Continuous-time models

The Black-Scholes model

As mentioned briefly before, the Black-Scholes model for a stock price process is defined as

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{1}$$

where W is a standard Brownian motion, and from Ito's lemma, we have seen that S_t satisfies the SDE

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

which is known as **Geometric Brownian motion**. μ is the **drift** of the process which describes the overall upward/downward trend of S, and σ is the **volatility**, which describes the variability of S.

The terminal stock price distribution

- Re-arranging (1) we see that

  $$\log S_t - \log S_0 = \log \frac{S_t}{S_0} = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\,.$$

  Thus we can calculate the distribution of $\log \frac{S_t}{S_0}$ as $\log \frac{S_t}{S_0} \sim N((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$ or equivalently $\log S_t \sim N(\log S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$, where we have used that $W_t \sim N(0,t)$ and for any random variable X, $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$.

- For some fixed $T > 0$, we can then compute $\mathbb{P}(S_T > K)$ as follows:

  $$\begin{aligned}\mathbb{P}(S_T > K) &= \mathbb{P}(\log \frac{S_T}{S_0} > \log \frac{K}{S_0}) \\ &= \mathbb{P}(\frac{\log \frac{S_T}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} > \frac{\log \frac{K}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\ &= \mathbb{P}(Z > \frac{\log \frac{K}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\ &= \Phi^c(z)\end{aligned}$$

  where $z = \frac{\log \frac{K}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, because $Z = \frac{\log \frac{S_T}{S_0} - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ is a standard $N(0,1)$ random variable, and $\Phi^c(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$, and we have used that standard result that $(X-\mu)/\sigma \sim N(0,1)$ if $X \sim N(\mu, \sigma^2)$.

- Example: set $S_0 = 1, K = 1.1, \sigma = .1, T = .25, \mu = .05$. Then we have $z = 1.681153596$ and $\mathbb{P}(S_T > K) = \Phi^c(z) = 1 - \Phi(z) = 0.046361687633$. We can Normsdist(.) to calculate Φ and Φ^c in Excel.

- Note that

  $$\mathbb{P}(S_t \le S) = \mathbb{P}(\log S_t \le \log S) = F(\log S)$$

  where F is the distribution function of $\log S_t$. Differentiating both sides with respect to S and using the chain rule, we see that the density $p_{S_t}(S)$ of S_t is given by

  $$p_{S_t}(S) = \frac{d}{dS}\mathbb{P}(S_t \le S) = \frac{1}{S}F'(\log S) = \frac{1}{S}p_{X_t}(x)$$

  where $x = \log S$ and $p_{X_t}(x)$ is the density of $X_t = \log S_t$ which is given by $p_{X_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(x - X_0 - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}}$ since $X_t \sim N(X_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)$, and the density of a general $N(\mu_1, \sigma_1^2)$ random variable is given by

  $$p(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

  so

  $$p_{S_t}(S) = \frac{1}{S\sigma\sqrt{2\pi t}}\, e^{-\frac{(\log \frac{S}{S_0} - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}}$$

  for $S > 0$. S_t has what is known as a **lognormal distribution**. Note the presence of the $\frac{1}{S}$ pre-factor, and this pdf is only defined for $S > 0$ because the stock price cannot go negative.

- $p_{S_t}(S)$ is a pdf and thus integrates to 1, i.e. $\int_0^\infty p_{S_t}(S)dS = 1$.

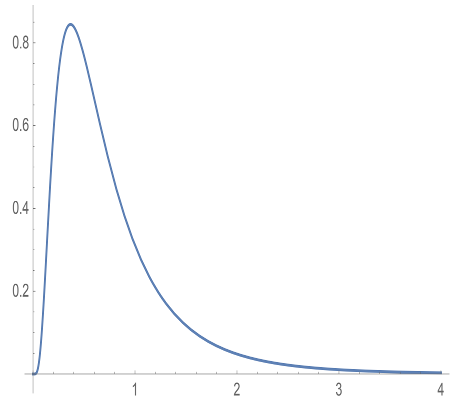

Figure 1: Here we have plotted the stock price density $f_{S_t}(S)$ for $\sigma = 1$ and $t = 1$.

