Lévy processes as weak limits of rough Heston models

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Abstract

We show weak convergence of the marginals for a re-scaled rough Heston model to a Normal Inverse Gaussian (NIG) Lévy process. This shows we can obtain such a limit without having to impose that the true Hurst exponent H for the model is $\frac{1}{2}$ as in [AC24],[1] or that $H \searrow -\frac{1}{2}$ as in [AAR25], so the result potentially has increased financial relevance. We later extend to the case when V has jumps, where we show weak convergence of the finite-dimensional distributions of the integrated variance to a deterministic time-change of the first passage time process to lower barriers for a more general class of spectrally positive Lévy processes.[2]

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1 Introduction

The Rough Heston stochastic volatility model was introduced in [JR16], and (using $C-$tightness arguments) they show that the model arises naturally as a weak large-time limit of a high-frequency market microstructure model driven by two nearly unstable Hawkes processes. [ALP19], [ER19] and [GK19] show that the characteristic function of the log stock price for Rough Heston-type models can be expressed in terms of the solution to a non-linear Volterra integral equation (VIE) (see also [EFR18], [ER18] and [BLP24], [BPS24], [CT20] for extensions to jumps in V), which allows for accurate option pricing for $H \ll 1$ (and even $H = 0$) using an Adams scheme to solve the VIE numerically. This avoids Monte Carlo (MC) techniques for which traditional MC schemes are notoriously inaccurate for $H \ll 1$ (both in terms of bias and sample variance). From the stochastic Fubini theorem, it is well known that $A_t = \int_0^t V_s ds$ satisfies an equation of the form: $A_t = G_0(t) + \int_0^t \kappa(t-s) W_{A_s} ds$ for some Brownian motion W, which is a non-linear VIE for A in terms of the a.s. $(\frac{1}{2}-\varepsilon)-$Hölder continuous function $W_{(\cdot)}$, and is still well defined even if κ is only in $L^1(0,T)$; this also allows us to consider the hyper-rough regime $H \in (-\frac{1}{2}, 0]$. If we discretize this SVE and re-write in terms of the final (discrete-time) increments of A and W, then by Eq. (E-3) in Appendix E, we can use an independent sequence of Inverse Gaussian random variables to perform an approximate Monte Carlo simulation of A (see Algorithm 1 in [AA25] for details) which is naturally suited to the regimes $H \ll 1$ and $H \in (-\frac{1}{2}, 0]$. The variance process V is $(H-\varepsilon)-$Hölder continuous like the fBM (see e.g. Theorem 3.2 in [JR16]) and the model exhibits power-law skew for implied volatility in the small-time limit (see e.g Theorem 3.1 in [FGS21]/Corollary 3.4 in [FSV21]). See also [ALP19], [Cuch22], [FG24], [FGS21], [FS21] for further related results on rough Heston-type models.

In related work, [?] considers two identical i.i.d. Hawkes processes $N^{\pm}$ with kernel $\phi \in L^1(0,\infty)$ and, assuming that the asset price $P_t = \text{const} \times \lim_{u\to\infty} \mathbb{E}[N_u^+ - N_u^- | \mathcal{F}_t]$, i.e. a constant times the conditional expected value of all future order flow, then P is clearly a martingale, but can also be written in the propagator form $P_t = \int_0^t \zeta(t-u)(dN_u^+ - dN_u^-)$. If $\zeta'(t) = -\zeta(0)\phi(t)$ and if $\zeta(\infty) > 0$, $\zeta(\infty)$ is the (non-transient) permanent price impact component of ζ, and we can compute the expected market impact of an exogenous metaorder executed at constant rate v over duration τ as $MI(t) = v\int_0^{t\wedge\tau} \zeta(t-s)ds$ (see also [JR20] for more on this).

[BL24] improve accuracy in estimating the tail part of the integral for Fourier inversion (for e.g. call options under the rough Heston model) using a sinh contour which goes outside the usual strip of analyticity for the mgf (but is admissible as long as it avoid poles of the characteristic function), and also discuss refinements to the usual Adams schemes for solving the rough Heston VIE. For articles on statistical estimation of H in more general setups, we refer the reader to [?], [?], [?] (and follow-up papers on the LAN property and minimax theorems), and the

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[1] The H parameter in [AC24] is not fixed to be $\frac{1}{2}$, but the Hurst exponent for their family of V^ε processes is of course $\frac{1}{2}$ since the models are all standard Heston for $\varepsilon > 0$.

[2] Note that for citations to results in [ALP19], [BLP24], [BPS24], [FGS21] and [GK19] we are referring to Lemma or Theorem numbers in the published, not the arXiv versions.

asymptotic normality result for $\sqrt{n}(\hat{H}_n - H)$, $\frac{\sqrt{n}}{\log n}(\hat{\sigma}_n - \sigma)$ in Theorem 1 in [?] for σB^H (where B^H is fBM with σ unknown) for the Whittle approximation $(\hat{H}_n, \hat{\sigma}_n)$ for the MLEs in terms of the Fisher information matrix in the high-frequency regime.

[AC24] show that a re-scaled standard Markov Heston model with fast mean-reversion and large vol-of-vol (via an H parameter which is not the Hurst exponent) tends weakly on path space to one of three different models (either Black-Scholes, a Normal Inverse Gaussian or a Normal Lévy model), depending on whether their H parameter is $>$, $=$, or $< -\frac{1}{2}$. [AAR25] obtain a similar result without any ε parameter but instead letting $H \searrow -\frac{1}{2}$ for the so-called *hyper-rough Heston model* (see also Section 5 in [FGS21] and Section 7 in [A21] for more on this model), and exploiting Dirac-type behaviour in their Lemma 2.4 (see Appendix E here for a short summary/formal derivation of their result).

In this note, we fill in the gap between [AC24] and [AAR25], by showing that a conceptually similar result is obtained for any $H \in (0, \frac{1}{2}]$ (in particular our regime for $H = \frac{1}{2}$ corresponds to the regime in Eq. (0.3) in [AC24] with their $H = -\frac{1}{2}$). In Section 2, we extend the model to allow positive jumps in V^ε, and in this case (using the Laplace transform of the hitting time to a lower barrier for a spectrally positive Lévy process), we find that the limiting process for the integrated variance is a deterministic time-change of the first passage time process to lower barriers for a more general class of Lévy processes (the convergence is proved using a compactness argument with the Kolmogorov-Riesz-Fréchet theorem).

1.1 Asymptotics for the terminal log stock price for a re-scaled rough Heston model

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ throughout with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

Proposition 1.1 *Consider a re-scaled rough Heston model for a log stock price process X_t^ε:*

$$\begin{aligned} dX_t^\varepsilon &= -\frac{1}{2}V_t^\varepsilon dt + \sqrt{V_t^\varepsilon} dB_t, \\ V_t^\varepsilon &= V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\frac{1}{\varepsilon}\lambda(\theta - V_s^\varepsilon) ds + \frac{1}{\varepsilon}\nu \sqrt{V_s^\varepsilon} dW_s \Big), \end{aligned} \tag{1}$$

where B and W are two Brownian motions with $dB_t dW_t = \rho dt$ with $\rho \in [-1, 0]$, $\alpha \in (\frac{1}{2}, 1]$ and $\lambda, \nu > 0$. Then (for $t > 0$ fixed) X_t^ε tends weakly to X_t as $\varepsilon \to 0$, where X is a Normal Inverse Gaussian Lévy process which does not depend on $H = \alpha - \frac{1}{2}$.

Proof. Let $I^\alpha(f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds$ denote the αth-order fractional integral of a function f for $\alpha \in (0, 1]$, and (without loss of generality), we assume $X_0^\varepsilon = 0$. Then for $p \in (0, 1)$ (which will be sufficient for our purposes when we invoke the [Bill86] weak convergence result below)

$$\mathbb{E}[e^{pX_t^\varepsilon}] = e^{V_0 I^{1-\alpha}\phi_\varepsilon(t) + \frac{1}{\varepsilon}\lambda\theta I^1 \phi_\varepsilon(t)} \tag{2}$$

where ϕ_ε is the unique solution of the non-linear Volterra integral equation (VIE):

$$\phi_\varepsilon(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\frac{1}{2}(p^2 - p) + \frac{1}{\varepsilon}(\rho p \nu - \lambda)\phi_\varepsilon(s) + \frac{1}{\varepsilon^2}\frac{1}{2}\nu^2 \phi_\varepsilon(s)^2 \Big) ds$$

(see e.g. Section 4 in [ER19] or Section 7 in [ALP19]). Now let $\phi_\varepsilon(t) = \varepsilon\psi(\varepsilon^q t)$. Then

$$\begin{aligned} \varepsilon\psi(\varepsilon^q t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\frac{1}{2}(p^2 - p) + (\rho p \nu - \lambda)\psi(\varepsilon^q s) + \frac{1}{2}\nu^2 \psi(\varepsilon^q s)^2 \Big) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\varepsilon^q t} (t - u\varepsilon^{-q})^{\alpha-1} \Big(\frac{1}{2}(p^2 - p) + (\rho p \nu - \lambda)\psi(u) + \frac{1}{2}\nu^2 \psi(u)^2 \Big) du\, \varepsilon^{-q} \\ &= \frac{\varepsilon^{-q(\alpha-1)}}{\Gamma(\alpha)} \int_0^{\varepsilon^q t} (\varepsilon^q t - u)^{\alpha-1} \Big(\frac{1}{2}(p^2 - p) + (\rho p \nu - \lambda)\psi(u) + \frac{1}{2}\nu^2 \psi(u)^2 \Big) du\, \varepsilon^{-q} \end{aligned}$$

where we set $\varepsilon^q s = u$ in the second line, so $\varepsilon^q ds = du$. Then setting $\varepsilon^q t \mapsto t$, we see that

$$\varepsilon\psi(t) = \frac{\varepsilon^{-q\alpha}}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} F(\psi(u)) du \tag{3}$$

where $F(w) = \frac{1}{2}(p^2 - p) + (\rho p \nu - \lambda)w + \frac{1}{2}\nu^2 w^2$. If now we let $q = -\frac{1}{\alpha}$, the VIE (3) is independent of ε, so

$$\phi_\varepsilon(t) = \varepsilon\psi(\frac{t}{\varepsilon^{\frac{1}{\alpha}}}).$$

Hence for every $t > 0$ (from Lemma 4.5 in [FGS21])

