Asymptotic behaviour of estimators for H and σ as $n \to \infty$

Let X be a real-valued stationary Gaussian process $(X_t)_{t=0}^{\infty}$ (i.e. X_t has the same Normal distribution for all $t \in \mathbb{N}$) with a summable **autocovariance function** $r(k) := \mathbb{E}(X_t X_{t+k})$, i.e. $\sum_{k=1}^{\infty} |r(k)| < \infty$ (our interest will be **fractional Gaussian noise** (fGN) $X_n = B_n^H - B_{n-1}^H$, where B^H is fBM so $X_t \sim N(0,1)$ for all t). The **spectral density** of a stationary process X is the function $f_\theta(\omega)$ whose **Fourier series** coefficients are equal to $(r(k))_{k=0}^{\infty}$, i.e.

$$r(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega j} f_\theta(\omega) d\omega$$

so $f_\theta(\omega) = \sum_{k=-\infty}^{\infty} r(k) e^{i\omega k}$ when the infinite series converges (which is the case when $r(k)$ is summable).

The **Whittle approximation** for the determinant and the Inverse of the Covariance matrix Σ of X is

$$\log(\det \Sigma) \sim \frac{n}{4\pi} \int_{-\pi}^{\pi} \log f_\theta(\omega \, d\omega \quad , \quad \Sigma_{ij}^{-1} \sim \int_{-\pi}^{\pi} \frac{1}{f_\theta(\omega)} e^{i(j-k)\omega} d\omega \tag{1}$$

as $n \to \infty$ (this is the so-called **Szegö** limit (or Grenander–Szegö theorem) for Toeplitz matrices), and for fGN there is an explicit formula for $f_\theta = f_H$ in Proposition 7.2.9 in [Taq02] (note the [Taq02] form for the spectral density is divided by 2π, and Eq 5.40 in [Ber94] has a spurious factor of $1/(2\pi)$)

Define the normalized **discrete Fourier transform** (DFT) at frequency $\omega_j = \frac{2\pi j}{n}$ as:

$$Z_n(\omega_j) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k e^{-i\omega_j k}.$$

Then can re-write the covariance part of the log likelihood (LL) of X in terms of $Z_n(\omega_j)$ as

$$\begin{aligned} \frac{1}{\pi} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} X_j \left(\int_{-\pi}^{\pi} \frac{1}{f_\theta(\omega)} e^{i(j-k)\omega} d\omega \right) X_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{f_\theta(\omega)} \sum_{j,k} e^{i(j-k)\omega} X_j X_k \, d\omega \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{f_\theta(\omega)} \sum_{j,k} e^{ij\omega} X_j e^{-ik\omega} X_k \, d\omega \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{f_\theta(\omega)} |Z_n(\omega)|^2 \, d\omega \end{aligned} \tag{2}$$

where $|Z_n(\omega)|^2 = I_n(\omega, y) = (\sum_{j=0}^{n-1} e^{ij\omega} X_j)(\sum_{j=0}^{n-1} e^{-ij\omega} X_j) = |\sum_{j=0}^{n-1} e^{ij\omega} X_j|^2$ is the **periodogram** of the random vector X (see plot below for fGN), and we can trivially verify that $I_n(\omega, y)$ is symmetric in ω.

Asymptotic independence and normality of the DFT

For fixed $\omega_j \in (0, \pi)$, as $n \to \infty$ $Z_n(\omega_j) \xrightarrow{d} N(0, f(\omega_j))$ and for $\omega_j \neq \omega_k$, the coefficients $Z_n(\omega_j)$ and $Z_n(\omega_k)$ are asymptotically uncorrelated (we numerically test this in Python here:
`https://colab.research.google.com/drive/1ruxvZX8brSOKGTi5RFTznRNjOwpQlVUc?usp=sharing)`

Sketch proof:

$$\mathrm{Cov}(Z_n(\omega_j), Z_n(\omega_k)) = \mathbb{E}[Z_n(\omega_j)\overline{Z_n(\omega_k)}] = \frac{1}{n} \sum_{s=1}^{n} \sum_{t=1}^{n} \mathbb{E}[X_s X_t] e^{-i\omega_j s} e^{i\omega_k t} = \frac{1}{n} \sum_{s=1}^{n} \sum_{t=1}^{n} r(t-s) e^{-i\omega_j s} e^{i\omega_k t}.$$

