Solving the FX cross-smiles problem - Schrödinger bridges, Sinkhorn convergence and duality with finite options/liquidity

Martin Forde

10th Sept 2025

Abstract

We adapt the static Schrödinger bridge approach in [Guy20] to the problem of constructing a joint law for two FX rates X and Y consistent with observed European option prices on all three cross-rates. As in [Guy20], we minimize relative entropy $H(\mu|\bar{\mu})$ with respect to a reference measure $\bar{\mu}$ over calibrated models, but here we give a one-line proof for the dualization using the Sion minimax theorem, since sub-level sets of $H(\mu|\bar{\mu})$ are compact. A variational argument as in [Guy20] (see also Remark 3.4 in [Nutz22]) leads to a first-order optimality condition with three coupled integral equations for the optimal European option portfolios u, v, w, and we prove convergence for the three marginals (in the total variation metric) for the Sinkhorn fixed point scheme under certain assumptions on the densitites of X,Y,Z, if an admissible joint density for μ_X, μ_Y and μ_Z exists with finite entropy. Two of the Sinkhorn equations are of the usual fixed-point type but the final equation requires numerical root finding (as is the case for the fourth and fifth equations in the first order optimality conditions for the SPX-VIX problem in [Guy20]). The primary application here is pricing e.g. Basket, Quanto, Best-of, or other varietals of Rainbow options on X and Y so as to be consistent with all three smiles, and we implement the Sinkhorn scheme numerically using recent cross-smile data for the EUR-USD-GBP and EUR-USD-JPY triangles. The Sinkhorn scheme typically converges quickly in practice (around 40 iterations). We also prove duality and give numerical results for the finite-option version of this problem, and similarly for the finite-option version with bid-ask spreads and finite liquidity, which leads to a new formulation for the primal problem with a finite penalty for mis-calibration. Moreover, the original constrained primal minimization problem, and its dual (concave maximization) problem can very often be solved directly (after appropriate discretization) using CVXPY with the MOSEK interior point solver[1], so in this sense the Sinkhorn scheme is often superfluous in practice but still of theoretical interest. [2]

1 Background

Regularized optimal transport methods have been a game changer for exotic option calibration problems in recent years, because it allows the inner inf in the minmax problem which arises from dualization to be computed explicitly (see e.g. pg 8 in [Guy20] or [GLOW22]). For regularized problems in the discrete-time setting, if we use an entropic penalty then can apply the Sinkorn fixed point scheme (with provable convergence in certain standard cases, see e.g.[Nutz22]), for which the most notable practical application has been a discrete-time solution to the fabled SPX-VIX calibration problem in [Guy20], and a partial continuous-time model in [BG24] (the solution in the latter gives rise to a fully continuous-time model if we define the price of the T_1-maturity VIX future at time $t \in [0, T_1]$ to be $\mathbb{E}(F^{-1}_{V|S_{T_1}}(\Phi(B_T/\sqrt{T}))|\mathcal{F}^{W,B}_t) = \mathbb{E}(\mathrm{VIX}_{T_1}|\mathcal{F}^{W,B}_t)$ in the notation of section 4.4 in [BG24], where B is another Brownian motion independent of their original Brownian motion W rather than using an independent standard Uniform random variable as they do).

In the continuous time setting, we can also solve calibration problems using a Markov local and/or stochastic volatility models with finite tradeable European and/or VIX or barrier options using more general penalty terms, by numerically solving a HJB equation which emerges from dualization (see e.g. [GLOW22] and further discussion below), although rigorous duality results are more cumbersome to establish. The cross-smile problem we consider in this article (as outlined in the abstract) falls under the general class of problems of the form $\inf_{\mu \in C} H(\mu|\bar{\mu})$ for a convex set C which is closed under the total variation metric, for which there is a general existence/uniqueness result in Theorem 1.10 in [Nutz22] (see also Proposition 1.17 and Example 1.18).

In principle, a bivariate [Carr09] Local Variance Gamma (LVG) model (see also [CN17]) can also give an exact fit to three cross-smiles, but in practice the implied local correlation function $\rho_{loc}(x, y) := (\sigma_X(x)^2 + \sigma_Y(y)^2 - \sigma_Z(x/y)^2)/(2\sigma_X(x)\sigma_Y(y))$ that comes out from the three smiles typically falls outside $[-1, 1]$ (similarly a two-maturity standard Carr LVG model can be fitted to a single SPX and VIX smile with the same maturity, but in practice the vol function for the 2nd epoch typically has a singularity, and a simpler (albeit somewhat unrealistic)

---

[1] We thank Joseph Sullivan for pointing this out

[2] We thank David Hobson and Amir Dembo for sharing their insights and Spyridon Pougkakiotis for introducing the author to MOSEK

model that achieves the same goal is described in Eq 6.3 in [Guy20]). Alternatively, we can use two-dimensional Bass local volatility model to extract a time-inhomogenous $\rho_{loc}(x,y,t)$ but one then needs to check that the implied correlation function $|\rho_{loc}(.,.,t)| \leq 1$ for all $t \in [0,T]$.

Guo et al. [GLW22],[GLOW22] (see also [HL19] and [Guy22]) show how to construct a generalized local/stochastic volatility model consistent with a finite number of European tradeable options at multiple maturities by minimizing a cost function over calibrated models which penalizes deviations from a standard reference model (e.g. Black-Scholes or Heston), and then re-casting the problem via dualization as an (unconstrained) minmax problem in terms of a non-linear HJB equation (so the cost function effectively regularizes the problem). If options at multiple maturities are used in the calibration set, the HJB equation unfortunately also includes Dirac source terms (but this can be avoided using a nested PDE, see subsection 2.12 below), and this method is extended to include VIX options in section 3.3 in [GLOW22], by re-expressing V_t for the reference model in terms of $\mathbb{E}(\int_t^T \sigma_s^2 ds|\mathcal{F}_t)$ (this analysis is simplified in subsection 2.12 here using that VIX_t^2 is just an affine function of V_t when the drift of V under the reference model has a affine drift). This approach is mathematically rich and exciting albeit numerically intensive since it requires numerically solving a non-linear HJB equation using e.g. very fiddly implicit policy-iteration finite difference schemes and then maximizing over the option weights vector. If path-dependent options are included in the calibration set we have the issue that we do not know whether such a consistent model exists to begin with. The [GLOW22] methodology can in principle be generalized to a rough reference model using a variational approach, but one ends up with a (seemingly) intractable non-standard FBSDE.

[JLO23] construct the two-maturity Bass martingale M, by considering a standard 1d Brownian motion W with initial distribution α_0 and $M_t = \mathbb{E}(F_1(W_T)|\mathcal{F}_t^W)$, and derive two coupled integral equations for α_0 and F_1 in terms of convolutions with the standard Brownian density to enforce that $M_0 \sim \mu_0$ and $M_1 \sim \mu_1$. They also show this formulation is equivalent to the fixed-point integral equation in [CL21] (which maps cdfs to cdfs), and they propose an alternating Sinkhorn-type scheme to solve for (α_0, F_1) (see section 3.2 of [JLO23]) which they call the measure preserving martingale Sinkhorn (MPMS) algorithm. Recall that M also minimizes $\mathbb{E}(\int_0^T (\sigma_t - 1)^2)dt$ (hence the name stretched Brownian motion) or equivalently maximizes $\mathbb{E}(\int_0^T |\sigma_t|dt)$ and $\mathbb{E}(M_T W_T)$ subject to the two marginal constraints, where $dM_t = \sigma_t dW_t$. Using the HJB/dualization approach we can obtain the dual representation for this problem in Eq 2 in [JLO23] using the Fenchel-Moreau theorem which can be re-written in a (possibly) more intuitive/familiar form as $\sup_\varphi(-\int \varphi(y)\mu_T(dy) + \int u^\varphi(x,0)\mu_0(dx))$ where u^φ satisfies the HJB equation $u_t + \inf_\sigma(\frac{1}{2}\sigma^2 u_{xx} + \frac{1}{2}(\sigma^2 - 1)) = 0$ with $u(x,T) = \varphi$, which is the same as the usual form for Guo-Loeper type problems with a single terminal target law μ_T except now we integrate $u^\varphi(x,0)$ over μ_0 since this is a two-marginals problem.

