

Homework 1

Let $(W_t)_{t \geq 0}$ be a standard Brownian motion throughout.

1. Write down a formula for $u(x, t) := \mathbb{E}(f(W_T) | W_t = x)$ for a general function f for $0 \leq t \leq T$.

Solution. From the definition of BM, the conditional distribution of W_T given $W_t = x$ for $0 \leq t \leq T$ is $N(x, T - t)$ (with density $\frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}}$), so

$$\mathbb{E}(f(W_T) | W_t = x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} f(y) dy = (p_{T-t} * f)(x)$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ is the density of W_t , and $*$ denotes **convolution** (the convolution of two functions f and g is $(f * g)(x) := \int_{-\infty}^{\infty} f(y)g(x-y)dy$. Note by setting $x-y = u$ so $y = x-u$ and $dy = -du$, we see that $(f * g)(x) = -\int_{\infty}^{-\infty} f(x-u)g(u)du = \int_{-\infty}^{\infty} f(x-u)g(u)du = (g * f)(x)$, i.e. $f * g = g * f$.

2. Let

$$B_t = (1-t)W_{\frac{t}{1-t}}$$

for $0 \leq t < 1$. Compute $\mathbb{E}(B_s B_t)$ for $0 < s < t < 1$. What do you notice at $t = 1$?

Solution. $f(t) = \frac{t}{1-t}$ is a strictly increasing function for $t \in [0, 1)$ and $\mathbb{E}(W_s W_t) = \min(s, t)$ (see Brownian motion chapter), so $\mathbb{E}(B_s B_t) = (1-s)(1-t) \min(\frac{s}{1-s}, \frac{t}{1-t}) = (1-s)(1-t) \frac{s}{1-s} = s(1-t)$.

We note that $\mathbb{E}(B_1^2) = 1 \times 0 = 0$, hence $B_1 = 0$ a.s. B is known as the **Brownian bridge** (see Figure 1 below), which is Brownian motion conditioned to be at zero at time 1.

3. Let $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ (this is the famous **Black-Scholes model**). Compute $\mathbb{E}(S_t^p)$ (hint: re-write S_t^p as $S_0^p e^{pX_t}$ where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$).

Solution.

$$\mathbb{E}(S_t^p) = S_0^p \mathbb{E}(e^{pX_t}) = S_0^p \mathbb{E}(e^{p((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)})$$

But $(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \sim N(\mu_1, \sigma_1^2)$, where $\mu_1 = (\mu - \frac{1}{2}\sigma^2)t$ and $\sigma_1^2 = \sigma^2 t$. Thus

$$\mathbb{E}(S_t^p) = S_0^p e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2} = S_0^p e^{(\mu - \frac{1}{2}\sigma^2)pt + \frac{1}{2}\sigma^2 p^2 t}$$

where we have used that the mgf of a general Normal $N(\mu_1, \sigma_1^2)$ random variable is $e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2}$ from Applied Probability Revision chapter. Note that $\mathbb{E}(S_t^p) < \infty$ for all $p \in \mathbb{R}$, i.e. all moments of S_t are finite.

4. Let $X_t = \sum_{i=1}^n (W_t^{(i)})^2$, where $W^{(i)}$ are n independent standard Brownian motions. Using that $\mathbb{E}(e^{-\lambda Z^2}) = \frac{1}{(1+2\lambda)^{\frac{1}{2}}}$ for $\lambda > 0$ where $Z \sim N(0, 1)$, compute $\mathbb{E}(e^{-\lambda X_t})$ for $\lambda > 0$. X is known as a **Bessel squared process** of dimension n .

Solution. Using that $B_t^{(i)} \sim \sqrt{t}Z$ and the independence of the δ -Brownian motions, we see that

$$\mathbb{E}(e^{-\lambda X_t}) = \prod_{i=1}^{\delta} \mathbb{E}(e^{-\lambda (B_t^{(i)})^2}) = \prod_{i=1}^{\delta} \mathbb{E}(e^{-\lambda (\sqrt{t}Z)^2}) = (\mathbb{E}(e^{-\lambda t Z^2}))^{\delta} = \frac{1}{(1+2\lambda t)^{\frac{1}{2}\delta}}.$$

5. Compute the conditional distribution of W_t given W_s , for $0 < t < s$. You may use that for two correlated Normal random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ with $\text{Corr}(X, Y) = \rho$,

$$Y|X \sim N(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(X - \mu_X), (1 - \rho^2)\sigma_Y^2)$$

and recall that the correlation of two random variables X and Y is defined as $\text{Corr}(X, Y) = \frac{\mathbb{E}((X - \mu_X)(Y - \mu_Y))}{\sigma_X \sigma_Y}$.