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\phi_\varepsilon(t) \quad=\quad \lim_{t\to\infty}\psi(t) \quad=\quad \psi(\infty) \quad=\quad U_1(p) \quad=\quad \frac{1}{\nu^2}[\lambda - p\nu\rho - \sqrt{\lambda^2 - 2\lambda\rho\nu p + \nu^2 p(1-p\bar{\rho}^2)}]$$

for $p \in (0,1)$, where $\psi(\infty)$ is the smallest root of F and $\bar{\rho}^2 = 1-\rho^2$, and the convergence to $\psi(\infty)$ is rapid for $\varepsilon \ll 1$ since $\frac{1}{\varepsilon}\phi_\varepsilon(t) = \psi(\frac{t}{\varepsilon^{\frac{1}{\alpha}}})$. More precisely, Lemma 4.5 in [FGS21] can be applied to $-\psi$, which solves the following VIE:

$$-\psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}F_1(-\psi(u))du, \quad \text{with } F_1(w) = \frac{1}{2}(p-p^2) + (\rho p\nu - \lambda)w - \frac{1}{2}\nu^2 w^2$$

on $\mathbb{R}_+$, since $F_1(0) > 0$ and F_1 is analytic and decreasing on $\mathbb{R}_+$, the aforementioned result in [FGS21] implies that $-\psi(t)$ converges as $t \to \infty$ to the positive root of F_1, which coincides with $-U_1(p)$. Then for the exponent in (2), we know that $\frac{1}{\varepsilon}\phi_\varepsilon(.) = \psi(\frac{t}{\varepsilon^{\frac{1}{\alpha}}})$ is continuous on $\mathbb{R}_+$ and admits finite limit at ∞, and thus is bounded), so (by the dominated convergence theorem) we see that

$$V_0 I^{1-\alpha}\phi_\varepsilon(t) + \frac{\lambda\theta}{\varepsilon}I^1\phi_\varepsilon(t) \quad=\quad \frac{V_0}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\phi_\varepsilon(s)ds + \lambda\theta\int_0^t \frac{1}{\varepsilon}\phi_\varepsilon(s)ds \quad\to\quad 0 + \lambda\theta U_1(p)t$$

as $\varepsilon \to 0$, and $\lambda\theta U_1(p)t$ is the log mgf of a Normal Inverse Gaussian Lévy process (see e.g. Remark 2.3 in [FJ11]). Then from e.g. the solution to Problem 30.4 on Page 573 in [Bill86] (which is also used in [GK19]), X_t^ε tends weakly to the marginal law of an NIG process. ■

Remark 1.1 The quadratic rough Heston (qRHeston) model has generally been more successful than the RHeston in jointly fitting SPX and VIX smiles,[3] but it seems unclear whether the stock price for the qRHeston model is a true martingale, since the drift term for the Z process explodes quadratically when Z is large under the share measure $\mathbb{P}^*$ (see e.g. [AP07], [Lew00], [Sin98] for background on this phenomenon). For pricing European options with Monte Carlo we can just modify the drift or volatility coefficient of the SVE for Z to circumvent this (but that prevents us from using the exact VIX sampling formula in Chapter 6.2 in [Rom22]); S will of course still be a martingale for a discrete-time Euler-scheme approximation. The Z process in the qRHeston model satisfies $Z_t = Z_0 + \int_0^t \kappa(t-s)\sqrt{aZ_s^2+c}\,dW_s$ for some $\kappa \in L^2$,[4] and for $\kappa(t) = e^{-\lambda t}t^{\alpha-1}$ i.e. the *Gamma kernel* as used in [BG25], $\mathbb{E}[Z_t^2] \geq \mathbb{E}[\tilde{Z}_t^2]$ where $\tilde{Z}$ satisfies the linear SVE $\tilde{Z}_t = Z_0 + \int_0^t \kappa(t-s)\sqrt{a}\tilde{Z}_s dW_s$. However, we find that $\mathbb{E}[\tilde{Z}_t^2] \to \infty$ as $t \to \infty$ when we solve the linear VIE for $\mathbb{E}[\tilde{Z}_t^2]$ using resolvents (because the solution involves a $E_{\alpha,\alpha}(\cdot)$ function with a positive argument tending to $+\infty$), so the Gamma kernel does not produce ergodic behaviour for Z (or V) (of course the presence of λ still "tempers" the behaviour of Z in some sense).

2 Adding jumps into V^ε

We now assume that the forward variance $\xi_t^\varepsilon(u) := \mathbb{E}[V_u^\varepsilon|\mathcal{F}_t]$ satisfies

$$d\xi_t^\varepsilon(u) \quad=\quad \kappa_\varepsilon(u-t)(\sigma\sqrt{V_t^\varepsilon}dW_t + d\tilde{J}_t^\varepsilon), \quad u > t, \tag{4}$$

where $d\tilde{J}_t^\varepsilon = \int_{\mathbb{R}_+} x(N^\varepsilon(dx,dt) - V_t^\varepsilon\nu(dx)dt)$ and $N^\varepsilon(dx,dt)$ is a (time-inhomogenous) Poisson random measure with (random) intensity $V_t^\varepsilon\nu(dx)dt$;[5] ν only has positive support with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}_+} x^2\nu(dx) < \infty$, so $\tilde{J}^\varepsilon$ has positive-only jumps. The kernel κ_ε is defined by $\kappa_\varepsilon(t) = \frac{1}{\varepsilon}t^{\alpha-1}E_{\alpha,\alpha}(-\frac{\lambda}{\varepsilon}t^\alpha)$ with $\alpha \in (\frac{1}{2},1)$ and $\lambda > 0$, where $E_{\alpha,\beta}(z)$ denotes the Mittag-Leffler function. We refer to Remark 2.1 below for relevant examples from the literature that fall within this setting.

The critical observation for the arguments that follow is that $f^{\alpha,\lambda}(t) = \lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ is a probability density, so $\lambda\kappa_\varepsilon(\cdot)$ has Dirac-type behaviour as $\varepsilon \to 0$. We also mention that $\int_t^\infty f^{\alpha,\lambda}(s)\,ds \underset{t\to\infty}{\sim} \frac{1}{\lambda\Gamma(1-\alpha)}t^{-\alpha}$, which implies that

$$\int_t^\infty \lambda\kappa_\varepsilon(s)ds \underset{\epsilon\to 0}{\sim} \frac{1}{\lambda\Gamma(1-\alpha)}\varepsilon t^{-\alpha} \tag{5}$$

for $t > 0$ (again see Appendix A.1 of [ER19] for details on these points).

The variance process $V^\varepsilon = (V_t^\varepsilon)_{t\geq 0}$ is predictable, non-negative, has trajectories in $L^1_{\text{loc}}(\mathbb{R}_+)$ (see e.g. [A21], [ACLP21] and [BLP24]) and satisfies the following SVE of convolution-type with jumps:

$$V_t^\varepsilon = \xi_0^\varepsilon(t) + \int_0^t \kappa_\varepsilon(t-s)(\sigma\sqrt{V_s^\varepsilon}dW_s + d\tilde{J}_s^\varepsilon), \quad \mathbb{P}\otimes dt - \text{a.e.}, \tag{6}$$

[3] see e.g. [BG25] and notes/code on the second author's website.

[4] note we can set the b parameter in the model to zero W.L.O.G, and the diffusion coefficient in the SVE for Z is Lipschitz so we can appeal to the existence and uniqueness result in Theorem 3.3 in [ALP19], and Lemma 3.1 of the same paper.

[5] see also Eq. (1) and the equation below it in [BPS24], Eq. (14) in [BLP24], Section 5 in [CT20] and Slide 6 in [Cuch22].

where $\xi_0^\varepsilon \in L^1_{\text{loc}}(\mathbb{R}_+)$ is the initial variance curve. In particular, $V_t^\varepsilon = \xi_t^\varepsilon(t)$, $\mathbb{P}-$a.s., for a.e. $t \in (0,\infty)$. Note that we require V^ε to be non-negative in order to consider the square root in (6), while the predictability of V^ε with locally integrable paths ensures that the stochastic integrals in (6) are properly defined. We refer to [A21] for weak existence results for (6), see also [ACLP21].

The process V^ε here is an affine Volterra process with jumps and falls under the framework of [BLP24] (see also [BPS24]). In particular, [BLP24, Lemma 1] establishes the following integrability property of V^ε that we will use for our analysis:

$$\mathbb{E}\left[\int_0^T V_t^\varepsilon \, dt\right] < \infty, \quad T > 0. \tag{7}$$

We also refer to [BLP24, Lemma 12] for a stronger L^2-type integrability result which applies to our dynamics in (6) when $\xi_0^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+)$. From (7), we see that

$$\mathbb{E}\left[\int_0^T \left(\int_{\mathbb{R}_+} |x|^2 \nu(dx)\right) V_t^\varepsilon dt\right] < \infty, \quad T > 0,$$

and hence

$$\tilde{J}^\varepsilon = \int_0^\cdot \int_{\mathbb{R}_+} x(N^\varepsilon(dx,dt) - V_t^\varepsilon \nu(dx)dt) \text{ is a square-integrable martingale in } [0,T], \text{ for every } T > 0. \tag{8}$$

Although we do not explicitly consider the stock price process S in this section, a proof of the martingale property for S is given in Section 3 of [BPS24].

Note we are now using σ, not ν, for the vol-of-vol term in (4), since ν is being used here for the Lévy density, and (in the absence of jumps) (4) is the usual equation for the forward variance under the standard rough Heston model, see e.g. [ER18] or Proposition 2.2 in [FGS21]. The model in (4) can be viewed as a generalized rough Heston model in the spirit of [BPS24], where the mean-reversion speed, vol-of-vol, and jump-intensity all scale as $\frac{1}{\varepsilon}$.

Assumption 2.1 *We assume that $\xi_0^\varepsilon(\cdot)$ is non-negative, uniformly bounded and continuous and $\xi_0^\varepsilon(\cdot)$ tends pointwise to a bounded continuous function $\xi_0^0(\cdot)$ as $\varepsilon \to 0$.*

In the following remark, we present two important and standard cases in which Assumption 2.1 is satisfied.

Remark 2.1 The SVE

$$V_t^\varepsilon \quad = \quad V_0 + \int_0^t K(t-s)\frac{1}{\varepsilon}\Big(\lambda(\theta - V_s^\varepsilon)ds + \sigma\sqrt{V_s^\varepsilon}dW_s \, + \, d\tilde{J}_s^\varepsilon\Big) \tag{9}$$

with $K(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ (with the same jump structure for $\tilde{J}^\varepsilon$ as in (4)) is a special case of Eq. (14) in [BLP24] with their $g_0 \equiv V_0$, $b_0 = \frac{1}{\varepsilon}\lambda\theta$, $B = b_1 = -\frac{\lambda}{\varepsilon}$, $A_0 = 0$, $A_1 = \frac{1}{\varepsilon^2}\sigma^2$, $\nu_0 = 0$, $\nu_1(dx) = \frac{1}{\varepsilon}\nu(dx)$. From the last equation on Page 28 in [BLP24], this process is equivalent to (6) (for the corresponding forward variance processes, compare Remark 5 in [BLP24] with (4)), for which their $R_B = \lambda\kappa_\varepsilon$,[6] $E_B = K - R_B * K = \varepsilon\kappa_\varepsilon$, and $\xi_0^\varepsilon(\cdot) = g_0 - R_B * g_0 + E_B * b_0 = V_0 - (V_0 - \theta)\int_0^\cdot f^{\alpha,\frac{\lambda}{\varepsilon}}(s)ds$ (which agrees with Proposition 2.1 in [FGS21]). In this case (from (5)) we find that $\xi_0^\varepsilon(u) \to \theta$ for $u > 0$ as $\varepsilon \to 0$, which we can also easily obtain from Proposition 2.1 in [FGS21] since $\tilde{J}^\varepsilon$ is a martingale, so the jumps do not affect $\xi_0^\varepsilon(t) = \mathbb{E}[V_t^\varepsilon]$.