In this article, we adapt the Schrödinger bridge approach in [Guy20] to the problem of constructing a joint density for two FX rates X and Y consistent with observed European option prices on all three cross-rates. This can of course be solved as a pure optimal transport (OT) (linear programming) problem where we compute the max or min price of a non-tradeable option given tradeable European options on all three cross-rates (as indeed can Guyon's SPX-VIX calibration problem[3], see e.g. numerical results in subsection 2.10), but the Schrödinger bridge solution is more realistic since the optimal coupling μ^* has a joint density unlike OT problems where the support of μ^* is typically just the graph of n-functions, e.g. the celebrated *left curtain coupling* in [BJ16] where $n = 2$ where $Y = T_u(X)$ or $T_d(X)$ for X in a certain range, where $T_u(x) \geq x$ is non-decreasing and $T_d(x) \leq x$ non-increasing (see explicit derivations of the non-linear coupled ODEs for $T_u(.)$ and $T_d(.)$ on page 12 in [HLT16], and the associated hedge quantities on page 18 in [HLT16]). Variants of the left curtain coupling are derived in the context of robust pricing/hedging of forward starter options in [HK15] ($n = 3$), and [HN12] ($n = 2$), and (as above) such problems can be efficiently approximated in MOSEK (one can also compute e.g. minimal price of an American puts with finite exercise dates given two marginals).

Under mild assumptions, we show that the upper (resp. lower) bound for options on X/Y which pay $(X - KY)^+$ in the original currency are attained by the lower (resp. upper) Fréchet-Hoeffding bound (see also Hobson et al. [HLW05],[HLW05b] for a more involved analysis with basket options where the weights are assumed to be non-negative or 1 unlike our example here).

Once one has verified strict convexity of $G(u,v,w)$ using Hölder's inequality), our Sinkhorn scheme can be viewed as a coordinate ascent scheme for the concave maximization problem in (6), see also page 52 in [Nutz24]; although proving convergence of (u_n, v_n, w_n) appears difficult due to non-compactness issues which don't arise in the standard simple proof in e.g. Proposition 6.5.1 in [Bert15].

Our Sinkhorn scheme typically converges quickly in practice (around 40 iterations). We also construct a continuous-time martingale model consistent with the three marginals using conditional sampling as in [BG24], and a rough extension of this model using a rough Bergomi Bass martingale as in section 4 in [F24], and we discuss more realistic choices for the reference measure e.g. using the Gaussian copula. We also show how to adapt the Sinkhorn scheme for the forward-starter calibration problem in the usual setup with one asset and two maturities, in this case it appears one can prove convergence for the three marginals and the martingale condition for the Sinkhorn scheme if we modify the scheme by running a Sinkhorn "subscheme" which just involves running the first

[3] We thank Rowan Austin for demonstrating this in MOSEK using sparse flattened matrices to deal with the constraints.

three Sinkhorn equations with fixed hedge amount Δ_n each time, and alternating this with the fourth equation for solves for Δ_n, but to prove convergence requires an infinite amount of conditions of the form in Assumption 2.1 which is impossible to check in practice. Finally we switch attention to continuous time Schrödinger bridges, and we formally simplify the analysis of the SPX-VIX calibration problem for a Markov stochastic volatility model discussed in [GLOW22]. **Rigorous proof using Fenchel-Moreau**?.

2 Outline of problem

For major currencies like the Euro, Dollar and Pound, European options are very actively traded on all three cross-rates. A natural question to ask is: how to we construct a model so as to be consistent with observed European option prices at multiple strikes (at a single maturity) on all three cross-rates. This problem is attempted in [Aus11] but the article does not check that the resulting joint density is non-negative.

Assuming our home currency is dollars, we let X denote the price of 1 Euro in dollars at time T in the future (known as the EUR/USD rate), and Y denote the price of 1 pound in dollars at time T (the GBP/USD rate). A European option on EUR/GBP (i.e. on the cross-rate $Z = X/Y$) pays $(Z-K)^+$ pounds at time T, or equivalently $Y(Z-K)^+$ dollars. If we assume interest rates are zero for simplicity (otherwise we work with forward rates instead), then given a risk-neutral measure $\mathbb{Q}$ associated to the US economy, the initial price of such an option (in pounds) is

$$\frac{1}{Y_0}\mathbb{E}^{\mathbb{Q}}(Y(\frac{X}{Y}-K)^+) \quad = \quad \mathbb{E}^{\tilde{\mathbb{Q}}}((Z-K)^+)$$

where $Y_0 = \mathbb{E}^{\mathbb{Q}}(Y)$, and we have re-written the expectation on the left side here using the probability measure

$$\tilde{\mathbb{Q}}(A) \quad := \quad \mathbb{E}^{\mathbb{Q}}(\frac{Y}{\mathbb{E}^{\mathbb{Q}}(Y)}A)$$

associated with using GBP as the home currency. From here, we will just use $\mathbb{E}(.)$ to denote expectations under $\mathbb{Q}$.

From the Breeden-Litzenberger formula, we can extract the law of X and Y under $\mathbb{Q}$ from European call prices at all strikes on X, and the law of X/Y under $\tilde{\mathbb{Q}}$, and in practice this would typically be done using an SVI-type parametrization to interpolate between tradeable strikes with parameters chosen so as to preclude butterfly arbitrage (see the many articles by Gatheral, Jacquier, Martini, Mingone et al. on this theme).

2.1 Formulation of the problem

x/y**,** $y=0$ **issue** . We use the convention that $\mathbb{R}_+ = [0,\infty)$ throughout. Now suppose we are given three target laws μ_X, μ_Y and μ_Z on $[0,\infty)$ and (without loss of generality) we assume that

$$\int_{[0,\infty)} x\mu_X(dx) \quad = \quad \int_{[0,\infty)} y\mu_Y(dy) \quad = \quad \int_{[0,\infty)} z\mu_Z(dz) \quad = \quad 1 \tag{1}$$

i.e. the initial rates $X_0 = Y_0 = 1$ and hence $Z_0 = X_0/Y_0 = 1$ as well. Let $\Pi(\mu,\nu)$ denote the space of joint probability measures on $[0,\infty)\times[0,\infty)$ with marginals μ and ν respectively. We wish to find a $\mu^* \in \Pi(\mu_X,\mu_Y)$ such that if X, Y have joint distribution μ^*, then $X/Y \sim \mu_Z$ under the probability measure $\tilde{\mathbb{Q}}$ defined above.

Remark 2.1 If such a μ^* exists, it can then be used to price quanto options like e.g. Best-of options which pay $(X-K_1)^+/K_1 \wedge (Y-K_2)^+/K_2$, or a quanto option which pays $(\frac{X}{Y}-K)^+$ dollars (as opposed to EUR).

This leads to the following definition:

Definition 2.1 *We define the convex set*

$$\mathcal{P}(\mu_X,\mu_Y,\mu_Z) \quad = \quad \{\mu \in \Pi(\mu_X,\mu_Y) : \int_{\mathbb{R}_+^2} (x-Ky)^+\mu(dx,dy) \; = \; \int_{[0,\infty)} (z-K)^+\mu_Z(dz) \text{ for all } K \geq 0\} \tag{2}$$

(note there is no $\frac{1}{Y_0}$ factor on the left here as we have already assumed $Y_0 = 1$).

2.2 Technical results

Let

$$P(\mu_X,\mu_Y,\mu_Z) \quad := \quad \inf_{\mu\in\mathcal{P}(\mu_X,\mu_Y,\mu_Z)} H(\mu|\bar{\mu}) \tag{3}$$

where $H(\mu|\bar{\mu}) := \mathbb{E}^{\mu}(\log \frac{d\mu}{d\bar{\mu}}) = \mathbb{E}^{\bar{\mu}}(\frac{d\mu}{d\bar{\mu}}\log \frac{d\mu}{d\bar{\mu}})$ is the entropy of μ with respect to $\bar{\mu}$. Since $h(z) = z\log z$ is convex, $\lim_{z\searrow 0} h(z) = 0$ and $\mathbb{E}^{\bar{\mu}}(\frac{d\mu}{d\bar{\mu}}) = 1$, we see that $H(\mu|\bar{\mu}) \geq 0$, and $H(\bar{\mu},\bar{\mu}) = 0$, and $H(.,\bar{\mu})$ is lower semicontinuous (l.s.c.), see e.g. Lemma 1.3 in [Nutz22]. This is a variant of the static Schrödinger bridge problem (see again [Nutz22] for more on this). At this point we make our final standing assumption for the problem that follows to be well defined:

Assumption 2.1 *There exists a* $\mu^* \in \mathcal{P}(\mu_X,\mu_Y,\mu_Z)$ *with* $H(\mu^*|\bar{\mu}) < \infty$.

