

The conditional law of the Bacry-Muzy and Riemann-Liouville log correlated Gaussian fields and their GMC, via Gaussian Hilbert and fractional Sobolev spaces

Martin Forde Benjamin Smith*

November 18, 2025

Abstract

We compute $\mathbb{E}(X_t | (X_s)_{0 \leq s \leq L})$ for the standard Bacry-Muzy log-correlated Gaussian field X with covariance $\log^+ \frac{T}{|t-s|}$, which corrects the finite-horizon prediction formula in Vargas et al.[DRV12]. The problem can be viewed as a linear filtering problem, and we solve the problem by showing that the $L^2(\mathbb{P})$ closure of $\{\int_{[0,L]} \phi(s) X_s ds : \phi \in \mathcal{S}, \text{supp}(\phi) \subseteq [0, L]\}$ is equal to $\{X(\phi) : \phi \in H^{-\frac{1}{2}}, \text{supp}(\phi) \subseteq [0, L]\}$, where $X(\phi)$ is defined as a continuous linear extension of X acting on $\mathcal{S} \subset H^s$, H^s denotes the fractional Sobolev space of order s and \mathbb{P} is the law of the field X on the space of tempered distributions. The explicit formula for the filter is obtained as the solution to a Fredholm integral equation of the first kind with logarithmic kernel. From this we characterize the conditional law of the Gaussian multiplicative chaos (GMC) M_γ generated by X , using that M_γ is measurable with respect to X . We also outline how one can adapt this result for the Riemann-Liouville GMC introduced in [FFGS19], which has a natural application to the Rough Bergomi volatility model in the $H \rightarrow 0$ limit.¹

1 Introduction

Originally pioneered by Kahane[Kah85], Gaussian multiplicative chaos (GMC) is a random measure on a domain of \mathbb{R}^d that can be formally written as

$$M_\gamma(dx) = e^{\gamma X_x - \frac{1}{2}\gamma^2 \mathbb{E}(X_x^2)} dx \quad (1)$$

where X is a Gaussian field with zero mean and covariance $K(x, y) := \mathbb{E}(X_x X_y) = \log^+ \frac{1}{|y-x|} + g(x, y)$ for some bounded continuous function g . X is not defined pointwise because there is a singularity in its covariance, rather X is a random tempered distribution, i.e. an element of the dual of the Schwartz space \mathcal{S} under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of (1) requires a regularizing sequence X^ε of Gaussian processes (with the singularity removed, see e.g. [BBM13] and [BM03] for a description of such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular region, which is summarized in Section 2.3 in [FFGS19], or page 17 in [RV10] and section 3.4 in [Sha16] for a general method in \mathbb{R}^d using a convolution to smooth X). In most of the literature on GMC, the choice of X^ε is a martingale in ε , from which we can then easily verify that $M_\gamma^\varepsilon(A) = \int_A e^{\gamma X_x^\varepsilon - \frac{1}{2}\gamma^2 \text{Var}(X_x^\varepsilon)} dx$ is a martingale, and then obtain a.s. convergence of $M_\gamma^\varepsilon(A)$ using the martingale convergence to a random variable $M_\gamma(A)$ with $\mathbb{E}(M_\gamma(A)) = \text{Leb}(A)$, and with a bit more work we can verify that $M_\gamma(.)$ defines a random measure (see the end of Section 4 on page 18 in [RV10]).

If $\gamma^2 < 2d$, $M_\gamma^\varepsilon(dx) = e^{\gamma X_x^\varepsilon - \frac{1}{2}\gamma^2 \mathbb{E}((X_x^\varepsilon)^2)} dx$ tends weakly to a multifractal random measure M_γ with full support a.s. which satisfies the multifractal property

$$\mathbb{E}(M_\gamma([0, t])^q) = c_q t^{\zeta(q)} \quad (2)$$

for $q \in (1, q^*)$ for some constant $c_q = \mathbb{E}(M_\gamma([0, 1])^q)$, where $q^* = \frac{2}{\gamma^2}$ ² and

$$\zeta(q) = q - \frac{1}{2}\gamma^2(q^2 - q)$$

*Dept. Mathematics, King's College London, Strand, London, WC2R 2LS (Benjamin.Smith@kcl.ac.uk)

¹We would like to thank Juhan Aru, Janne Junnila, Vincent Vargas and Lauri Viitasaari for helpful discussions.

²see Lemma 3 in [BM03] to see why the critical q value is q^*

and $\mathbb{E}(M_\gamma([0, t])^q) = \infty$ if $q > \frac{2}{\gamma^2}$, see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). Moreover, we can show that the support of M_γ is a so-called γ -thick points of X , i.e. points such that $\lim_{\varepsilon \rightarrow 0} \frac{X_\varepsilon^\varepsilon}{\log \frac{1}{\varepsilon}} = \gamma$ (see e.g. section 2 in [Aru17], [Ber17] and page 7 in [RV16] for more on this), and for $g \equiv 0$, explicit expressions are known for the Mellin transform of the law of $M_\gamma([0, 1])$ (see e.g. [Ost09], [Ost13], [Ost18]), which show that $\log M_\gamma([0, 1])$ has an infinitely divisible law, and an explicit formula for sampling the law of the total mass of the GMC on the interval is given in [RZ17].

M_γ is the zero measure for $\gamma^2 = 2d$ and $\gamma^2 > 2d$; in these cases a different re-normalization is required to obtain a non-trivial limit. Specifically, for $\gamma^2 = 2d$, we obtain a non-trivial limit by considering $\sqrt{\log \frac{1}{\varepsilon}} \cdot M_\varepsilon^{\gamma=2}$ as $\varepsilon \rightarrow 0$ or the “derivative measure” $\frac{d}{d\gamma} e^{\gamma X_\varepsilon^\varepsilon - \frac{1}{2}\gamma^2 \text{Var}(X_\varepsilon)}|_{\gamma=\sqrt{2d}}$. [DRSV14] show that both these objects tend weakly to the same measure μ' as $\varepsilon \rightarrow 0$, and in 2d Aru et al.[APS19] have shown that $\frac{M_\gamma}{2-\gamma} \rightarrow 2\mu'$ in probability as γ tends to the critical value of 2, and the critical γ -value is particularly important in Liouville quantum gravity (again see [DRSV14] for further discussion). One can also construct a GMC for the super-critical phase, using an independent stable subordinator time-changed by a sub-critical GMC (see section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for γ -values greater than $\sqrt{2}$, which is closely related to the non-standard branch of gravity in conformal field theory.

In the sub-critical case, using a limiting argument it can be shown that M_γ satisfies the “master equations”: $M(X + f, dz) = e^{\gamma f(z)} M(X, dz)$ and $\mathbb{E}(\int_D F(X, z) M_\gamma(dz)) = \mathbb{E}(\int_D F(X + \gamma K(z, .), z) dz)$ for any measurable function F and any interval D , which comes from the Cameron-Martin theorem for Gaussian measures and the notion of *rooted measures* and the disintegration theorem (see section 2.1 in [Aru17] for a nice discussion on this). Moreover, either of these two can equations can be taken as the definition of GMC, and they uniquely determine M_γ as a measurable function of X , and hence also uniquely fix its law (see also Theorem 6 in Shamov[Sha16]).

GMC also has a natural and important application in Liouville Quantum Field Theory; LQFT is a 2d model of random surfaces, which (formally) we can view as a random metric in the context of quantum gravity, where we weight the classical free field action with an interaction term given by the exponential of a GMC and can be viewed as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time.

