Alternate estimators for H

The $m(q, \Delta)$ estimator from [GJR18]

For the first task, let $SS_n^{(q)} := \frac{1}{n} \sum_{i=1}^{n} |B_{i\Delta}^H - B_{(i-1)\Delta}^H|^q$. Then

$$\mathbb{E}(SS_n^{(q)}) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^{n} |B_{i\Delta}^H - B_{(i-1)\Delta}^H|^q) \;=\; \Delta^{qH}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(|B_i^H - B_{i-1}^H|^q) \;=\; \mathbb{E}(|Z|^q)\Delta^{qH} \;=\; K_q\Delta^{qH} \quad (1)$$

where $K_q = \mathbb{E}(|Z|^q) = \frac{2^{q/2}}{\sqrt{\pi}}\Gamma(\frac{q+1}{2})$, with $q > -1$ and $Z \sim N(0,1)$. From this we can then derive the estimator for the first task. Alternatively, if let $X = \sigma B_t^H$ and now define $SS_n^{(q)} = \frac{1}{n}\sum_{i=1}^{n} |X_{i\Delta}^H - X_{(i-1)\Delta}^H|^q$, and assume H and σ are unknown, then (1) changes to

$$\mathbb{E}(SS_n^{(q)}) \;=\; \sigma^q K_q \Delta^{qH}$$

which leads to the estimates $(\hat{H}_n, \hat{\sigma}_n)$ for (H, σ) defined by

$$SS_n^{(q)} \;=\; \hat{\sigma}_n^q \, K_q \Delta^{q\hat{H}_n}$$

if we have computed $SS_n^{(q)}$ for at least two Δ-values. Taking logs we see that

$$\log SS_n^{(q)} \;=\; q\log\hat{\sigma}_n + \log K_q \,+\, q\hat{H}_n \log\Delta$$

so we can perform **linear regression** on $\log SS_n^{(q)}$ vs $\log\Delta$ for a range of Δ-values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal $q\hat{H}_n$ (q is chosen by you, e.g. $q = 1, 2, 2.5, 3$ etc), and the **intercept** at $\log\Delta = 0$ is $q\log\hat{\sigma}_n + \log K_q$, from which we can compute $\hat{\sigma}_n$ since K_q has an explicit formula above. This is the $m(q, \Delta)$ estimator discussed in [GJR18]. One can then also compute the R^2-**statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of $\hat{H}_n$ and $\hat{\sigma}_n$.

The Han-Schied [HS21] estimator

Let $X_t = \sigma B_t^H$ and let

$$\theta_{m,k} \;=\; 2^{\frac{m}{2}}\left(2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}\right) \;=\; -2^{\frac{m}{2}}\left(X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{2k}{2^{m+1}}}\right)$$

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for $R(s,t) = \mathbb{E}(B_s^H B_t^H)$, one can check that

$$\mathbb{E}(\theta_{m,k}^2) \;=\; \sigma^2 2^{m-2H(1+m)}(4 - 4^H). \quad (2)$$

Then setting $s_n^2 = \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\theta_{m,k}^2$, we see that $\mathbb{E}(s_n^2) = \sum_{m=0}^{n-1} 2^m \mathbb{E}(\theta_{m,k}^2)$ (since (2) does not depend on k) which simplifies to

$$\mathbb{E}(s_n^2) \;=\; \sigma^2(4^{n(1-H)} - 1) \;\sim\; \sigma^2 4^{n(1-H)} \;=\; \sigma^2 2^{2n(1-H)} \quad (3)$$

as $n \to \infty$, which suggests an estimator $\hat{H}_n$ defined by $s_n = \hat{\sigma}_n \, 2^{n(1-\hat{H}_n)}$ which (assuming $\hat{\sigma}_n = O(1)$ as $n \to \infty$) we can re-arrange as

$$\hat{H}_n \;=\; 1 - \frac{1}{n}\log_2\left(\frac{s_n}{\hat{\sigma}_n}\right) \;=\; 1 - \frac{1}{n}\log_2 s_n \,+\, O\left(\frac{1}{n}\right)$$

where $\log_2$ denotes the base-2 logarithm, so (ignoring the $O(\frac{1}{n})$ remainder term), we recover the Han-Schied[HS21] estimator $\hat{H}_n = 1 - \frac{1}{n}\log_2 s_n$. Then

$$\begin{aligned}
\mathbb{E}(\hat{H}_n) &= 1 - \frac{1}{n}\mathbb{E}(\log_2(s_n)) = 1 - \frac{1}{2n}\mathbb{E}(\log_2(s_n^2)) \geq 1 - \frac{1}{2n}\log_2\mathbb{E}(s_n^2) = 1 - \frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)} - 1)) \\
&\geq 1 - \frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)})) \\
&= 1 - \frac{1}{2n}\log_2(\sigma^2) - \frac{1}{2n}\log_2(4^{n(1-H)})) \\
&= H - \frac{1}{2n}\log_2(\sigma^2)
\end{aligned}$$

