

# Formula sheet

Note: any characteristic function for a Lévy process will be given to you in a class test/exam question

- Let  $X_t = \mu t + \sigma W_t$ , where  $W$  is Brownian motion. Then

$$F(b|x) = \mathbb{P}(\bar{X}_t \leq b | X_t = x) = 1 - e^{-\frac{2b(b-x)}{\sigma^2 t}} \quad (1)$$

for  $b \geq \max(x, 0)$ , where  $\bar{X}_t = \max_{0 \leq s \leq t} X_s$ . This allows us to sample  $\bar{X}_t$  given  $X_t$  as  $F^{-1}(U|X_t)$ , where

$$F^{-1}(u|x) = \frac{1}{2}(x + \sqrt{x^2 - 2t\sigma^2 \log(1-u)})$$

with  $U \sim U[0, 1]$  and  $F^{-1}(\cdot|x) : (0, 1) \rightarrow [\max(x, 0), \infty]$  is the inverse of the function on the right in (1) (viewed as a function of  $b$ ), which is very useful for Monte Carlo pricing of **barrier options**. Note  $F(b|x)$  (and hence  $F^{-1}(u|x)$ ) do not depend on  $\mu$ , but obviously the distribution of  $X_t$  depends on  $\mu$ .

- Let  $(Y_i)$  be a sequence of i.i.d. random variables with density  $\mu(x)$  and  $(N_t)_{t \geq 0}$  be a **Poisson process** for which  $\mathbb{E}(e^{pN_t}) = e^{\lambda t(e^p - 1)}$  (and recall that  $N_t \in \mathbb{N}$ ), and assume  $N_t$  is independent of  $(Y_i)$ , and let  $X_t = \sum_{i=1}^{N_t} Y_i$ .  $X$  is known as a **compound Poisson process**, and

$$\mathbb{E}(e^{iuX_t - q[X, X]_t}) = e^{\lambda t \int_{-\infty}^{\infty} (e^{iux - qx^2} - 1) \mu(x) dx} \quad (2)$$

for  $u \in \mathbb{R}$ ,  $q \geq 0$ .

- Let  $X$  be a random variable with  $\int_{-\infty}^{\infty} |\mathbb{E}(e^{iuX})| du < \infty$ . Then  $X$  has a density  $f_X(x)$ , which can be computed using the **inverse Fourier transform**:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \mathbb{E}(e^{iuX}) du.$$

Similarly, for the joint density  $f_{X,Y}(x, y)$  of two random variables  $X, Y$  is given by the double IFT:

$$f_{X,Y}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iux - ivy} \mathbb{E}(e^{iuX + ivY}) dudv$$

so e.g. we can apply this to (2) with  $[X, X]_t = Y$  and  $q = iu$ .

- General definition of a Lévy process is here: [https://en.wikipedia.org/wiki/L%C3%A9vy\\_process](https://en.wikipedia.org/wiki/L%C3%A9vy_process) Brownian motion, Poisson processes, compound Poisson processes and  $\alpha$ -stable processes are all examples of Levy processes.
- A **symmetric  $\alpha$ -stable process**  $X$  with parameters  $\alpha \in (0, 2]$ ,  $\sigma > 0$  is a generalization of Brownian motion, which has **independent stationary increments** like Brownian motion but  $\mathbb{E}(e^{iuX_t}) = e^{-t\sigma^\alpha |u|^\alpha}$  for  $u \in \mathbb{R}$ , and  $\mathbb{E}(X_t^2) = \infty$  if  $\alpha < 2$ . For the special case  $\alpha = 2$ ,  $X_t = \sqrt{2}\sigma W_t$  where  $W$  is a standard Brownian motion.
- A general Lévy process with no Brownian component has the **Lévy-Khintchine** representation

$$\mathbb{E}(e^{iuX_t}) = \exp(t(i\gamma u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{|x| \leq 1}) \nu(x) dx)) \quad (3)$$

where the Lévy density  $\nu(x) \geq 0$  must satisfy  $\int_{-\infty}^{\infty} \nu(x) \min(1, x^2) dx < \infty$ . If the  $iux$  integral here is finite, it can be absorbed into a modified  $\gamma$  term to get

