Homework 3

1. There exists a function $k(s,t)$ such that fBM B^H can be realized as $B^H_t = \int_0^t k(s,t)dB_s$ where B is standard Brownian motion, and $k(s,t) \sim \textit{const.}(t-s)^{H-\frac{1}{2}}$ as $s \nearrow t$, so k blows up as $s \nearrow t$ when $H \in (0,\frac{1}{2})$. Use this to decompose B^H in terms of what happens before and after time t, and compute the conditional distribution of $B^H_u|\mathcal{F}^B_t$. Is B^H a martingale or a Markov process?

Solution. For $0 < t < u$:

$$B^H_u \;=\; \int_0^t k(s,u)dB_s + \int_t^u k(s,u)dB_s\,.$$

The two expressions on the right hand side are independent, and (conditioned on $(B_s)_{0\le s\le t}$), we see that

$$B^H_u|\mathcal{F}^B_t \;\sim\; N(\int_0^t k(s,u)dB_s, \int_t^u k(s,u)^2 ds)$$

for $0 \le t \le u$. In particular, since $\int_0^t k(s,u)dB_s \neq B^H_t$ in general when $H \neq \frac{1}{2}$ (and $\int_0^t k(s,u)dB_s$ is not just a function of B_t), we see that B^H isn't a Markov process (and hence has **memory**), and is not a martingale when $H \neq \frac{1}{2}$.

2. A commonly used variant of fBM is the **Riemann-Liouville** process $Z^H_t = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dB_s$, which is also self-similar and has the same roughness as fBM, but no longer has stationary increments. Compute $\mathbb{E}((Z^H_t)^2)$, and write down an integral expression for the covariance of Z^H.

Solution. From the Ito isometry

$$\mathbb{E}((Z^H_t)^2) \;=\; 2H\int_0^t (t-s)^{2H-1}ds \;=\; t^{2H}$$

so $\mathbb{E}((Z^H_t)^2) = \mathbb{E}((B^H_t)^2)$, and

$$R(s,t) \;=\; \mathbb{E}(Z^H_s Z^H_t) \;=\; 2H\int_0^s (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du$$

for $0 \le s \le t$, which evaluates to a more complex covariance function than fBM in terms of the confluent hypergeometric function (we omit the details here).

3. Consider a one-sided CGMY process X for which $\mathbb{E}(e^{iuX_t}) = e^{C\Gamma(-Y)((M-iu)^Y - M^Y)t}$ for $u \in \mathbb{R}$ and $Y \in (0,2)\setminus\{1\}$. Show that X can still go negative when $Y \in (1,2)$ despite having positive-only jumps.

Solution.

$$i\mathbb{E}(X_t) \;=\; \frac{d}{du}\mathbb{E}(e^{iuX_t})|_{u=0}$$

from which we find that $\mathbb{E}(X_t) = \gamma := -CM^{Y-1}Y\Gamma(-Y)t$, which is negative for $Y \in (1,2)$, which cannot be true unless $\mathbb{P}(X_t < 0) > 0$.

Remark 0.1 One can also show that

$$\mathbb{E}(e^{iuX_t}) \;=\; \exp(t(i\gamma u + \int_0^\infty (e^{iux} - 1 - iux)\frac{Ce^{-Mx}}{x^{1+Y}}dx)) \tag{1}$$

which confirms that $i\mathbb{E}(X_t) = \frac{d}{du}\mathbb{E}(e^{iuX_t})|_{u=0} = i\gamma t$, so $\mathbb{E}(X_t) = \gamma t$, and hence $M_t = X_t - \gamma t$ is an $\mathcal{F}^X$-martingale because $\mathbb{E}(M_t) = 0$, and M is also a Lévy process and hence has stationary i.i.d. increments so $\mathbb{E}(M_t|\mathcal{F}_s) = M_s + \mathbb{E}(M_{t-s}|\mathcal{F}_s) = M_s + \mathbb{E}(M_{t-s}) = M_s$. Note for $Y \in (1,2)$ the $-iux$ term is required in the integrand for the integral to be finite so $e^{iux} - 1 - iux = O(x^2)$ (if we exclude this term, the integrand in (1) is $O(\frac{x}{x^{1+Y}}) = O(\frac{1}{x^Y})$ as $x \downarrow 0$, and hence not integrable at zero when $Y \in (1,2)$).