Now let $h = t - s$, so $t = s + h$. Then we see that

$$\mathrm{Cov}(Z_n(\omega_j), Z_n(\omega_k)) = \frac{1}{n} \sum_{s=1}^{n} \sum_{h=-(s-1)}^{n-s} r(h) e^{-i\omega_j s} e^{i\omega_k (s+h)} = \frac{1}{n} \sum_{s=1}^{n} e^{i(\omega_k - \omega_j)s} \sum_{h=-(s-1)}^{n-s} r(h) e^{i\omega_k h}.$$

As $n \to \infty$, and since $r(h)$ is absolutely summable, we see that

$$\sum_{h=-(s-1)}^{n-s} r(h) e^{i\omega_k h} \to \sum_{h=-\infty}^{\infty} r(h) e^{i\omega_k h} = f_\theta(\omega_k).$$

Thus $\mathrm{Cov}(Z_n(\omega_j), Z_n(\omega_k)) \approx \frac{1}{n} \sum_{s=1}^{n} e^{i(\omega_k - \omega_j)s} f_\theta(\omega_k)$. This is a geometric series with partial sum $\frac{1}{n} \sum_{s=1}^{n} e^{i(\omega_k - \omega_j)s} = \frac{1}{n} \cdot \frac{e^{i(\omega_k - \omega_j)}(1 - e^{in(\omega_k - \omega_j)})}{1 - e^{i(\omega_k - \omega_j)}}$ which is clearly 1 if $j = k$, or tends to zero if $j \neq k$, as claimed.

Using the Whittle approximation to derive the Local Asymptotic Normality (LAN) property

Using (1) and (2), the Whittle approximation for the LL of X is

$$\ell_n(\theta) \quad = \quad -\frac{n}{4\pi}\int_{-\pi}^{\pi}(\log f_\theta(\omega) + \frac{|Z_n(\omega)|^2}{f_\theta(\omega)})d\omega \; + \; const.$$

where the log part approximates the determinant part, the Z_n part approximates the covariance part, and the constant is unimportant as it doesn't depend on θ.

The **Whittle estimator** is then defined as argmax of the Whittle approximation for the LL, which we can re-write (using similar notation to [FTW21] and [Syz23] as $\hat{\theta}_n = \arg\min_\theta U_n(\theta)$, where

$$U_n(\theta) \quad = \quad \frac{1}{4\pi}\int_{-\pi}^{\pi}(\log f_\theta(\omega) + \frac{|Z_n(\omega)|^2}{f_\theta(\omega)})d\omega \tag{3}$$

since the $\frac{n}{4\pi}$ prefactor doesn't affect the maximizer(s), and removing the minus sign here just transforms this to a minimization problem.

Remark 0.1 $U_n(\theta) \to U_\infty(\theta) := \frac{1}{4\pi}\int_{-\pi}^{\pi}(\log(\sigma^2 f_H(\omega)) + \frac{\sigma^2 f_H(\theta_0)}{\sigma^2 f_H(\omega)})d\omega$ under $\mathbb{P}_{\theta_0}$, and this expression is minimized at $\theta = \theta_0$ and if $\arg\min U_n(\theta) \to \arg\min U_\infty(\theta) = \theta_0$, the Whittle estimator $\hat{\theta}_n$ is **consistent**, i.e. $\hat{\theta}_n \to \theta$ in probability.

Then the **score** is

$$\ell_n'(\theta) \quad = \quad -\frac{n}{2\pi}\int_{-\pi}^{\pi}\frac{f_\theta(\omega) - Z_n(\omega)^2}{f_\theta(\omega)^2}\nabla_\theta f_\theta(\omega)d\omega\,.$$

Above we noted that $Z_n(\omega)$ is symmetric in ω, and approximating this integral as a sum over each $\omega_j = \frac{2\pi j}{n}$ we get

$$\ell_n'(\theta) \quad \approx \quad -\frac{2n\pi}{4n\pi}\sum_{j=1}^{n}\frac{f_\theta(\omega_j) - Z_n(\omega_j)^2}{f_\theta(\omega_j)^2}\nabla_\theta f_\theta(\omega_n)\,.$$

