Mock Class Test

1. Let $X_t = \gamma t + W_t$ and $e_q \sim \text{Exp}(q)$ independent of X, where W is a standard Brownian motion. Which of the following statements is true:

- $\bar{X}_t$ and $X_t - \bar{X}_t$ are independent
- $\bar{X}_{e_q}$ and $\underline{X}_{e_q}$ are independent
- $\bar{X}_{e_q}$ and $\bar{X}_{e_q} - X_{e_q}$ are independent if and only if $\gamma = 0$
- $\bar{X}_{e_q}$ and $X_{e_q} - \bar{X}_{e_q}$ are independent; note this is true for any Lévy process, not just BM. At-the-moment, this is just a fact to know without proof.

2. Let X be a Lévy process and $e_q \sim \text{Exp}(q)$ independent of X. Which of the following statements is true:

- $\bar{X}_{e_q} - X_{e_q} \sim -\underline{X}_{e_q}$. Also just a fact to know without proof atm, although not too hard to prove for BM.
- $\bar{X}_{e_q} \sim -\underline{X}_{e_q}$
- $\bar{X}_t - X_t \sim -\underline{X}_t$
- None of the above

3. Let X be a random variable with $\int_{-\infty}^{\infty} |\mathbb{E}(e^{iuX})| du < \infty$. The density $f_X(x)$ of X at x can be computed as

- $f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \mathbb{E}(e^{iuX}) du$. Also just a fact to know without proof.
- $f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iux} \mathbb{E}(e^{iuX}) du$
- $f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iut} \mathbb{E}(e^{iuX}) du$
- $f_X(x) = \frac{1}{\pi} \int_{0}^{\infty} e^{-iux} \mathbb{E}(e^{iuX}) du$

4. Let B^H denote fBM. Which of the following statements is true:

- $\mathbb{E}((B_t^H)^2) = 2H\, t^{2H}$
- $\text{Var}((B_t^H)) = t^{2H}$. This comes from computing $R_H(t,t)$, where $R_H(s,t)$ is the covar function of fBM.
- $\mathbb{E}((B_t^H)^2) = t^H$
- $\mathbb{E}((B_t^H)^2) = t$

5. Let $X_t = \sigma W_t$, and recall the Garman-Klass unbiased estimator for σ^2 given by $\hat{\sigma}^2_{GK} = \frac{1}{2} R_1^2 - (2 \log 2 - 1) X_1^2$ where $R_1 = \bar{X}_1 - \underline{X}_1$. Can we improve on this estimator using antithetic sampling?

- No because the antithetic version of $\hat{\sigma}^2_{GK}$ is equal to $\hat{\sigma}^2_{GK}$
- Yes because antithetic sampling always lowers the sample variance
- No because the antithetic version will be biased
- Cannot say

6. Let X be a Lévy process, and let $\bar{X}_t^{(n)} = \max_{0 \le k \le n} X_{kt/n}$ denote the **discretely sampled** maximum of X. It is known that $\mathbb{E}(\bar{X}_t^{(n)}) = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}(X_{kt/n}^+)$. Which of the following statements is true:

- $\sum_{k=1}^{n} \frac{1}{k} X_{kt/n}^+$ is an unbiased estimator for $\bar{X}_t$ No, because this has expectation $\mathbb{E}(\bar{X}_t^{(n)}) < \mathbb{E}(\bar{X}_t)$
- $\mathbb{E}(\bar{X}_t^{(n)}) \le \mathbb{E}(X_{t/n}^+)$ no, because the right hand side is just the first sum in the sum above
- $\mathbb{E}(\bar{X}_t^2) = \infty$ because a Lévy process has fat tails. Some Lévy process have fat tails, but not all, e.g. Brownian motion or a standard Poisson process.
- If X is Brownian motion, we get a better estimate for $\bar{X}_t$ by sampling from the conditional distribution of the relative maximum over each time step after initially sampling the change of X over each time step.