2 Construction of the standard Bacry-Muzy GMC on the line

Define the Gaussian process $\omega_\varepsilon(t)$ as in Eq 7 in [BBM13] with $\lambda = 1$ (except here use ε instead of l), and set $X_t^\varepsilon := \omega_\varepsilon(t) - \mathbb{E}(\omega_\varepsilon(t))$, so

$$X_t^\varepsilon = \int_{(u,s) \in \mathcal{A}_\varepsilon(t)} dW(u, s) \quad (3)$$

where $dW(u, s)$ is 2-dimensional Gaussian white noise with variance $s^{-2}duds$, and $\mathcal{A}_\varepsilon(t)$ is triangular region defined in Eq 8 (and Figure 1) in [BBM13]. Then

$$R_\varepsilon(s, t) := \mathbb{E}(X_s^\varepsilon X_t^\varepsilon) = \begin{cases} \log \frac{T}{\tau} & \varepsilon \leq \tau \leq T \\ \log \frac{T}{\varepsilon} + 1 - \frac{\tau}{\varepsilon} & \tau \leq \varepsilon \\ 0 & \tau > T \end{cases} \quad (4)$$

where $\tau = |t - s|$ (see Eq 10 in [BBM13]), and one can easily verify that

$$R_\varepsilon(s, t) \leq \log \frac{T}{\tau} = R(s, t) \quad (5)$$

for $s, t \in [0, T]$ (see Eq 25 in [BM03]). Using (3), we also see that

$$\mathbb{E}(X_t^\varepsilon X_s^{\varepsilon'}) = \mathbb{E}\left(\int_{(u,v) \in \mathcal{A}_\varepsilon(t)} dW(u, v) \int_{(u,v) \in \mathcal{A}_{\varepsilon'}(s)} dW(u, v)\right) = \int_{\mathcal{A}_\varepsilon(t) \cap \mathcal{A}_{\varepsilon'}(s)} \frac{1}{v^2} dudv = \mathbb{E}(X_s^\varepsilon X_t^\varepsilon)$$

for $0 < \varepsilon' \leq \varepsilon$ (i.e. the answer does not depend on ε'). We now define the measure

$$M_\gamma^\varepsilon(dt) = e^{\gamma X_t^\varepsilon - \frac{1}{2}\gamma^2 \text{Var}(X_t^\varepsilon)} dt.$$

One can easily verify that $M_\gamma^\varepsilon(A)$ is a backwards martingale with respect to the filtration $\mathcal{F}_\varepsilon := \sigma(W(A, B) : A \subset \mathbb{R}^+, B \subseteq [\varepsilon, \infty))$ (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and

$$\sup_{\varepsilon > 0} \mathbb{E}(M_\gamma^\varepsilon(A)^q) < \infty \quad (6)$$

(Lemma 3 i) in [BM03]), so from the martingale convergence theorem, $M_\gamma^\varepsilon(A)$ converges to some random variable $M_\gamma(A)$ in L^q for $q \in (1, q^*)$, and from the reverse triangle inequality this implies that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}((M_\gamma^\varepsilon(A))^q) = \mathbb{E}(M_\gamma(A)^q) \quad (7)$$

Moreover, one can show that $M_\gamma(\cdot)$ defines measure (see e.g. end of Section 4 on page 18 in [RV10]), and since $M_\gamma^\varepsilon(A) \rightarrow M_\gamma(A)$ a.s. for any Borel set A this implies weak convergence of M_γ^ε to M_γ a.s. (from e.g. Theorem 3.1 parts a) and f) in Ethier&Kurtz[EK86]).

Moreover M_γ is multifractal, i.e. $\mathbb{E}(|M_\gamma([0, t])|^q) = c_{q,T} t^{\zeta(q)}$ (see e.g. Lemma 4 in [BM03]) for some finite constant $c_{q,T} > 0$, depending only on q and T . For integer $q \geq 1$, we also note that

$$\begin{aligned} \mathbb{E}(M_\gamma(A)^q) &= \int_A \dots \int_A e^{\gamma^2 \sum_{1 \leq i < j \leq q} \log \frac{T}{|u_i - u_j|}} du_i \dots du_q \\ &= \int_A \dots \int_A e^{\gamma^2 q(q-1) \log T + \sum_{1 \leq i < j \leq q} \log \frac{1}{|u_i - u_j|}} du_i \dots du_q \\ &= T^{\gamma^2 q(q-1)} \int_A \dots \int_A e^{\sum_{1 \leq i < j \leq q} \log \frac{1}{|u_i - u_j|}} du_i \dots du_q \end{aligned} \quad (8)$$

so we see that

$$c_{q,T} = c_q T^{\gamma^2 q(q-1)}$$

where $c_q := c_{q,1}$, and this also holds for non-integer q (see e.g. Theorem 3.16 in [Koz06]).

3 The conditional law of the standard log correlated Gaussian field

Consider a standard log-correlated Gaussian field Z on \mathbb{R} with covariance $R(s, t) = \log^+ \frac{T}{|t-s|}$. From the Minlos-Bochner theorem, we know that the law of Z is a Gaussian measure on the space \mathcal{S}' of *tempered distributions* (see e.g. [DRSV17] and Appendix A in [FFGS19] for more details on tempered distributions) which is the dual of the Schwartz space \mathcal{S} (see e.g. section 2.2 in [DRSV14] and Theorem 2.1 in [BDW17]). Moreover, \mathcal{S} is a Montel space and thus is reflexive, i.e. $(\mathcal{S}')'$ is isomorphic to \mathcal{S} using the canonical embedding of \mathcal{S} into its bi-dual $(\mathcal{S}')'$. From here on, we are only concerned with the restriction of Z to $[0, T]$ (on which the covariance of Z is just $\log \frac{T}{|t-s|}$, so we set Z equal to zero outside this interval for simplicity).

Proposition 3.1 X^ε tends to X in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definitions), where X has the same law as the field Z defined above.

Proof. $0 \leq R_\varepsilon(s, t) \leq R(s, t)$ for $s, t \in [0, T]$ (see (5)), so from the dominated convergence theorem, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{[0, T]^2} \phi_1(s) \phi_2(t) R_\varepsilon(s, t) ds dt = \int_{[0, T]^2} \phi_1(s) \phi_2(t) R(s, t) ds dt \quad (9)$$

for any $\phi_1, \phi_2 \in \mathcal{S}$, where $R_\varepsilon(s, t)$ is defined as in (4). Similarly, for any sequence $\phi_k \in \mathcal{S}$ with $\|\phi_k\|_{m,j} \rightarrow 0$ for all $m, j \in \mathbb{N}_0^n$ for any $n \in \mathbb{N}$ (i.e. under the Schwartz space semi-norm defined in Eq 1 in [BDW17])

$$\lim_{k \rightarrow \infty} \int_{[0, T]^2} \phi_k(s) \phi_k(t) R(s, t) ds dt = 0 \quad (10)$$

since $\nu(A) := \int_A R(s, t) ds dt$ is a bounded non-negative measure (since $\int_0^T \int_0^t R(s, t) ds dt < \infty$), and the convergence here implies in particular that ϕ_k tends to ϕ pointwise, so we can use the bounded convergence theorem. Thus if we define

$$\begin{aligned} \mathcal{L}_{X^\varepsilon}(f) &:= \mathbb{E}(e^{i(f, X^\varepsilon)}) = e^{-\frac{1}{2} \int_{[0, T]^2} f(s) f(t) R_\varepsilon(s, t) ds dt} \\ \mathcal{L}(f) &:= e^{-\frac{1}{2} \int_{[0, T]^2} f(s) f(t) R(s, t) ds dt} \end{aligned}$$

for $f \in \mathcal{S}$, then from (9) and (10) and Lévy's continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW17]), we see that $\mathcal{L}_{X^\varepsilon}(f)$ tends to $\mathcal{L}(f)$ pointwise and $\mathcal{L}(\cdot)$ is continuous at zero, then there exists a generalized random field

