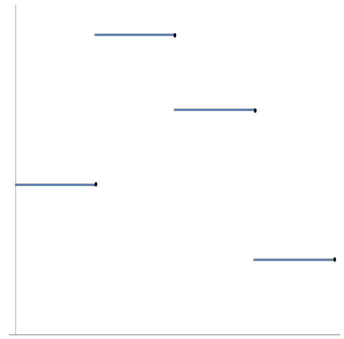

Figure 1: Example of a simple random process α_t

Stochastic integrals with respect to Brownian motion

Let W be a standard Brownian motion as above. We now want to make sense of an integral of the form $\int_0^t \alpha_s dW_s$ where α is a stochastic (i.e. random) process. It is natural to only consider **adapted** process α, i.e. such that the value of α_t for each t depends only on the history of W up to time t, i.e. α **cannot see into the future**, which will be natural in a financial context later when we consider **dynamic trading strategies** under the famous Black-Scholes model. So e.g. $\alpha_t = \int_0^t f(W_s)ds$ is an adapted process but $\alpha_t = W_{t+\delta}$ or $\alpha_t = \int_0^{t+\delta} W_s ds$ is not if $\delta > 0$. An integral of the form $\int_0^t \alpha_s dW_s$ is known as a **stochastic integral**. These cannot be defined as a conventional integral because W has **infinite variation**, which means that

$$\lim_{\|\Pi_n\|\to 0} \sum_{i=0}^{n-1} |W_{t_{i+1}^n} - W_{t_i^n}| \quad = \quad \infty \tag{1}$$

where Π_n denotes a sequence of partitions with $|\Pi_n| \to 0$ as before. Note that we computing the sum of absolute increments $|W_{t_i} - W_{t_{i-1}}|$ not square increments $|W_{t_i} - W_{t_{i-1}}|^2$ here.

To show this is true for a uniform partition, we note that

$$\sum_{i=0}^{n-1} |W_{t_{i+1}^n} - W_{t_i^n}| \quad \sim \quad \sum_{i=0}^{n-1} \sqrt{\Delta t}\,|Z_i| \quad = \quad n\sqrt{\frac{t}{n}} \cdot \frac{1}{n} \sum_{i=1}^{n} |Z_i|$$

where $\Delta t = t/n$, but we know that $\frac{1}{n}\sum_{i=1}^n |Z_i| \to \mathbb{E}(|Z_i|)$ by the SLLN, so the right hand side is $O(\sqrt{n})$.

Let $0 < t_1 < t_2 < \ldots . t_N = T$. Then we can define the **stochastic integral** of a **simple random process** of the form $\alpha_t = \sum_{i=1}^N \alpha_i 1_{t\in(t_{i-1},t_i]}$ (where α_i is random but can only depend on the history of W up to time t_{i-1} and $\sum_{i=1}^\infty \alpha_i^2 < \infty$), in the natural way as

$$\int_0^T \alpha_s dW_s \quad = \quad \sum_{i=1}^N \alpha_i (W_{t_i} - W_{t_{i-1}}) \,.$$

To generalize the stochastic integral to more general (non-simple) process α_t, we approximate the process to arbitrary accuracy with a simple process and use arguments involving L^2-convergence.

Stochastic differential equations and Ito's lemma

We now consider a **stochastic differential equation** (SDE) which we can write informally as

$$dX_t \quad = \quad b(X_t)dt + \sigma(X_t)dW_t \tag{2}$$

and enquire, can we assign a rigorous meaning to this equation? Note that if $b(x) = 0$ and $\sigma(x) = 1$, then $dX_t = dW_t$, so $X_t = X_0 + W_t$.

Definition 0.1 *A **strong solution** to the SDE (2) with $X_0 = x_0$ is a process X_t with a continuous sample path which satisfies the integral equation*

$$X_t \quad = \quad x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \,. \tag{3}$$

*b is known as the **drift**, and σ is known as the **volatility** (or diffusion coefficient).*

Remark 0.1 Note if $\sigma \equiv 0$, then (2) reduces to the Ordinary Differential Equation (ODE):

$$\frac{dx(t)}{dt} = b(x(t))$$

which we can solve be re-writing as $\frac{dx}{b(x)} = dt$ and then integrating both sides.