From the result in previous section, $(Z_n(\omega))_{n=1}^\infty$ is a sequence of asymptotically independent Normal random variables and hence $f_\theta(\omega_j) - Z_n(\omega_j)^2$ is a sequence of approximately independent shifted $\chi^2(df = 1)$ random variables with zero expectation and variance $2f_\theta(\omega_j)^2$ (using that $\mathrm{Var}(Z^2) = 2$ when $Z \sim N(0,1)$);

Hence using a Lyapunov-type CLT, this sum has expectation zero, and variance equal to

$$(\frac{2}{4})^2\sum_{j=1}^{n}\frac{2f_\theta(\omega_j)^2}{f_\theta(\omega_j)^4}f_\theta(\omega_j)^2 \quad \approx \quad \frac{1}{2}\cdot\frac{n}{\pi}\int_0^\pi(...)d\omega \quad = \quad \frac{n}{4\pi}\int_{-\pi}^{\pi}\frac{\nabla_\theta f_\theta(\omega)^\top\nabla_\theta f_\theta(\omega)}{f_\theta(\omega)^2}d\omega \quad = \quad nI(\theta)$$

where $I(.)$ is the **Fisher information matrix**. n term here then cancels with $(\frac{u}{\sqrt{n}})^2$ when we make perturbation to get the LAN property, as claimed in Cohen et. al.

Asymptotic normality of MLEs

Using the Taylor remainder theorem and setting the answer to zero, we see that

$$\nabla\ell_n(\hat{\theta}_n) \quad = \quad \nabla\ell_n(\theta_0) \; + \; \nabla^2\ell_n(\tilde{\theta}_n)(\hat{\theta}_n - \theta_0) \quad = \quad 0$$

for some $\tilde{\theta}_n \in [\theta, \theta_n]$, which we can re-arrange as

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \quad = \quad -(\frac{1}{n}\nabla^2\ell_n(\tilde{\theta}_n))^{-1}\frac{1}{\sqrt{n}}\nabla\ell_n(\theta_0)\,.$$

Asymptotic normality of the score

The (approximate) score function under the Whittle likelihood is:

$$\nabla\ell_n(\theta_0) \quad = \quad \int_{-\pi}^{\pi}[\frac{\partial\log f_{\theta_0}(\omega)}{\partial\theta}(\frac{I_n(\omega)}{f_{\theta_0}(\omega)} - 1)]d\omega \tag{4}$$

where $I_n = |Z_n(\omega)|^2$ as before. Recall from above that $\frac{1}{\sqrt{n}}\nabla\ell_n(\theta_0) \xrightarrow{d} N(0, I(\theta_0))$, where the Whittle likelihood approximation for $I(\theta_0)$ is $I(\theta_0) = \frac{1}{4\pi}\int_{-\pi}^{\pi}(\frac{\partial\log f_{\theta_0}(\omega)}{\partial\theta})(\frac{\partial\log f_{\theta_0}(\omega)}{\partial\theta})^\top d\omega$

Hessian convergence to a constant

The Hessian of the Whittle log-likelihood is

$$\nabla^2\ell_n(\theta_0) \quad = \quad -\int_{-\pi}^{\pi}[\frac{\partial^2 \log f_{\theta_0}(\omega)}{\partial\theta\partial\theta^\top}(1-\frac{I_n(\omega)}{f_\theta(\omega)})+(\frac{\partial}{\partial\theta}\log f_{\theta_0}(\omega))^\top(\frac{\partial}{\partial\theta}\log f_{\theta_0}(\omega))\frac{I_n(\omega)}{f_{\theta_0}(\omega)}]\,d\omega\,.$$

Using that $\mathbb{E}_{\theta_0}(I_n(\omega)) = f_{\theta_0}(\omega)$, the expectation of this expression is just $-I(\theta_0)$, and (under ergodicity and mixing condition) we have the convergence $\frac{1}{n}\nabla^2\ell_n(\tilde{\theta}_n) \xrightarrow{p} -I(\theta_0)$ where $\tilde{\theta}_n$ lies between $\hat{\theta}_n$ and θ_0, and the Fisher information $I(\theta_0)$ is as defined above. By Slutsky's theorem, we conclude that $\sqrt{n}(\hat{\theta}_n-\theta_0) \xrightarrow{d} N(0, I^{-1}(\theta_0))$.

