

# A simplified unified propagator model for signed/unsigned order flow, concave price impact and rough volatility

Let  $B$  denote a standard Brownian motion with  $B_0 = 0$ , and consider a hyper-rough Heston process  $(F_t^+)_{t \geq 0}$  which satisfies

$$F_t^+ = g(t) + \int_0^t f(t-s)B_{F_s^+}ds$$

where  $f(t) = e^{-\theta t}t^{H-\frac{1}{2}}$  is the usual Gamma kernel with  $H \in (-\frac{1}{2}, \frac{1}{2})$ ,  $\theta > 0$  so  $f \in L^1$  and we assume  $g \in C^1$ , and we know that  $F_t^+$  is a.s.  $(1+H-\varepsilon) \wedge 1$ -Hölder continuous. Then if  $h * f \equiv 1$ , we see that

$$\begin{aligned} \int_0^t h(t-s)(F_s^+ - g(s))ds &= \int_0^t h(t-s) \int_0^s f(s-u)B_{F_u^+}duds = \int_0^t \int_u^t h(t-s)f(s-u)dsB_{F_u^+}du \\ &= \int_0^t \int_0^{t-u} h(t-s-u)f(s)dsB_{F_u^+}du \\ &= \int_0^t (h * f)(t-u)B_{F_u^+}du = \int_0^t B_{F_u^+}du \end{aligned}$$

where Fubini is justified since  $F_t^+ < \infty$  a.s. for finite  $t$  and  $B$  is a.s. continuous, and thus

$$\frac{d}{dt} \int_0^t h(t-s)(F_s^+ - g(s))ds = B_{F_t^+}.$$

Taking Laplace transforms, the condition  $h * f \equiv 1$  becomes  $\hat{f}(\lambda)\hat{h}(\lambda) = \frac{1}{\lambda}$ , from which we find that

$$h(t) = \frac{1}{\Gamma(\frac{1}{2}+H)}\theta^{\frac{1}{2}+H}\left(1 - \frac{\Gamma(-\frac{1}{2}-H, t\theta)}{\Gamma(-\frac{1}{2}-H)}\right)$$

where  $\Gamma(a, z) = \int_z^\infty s^{a-1}e^{-s}ds$  is the incomplete Gamma function, and in particular  $h(t) = O(t^{-H-\frac{1}{2}})$  as  $t \rightarrow 0$ .

We now assume the *signed order flow* for an asset is

$$V_t = F_t^+ - F_t^- \tag{1}$$

where  $F_t^- = g(t) + \int_0^t f(t-s)W_{F_s^-}ds$  and  $W$  is another Brownian motion independent of  $B$  (so  $F^-$  is an i.i.d. copy of  $F^+$ ). Then we see that

$$\frac{d}{dt} \int_0^t h(t-s)V_sds = B_{F_t^+} - W_{F_t^-}$$

since the  $g$  terms cancel, and for  $0 \leq t \leq u$

$$\mathbb{E}(F_u^+ | \mathcal{F}_{F_t^+}^B) = g(u) + \mathbb{E}\left(\int_0^u f(u-s)B_{F_s^+}ds | \mathcal{F}_{F_t^+}^B\right) = g(u) + \int_0^t f(u-s)B_{F_s^+}ds + B_{F_t^+} \int_t^u f(u-s)ds$$

(and similarly for  $\mathbb{E}(F_u^- | \mathcal{F}_{F_t^-}^W)$ ), where we have used that  $(B_{F_t^+})_{t \geq 0}$  is a  $\mathcal{G}_t^+$ -martingale (give reference), where  $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$ .

Making the usual assumption that the asset price  $P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u | \mathcal{G}_t)$  where  $F_t = F_t^+ - F_t^-$  and  $\mathcal{G}_t = \sigma(\mathcal{G}_t^+, \mathcal{G}_t^-)$  with  $\mathcal{G}_t^- = \mathcal{F}_{F_t^-}^W$  (for some  $\kappa > 0$ )<sup>1</sup>, we find that  $f(u-s) \rightarrow 0$  as  $u \rightarrow \infty$  since  $\theta > 0$ , but  $\lim_{u \rightarrow \infty} \int_t^u f(u-s)ds = c_{H,\theta}$  where  $c_{H,\theta} = \theta^{-\frac{1}{2}-H}\Gamma(\frac{1}{2}+H)$ , so

$$P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(F_u^+ - F_u^- | \sigma(\mathcal{F}_{F_t^+}^B, \mathcal{F}_{F_t^-}^W)) = \kappa c_{H,\theta}(B_{F_t^+} - W_{F_t^-})$$

i.e.  $P$  is the difference of two i.i.d. hyper-rough Heston models (each with correlation  $\rho = 1$  since each process only has one driving Brownian motion), and

$$P_t = \kappa c_{H,\theta} \frac{d}{dt} \int_0^t h(t-s)V_sds$$

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<sup>1</sup>note we are assuming  $P_0 = 0$  without loss of generality since we can easily add a  $P_0$  term

which is a weak formulation of the usual Propagator model, and the *unsigned* order flow is  $U_t = F_t^+ + F_t^-$ . In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 is given by

$$MI(t) = \frac{d}{dt} \int_0^t h(t-s) s ds = \frac{\theta^{-\frac{1}{2}+H}(t\theta\Gamma(-\alpha_+) - t\theta\Gamma(-\alpha_+, t\theta) - \Gamma(\alpha_-, 0) + \Gamma(\alpha_-, t\theta))}{\Gamma(-\alpha_+)\Gamma(\alpha_+)}$$

(with  $\alpha_{\pm} = \frac{1}{2} \pm H$ ) which is  $O(t^{\frac{1}{2}-H})$  as  $t \rightarrow 0$  and globally concave in  $t$  for  $H \in (-\frac{1}{2}, \frac{1}{2})$ , consistent with empirical evidence (note the usual square-root impact law corresponds to  $H = 0$  here). Recall from above that  $F_t^+$  has Hölder regularity  $(H + 1 - \varepsilon) \wedge 1$ , and empirical evidence in [MORS26] suggests  $H \in (-\frac{1}{2}, 0)$ , so  $(H + 1 - \varepsilon) \wedge 1 \in (\frac{1}{2}, 1)$ .

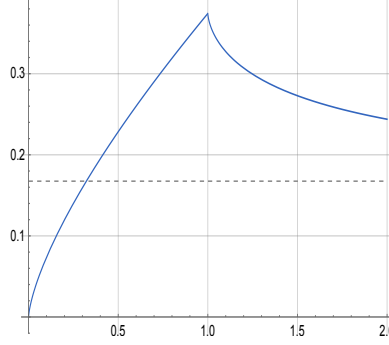


Figure 1: Concave price impact of an exogenous metaorder executed at constant trading speed 1 over  $[0, 1]$  with  $H = -\frac{1}{10}$ ,  $\theta = \frac{1}{10}$  and  $\kappa = 1$ .

## References

- [MORS26] Muhle-Karbe, J., Y.Chahdi, M.Rosenbaum, and G.Szymanski, “A unified theory of order flow, market impact, and volatility”, preprint.