0. Let X be a random variable with $\mathbb{E}(|X|) < \infty$, and let $\mathcal{F}_t$ be a filtration. Show that $M_t = \mathbb{E}(X|\mathcal{F}_t)$ is a martingale

$$\mathbb{E}(M_t|\mathcal{F}_s) \;=\; \mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) \;=\; \mathbb{E}(X|\mathcal{F}_s) \;=\; M_s$$

as required.

1. Let $(X_t)_{t\geq 0}$ denote a martingale. Show that $\mathbb{E}((X_{T_2}-K)^+) \geq \mathbb{E}((X_{T_2}-K)^+)$ if $0 \leq T_1 \leq T_2$.

Solution.

$$\begin{aligned}
\mathbb{E}((X_{T_2}-K)^+) &= \mathbb{E}(\mathbb{E}((X_{T_2}-K)^+|X_{T_1})) \quad \text{(from the \textbf{tower property}}\\
&\geq \mathbb{E}(\mathbb{E}(X_{T_2}-K|X_{T_1})^+)\\
&\quad \text{(from the \textbf{conditional Jensen inequality} applied to the convex function } f(x)=(x-K)^+)\\
&= \mathbb{E}((X_{T_1}-K)^+)\,.
\end{aligned}$$

Hence we see that call option prices with maturity T_2 are $\geq$ call option prices with maturity T_1. This is known as the **convex ordering** condition, which we can write as $\mu_{T_1} \preceq \mu_{T_2}$, where μ_t denotes the density of X_t.

2. Let B^α denote a Brownian motion with $B_0^\alpha \sim \alpha$ (i.e. a random initial starting point with density α with $B^\alpha - B_0^\alpha$ independent of B_0^α). Write down an integral expression for the density of B_t^α.

Solution.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \alpha(x)dx \;=\; \int_{-\infty}^{\infty} R_t(y-x)\alpha(x)dx \;=\; (R_t * \alpha)(y)$$

where $R_t(x)$ denotes the density of Brownian motion.

3. Write an expression for $\mathbb{E}(F(B_1^\alpha)|B_t^\alpha = x)$.

Solution.

$$\mathbb{E}(F(B_1^\alpha)|B_t^\alpha = x) \;=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{(x-y)^2}{2(1-t)}} F(y)dy \;=\; \int_{-\infty}^{\infty} R_{1-t}(y-x)F(y)dy \;=\; (R_{1-t} * F)(x)\,.$$

Now let $M_t = \mathbb{E}(F(B_1^\alpha)|B_t^\alpha)$ for $t \in (0,1]$. Then replacing x with the random B_t^α, we see that

$$M_t \;=\; (R_{1-t} * F)(B_t^\alpha)\,.$$

We wish to choose α so that $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$, for two given distributions μ_0 and μ_1 (both with zero expectations), with μ_0, μ_1 in convex order.

Then

$$M_t \;=\; (R_{1-t} * F)(B_t^\alpha)\,. \tag{1}$$

Let μ be a probability density. We define the **push-forward** $F_\# \mu$ of μ by F as the distribution of $F(X)$ if $X \sim \mu$, so

$$\mathbb{P}(F(X) \leq x) \;=\; \mathbb{P}(X \leq F^{-1}(x)) \;=\; \int_{-\infty}^{F^{-1}(x)} \mu(y)dy\,.$$