4. Suppose we wish to estimate $\frac{d}{dx}\mathbb{E}((x+W_t)^2) = \frac{d}{dx}(x^2+t) = 2x$ (which is the initial Delta of a contract which pays $(x+W_t)^2$) using **Monte Carlo** with a **first-order finite difference** approximation to the derivative. Is it better to use the same Brownian motion or two independent Brownian motions for each part of the finite difference computation?

Solution. If we use the same Brownian motion W for both terms, we see that

$$\frac{1}{\varepsilon}((x+\varepsilon+W_t)^2-(x+W_t)^2) \quad = \quad 2W_t+2x+\varepsilon$$

which has expectation $2x+\varepsilon$ (and hence asymptotically unbiased as $\varepsilon \to 0$), and variance $4t$. Note that the sample variance comes down to $\frac{4t}{n}$ if we sample n iid copies of W_t as we typically do in Monte Carlo.

For the other case, we first note that

$$\begin{aligned}\mathbb{E}((x+W_t)^2) &= \mathbb{E}(W_t^2+2W_tx+x^2) \;=\; t+x^2\\ \mathbb{E}((x+W_t)^4) &= 3t^2+6x^2t+x^4\\ \mathrm{Var}((x+W_t)^2) &= \mathbb{E}((x+W_t)^4)-\mathbb{E}((x+W_t)^2)^2 \;=\; 2t(t+2x^2)\end{aligned}$$

so if W and B are independent Brownians,

$$\begin{aligned}\mathbb{E}(\frac{1}{\varepsilon}((x+\varepsilon+B_t)^2-(x+W_t)^2)) &= \frac{1}{\varepsilon}((x+\varepsilon)^2-x^2) \;=\; 2x+\varepsilon\\ \mathrm{Var}(\frac{1}{\varepsilon}((x+\varepsilon+B_t)^2-(x+W_t)^2)) &= \frac{1}{\varepsilon^2}[2t(t+2(x+\varepsilon)^2)+2t(t+2x^2)] \;=\; O(\frac{1}{\varepsilon^2})\end{aligned}$$

as $\varepsilon \to 0$, so the estimate $\frac{1}{\varepsilon}((x+\varepsilon+B_t)^2-(x+W_t)^2)$ is also asymptotically unbiased, but its variance blows up as $\varepsilon \to 0$, so the first method is more stable. More generally, it's usually better to numerically estimate Greeks using the same initial random seed (i.e. common random numbers) rather than two different seeds, for essentially the same reason that we have seen here, i.e. to get an asymptotically unbiased estimate for the Greek with $O(1)$ variance.

5. Let X be Lévy process and let $\tau_b=\inf\{t>0: X_t>b\}$ for $b>0$, and $O_b=X_{\tau_b}-b$ denote the **overshoot** of X at the level b. For a symmetric α-stable process, the density of O_b is

$$f_Y(y) \quad = \quad \frac{(b/y)^{\frac{1}{2}\alpha}\sin(\pi\frac{1}{2}\alpha)}{\pi(b+y)}.$$

What is the range of O_b? Compute $\mathbb{E}(O_b)$.

Solution. Range is $(0,\infty)$, $\mathbb{E}(O_b)=\infty$ since $yf_Y(y)\sim\frac{1}{y^{\frac{1}{2}\alpha}}$, which isn't integrable at $y=\infty$ since $\alpha\in(0,2)$.