Solution. For our case here, $X = W_s$, $Y = W_t$, $\mu_X = 0$, $\mu_Y = 0$, $\sigma_X = \sqrt{s}$, $\sigma_Y = \sqrt{t}$ and $\rho = \frac{\min(s, t)}{\sqrt{st}} = \frac{\sqrt{t}}{\sqrt{s}}$, and recall that we have shown in the lecture notes that $\mathbb{E}(W_s W_t) = \min(s, t)$. Thus

$$W_t|W_s \sim N(\rho \frac{\sqrt{t}}{\sqrt{s}} W_s, (1 - \rho^2)t) = N(\frac{t}{s} W_s, t(1 - \frac{t}{s})).$$

6. Compute $\mathbb{E}(W_t^3|W_s = x)$ for $0 \leq s \leq t$.

Solution. $W_t - W_s \sim N(0, t - s)$, so

$$\begin{aligned}\mathbb{E}((x + W_t - W_s)^3|W_s = x) &= \mathbb{E}(x^3 + 3x^2(W_t - W_s) + 3x(W_t - W_s)^2 + (W_t - W_s)^3|W_s = x) \\ &= x^3 + 3x(t - s).\end{aligned}$$

We can generalize this computation to compute $\mathbb{E}(W_t^n|W_s = x)$ for any $n \in \mathbb{N}$, since (from the **binomial theorem**) we know that $(x + W_t - W_s)^n = \sum_{i=0}^n x^{n-i}(W_t - W_s)^i \binom{n}{i}$, and we also know that all odd moments of $W_t - W_s$ are zero.

7. Let $C(u, v) = \frac{uv}{1-\theta(1-u)(1-v)}$ for $\theta \in (-1, 1)$. Show that $C(u, v)$ is a joint distribution function of two $U[0, 1]$ random variables U and V . You may use that $C_{uv}(u, v) \geq 0$ without proof.

What happens when $\theta = 0$? How can we use this to sample two correlated random variables X and Y with distribution functions $F_X(x)$ and $F_Y(y)$? Compute the conditional cdf of $V|U = u$.

Solution. $C(u, 0) = \mathbb{P}(U \leq u, V \leq 0) = 0$, $C(0, v) = \mathbb{P}(U \leq 0, V \leq v) = 0$, $C(u, 1) = \mathbb{P}(U \leq u, V \leq 1) = u$, $C(1, v) = \mathbb{P}(U \leq 1, V \leq v) = v$, and the joint density $C_{uv}(u, v) \geq 0$.

For $\theta = 0$, $C(u, v) = uv$, so the joint density is $C_{u,v}(u, v) = 1$, i.e. U and V are independent $U[0, 1]$ in this case (but not for general $\theta \neq 0$).

For the penultimate part, we can sample U and V (using the trick in Hwk 6 q3), and then set $X = F_X^{-1}(U)$ and $Y = F_Y^{-1}(V)$. For the final part, (from Bayes' rule) the conditional density is

$$\frac{\mathbb{P}(U \leq u, V \leq v)}{\mathbb{P}(U \leq u)} = \frac{C(u, v)}{u}.$$

A joint distribution function $C(u, v)$ with $U[0, 1]$ marginals is known as a **copula**, often used to simulate correlated default events in credit modelling.

8. **Portfolio optimization, relevant for two of the summer projects.** Consider a financial market defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with d assets with **random payoffs** (S^1, \dots, S^d) at time T (which are **linearly independent**) with market prices π_i at $t = 0$ (these assets can include **European call/put options**). Let $Y_i = S_i - \pi_i$, and assume a financial agent can only trade at time zero.

Derive the first order optimality condition for an agent to maximize their **expected utility** $\mathbb{E}(U(b \cdot Y))$ over $b \in \mathbb{R}^d$, where $U : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function with $U''(x) < 0$ and b_i is the position in the i 'th asset and we assume that $\mathbb{E}(U(b \cdot Y)) < \infty$ for all $b \in \mathbb{R}^d$.