Estimating (H,σ) under high-frequency observations

The high-frequency (HF) regime corresponds to observations of the process $X = (\sigma B^H_{T/n}, \sigma B^H_{2T/n}, ..., \sigma B^H_T)$ (with $\theta = (H,\sigma)$ unknown (as in Part 2, where ν plays the role of σ), and W.L.O.G. we set $T = 1$. Note that $Y_j = \sigma n^H(B^H_{j/n} - B^H_{(j-1)/n})$ is a standard fGN, but the issue now is that the true H is unknown, so the Y process here is unobserved.

Without any add-on noise, we can easily verify that the true MLE is **scale-independent** (since there is an explicit expression for the MLE for $\hat{\sigma}$, see FM14 2023 chapter2, and the **Whittle estimator** for H (defined above, which we will henceforth denote by $\hat{H}_n$) is also scale-independent (assuming σ is unknown)[1].

To see this, for fGN we can re-write $U_n(\theta)$ in (3) in the form

$$U_n(\theta) \quad = \quad \frac{1}{4\pi}\int_{-\pi}^{\pi}(\log(\sigma^2 f_H(\omega)) + \frac{|Z_n(\omega)|^2}{\sigma^2 f_H(\omega)})d\omega\,.$$

Minimizing the integrand in σ we find that $\hat{\sigma}^2_n = \int_{-\pi}^{\pi}\frac{|Z_n(\omega_1)|^2}{f_H(\omega_1)}d\omega_1$, and evaluating the integrand at $\hat{\sigma}_n$, we see that

$$U_n(H,\hat{\sigma}^2_n) \quad = \quad \frac{1}{4\pi}\int_{-\pi}^{\pi}d\omega\cdot\log(\int_{-\pi}^{\pi}\frac{|Z_n(\omega_1)|^2}{f_H(\omega_1)}d\omega_1) \;+\; \frac{1}{4\pi}\int_{-\pi}^{\pi}(\log f_H(\omega) + \frac{|Z_n(\omega)|^2}{\int_{-\pi}^{\pi}\frac{|Z_n(\omega_1)|^2}{f_H(\omega_1)}d\omega_1\cdot f_H(\omega)})d\omega$$

which only changes by constant (which doesn't depend on H) if we multiply Z_n by a constant; thus we still obtain the same $\hat{H}_n$ when we minimize this expression over H, so $\hat{H}_n$ is scale-independent as claimed.

Thus for the high-frequency regime, $\hat{H}_n$ has same behaviour as for original regime, and in particular $\hat{H}_n$ tends asymptotically to a Normal RV with variance equal to the $(1,1)$ component of the inverse of the Fischer information matrix:

$$I(H,\sigma) \quad = \quad \frac{1}{4\pi}\begin{bmatrix}\int_{-\pi}^{\pi}(\frac{\partial}{\partial H}\log f_\theta(\omega))^2 d\omega & \int_{-\pi}^{\pi}\frac{\partial}{\partial H}\log f_H(\omega)\frac{\partial}{\partial\sigma}\log f_\theta(\omega)d\omega \\ \int_{-\pi}^{\pi}\frac{\partial}{\partial H}\log f_H(\omega)\frac{\partial}{\partial\sigma}\log f_\theta(\omega)d\omega & \int_{-\pi}^{\pi}(\frac{\partial}{\partial\sigma}\log f_H(\omega))^2 d\omega\end{bmatrix}.$$

Then using that $f_\theta(\omega) = f_\theta((H,\sigma)) = \sigma^2 f_H(\omega)$, this simplifies to

$$I(H,\sigma) \quad = \quad \frac{1}{4\pi}\begin{bmatrix}\int_{-\pi}^{\pi}(\frac{\frac{\partial}{\partial H}f_H(\omega)}{f_H(\omega)})^2 d\omega & \frac{1}{2\sigma}\int_{-\pi}^{\pi}\frac{\frac{\partial}{\partial H}f_H(\omega)}{f_H(\omega)}d\omega \\ \frac{1}{2\sigma}\int_{-\pi}^{\pi}\frac{\frac{\partial}{\partial H}f_H(\omega)}{f_H(\omega)}d\omega & \frac{2}{\sigma^2}\end{bmatrix}.$$

and we recommend computing these integrals in Mathematica with the `NIntegrate` command. Using that the inverse of a symmetric 2×2 matrix $A = \begin{bmatrix}a & b\\ b & d\end{bmatrix}$, is $A^{-1} = \frac{1}{ad-b^2}\begin{bmatrix}d & -b\\ -b & a\end{bmatrix}$ we see in particular that