X in (3) is **implicitly defined** in terms of itself, so how do we construct a solution rigorously? We first simulate the Brownian sample path W. We then start with the constant solution $X_t^0 = x_0$, feed it into the SDE, and then we keep feeding the answer back into the SDE as follows:

$$\begin{aligned} X_t^{(1)} &= x_0 + \int_0^t b(x_0)ds + \int_0^t \sigma(x_0)dW_s\,, \\ X_t^{(2)} &= x_0 + \int_0^t b(X_s^{(1)})ds + \int_0^t \sigma(X_s^{(1)})dW_s \\ &\dots \\ X_t^{(n+1)} &= x_0 + \int_0^t b(X_s^{(n)})ds + \int_0^t \sigma(X_s^{(n)})dW_s \\ &= \dots \end{aligned}$$

Using the Borel-Cantelli lemma we can then show that the sequence of sample paths $\{X_s^{(n)}; 0 \le s \le t\}$ converges to some continuous function X_t which we call the solution to (2), i.e. $\lim_{n\to\infty} \max_{0\le s\le t}|X_s^n - X_s)| \to 0$ (see FM04 lecture notes for proof). This is called the **Picard iteration method**.

A function f is **Lipschitz** continuous if $|f(x) - f(y)| \le K|y-x|$ for some non-negative constant $K < \infty$.

Proposition 0.1 *If b and σ are bounded and Lipschitz continuous, then a strong solution exists for T.*

On a computer, we typically do not use the Picard method to numerically approximate the SDE, but rather we use an extension of the Euler scheme where we essentially "freeze" the coefficients over each time step as follows:

$$X_{t+\Delta t}^{(n)} = X_t^{(n)} + b(X_t^{(n)})\Delta t + \sigma(X_t^{(n)})\sqrt{\Delta t}\, Z_i \tag{4}$$

where Z_i is a sequence of i.i.d. standard Normals and $\Delta t = \frac{1}{n}$ as before, and under certain conditions we can show that the approximate solution $X^{(n)}$ tends to true solution X to the SDE as $n \to \infty$ (i.e. as the step size $\Delta t = \frac{1}{n} \to 0$).

Ito's lemma

From basic algebra we find that

$$W_t^2 - W_s^2 = 2W_s(W_t - W_s) + (W_t - W_s)^2$$

Similarly

$$W_t^2 - W_0^2 = W_{t_n}^2 - W_{t_{n-1}}^2 + (W_{t_{n-1}}^2 - W_{t_{n-2}}^2) + \dots + \dots(W_{t_1}^2 - W_0^2) = 2\sum_{i=1}^n W_{\frac{(i-1)t}{n}} \Delta W_i + \sum_{i=1}^n (\Delta W_i)^2\,.$$

Hence as $\Delta t \to 0$, and using that $,\sum_{i=1}^n (\Delta W_i)^2 \to t$ (from Brownian motion chapter) and the definition of the stochastic integral above as a limit, we see that this tends to

$$W_t^2 - W_0^2 = 2\int_0^t W_s dW_s + t\,.$$

Note we have the additional term here (the t term on the right) which we would not get if W were a differentiable function.

This is a special case of the following hugely important result which is central to the whole course:

Theorem 0.2 *For a process X satisfying the SDE in (2) and a function $f(x,t)$ which is twice differentiable in x and once in t, we have*

$$\begin{aligned} f(X_t, t) &= f(X_0, 0) + \int_0^t f_x(X_s, s)dX_s + \int_0^t [f_t(X_s, s) + \frac{1}{2} f_{xx}(X_s, s)\sigma(X_s)^2]ds \\ &= f(X_0, 0) + \int_0^t f_x(X_s, s)[b(X_s)ds + \sigma(X_s)dW_s] + \int_0^t [f_t(X_s, s) + \frac{1}{2} f_{xx}(X_s, s)\sigma(X_s)^2]ds\,. \end{aligned} \tag{5}$$

As shorthand, we write (5) in the **differential form** as

$$df(X_t,t) \;=\; f_t(X_t,t)dt \,+\, f_x(X_t,t)dX_t \,+\, \frac{1}{2}f_{xx}(X_t,t)\sigma(X_t)^2dt\,.$$

Note again that we get a surprising additional second order term here (the term involving f_{xx}) which we do not get if X_t is just a non-random function which differentiable in x and in t), and this extra terms is the main difference between ordinary calculus and **stochastic calculus**.