(see also Theorem 1.10, Proposition 1.17 and Example 1.18 in [Nutz22]).

Recall that $H(\mu|\bar{\mu})$ is the large deviation rate function in Sanov's theorem and is jointly l.s.c. under the weak topology and jointly convex (see e.g. Lemma 1.3 in [Nutz22]), hence the sub-level sets

$$\mathcal{P}_c(\mathbb{R}_+^2) \quad = \quad \{\mu \in \mathcal{P}(\mathbb{R}_+^2) \,:\, H(\mu|\bar{\mu}) \leq c\}$$

are also convex, and moreover are compact in the weak topology since $H(.|\bar{\mu})$ is a good rate function (see e.g. Corollary 6.2.3, discussion at the top of page 263 and Lemmas 6.2.12 and 6.2.13 in [DZ10]).

We now choose $c > 0$ sufficiently large so that $H(\mu^*|\bar{\mu}) < c$, so we can re-write (3) as $P(\mu_X,\mu_Y,\mu_Z) = \inf_{\mu\in\mathcal{P}_c(\mu_X,\mu_Y,\mu_Z)} H(\mu|\bar{\mu})$, which will be key for all the duality results that follow.

Define

$$f(u,v,w,\mu) \quad := \quad H(\mu|\bar{\mu}) \,-\, \mathbb{E}^{\mu}(u(X)+v(Y)+Yw(Z)) \,+\, \mathbb{E}^{\mu_X}(u(X))+\mathbb{E}^{\mu_Y}(v(Y))+\mathbb{E}^{\mu_Z}(w(Z))\,. \tag{4}$$

Then f is convex and l.s.c. in μ under the weak topology (lower semicontinuity in μ for the $\mathbb{E}^{\mu}(Yw(Z))$ term follows from the Portmanteau theorem because $yw(x/y)$ is l.s.c. on $\mathbb{R}_+^2$), and f is concave and continuous in (u,v,w). We have also seen thar $\mathcal{P}_c(\mathbb{R}_+^2)$ is compact and convex, and we know that $C_b(\mathbb{R}_+^2)$ is convex. Thus (under Assumption 2.1), using the Sion minimax theorem, we see that

$$\begin{aligned} P(\mu_X,\mu_Y,\mu_Z) \quad &= \inf_{\mu\in\mathcal{P}_c(\mathbb{R}_+^2)}\ \sup_{u,v,w\in C_b(\mathbb{R}_+^2)} f(u,v,w,\mu) \quad = \inf_{\mu\in\mathcal{P}_c(\mathbb{R}_+^2)}\ \sup_{u,v,w\in C_b(\mathbb{R}_+^2)} f(u,v,w,\mu) \\ &= \sup_{u,v,w\in C_b}\ \inf_{\mu\in\mathcal{P}_c(\mathbb{R}_+^2)} f(u,v,w,\mu)\,. \end{aligned} \tag{5}$$

Moreover, from the Donsker-Varadhan/Gibbs variational formula (see also pg 8 in [Guy20]), we know that for any random variable X

$$\inf_{\mu\in\mathcal{P}}\{H(\mu|\bar{\mu}) - \mathbb{E}^{\mu}(X)\} \quad = \quad -\log\mathbb{E}^{\bar{\mu}}(e^X)$$

(see e.g. Eq 1.1 in Dupuis et al.[ACD15]), and in Appendix A we verify this is still true even when $\mathbb{E}^{\bar{\mu}}(e^X) = \infty$ using a mollification argument. Hence the inner inf in (5) can be computed explicitly as a concave maximization problem:

$$P(\mu_X,\mu_Y,\mu_Z) \quad = \quad \sup_{u,v,w\in\mathcal{C}^1} G(u,v,w) \tag{6}$$

where

$$G(u,v,w) \quad = \quad \mathbb{E}^{\mu_X}(u(X))+\mathbb{E}^{\mu_Y}(v(Y))+\mathbb{E}^{\mu_Z}(w(Z)) - \log\mathbb{E}^{\bar{\mu}}(e^{u(X)+v(Y)+Yw(X/Y)}) \tag{7}$$

(see also Theorem 3.2 in [Nutz22]), and (again from pg 8 in [Guy20] or Appendix A) we know that the optimal μ (for each $u,v,w\in\mathcal{C}^1$ fixed) for the inner inf in (5) is of the form

$$\frac{e^{u(x)+v(y)+yw(\frac{x}{y})}}{\mathbb{E}^{\bar{\mu}}(e^{u(X)+v(Y)+Yw(\frac{X}{Y})})}\,\bar{\mu}(x,y) \tag{8}$$

Remark 2.2 We can easily adapt the Sion minimax argument above with finite level sets to obtain the corresponding concave maximization problem in Eq 5.4 in [Guy20] for the SPX-VIX calibration problem, using C_b test functions to deal with the additional martingale and dispersion constraints.

Proposition 2.2 *A unique minimizer for* (3) *exists.*

Proof. See Appendix B. ■

Since a unique minimizer μ^* for exists, from Theorem 3.1 in [Csi75] (see also end of page 15 in [Nutz22]), the minimizer is of the form in (8).

In the next section, we use (8) as our ansatz for the optimal μ^*, and if this is the case we can assume that the denominator here is 1, i.e. that

$$\mu^*(x,y) \quad = \quad e^{u^*(x)+v^*(y)+yw^*(\frac{x}{y})}\bar{\mu}(x,y) \tag{9}$$

because when we enforce that μ^* has marginals μ_X and μ_Y below, this forces $\mathbb{E}(e^{u^*(x)+v^*(y)+yw^*(\frac{x}{y})})$ to be 1, or else the marginals will be out by a multiplicative factor (u,v,w are known as the *Schrödinger potentials*).

Remark 2.3 Given this normalization using a variational argument as in [Guy20], we can remove the log term in (6) and we still end up with the same equations for u, v and w below (see Remark 7 in [Guy20]; this is also the formulation used in section 3 in [Nutz22]), so we can re-write the maximization problem in (6) as

$$P(\mu_X, \mu_Y, \mu_Z) \quad = \quad \sup_{u,v,w \in \mathcal{C}^1} [\mathbb{E}^{\mu_X}(u(X)) + \mathbb{E}^{\mu_Y}(v(Y)) + \mathbb{E}^{\mu_Z}(w(Z)) - \mathbb{E}^{\bar{\mu}}(e^{u(X)+v(Y)+Yw(X/Y)}) \, + \, 1] \, .$$

We will not work with this formulation in this article.

2.3 Coupled integral equations for u, v, w

From here on we assume μ_X, μ_Y and μ_Z have densities.

Let $\bar{\mu}$ denote a reference probability measure, typically chosen to be the product measure of μ_X and μ_Y, so $\bar{\mu}$ already satisfies two of the three marginal constraints. Then using the ansatz in (9) the marginal constraints can be written

$$\begin{aligned} \mu_X(x) &= e^{u(x)} \int_0^\infty e^{v(y)+yw(\frac{x}{y})} \bar{\mu}(x,y) dy \\ \mu_Y(y) &= e^{v(y)} \int_0^\infty e^{u(x)+yw(\frac{x}{y})} \bar{\mu}(x,y) dx \\ \mu_Z(K) &= \int_0^\infty \int_0^\infty y \delta(\frac{x}{y} - K) e^{u(x)+v(y)+yw(\frac{x}{y})} \bar{\mu}(x,y) dy dx \\ &= \int_0^\infty \int_0^\infty \frac{x}{z} \delta_K(dz) e^{u(x)+v(\frac{x}{z})+\frac{x}{z}w(z)} \frac{x}{z^2} \bar{\mu}(x, \frac{x}{z}) dz dx \\ &= \int_0^\infty \frac{x}{K} e^{u(x)+v(\frac{x}{K})+\frac{x}{K}w(K)} \frac{x}{K^2} \bar{\mu}(x, \frac{x}{K}) dx \end{aligned} \tag{10}$$

where we have made the transformation $z = x/y$ (i.e. $y = x/z$), so $dy = -\frac{x}{z^2} dz$ in the inner integral. Or we can obtain the same three equations using a variational argument as on pg 8 in [Guy20] (or Remark 3.4 in [Nutz22]), so we can intepret these three coupled integral equations as Euler-Lagrange equations for the concave maximization problem in (6).