Conversely, if ξ_0 in (6) is independent of ε and given exogenously, then we can find a g_0 in [BLP24] consistent with ξ_0 by solving the linear VIE $g_0 - R_B * g_0 + E_B * b_0 = \xi_0(\cdot)$; specifically, letting $f = \xi_0(\cdot) - E_B * b_0$, we can re-write the VIE as $g_0 - R_B * g_0 = f$, which has solution $g_0 = \; f - f * r$. Here r is the resolvent of the 2nd kind of $-R_B$ (which will depend on ε in general).[7]

Let $V_1(p) = \int_{\mathbb{R}_+}(e^{px} - 1 - px)\nu(dx)$.[8] Notice that, for every $p \le 0$, $V_1(p) < \infty$ since $|e^{px} - 1 - px| \le p^2|x|^2$ for any $x \in \mathbb{R}_+$.

We now state the main result for this section:

Theorem 2.2 *The finite-dimensional distributions of $A_\cdot^\varepsilon = \int_0^{(\cdot)} V_s^\varepsilon ds$ tend weakly to those of a time-changed Lévy process $X_{g(\cdot)}$, where $X_t = \inf\{s : Z_s < -t\}$, Z is a Lévy process with Lévy triple $(-\lambda, \sigma^2, \nu)$, and $g(t) = \lambda\int_0^t \xi_0^0(u)du$.*

[6] R_B is the resolvent of the second kind of $-KB$, see e.g. Page 13 of [BLP24] for definition.

[7] A general linear VIE of the form $x(\tau) + (k * x)(\tau) = f(\tau)$ has solution $x(\tau) = f(\tau) - (r * f)(\tau)$ where r is the **resolvent** of the 2nd kind of k, which is the unique function r which satisfies $r + r * k = k$ (the resolvent exists if k is locally integrable). To see this, we substitute $x(\tau)$ into the VIE to get $x \, + \, k * x = x + k * (f - r * f) = x + (k - k * r) * f = x + r * f = f$.

[8] an instructive example to keep in mind here is the case when V_1 is the cgf for a one-sided tempered stable (CGMY) process with $\nu(dx) = \frac{Ce^{-Mx}}{x^{1+Y}}1_{\{x>0\}}$ for $C, M > 0$ and $Y \in (0,2) \setminus \{1\}$, for which $V_1(p) = C(M(M-p)^Y + M^Y(-M + pY))\Gamma(-Y))/M$.

Remark 2.2 X is a Lévy subordinator, see e.g. Theorem 46.2 in [Sato99]; in particular, if we let $(\tilde{\lambda},0,\tilde{\nu})$ denote the Lévy triple of X with respect to the truncation function $h(x)\equiv 0$, then X has no Gaussian component, $\tilde{\nu}(\mathbb{R}_-)=0$ and $\int_{[0,1]}x\tilde{\nu}(dx)<\infty$, see e.g. Theorem 21.5 in [Sato99]. Moreover, if $\tilde{\nu}(dx)$ has a density with respect to Lebesgue measure, i.e. $\tilde{\nu}(dx)=\tilde{\nu}(x)dx$, then the Lévy Khintchine-type formula in Theorem 25.17 in [Sato99] for X_T is:

$$\mathbb{E}[e^{-pX_T}]=e^{T(-\tilde{\lambda}p+\int_0^\infty(e^{-px}-1)\tilde{\nu}(x)dx)}$$

for $p\geq 0$. Thus $\tilde{V}(-p):=\log\mathbb{E}[e^{-pX_1}]=-\tilde{\lambda}p+\int_0^\infty(e^{-px}-1)\tilde{\nu}(x)dx$, and differentiating both sides with respect to p, we obtain:

$$\tilde{V}'(-p)\quad=\quad\tilde{\lambda}+\int_0^\infty xe^{-px}\tilde{\nu}(x)dx$$

for $p\geq 0$, so in principle we can recover $\tilde{\nu}$ from $\tilde{V}$ by Laplace inversion.

Proof. Recall that the dynamics of the variance process V^ε are given in (6). Let $f:[0,\infty)\to(-\infty,0]$ be a locally bounded function on $\mathbb{R}_+$ (i.e., $f\in L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}_-)$). Then from Appendix A (see also [BLP24] and [BPS24]), we know that for every $T>0$

$$M_t\quad:=\quad e^{\int_0^t f(T-s)V_s^\varepsilon ds+G_t},\quad t\in[0,T]\tag{10}$$

is a martingale if [9]

$$G_t\quad=\quad\int_t^T g_\varepsilon(T-s)\xi_t^\varepsilon(s)ds\quad=\quad\int_0^{T-t}G(u,\psi_\varepsilon(u))\,\xi_t^\varepsilon(T-u)du,$$

where g_ε is a locally bounded function and $\psi_\varepsilon(\tau):=\int_0^\tau\kappa_\varepsilon(\tau-r)g_\varepsilon(r)dr$ satisfies the non-linear Riccati-Volterra integral equation

$$\psi_\varepsilon(t)\quad=\quad\int_0^t\kappa_\varepsilon(t-s)G(s,\psi_\varepsilon(s))ds\tag{11}$$

with

$$G(s,w)\quad=\quad f(s)+\frac{1}{2}\sigma^2w^2+V_1(w).\tag{12}$$

Existence and uniqueness for this equation is established in Lemma 2.3 below. Since $G_T=0$ we see that

$$M_t\quad=\quad\mathbb{E}[M_T|\mathcal{F}_t]\quad=\quad\mathbb{E}\Big[e^{\int_0^T f(T-s)V_s^\varepsilon ds}|\mathcal{F}_t\Big].$$

In particular, at $t=0$, taking the expected value we have

$$\mathbb{E}\Big[e^{\int_0^T f(T-s)V_s^\varepsilon ds}\Big]\quad=\quad e^{\int_0^T G(s,\psi_\varepsilon(s))\xi_0^\varepsilon(T-s)ds},\tag{13}$$

which is the main equation we need.

Lemma 2.3 *There exists a unique continuous solution $\psi_\varepsilon:\mathbb{R}_+\to\mathbb{R}_-$ of Eq.* (11).

Proof. See Appendix B. ■

Remark 2.3 One can give a simpler proof of Lemma 2.3 using a standard fixed point argument if we assume that λ is sufficiently large.[10]

Formally, the asymptotic solution to (11) comes from considering its Dirac limit as $\varepsilon\to 0$:

$$\psi_0(t)\quad=\quad\frac{1}{\lambda}G(t,\psi_0(t))$$

(recall from above that $\lambda\kappa_\varepsilon(\cdot)$ has Dirac-type behaviour as $\varepsilon\to 0$ so $\lambda\kappa_\varepsilon(t-s)$ will be concentrated at $s=t$). Re-arranging terms here, we obtain our conjecture limit equation:

$$f(t)-\lambda\psi_0(t)+\bar{G}(\psi_0(t))=-\lambda\psi_0(t)+G(t,\psi_0(t))=0,\quad t\geq 0,\tag{14}$$

where G is defined as in (12) and $\bar{G}(w)=\frac{1}{2}\sigma^2w^2+\int_{\mathbb{R}_+}(e^{xw}-1-xw)\nu(dx)$ for $w\leq 0$. Since $G(t,0)=f(t)\leq 0$ and the function $w\mapsto G(t,w)-\lambda w$ is continuous and decreasing on $\mathbb{R}_-$, with $G(t,w)-\lambda w\to\infty$ as $w\to-\infty$, we see that there exists a unique non-positive solution $\psi_0:\mathbb{R}_+\to\mathbb{R}_-$ to (15). In particular, since f is locally bounded on $\mathbb{R}_+$, ψ_0 is locally bounded on $\mathbb{R}_+$, as well (i.e., $\psi_0\in L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}_-)$). We now give the main technical lemma of the theorem:

[9] the final expression in the next equation just follows from setting $u=T-s$ in the left integral; then $s=T-u$ and $ds=-du$.
[10] proof available on request.

Lemma 2.4 *For every $T > 0$,*

$$\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi_0 \quad \text{in } L^1(0,T).$$

Proof. See Appendix C. ■

Remark 2.4 As with existence and uniqueness above, one can again give a simpler proof here (specifically *pointwise* convergence for ψ_ε) if λ is sufficiently large, using a Lipschitz argument with a comparison result for linear VIEs using resolvents.[11]

As a corollary of Lemma 2.4, we see that

$$\begin{aligned}\int_0^T |G(s,\psi_\varepsilon(s)) - G(s,\psi_0(s))|ds &\le \int_0^T \Big(\frac{1}{2}\sigma^2|\psi_\varepsilon(s)+\psi_0(s)| + |\tilde{h}(\psi_\varepsilon(s),\psi_0(s))|\Big)|\psi_\varepsilon(s)-\psi_0(s)|ds \\ &\le K_1 \int_0^T |\psi_\varepsilon(s)-\psi_0(s)|ds \underset{\varepsilon\to 0}{\longrightarrow} 0,\end{aligned} \tag{15}$$

where $\tilde{h}$ is the function defined in (B-6), and

$$K_1 := \sup_{\varepsilon>0} \Big\|\frac{1}{2}\sigma^2|\psi_\varepsilon+\psi_0| + |\tilde{h}(\psi_\varepsilon,\psi_0)|\Big\|_{L^\infty(0,T)};$$

since ψ_0 is locally bounded on $\mathbb{R}_+$, K_1 is finite by (C-1) and $\tilde{h}$ is continuous, see Appendix B. Then

$$\begin{aligned}\left|\int_0^T \Big(G(s,\psi_\varepsilon(s))\xi_0^\varepsilon(T-s) - G(s,\psi_0(s))\xi_0^0(T-s)\Big)ds\right| \;&\le\; \int_0^T |G(s,\psi_\varepsilon(s)) - G(s,\psi_0(s))|\,\xi_0^0(T-s)ds \\ &+\; \Big(\sup_{\varepsilon>0}\|G(\cdot,\psi_\varepsilon)\|_{L^\infty(0,T)}\Big)\int_0^T |\xi_0^\varepsilon(T-s)-\xi_0^0(T-s)|ds\end{aligned}$$

where the last two terms tend to zero as $\varepsilon\to 0$ by (16) and Assumption 2.1. Hence from (13) and (15) we see that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\Big[e^{\int_0^T f(T-s)V_s^\varepsilon ds}\Big] &= \lim_{\varepsilon\to 0} e^{\int_0^T G(s,\psi_\varepsilon(s))\xi_0^\varepsilon(T-s)ds} \\ &= e^{\int_0^T G(s,\psi_0(s))\xi_0^0(T-s)ds} \;=\; e^{\lambda\int_0^T \psi_0(s)\xi_0^0(T-s)ds}.\end{aligned} \tag{16}$$