$$\mathbb{E}(e^{iuX_t}) = \exp(t(i\tilde{\gamma}u + \int_{-\infty}^{\infty} (e^{iux} - 1) \nu(x) dx))$$

where  $\tilde{\gamma} = \gamma - \int_{-\infty}^{\infty} x1_{|x| \leq 1} \nu(x) dx$ , which agrees with the form for the compound Poisson case when  $\int_{-\infty}^{\infty} \nu(x) dx < \infty$ . A compound Poisson process is a Lévy process with  $\nu(x) = \lambda \mu(x)$  where  $\mu(x)$  is the jump density as above.

- A general **KoBoL**/CGMY-type Lévy process has  $\nu(x) = \frac{c_+ e^{-Mx}}{x^{1+Y_+}} 1_{x>0} + \frac{c_- e^{-G|x|}}{|x|^{1+Y_-}} 1_{x<0}$  for  $Y_+, Y_- \in (0, 2)$ , for which

$$\mathbb{E}(e^{iuX_t}) = \exp(t(i\gamma u + c_+ \Gamma(-Y_+)((M - iu)^{Y_+} - M^{Y_+}) + c_- \Gamma(-Y_-)((G + iu)^{Y_-} - G^{Y_-}))) \quad (4)$$

for  $Y_\pm \neq 1$  and  $u \in \mathbb{R}$  and we take the principal branch for the power functions (or if we set  $u = -ip$  so  $iu = p$  with  $p \in \mathbb{R}$ , then the right hand side is valid for  $p \in [-G, M]$  and  $\mathbb{E}(e^{pX_t}) < \infty$  in this range). An  $\alpha$ -stable process has  $G = M = 0$  and  $\alpha = Y_+ = Y_-$ ; a symmetric  $\alpha$ -stable process has  $c_+ = c_-$  and  $\gamma = 0$ , and an  $\alpha$ -stable process with positive-only jumps has  $c_- = 0$ , so in this case  $\mathbb{E}(e^{iuX_t}) = e^{iu\gamma t + c_+ \Gamma(-Y_+)(-iu)^{Y_+} t}$ , and this process is non-decreasing if  $Y_+ \in (0, 1)$  and  $\gamma \geq 0$  (note we usually set  $\gamma = 0$  for this case).

- Let  $X$  be a Lévy process and  $e_q$  an  $\text{Exp}(q)$  random variable independent of  $X$ . Then  $\bar{X}_{e_q}$  and  $\bar{X}_{e_q} - X_{e_q}$  are independent, and  $\bar{X}_{e_q} - X_{e_q} \sim -\underline{X}_{e_q}$ , and if  $X$  has a density  $\rho_t(x)$ , then

$$\Phi_q^+(z) := \mathbb{E}(e^{iz\bar{X}_{e_q}}) = \exp\left(\int_0^\infty t^{-1} e^{-qt} \int_0^\infty (e^{izx} - 1) \rho_t(x) dx dt\right) \quad (z \in \mathbb{R}).$$

- From this we find in Hwk 2 that

$$\mathbb{E}(\bar{X}_t) = \int_0^t \frac{\mathbb{E}(X_s^+)}{s} ds.$$

- A zero-mean Gaussian process  $B_t^H$  is called standard **fractional Brownian motion** (fBM) with **Hurst exponent**  $H \in (0, 1)$  if

$$R_H(s, t) = \mathbb{E}(B_t^H B_s^H) - \mathbb{E}(B_t^H) \mathbb{E}(B_s^H) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - (t - s)^{2H})$$

for  $0 \leq s \leq t$ . When  $H \in (0, \frac{1}{2})$ ,  $B^H$  is **rougher** than standard BM, and when  $H \in (\frac{1}{2}, 1)$ ,  $B^H$  is **smoother** than standard BM; more specifically  $B^H$  is  $H - \varepsilon$  **Hölder continuous** which means that  $|B_t^H - B_s^H| \leq c_1(\omega)|t - s|^{H-\varepsilon}$  a.s. for any  $\varepsilon \in (0, H)$  where  $c_1(\omega)$  is a (in general random) constant.