$$(I(H,\sigma)^{-1})_{1,1} \quad = \quad \frac{1}{a-\frac{b^2}{d}} \quad = \quad (\frac{1}{4\pi}\int_{-\pi}^{\pi}(\frac{\frac{\partial}{\partial H}f_H(\omega)}{f_H(\omega)})^2 d\omega - \frac{1}{8\pi^2}(\int_{-\pi}^{\pi}\frac{\frac{\partial}{\partial H}f_H(\omega)}{f_H(\omega)}d\omega)^2)^{-1} \tag{5}$$

which agrees with Theorem 1 in [Syz23]. Note $(I(H,\sigma)^{-1})_{1,1} > \frac{1}{I(H,\sigma)_{1,1}} = \frac{1}{\frac{1}{4\pi}\int_{-\pi}^{\pi}(\frac{\frac{\partial}{\partial H}f_H(\omega)}{f_H(\omega)})^2 d\omega}$ which is the asymptotic variance we would get for $\hat{H}$ if σ were known, so σ being unknown adds to the variance of $\hat{H}_n$ as we would intuitively expect.

[1]As an aside we remark the Han-Schied [HS21] estimator for H is not scale-independent but has the advantage of being model agnostic

Estimators for H and ν in Theorem 1 in Syzmanski [Syz23]

We can define a re-scaled estimator $\hat{\nu}_n$ as

$$\hat{\sigma}_n \quad = \quad n^{\hat{H}_n}\hat{\nu}_n$$

and set $\tilde{\sigma}_n = n^H\hat{\nu}_n$, where $\hat{H}_n$ is the (scale-independent) Whittle estimator for H for X and $\hat{\nu}_n$ is the Whittle estimator of the multiplicative constant for the observed "time-stretched" process $\tilde{Y}_j = X_{j/n} = \sigma(B^H_{j/n} - B^H_{(j-1)/n})$ (note $\tilde{Y} = \nu\tilde{X}$ where $\tilde{X}$ is an fGN with $\nu \ll 1$ for n large), and we note that $\hat{\sigma}_n$ *is* computable from observations of X. Moreover, we can re-write $\hat{\sigma}_n$ as $\hat{\sigma}_n = n^{\hat{H}_n - H}\tilde{\sigma}_n$ and we know that

$$\sqrt{n}(\hat{H}_n - H) \quad \to \quad N(0, (I^{-1})_{1,1})$$

from the standard theory for the non high-frequency regime above since (as discussed in previous section) $\hat{H}$ is unaffected by switching between the HF and the original regime. Hence (following the top of page 13 in [Syz23], after correcting some typos there) we see that

$$\begin{aligned}
\frac{\sqrt{n}}{\log n}(\hat{\sigma}_n - \sigma) \quad &= \quad \frac{\sqrt{n}}{\log n}(\tilde{\sigma}_n - \sigma) \,+\, \frac{\sqrt{n}}{\log n}(\hat{\sigma}_n - \tilde{\sigma}) \\
&= \quad \frac{\sqrt{n}}{\log n}(\tilde{\sigma}_n - \sigma) \,+\, \frac{\sqrt{n}}{\log n}\tilde{\sigma}_n(n^{\hat{H}_n - H} - 1) \\
&= \quad \frac{\sqrt{n}}{\log n}(\tilde{\sigma}_n - \sigma) \,+\, \frac{\sqrt{n}}{\log n}\tilde{\sigma}_n \log n\,(\hat{H}_n - H) \,+\, O((\hat{H}_n - H)^2) \\
&= \quad \frac{\sqrt{n}}{\log n}(\tilde{\sigma}_n - \sigma) \,+\, \tilde{\sigma}_n\sqrt{n}\,(\hat{H}_n - H) \,+\, O((\hat{H}_n - H)^2)
\end{aligned}$$

where we have used the Taylor expansion $n^x - 1 = x\log n + O(x^2)$ in the penultimate line, with $x = \hat{H}_n - H$ here. Then we see that the first term here is a higher order (i.e. smaller) term than the second term because $\sqrt{n}(\tilde{\sigma}_n - \sigma)$ is asymptotically Normal from the standard theory above for the non high-frequency regime because $\tilde{\sigma}_n$ behaves the same as the usual Whittle estimator $\hat{\sigma}_n$ in the non-HF regime. Moreover, since $\tilde{\sigma}_n$ is a **consistent estimator**, $\tilde{\sigma}_n = \sigma(1 + o(1))$, so (at leading order) we find that

