The left curtain martingale coupling

We conjecture that if $X \leq m$, then Y = X, otherwise $Y = T_u(X)$ with probability q(X) or $T_d(X)$ with probability q(X), with $T_d(x) \leq x \leq T_u(x)$, with $q(x) = \frac{x - T_d(x)}{T_u(x) - T_d(x)}$ to respect the martingale condition, and we guess that $T_d: (m, \infty) \to (-\infty, m)$ is decreasing and $T_u: (m, \infty) \to (m, \infty)$ increasing for the left curtain coupling.

• For y > m with m in the support of ν , then $y = T_u(x)$ for some x > m. Then

$$1 - F_{\nu}(y) = \mathbb{P}(Y > y) = \mathbb{P}(\mathbb{P}(Y > y | X)) = \mathbb{E}(q(X) 1_{T_{n}(X) > y})$$

SO

$$1 - F_{\nu}(T_u(x)) = \mathbb{E}(q(X)1_{T_u(X) > T_u(x)}) = \mathbb{E}(q(X)1_{X > x})$$

and differentiating we find that

$$\frac{d}{dx}F_{\nu}(T_{u}(x)) = T'_{u}(x)F'_{\nu}(T_{u}(x)) = q(x)F'_{\mu}(x)$$

for $x > T_u^{-1}(m) = m$, so we see that $T_u'(x) > 0$ as required for the left curtain coupling.

• Conversely for $y \le m$, then $y = T_d(x)$ for some x > m, but there are two ways for Y to get to y: either we came from X = y, or $y = T_d(x)$ with x > m, so

$$F_{\nu}(y) = \mathbb{P}(\mathbb{P}(Y \le y|X)) = \mathbb{P}(X \le y) + \mathbb{E}((1 - q(X))1_{T_d(X) \le y})$$
$$= F_{\mu}(y) + \mathbb{E}((1 - q(X))1_{T_d(X) \le y})$$

so

$$F_{\nu}(T_d(x)) - F_{\mu}(T_d(x)) \quad = \quad \mathbb{E}((1 - q(X)) \mathbf{1}_{T_d(X) \le T_d(x)}) \quad = \quad \mathbb{E}((1 - q(X)) \mathbf{1}_{X \ge x})$$

(note the change of inequality in the final indicator since T_d is decreasing), then differentiating we find that

$$\frac{d}{dx}(F_{\nu}(T_d(x)) - F_{\mu}(T_d(x))) = T'_d(x)(F'_{\nu}(T_d(x)) - F'_{\mu}(T_d(x))) = -(1 - q(x))F'_{\mu}(x)$$
(1)

for $x>T_u^{-1}(m)=m$, so we have recovered Eqs 3.12 and 3.13 in [HT15], and we see that $T_d'(x)>0$ in the region where $F_\nu'< F_\mu'$ (which is [-1,1] for the simple uniform example below). T_d is increasing????

Example

For the case when μ is U[-1,1] and ν is U[-2,2], sees we don't need an m, so there is no F'_{μ} in (1), in which case $T'_d(x) < 0$, and we seemingly find that $T_u(x) = \frac{3}{2}(x+1) - 1$, and $T_d(x) = -\frac{1}{2}(x+1) - 1$