If the integrals in the expressions for $\mu_X(x)$ and $\mu_Y(y)$ are finite and non-zero, the first two equations can be re-arranged as

$$\begin{aligned} u(x) &= \log \mu_X(x) \, - \, \log \int_0^\infty e^{v(y)+yw(\frac{x}{y})} \bar{\mu}(x,y) dy \\ v(y) &= \log \mu_Y(y) \, - \, \log \int_0^\infty e^{u(x)+yw(\frac{x}{y})} \bar{\mu}(x,y) dx \end{aligned}$$

and from here on we make the usual choice that $\bar{\mu}(x,y) = \mu_X(x)\mu_Y(y)$, so these simplify further to

$$\begin{aligned} u(x) &= - \log \int_0^\infty e^{v(y)+yw(\frac{x}{y})} \mu_Y(y) dy \\ v(y) &= - \log \int_0^\infty e^{u(x)+yw(\frac{x}{y})} \mu_X(x) dx \end{aligned}$$

and setting $K = z$, we see that

$$H(z; u, v, w) \quad = \quad \int_0^\infty e^{u(x)+v(\frac{x}{z})+\frac{x}{z}w} \frac{x^2}{z^3} \mu_X(x) \mu_Y(\frac{x}{z}) dx - \mu_Z(z) \quad = \quad 0 \tag{11}$$

where $H : \mathbb{R}_+ \times L_0(\mathbb{R}_+) \times L_0(\mathbb{R}_+) \times \mathbb{R} \to \mathbb{R}$.

2.4 The Sinkhorn scheme

The Sinkhorn-type fixed point iterative scheme to solve these coupled equations is then given by

$$u_{n+1}(x) \quad := \quad - \log \int_0^\infty e^{v_n(y)+yw_n(\frac{x}{y})} \mu_Y(y) dy \tag{12}$$

$$v_{n+1}(y) \quad := \quad - \log \int_0^\infty e^{u_{n+1}(x)+yw_n(\frac{x}{y})} \mu_X(x) dx \tag{13}$$

$$0 \quad = \quad H(z; u^{n+1}, v^{n+1}, w^{n+1}) \tag{14}$$

and we typically set $u^0 \equiv v^0 \equiv w^0 \equiv 0$.

Assumption 2.3 *$\text{supp}(\mu_X) = [0, \bar{X}]$, $\text{supp}(\mu_Y) = [\underline{Y}, \bar{Y}]$, $\text{supp}(\mu_Z) = [\underline{Z}, \bar{Z}]$ and $\mu_Y(y) \in [\underline{\mu}_Y, \bar{\mu}_Y]$ and $\mu_Z(y) \in [\underline{\mu}_Z, \bar{\mu}_Z]$ where all constants here are positive and finite.*

Then if v_n and w_n lie in finite intervals $[\underline{v}_n, \bar{v}_n]$ and and $[\underline{w}_n, \bar{w}_n]$, we see that

$$\begin{aligned} u_{n+1}(x) &\in [-\bar{v}_n - \bar{Y}\bar{w}_n, -\underline{v}_n - \underline{Y}\,\underline{w}_n] \\ v_{n+1}(y) &\in [-\bar{u}_{n+1} - \bar{Y}\bar{w}_n, -\underline{u}_{n+1} - \underline{Y}\,\underline{w}_n] \end{aligned}$$

and hence u_{n+1} and v_{n+1} are also bounded from below.

Let $h(x; w, z) = e^{u(x)+v(\frac{x}{z})+\frac{x}{z}w}\frac{x^2}{z^3}\mu_X(x)\mu_Y(\frac{x}{z})$ denote the integrand in the expression for $I_3(w, z) := H(z; u, v, w)$ where we have dropped the dependence on u and v to ease notation. $I_3(w, z)$ is clearly strictly monotonically increasing in w. Moreover (under Assumption 2.3) if u, v are bounded then $H(z; u, v, w)$ is finite for all $w \in \mathbb{R}$, and $h(x; w, z)$ tends monotonically to $+\infty$ as $w \to \infty$ and 0 as $w \to -\infty$. Hence (from the monotone convergence theorem) $I_3(w, z)$ tends to $+\infty$ as $w \to \infty$ and tends to $-\mu_Z(z) < 0$ as $w \to -\infty$. Then considering an arbitrary sequence $w^{(k)} \to w$, we see that $I_3(., z)$ is continuous (from the bounded convergence theorem), so a unique finite root $w(z)$ exists (where we have emphasized the dependence on z here).

If u_{n+1}, v_{n+1} are bounded from above, then from the third integral equation

$$\int_0^\infty e^{u(x)+v(\frac{x}{z})+\frac{x}{z}w}\frac{x^2}{z^3}\mu_Y(\frac{x}{z})\mu_X(x)dx - \mu_Z(z) \quad = \quad 0$$

we see that $w_{n+1} \le \bar{w}$, where $e^{\underline{X}\bar{w}/\bar{Z}}e^{\underline{u_{n+1}}+\underline{v_{n+1}}}\frac{\underline{X}^2}{\bar{Z}^3}\underline{\mu}_Y = \bar{\mu}_Z$, and $w_{n+1} \ge \underline{w}$ where $e^{\bar{X}\underline{w}/\underline{Z}}e^{\overline{u_{n+1}}+\overline{v_{n+1}}}\frac{\bar{X}^2}{\underline{Z}^3}\bar{\mu}_Y = \underline{\mu}_Z$. Hence by induction, $(u_{n+1}, v_{n+1}, w_{n+1})$ are bounded for each n so the Sinkhorn scheme doesn't blow up at any finite n. This does not show that (u_n, v_n, w_n) are uniformly bounded in n, but in the next section we prove convergence of the marginals for the scheme in the total variation metric.

The Sinkhorn scheme typically converges very quickly in practice, and note the third equation here requires numerical root-finding as for the SPX-VIX calibration problem discussed in [Guy20], but we have one less dimension here since we only have to compute $w(z)$ for a range of values for the one argument z not a two-variable function of the form $\Delta(s_1, v)$ as in [Guy20].

2.4.1 Convergence of the Sinkhorn scheme

Set $\mu_{i,j,k}(x, y) = e^{u_i(x)+v_j(y)+yw_k(x/y)}\bar{\mu}(x, y)$, and let $\mu^X_{i,j,k}(x)$ (resp. $\mu^Y_{i,j,k}(y)$) denote the first (resp. second) marginal of $\mu_{i,j,k}$, and $\mu^Z_{i,j,k}(z)$ denote the $Z-$marginal of $\mu_{i,j,k}$ under $\tilde{\mathbb{Q}}$.

Proposition 2.4 *Under our standing Assumptions 2.2-2.4 [4] and 2.3, the marginals for the Sinkhorn scheme converge in the following sense: $\|\mu^X_{n,n,n} - \mu_X\|_{TV} \to 0$ and $\|\mu^Y_{n,n,n} - \mu_Y\|_{TV} \to 0$ as $n \to \infty$ (where $\|.\|_{TV}$ denotes the total variation distance, see [Nutz22]) with $\mu^Z_{n,n,n} = \mu_Z$ for all $n \ge 1$, and $\mathbb{E}^{\mu_X}(u_n(X)) + \mathbb{E}^{\mu_Y}(v_n(Y)) + \mathbb{E}^{\mu_Z}(w_n(Z)) \le H(\mu^*|\bar{\mu}) < \infty$.*

Remark 2.4 If Assumptions 2.2 and 2.3 hold (which just relate to the three given marginals), then the proposition implies that if the Sinkhorn scheme fails to converge, then Assumption 2.4 cannot hold. The result does not discount the possibility that there may exist a $\mu^* \in \mathcal{P}(\mu_X, \mu_Y, \mu_Z)$ with $H(\mu^*|\bar{\mu}) = \infty$. The result does not tell us about convergence (or otherwise) of (u_n, v_n, w_n) in $\mathcal{C}$.