X (i.e. a random tempered distribution, such that $L_X = L$ and X^ε tends to X in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definition). ■

In general, the conditional expectation of a random variable is equal to its projection onto the Gaussian Hilbert space (sub-Hilbert space of $L^2(\Omega, \mathcal{F}, \mathbb{P})$) generated by the variables on which we are conditioning. To this end, we let \bar{F} denote the Hilbert space given by the $L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$ closure of

$$F = \{X(\phi) : \phi \in \mathcal{S}, \text{supp}(\phi) \subseteq [0, L]\}$$

where $\mathcal{F}_L = \sigma((X_u)_{0 \leq u \leq L})$. The closure here is necessary because the notion of orthogonal projection requires the Hilbert space structure, and there is no guarantee that the conditional expectation $\mathbb{E}(X(\psi)|\mathcal{F}_L)$ will be a random variable of the form $\int_{[0, L]} X_s \phi(s) ds$ with $\phi \in \mathcal{S}$.

In order to characterize \bar{F} , we first note that

$$\mathbb{E}\left[\left(\int X_s \phi(s) ds\right)^2\right] = \int \int R(s, t) \phi(s) \phi(t) ds dt.$$

From Eqs 2.1 in [DRV12], we also know that

$$c \|\phi\|_{H^{-\frac{1}{2}}}^2 \leq \int \int R(s, t) \phi(s) \phi(t) ds dt \leq C \|\phi\|_{H^{-\frac{1}{2}}}^2 \quad (11)$$

where $0 < c < C < \infty$. Let H^s denotes the fractional Sobolev space of order s (see e.g. Section 2.2 in [JSW18] for definitions). Then we can put two inner products on the linear space \mathcal{S} of Schwarz functions:

1. $\langle \phi, \psi \rangle_{H^{-\frac{1}{2}}} := \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}} \hat{\phi}(k) \bar{\hat{\psi}}(k) dk$ (i.e. the standard inner product on $H^{-\frac{1}{2}}$)
2. $\langle \phi, \psi \rangle := \mathbb{E}[X(\phi)X(\psi)] = \int \int \phi(s) \psi(t) R(s, t) ds dt$

Eq 2.2 in [DRV12] shows that these two inner products are equivalent and thus generate the same topologies on \mathcal{S} .

We now make the following observations:

- Let $\phi \in H^{-\frac{1}{2}}$, with $\text{supp}(\phi) \subseteq [0, L]$. \mathcal{S} is dense in $H^{-\frac{1}{2}}$, so there exists a sequence $\phi_n \in \mathcal{S}$ with $\text{supp}(\phi_n) \subseteq [0, L]$ such that $\|\phi_n - \phi\|_{H^{-\frac{1}{2}}} \rightarrow 0$, and ϕ is a Cauchy sequence in $H^{-\frac{1}{2}}$ so (by the equivalence of norms) $X(\phi_n)$ is a Cauchy sequence in \bar{F} , and thus converges to some Y in \bar{F} . This defines $X(\phi) := Y$ as a continuous linear extension of X from \mathcal{S} to the larger space $H^{-\frac{1}{2}}$, which we will also often write as $\int \phi(t) X_t dt$. To check that $X(\phi)$ is uniquely specified, consider two such sequences ϕ_n and ϕ'_n . Then from the triangle inequality

$$\|\phi_n - \phi'_n\|_{H^{-\frac{1}{2}}} \leq \|\phi_n - \phi\|_{H^{-\frac{1}{2}}} + \|\phi - \phi'_n\|_{H^{-\frac{1}{2}}} \rightarrow 0$$

and thus (by the equivalence of norms) we have $\|X(\phi_n) - X(\phi'_n)\|_{L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})} = \|X(\phi_n) - X(\phi'_n)\|_{\bar{F}} \rightarrow 0$.

- Conversely, for any $Z \in \bar{F}$, there exists a sequence $\phi_n \in \mathcal{S}$ such that $X(\phi_n)$ converges to $Z \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$, so ϕ_n is a Cauchy sequence with respect to the second norm defined above, and hence also a Cauchy sequence with respect to the $H^{-\frac{1}{2}}$ norm (by the equivalence of the two norms). $H^{-\frac{1}{2}}$ is a Hilbert space so Cauchy sequences in $H^{-\frac{1}{2}}$ converge i.e. there exists a ϕ in $H^{-\frac{1}{2}}$ such that $\phi_n \rightarrow \phi \in H^{-\frac{1}{2}}$.

Thus we have shown that

$$\bar{F} = \{X(\phi) : \phi \in H^{-\frac{1}{2}}, \text{supp}(\phi) \subseteq [0, L]\}$$

where we are using the extension of X to $H^{-\frac{1}{2}}$ on the right hand side here as defined in the first bullet point above.

Moreover (since $\mathbb{E}(X(\psi)|\mathcal{F}_L) \in \bar{F}$) we see that for any $\psi \in \mathcal{S}$

$$\mathbb{E}(X(\psi)|\mathcal{F}_L) = \int_{[0, L]} X_s k_\psi(s) ds := X(k_\psi)$$

for some $k_\psi(s) \in H^{-\frac{1}{2}}([0, L])$, where $X(\cdot)$ in the final expression is the linear extension we have just defined. This analysis shows that \bar{F} is isometrically isomorphic to the set of functions in $H^{-\frac{1}{2}}$ with support in $[0, L]$.

Moreover, we can now extend the inner product to $H^{-\frac{1}{2}}$ as

$$\langle \phi, \psi \rangle = \lim_{n \rightarrow \infty} \mathbb{E}[X(\phi_n)X(\psi_n)] = \lim_{n \rightarrow \infty} \int \int \phi_n(s)\psi_n(t)R(s,t)dsdt$$

where $\phi_n, \psi_n \in \mathcal{S}$ and $\phi_n \rightarrow \phi$ in $H^{-\frac{1}{2}}$ and $\psi_n \rightarrow \psi$ in $H^{-\frac{1}{2}}$.

Proposition 3.2 $X \in H^{-\frac{1}{2}-\delta}$ a.s. for any $\delta > 0$.

Proof. The proof is almost identical to Proposition 2.1 in [FFGS19], but since some of its arguments are needed for the next Proposition as well, we have put a proof in Appendix A. ■

Remark 3.1 One can actually show the stronger result that $X \in H^{-\delta} \subset H^{-\frac{1}{2}}$ a.s. for any $\delta > 0$, but we will not need this here (see also [BDW17]).

Proposition 3.3 $X^\varepsilon \rightarrow X$ in $H^{-\frac{1}{2}-\delta}$ in probability for any $\delta > 0$, where X^ε is defined as in (3).