Solution. As in first year calculus, we set compute derivatives wrt each b_i and then set the answer to zero:

$$\frac{\partial}{\partial b_i} \mathbb{E}(U(b \cdot Y)) = \mathbb{E}(Y_i U'(b \cdot Y)) = 0$$

for $i = 1..d$, i.e. we have d equations for the d unknowns b_1^*, \dots, b_d^* for the optimal portfolio allocation b^* . Note we can re-write this as

$$\mathbb{E}^{\mathbb{Q}}(Y_i) = 0 \tag{1}$$

where we define a **new probability measure** as $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(\frac{U'(b^* \cdot Y)}{\mathbb{E}^{\mathbb{P}}(U'(b^* \cdot Y))} 1_A)$ for events $A \in \mathcal{F}$, and (as a sanity check) we note that $\mathbb{Q}(\Omega) = 1$.

Under the moment condition stated in the question, it turns out that a unique solution b^* exists if the **no-arbitrage** condition is satisfied: if $\mathbb{P}(b \cdot Y > 0) > 0$ then $\mathbb{P}(b \cdot Y < 0) > 0$. In this case, from (1), we see that under \mathbb{Q} , all contracts are priced according to the market, i.e. $\mathbb{E}^{\mathbb{Q}}(S_i) = \pi_i$. \mathbb{Q} is known as a **risk-neutral measure**. We can solve this maximization problem numerically using e.g. MOSEK convex optimization package in Python (used for summer project).

A common choice is the **exponential utility function** $U(x) = -e^{-\lambda x}$, in which case (for $\lambda = 1$) we are computing $\max_b (-\mathbb{E}(e^{-b \cdot Y})) = \max_b (-\mathbb{E}(e^{b \cdot Y})) = -\min_b \mathbb{E}(e^{b \cdot Y})$, i.e. minus the **minimum of the mgf** of Y .

For the 1d case $d = 1$, if $\pi_1 = p = f(S)$ with market price p and S has density $q(S)$, then we can re-write expected utility as

$$\mathbb{E}(U(f(S) - p)) = \mathbb{E}(-e^{-\lambda(f(S) - p)}) = - \int_0^\infty e^{-\lambda b(f(S) - p)} q(S) dS.$$

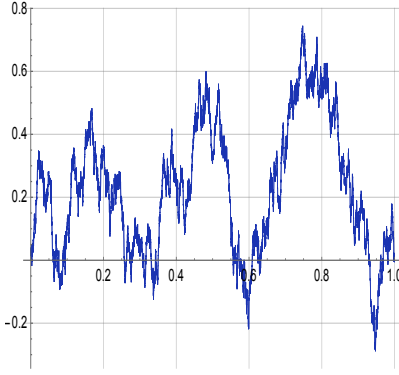


Figure 1: Simulation of a Brownian bridge on $[0, 1]$.

We can evaluate this integral explicitly in certain cases, e.g. if $S \sim \text{Exp}(1)$ and $f(S) = S$, we see that

$$\mathbb{E}(U(b(S-p))) = \mathbb{E}(-e^{-\lambda b(S-p)}) = \int_0^\infty -e^{-\lambda b(S-p)} e^{-S} dS = -\frac{e^{pb\lambda}}{1+b\lambda}$$

if $\lambda b + 1 > 0$, and $-\infty$ otherwise. Differentiating this expression wrt b and setting the answer to zero, we find that

$$b^* = \frac{1-p}{p\lambda} \quad (2)$$

and the risk-neutral density \mathbb{Q} corresponding to b^* is the density of a $\text{Exp}(\frac{1}{p})$ random variable, under which $\mathbb{E}^{\mathbb{Q}}(S) = p$ as claimed.

Note the “fair price” of the stock is $\int_0^\infty S e^{-S} dS = \mathbb{E}(S) = 1$, so (2) says that we **buy** stock when the stock is **underpriced**, and **sell** when the stock is **overpriced**, and $|b^*|$ is smaller when λ is larger i.e. when the trader is more risk-averse.

We can also modify the problem to account for interest rates, **bid-ask spreads**, **finite liquidity** or a **limit order book** structure.