Proof. (of Proposition 2.4). Following Lemma 6.4 in [Nutz22], we see that

$$\begin{aligned} H(\mu_{n+1,n,n}|\mu_{n,n,n}) &= \int \log\frac{d\mu_{n+1,n,n}}{d\mu_{n,n,n}}\mu_{n+1,n,n}(x, y)dxdy \quad = \quad \int\int(u_{n+1} - u_n)\mu_{n+1,n,n}(x, y) \\ &= \int(u_{n+1} - u_n)d\mu_X \qquad (15) \end{aligned}$$

since $\mu_{n+1,n+1,n}$ has the correct marginal for X by construction (see also Eq 6.5 in [Nutz22]), and similarly

$$\begin{aligned} H(\mu_{n+1,n+1,n}|\mu_{n+1,n,n}) &= \int(v_{n+1} - v_n)d\mu_{n+1,n+1,n} \quad = \quad \int(v_{n+1} - v_n)d\mu_Y \\ H(\mu_{n+1,n+1,n+1}|\mu_{n+1,n+1,n}) &= \int y(w_{n+1} - w_n)d\mu_{n+1,n+1,n+1} \quad = \quad \int(w_{n+1} - w_n)d\mu_Z \qquad (16) \end{aligned}$$

for all $n \ge 0$. From Assumption 2.1, an admissible μ^* exists with $H(\mu^*|\bar{\mu}) < \infty$, so (from Lemma 1.4b) in [Nutz22]) we see that

$$H(\mu^*|\bar{\mu}) - H(\mu^*|\mu_{n,n,n}) \quad = \quad \mathbb{E}^{\mu^*}(\log\frac{d\mu_{n,n,n}}{d\bar{\mu}}) \qquad (17)$$

[4] recall that Assumption 2.4 implies Assumption 2.1

which (from the definition of the scheme) we can re-write as

$$\mathbb{E}^{\mu^*}(\log\frac{d\mu_{n,n,n}}{d\bar{\mu}}) \quad = \quad \mathbb{E}^{\mu^*}(u_n(X)+v_n(Y)+Yw_n(X/Y)) \quad = \quad \mathbb{E}^{\mu_X}(u_n(X))+\mathbb{E}^{\mu_Y}(v_n(Y))+\mathbb{E}^{\mu_Z}(w_n(Z)).\tag{18}$$

Re-writing the right hand side here as a telescoping sum using the three equations at the beginning of the proof, we see that

$$H(\mu^*|\mu_{n,n,n}) \quad = \quad H(\mu^*|\bar{\mu}) - \sum_{i=0}^{n-1}(H(\mu_{i+1,i,i}|\mu_{i,i,i})+H(\mu_{i+1,i+1,i}|\mu_{i+1,i,i})+H(\mu_{i+1,i+1,i+1}|\mu_{i+1,i+1,i})).\tag{19}$$

so

$$0 \quad \leq \quad H(\mu^*|\mu_{n,n,n}) \quad \leq \quad H(\mu^*|\bar{\mu})$$

and we see that $H(\mu^*|\mu_{n,n,n})$ is non-increasing. Hence from (17) and (18) we see that

$$\mathbb{E}^{\mu^*}(\log\frac{d\mu_{n,n,n}}{d\bar{\mu}}) \quad = \quad \mathbb{E}^{\mu_X}(u_n(X))+\mathbb{E}^{\mu_Y}(v_n(Y))+\mathbb{E}^{\mu_Z}(w_n(Z)) \quad \leq \quad H(\mu^*|\bar{\mu}) \quad < \quad \infty$$

(recall $H(\mu^*|\bar{\mu})$ is finite by assumption).

The sum on the right in (19) is bounded by $H(\mu^*|\bar{\mu})$, so the summand tends to zero. Then from the Data Processing Inequality (Example 1.7 in [Nutz22]), for $i \geq 1$, the third term in the summand in (19) dominates $H(\mu^Y_{i+1,i+1,i+1}|\mu_Y)$ and the first term dominates $H(\mu_X|\mu^X_{i,i,i})$ (note the ordering is now the other way round, but this won't matter when we apply Pinsker's inequality below) so these two quantities also tend to zero as $i \to \infty$ (so clearly $H(\mu^Y_{i,i,i}|\mu_Y) \to 0$ as well), and $\mu^Z_{i,i,i}$ satisfies the third marginal condition exactly by construction for all $i \geq 1$. Then from Pinsker's inequality (Lemma 1.2 in [Nutz22])

$$\begin{aligned}
\|\mu^X_{i,i,i}-\mu_X\|_{TV} \quad &\leq \quad H(\mu_X|\mu^X_{i,i,i}) \quad \to \quad 0 \\
\|\mu^Y_{i,i,i}-\mu_Y\|_{TV} \quad &\leq \quad H(\mu^Y_{i,i,i}|\mu_Y) \quad \to \quad 0
\end{aligned}$$

■

Remark 2.5 See Section 6.2 in [Nutz22] for discussion on the rate of convergence for Sinkhorn schemes.

2.5 Finite-option duality

In this subsection, we consider the n-option analogue of Eq (5). More specifically, we let C^X, C^Y, C^Z denote random vectors with components $C^X_j=(X-K^X_j)^+$, $C^Y_j=(Y-K^Y_j)^+$, $C^Z_j=(X-K^Z_jY)^+$ for $j=0...m$ with corresponding market price vectors P^X, P^Y and P^Z, and let $\mathcal{P}(P^X,P^Y,P^Z)=\{\mu\in\mathcal{P}(\mathbb{R}^2_+):\mathbb{E}^\mu((X-K^X_j)^+)=P^X_j,\ \mathbb{E}^\mu((Y-K^Y_j)^+)=P^Y_j,\ \mathbb{E}^\mu((X-K^Z_jY)^+)=P^Z_j\}$, and we further assume that the zeroth strikes are zero, i.e. $K^X_0=0$, $K^Y_0=0$ and $K^Z_0=0$ which correspond to the tradeable forward contracts. Then assuming $\mathcal{P}(P^X,P^Y,P^Z)$ is non-empty and includes an element with $H(\mu|\bar{\mu})\leq c$, again (by Sion minimax theorem) we see that

$$\begin{aligned}
\inf_{\mu\in\mathcal{P}(P^X,P^Y,P^Z)} H(\mu|\bar{\mu}) \quad &= \quad \inf_{\mu\in\mathcal{P}_c(\mathbb{R}^2_+)}\sup_{u,v,w\in\mathbb{R}^{m+1}} f(u,v,\mu) \\
&= \quad \sup_{u,v,w\in\mathbb{R}^{m+1}}\inf_{\mu\in\mathcal{P}(\mathbb{R}^2_+)} f(u,v,\mu) \\
&= \quad \sup_{u,v,w\in\mathbb{R}^{m+1}} (u.P^X+v.P^Y+w.P^Z-\log\mathbb{E}^{\bar{\mu}}(e^{u.C^X+v.C^Y+w.C^Z}))
\end{aligned}\tag{20}$$

where $f(u,v,w,\mu):=H(\mu|\bar{\mu})-\mathbb{E}^\mu(u.C^X+v.C^Y+w.C^Z)+u.P^X+v.P^Y+w.P^Z$, and again we have made the natural assumption here that $\mathcal{P}(P^X,P^Y,P^Z)$ is non-empty and has an element μ with $H(\mu|\bar{\mu})<c$, where the optimal μ for each (u,w,v) takes the form

$$\mu(x,y) \quad = \quad \frac{e^{u.C^X+v.C^Y+w.C^Z}}{\mathbb{E}^{\bar{\mu}}(e^{u.C^X+v.C^Y+w.C^Z})}\,\bar{\mu}(x,y)\,.$$

Hence we see that u,v,w maximizes expected utility under $\bar{\mu}$ using European options and forwards on all three cross-rates under $\bar{\mu}$, if the u,v,w contracts are priced according to the market, so clearly the dual problem is more relevant if $\bar{\mu}$ is the real-world (physical) measure, in which case this coincides with the usual portfolio optimization problem of maximizing expected exponential utility using available tradeable assets, see numerical results in Figure . (see also Remark 13 in [Guy24]). Note we can also add additional budget constraints into this problem, see e.g. [APR18].