Proof. See Appendix B. ■

We know that for any $\psi \in \mathcal{S}$ with $\text{supp}(\psi) \subseteq [L, T]$, the conditional expectation $\mathbb{E}(X(\psi)|\mathcal{F}_L) = X(k_\psi)$ minimizes

$$\mathbb{E}((X(\psi) - Y)^2)$$

over all $Y \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$, and $\mathbb{E}((\int_{[L,T]} X_t \psi(t) dt - \mathbb{E}(\int_{[L,T]} X_t \psi(t) dt | \mathcal{F}_L))Z) = 0$ for all $Z \in \mathcal{F}_L$, so in particular setting $Z = \int_{[0,L]} \psi_2(s) X_s ds$ for $\psi_2 \in \mathcal{S}$ with $\text{supp}(\psi_2) \subseteq [0, L]$, we see that

$$\begin{aligned} 0 &= \mathbb{E}((X(\psi) - X(k_\psi))X(\psi_2)) \\ &= \mathbb{E}\left(\left(\int_{[L,T]} \psi(t) X_t dt - \int_{[0,L]} k_\psi(u) X_u du\right) \int_{[0,L]} \psi_2(s) X_s ds\right) \\ &= \int_{[L,T]} \int_{[0,L]} \psi(t) \psi_2(s) R(t-s) ds dt - \int_{[L,T]} \int_{[0,L]} R(s-u) k_\psi(u) \psi_2(s) du ds. \end{aligned} \quad (12)$$

In (14) below we construct an explicit solution $k_t(\cdot)$ to

$$0 = \mathbb{E}\left(\left(X_t - \int_{[0,L]} k(u) X_u du\right) X_s\right) = R(s, t) - \int_{[0,L]} R(u, s) k_t(u) du \quad (13)$$

for $s \in [0, L]$, with $k_t \in \text{supp}(\psi) \subseteq [t, T]$, which implies that (12) holds if we set $k_\psi(u) = \int_{[L,T]} \psi(t) k_t(u) dt$.

Proposition 3.4 The covariance operator $R\phi = \int_0^T R(s, t)\phi(s)ds$ acting on $H^{-\frac{1}{2}}$ is positive definite, and $\int_0^T R(s, t)\phi(s)ds = 0$ if and only if $\phi \equiv 0$ Lebesgue a.e.

Proof. From the discussion on page 4, we know that bilinear form R is (up to an equivalence) the inner product on $H^{-\frac{1}{2}}$ so it has to be positive definite (from the definition of a norm), and thus $\int_0^T R(s, t)\phi(s)ds \neq 0$ if $\phi \neq 0$, since otherwise $R(\phi, \phi) = \int_0^T \int_0^T R(s, t)\phi(s)ds\phi(t)dt = 0$. ■

The integral equation in (13) (with t fixed) is the well known *Wiener-Hopf equation*. We refer the reader to [Poor94] for more details on the Wiener-Hopf equation in the context of ordinary Gaussian processes.

Corollary 3.5 Proposition 3.4 shows that the Wiener-Hopf equation in (13) has a unique solution.

If $t \leq T$ (so we can replace \log^+ with \log), we can re-write (12) as

$$\int_{[0,L]} k_t(u) \log \frac{T}{|s-u|} du = f(s) := \log \frac{T}{t-s}$$

and we see that this is now a Fredholm integral equation of the 1st kind with logarithmic kernel, which can be solved explicitly by a minor extension of page 299 in [EK00] (who consider $T = 1$) to give

$$k_t(u) = \frac{1}{\pi^2} \int_0^L \frac{\sqrt{v(L-v)}}{\sqrt{u(L-u)}} \frac{f'(v)}{u-v} dv + \frac{c_t}{\pi \sqrt{u(L-u)}} = \frac{(c_t - 1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}} \quad (14)$$

where the integral in the second expression is understood in the principal value sense, and

$$c_t = \int_0^L k_t(u) du = \frac{1}{\pi(\log(\frac{1}{4}L) - \log T)} \int_0^L \frac{\log \frac{t-v}{T}}{\sqrt{v(L-v)}} dv < \infty.$$

We now verify that $k_\psi(u) \in H^{-\frac{1}{2}}$. To this end, we first note that

$$\pi \log \frac{L}{4} - \pi \log T = \int_0^L \frac{\log \frac{L-v}{T}}{\sqrt{v(L-v)}} dv \leq \int_0^L \frac{\log \frac{t-v}{T}}{\sqrt{v(L-v)}} dv \leq \int_0^L \frac{\log \frac{t}{T}}{\sqrt{v(L-v)}} dv \leq \pi \log t - \pi \log T.$$

$$\frac{(c_t - 1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}} 1_{u \in [0,L]} 1_{t \in [L,T]} \leq h(u,t) = \frac{c_1}{(t-u)\sqrt{u(L-u)}} 1_{u \in [0,L]} 1_{t \in [L,T]}$$

for some constant c_1 . We know that

$$\int_{[L,T]} \left(\int_{[0,L]} |\psi(t)h(u,t)|^p du \right)^{\frac{1}{p}} dt \leq \|\psi\|_{L^\infty} \int_{[L,T]} \left(\int_{[0,L]} |h(u,t)|^p du \right)^{\frac{1}{p}} dt \quad (15)$$

and setting $p = \frac{3}{2}$ we find that

$$\int_{[0,L]} |h(u,t)|^p du = G(t) := \text{const.} \times \frac{2t-L}{t(t-L)^{\frac{5}{4}}}$$

which implies that

$$\int_{[L,T]} G(t)^{\frac{1}{p}} dt < \infty$$

so for $p = \frac{3}{2}$ the double integral in (15) is finite, so (from the Minkowski integral inequality) $\int_{[L,T]} h(.,t) dt$ and thus $\int_{[L,T]} \psi(t)k_t(.) dt \in L^p$, and hence its Fourier transform is in $L^q = L^3$ where $1/p + 1/q = 1$, and thus is $O(|\xi|^{-\frac{1}{3}-\varepsilon})$ for $\xi \gg 1$ and $O(|\xi|^{-\frac{1}{3}+\varepsilon})$ for $\xi \ll 1$.

Hence

$$\begin{aligned} \|k_\psi\|_{H^{-\frac{1}{2}}} &= \int_{-\infty}^{\infty} (1 + |\xi^2|)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\xi u} \int_{[L,T]} \psi(t)k_t(u) 1_{u \in [0,L]} dt du d\xi \\ &= \int_{-\infty}^{\infty} (1 + |\xi^2|)^{-\frac{1}{2}} \int_{[0,L]} e^{i\xi u} \int_{[L,T]} \psi(t)k_t(u) dt du d\xi < \infty \end{aligned}$$

which verifies the validity of our explicit solution for $k_u(t)$.

Remark 3.2 Corollary 3.3 in [DRV12] gives the following nice prediction formula for a log-correlated Gaussian field X with covariance $\log \frac{T}{|t-s|}$:³

$$\mathbb{E}(X_t | (X_s)_{s \leq 0}) = \frac{1}{\pi} \int_{-\infty}^0 \frac{\sqrt{t}}{(t-s)\sqrt{-s}} X_s ds$$

which we can verify satisfies the associated Wiener-Hopf equation (and is also very similar to the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the $H \rightarrow 0$ limit). However the prediction formula for the finite history case stated in Theorem 3.5 in [DRV12] appears to be wrong since numerical tests confirm that it does not satisfy the Wiener-Hopf equation. Our linear filter $\int_{[0,L]} k_t(u) X_u du$ corrects this formula for the case when $L+t \leq T$.