We now characterize the process which has (17) as its characteristic function.
For a Lévy process Z with Lévy triple $(-\lambda,\sigma^2,\nu)$, $e^{uZ_t-\Lambda(u)t}$ is an $\mathcal{F}_t^Z$−martingale for $u\le 0$, where $\Lambda(u) = -\lambda u + \frac{1}{2}\sigma^2u^2 + V_1(u)$; this is a consequence of the stationary and independent increments property, together with Theorem 25.17 in [Sato99] (see (D-4) below for an analogous argument). From Proposition D.1 in Appendix D applied to the spectrally negative process $-Z$ (see also e.g. Theorem 46.3 in [Sato99] or Eq. (2.5) in [KKR13] for an alternative proof), we know that

$$\mathbb{E}[e^{-qH_a}] \;=\; e^{-\Lambda^{-1}(q)a} \;=\; e^{\Lambda^{-1}(q)|a|} \tag{17}$$

for $q\ge 0$ and $a\le 0$, where $H_a := \inf\{t\ge 0 : Z_t < a\}$ and $\Lambda^{-1}(q)\le 0$ denotes the inverse function of Λ.

Now let $X_t = H_{-t}$ (not the same X process as in Subsection 1.1), which is a Lévy subordinator, see Remark 2.2 and the references therein. Then from the i.i.d. property for Lévy processes, (18) and the fundamental theorem of calculus, considering a right continuous non-positive piecewise constant function f we have, for any continuously differentiable nondecreasing function g starting from 0,

$$\mathbb{E}\Big[e^{\int_0^T f(T-s)d(X_{g(s)})}\Big] \;=\; e^{\int_0^T \Lambda^{-1}(|f(T-s)|)g'(s)ds} \;=\; e^{\int_0^T \Lambda^{-1}(-f(s))g'(T-s)ds} \tag{18}$$

(see also Lemma 15.1 in [CT04], which is used in [AAR25]). But $\Lambda^{-1}(-f(\cdot))$ is $\psi_0(\cdot)$ from Eq. (15). Hence for $0\le s_1<\cdots<s_n<T$, choosing a right continuous map f such that

$$f(T-s) = (u_1+u_2+...+u_n)1_{\{0\le s\le s_1\}} + (u_2+...+u_n)1_{\{s_1<s\le s_2\}} + ... + u_n1_{\{s_{n-1}<s\le s_n\}}, \quad s\in[0,T]$$

with $u_1,u_2,\dots,u_n\le 0$, we see that

$$\int_0^T f(T-s)V_s^\varepsilon ds \;=\; u_1A_{s_1}^\varepsilon + ... + u_nA_{s_n}^\varepsilon, \qquad \int_0^T f(T-s)dX_{g(s)} \;=\; u_1X_{g(s_1)} + ... + u_nX_{g(s_n)},$$

where $A_t^\varepsilon = \int_0^t V_s^\varepsilon ds$. Therefore by (17) and (19) (again by Problem 30.4 in [Bill86]) we deduce that the finite-dimensional distributions of $A_\cdot^\varepsilon = \int_0^{(\cdot)} V_s^\varepsilon ds$ converge weakly to those of the time-changed Lévy process $X_{g(t)}$ with $g'(t) = \lambda\xi_0^0(t)$ and $g(0)=0$. The proof is now complete. ■

[11] proof available on request.

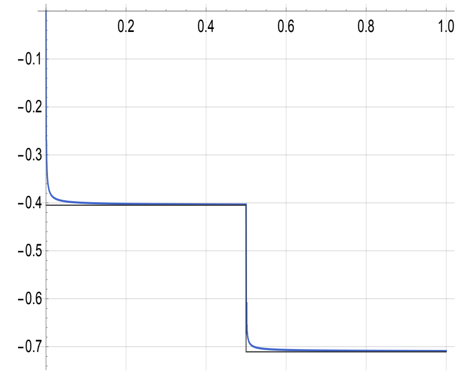

Figure 1: Here we have plotted ψ_ε in Eq. (20) (in blue) and ψ_0 in Eq. (15) (grey dashed), using an Adams scheme with 2000 time steps with $\varepsilon = .01$, $H = 0.2$, $\nu = .4$, $\lambda = 1$, $T = 1$, $f(s) = -\frac{1}{2}(1_{\{s<\frac{1}{2}\}} + 1_{\{s\leq 1\}})$ and $\nu(x) = \frac{Ce^{-Mx}}{x^{1+Y}} 1_{\{x>0\}}$ for $C = 1$, $M = 3$ and $Y = 1.5$, and we see convergence to ψ_0 (see e.g. [BL24] for details on refinements to Adams schemes). Numerically solving the VIE in Eq. (11) for $\varepsilon \ll 1$ appears to be much harder due to the Dirac nature of the kernel.

Remark 2.5 For the process V^ε in (9), if we instead set $M_t = e^{\int_0^t f(T-s)V_s^\varepsilon ds + G_t}$ with $G_t = \int_t^T \tilde{g}_\varepsilon(T-s) g_t^\varepsilon(s) ds$ and we define the "adjusted forward process" $g_t^\varepsilon(s)$ as in Eq. (3) in [BPS24] (see also Eq. (37) in [BLP24]) by

$$g_t^\varepsilon(s) \quad = \quad V_0 + \frac{\lambda\theta}{\varepsilon}\int_0^s K(r)dr + \int_0^t K(s-r)\frac{1}{\varepsilon}\Big(-\lambda V_r^\varepsilon dr + \sigma\sqrt{V_r^\varepsilon} dW_r + d\tilde{J}_r^\varepsilon\Big)$$

for $s > t$, so $g_t^\varepsilon(t) = V_t^\varepsilon$, $\mathbb{P}-$a.s., for a.e. $t > 0$ (with $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$), then following arguments analogous to those in Appendix A we can check that ψ_ε in (11) satisfies

$$\varepsilon\psi_\varepsilon(\tau) \quad = \quad \int_0^\tau K(\tau-s)\Big(f(s) - \lambda\psi_\varepsilon(s) + \frac{1}{2}\sigma^2\psi_\varepsilon(s)^2 + V_1(\psi_\varepsilon(s))\Big)ds, \quad \tau \geq 0. \tag{19}$$

We refer to Remark 5 in [BLP24] (see also Lemma 4.4 in [ALP19]) for a variation of constants argument showing the equivalence between (11) and (20). According to Lemma 2.4, the limiting solution as $\varepsilon \to 0$ is ψ_0 (we test this numerically in Figure **??**).

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Appendix

A Derivation of the VIE for ψ_ε

We set $\varepsilon = 1$ and drop the ε superscripts to ease notation since the arguments will be exactly the same for general $\varepsilon > 0$. Fix $T > 0$. Recall that $G_t = \int_t^T g(T-s)\xi_t(s)ds$ for a locally bounded function g, $d\xi_t(u) = \kappa(u-t)(\sigma\sqrt{V_t}dW_t + d\tilde{J}_t)$ and $M_t = e^{\int_0^t f(T-s)V_s ds + G_t}$. In integral form

$$\xi_t(u) = \xi_0(u) + \int_0^t \kappa(u-s)(\sigma\sqrt{V_s}dW_s + d\tilde{J}_s), \quad u > t;$$

since $V_u = \xi_0(u) + \int_0^u \kappa(u-s)(\sigma\sqrt{V_s}dW_s + d\tilde{J}_s)$, it follows that

$$V_u 1_{\{u\le t\}} + \xi_t(u)1_{\{u>t\}} = \xi_0(u) + \int_0^t 1_{\{s\le u\}}\kappa(u-s)(\sigma\sqrt{V_s}dW_s + d\tilde{J}_s). \tag{A-1}$$

Then we can re-write $G_t = \int_t^T g(T-s)\xi_t(s)ds$ as

$$\begin{aligned}
G_t &= \int_t^T g(T-s)\xi_0(s)ds + \int_t^T g(T-s)\left(\xi_t(s)-\xi_0(s)\right)ds \\
&= \int_0^T g(T-s)\xi_0(s)ds - \int_0^t g(T-s)\xi_0(s)ds + \int_t^T g(T-s)\left(\xi_t(s)-\xi_0(s)\right)ds \\
&= \int_0^T g(T-s)\xi_0(s)ds - \int_0^t g(T-s)V_s ds + \int_0^t g(T-s)(V_s-\xi_0(s))ds + \int_t^T g(T-s)\left(\xi_t(s)-\xi_0(s)\right)ds \\
&= \int_0^T g(T-s)\xi_0(s)ds - \int_0^t g(T-s)V_s ds + \int_0^T g(T-s)\Big((V_s-\xi_0(s))1_{\{s\le t\}} + (\xi_t(s)-\xi_0(s))1_{\{s>t\}}\Big)ds \\
&= \int_0^T g(T-s)\xi_0(s)ds - \int_0^t g(T-s)V_s ds + \int_0^T g(T-s)\Big(\int_0^t 1_{\{r\le s\}}\kappa(s-r)(\sigma\sqrt{V_r}dW_r + d\tilde{J}_r)\Big)ds \\
&= \int_0^T g(T-s)\xi_0(s)ds - \int_0^t g(T-s)V_s ds + \int_0^t \Big(\int_r^T \kappa(s-r)g(T-s)ds\Big)(\sigma\sqrt{V_r}dW_r + d\tilde{J}_r).
\end{aligned}$$

Here, in the second [resp., third] equality we add and subtract $\int_0^t g(T-s)\xi_0(s)ds$ [resp., $\int_0^t g(T-s)V_s ds$], in the fourth we combine the last two integrals on the previous line using indicator functions, in the fifth we apply (A-1) and the last equality holds by the stochastic Fubini theorem, see [Prot05, Theorem IV.65]. More specifically, the application of this version of the stochastic Fubini theorem is justified because

$$\int_0^T 1_{\{s>r\}}|\kappa(s-r)|^2|g(T-s)|^2 ds \le \|\kappa\|^2_{L^2(0,T)}\|g\|^2_{L^\infty(0,T)}, \quad r\in[0,t].$$