The Black-Scholes PDE

Proposition 0.1 *There is a unique solution $C(S,t)$ to the Black-Scholes partial differential equation:*

$$C_t(S,t) + rSC_S(S,t) + \frac{1}{2}\sigma^2 S^2 C_{SS}(S,t) = rC(S,t) \tag{2}$$

with terminal boundary condition $C(S,T) = f(S)$.

We will not prove this on this course, but the solution for the case when $f(S) = \max(S-K,0)$ is given below as the famous **Black-Scholes formula**.

Now suppose a stock follows the Black-Scholes model in (1) and a trader begins with **initial wealth** X_0 and at each time instant t, holds ϕ_t shares of stock. ϕ_t can be random but must be $\mathcal{F}_t^S-$**adapted**, which means its value can only depend on what happened up to time t, i.e. the trader cannot look into the future to determine the value of ϕ_t, as we would expect. The trader's remaining wealth $X_t - \phi_t S_t$ is placed in a **riskless bank account** which earns $(X_t - \phi_t S_t)rdt$ in each infinitesimal time instant dt (note that if $X_t - \phi_t S_t$ is negative then we are **borrowing** money so are paying $(X_t - \phi_t S_t)rdt$ in interest). This is what we call a **self-financing** trading strategy for a continuous model - there are no **injections** or **withdrawals of funds** and no transaction costs. Then the trader's total wealth X_t evolves as

$$\begin{aligned} dX_t &= \phi_t dS_t + (X_t - \phi_t S_t)rdt && (3)\\ &= \phi_t S_t(\mu dt + \sigma dW_t) + (X_t - \phi_t S_t)rdt \\ &= (rX_t + \phi_t S_t(\mu - r))dt + \phi_t S_t \sigma dW_t\,. && (4) \end{aligned}$$

Recall that the first line is shorthand for

$$X_t - X_0 = \int_0^t \phi_u dS_u + \int_0^t (X_u - \phi_u S_u)rdu$$

Applying Ito's lemma to $C(S_t,t)$, where $C(S,t)$ is the solution to the PDE (2), we see that

$$\begin{aligned} dC(S_t,t) &= C_t(S_t,t)dt + C_S(S_t,t)dS_t + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 dt \\ &= C_t(S_t,t)dt + C_S(S_t,t)S_t(\mu dt + \sigma dW_t) + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 dt \\ &= (C_t(S_t,t) + \mu C_S(S_t,t)S_t + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2)dt + C_S(S_t,t)S_t\sigma dW_t\,. && (5) \end{aligned}$$

We now wish to find a ϕ_t such that its corresponding wealth process $X_t = X_t^\phi$ satisfies $X_T = f(S_T)$, i.e. our self-financing trading strategy ϕ_t **perfectly replicates** the payoff $f(S_T)$ at time T. To this end, we guess that $\phi_t = C_S(S_t,t)$ for all $t \in [0,T]$. Then if we **compare the drift terms** in Eqs (4) and (5) and guess that $\phi_t, X_t)$ and $X_t = C(S_t,t)$ we see that

$$C_t(S_t,t) + \mu C_S(S_t,t)S_t + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 = rC(S_t,t) + C_S(S_t,t)S_t(\mu - r)$$

which (after cancelling the μ terms) we can re-write as

$$C_t(S_t,t) + rC_S(S_t,t)S_t + \frac{1}{2}C_{SS}(S_t,t)S_t^2\sigma^2 = rC(S_t,t)$$

Then (from the Black-Scholes PDE), we see that this equation is in fact satisfied, i.e. left and right hand sides agree.

Thus we see that $X_t = C(S_t, t)$ and in particular $X_T = C(S_T, T) = f(S_T)$ (from the boundary condition of the Black-Scholes PDE).

