A simplified unified propagator model for signed/unsigned order flow, concave price impact and rough volatility

Similar to Theorem 3.1 in [MORS26], we consider a hyper-rough Heston process $(F_t^+)_{t\geq 0}$ which satisfies

$$F_t^+ \quad = \quad g(t) + \int_0^t f(t-s) B_{F_s^+} ds$$

where B is a standard Brownian motion with $B_0 = 0$ and $f(t) = e^{-\theta t} t^{H-\frac{1}{2}}$ is the usual Gamma kernel with $H \in (-\frac{1}{2}, \frac{1}{2})$, $\theta > 0$ so $f \in L^1$ and we assume $g \in C^1$, and we know that F_t^+ is a.s. $(2\alpha - \varepsilon) \wedge 1$-Hölder continuous where $\alpha = H + \frac{1}{2}$ (see [JM20] for details) [1]; F^+ is an increasing process which is non-differentiable if $H \leq 0$. Then if $h * f \equiv 1$, we see that

$$\begin{aligned}
\int_0^t h(t-s)(F_s^+ - g(s)) ds \quad &= \quad \int_0^t h(t-s) \int_0^s f(s-u) B_{F_u^+} du ds \quad = \quad \int_0^t \int_u^t h(t-s) f(s-u) ds B_{F_u^+} du \\
&= \quad \int_0^t \int_0^{t-u} h(t-s-u) f(s) ds B_{F_u^+} du \\
&= \quad \int_0^t (h * f)(t-u) B_{F_u^+} du \quad = \quad \int_0^t B_{F_u^+} du
\end{aligned}$$

where Fubini is justified since $F_t^+ < \infty$ a.s. for finite t and B is a.s. continuous, and thus

$$\frac{d}{dt} \int_0^t h(t-s)(F_s^+ - g(s)) ds \quad = \quad B_{F_t^+}.$$

Taking Laplace transforms, the condition $h * f \equiv 1$ becomes $\hat{f}(\lambda)\hat{h}(\lambda) = \frac{1}{\lambda}$, from which we find that

$$h(t) \quad = \quad \frac{1}{\Gamma(\frac{1}{2} + H)} \theta^{\frac{1}{2}+H} \left(1 - \frac{\Gamma(-\frac{1}{2} - H, t\theta)}{\Gamma(-\frac{1}{2} - H)}\right)$$

where $\Gamma(a, z) = \int_z^\infty s^{a-1} e^{-s} ds$ is the incomplete Gamma function, and in particular $h(t) = O(t^{-H-\frac{1}{2}})$ as $t \to 0$.

Again following [MORS26], we now assume the *signed order flow* for an asset is

$$V_t \quad = \quad F_t^+ - F_t^- \tag{1}$$

where $F_t^- = g(t) + \int_0^t f(t-s) W_{F_s^-} ds$ and W is another Brownian motion independent of B (so F^- is an i.i.d. copy of F^+). Then we see that

$$\frac{d}{dt} \int_0^t h(t-s) V_s ds \quad = \quad B_{F_t^+} - W_{F_t^-}$$

since the g terms cancel, and for $0 \leq t \leq u$

$$\mathbb{E}(F_u^+ | \mathcal{F}^B_{F_t^+}) \quad = \quad g(u) + \mathbb{E}\left(\int_0^u f(u-s) B_{F_s^+} ds | \mathcal{F}^B_{F_t^+}\right) \quad = \quad g(u) + \int_0^t f(u-s) B_{F_s^+} ds + B_{F_t^+} \int_t^u f(u-s) ds$$

(and similarly for $\mathbb{E}(F_u^- | \mathcal{F}^W_{F_t^-})$), where we have used that $(B_{F_t^+})_{t\geq 0}$ is a $\mathcal{G}_t^+$−martingale (give reference), where $\mathcal{G}_t^+ = \mathcal{F}^B_{F_t^+}$.