$$\frac{\sqrt{n}}{\log n}(\hat{\sigma}_n - \sigma) \quad = \quad \sigma\sqrt{n}\,(\hat{H}_n - H) \to N(0, \sigma^2(I^{-1})_{1,1})$$

where $(I^{-1})_{1,1} = (I(H,\sigma)^{-1})_{1,1}$ was computed in (5), which also agrees with Theorem 1 in [Syz23] (note for us the $\gamma(H)$ function in [Syz23] is just $\gamma(H) = H$, see Remark just above Section 1.2 in [Syz23]). Note that $\hat{\sigma}_n - \sigma$ now has standard deviation which is $O(\frac{\log n}{\sqrt{n}})$ as opposed to just $O(\frac{1}{\sqrt{n}})$, due to the HF regime, and specifically the fact that $n^{\hat{H}_n}(\sigma B^H_{\frac{1}{n}}, ..., B^H_1)$ is a noisy observation of an fGN since $\hat{H}_n \neq H$ in general.

Adding additive noise

To incorporate the **microstructure noise** which arises from using realized variance to estimate B^H under high-frequency observations, we add $\sqrt{\frac{2}{n}}$ times a standard Gaussian to the original observed νB^H (see CLT part of Brownian motion chapter in FM02 to see where the $\frac{2}{n}$ comes from; this has nothing to do with fBM), or equivalently (if we work with increments instead as we do above) we add $\sqrt{\frac{2}{n}}Y$ to fGN observed on $[0,1]$, where $Y_t = \varepsilon_t - \varepsilon_{t-1}$ and the $\varepsilon_t's$ are i.i.d. standard Normals (note Y is an MA(1) process of the form $Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}$ with $\theta = -1$ here). Then

$$\mathbb{E}(Y_sY_t) \quad = \quad \mathbb{E}((X_t - X_{t-1})(X_s - X_{s-1}) \quad = \quad 2r(t-s) - r(t-1-s) - r(t+1-s)$$

so Y is also a stationary Gaussian process.

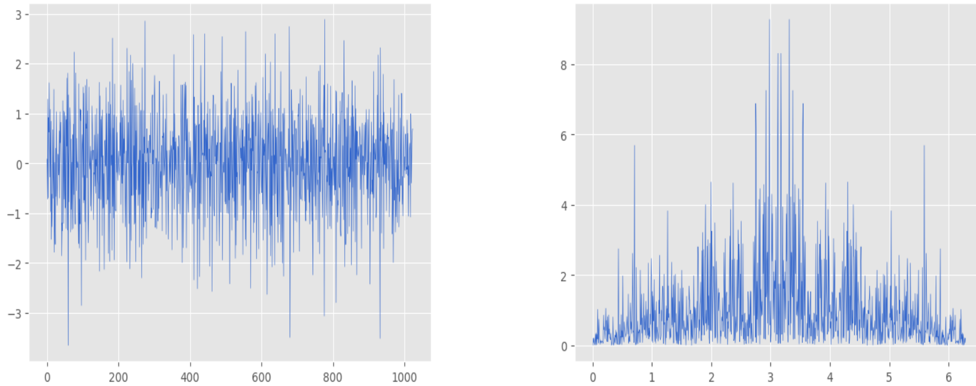

Figure 1: Simulation of fGN (left) and its periodogram (right) for $H = .25$

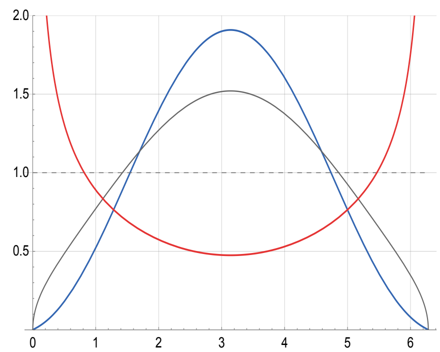

Figure 2: Spectral density $f_H(\omega)$ of fGN for $H = .1$ (blue), $H = .25$ (grey), $H = .5$ (grey dashed) and $H = .75$ (red)

.

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