2.6 Bid/ask spreads, finite liquidity and the dual portfolio optimization problem with exponential utility

To allow for finite liquidity, we modify the problem above to

$$\inf_{\mu\in\mathcal{P}_c(\mathbb{R}^2_+)}\sup_{u\in[q_b^X,q_a^X],v\in[q_b^Y,q_a^Y],w\in[q_b^Z,q_a^Z]}(\frac{1}{\alpha}H(\mu|\bar{\mu})+\mathbb{E}^\mu(u.C^X+v.C^Y+z.C^Z)\,+\,\chi(u,v,w)) \tag{21}$$

where $\chi(u,v,w)=-P_a^X.u^++P_b^X.u^--P_a^Y.v^++P_b^Y.v^--P_a^Z.w^++P_b^Z.w^-$, and $\alpha>0$ is a risk aversion parameter (the reason the primal takes this form will become clear below when we compute the dual). As before, we can apply the Sion minimax theorem to interchange the inf and sup, and then the inner inf and the optimal μ can be computed explicitly, so we can further re-write the sup inf as a portfolio optimization problem

$$\sup_{u\in[q_b^X,q_a^X],v\in[q_b^Y,q_a^Y],w\in[q_b^Z,q_a^Z]}(-\frac{1}{\alpha}\log\mathbb{E}^{\bar{\mu}}(e^{-\alpha(u.C^X+v.C^Y+z.C^Z)})\,+\,\chi(u,v,w)) \tag{22}$$

with liquidity constraints, where the optimal μ is $\mu(x,y)=\frac{e^{-\alpha(u.C^X+v.C^Y+z.C^Z)}}{\mathbb{E}^{\bar{\mu}}(e^{-\alpha(u.C^X+v.C^Y+z.C^Z)})}\bar{\mu}(x,y)$.

Since $H(\mu|\bar{\mu})$ does not depend on u,v,w in (21), we can easily evaluate the inner sup to re-write (21) more explicitly as

$$\begin{aligned}\inf_{\mu\in\mathcal{P}_c(\mathbb{R}_+)}[\frac{1}{\alpha}H(\mu|\bar{\mu})\quad&+\quad q_a^X.(\mathbb{E}^\mu(C^X)-P_a^X)^+\,+\,|q_b^X|.(P_b^X-\mathbb{E}^\mu(C^X))^+\\&+\quad q_a^Y.(\mathbb{E}^\mu(C^Y)-P_a^X)^+\,+\,|q_a^Y|.(P_b^X-\mathbb{E}^\mu(C^Y))^+\\&+\quad q_a^Z.(\mathbb{E}^\mu(C^Z)-P_a^Z)^+\,+\,|q_b^Z|.(P_b^X-\mathbb{E}^\mu(C^Z))^+\end{aligned} \tag{23}$$

i.e. we minimize entropy over models which fall within the bid-offer spread, and models which don't (but the latter incur an additional finite penalty for each option which falls outside which is proportional to the available liquidity), which we can interpret as the amount that the market market is "picked off" by mispricing. We can easily verify that (23) and (22) agree numerically using the MOSEK interior-point optimizer, see results below.[5]

Note if we remove the liquidity constraints and set $\alpha=1$, we can re-write (22) as

$$\sup_{u,v,w\in\mathbb{R}^n}(-\log\mathbb{E}^{\bar{\mu}}(e^{-(u.C^X+v.C^Y+z.C^Z)})\,+\,\chi(u,v,w))\quad=\quad\sup_{u,v,w\in\mathbb{R}^n}(-\log\mathbb{E}^{\bar{\mu}}(e^{u.C^X+v.C^Y+z.C^Z})\,+\,\chi(-u,-v,-w))$$

which reduces to the concave maximization problem in (20), but with bid/ask spreads.

Remark 2.6 If we modify the problem to $P(\mu_X,\mu_Y,\mu_Z):=\inf_{\mu\in\mathcal{P}(\mu_X,\mu_Y,\mu_Z)}\mathbb{E}^\mu(\frac{1}{\varepsilon}c(X,Y)+H(\mu|\bar{\mu}))$ for $\varepsilon>0$ and some measurable cost function $c(x,y)$, then $H(\mu|\bar{\mu})$ changes to $H(\mu|\bar{\mu})+c(X,Y)$ in (5), so (6) and (8) change to

$$\begin{aligned}P(\mu_X,\mu_Y,\mu_Z)\quad&=\quad\sup_{u,v,w\in\mathcal{C}^1}\mathbb{E}^{\mu_X}(u(X))+\mathbb{E}^{\mu_Y}(v(Y))+\mathbb{E}^{\mu_Z}(w(Z))-\log\mathbb{E}^{\bar{\mu}}(e^{u(X)+v(Y)+Yw(X/Y)-\frac{1}{\varepsilon}c(X,Y)}))\\\mu(x,y)\quad&=\quad e^{u(x)+v(y)+yw(\frac{x}{y})}e^{-\frac{1}{\varepsilon}c(x,y)}\bar{\mu}(x,y)\end{aligned}$$

i.e. $\frac{1}{\varepsilon}c(x,y)$ gets absorbed into $\bar{\mu}$ (see e.g. [Nutz22],[Nutz21]). We do not pursue this approach further in this article.

2.7 Using a copula for $\bar{\mu}$

A more flexible choice for $\bar{\mu}$ would be to e.g. use the Gaussian copula, and fit the correlation parameter ρ to e.g. historical data or use the *Margrabe formula*: $\sigma_Z^2=\sigma_X^2+\sigma_Y^2-2\rho\sigma_X\sigma_Y$ (used for pricing EUR/GBP options under a bivariate Black-Scholes model) to back out ρ using implied volatilities, although this would require non-trivial amendments to our current implementation.

To compute $\bar{\mu}$ for the Gaussian copula, we set $X=F_X^{-1}(\Phi(Z_1))$ and $Y=F_Y^{-1}(\Phi(Z_2))$ where $Z_1,Z_2\sim N(0,1)$ with $\mathbb{E}(Z_1Z_2)=\rho$, then $X\sim\mu_X$ and $Y\sim\mu_Y$ as required, and

$$\mathbb{P}(X\le x,Y\le y)\quad=\quad\mathbb{P}(F_X^{-1}(\Phi(Z_1))\le x,F_Y^{-1}(\Phi(Z_2))\le y)\quad=\quad\mathbb{P}(Z_1\le\Phi^{-1}(F_X(x)),Z_2\le\Phi^{-1}(F_Y(y)))\,.$$

$\bar{\mu}$ is then given by

$$\bar{\mu}(x,y)\quad=\quad\frac{\partial^2}{\partial x\partial y}\mathbb{P}(X\le x,Y\le y)\quad=\quad F(x)G(x)f(\Phi^{-1}(F_X(x)),\Phi^{-1}(F_Y(y)))$$

where f is the joint density of Z_1 and Z_2, and $F(x)=\frac{d}{dx}(\Phi^{-1}(F_X(x)))$ and $G(x)=\frac{d}{dy}(\Phi^{-1}(F_Y(y)))$ (see numerical results in Figure 1).

[5] We thank Zhuoran Li for providing code for this.

Remark 2.7 Using the Magrabe formula with all combinations of the 5×5×5 (mid) implied vols in the table below for the EUR-USD-GBP triangle leads to a range for the implied correlation for EUR/USD and GBP/USD of $[0.7445, 0.8156]$, and the same computation for the EUR-USD-JPY triangle leads to range for the implied correlation between EUR/JPY and USD/JPY of $[0.6074, 0.8677]$.