Remark 3.3 Clearly if $t-L > T$, the history of X over $[0,L]$ is of no use for prediction since in this case $\mathbb{E}(X_s X_t) = 0$ for $s \in [0,L]$, and the conditioned process then has the same law as the unconditioned process.

³i.e. \log not \log^+

3.1 The conditional covariance

We use $\mathbb{E}_L(\cdot)$ as shorthand for $\mathbb{E}(\cdot | (X_u)_{0 \leq u \leq L})$. Then from the tower property we see that

$$\begin{aligned} & \mathbb{E}((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \\ &= \mathbb{E}(\mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s)))) \\ &= \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \end{aligned}$$

and the final equality follows since the conditional covariance of a Gaussian process or field is deterministic, and does not depend on its history. Given $k_t(u)$, we can now compute the conditional covariance in the final line explicitly (for $s, t \in [L, T]$) as

$$\begin{aligned} R_L(s, t) &:= \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \\ &= \mathbb{E}((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \\ &= \mathbb{E}((X_t - \int_{[0, L]} k_t(u) X_u du)(X_s - \int_{[0, L]} k_s(v) X_v dv)) \\ &= R(s, t) - \int_{[0, L]} k_t(u) R(u, s) du - \int_{[0, L]} k_s(v) R(v, t) dv + \int_{[0, L]} \int_{[0, L]} k_t(u) k_s(v) R(u, v) dudv. \end{aligned}$$

4 Application to Gaussian multiplicative chaos

4.1 Rooted measures

Proposition 4.1 (see also Lemma 2.1 in [Aru17] and Theorems 4 and 17 in [Sha16]). We have the following “master equation” for any bounded continuous function F on $H^{-\frac{1}{2}-\delta} \times [0, T]$ (under the product topology induced by the Hilbert space norm on $H^{-\frac{1}{2}-\delta}$ and the usual Euclidean metric on $[0, T]$):

$$\frac{1}{T} \mathbb{E} \left(\int_0^T F(X, t) M_\gamma(dt) \right) = \frac{1}{T} \mathbb{E} \left(\int_0^T F(X + \gamma R(t, .), t) dt \right). \quad (16)$$

Proof. See Appendix C.⁴ ■

Corollary 4.2 M_γ is measurable with respect to X .

Proof. $\mathcal{H} = H^{-\frac{1}{2}-\delta} \times [0, T]$ is a metric space, so if μ and ν are two finite Borel measures on \mathcal{H} then $\int f d\mu = \int f d\nu$ for all $f \in C_b(\mathcal{H})$ means that $\mu = \nu$, so the left hand side of (16) uniquely defines a measure \mathbb{P}^* on $\mathcal{H} \times [0, D]$ which satisfies

$$\frac{1}{T} \mathbb{E} \left(\int_0^T F(X, t) M_\gamma(dt) \right) = \int \int F(\omega, t) \mathbb{P}^*(d\omega, dt)$$

where

$$\begin{aligned} \mathbb{P}^*(d\omega, dt) &:= \frac{1}{T} \mathbb{E}(1_{X \in d\omega} M_\gamma(\omega, ds)) = \frac{1}{T} \mathbb{Q}^X(d\omega) M_\gamma(\omega, dt) = \frac{1}{T} \mathbb{E}(1_{X + \gamma R(t, .) \in d\omega}) dt \\ &= \mathbb{P}(X + \gamma R(t, .) \in d\omega) \frac{1}{T} dt \end{aligned}$$

where \mathbb{Q}^X denotes the law of X on $H^{-\frac{1}{2}-\delta}$.

Moreover, if $F \equiv 1$, $\frac{1}{T} \mathbb{E} \left(\int_0^T F(X, t) M_\gamma(dt) \right) = 1$, so $\mathbb{P}^*(d\omega, dt)$ is a probability measure, known as a *rooted* or *Peyrière measure* (see [Aru17] and [Sha16] for more on this). Moreover, using a similar argument to the third bullet point in Appendix C, we know that the conditional law of \mathbb{P}^* given X is $M_\gamma(dt)/M_\gamma([0, T])$ and from the disintegration theorem, we know that this (probability) measure is a measurable with respect to X . Then using a similar argument to the second bullet point in Appendix C, if we take the sample space Ω to be $H^{-\frac{1}{2}-\delta}$ with σ -algebra $\sigma(H^{-\frac{1}{2}-\delta})$, then the “tilted” probability measure $\mathbb{Q}_\gamma^X(d\omega) := \frac{1}{T} M_\gamma([0, T]) \mathbb{Q}^X(d\omega)$ on (Ω, \mathcal{F}) is the marginal law of \mathbb{P}^* on $H^{-\frac{1}{2}-\delta}$ (where \mathbb{Q}^X is the law of X on $H^{-\frac{1}{2}-\delta}$) and $\mathbb{Q}_\gamma^X \ll \mathbb{Q}^X$, so $\frac{1}{T} M_\gamma([0, T])(\omega)$ is the (a.s.) unique Radon-Nikodym derivative of \mathbb{Q}_γ^X with respect to \mathbb{Q}^X , which is a measurable function of ω . Thus we have shown that $M_\gamma(dt)/M_\gamma([0, T])$ and $M_\gamma([0, T])$ are measurable wrt X and thus so is M_γ . ■

⁴We thank Juhani Aru for his help with multiple parts of this proof.

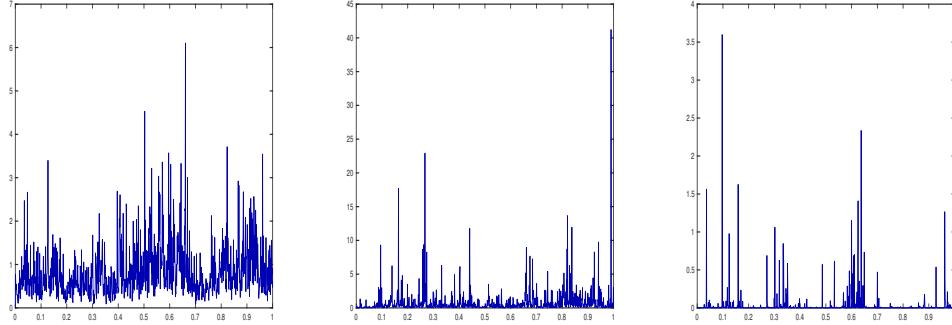


Figure 1: Here we have plotted a Monte Carlo simulation of the multifractal random measure $M_\gamma(dt)$ on $[0, 1]$ with $\gamma = 0.20, 0.45$ and 1 using the regularized autocovariance $\log^+ \frac{T}{|t|+\varepsilon}$ for $\varepsilon = .000001$, and we see greater intermittency as γ increases.

