# Homework 3

**1**. (Estimating volatility). Let $X_t = \mu t + \sigma W_t$. Show that $\hat{\sigma}_n^2 = \sum_{i=1}^n (X_{(i+1)/n} - X_{i/n})^2$ is a consistent estimator for $\sigma^2$. Is $\hat{\sigma}_n^2$ an unbiased estimator?

**Solution**.

$$\begin{aligned}\hat{\sigma}_n^2 &= \sum_{i=1}^n (X_{(i+1)/n} - X_{i/n})^2 = \sum_{i=1}^n (\frac{\mu}{n} + \sigma(W_{(i+1)/n} - W_{i/n}))^2 \overset{(\text{law})}{=} \sum_{i=1}^n (\frac{1}{n}\mu + \frac{\sigma}{\sqrt{n}} Z_i)^2 \\ &= \frac{\mu^2}{n} + \frac{2\mu\sigma}{n} \sum_{i=1}^n \frac{Z_i}{\sqrt{n}} + \frac{\sigma^2}{n} \sum_{i=1}^n Z_i^2 \\ &= \frac{\mu^2}{n} + \frac{2\mu\sigma}{\sqrt{n}} \cdot \sum_{i=1}^n \frac{Z_i}{n} + \frac{\sigma^2}{n} \sum_{i=1}^n Z_i^2\end{aligned}$$

where $Z_i$ are i.i.d. $N(0,1)$, so $\hat{\sigma}_n^2 \to \sigma^2$ as $n \to \infty$ from the SLLN so $\hat{\sigma}_n^2$ is a consistent estimator for $\sigma^2$.

Note this applies to the log stock price $X_t = \log S_t$ for the Black-Scholes model if we just replace $\mu$ here with $\mu - \frac{1}{2}\sigma^2$, and the final limit does not depend on $\mu$.

For the second part, for $n$ finite, we see that $\mathbb{E}(\hat{\sigma}_n^2) = \frac{\mu^2}{n} + \sigma^2$, and hence is only unbiased when $\mu = 0$.

**2.** Using the expression for $\mathbb{P}(S_T > K)$ in the Black-Scholes chapter, what can we deduce about convergence of $S_t$ as $t \to \infty$ when $\mu = 0$.

**Solution**. For $\mu = 0$

$$\mathbb{P}(S_T > K) = \Phi^c(\frac{\log \frac{K}{S_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) \to 0$$

as $T \to \infty$, because $\frac{\log \frac{K}{S_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \sim \frac{1}{2}\sigma\sqrt{T} \to +\infty$ as $T \to \infty$. Hence $\mathbb{P}(S_T > K) = \mathbb{P}(|S_T - 0| > K) \to 0$ for any $K > 0$, so $S_t \to 0$ **in probability** under $\mathbb{P}$ as $T \to \infty$.

**3.**(Quadratic co-variation of two correlated Brownian motions). Let $W$ be a Brownian motion, and let $B_t = \rho W_t + \bar{\rho}\tilde{W}_t$ where $\bar{\rho} = \sqrt{1-\rho^2}$ and $\tilde{W}_t$ is another BM independent of $W$. Then it can be shown that $B$ is also a Brownian motion and $\mathbb{E}(W_t B_t) = \rho t$. Compute

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (W_{(i+1)/n} - W_{i/n})(B_{(i+1)/n} - B_{i/n}) \,.$$

**Solution**. The sum here has the same distribution as

$$\sum_{i=0}^{n-1} \sqrt{\Delta t} Z_i \cdot \sqrt{\Delta t}(\rho Z_i + \bar{\rho}\tilde{Z}_i) = \frac{1}{n} \sum_{i=0}^{n-1} Z_i(\rho Z_i + \bar{\rho}\tilde{Z}_i) \to \rho$$

where $\Delta t = \frac{1}{n}$, and $Z_i$ and $\tilde{Z}_i$ are two independent sequences of i.i.d. standard Normals. The convergence then follows from the SLLN.

**4.** (Estimating volatility). Let $X_t = \mu t + \sigma W_t$ and let $\bar{X}_t = \max_{0\le s\le t} X_s$ and $\underline{X}_t = \min_{0\le s\le t} X_s$. Using that

$$\mathbb{E}^{\mathbb{P}}(\bar{X}_t(\bar{X}_t - X_t) + \underline{X}_t(\underline{X}_t - X_t)) = \sigma^2 t \tag{1}$$

(we will see a proof of this later in the course) derive an unbiased estimate for $\sigma^2$ from $n$ daily observations of $X = \log S$ using the daily returns $r_i := X_{i\Delta t} - X_{(i-1)\Delta t}$, daily highs $H_i = \max_{s\in[(i-1)\Delta t, i\Delta t]}(X_s - X_{(i-1)\Delta t})$, and daily lows $L_i = \min_{s\in[(i-1)\Delta t, i\Delta t]}(X_s - X_{(i-1)\Delta t})$ for $i \in \mathbb{N}$, where $\Delta t = 1$ day.

**Solution**. From the i.i.d. increments property of $X$, $r_i \sim X_1$, $H_i \sim \bar{X}_1$, $L_i \sim \underline{X}_1$. Combining this with Eq (1), we see that

$$\mathbb{E}^{\mathbb{P}}(\frac{1}{n}\sum_{i=1}^n (H_i(H_i - r_i) + L_i(L_i - r_i))) = \sigma^2 \Delta t$$

so $\hat{\sigma}^2 := \frac{1}{n\Delta t}\sum_{i=1}^n (H_i(H_i - r_i) + L_i(L_i - r_i))$ is an unbiased estimate for $\sigma^2$, which is robust to unknown $\mu$.

**5**. (Double barrier computation). Let $X_t = \gamma t + W_t$, $M_t := \max_{0\leq s\leq t} X_s$ and $m_t := \min_{0\leq s\leq t} X_s$. Using that

$$\mathbb{P}(X_t \in dx, M_t < b, m_t > a) \quad = \quad -\frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\lambda_n t} e^{\gamma x - \frac{1}{2}\gamma^2 t} \sin(\frac{n\pi(x-a)}{b-a}) \sin(\frac{n\pi a}{b-a}) dx$$

for $a < 0 < b$ where $\lambda_n = \frac{n^2\pi^2}{2(b-a)^2}$, explain how you would use this to compute the cdf of $R_t := \max_{0\leq s\leq t}|X_s|$.

**Solution**.

$$\mathbb{P}(R_t < r) \quad = \quad \mathbb{P}(M_t < r, m_t > -r)\,.$$

We compute this by integrating each term of the series from $x = a$ to $b$ to compute the right hand side (assume we can interchange integral and series without proof)