So to summarize what we have done this far: we have shown how **replicate** a **terminal** payoff of $f(S_T)$ under the Black-Scholes model by:

- Holding $X_0 = C(S_0, 0)$ **in cash at time zero**. Recall that $C(S,t)$ is the solution to the PDE in (2), so we need to solve this PDE (see below for more on this).
- **Dynamically trading** the stock from 0 to T, holding $\phi_t = C_S(S_t, t)$ units of stock at each time instant t. $C_S(S_t, t)$ is known as the **Delta** (Δ) of the option at time t.
- **Hold our remaining wealth** $X_t - C_S(S_t, t)S_t$ at each time instant t in the risk-free bank account, so our total wealth X_t evolves as in (4) with $X_T = f(S_T)$.

Thus $X_0 = C(S_0, 0)$ is the **cost of replicating** $f(S_T)$, since everything we do after time zero is self-financing. If the option price in the market $C^{mkt} > X_0 = C(S_0, 0)$, then there is **arbitrage**. Specifically, we can sell the option in the market for C^{mkt} and replicate it using the arguments above at a cost of $C(S_0, 0)$ to realize a riskless profit of $C^{mkt} - C(S_0, 0) > 0$. Conversely, if the option is too cheap in the market, we can buy the option, and replicate $-f(S_T)$ at a cost of $-C(S,t)$, and again realize a riskless profit of $C(S_0, 0) - C^{mkt} > 0$. Thus $C(S_0, 0)$ is the **unique no-arbitrage price** of the option at time 0. Similarly, $C(S_t, t)$ is the **unique no-arbitrage price** of the option at time t, if we re-do the argument above but start at time t rather than time zero.

IMPORTANT: Note that the Black-Scholes PDE and boundary condition are **independent of** μ, and hence the no-arbitrage price of the option is also independent of μ, as for the binomial model.

$C_{SS}(S_t, t)$ is known as the **Gamma** (Γ) and $C_t(S_t, t)$ is known as the **Theta** (Θ) of the option, and as we shall see, these partial derivatives have explicit formulae for the case when $f(S) = \max(S - K, 0) = (S - K)^+$, i.e. a European call option.

Solving the Black-Scholes PDE - the Black-Scholes formula

For a standard European call option, we know that $f(S) = \max(S - K, 0) = (S - K)^+$ and in this case, the Black-Scholes PDE has an explicit solution given by the famous **Black-Scholes formula**:

$$C(S,t) \quad = \quad C^{BS}(S, K, \sigma, T-t, r) \quad = \quad S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where $\tau = T - t$ is the time-to-maturity and

$$d_1 \quad = \quad \frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad , \quad d_2 \quad = \quad d_1 - \sigma\sqrt{\tau}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$ is the standard cumulative Normal distribution function. Note that the call price C^{BS} actually depends on five parameters, but only S and t dynamically change over time.

Thus $C(S_t, t) = C^{BS}(S_t, K, \sigma, T-t, r)$ and $\phi_t = C_S(S_t, t)$, which can be calculated explicitly using the formula in (6) below. We can also compute $C(S,t)$ explicitly if the terminal payoff is $\log S_T$, $(\log S_T)^2$ or S_T^p (see mock exams and homeworks), in which case the argument is exactly the same except the boundary condition for the PDE will change and thus $C(S,t)$ will change, and hence so will $\phi_t = C_S(S_t, t)$.

Numerical example

Assume current stock price is 1, volatility is .10 and interest rate is .05. Price a call option at time zero with strike 1.1 with maturity 1: take $S = 1, K = 1.1, \sigma = .1, \tau = 1, r = .05$ and $t = 0$ and $\tau = T$. Plugging these numbers into the Black-Scholes formula we obtain

$$\begin{aligned} d_1 &= -0.403101798043249\,, \\ d_2 &= -0.503101798043249 \end{aligned}$$

and the call price

$$C \quad = \quad 0.021739503382137$$

(Python notebook for the Black-Scholes model and formula available at
`https://colab.research.google.com/drive/1kN8mahKStvATkQm2sqVMdxeJgyIdUOXv?usp=sharing`, and the Excel sheet `BlackScholesFormula+Greeks.xls` on KEATS implements this formula in Visual Basic.