6. $U_b=b-X_{\tau_b-}$ denote the **undershoot** of X, where X_{τ_b-} denotes the left limit of X at τ_b i.e. the value of X just before the jump. For a one-sided α-stable process with $\alpha\in(0,1)$, U_b/b has density

$$\frac{\sin(\pi\alpha)}{\pi y^\alpha}(1-y)^{\alpha-1}$$

What is the range of U_b?

Solution. Range is $(0,b]$ (zero is not included since the process jumps over b a.s. as it does not have a Brownian component).

7. Let X be a general α-stable process with $\rho=\mathbb{P}(X_1\geq 0)$. Then the joint density of the overshoot, undershoot $b-X_{\tau_b-}$ and $b-\bar{X}_{\tau_b-}$ is:

$$f(u,v,y) \quad = \quad c_1\frac{(b-y)^{\alpha\rho-1}(v-y)^{\alpha(1-\rho)-1}}{(v+u)^{1+\alpha}}$$

for some constant c_1. Write an integral expression for the joint density of the overshoot and undershoot.

Solution. Answer is $\int_0^{\min(b,v)} f(u,v,y)dy$, since $\bar{X}_{\tau_b-}\geq X_{\tau_b-}$ and $\bar{X}_{\tau_b-}\geq 0$.

8. Let U_1,U_2 be i.i.d. $U[0,1]$ random variables. Then Z defined by

$$\Theta \;=\; \pi(U_1-\frac{1}{2}), \quad E \;=\; -\log U_2, \quad Z \;=\; \frac{\sin(\alpha\Theta)}{\cos(\Theta)^{\frac{1}{\alpha}}}(\frac{\cos((\alpha-1)\Theta)}{E})^{\frac{1-\alpha}{\alpha}} \tag{2}$$

has the same distribution as X_1, where X is a symmetric α-stable process with $\mathbb{E}(e^{iuX_t})=e^{-t|u|^\alpha}$ (note that $\Theta\sim U[-\frac{1}{2}\pi,\frac{1}{2}\pi]$ and $E\sim\mathrm{Exp}(1)$). Explain how to sample X_t using Z. How do we use **antithetic** sampling to simulate paths of X?

Remark 0.2 (2) is known as the **Chambers,Mallows&Stuck** method, see short Python implementation at `https://colab.research.google.com/drive/1vFp3doGahvYU5tI7XXm3XEYOonH_jlxh?usp=sharing`, which also samples the overshoot and undershoot.

Solution. $\mathbb{E}(e^{iuX_t}) = \mathbb{E}(e^{iut^{\frac{1}{\alpha}}X_1})$, so we see that $X_t \sim t^{\frac{1}{\alpha}}X_1$ (i.e. X is **self-similar**, and recall that any Lévy process has i.i.d. stationary increments, so $X_{t+\Delta t} - X_t \sim (\Delta t)^{\frac{1}{\alpha}}Z$. For antithetic sampling, we can sample four different versions Z_1, Z_2, Z_3, Z_4 of Z using the four combinations (U_1, U_2), $(U_1, 1-U_2)$, $(1-U_1, U_2)$ and $(1-U_1, 1-U_2)$, since if $U \sim U[0,1]$ then so is $1-U$.

9. For a compound Poisson process X with rate λ and jump size distribution $\nu(x)$, we have seen that

$$\log \mathbb{E}(e^{-q[X,X]_t}) = \lambda \int_{-\infty}^{\infty} (e^{-qx^2} - 1)\nu(x)\,. \tag{3}$$

What is the distribution of the square of the jump sizes?

