Homework 2

1. Let X be a Lévy process, and e_q an $\text{Exp}(q)$ random variable independent of X, and let $\rho_t(x)$ denote the density of X_t. Using that

$$\Phi_q^+(z) \quad := \quad \mathbb{E}(e^{iz\bar{X}_{e_q}}) \quad = \quad \exp(\int_0^\infty t^{-1}e^{-qt}\int_0^\infty (e^{izx}-1)\rho_t(x)dxdt) \tag{1}$$

for $z \in \mathbb{R}$, compute $\mathbb{E}(\bar{X}_t)$ (this question requires no knowledge of Lecture 3).

Solution.

$$(\Phi_q^+)'(0) \quad = \quad i\mathbb{E}(\bar{X}_{e_q}) \quad = \quad i\int_0^\infty e^{-qt}\frac{\mathbb{E}(X_t^+)}{t}dt\,.$$

But we also know that $i\mathbb{E}(\bar{X}_{e_q}) = i\int_0^\infty qe^{-qt}\mathbb{E}(\bar{X}_t)dt$, so (dividing both expressions by iq) we see that

$$\int_0^\infty e^{-qt}\mathbb{E}(\bar{X}_t)dt \quad = \quad \frac{1}{q}\int_0^\infty e^{-qt}\frac{\mathbb{E}(X_t^+)}{t}dt\,.$$

But by a standard simple property of Laplace transforms, $\frac{1}{q}\mathcal{L}f = \mathcal{L}F$, where $\mathcal{L}$ denotes the Laplace transform operator and $F(t) = \int_0^t f(s)ds$ (see e.g. `https://en.wikipedia.org/wiki/Laplace_transform#Properties_and_theorems`). Hence (comparing both sides) we see that

$$\mathbb{E}(\bar{X}_t) \quad = \quad \int_0^t \frac{\mathbb{E}(X_s^+)}{s}ds$$

(we have seen this formula before in FM02 without proof).

2. The general **Wiener-Hopf** formula for a Lévy process states that

$$\mathbb{E}(e^{izX_{e_q}}) \quad = \quad \mathbb{E}(e^{iz\bar{X}_{e_q}})\,\mathbb{E}(e^{iz\underline{X}_{e_q}})$$

for $z \in \mathbb{R}$. Using (1), show that $\bar{X}_{e_q}$ has a **Lévy-Khintchine** representation of the form $\log\Phi_q^+(z) = \int_0^\infty(e^{izx}-1)\nu(x)dx$ for some non-negative function $\nu(x)$ (recall we refer to $\nu(x)$ as the Lévy density).

Solution. Interchanging integrals in (1), we see that

$$\log\Phi_q^+(z) \quad := \quad \log\mathbb{E}(e^{iz\bar{X}_{e_q}}) \quad = \quad \int_0^\infty (e^{izx}-1)\int_0^\infty t^{-1}e^{-qt}\rho_t(x)dtdx \quad = \quad \int_0^\infty (e^{izx}-1)\nu(x)dx$$

where $\nu(x) = \int_0^\infty t^{-1}e^{-qt}\rho_t(x)dt$.

Remark 0.1 If X is standard Brownian motion, using the known density $\rho_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2}x^2/t}$ of W_t, we can easily check that $\nu(x) = \frac{e^{-x\sqrt{2q}}}{x}1_{x>0}$, and we have seen in the FM02 Mock that $\bar{X}_{e_q} \sim \text{Exp}(\sqrt{2q})$. A Lévy process with this $\nu(x)$ function is known as a **Gamma process**, which is essentially a **one-sided CGMY process** with $Y=0$.

3. Let X be a **Cauchy process** (i.e. an α-stable process with $\alpha = 1$ so $\nu(x) = \frac{1}{\pi x^2}$) for which the density of X_t given $X_s = x$ (for $0 \le s \le t$) is a Cauchy distribution with density

$$p(x,y;\tau) \quad = \quad \frac{1}{\pi}\frac{\tau}{\tau + (y-x)^2}$$

where $\tau = t - s$. Compute the density of $\hat{X}_t$ given $\hat{X}_s = x$ (for $0 \le s \le t < 1$), where $\hat{X}$ is a Cauchy **bridge process**, i.e. X conditioned to be 0 at time 1.

Solution. Using Bayes' formula, the conditional density is

$$\frac{p(x,y;t-s)p(y,0;1-t)}{p(x,0;1-s)}$$

which evaluates to

$$\frac{(1-t)(t-s)((1-s)^2+x^2)}{\pi(1-s)((s-t)^2+(x-y)^2)((1-t)^2+y^2)}\,.$$

This can be used to simulate the bridge process using the usual $F^{-1}(U)$ method.

4. Let $X_t = \sigma B_t^H$ where B^H is fBM, and let $\theta_{m,k} = -2^{\frac{m}{2}}(X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} + X_{\frac{2k}{2^{m+1}}})$ and $s_n^2 = \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\theta_{m,k}^2$. Using that

$$\mathbb{E}(s_n^2) \quad = \quad \sigma^2(4^{n(1-H)} - 1) \quad \sim \quad \sigma^2 4^{n(1-H)} \quad = \quad \sigma^2 2^{2n(1-H)}$$

and $s_n^2 \sim \sigma^2 2^{2n(1-H)}$ as $n \to \infty$ (i.e. the ratio of both sides tends to 1), derive an estimator $\hat{H}_n$ for H using a ratio of s_n terms at different resolutions. Is $\hat{H}$ scale-invariant? You may assume that $\mathbb{E}(\theta_{m,k}^2)$ is independent of k.

Solution. $\frac{s_n^2}{2^{2n(1-H)}} \to \sigma^2$ as $n \to \infty$, so $\frac{s_{n-1}^2}{2^{2(n-1)(1-H)}} \to \sigma^2$ and the ratio of these two terms satisfies

$$\frac{s_n^2}{s_{n-1}^2} \cdot 2^{-2(1-H)} \quad \to \quad 1\,.$$

Setting $\frac{s_n^2}{s_{n-1}^2} = 2^{2(1-\hat{H}_n)}$, we see that $\frac{2^{2(1-\hat{H}_n)}}{2^{2(1-H)}} \to 1$, so $2(1-\hat{H}_n) - 2(1-H) = 2(H - \hat{H}_n) \to 0$, i.e. $\hat{H}_n$ is a consistent estimator, and we can write $\hat{H}_n$ explicitly as

$$\hat{H}_n \quad = \quad 1 - \frac{1}{2}\log_2 \frac{s_n^2}{s_{n-1}^2}\,.$$

$\hat{H}_n$ is scale-invariant since if we multiply the path X by λ, the ratio $\frac{s_n^2}{s_{n-1}^2}$ remains un-changed because the λ terms cancel.