$C(S,t) \to \max(S-K,0)$ as $t \to T$, and (less obvious) $C(S,t) \to S$ if $T \to \infty$. It can be shown from something called **Jensen's inequality** that $C(S,t)$ is increasing in $\tau = T-t$.

Remark 0.1 The initial (i.e. $t=0$) cost of the replicating strategy at time zero is $C(S_0,0)$, and at each time instant, $C(S_t,t)$ is the unique no-arbitrage price of the call option (see next subsection to see why this is so).

The Greeks

We can compute **partial derivatives** of the BS formula with respect to each of the parameters:

$$\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} = \Phi(d_1) > 0, \qquad (6)\\
\Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{\tau}} > 0\,,\\
\Lambda &= \frac{\partial C}{\partial \sigma} = S n(d_1)\sqrt{\tau} > 0\\
\Theta &= \frac{\partial C}{\partial \tau} = -\frac{\partial C}{\partial t} > 0
\end{aligned}$$

where $n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard Normal density. Δ, Γ, Λ and Θ are known as the Delta, Gamma, Vega and Theta respectively of the option. The proof of these expressions for the Greeks are very tedious and not examinable.

- Delta measures the sensitivity of the call option price to small changes in the underlying stock price.
- Vega measures the sensitivity of the call option price to small changes in the volatility.
- Gamma measures the sensitivity of the Delta to small changes in the underlying stock price.
- Theta measures the sensitivity of the call option price to small changes in the time-to-maturity.

For the numerical example above, we obtain $\Delta = 0.343435379743$, $\Gamma = 3.677756432729$ and $\Lambda = 0.367811626137$.

The Feynman-Kac formula

It turns out that $C(S,t)$ (the price of a option which pays $f(S_T)$ at time T under the Black-Scholes model) also has the following *probabilistic* representation:

$$C(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(f(S_T)\,|\,S_t = S)$$

where $\mathbb{Q}$ is a **new probability measure** under which S satisfies

$$dS_t = S_t(r dt + \sigma dW_t^{\mathbb{Q}}) \qquad (7)$$

and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$ (this is a special case of a more general result called the **Feynman-Kac** formula). In words: **the option price (i.e. the cost of replicating the option) is the discounted expected value of $f(S_T)$ in the risk-neutral world $\mathbb{Q}$ where the drift is r not μ**. We refer to this world as the **risk neutral measure**. Note that (7) this is the same as the Black-Scholes SDE $dS_t = S_t(\mu dt + \sigma dW_t)$, but the real-world μ has been replaced by r, as for the discrete case.

Remark 0.2 The explicit formula for computing probabilities under $\mathbb{Q}$ in terms of $\mathbb{P}$ comes from **Girsanov's theorem**, and is given as follows: for any event $A \in \mathcal{F}_T$

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(e^{\gamma W_T - \frac{1}{2}\gamma^2 T} 1_A)$$

where $\gamma = (r-\mu)/\sigma$.

As a sanity check, we note that $\mathbb{Q}(\Omega) = \mathbb{P}(e^{\gamma W_T - \frac{1}{2}\gamma^2 T}) = 1$, since $e^{\gamma W_t - \frac{1}{2}\gamma^2 t}$ is a martingale as we have checked before.

Example. Price a digital call option under the Black-Scholes model which pays 1 if $S_T > K$ and zero otherwise.