Making the usual assumption that the asset price $P_t = \kappa \lim_{u\to\infty} \mathbb{E}(F_u | \mathcal{G}_t)$ where $F_t = F_t^+ - F_t^-$ and $\mathcal{G}_t = \sigma(\mathcal{G}_t^+, \mathcal{G}_t^-)$ with $\mathcal{G}_t^- = \mathcal{F}^W_{F_t^-}$ (for some $\kappa > 0$)[2], we find that $f(u-s) \to 0$ as $u \to \infty$ since $\theta > 0$, but $\lim_{u\to\infty} \int_t^u f(u-s) ds = c_{H,\theta}$ where $c_{H,\theta} = \theta^{-\frac{1}{2}-H} \Gamma(\frac{1}{2} + H)$, so

$$P_t \quad = \quad \kappa \lim_{u\to\infty} \mathbb{E}(F_u^+ - F_u^- \,|\, \sigma(\mathcal{F}^B_{F_t^+}, \mathcal{F}^W_{F_t^-})) \quad = \quad \kappa\, c_{H,\theta} (B_{F_t^+} - W_{F_t^-})$$

i.e. P is the difference of two i.i.d. hyper-rough Heston models (each with correlation $\rho = 1$ since each process only has one driving Brownian motion), and

$$P_t \quad = \quad \kappa\, c_{H,\theta} \frac{d}{dt} \int_0^t h(t-s) V_s ds$$

[1] we can simulate this process using the Monte Carlo scheme in [AA25] via Normal Inverse Gaussian variates
[2] note we are assuming $P_0 = 0$ without loss of generality since we can easily add a P_0 term

which is a weak formulation of the usual Propagator model, and the *unsigned* order flow is $U_t = F_t^+ + F_t^-$. In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 up time t_0 is given by

$$MI(t) = \frac{d}{dt}\int_0^t h(t-s)sds = \frac{\theta^{-\frac{1}{2}+H}(t\theta\Gamma(-\alpha_+) - t\theta\Gamma(-\alpha_+, t\theta) - \Gamma(\alpha_-, 0) + \Gamma(\alpha_-, t\theta))}{\Gamma(-\alpha_+)\Gamma(\alpha_+)}$$

for $0 \le t \le t_0$ (with $\alpha_\pm = \frac{1}{2} \pm H$) which is $O(t^{\frac{1}{2}-H})$ as $t \to 0$ and globally concave in t for $H \in (-\frac{1}{2}, \frac{1}{2})$, consistent with empirical evidence (note the usual square-root impact law corresponds to $H = 0$ here), and we can show that the asymptotic (i.e. permanent) impact as $t \to \infty$ (given no trading after t_0) is

$$\frac{t_0\theta^\alpha}{\Gamma(\alpha)}$$

(this is the asymptotic grey line in the plot below). Recall from above that F_t^+ has Hölder regularity $(2\alpha - \varepsilon) \wedge 1$, and empirical evidence in [MORS26] suggests $H \in (-\frac{1}{4}, 0)$, so $2\alpha \in (\frac{1}{2}, 1)$.

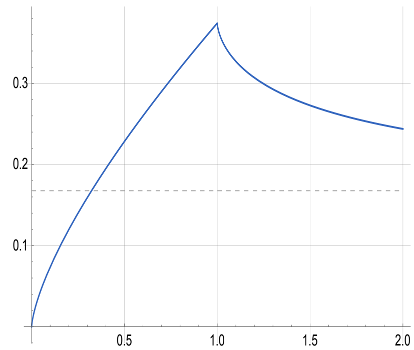

Figure 1: Price impact of an exogenous metaorder executed at constant trading speed 1 over $[0, 1]$ with $H = -\frac{2}{10}$, $\theta = \frac{1}{10}$ and $\kappa = 1$. The blue line asymptotes to the horizontal grey line as $t \to \infty$ which represents the permanent price impact.

References

[AA25] Abi Jaber, E. and E.Attal, "Simulating integrated Volterra square-root processes and Volterra Heston models via Inverse Gaussian", Preprint, 2025

[JM20] P.Jusselin and M.Rosenbaum, "No-arbitrage implies power-law market impact and rough volatility", 2018, to appear in Mathematical Finance, *Mathematical Finance*, Volume 30, Issue 4 pp. 1309-1336

[MORS26] Muhle-Karbe, J., Y.Chahdi, M.Rosenbaum, and G.Szymanski, "A unified theory of order flow, market impact, and volatility", preprint.