THE RULE FOR THE GENERALIZED ITO LEMMA IS: WHATEVER IS IN FRONT OF dW_t IN THE ORIGINAL SDE GETS SQUARED AND MULTIPLIED BY ONE-HALF

Important examples

Note that W satisfies the trivial SDE:

$$dX_t \;=\; 0dt \,+\, 1dW_t\,.$$

1. Let $f(x,t)=x^2-t$ so $f(W_t,t)=W_t^2-t$. Then $f_x(x,t)=2x$, $f_{xx}(x,t)=2$ and $f_t(x,t)=-1$. Thus from Ito's lemma we have

$$\begin{aligned} df(W_t,t) &= f_t(W_t,t)dt+f_x(W_t,t)dW_t+\frac{1}{2}f_{xx}(W_t,t)dt \\ &= -dt \,+\, 2W_tdW_t \,+\, \frac{1}{2}\cdot 2dt \\ &= 2W_tdW_t\,. \end{aligned}$$

Writing this in integrated form we have $f(W_t,t)-f(W_0,0)=W_t^2-t=\int_0^t 2W_sdW_s$, and we can easily verify that $M_t=W_t^2-t$ is an $\mathcal{F}_t^W$-martingale, i.e. $\mathbb{E}(M_t|(W_u)_{0\le u\le s})=M_s$.

2. Let $f(x,t)=S_0e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}$, so $S_t:=f(W_t,t)=S_0e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}$. This is the famous **Black-Scholes** model for a stock price process. Then $f_x(x,t)=\sigma f(x,t)$, $f_{xx}(x,t)=\sigma^2f(x,t)$ and $f_t(x,t)=(\mu-\frac{1}{2}\sigma^2)f(x,t)$. Thus from Ito's lemma we have

$$\begin{aligned} dS_t &= f_t(W_t,t)dt+f_x(W_t,t)dW_t+\frac{1}{2}f_{xx}(W_t,t)dt \\ &= (\mu-\frac{1}{2}\sigma^2)S_tdt \,+\, \sigma S_tdW_t \,+\, \frac{1}{2}\sigma^2S_tdt \\ &= \mu S_tdt \,+\, \sigma S_tdW_t \qquad (6) \\ &= b(S_t)dt \,+\, \tilde{\sigma}(S_t)dW_t \end{aligned}$$

where $b(S)=\mu S$ and $\tilde{\sigma}(S)=\sigma S$. Note that b and σ are both linear functions of S, so this process is known as a **Geometric Brownian motion**.

We can show that S_t is a $\mathcal{F}_t^W$-martingale iff $\mu=0$, and this is an example of a stochastic differential equation. μ describes the overall trend of the process, i.e. its tendency to go up or down in the long run (it can be shown using the mgf of a Normal distribution that $\mathbb{E}(S_t)=S_0e^{\mu t}$, and σ is the volatility which controls the variability of the stock price (see Excel sheet on website). Note that S_t is always positive.

3. **pth power of stock price for Black-Scholes model**. Assume that $dS_t=S_t\sigma dW_t$ and let $M_t=S_t^p$ for $p>1$. Apply Ito's lemma to M_t, and write dM_t as an SDE in terms of M_t itself.

Set $M_t=f(S_t,t)$, where $f(S,t)=S^p$. Applying Ito's lemma to $dS_t=S_t\sigma dW_t$, we have that $f_t(S,t)=0$, $f_S(S,t)=pS^{p-1}$, $f_{SS}(S,t)=p(p-1)S^{p-2}$ and

$$\begin{aligned} dM_t \;=\; df(S_t,t) &= f_t(S_t,t)dt \,+\, f_S(S_t,t)dS_t \,+\, \frac{1}{2}f_{SS}(S_t,t)S_t^2\sigma^2dt \\ &= pS_t^{p-1}dS_t \,+\, \frac{1}{2}p(p-1)S_t^{p-2}S_t^2\sigma^2dt \\ &= pS_t^{p-1}S_t\sigma dW_t \,+\, \frac{1}{2}p(p-1)S_t^p\sigma^2dt \\ &= \frac{1}{2}p(p-1)\sigma^2M_tdt \,+\, p\sigma M_tdW_t \\ &= M_t(\frac{1}{2}p(p-1)\sigma^2dt \,+\, p\sigma dW_t) \end{aligned}$$

i.e. the drift and volatility for M_t are linear functions of M_t, so M is a Geometric Brownian motion like S in (6).