Recalling that $d\tilde{J}_t = \int_{\mathbb{R}_+} x(N(dx,dt) - V_t\nu(dx)dt)$ is a square-integrable martingale (see (8)), the boundedness of the process $r \mapsto \int_0^T 1_{\{s>r\}}|\kappa(s-r)|^2|g(T-s)|^2 ds$ in $[0,t]$ ensures that

$$\left(\int_0^T 1_{\{s>\cdot\}}|\kappa(s-\cdot)|^2|g(T-s)|^2 ds\right)^{\frac{1}{2}} \text{ is integrable in the semimartingale } \sigma\int_0^{\cdot}\sqrt{V_r}dW_r + \tilde{J}.$$

Additionally, for a.e. $s \in [0,T]$, the process

$$g(T-s)\int_0^u 1_{\{r<s\}}\kappa(s-r)(\sigma\sqrt{V_r}dW_r + d\tilde{J}_r), \quad u \in [0,t] \tag{A-2}$$

is càdlàg in $[0,t]$. Indeed, by Tonelli's theorem

$$\begin{aligned}
&\int_0^T |g(T-s)|^2\mathbb{E}\left[\int_0^t 1_{\{r<s\}}|\kappa(s-r)|^2\left(\sigma^2 + \int_{\mathbb{R}_+}|x|^2\nu(dx)\right)V_r dr\right]ds \\
&\quad\leq \|g\|^2_{L^\infty(0,T)}\left(\sigma^2 + \int_{\mathbb{R}_+}|x|^2\nu(dx)\right)\mathbb{E}\left[\int_0^t\left(\int_0^T 1_{\{r<s\}}|\kappa(s-r)|^2 ds\right)V_r dr\right] \\
&\quad\leq \|\kappa\|^2_{L^2(0,T)}\|g\|^2_{L^\infty(0,T)}\left(\sigma^2 + \int_{\mathbb{R}_+}|x|^2\nu(dx)\right)\mathbb{E}\left[\int_0^t V_r dr\right] < \infty,
\end{aligned}$$

where for the final estimate we use that $V \in L^1(\Omega\times[0,T])$, see (7). Therefore

$$\mathbb{E}\left[\int_0^t 1_{\{r<s\}}|\kappa(s-r)|^2\left(\sigma^2 + \int_{\mathbb{R}_+}|x|^2\nu(dx)\right)V_r dr\right] < \infty, \quad \text{for a.e. } s \in [0,T].$$

This demonstrates that, for a.e. $s \in [0,T]$, the process $r \mapsto g(T-s)1_{\{r<s\}}\kappa(s-r)$ is integrable with respect to the (semi)martingale $\sigma\int_0^{\cdot}\sqrt{V_r}dW_r + \tilde{J}$ in $[0,t]$. Thus, the desired càdlàg property of the stochastic integrals in (A-2) follows by construction.

By Ito's lemma, supposing that $t \mapsto \int_t^T \kappa(s-t)g(T-s)ds$ is non-positive and denoting by $N(dx,dt)$ the random measure with compensator $V_t\nu(dx)dt$, we infer that

$$\begin{aligned}
\frac{dM_t}{M_{t-}} &= f(T-t)V_t dt - g(T-t)V_t dt + \left(\int_t^T \kappa(s-t)g(T-s)ds\right)(\sigma\sqrt{V_t}dW_t + d\tilde{J}_t) \\
&\quad + \frac{1}{2}\sigma^2\left(\int_t^T \kappa(s-t)g(T-s)ds\right)^2 V_t dt \\
&\quad + \int_{\mathbb{R}_+}\left(e^{x\int_t^T\kappa(s-t)g(T-s)ds} - 1 - x\left(\int_t^T\kappa(s-t)g(T-s)ds\right)\right)N(dx,dt) \\
&= V_t\left(f(T-t) - g(T-t) + \frac{1}{2}\sigma^2\left(\int_t^T\kappa(s-t)g(T-s)ds\right)^2 + V_1\left(\int_t^T\kappa(s-t)g(T-s)ds\right)\right)dt \\
&\quad + \text{loc. martingale term},
\end{aligned} \tag{A-3}$$

where in the last equality we add and subtract $V_t V_1(\int_t^T \kappa(s-t)g(T-s)ds)dt$ (recall that the map V_1 is defined before the statement of Theorem 2.2). Then we see that M_t is a local martingale if g satisfies the VIE:

$$g(T-t) = f(T-t) + \frac{1}{2}\sigma^2\left(\int_t^T g(T-s)\kappa(s-t)ds\right)^2 + V_1\left(\int_t^T\kappa(s-t)g(T-s)ds\right), \quad \text{for a.e. } t \in [0,T]. \tag{A-4}$$

Setting $\tau = T-t$ and $r = T-s$ (so $s = T-r$ and $s-t = T-r-(T-\tau) = \tau - r$), we can re-write $\int_{T-\tau}^T g(T-s)\kappa(s-t)ds = \int_0^\tau \kappa(\tau-r)g(r)dr$, so

$$g(\tau) \quad = \quad f(\tau) + \frac{1}{2}\sigma^2\left(\int_0^\tau \kappa(\tau-r)g(r)dr\right)^2 + V_1\left(\int_0^\tau\kappa(\tau-r)g(r)dr\right).$$

Setting $\psi(\tau) = \int_0^\tau \kappa(\tau-r)g(r)dr$ and taking the convolution of both sides with κ, we can re-cast this in terms of ψ as

$$\psi(\tau) \quad = \quad \int_0^\tau \kappa(\tau-s)\left(f(s) + \frac{1}{2}\sigma^2\psi(s)^2 + V_1(\psi(s))\right)ds\,.$$

We now argue that, when g satisfies (A-4), M_t is in fact a true martingale.[12] From (A-3), since the drift term vanishes by (A-4), expanding the expression of the local martingale term yields

$$\frac{dM_t}{M_{t-}} = \left(\int_t^T\kappa(s-t)g(T-s)ds\right)\sigma\sqrt{V_t}dW_t + \int_{\mathbb{R}_+}\left(e^{x\int_t^T\kappa(s-t)g(T-s)ds} - 1\right)(N(dx,dt) - V_t\nu(dx)dt),$$

[12] see also related discussion below Eq. (2.19) in [GK19].

which means that, denoting by $\mathcal{E}$ the Doléans-Dade exponential,

$$
\begin{aligned}
M_t =& e^{\int_0^T g(T-s)\xi_0(s)ds} \\
&\times \mathcal{E}\Big(\int_0^t \Big(\int_r^T \kappa(s-r)g(T-s)ds \Big) \sigma\sqrt{V_r}dW_r + \int_0^t \int_{\mathbb{R}_+} \Big(e^{x\int_r^T \kappa(s-r)g(T-s)ds} - 1 \Big)(N(dx,dr) - V_r\nu(dx)dr) \Big).
\end{aligned}
$$

Since $t \mapsto \int_t^T \kappa(s-t)g(T-s)ds$ is non-positive and bounded, the martingale property of M_t is a consequence of Lemma 6.1 in [A21], see also Lemma 3.2 in [BPS24].

B Existence and uniqueness for the VIE for ψ_ε

We first recall the Riccati-Volterra integral equation in (11), which we re-write as

$$
\begin{aligned}
\psi_\varepsilon(t) &= \int_0^t \kappa_\varepsilon(t-s)f(s)ds + \int_0^t \kappa_\varepsilon(t-s)\Big(\frac{1}{2}\sigma^2\psi_\varepsilon^2(s) + \int_{\mathbb{R}_+} (e^{x\psi_\varepsilon(s)} - 1 - x\psi_\varepsilon(s))\nu(dx) \Big) ds \\
&= (\kappa_\varepsilon * f)(t) + \Big(\kappa_\varepsilon * \Big(\frac{1}{2}\sigma^2\psi_\varepsilon^2 + V_1(\psi_\varepsilon) \Big) \Big)(t), \quad t \geq 0. \qquad \text{(B-1)}
\end{aligned}
$$

By Theorem 3.1, Chapter 5 in [GLS90], κ_ε is completely monotone, and by Theorem 2.2, Chapter 2 in [GLS90], $t \mapsto (\kappa_\varepsilon * f)(t)$ is continuous on $\mathbb{R}_+$ (as $f \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}_-)$). Moreover, the function $\widetilde{G}$ defined by the relation

$$
\widetilde{G}(w) - \frac{1}{2}\sigma^2 w^2 = \begin{cases} 0, & w > 0 \\ \int_{\mathbb{R}_+} (e^{xw} - 1 - xw)\nu(dx), & w \leq 0 \end{cases}
$$

is continuous and non-negative on $\mathbb{R}$. Then by Theorem 1.1, Chapter 12 in [GLS90], there exists a continuous noncontinuable[13] local solution $\widetilde{\psi}_\varepsilon$ of the equation

$$
\widetilde{\psi}_\varepsilon(t) = (\kappa_\varepsilon * f)(t) + (\kappa_\varepsilon * \widetilde{G}(\widetilde{\psi}_\varepsilon))(t), \quad t \in [0, T_{\max}), \qquad \text{(B-2)}
$$

for some $T_{\max} > 0$. Then, noting that $\widetilde{G} = \frac{1}{2}\sigma^2(\cdot)^2 + V_1(\cdot)$ on $\mathbb{R}_-$, from the following lemma we know that $\widetilde{\psi}_\varepsilon$ also solves (B-1) on $[0, T_{\max})$:

Lemma B.1 *$\widetilde{\psi}_\varepsilon$ is non-positive.*

Proof. For convenience we define $h\colon \mathbb{R} \to \mathbb{R}_-$ by

$$
h(w) = \begin{cases} \frac{1}{w}\int_{\mathbb{R}_+} (e^{wx} - 1 - wx)\nu(dx), & w < 0, \\ 0, & w \geq 0; \end{cases} \qquad \text{(B-3)}
$$

(which is continuous and non-positive), so $\frac{1}{2}\sigma^2 w^2 + w \cdot h(w) = \widetilde{G}(w)$ for $w \in \mathbb{R}$.

Then, for every $T \in (0, T_{\max})$, from (B-2)

$$
\widetilde{\psi}_\epsilon(t) = (\kappa_\varepsilon * f)(t) + \int_0^t \kappa_\varepsilon(t-s)\Big(\frac{1}{2}\sigma^2\widetilde{\psi}_\varepsilon(s) + h(\widetilde{\psi}_\varepsilon(s)) \Big)\widetilde{\psi}_\varepsilon(s)ds, \quad t \in [0, T].
$$

By Remark B.6 in [AE19] (which allows to consider a possibly discontinuous function f) and recalling that $f \leq 0$, this reformulation enables us to use Theorem C.1 in [AE19] and conclude that $\widetilde{\psi}_\varepsilon \leq 0$ in $[0, T_{\max})$, as T is arbitrary. ■

To prove that a noncontinuable $\mathbb{R}_-$−valued solution ψ_ε of (B-1) is global, we note that $\frac{1}{2}\sigma^2\psi_\varepsilon^2 + V_1(\psi_\varepsilon) \geq 0$ in $[0, T_{\max})$, so (B-1) yields

$$
(\kappa_\varepsilon * f)(t) \leq \psi_\varepsilon(t) \leq 0, \quad t \in [0, T_{\max}). \qquad \text{(B-4)}
$$

Since $\kappa_\varepsilon * f$ is continuous and dominates ψ_ε on $[0, T_{\max})$, ψ_ε cannot explode to $-\infty$ at $T_{\max}$. Then, given that by Theorem 1.1, Chapter 12 in [GLS90],

$$
\limsup_{t \to T_{\max}} |\psi_\varepsilon(t)| = \infty \quad \text{if } T_{\max} < \infty,
$$

we conclude that $T_{\max} = \infty$.