EUR/USD	1.0567	1.0680	1.0798	1.0950	1.1025
Implied vol	5.69/6.315%	5.621/5.966%	5.54/5.815%	5.509/5.854%	5.45/6.075%
a, b, σ, ρ, m	-0.0005100	0.009510	0.08579	0.30719	0.03433
GBP/USD	1.2331	1.2480	1.2632	1.2718	1.2919
Implied vol	6.148/7.239%	6.125/6.727%	5.985/6.465%	5.873/6.475%	5.681/6.772%
a, b, σ, ρ, m	0.0002773	0.002254	0.01867	-0.3272	0.003880
EUR/GBP	0.84261	0.84852	0.85478	0.86142	0.8681
Implied vol	3.414/4.584%	3.59/4.232%	3.66/4.17%	3.735/4.373%	3.681/4.841%
a, b, σ, ρ, m	0.0001047	0.001959	0.01139	0.2032	-0.001484

Table of (bid-ask) implied volatilities with 1-month maturity (with strikes going horizontally in the 1st, 3rd and 5th rows) on 11th Feb 2024, and SVI parameters fitted to the mid-implied vols. The forward prices here are 1.0796 for EUR/USD, 1.2630 for GBP/USD and .85483 for EUR/GBP (data obtained from Bloomberg), and these SVI parameters are used for the Sinkhorn scheme.

2.8 A continuous martingale consistent with (μ_X, μ_Y, μ_Z)

We can construct a Markov functional-type continuous martingale model $(X_t, Y_t)_{t\geq 0}$ consistent with the three marginals (μ_X, μ_Y, μ_Z) using conditional sampling as in [BG24]. Specifically, we let

$$\begin{aligned} X_t &= \mathbb{E}(F_X^{-1}(\Phi(\frac{W_T}{\sqrt{T}}))|\mathcal{F}_t^W) &= \mathbb{E}(f(W_T)|\mathcal{F}_t^W) \\ Y_t &= \mathbb{E}(F_{Y|X}^{-1}(\Phi(\frac{B_T}{\sqrt{T}}), X_T)|\mathcal{F}_t^{W,B}) &= \mathbb{E}(g(B_T, W_T)|\mathcal{F}_t^{W,B}) \end{aligned}$$

for $t \in [0, T]$ (setting $X = X_T$ and $Y = Y_T$), where W, B are two independent Brownian motions, Φ is the standard Normal cdf, F_X is the distribution function of μ_X, and $F_{Y|X}(., x)$ is the conditional distribution function of Y given X (which comes from the solution $\mu^*(x, y)$ to the Sinkhorn equations). f and $g(., .)$ are shorthand for the functions which appear inside the expectations in the middle eqs in each line but with the second function re-expressed in terms of W_T. To see that Y has the correct conditional law given X at time zero, we note that

$$Y_T = F_{Y|X}^{-1}(\Phi(\frac{B_T}{\sqrt{T}}), X) = F_{Y|X}^{-1}(V, X)$$

and B and W are independent, so the $U[0, 1]$ random variables $U = \Phi(\frac{W_T}{\sqrt{T}})$ and $V = \Phi(\frac{B_T}{\sqrt{T}})$ are independent, so we are sampling $Y|X$ correctly.

Note this approach is somewhat antisymmetic in so far as the Y process is more complicated that the X process; a more symmetric approach would to be use the true Bass martingale where $(X, Y) = (V_x(W_T, B_T), V_y(W_T, B_T))$ for some convex function V characterized in terms of the standard stretched Brownian motions s^{2}BM (see [BBHK20],[BST23],[AMP25 see e.g. Theorem 1.4 in [BST23].

2.9 A bivariate rough model consistent with (μ_X, μ_Y, μ_Z)

We can construct a (potentially) more realistic rough volatility model consistent with (μ_X, μ_Y, μ_Z) with a two-dimensional version of the Bass-type martingale introduced in section 4 of [F24]:

$$\begin{aligned} X_t &= \mathbb{E}(\phi(X_T)|\mathcal{F}_t^{W,B}) \\ Y_t &= \mathbb{E}(F_{Y|X}^{-1}(F_Y(\tilde{\phi}(\tilde{X}_T)), X_T)|\mathcal{F}_t^{W,B,\tilde{W},\tilde{B}}) \end{aligned}$$

where $F_{Y|X}(y, x)$ is the conditional distribution function of $Y|X$, and X and $\tilde{X}$ are the log stock price process for two (independent) generalized rough Bergomi models of the form in Eq 5 in [F24] driven by two independent Brownian motions W, B and another two independent Brownians $\tilde{W}, \tilde{B}$ (with the latter two also independent of the first two), and $\phi, \tilde{\phi}$) are chosen so that $\phi(X_T) \sim \mu_X$ and $\tilde{\phi}(\tilde{X}_T) \sim \mu_Y$.

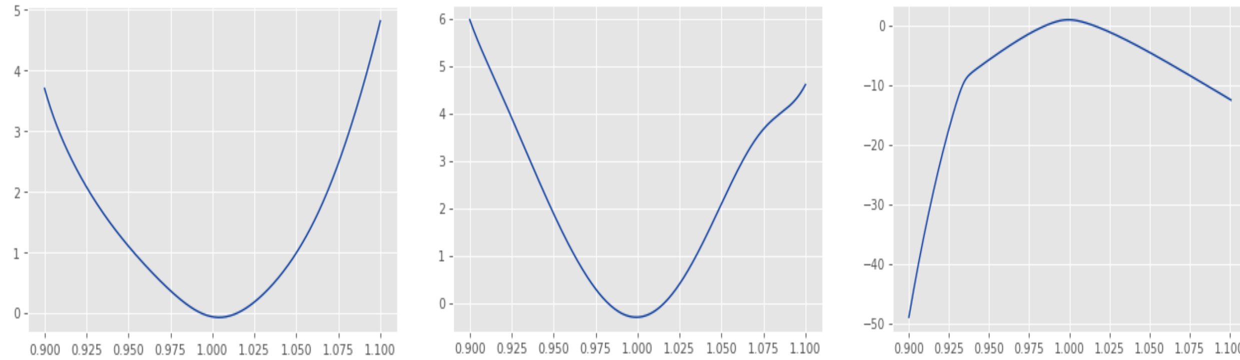


Figure 1: Here we have plotted the maximizing u, v and w respectively after 40 iterations of the Sinkhorn algorithm, using $[.9, 1.1]^2$ as the domain for numerical integration for X, Y.

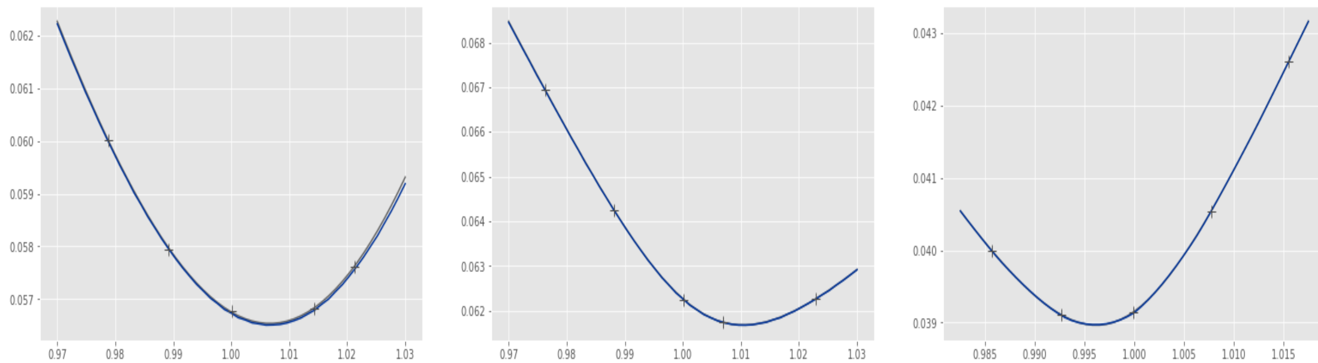


Figure 2: Here we have plotted the Sinkhorn smiles (blue) as a function of moneyness K/F_0 associated with the u, v and w in the plot above verses the original (mid)-market smiles (grey crosses) and the SVI interpolated smiles (grey line, which can barely be seen as it's very close to the blue curve) for EUR/USD, GBP/USD and EUR/GBP on 11th Feb 2024. Five options were used for each cross-rate (At-the-money, and .10, .25, .75, and .90 Delta calls, as is customary in FX options markets), using the standard SVI parametrization to interpolate between them and 400 point Gaussian quadrature for the single and double integrals which appear in the Sinkorn equations, and the bisection method for the root-finding.