Solution. $\mathbb{P}(J^2 \le x) = \mathbb{P}(-\sqrt{x} \le J \le \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} \nu(u)du$, so (from the chain rule) the density of J^2 is $\nu_{[X,X]}(x) = \frac{1}{2}x^{-\frac{1}{2}}(\nu(\sqrt{x}) + \nu(-\sqrt{x})$ for $x > 0$, and zero otherwise. Hence we can re-write (3) as

$$\log \mathbb{E}(e^{-q[X,X]_t}) = \lambda \int_{-\infty}^{\infty} (e^{-qx} - 1)\nu_{[X,X]}(x)\,. \tag{4}$$

Remark 0.3 (4) also holds for a general Lévy process X with the λ term removed, so in particular for an α-stable process which has $\nu(x) = \frac{c_1}{|x|^{1+\alpha}}$, we find that the Lévy density of $[X,X]$ is $\nu_{[X,X]}(x) = \frac{c_1}{x^{1+\frac{1}{2}\alpha}}$ for $y > 0$ and zero otherwise, i.e. a one-sided α-stable process with exponent $\frac{1}{2}\alpha$, and the Lévy density of $[X,X]$ for a symmetric CGMY process is $\nu_{[X,X]}(x) = \frac{c_1 e^{-M\sqrt{x}}}{x^{1+\frac{1}{2}\alpha}}$ for $x > 0$.

10. Consider an α-stable Lévy process X with $\alpha = Y$ and positive-only jumps for which $\mathbb{E}(e^{iuX_t}) = e^{C\Gamma(-Y)(-iu)^Y t}$ for $u \in \mathbb{C}$ with $\mathrm{Im}(u) \ge 0$ and $Y \in (0,2) \setminus 1$. Using that

$$\frac{\mathbb{E}(e^{(iu-M)X_t})}{\mathbb{E}(e^{-MX_t})} = e^{C\Gamma(-Y)((M-iu)^Y - M^Y)t}$$

for $M \ge 0$ and $u \in \mathbb{R}$, we see that the right hand side is the characteristic function of a one-sided CGMY process. What does this tell us about the law of this process in terms of the original α-stable process?

Solution. Define the new probability measure $\mathbb{Q}$ as

$$\mathbb{Q}(A) = \frac{\mathbb{E}(e^{-MX_t}1_A)}{\mathbb{E}(e^{-MX_t})}$$

for $A \in \mathcal{F}_t^X$. Then combining the last two equations, we see that $\mathbb{E}^{\mathbb{Q}}(e^{iuX_t}) = \frac{\mathbb{E}(e^{(iu-M)X_t})}{\mathbb{E}(e^{-MX_t})} = e^{C\Gamma(-Y)((M-iu)^Y - M^Y)t}$, i.e. the CF of a one-sided CGMY process .

Remark 0.4 This tells us that X is a one-sided CGMY process under $\mathbb{Q}$. This is very useful for **Monte Carlo**, since we can use the CMS method above to sample X_t under the original measure, then the price of a European call option under the CGMY model is

$$\mathbb{E}^{\mathbb{Q}}(g(X_t)) = \frac{\mathbb{E}(e^{-MX_t}g(X_t))}{\mathbb{E}(e^{-MX_t})} = \frac{\mathbb{E}(e^{-MX_t}g(X_t))}{e^{tC\Gamma(-Y)M^Y}}\,.$$

In practice, we would typically define a log stock price to be the **difference of two independent** CGMY processes X and $\tilde{X}$ under a measure $\mathbb{Q}^*$ with decay parameters G and M respectively; then we can price a contract paying $g(X_t - \tilde{X}_t)$ as

$$\mathbb{E}^{\mathbb{Q}^*}(g(X_t - \tilde{X}_t)) = \frac{\mathbb{E}(e^{-MX_t - G\tilde{X}_t}g(X_t - \tilde{X}_t))}{\mathbb{E}(e^{-MX_t - G\tilde{X}_t})} = \frac{\mathbb{E}(e^{-MX_t - G\tilde{X}_t}g(X_t - \tilde{X}_t))}{\mathbb{E}(e^{-MX_t})\mathbb{E}(e^{-G\tilde{X}_t})}$$

where X and $\tilde{X}$ are one-sided α-stable with $\alpha = Y$ under the original measure being used on the right.