To establish uniqueness, let ψ_1, ψ_2 be two $\mathbb{R}_-$−valued solutions of (B-1) defined on $\mathbb{R}_-$. Then setting $\delta = \psi_1 - \psi_2$,

$$
\delta(t) = \int_0^t \kappa_\varepsilon(t-s)\Big(\frac{1}{2}\sigma^2(\psi_1(s) + \psi_2(s))\delta(s) + \int_{\mathbb{R}_+} \Big(e^{x\psi_1(s)} - e^{x\psi_2(s)} - x(\psi_1(s) - \psi_2(s)) \Big)\nu(dx) \Big) ds. \qquad \text{(B-5)}
$$

[13] see Page 343 in [GLS90] for definition.

Now let $\mathbb{R}_-^2 = \{(w_1,w_2) \in \mathbb{R}^2, w_1 \le 0 \text{ and } w_2 \le 0\}$ denote the negative quadrant of the plane, and define the auxiliary function $\tilde{h}\colon \mathbb{R}_-^2 \to \mathbb{R}$ by

$$\tilde{h}(w_1,w_2) = \begin{cases} \frac{1}{w_1-w_2}\int_{\mathbb{R}_+}(e^{xw_1}-e^{xw_2}-x(w_1-w_2))\nu(dx), & w_1 \neq w_2, \\ \int_{\mathbb{R}_+} x(e^{xw_1}-1)\nu(dx), & \text{otherwise.} \end{cases} \tag{B-6}$$

The map $\tilde{h}$ is non-positive on its domain $\mathbb{R}_-^2$, because $w \mapsto e^{xw} - xw$ is nonincreasing on $\mathbb{R}_-$ for every $x \in \mathbb{R}_+$. By the dominated convergence theorem $\tilde{h}$ is continuous on $\mathbb{R}_-^2 \setminus \{(w_1,w_2), w_1 \neq w_2\}$. To prove the continuity at the points $(w_1,w_2) \in \mathbb{R}_-^2$ with $w_1 = w_2$, consider two sequences $(w_{1,n})_n, (w_{2,n})_n \subset \mathbb{R}_-$ such that

$$w_{1,n} \neq w_{2,n} \quad \text{and} \quad \lim_{n\to\infty} w_{1,n} = w_1 = w_2 = \lim_{n\to\infty} w_{2,n}.$$

Without loss of generality, suppose that $w_{1,n} > w_{2,n}$ for every $n \in \mathbb{N}$. We then compute, using the inequality $|e^u - 1 - u| \le |u|^2$ for $u \in \mathbb{R}_-$,

$$\begin{aligned} &\left|\tilde{h}(w_{1,n},w_{2,n}) - \tilde{h}(w_1,w_2)\right| \\ &\quad \le \frac{1}{|w_{1,n}-w_{2,n}|}\int_{\mathbb{R}_+} e^{xw_{1,n}}\left|e^{x(w_{2,n}-w_{1,n})} - 1 - x(w_{2,n}-w_{1,n})\right|\nu(dx) + \int_{\mathbb{R}_+} x\left|e^{xw_{1,n}} - e^{xw_1}\right|\nu(dx) \\ &\quad \le \left(\int_{\mathbb{R}_+} |x|^2\nu(dx)|\right)|w_{2,n}-w_{1,n}| + \mathrm{o}(1) \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Considering that $\tilde{h}(w_{1,n},w_{1,n}) \to \tilde{h}(w_1,w_2)$ by dominated convergence, the previous computations prove the continuity of $\tilde{h}$ at (w_1,w_2) with $w_1 = w_2$.

From the definition of $\tilde{h}$, we have

$$\int_{\mathbb{R}_+}\left(e^{x\psi_1(s)} - e^{x\psi_2(s)} - x(\psi_1(s)-\psi_2(s)\right)\nu(dx) = \tilde{h}(\psi_1(s),\psi_2(s))\delta(s), \quad s \ge 0,$$

hence (from (B-5)) δ solves the linear VIE:

$$\delta(t) = \int_0^t \kappa_\varepsilon(t-s)\left(\frac{1}{2}\sigma^2(\psi_1(s)+\psi_2(s)) + \tilde{h}(\psi_1(s),\psi_2(s))\right)\delta(s)ds.$$

This equation admits $\delta \equiv 0$ as its unique solution by the first part of Theorem C.1 in [AE19], whence we conclude that $\psi_1 = \psi_2$. This proves that (B-1) has a unique continuous global $\mathbb{R}_-$-valued solution.

C L^1-convergence for ψ_ε

We recall that f is a locally bounded non-positive function defined on $\mathbb{R}_+$, which we also write as $f \in L^\infty_{\text{loc}}(\mathbb{R}_+;\mathbb{R}_-)$.

C.1 Relative compactness in L^1

Fix $T > 0$ and a sequence $(\varepsilon_n)_n \subset (0,\infty)$ which converges to 0.

Lemma C.1 *$(\psi_{\varepsilon_n})_n$ admits a convergent subsequence in $L^1(0,T)$.*

Proof. $(\psi_{\varepsilon_n})_n$ is bounded in $L^1(0,T)$ because (from (B-4)) we know that

$$\|\psi_\varepsilon\|_{L^\infty(0,T)} \le \|\kappa_\varepsilon * f\|_{L^\infty(0,T)} \le \|\kappa_\varepsilon\|_{L^1(0,T)}\|f\|_{L^\infty(0,T)} \le \frac{1}{\lambda}\|f\|_{L^\infty(0,T)}. \tag{C-1}$$

Now define the extended maps $\bar{\psi}_n\colon \mathbb{R} \to \mathbb{R}_-$ by

$$\bar{\psi}_n(t) = \begin{cases} \psi_{\varepsilon_n}(t), & t \in [0,T], \\ 0, & \text{otherwise.} \end{cases}$$

To prove the lemma, by the Kolmogorov-Riesz-Fréchet theorem (see e.g. Theorem 4.26 in [Brez11]), it suffices to show that

$$\lim_{h\to 0}\|\tau_h\bar{\psi}_n - \bar{\psi}_n\|_{L^1(\mathbb{R})} = 0 \quad \text{uniformly in } n, \tag{C-2}$$

where τ_h is the translation operator defined by $\tau_h g(x) = g(x+h)$ for an arbitrary function $g\colon \mathbb{R} \to \mathbb{R}$.

Consider the case $h>0$; when $h<T$, by (B-1), recalling (B-3) and (B-6),

$$\begin{aligned}\tau_h\bar\psi_n(t)-\bar\psi_n(t)&=(\kappa_{\varepsilon_n}*f)(t+h)-(\kappa_{\varepsilon_n}*f)(t)\\&\quad+\int_t^{t+h}\kappa_{\varepsilon_n}(s)\Big(\frac12\sigma^2\bar\psi_n(t+h-s)+h(\bar\psi_n(t+h-s))\Big)\bar\psi_n(t+h-s)ds\\&\quad+\int_0^t\kappa_{\varepsilon_n}(s)\phi_n(t-s;h)(\tau_h\bar\psi_n(t-s)-\bar\psi_n(t-s))ds\\&:=\mathbf{I}_{n,h}(t)+\mathbf{II}_{n,h}(t)+(\kappa_{\varepsilon_n}*(\phi_n(\cdot;h)(\tau_h\bar\psi_n-\bar\psi_n)))(t),\quad t\in[0,T-h],\end{aligned}$$

where $\mathbf{I}_{n,h}(t)$ and $\mathbf{II}_{n,h}(t)$ refer to the first and second lines respectively on the right hand side here, and

$$\phi_n(t;h)=\frac12\sigma^2(\tau_h\bar\psi_n(t)+\bar\psi_n(t))+\tilde h(\tau_h\bar\psi_n(t),\bar\psi_n(t)),\quad t\in\mathbb{R}. \tag{C-3}$$

Hence $\chi=\tau_h\bar\psi_n-\bar\psi_n-\mathbf{I}_{n,h}-\mathbf{II}_{n,h}$ solves the linear VIE

$$\chi=\kappa_{\varepsilon_n}*(\phi_n(\cdot;h)(\tau_h\bar\psi_n-\bar\psi_n))=\kappa_{\varepsilon_n}*(\phi_n(\cdot;h)\chi+\phi_n(\cdot;h)(\mathbf{I}_{n,h}+\mathbf{II}_{n,h}))$$

on the interval $[0,T-h]$. $\mathbf{I}_{n,h}$, $\mathbf{II}_{n,h}$ and $\phi_n(\cdot;h)$ are continuous on $[0,T-h]$, so (given that $\phi_n(\cdot;h)$ is non-positive), Theorem C.3 in [AE19] implies that

$$|(\tau_h\bar\psi_n-\bar\psi_n)(t)|\le|\mathbf{I}_{n,h}(t)|+|\mathbf{II}_{n,h}(t)|+(\kappa_{\varepsilon_n}*|\phi_n(\cdot;h)(\mathbf{I}_{n,h}+\mathbf{II}_{n,h})|)(t),\quad t\in[0,T-h]. \tag{C-4}$$

