad \text{(from the \textbf{conditional Jensen inequality} applied to the convex function } f(x)=(x-K)^+)\\
&= \mathbb{E}((X_{T_1}-K)^+)\,.
\end{aligned}$$

Hence we see that call option prices with maturity T_2 are $\geq$ call option prices with maturity T_1. This is known as the **convex ordering** condition, which we can write as $\mu_{T_1} \preceq \mu_{T_2}$, where μ_t denotes the density of X_t.

Bass martingale with random initial starting distribution

Let B^α denote a Brownian motion with $B_0^\alpha \sim \alpha$ (i.e. a random initial starting point with density $\alpha(x)$, and assume the process $B^\alpha_{(.)} - B^\alpha_0$ is independent of B^α_0). Then the density of B^α_t is

$$\int_{-\infty}^{\infty} R_t(y-x)\alpha(x)dx \quad = \quad (R_t * \alpha)(y)\,.$$

Moreover

$$\mathbb{E}(F(B_1^\alpha)|B_t^\alpha = x) \quad = \quad \int_{-\infty}^{\infty} R_{1-t}(y-x)F(y)dy \quad = \quad (R_{1-t}*F)(x)\,.$$

Now let $M_t = \mathbb{E}(F(B_1^\alpha)|B_t^\alpha)$ for $t \in (0,1]$. We wish to choose F and α so that $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$, for two given distributions μ_0 and μ_1 (both with zero expectations), with μ_0, μ_1 in convex order.

Then

$$M_t \quad = \quad (R_{1-t}*F)(B_t^\alpha)\,. \tag{1}$$

Let μ be a probability density. The **push-forward** $F_\#\mu$ of μ by F is the distribution of $F(X)$ if $X \sim \mu$, so

$$\mathbb{P}(F(X)\leq x) \quad = \quad \mathbb{P}(X \leq F^{-1}(x)) \quad = \quad \int_{-\infty}^{F^{-1}(x)} \mu(y)dy\,. \tag{2}$$

Thus if μ_t denotes the density of M_t, (1) implies that

$$\mu_t \quad = \quad (R_{1-t}*F)_\#(R_t*\alpha)$$

since the distribution of B_t is $R_t * \alpha$. In particular

$$\begin{aligned}
\mu_0 &= (R_1 * F)_\#(\alpha) \\
\mu_1 &= F_\#(R_1*\alpha)
\end{aligned} \tag{3}$$

since $R_0 * f = f$ for any f. This suggests an **alternating iterative scheme**:

$$\mu_0 \quad = \quad (R_1*F^n)_\#(\alpha^{n+1}), \tag{4}$$
$$\mu_1 \quad = \quad F^{n+1}_\#(R_1*\alpha^{n+1}) \tag{5}$$

to solve for (α, F), with $F^0(x) = x$ as the initial guess. For any probability distribution μ, let G_μ denote the cdf associated with μ. Then to compute α^{n+1}, we use (2) applied to (3):

$$\begin{aligned}
G_{\mu_0}(x) &= \int_{-\infty}^{(R_1*F^n)^{-1}(x)} \alpha^{n+1}(y)dy \\
\Rightarrow \quad G_{\mu_0}((R_1*F^n)(x)) &= \int_{-\infty}^{x} \alpha^{n+1}(y)dy \quad = \quad G_{\alpha^{n+1}}(x)
\end{aligned}$$

and we can differentiate the final line wrt x to get the density $\alpha^{n+1}(x)$ using the fundamental theorem of calculus. Using this α^{n+1}, the solution to the second equation in (5) is then $F^{n+1}(x) = G_{\mu_1}^{-1}(G_{R_1*\alpha^{n+1}}(x))$, and we then repeat this iterative process.

Deriving the Conze-Henry-Labordere fixed point equation for the distribution function of α

Sampling: draw from α, and then set $X_0 = G_{\mu_0}^{-1} \circ G_\alpha(B_0) = (R_1 * F)(B_0)$

Recall that for two probability densities ν_1 and ν_2, for $h = G_{\nu_1}^{-1}(G_{\nu_2})$, $\nu_1 = h_{\#}\nu_2$. Applying this to (3) (assuming F is strictly increasing) we see that

$$\begin{aligned} R_1 * F &= G_{\mu_0}^{-1} \circ G_\alpha \\ &= G_{\mu_1}^{-1} \circ G_{R_1 * \alpha} \;=\; G_{\mu_1}^{-1} \circ (R_1 * G_\alpha) \end{aligned} \tag{6}$$

using that $G_{R_1*\alpha} = R_1 * G_\alpha$. To check this identity, we take derivatives of the right hand side wrt x to get

$$\frac{d}{dx}\int_{-\infty}^{\infty} R_1(y) G_\alpha(x-y)dy \;=\; \int_{-\infty}^{\infty} R_1(y) G'_\alpha(x-y)dy \;=\; \int_{-\infty}^{\infty} R_1(y)\alpha(x-y)dy \;=\; (R_1 * \alpha)(x)\,.$$

Then using (6) and then (7), we see that

$$G_\alpha \;=\; G_{\mu_0} \circ (R_1 * F) \;=\; G_{\mu_0} \circ (R_1 * (G_{\mu_1}^{-1} \circ (R_1 * G_\alpha))) \;=\; \Phi(G_\alpha)$$

where Φ is shorthand for the all operators successively being applied to G_α on the right hand side.

This is conceptually similar to a simple non-linear 1d equation of the form $x = g(x)$ in first year Numerical analysis, which we can solve using the fixed point method $x_{n+1} = g(x_n)$ if $|g'(x)| < 1$. We can use the same method here except now the scalar x_n is replaced by a function G_α^n, so the iterative scheme becomes

$$G_\alpha^{n+1} \;=\; \Phi(G_\alpha^n) \tag{7}$$

which (under suitable conditions) converges to a function $G_\alpha^\infty(.)$, which is the desired cdf for α so as to make $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$. Note once we have G_α can compute the distribution of B_1^α and hence the required function F to make $F(B_1^\alpha)) \sim \mu_1$.

Short piece of Python code to implement (7) available at
`https://colab.research.google.com/drive/1neZR8aRTJnJ4PspoHCMt6kroGJEP87HL?usp=sharing`