7. Let W be a standard Brownian motion and $X_t = W_t + \gamma t$ for $\gamma \geq 0$. Then it can be shown that $H_b = \min\{t : X_t = b\}$ for $b > 0$ satisfies

$$\mathbb{E}(e^{pH_b}) \quad = \quad e^{b(\gamma - \sqrt{\gamma^2 - 2p})} \quad = \quad e^{b\int_0^\infty (e^{px}-1)\frac{e^{-Mx}}{\sqrt{2\pi x^3}}dx}$$

if $2p \leq \gamma^2$ and $+\infty$ otherwise, where $M = \frac{1}{2}\gamma^2$. What does this tell us about the process $(H_b)_{b\geq 0}$?

- From the Levy-Khintchine representation on the right hand side, we can read off that H_b is a one-sided CGMY Lévy process with Lévy density $\nu(x) = \frac{e^{-Mx}}{\sqrt{2\pi x^3}}$.
- H_b is a one-sided α-stable process (this is only true if $\gamma = 0$).
- H_b is a Lévy process with Lévy density $\frac{e^{-Mx}}{\sqrt{2\pi x^3}}$ with positive and negative jumps ($\nu(x)$ is only defined for non-negative x, so there cannot be negative jumps.)
- $\mathbb{E}(H_b) = \infty$ for $\gamma > 0$. $\mathbb{E}(H_b) = \frac{d}{dp}\mathbb{E}(e^{pH_b})|_{p=0} = \frac{b}{\gamma}$, so $\mathbb{E}(H_b) = \infty$ is only true if $\gamma = 0$.

Remark 0.1 If $\gamma < 0$ then we have the modified formula:

$$\mathbb{E}(e^{pH_b} 1_{H_b<\infty}) \quad = \quad e^{b(\gamma - \sqrt{\gamma^2 - 2p})} \quad = \quad e^{-2|\gamma|b}\, e^{b\int_0^\infty (e^{px}-1)\frac{e^{-Mx}}{\sqrt{2\pi x^3}}dx}$$

for $\gamma^2 \leq 2p$, and letting $p \downarrow 0$, we see that $\mathbb{P}(H_b < \infty) = e^{-2|\gamma|b}$, and we can re-write the LHS as $\mathbb{E}(e^{pH_b}|H_b < \infty)\mathbb{P}(H_b < \infty)$.

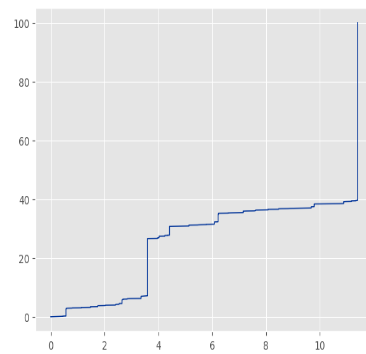

Figure 1: H_b process for $X_t = \gamma t + W_t$ with $\gamma = -\frac{1}{2}$, which jumps to infinity around $b \approx 11.4$

8. Let $X_1, X_2, \ldots, X_n$ denote n observations of a Lévy process X at unit intervals (with $X_0 = 0$), and $\mathbb{E}(e^{iuX_t}) = e^{t\psi(u;\theta)}$ for $u \in \mathbb{R}$ for some model parameters $\theta = (\theta_1, ..., \theta_k)$. The MLE of X for θ computed as

- $\text{argmax}_\theta \prod_{i=1}^n (\frac{1}{2\pi}\int_{-\infty}^\infty e^{-iuX_i}e^{t\psi(u;\theta)}du)$
- $\text{argmax}_\theta \prod_{i=1}^n (\int_{-\infty}^\infty e^{-iu\Delta X_i}e^{t\psi(u;\theta)}du)$, where $\Delta X_i = X_i - X_{i-1}$, since the $\Delta X_i's$ are i.i.d. We have omitted the $\frac{1}{2\pi}$ factor here but it doesn't matter because the MLE is the argmax not the max.
- $\text{argmax}_\theta (\frac{1}{2\pi}\int_{-\infty}^\infty e^{-iuX_1}e^{t\psi(u;\theta)}du)$
- $\max_\theta \prod_{i=1}^n (\frac{1}{2\pi}\int_{-\infty}^\infty e^{-iu\Delta X_i}e^{t\psi(u;\theta)}du)$

Hint: look at q3