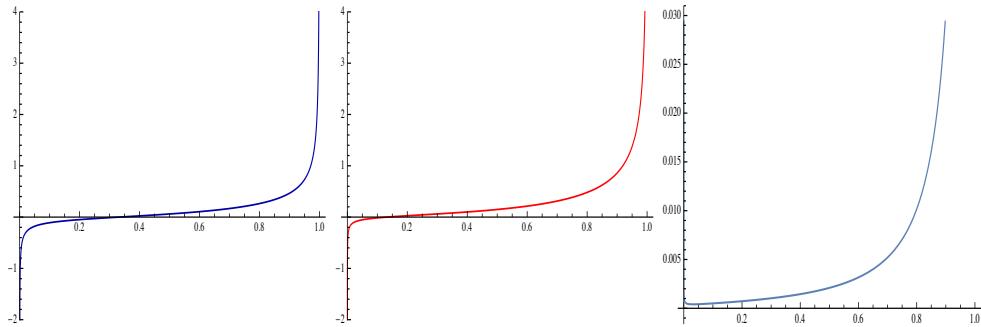


Figure 2: In the first three graphs we have plotted the optimal linear filter $k(u)$ in (14) associated with the multifractal random walk with $L = 1, T = 2$ for $t = 2, 1.5$ and 1.00001 respectively, and the numerics confirm that the Wiener-Hopf equation is satisfied (Mathematica code available on request), and $k(u)$ is U-shaped and strictly positive for all $u \in [0, L]$ for t sufficiently small

4.2 The conditional law of M_γ

From the Corollary above, $M_\gamma(dt)$ is a measurable wrt X , so M_γ given $(X)_{0 \leq s \leq L}$ is just obtained as

$$M_\gamma((X)_{0 \leq s \leq L} \oplus X', dt) \quad (17)$$

where \oplus denotes concatenation, and X' is a Gaussian field (which is also a random element of \mathcal{S}') on $[L, T]$ with mean $\mathbb{E}_L(X_t)$ and covariance $R_L(s, t)$. This then uniquely specifies the law of M_γ conditioned on its history over $[0, L]$.

4.3 Conditional law of the Riemann-Liouville field

Formally letting $H \rightarrow 0$ in the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the $H \rightarrow 0$ limit, we obtain the following conditional law for the Riemann-Liouville field Z defined in section 2 in [FFGS19]:

Proposition 4.3 Z has conditional mean and covariance given by

$$\begin{aligned} \mathbb{E}(Z_u | (Z_v)_{0 \leq v \leq t}) &= \int_0^t \bar{k}(s) Z_s ds \\ \text{Cov}(Z_s, Z_u | (Z_v)_{0 \leq v \leq t}) &= \int_t^{s \wedge u} (u-v)^{-\frac{1}{2}} (s-v)^{-\frac{1}{2}} dv \end{aligned} \quad (18)$$

for $u \geq t$, where $\bar{k}(s) = \frac{1}{\pi} \left(\frac{u-t}{t-s} \right)^{\frac{1}{2}} \frac{1}{u-s}$.

Remark 4.1 This is essentially the same type of linear filter that we have obtained in section 3 for the Bacry-Muzry field. To make this rigorous, we can consider $Y_t = e^{Z_t}$; then one can verify that

Y is a strictly stationary Gaussian field with covariance $R_Y(s, t) = R(\tau) := 2\tanh^{-1}(e^{-\frac{1}{2}|\tau|})$ where $\tau = t - s$, and from Parseval's theorem (similar to Eq 2.1 in [DRV12]) we obtain

$$\int \int \phi(t)\phi(s)R_Y(s, t)dsdt = \int \hat{R}(k) |\hat{\phi}(k)|^2 dk$$

where $\hat{R}(k) = \frac{-iH_{-\frac{1}{2}-ik}+iH_{-\frac{1}{2}+ik}+2\pi\tanh(k\pi)}{k\sqrt{2\pi}}$ and H_n denotes the n th harmonic number. Then $\hat{R}(|k|)$ is continuous, strictly positive and decreasing with $\hat{R}(0) < \infty$ and $\hat{R}(|k|) \sim \frac{\sqrt{\pi}}{|k|\sqrt{2}} \sim \text{const.} \times (1 + |k|^2)^{-\frac{1}{2}}$ as $|k| \rightarrow \infty$. Hence (11) still holds with R replaced by R_Y and we can then repeat our previous arguments to make (18) rigorous (after transforming back from Y to Z). In [FFGS19] we define the GMC associated with Z (which we call ξ_γ) and one can show that ξ_γ is also measurable with respect to Z so (17) still holds with M_γ replaced by ξ_γ and X replaced by Z .

References

- [APS19] Aru, J., E.Powell and A.Sepulveda, “Critical Liouville Measure as a limit of Sub-critical measures”, *Electron. Commun. Probab.*, Volume 24 (2019), paper no. 18, 16 pp.
- [Aru17] Aru, J., “Gaussian Multiplicative Chaos through the lens of the 2D Gaussian Free Field”, to appear in IRS 2017 special issue in *Markov. Process Relat.*
- [BDM01] Bacry, E., J.Delour, and J.Muzy, “Multifractal Random Walks”, *Phys. Rev. E*, 64, 026103-026106, 2001.
- [BDM01b] Bacry, E., J.Delour, and J.Muzy, “Modelling Financial time series using multifractal random walks”, *Physica A*, 299, 84-92, 2001.
- [BBM13] Bacry, E., Rachel Baile, and J.Muzy, “Random cascade model in the limit of infinite integral scale as the exponential of a nonstationary $1/f$ noise: Application to volatility fluctuations in stock markets”, *PHYSICAL REVIEW E* 87, 042813, 2013.
- [BM03] Bacry, E., and J.Muzy, “Log-Ininitely Divisible Multifractal Process”, *Commun. Math. Phys.*, 236, 449-475, 2003.
- [Ber17] Berestycki, N., “Introduction to the Gaussian Free Field and Liouville Quantum Gravity”, draft lecture notes, updated version, Dec 2017.
- [BJRV14] Barral, J., Jin, X., R.Rhodes, SV.Vargas, “Gaussian multiplicative chaos and KPZ duality”, *Communications in Mathematical Physics*, 2014.
- [BDW17] Bierme, H., O.Durieu, and Y.Wang, “Generalized Random Fields and Lévy’s continuity Theorem on the space of Tempered Distributions”, preprint, 2017.
- [DRV12] Duchon, J., R.Robert and V.Vargas, “Forecasting Volatility With The Multifractal Random Walk Model”, *Mathematical Finance*, 22,1, 83-108.
- [DRSV14] Duplantier, D., R.Rhodes, S.Sheffield, and V.Vargas, “Renormalization of Critical Gaussian Multiplicative Chaos and KPZ Relation”, *Communications in Mathematical Physics*, August 2014, Volume 330, Issue 1, pp 283-330.
- [DRSV17] Duplantier, D., R.Rhodes, S.Sheffield, and V.Vargas, “Log-correlated Gaussian Fields: An Overview”, *Geometry, Analysis and Probability*, August pp 191-216, 2017.
- [DS99] Decreusefond, L. and A.S. Ustunel, “Stochastic analysis of the fractional Brownian motion”, *Potential Anal.*, 10: 177-214, 1999.
- [EK00] Estrada R., Kanwal R.P., “Integral Equations with Logarithmic Kernels”, in *Singular Integral Equations*, Birkhauser, Boston, MA, 2000.
- [EK86] Ethier, S.N. and T.G.Kurtz, “Markov Processes: characterization and convergence”, John Wiley and Sons, New York, 1986.
- [FFGS19] Forde, M., M.Fukasawa, S.Gerhold and B.Smith, “Sub and super-critical Gaussia multiplicative chaos for the the Riemann-Liouville process as $H \rightarrow 0$, and skew flattening/blow up for the Rough Bergomi model”, preprint, 2019.
- [FSV19] Forde, M., B.Smith and L.Viitasaari, “Rough volatility and CGMY jumps with a finite history and the Rough Heston model - small-time asymptotics in the $k\sqrt{t}$ regime”, preprint, 2019.