Solution:

$$P(S,t) \quad = \quad e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(1_{S_T>K} \,|\, S_t = S) \quad = \quad e^{-r(T-t)}\mathbb{Q}(S_T > K \,|\, S_t = S)$$

where (again) S_t satisfies the SDE $dS_t = S_t(rdt + \sigma dW_t)$ under the probability measure $\mathbb{Q}$, and $\mathbb{Q}(A)$ denotes the probability of an event A under the probability measure $\mathbb{Q}$, and we have used that for any random variable X, $\mathbb{P}(X > K) = \mathbb{E}(1_{X>K})$. If we now let $t = 0$, then we can compute $\mathbb{Q}(S_T > K)$ similar to before as:

$$\mathbb{Q}(S_T > K) \quad = \quad \Phi^c(z)$$

where now $z = \frac{\log \frac{K}{S_0} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ (note that we have now just replaced μ with r). Similarly

$$\mathbb{Q}(S_T > K \,|\, S_t = S) \quad = \Phi^c(z)$$

where $z = \frac{\log \frac{K}{S_t} - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Monotonicity of call prices as a function of maturity

The **tower property** says that for any two random variables X and Y:

$$\mathbb{E}(Y) \quad = \quad \mathbb{E}(\mathbb{E}(Y|X))\,.$$

For any random variable Y, **Jensen's inequality** says $\mathbb{E}(g(Y)) \geq g(\mathbb{E}(Y))$ for any convex function g (if g is twice differentiable, convex just means that $g''(y) \geq 0$). Similarly for any other random variable X, the conditional Jensen inequality says that $\mathbb{E}(g(Y)|X) \geq g(\mathbb{E}(Y|X))$.

Now let $(X_t)_{t\geq 0}$ denote a martingale and $g(x) = (x-K)^+$ denote the call option payoff, which is a convex function of x. Then for $0 \leq T_1 \leq T_2$, we see that

$$C(K,T_2) \quad := \quad \mathbb{E}(g(X_{T_2})) \quad = \quad \mathbb{E}(\mathbb{E}(g(X_{T_2})|X_{T_1}) \quad \geq \quad \mathbb{E}(g(\mathbb{E}(X_{T_2}|X_{T_1})) \quad = \quad \mathbb{E}((X_{T_1} - K)^+) \quad = \quad C(K,T_1)$$

where we have used the tower property for the second equality and conditional Jensen to obtain the inequality, and that $\mathbb{E}(X_{T_2}|X_{T_1}) = X_{T_1}$.

In particular, for the Black-Scholes model with $r = 0$, this means that $\mathbb{E}^{\mathbb{Q}}((S_{T_2} - K)^+) \geq \mathbb{E}^{\mathbb{Q}}((S_{T_1} - K)^+)$, i.e. call option prices with maturity T_2 are greater than call option prices with maturity T_1, since S is a martingale under $\mathbb{Q}$ as we have proved below.

The put-call parity

We now return to European call and put options. By considering both cases $S_T > K$ and $S_T < K$ we see that

$$(S_T - K)^+ \, + \, K \quad = \quad \max(K - S_T, 0) \, + \, S_T\,.$$

A portfolio of 1 call option and Ke^{-rT} dollars will be equal in value to the left hand side at T. Similarly, a portfolio of 1 put option and 1 share will be equal in value to the right hand side at T. Thus we obtain the **put-call parity**:

$$C + Ke^{-rT} \quad = \quad P + S_0$$

where C is the initial price of the call, where P is the initial price of the put.

Implied volatility

The Vega $\frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{\tau}$ of a call option under Black-Scholes is positive, so C is monotonically increasing as a function of σ. Thus, given an observed call price C^{obs} in the market, we can extract a unique $\hat{\sigma}$ value consistent i.e. such that

$$C(S, K, \hat{\sigma}, \tau, r) \quad = \quad C^{obs}\,.$$

if $\max(S_0 - Ke^{-rT}, 0) \leq C^{obs} < S_0$. This σ is known as the **implied volatility** of the option, and is a very important concept in practice. Note $\hat{\sigma}$ cannot be computed explicitly, so have to compute it numerically as a root finding exercise using e.g. the bisection or Newton-Raphson method.

Continuous-time martingales

A continuous-time **martingale** $(X_t)_{t\geq 0}$ is a stochastic process which satisfies the following two conditions:

(i) $\mathbb{E}(X_t \,|\, \mathcal{F}_s) = X_s$ for $0 \leq s \leq t$, where $\mathcal{F}_s$ is (informally) the information available at time s. For our purposes on this course, $\mathcal{F}_s$ just means the historical sample path of the process X from time 0 to time s (this concept is made rigorous using the concept of filtrations in stochastic analysis, which is beyond the scope of this course).