4. The Ornstein-Uhlenbeck process. Consider the Ornstein-Uhlenbeck (OU) process which satisfies

$$dY_t \;=\; -\kappa Y_t dt \,+\, \sigma dW_t \tag{7}$$

for $\kappa, \sigma > 0$. The drift term means the process tends to "mean-revert" back to zero if Y goes too far away from zero, and more generally we can let $dY_t = \kappa(\theta - Y_t)dt + \sigma dW_t$, and in this case Y will mean-revert around the level θ. For this example we keep $\theta = 0$ for simplicity, and let $Z_t = e^{\kappa t} Y_t = f(Y_t, t)$ where $f(y,t) = e^{\kappa t} y$. Then $f_t = \kappa f$, $f_y = e^{\kappa t}$ and $f_{yy} = 0$, and applying Ito's lemma to Y_t directly we see that

$$\begin{aligned} dZ_t \;&=\; f_t dt + f_y dY_t \,+\, \frac{1}{2} f_{yy} \sigma^2 dt \;=\; \kappa e^{\kappa t} Y_t dt \,+\, e^{\kappa t} dY_t + 0 \\ &=\; \kappa e^{\kappa t} Y_t dt \,+\, e^{\kappa t}(-\kappa Y_t dt + \sigma dW_t) \\ &=\; \sigma e^{\kappa t} dW_t \,. \end{aligned}$$

Integrating this equation, we see that $Z_t = Z_0 \,+\, \int_0^t \sigma e^{\kappa s} dW_s$. Multiplying by $e^{-\kappa t}$ and noting that $Z_0 = Y_0$, we obtain

$$Y_t \;=\; e^{-\kappa t} Z_0 \,+\, e^{-\kappa t} \int_0^t \sigma e^{\kappa s} dW_s \;=\; e^{-\kappa t} Y_0 \,+\, \int_0^t \sigma e^{-\kappa(t-s)} dW_s \,.$$

This is the solution to the SDE in (7). It turns out that for any non-random continuous function ϕ, $\int_0^t \phi(s) dW_s \sim N(0, \int_0^t \phi(s)^2 ds)$ (proof not required, but can done by first considering the case when σ is piecewise constant in which case the result is obvious). In this case, setting $\phi(s) = \sigma e^{-\kappa(t-s)}$, we find that

$$\begin{aligned} Y_t \;&\sim\; e^{-\kappa t} Y_0 \,+\, N(0, \int_0^t \phi(s)^2 ds) \\ &=\; e^{-\kappa t} Y_0 \,+\, N(0, \sigma^2 \frac{1 - e^{-2\kappa t}}{2\kappa}) \end{aligned}$$

From this we see that $Y_\infty := \lim_{t \to \infty} Y_t \sim N(0, \frac{\sigma^2}{2\kappa})$. We call this the **stationary distribution** of Y. The OU process is often used in practice to model volatility or an interest rate process (this is the well known **Vasicek model**, see FM07 for more details on this and the Cox-Ingersoll-Ross (**CIR**) process).

5. **The Black-Scholes model, and foreign exchange rates**. Set $Y_t = f(S_t)$ where S_t satisfies the Black-Scholes SDE $dS_t = \sigma S_t dW_t$ with $\mu = 0$ and $f(S) = 1/S$. Applying the generalized Ito lemma to $dS_t = \sigma S_t dW_t$ we have $f_S = -\frac{1}{S^2} = -Y^2, f_{SS} = \frac{2}{S^3} = 2Y^3$ and

$$dY_t \;=\; -\frac{1}{S_t^2} dS_t + \frac{1}{2} \frac{2}{S_t^3} S_t^2 \sigma^2 dt \;=\; Y_t[\sigma^2 dt \,-\, \sigma dW_t] \,.$$

If S_t is e.g. the GBP/USD Exchange rate i.e. the cost of a pound in dollars, then $Y_t = 1/S_t$ is the USD/GBP exchange rate, i.e. the cost of a dollar in pounds.