We compute

$$\begin{aligned}\int_0^{T-h}\Big(\int_t^{t+h}\kappa_{\varepsilon_n}(s)|f(t+h-s)|ds\Big)dt&=\int_0^{T-h}\Big(\int_0^T 1_{\{s<t+h\}}1_{\{s>t\}}\kappa_{\varepsilon_n}(s)|f(t+h-s)|ds\Big)dt\\&=\int_0^T\kappa_{\varepsilon_n}(s)\Big(\int_0^{T-h}1_{\{t<s\}}1_{\{t>s-h\}}|f(t+h-s)|dt\Big)ds\quad\text{(by Tonelli)}\\&\le\int_0^h\kappa_{\varepsilon_n}(s)\Big(\int_0^s|f(t+h-s)|dt\Big)ds+\int_h^T\kappa_{\varepsilon_n}(s)\Big(\int_{s-h}^s|f(t+h-s)|dt\Big)ds\le\frac{2}{\lambda}h\|f\|_{L^\infty(0,T)},\end{aligned} \tag{C-5}$$

where h appears in the final term since both inner integrals have range $\le h$. Thus, denoting by $\bar f=f1_{[0,T]}\in L^\infty(\mathbb{R})$ and using Theorem 2.2, Chapter 2 in [GLS90],

$$\begin{aligned}\int_0^{T-h}|\mathbf{I}_{n,h}(t)|dt&\le\int_0^{T-h}\int_0^t\Big(\kappa_{\varepsilon_n}(s)|(\tau_hf-f)(t-s)|ds\Big)dt+\int_0^{T-h}\Big(\int_t^{t+h}\kappa_{\varepsilon_n}(s)|f(t+h-s)|ds\Big)dt\\&\le\Big(\int_0^{T-h}\kappa_{\varepsilon_n}(s)ds\Big)\|\tau_h\bar f-\bar f\|_{L^1(\mathbb{R})}+\frac{2}{\lambda}h\|f\|_{L^\infty(0,T)}\le\frac1\lambda\|\tau_h\bar f-\bar f\|_{L^1(\mathbb{R})}+\frac2\lambda h\|\bar f\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Since these estimates do not depend on $n\in\mathbb{N}$, the continuity of the translation in $L^1(\mathbb{R})$ (see, for instance, Lemma 4.3 in [Brez11]) yields that

$$\lim_{h\to0+}\int_0^{T-h}|\mathbf{I}_{n,h}(t)|dt=0\quad\text{uniformly in }n. \tag{C-6}$$

Given that $(\bar\psi_n)_n$ is bounded in $L^\infty(\mathbb{R})$ by (C-1) and $h(\cdot)$ (defined in (B-3)) is continuous on $\mathbb{R}$, the same computations as in (C-5) (but with 1 in place of f) show that

$$\lim_{h\to0+}\int_0^{T-h}|\mathbf{II}_{n,h}(t)|dt=0\quad\text{uniformly in }n. \tag{C-7}$$

Then (again by (C-1) and the continuity of $\tilde h$), there exists a constant $C>0$ such that

$$|\phi_n(t;h)|\le C,\quad t,\,h\in\mathbb{R},\,n\in\mathbb{N},$$

where $\phi_n(\cdot;h)$ is defined in (C-3). Therefore

$$\int_0^{T-h}|(\kappa_{\varepsilon_n}*|\phi_n(\cdot;h)(\mathbf{I}_{n,h}+\mathbf{II}_{n,h})|)(t)|dt\le C\frac1\lambda\int_0^{T-h}(|\mathbf{I}_{n,h}(t)|+|\mathbf{II}_{n,h}(t)|)dt,$$

whence (by (C-6) and (C-7)),

$$\lim_{h\to0+}\int_0^{T-h}|(\kappa_{\varepsilon_n}*|\phi_n(\cdot;h)(\mathbf{I}_{n,h}+\mathbf{II}_{n,h})|)(t)|dt=0\quad\text{uniformly in }n. \tag{C-8}$$

Combining (C-6), (C-7) and (C-8) in (C-4) we deduce that

$$\lim_{h\to 0+} \|\tau_h\bar\psi_n - \bar\psi_n\|_{L^1(0,T-h)} = 0 \quad \text{uniformly in } n. \tag{C-9}$$

On the interval $[-h,0]$ the maps $\bar\psi_n$ equal 0, hence, by (C-1),

$$\|\tau_h\bar\psi_n - \bar\psi_n\|_{L^1(-h,0)} = \int_{-h}^{0} |\tau_h\bar\psi_n(t)|dt \le \Big(\sup_n \|\bar\psi_n\|_{L^\infty(\mathbb{R})}\Big) h \underset{h\to 0+}{\longrightarrow} 0 \quad \text{uniformly in } n.$$

In a similar way, on the interval $[T-h,T]$

$$\|\tau_h\bar\psi_n - \bar\psi_n\|_{L^1(T-h,T)} = \int_{T-h}^{T} |\bar\psi_n(t)|dt \le \Big(\sup_n \|\bar\psi_n\|_{L^\infty(\mathbb{R})}\Big) h \underset{h\to 0+}{\longrightarrow} 0 \quad \text{uniformly in } n.$$

The three previous equations yield

$$\lim_{h\to 0+} \|\tau_h\bar\psi_n - \bar\psi_n\|_{L^1(\mathbb{R})} = 0 \quad \text{uniformly in } n.$$

When $h<0$, assuming without loss of generality that $|h|<T$ we can simply write

$$\begin{aligned}\|\tau_h\bar\psi_n - \bar\psi_n\|_{L^1(\mathbb{R})} &= \|\bar\psi_n\|_{L^1(0,|h|)} + \|\tau_h\bar\psi_n\|_{L^1(T,T+|h|)} + \|\bar\psi_n - \tau_h\bar\psi_n\|_{L^1(|h|,T)} \\ &\le 2\Big(\sup_n \|\bar\psi_n\|_{L^\infty(\mathbb{R})}\Big)|h| + \|\tau_{|h|}\bar\psi_n - \bar\psi_n\|_{L^1(0,T-|h|)} \underset{h\to 0-}{\longrightarrow} 0 \quad \text{uniformly in } n,\end{aligned}$$

where we use (C-9) for the last limit. Therefore (C-2) is verified and the proof is complete. ∎

C.2 Characterization of the limit points of ψ_ε

Lemma C.2 *For every $T>0$ and $g\in L^1(0,T)$,*

$$\lim_{\varepsilon\to 0} \int_0^T \Big|(\kappa_\varepsilon * g)(t) - \frac{1}{\lambda} g(t)\Big| dt = 0,$$

i.e. $\kappa_\varepsilon * g$ converges to $\frac{1}{\lambda}g$ in $L^1(0,T)$ as $\varepsilon\to 0$.

Proof. Let $c>0$. By the continuity of the translation in $L^1(\mathbb{R})$ (see e.g. Lemma 4.3 in [Brez11]), there exists an $\eta=\eta(c)\in(0,T)$ such that, defining $\bar g = g 1_{[0,T]} \in L^1(\mathbb{R})$, $\int_0^T |\bar g(t-s) - \bar g(t)|dt < c$, and hence

$$\int_s^T |g(t-s) - g(t)|dt < c,$$

for $s\in(0,\eta)$. Then from Tonelli's theorem and some straightforward manipulations,

$$\begin{aligned}\int_0^T \Big|(\kappa_\varepsilon * g)(t) - \frac{1}{\lambda} g(t)\Big| dt &\le \int_0^T \bigg(\int_0^t \kappa_\varepsilon(s)|g(t-s)-g(t)|ds + |g(t)|\Big(\frac{1}{\lambda} - \int_0^t \kappa_\varepsilon(s)ds\Big)\bigg) dt \\ &= \bigg\{ \int_0^\eta + \int_\eta^T \bigg\} \kappa_\varepsilon(s) \Big(\int_s^T |g(t-s)-g(t)|dt\Big) ds + \int_0^T |g(t)|\Big(\frac{1}{\lambda} - \int_0^t \kappa_\varepsilon(s)ds\Big) dt \\ &\le \frac{1}{\lambda}c + 2\|g\|_{L^1(0,T)} \int_\eta^T \kappa_\varepsilon(s)ds + \int_0^T |g(t)|\Big(\frac{1}{\lambda} - \int_0^t \kappa_\varepsilon(s)ds\Big) dt,\end{aligned}$$

and hence (by the dominated convergence theorem and (5)),

$$\limsup_{\varepsilon\to 0} \int_0^T \Big|(\kappa_\varepsilon * g)(t) - \frac{1}{\lambda} g(t)\Big| dt \le \frac{1}{\lambda}c.$$

Since c can be chosen arbitrarily small the proof is complete. ∎

Now recall the $\varepsilon=0$ solution ψ_0 defined in (15). Then we have the following:

Lemma C.3 *Consider $T>0$ and a sequence $(\epsilon_n)_n \subset (0,\infty)$ which converges to 0. Suppose that there exists a non-positive function $\bar\psi \in L^1(0,T)\cap L^\infty(0,T)$ such that $\psi_{\varepsilon_n} \to \bar\psi$ in $L^1(0,T)$. Then $\bar\psi = \psi_0$ a.e. in $(0,T)$.*

Proof. Recall our original VIE in (B-1): $\psi_\varepsilon = \kappa_\varepsilon * f + \kappa_\varepsilon * \bar{G}(\psi_\varepsilon)$. Multiplying by λ, taking the difference with (15), and adding and subtracting $\lambda(\kappa_{\varepsilon_n} * \bar{G}(\bar{\psi}))(t)$, we see that

$$\begin{aligned}
&\lambda(\psi_{\varepsilon_n}(t) - \psi_0(t)) \\
&\quad = \lambda(\kappa_{\varepsilon_n} * f)(t) - f(t) + \lambda(\kappa_{\varepsilon_n} * (\bar{G}(\psi_{\varepsilon_n}) - \bar{G}(\bar{\psi})))(t) + \lambda(\kappa_{\varepsilon_n} * \bar{G}(\bar{\psi}))(t) - \bar{G}(\psi_0(t)), \quad t \in [0,T]. \qquad \text{(C-10)}
\end{aligned}$$

By Lemma C.2, considering that $\bar{G}(\bar{\psi}(\cdot))$ belongs to $L^1(0,T)$ because $\bar{\psi} \in L^1(0,T) \cap L^\infty(0,T) = L^\infty(0,T)$,

$$\lim_{n\to\infty} \|\lambda(\kappa_{\varepsilon_n} * f) - f\|_{L^1(0,T)} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\lambda(\kappa_{\varepsilon_n} * \bar{G}(\bar{\psi})) - \bar{G}(\bar{\psi}(\cdot)\|_{L^1(0,T)} = 0.$$

and

$$\begin{aligned}
\lambda\|\kappa_{\varepsilon_n} * (\bar{G}(\psi_{\varepsilon_n}) - \bar{G}(\bar{\psi}))\|_{L^1(0,T)} &\leq \lambda\|\kappa_{\varepsilon_n}\|_{L^1(0,T)}\|\bar{G}(\psi_{\varepsilon_n}) - \bar{G}(\bar{\psi})\|_{L^1(0,T)} \\
&\leq \sup_{n\in\mathbb{N}} \left(\frac{1}{2}\sigma^2\|\psi_{\varepsilon_n} + \bar{\psi}\|_{L^\infty(0,T)} + \|\tilde{h}(\psi_{\varepsilon_n}, \bar{\psi})\|_{L^\infty(0,T)}\right)\|\psi_{\varepsilon_n} - \bar{\psi}\|_{L^1(0,T)} \\
&\leq \left(\frac{1}{2}\sigma^2\Big(\frac{1}{\lambda}\|f\|_{L^\infty(0,T)} + \|\bar{\psi}\|_{L^\infty(0,T)}\Big) + \sup_{n\in\mathbb{N}}\|\tilde{h}(\psi_{\varepsilon_n}, \bar{\psi})\|_{L^\infty(0,T)}\right)\|\psi_{\varepsilon_n} - \bar{\psi}\|_{L^1(0,T)} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Note that $\tilde{h}$ is continuous, and hence bounded in compact sets, and since ψ_{ε_n} and $\bar{\psi}$ are (uniformly) bounded, they take value in a compact set (ball), a.e., so the supremum in the final line is finite. Thus, from (C-10) we deduce that

$$\lambda(\bar{\psi}(t) - \psi_0(t)) = \bar{G}(\bar{\psi}(t)) - \bar{G}(\psi_0(t)) = \left(\frac{1}{2}\sigma^2(\bar{\psi}(t) + \psi_0(t)) + \tilde{h}(\bar{\psi}(t), \psi_0(t))\right)(\bar{\psi}(t) - \psi_0(t)), \quad \text{for a.e. } t \in (0,T).$$