(ii) $\mathbb{E}(|X_t|) < \infty$ for all finite $t \geq 0$.

We now describe some well known examples of continuous-time martingales:

- **Example: Brownian motion**. For standard Brownian motion, we know that $W_t - W_s \sim N(0, t-s)$. Thus

$$\begin{aligned}\mathbb{E}(W_t \,|\, \mathcal{F}_s) &= \mathbb{E}(W_t - W_s + W_s \,|\, \mathcal{F}_s) \\ &= W_s \,+\, \mathbb{E}(W_t - W_s \,|\, \mathcal{F}_s) \\ &= W_s \,+\, \mathbb{E}(W_t - W_s) \;=\; W_s\,.\end{aligned}$$

  Moreover, $\mathbb{E}(|W_t|) = \int_{-\infty}^{\infty} |x| \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx < \infty$ because for $|x|$ large, the decaying exponential term decays much faster than the $|x|$ term is growing. Thus W_t is a martingale.

- **Example 2**. Let $M_t = W_t^2 - t$. Then

$$\begin{aligned}\mathbb{E}(M_t - M_s \,|\, \mathcal{F}_s) &= \mathbb{E}(W_t^2 - t - (W_s^2 - s) \,|\, \mathcal{F}_s) \\ &= \mathbb{E}(W_t^2 - W_s^2 \,|\, \mathcal{F}_s) \,-\, (t-s) \\ &= \mathbb{E}((W_s + W_t - W_s)^2 - W_s^2 \,|\, \mathcal{F}_s) \,-\, (t-s) \\ &= \mathbb{E}(W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2 - W_s^2 \,|\, \mathcal{F}_s) \,-\, (t-s) \\ &= \mathbb{E}(2W_s(W_t - W_s) + (W_t - W_s)^2 \,|\, \mathcal{F}_s) \,-\, (t-s) \\ &= 2W_s\mathbb{E}(W_t - W_s) \,|\, \mathcal{F}_s) + \mathbb{E}((W_t - W_s)^2 \,|\, \mathcal{F}_s) \,-\, (t-s) \\ &\quad \text{(using that } W_s \text{ is known at time } s) \\ &= 2W_s\mathbb{E}(W_t - W_s) + \mathbb{E}((W_t - W_s)^2) \,-\, (t-s) \\ &= 0 \,+\, t - s - (t-s) \\ &\quad \text{(using that } W_t - W_s \sim N(0, t-s)) \\ &= 0\end{aligned}$$

  and hence $\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(M_s|\mathcal{F}_s) = M_s$. Moreover, $\mathbb{E}(|M_t|) \leq \mathbb{E}(W_t^2 + t) = 2t < \infty$, so $M_t = W_t^2 - t$ also satisfies the second condition, and hence M_t is a martingale.

- **Example 3: the Black-Scholes Stock price process with** $\mu = 0$. Let $S_t = S_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$. Then

$$\mathbb{E}(S_t \,|\, \mathcal{F}_s) \;=\; \mathbb{E}(S_0 e^{\sigma W_s - \frac{1}{2}\sigma^2 s} e^{\sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)} \,|\, \mathcal{F}_s) \;=\; S_s \mathbb{E}(e^{\sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)} \,|\, \mathcal{F}_s)$$

  where we have used that $S_s = S_0 e^{\sigma W_s - \frac{1}{2}\sigma^2 s}$, and the value of S_s is known at time s. But for any normal random variable $X \sim N(\mu, \nu^2)$, the moment generating function of X is given by $\mathbb{E}(e^{pX}) = e^{\mu p + \frac{1}{2}\nu^2 p^2}$. In our case here, $X = \sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)$ so $\mu = -\frac{1}{2}\sigma^2(t-s)$ and $\nu^2 = \sigma^2(t-s)$ and $p = 1$, so we see that

$$\begin{aligned}\mathbb{E}(S_t|\mathcal{F}_s) \;=\; \mathbb{E}(S_s e^{\sigma(W_t - W_s) - \frac{1}{2}\sigma^2(t-s)} \,|\, \mathcal{F}_s) &= S_s e^{\mu p + \frac{1}{2}\nu^2 p^2} \\ &= S_s e^{-\frac{1}{2}\sigma^2(t-s) + \frac{1}{2}\sigma^2(t-s)} \\ &= S_s\,.\end{aligned}$$