- [Jan09] Jost, C., “Gaussian Hilbert Spaces”, Cambridge University Press, 2009.
- [Jost06] Jost, C., “Transformation formulas for fractional Brownian motion”, *Stochastic Processes and their Applications*, 116, 1341-1357, 2006.
- [JSW18] Junnila, J., E.Saksman, C.Webb, “Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model”, preprint.
- [Kah85] Kahane, J.-P., “Sur le chaos multiplicatif”, *Ann. Sci. Math.*, Québec, 9 (2), 105-150, 1985.
- [Koz06] Kozhemyak, A., “Modélisation de séries financières à l'aide de processus invariants d'échelle. Application à la prediction du risque.”, thesis, Ecole Polytechnique, 2006,
- [LSSW16] Lodhia, A., S.Sheffield, X.Sun and S.S. Watson, “Fractional Gaussian fields: A survey”, *Probability Surveys*, Vol. 13, 1-56, 2016.
- [Ost09] Ostrovsky, D., “Mellin Transform of the Limit Lognormal Distribution Dmitry Ostrovsky”, *Commun. Math. Phys.*, 288, 287-310.
- [Ost13] Ostrovsky, D., “Selberg Integral as a Meromorphic Function”, *International Mathematics Research Notices*, Vol. 2013, No. 17, pp. 3988–4028, 2013.
- [Ost18] Ostrovsky, D., “A review of conjectured laws of total mass of Bacry–Muzy GMC measures on the interval and circle and their applications”, *Reviews in Mathematical Physics*, Vol. 30, No. 10, 1830003, 2018.
- [Poor94] Poor, H.V., “An Introduction to Signal Estimation and Detection”, 1994, Springer.
- [RV10] Robert, R. and V.Vargas, “Gaussian Multiplicative Chaos Revisited”, *Annals of Probability*, 38, 2, 605-631, 2010.
- [RV14] Rhodes, R. and V.Vargas, “Gaussian multiplicative chaos and applications: a review”, *Probab. Surveys*, 11, 315-392, 2014.
- [RV16] Rhodes, R. and V.Vargas, “Lecture notes on Gaussian multiplicative chaos and Liouville Quantum Gravity”, preprint.
- [RZ17] Remy, G. and T.Zhu, “The distribution of Gaussian multiplicative chaos on the unit interval”,
- [Sha16] Shamov, A., “On Gaussian multiplicative chaos”, *Journal of Functional Analysis*, 270(9):3224-3261, 2016.
- [Var17] Vargas, V., “Lecture notes on Liouville theory and the DOZZ formula”, 2017.
- [Wil91] Williams, D., “Probability with Martingales”, *Cambridge Mathematical Textbooks*, 1991.

A Proof of Proposition 3.2

$$\begin{aligned}
\mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) &= \mathbb{E}\left(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}|\hat{X}_k|^2dk\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\hat{X}_k\bar{\hat{X}}_kdk\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T e^{ikt}X_t dt \int_0^T e^{-iks}X_s ds dk\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T \int_0^T e^{ik(t-s)}X_s X_t ds dt dk\right) \\
&= \int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T \int_0^T e^{ik(t-s)}R(s,t) ds dt dk
\end{aligned}$$

Using that $R \in L^1([0,T]^2)$, we see that $\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T \int_0^T \mathbb{E}(X_s X_t) ds dt dk = \int_0^T \int_0^T R(s,t) ds dt \cdot \int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}dk < \infty$ iff $\delta > 0$, so by Fubini we have

$$\begin{aligned}
\mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) &= \mathbb{E}\left(\int_0^T \int_0^T R(s,t) \int_{-\infty}^{\infty} e^{ik(t-s)}(1+|k|^2)^{-\frac{1}{2}-\delta}dk ds dt\right) \\
&= 2c_{\delta} \int_0^T \int_0^t R(s,t)(t-s)^{\delta} \text{BesselK}(\delta, t-s) ds dt \\
&\leq c_{\delta} \int_{[0,T]^2} R(s,t) ds dt < \infty
\end{aligned} \tag{A-1}$$

where we have used that the Fourier transform of $\hat{f}(k) := (1 + |k|^2)^{-\frac{1}{2} - \delta}$ is $f(t) = c_\delta |t|^\delta \text{BesselK}(\delta, |t|)$ for some real constant c_δ , and that $t^\delta \text{BesselK}(\delta, t)$ is bounded on $[0, T]$ if $\delta > 0$. For $\delta \leq 0$, the integrand in the triple integral in the first line is not absolutely integrable.

B Proof of Proposition 3.3

Using that

$$\begin{aligned}\chi(s, t, \varepsilon, \varepsilon_2) &:= \mathbb{E}((X_t^{\varepsilon_2} - X_t^\varepsilon)(X_s^{\varepsilon_2} - X_s^\varepsilon)) = R_{\varepsilon_2}(s, t) - \mathbb{E}(X_s^{\varepsilon_2} X_t^\varepsilon) - \mathbb{E}(X_s^\varepsilon X_t^{\varepsilon_2}) + R_\varepsilon(s, t) \rightarrow 0 \\ &= R_{\varepsilon_2}(s, t) - \mathbb{E}(X_s^{\varepsilon \vee \varepsilon_2} X_t^{\varepsilon \wedge \varepsilon_2}) - \mathbb{E}(X_s^{\varepsilon \wedge \varepsilon_2} X_t^{\varepsilon \vee \varepsilon_2}) + R_\varepsilon(s, t)\end{aligned}$$

as $\varepsilon, \varepsilon_2 \rightarrow 0$ and that $|\chi(s, t, \varepsilon, \varepsilon_2)| \leq 4R(s, t)$, we can use a similar argument to (A-1) and the dominated convergence theorem to show that

$$\mathbb{E}(\|X^{\varepsilon_2} - X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}}^2) \leq c_\varepsilon \int_{[0, T]^2} \chi(s, t, \varepsilon, \varepsilon_2) ds dt \rightarrow 0 \quad (\text{B-1})$$

as $\varepsilon, \varepsilon_2 \rightarrow 0$, so X^ε is a Cauchy sequence in the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{-\frac{1}{2}-\delta})$ of $H^{-\frac{1}{2}-\delta}$ -valued random variables X with $\mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) < \infty$, and thus converges in this space. Using that

$$\mathbb{P}(\|X^{\varepsilon_2} - X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}} > k) \leq \frac{1}{k^2} \mathbb{E}(\|X^{\varepsilon_2} - X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}}^2)$$

the claim is proved.

C Proof of Proposition 4.1

Similar to the analysis before Lemma 2.1 in [Aru17] with rooted measures, we let $D = [0, T]$ and we can define a sequence of approximate “rooted” probability measures \mathbb{P}_ε^* on $D \times H^{-\frac{1}{2}-\delta}$ as

$$\mathbb{P}_\varepsilon^*(dt, d\omega) = \frac{dt}{\text{Leb}(D)} e^{\gamma\omega(t) - \frac{1}{2}\gamma^2 \mathbb{E}(X_\varepsilon^2)} \mathbb{Q}^{X^\varepsilon}(d\omega)$$

where $\mathbb{Q}^{X^\varepsilon}$ denotes the law of X^ε on $H^{-\frac{1}{2}-\delta}$, and X^ε is defined as in (3). Then

- The marginal law on D is

$$\frac{dt}{\text{Leb}(D)} \mathbb{E}^{\mathbb{Q}^{X^\varepsilon}}(e^{\gamma\omega(t) - \frac{1}{2}\gamma^2 \mathbb{E}(X_\varepsilon^2)}) = \frac{dt}{\text{Leb}(D)}$$

i.e. the uniform probability measure on D .