This implies that $\bar{\psi} = \psi_0$ a.e. in $(0,T)$. Indeed, if there exists a subset $N \subset (0,T)$ with positive Lebesgue measure where $\bar{\psi} \neq \psi_0$, then dividing the previous equation by $\bar{\psi} - \psi_0$ gives

$$\lambda = \frac{1}{2}\sigma^2(\bar{\psi}(t) + \psi_0(t)) + \tilde{h}(\bar{\psi}(t), \psi_0(t)) < 0 \quad \text{a.e. in } N,$$

which is a contradiction since $\lambda > 0$. The proof is now complete. ■

C.3 Conclusion

From Lemma C.1, we know that every sequence $(\psi_{\varepsilon_n})_n$ of solutions to (B-1) (where $(\varepsilon_n)_n \subset (0,\infty)$ converges to 0 as $n \to \infty$) admits a convergent subsequence $(\psi_{\varepsilon_{n_k}})_k$ in $L^1(0,T)$. Since $(\psi_\varepsilon)_{\varepsilon>0}$ is a bounded family of (continuous) non-positive functions in $L^\infty(0,T)$, see (C-1), the limit point of this subsequence belongs to $L^1(0,T) \cap L^\infty(0,T)$ and is non-positive, as well.
By Lemma C.3 in Subsection C.2, there exists a unique possible non-positive $L^1(0,T)-$limit point for $(\psi_{\varepsilon_{n_k}})_k$ in $L^1(0,T) \cap L^\infty(0,T)$: ψ_0, the unique non-positive solution of (15). Therefore, by the subsequence convergence principle we conclude that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi_0 \quad \text{in } L^1(0,T).$$

D Laplace transform of hitting time to an upper barrier for a spectrally negative Lévy process

Let X be a spectrally negative one-dimensional Lévy process, i.e. $\nu_X(0,\infty) = 0$, where ν_X is the Lévy measure associated with X, and assume $X_0 = 0$. Suppose that ν_X satisfies

$$\int_{(-\infty,-1)} |x|\nu_X(dx) < \infty; \qquad \text{(D-1)}$$

by general properties of Lévy processes (see e.g. Theorem 25.3 in [Sato99]), (D-1) ensures that $\mathbb{E}[|X_t|] < \infty$ for every $t > 0$.
In the next proposition, we establish a formula for the Laplace transform of the first hitting time of X to upper (non-negative) barriers.

Proposition D.1 *Consider a spectrally negative one-dimensional Lévy process X with Lévy measure ν_X satisfying* (D-1)*. Suppose that*

$$\gamma := \mathbb{E}[X_1] \geq 0.$$

For every $b \geq 0$, denote by τ_b the first hitting time of X to b, i.e. $\tau_b = \inf\{t \geq 0 : X_t > b\}$, and define the function $V : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$V(p) := \frac{1}{2}(\sigma_X)^2p^2 + \gamma p + \int_{\mathbb{R}_-} (e^{px} - 1 - px)\nu_X(dx), \quad p \geq 0,$$

where $\sigma_X^2 \geq 0$ denotes the Gaussian component of X. Then for all $q \geq 0$

$$\mathbb{E}[e^{-q\tau_b}] = e^{-bV^{-1}(q)}, \tag{D-2}$$

where V^{-1} is the inverse of V.

Proof. By Theorem 25.17 in [Sato99], for all $p \geq 0$ we have

$$\log \mathbb{E}[e^{pX_t}] = tV(p), \quad t \geq 0. \tag{D-3}$$

Thus, V is the logarithmic moment generating function (or cgf) of X_1 on $\mathbb{R}_+$. It then follows from Lemma 2.2.5 in [DZ98] that V is convex. Moreover, V is continuous and differentiable, with

$$V'(p) = \gamma + (\sigma_X)^2 p + \int_{\mathbb{R}_-} x(e^{px} - 1)\nu_X(dx), \quad p \geq 0.$$

Since $V' > 0$ on $(0, \infty)$, V is increasing on $\mathbb{R}_+$ and $\lim_{p\to\infty} V(p) = \infty$.

From the stationary and independent increments property one can easily verify that $M_t := e^{pX_t - V(p)t}$ is an $\mathcal{F}_t^X$−martingale. Indeed, for $0 \leq s \leq t$,

$$\mathbb{E}[M_t|\mathcal{F}_s^X] = \mathbb{E}\Big[e^{p(X_t - X_s)}|\mathcal{F}_s^X\Big]e^{pX_s - V(p)t} = \mathbb{E}\Big[e^{pX_{t-s}}\Big]e^{pX_s - V(p)t} = M_s, \tag{D-4}$$

where we use (D-3) for the third equality.

Now choose $p > 0$. Then applying the Optional Stopping Theorem to the bounded stopping time $t \wedge \tau_b$ we have

$$1 \quad = \quad \mathbb{E}[M_{t\wedge\tau_b}(1_{\{\tau_b \leq t\}} + 1_{\{\tau_b > t\}})] \quad = \quad \mathbb{E}[e^{pb - V(p)\tau_b}1_{\{\tau_b \leq t\}}] + \mathbb{E}[e^{pX_t - V(p)t}1_{\{\tau_b > t\}}].$$

Here for the second equality we use that $X_{\tau_b} = b$ when $\tau_b < \infty$ ($\mathbb{P}$−a.s.), because X can only have negative jumps. Using the monotone convergence theorem and that $\lim_{t\to\infty} 1_{\{\tau_b \leq t\}} = 1_{\{\tau_b < \infty\}}$ for the left term, and the bounded convergence theorem for the right term (with the bound e^{pb}, since $V > 0$ on $(0, \infty)$), we can take the limit as $t \to \infty$ and take the limit inside the expectation to obtain

$$1 = \mathbb{E}[e^{pb - V(p)\tau_b}1_{\{\tau_b < \infty\}}].$$

V is a bijection from $\mathbb{R}_+$ onto itself (since $\gamma \geq 0$), so we can re-write this as

$$\mathbb{E}[e^{-q\tau_b}1_{\{\tau_b < \infty\}}] = e^{-bV^{-1}(q)}, \quad q > 0. \tag{D-5}$$

Letting $q \searrow 0$ and using the bounded convergence theorem again, we see that

$$\mathbb{P}(\tau_b < \infty) = \mathbb{E}[1_{\{\tau_b < \infty\}}] = e^{-bV^{-1}(0+)},$$

where $V^{-1}(0+) = \lim_{q\searrow 0} V^{-1}(q)$. Considering that V^{-1} is continuous on $\mathbb{R}_+$, we deduce that $V^{-1}(0+) = V^{-1}(0) = 0$. Consequently, $\tau_b < \infty$ $\mathbb{P}$−a.s. and (D-5) becomes (D-2), completing the proof. ■

Remark D.1 When $\gamma > 0$, for every $b \geq 0$ the finiteness of the stopping time τ_b can be directly inferred from the LLN in Theorem 36.5 of [Sato99].

E Brief formal derivation of the main idea in [AAR25]

Consider a family of hyper-rough Heston models[14] (with zero mean-reversion for simplicity) for which the quadratic variation of the log stock price satisfies

$$\langle \log S^n \rangle_t \quad = \quad X_t^n \quad = \quad V_0 t + \Big(H_n + \frac{1}{2}\Big)\sigma \int_0^t (t-s)^{H_n - \frac{1}{2}} W_{X_s^n} ds$$

for $H_n \in (-\frac{1}{2}, 1)$. From Lemma 2.4 in [AAR25],[15] we formally expect that

$$\lim_{H_n \searrow -\frac{1}{2}} \Big(H_n + \frac{1}{2}\Big)\sigma \int_0^t (t-s)^{H_n - \frac{1}{2}} W_{X_s^n} ds \quad = \quad \sigma W_{X_t},$$

where X is the weak limit of X^n, so we expect X to satisfy

$$X_t \quad = \quad V_0 t + \sigma W_{X_t}. \tag{E-1}$$

[14] see [JR20], Section 7 in [A21] and Section 5 in [FGS21] for more on this model.

[15] this lemma is particularly easy to check when f is a polynomial.

Now let

$$Y_t \;=\; -t+\sigma W_t \tag{E-2}$$

and set $\tilde{X}_t = H_{-V_0 t}$, where $H_b = \inf\{t : Y_t = b\}$. Then setting $t \mapsto \tilde{X}_t$ in (E-2), we see that

$$-V_0 t \;=\; -\tilde{X}_t \,+\, \sigma W_{\tilde{X}_t} \tag{E-3}$$

i.e. $\tilde{X}$ satisfies the same equation as X_t in (E-1). Hence (using the notation/setup in Lemma 2.3 in [AAR25], i.e. $c=-V_0$, $b=\sigma$ and $a=-1$), we deduce that X is an Inverse Gaussian Lévy process with parameters $(V_0, \frac{V_0^2}{\sigma^2})$.

To analyze this process with VIEs, using that $\frac{1}{\Gamma(\alpha)} = \alpha + O(\alpha^2)$ as $\alpha \to 0$ (i.e. as $H \to -\frac{1}{2}$), we see that the usual rough Heston VIE (with $\rho = 0$) takes the form

$$\begin{aligned}
\phi(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(-\frac{1}{2}(u^2+iu) \,+\, \frac{1}{2}\sigma^2\phi(s)^2\Big)ds \\
&= (1+O(\alpha))\alpha\int_0^t (t-s)^{\alpha-1}\Big(-\frac{1}{2}(u^2+iu) \,+\, \frac{1}{2}\sigma^2\phi(s)^2\Big)ds \;\;\to\;\; -\frac{1}{2}(u^2+iu) \,+\, \frac{1}{2}\sigma^2\phi(t)^2
\end{aligned}$$

as $\alpha \to 0$ (again using Lemma 2.4 in [AAR25]), which is just an algebraic equation for ϕ. If we ignore the linear term in u for simplicity (i.e. ignore the drift of the log stock price), then the (relevant) solution to this equation is $\phi(t) = \frac{1}{\sigma^2}(1-\sqrt{1+\sigma^2 u^2})$, i.e. the smaller root as in the proof of Proposition 1.1.