and the final line is $> H$ if $\sigma < 1$, so $\mathbb{E}(\hat{H}_n) > H$ if $\sigma < 1$. See also discussion on optimal scaling factors in [HS21].
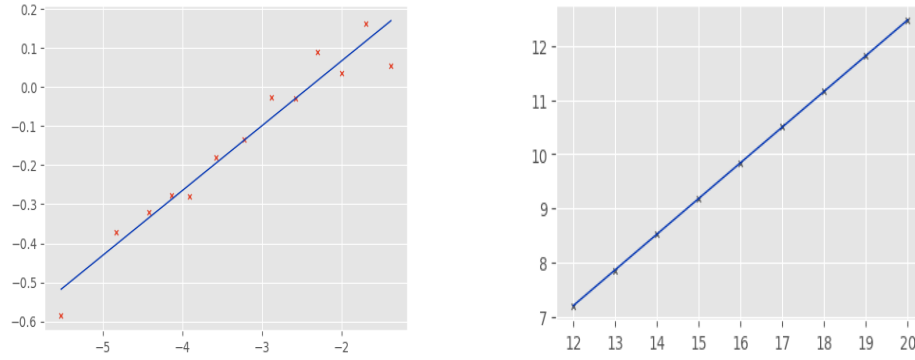
Figure 1: On the left, we see estimates of H for the SPX using the $m(q, \Delta)$ method for the SPX from 3rdJan22-15thJul24 for $q = 2$ (see similar plots in [GJR18]. On the right we see the linear regression in (4) for the Han-Schied method ($\log s_n$ vs n) for a true fBM path with 2^{20} time points, for which $\hat{H} = 0.0508$, and $\hat{\sigma} = 1.010$.

For more a general process X, (under certain conditions) [HS21] show that $\hat{H}_n \to R$ as $n \to \infty$, where $R = 1/q$, where q is the critical p-value at which $\lim_{n\to\infty} \sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}^H|^p$ switches from being zero (for $p > q$) to $+\infty$ (for $p < q$) (see Eq 2.2 in [HS21]).

Or we can jointly estimate H and σ by performing linear regression since

$$\log s_n \ = \ \log \hat{\sigma}_n \ + \ n(1 - \hat{H}_n) \log 2 \tag{4}$$

but we now have to compute $\log s_n$ for a range of different n-values to get a line of best fit, for which the slope is $(1 - \hat{H}_n) \log 2$ and the intercept is $\log \hat{\sigma}_n$.

You can then draw histograms of $\hat{H}_n$ if you simulate M fBM paths and compute the sample variance for $\hat{H}_n$ (or a confidence interval), or compute $\hat{H}_n$ for real data, e.g. using the SPX data file or data from yahoo finance.

Discussion on the rough Heston model, and the Riemann-Liouville process

The driftless rough Heston model satisfies

$$V_t \ = \ V_0 \ + \ \nu \int_0^t (t - u)^{H - \frac{1}{2}} \sqrt{V_u} dW_u \,.$$

Then $\mathbb{E}(V_t) = V_0$, and V has covariance function:

$$\mathbb{E}((V_s - V_0)(V_t - V_0)) \ = \ \nu^2 \mathbb{E}\left(\int_0^s (s - u)^{H - \frac{1}{2}} \sqrt{V_u} dW_u \cdot \int_0^t (t - r)^{H - \frac{1}{2}} \sqrt{V_r} dW_r \right)$$

$$= \ \nu^2 \int_0^s (s - u)^{H - \frac{1}{2}} (t - u)^{H - \frac{1}{2}} \mathbb{E}(V_u) du$$

$$= \ \nu^2 V_0 \int_0^s (s - u)^{H - \frac{1}{2}} (t - u)^{H - \frac{1}{2}} du \ = \ V_0^2 \nu^2 \bar{R}(s, t)$$

for $0 \le s \le t$, where $\bar{R}(s, t)$ is the covariance function for the Riemann-Liouville (RL) process $Z_t = \int_0^t (t - u)^{H - \frac{1}{2}} dW_u$ used for the rough Bergomi model, (note Z is a Gaussian process but V is not), but the explicit formula for $\bar{R}(s, t)$ is more complicated than the $R(s, t)$ formula for fBM.

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H \ = \ c_H \left(\int_{-\infty}^0 ((t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}}) dW_s \ + \ \int_0^t (t - s)^{H - \frac{1}{2}} dW_s \right) \ = \ c_H (A_t + Z_t)$$

for $t \in \mathbb{R}$, in terms of the RL process Z above (and note that A_t is known at time zero for all $t \ge 0$), and $c_H = \left(\frac{2H \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{\frac{1}{2}}$. Note also that A_t and Z_t are independent.

References

[BFG16] Bayer, C., P.K.Friz, and J.Gatheral, "Pricing under rough volatility", *Quantitative Finance*, 16(6), pp. 887–904, 2016.

[Cha14] Chang, Y-C., "Efficiently Implementing the Maximum Likelihood Estimator for Hurst Exponent", *Mathematical Problems in Engineering*, vol. 2014

[CD22] Cont, R. and P.Das, "Rough Volatility: Fact or Artefact?", preprint, 2022

[F23] Forde, M., "Statistical issues and calibration problems under rough and Markov volatility", `https://martinforde.github.io/Talk.pdf`

[GJR18] Gatheral, J., T.Jaisson and M.Rosenbaum, "Volatility is rough", *Quantitative Finance*, 18(6), 2018

[HS21] Han, X. and A.Schied, "The Hurst roughness exponent and its model-free estimation", preprint, 2021