  Moreover, setting $s = 0$ we see that $\mathbb{E}(|S_t|) = \mathbb{E}(S_t) = S_0 < \infty$, so S_t is a martingale.

Monte Carlo simulation and variance reduction with antithetic sampling

We can approximate a call price using Monte Carlo (MC) in the usual way using the sample average terminal payoff over M i.i.d stock price paths:

$$P_M \;=\; \frac{1}{M}\sum_{i=1}^{M}(S_T^i - K)^+$$

where S^i is the i'th Monte Carlo stock price path (which are all i.i.d.), and M is the total number of MC paths, and by the **strong law of large numbers** (SLLN) $P_M \to P$ a.s. as $M \to \infty$. Then (using that the MC paths are i.i.d), we see that

$$\mathrm{Var}^{\mathbb{Q}}(P_M) \;=\; \frac{1}{M^2} M\,\mathrm{Var}^{\mathbb{Q}}(e^{-rT}(S_T^i - K)^+) \;=\; \frac{1}{M}\,\mathrm{Var}^{\mathbb{Q}}(e^{-rT}(S_T^i - K)^+) \;=\; \frac{\sigma^2}{M}$$

where $\sigma^2 = \mathrm{Var}^{\mathbb{Q}}(e^{-rT}(S_T^i - K)^+)$, so the **sample standard deviation** of the MC call price estimate scales like $\frac{1}{\sqrt{M}}$, which we can also use to construct e.g. confidence intervals for the call price, since (by the central limit theorem) $\sqrt{M}(P - P_M) \sim N(0, \sigma^2)$.

If we flip the sign of W, and re-compute the S path, we call this an **antithetic** path. If we do this for all paths, we can get a better MC estimate for the option by averaging the payoff for the original and antithetic paths. We now show why antithetic sampling lowers the sample variance of a Monte Carlo estimate for the price of any option: let C_T denote the terminal value of an option payoff for a **single stock price path** S, and let $\tilde{C}_T$ denote the corresponding antithetic estimate. Then

$$\begin{aligned}
\mathrm{Var}(\tfrac{1}{2}C_T + \tfrac{1}{2}\tilde{C}_T)) &= \frac{1}{4}\mathrm{Var}(C_T) + \frac{1}{4}\mathrm{Var}(\tilde{C}_T) + \frac{1}{2}\mathrm{Cov}(C_T, \tilde{C}_T) \\
&= \frac{1}{2}\mathrm{Var}(C_T) + \frac{1}{2}\mathrm{Cov}(C_T, \tilde{C}_T) \\
&\leq \frac{1}{2}\mathrm{Var}(C_T) + \frac{1}{2}\mathrm{Var}(C_T)^{\frac{1}{2}}\mathrm{Var}(\tilde{C}_T)^{\frac{1}{2}} \\
&= \frac{1}{2}\mathrm{Var}(C_T) + \frac{1}{2}\mathrm{Var}(C_T) \;=\; \mathrm{Var}(C_T)
\end{aligned}$$

where the penultimate line follows from the **Cauchy-Schwarz inequality**, and the second and final lines follow since $C_T \sim \tilde{C}_T$, because $W \sim -W$ and hence $S \sim \tilde{S}$. Hence the variance of the antithetic estimate is no worse than the basic MC estimate, and for most payoffs is better in practice.

It is important to check the convergence of a Monte Carlo estimate for a call price by varying the Number of time steps (with a fixed random seed) and the Number of paths (without fixing, since convergence is typically quite slow and large bias for e.g. more advanced **rough volatility** models with realistic parameter values.