- The marginal law on $H^{-\frac{1}{2}-\delta}$ is $\frac{\int_D e^{\gamma\omega(t) - \frac{1}{2}\gamma^2 \mathbb{E}(X_\varepsilon^2)} dt}{\text{Leb}(D)} \mathbb{Q}^{X^\varepsilon}(d\omega) = \frac{M_\gamma^\varepsilon(D)}{\text{Leb}(D)} \mathbb{Q}^{X^\varepsilon}(d\omega)$, i.e. the law of X^ε tilted by $M_\gamma^\varepsilon(D)/\text{Leb}(D)$.
- The conditional law on D given ω is the probability measure: $\frac{e^{\gamma\omega(t) - \frac{1}{2}\gamma^2 \mathbb{E}(X_\varepsilon^2)}}{M_\gamma^\varepsilon(D)} dt = \frac{M_\gamma^\varepsilon(dt)}{M_\gamma^\varepsilon(D)}$.
- The conditional law on $H^{-\frac{1}{2}-\delta}$ given t is $e^{\gamma\omega(t) - \frac{1}{2}\gamma^2 \mathbb{E}(X_\varepsilon^2)} \mathbb{Q}^{X^\varepsilon}(d\omega)$. From Girsanov’s theorem (see e.g. section 6.1 in [Var17]), we can re-write this as

$$\mathbb{Q}(X^\varepsilon + \gamma R_\varepsilon(., t) \in d\omega) \quad (\text{C-1})$$

Thus we can sample from \mathbb{P}_ε^* by either (i) sampling from $\frac{M_\gamma^\varepsilon(D)}{\text{Leb}(D)} \mathbb{Q}^{X^\varepsilon}(d\omega)$ and then sampling a point according to $M_\gamma^\varepsilon(dt)/M_\gamma^\varepsilon(D)$, or ii) sampling t from the uniform measure on $[0, T]$, and then sampling $X^\varepsilon + \gamma R_\varepsilon(., t)$, with X^ε independent of t . Combining these two prescriptions, we see that

$$\mathbb{E}\left(\frac{M_\gamma^\varepsilon(D)}{\text{Leb}(D)} \int_0^T F(X^\varepsilon, t) \frac{M_\gamma^\varepsilon(dt)}{M_\gamma^\varepsilon(D)}\right) = \frac{1}{\text{Leb}(D)} \mathbb{E}\left(\int_0^T F(X^\varepsilon + \gamma R_\varepsilon(., t), t) dt\right)$$

which we can re-write as

$$\mathbb{E}\left(\int_0^T F(X^\varepsilon, t) M_\gamma^\varepsilon(dt)\right) = \mathbb{E}\left(\int_0^T F(X^\varepsilon + \gamma R_\varepsilon(., t), t) dt\right). \quad (\text{C-2})$$

We first consider the left hand side of this expression as $\varepsilon \rightarrow 0$. To begin with, we note that

$$\begin{aligned} & |\mathbb{E}\left(\int_0^T F(X^\varepsilon, t) M_\gamma^\varepsilon(dt) - \int_0^T F(X, t) M_\gamma(dt)\right)| \\ & \leq |\mathbb{E}\left(\int_0^T (F(X^\varepsilon, t) - F(X, t)) M_\gamma^\varepsilon(dt)\right)| + |\mathbb{E}\left(\int_0^T F(X, t)(M_\gamma^\varepsilon(dt) - M_\gamma(dt))\right)| \end{aligned} \quad (\text{C-3})$$

and we can bound the first term in the final expression using Hölder's inequality as

$$\begin{aligned} \mathbb{E}\left(\int_0^T (F(X^\varepsilon, t) - F(X, t)) M_\gamma^\varepsilon(dt)\right) & \leq \mathbb{E}\left(\sup_{t \in [0, T]} |F(X^\varepsilon, t) - F(X, t)| \cdot M_\gamma^\varepsilon([0, T])\right) \\ & \leq \mathbb{E}\left(\left(\sup_{t \in [0, T]} |F(X^\varepsilon, t) - F(X, t)|\right)^p\right)^{\frac{1}{p}} \cdot \mathbb{E}\left((M_\gamma^\varepsilon([0, T]))^q\right)^{\frac{1}{q}} \end{aligned} \quad (\text{C-4})$$

for $1/p + 1/q = 1$, and from (2) we know that

$$\mathbb{E}((M_\gamma^\varepsilon([0, T]))^q) = c_q T^{\zeta(q)} < \infty$$

for any $q \in (1, q^*) = \frac{2}{\gamma^2}$.

We claim that $\sup_{t \in [0, T]} |F(X^\varepsilon, t) - F(X, t)| \rightarrow 0$ a.s. Indeed, suppose to the contrary. Let $f_\varepsilon(t) := F(X^\varepsilon, t)$ and $f(t) := F(X, t)$. If the claim is false, f_ε does not tend to f uniformly on $[0, T]$, so there exists a sequence $\varepsilon_n \rightarrow 0$, a $\delta > 0$ and a sequence $t_n \in [0, T]$ such that

$$|f_{\varepsilon_n}(t_n) - f(t_n)| \geq \delta \quad (\text{C-5})$$

for all $n \in \mathbb{N}$. But by Bolzano-Weierstrass, we can choose a convergent subsequence (t_{n_k}) of (t_n) with $t_{n_k} \rightarrow t_\infty \in [0, T]$. Then $f_{\varepsilon_{n_k}}(t_{n_k}) = F(X^{\varepsilon_{n_k}}, t_{n_k})$ and $f(t_{n_k}) = F(X, t_{n_k})$. From Proposition 3.3 we know that X^ε tends to X in $H^{-\frac{1}{2}-\delta}$ in probability, and thus almost surely along a further subsequence $\varepsilon_{n_{k_j}}$, thus (by continuity of F in both arguments) $F(X^{\varepsilon_{n_{k_j}}}, t_{n_{k_j}}) \rightarrow F(X, t_\infty)$ a.s. and hence

$$|F(X^{\varepsilon_{n_{k_j}}}, t_{n_{k_j}}) - F(X, t_{n_{k_j}})| = |f_{\varepsilon_{n_{k_j}}}(t_{n_{k_j}}) - f(t_{n_{k_j}})| \rightarrow 0 \quad (\text{C-6})$$

a.s., which violates (C-5). Hence the right hand side of (C-4) tends to zero (along *any* subsequence) for $q \in (1, q^*)$

The term $\int_0^T F(X, t)(M_\gamma^\varepsilon(dt) - M_\gamma(dt))$ inside the expectation on the right hand side of (C-3) converges to zero a.s. since M_γ^ε tends weakly to M_γ a.s. (see top of page 3 for details) and the random $F(X, t)$ is continuous in t for each ω . Moreover

$$\int_0^T F(X, t)(M_\gamma^\varepsilon(dt) - M_\gamma(dt)) \leq \|F\|_\infty(M_\gamma^\varepsilon([0, T]) + M_\gamma([0, T])).$$

From (6) we also know that $M_\gamma^\varepsilon([0, T])$ is uniformly integrable, so by e.g. the Theorem in section 13.7 in [Wil91], the rightmost term of (C-3) tends to zero

Finally, the right hand side of (C-2) converges by the a.s. convergence of X^ε to X in Proposition 3.3 and the bounded convergence theorem.