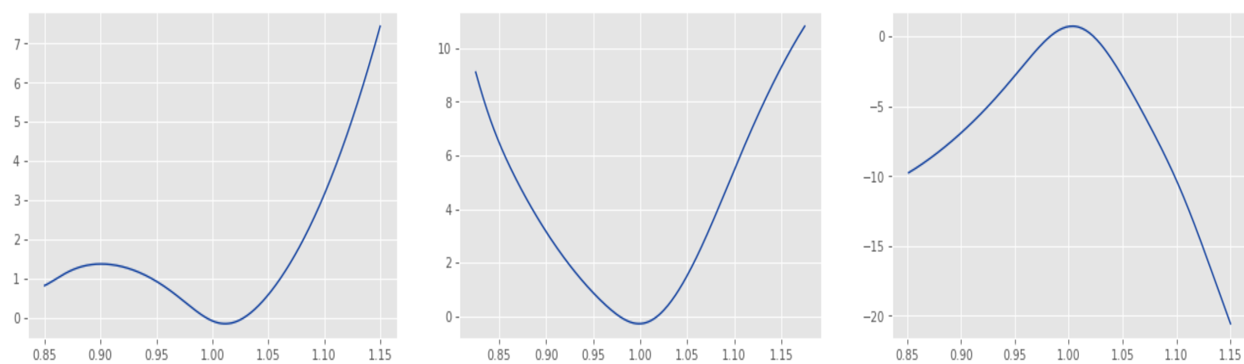
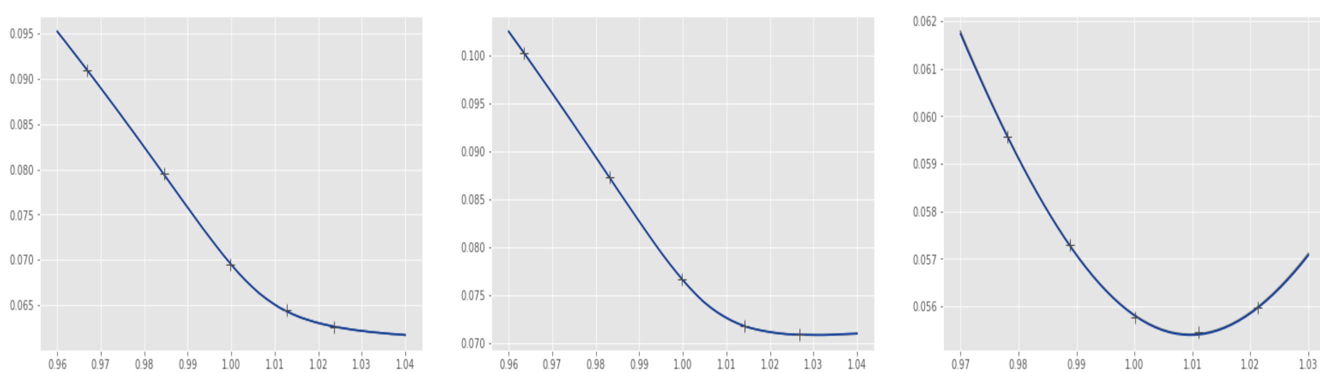


Figure 3: Here we have the same plots for EUR/JPY, USD/JPY and EUR/USD on 3rd Mar 2024



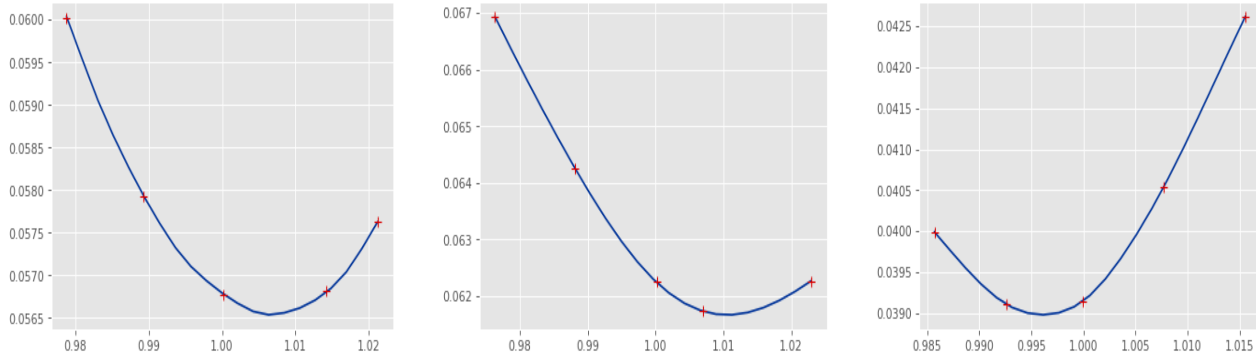


Figure 4: EUR/USD, GBP/USD and EUR/GBP smiles obtained using the finite-dimensional concave maximization in (20) using a Gaussian copula for the reference density $\bar{\mu}$ with correlation $\rho = .75$ to couple the target SVI densities for X and Y.

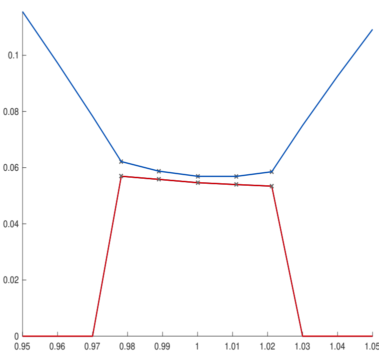


Figure 5: Here we compute extremal solutions to the FX cross-rate problem as an pure optimal transport problem; specifically, we compute the maximal (blue) and minimal (red) implied volatility of a single EUR-USD call option for each strike on the horizontal axis, where we just include the five tradeable calls on each cross-rate EUR/JPY, USD/JPY and EUR/USD in the calibration set (using the market bid and ask prices for each rate as inequality constraints in the LP problem and no use of smile interpolation), plus the forward prices (with no bid-ask spread assumed for the latter since this would be an order of magnitude smaller in practice).

2.10 The cross-smiles calibration problem as an optimal transport problem, finite tradeable options and linear programming

If we are happy to work with a discrete joint target law $\mu_{ij} = \mathbb{P}(X_i = x_i, Y_j = y_j)$, we can re-cast our cross-smile calibration problem as a finite-dimensional linear programming (LP) problem:

$$P = \max_\mu \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{ij} c(x_i, y_j)$$

for some payoff function $c(x, y)$, subject to market constraints $\sum_{i=1}^m \sum_{j=1}^n \mu_{ij}(x_i - K)^+ = c_i^{X,mkt}$, $\sum_{i=1}^m \sum_{j=1}^n \mu_{ij}(y_j - K)^+ = c_i^{Y,mkt}$ and $\sum_{i=1}^m \sum_{j=1}^n \mu_{ij}(x_i - Ky_j)^+ = c_i^{Z,mkt}$ plus forward constraints and $\mu_{ij} \in [0, 1]$ (or we can work with bid/offer inequality constraints). By stacking μ into a column vector (as opposed to its original formulation as a matrix), we can transform this problem to the standard form for LP problems: $\max_{x:Ax \leq b; 0 \leq x} c^T x$; then from the strong duality theorem for linear programming problems (see e.g. [Burke]), if P (or its dual problem $D = \min_{y:A^T y \geq b, 0 \leq y} b^T y$, i.e. to compute the cheapest superhedge for $c(X, Y)$) has a finite optimal value, then so does the other, and these optimal values coincide.

We can compute P and D numerically in e.g. MATLAB or Python using the `linprog` command, and see [HN12], [HK15] for explicit solutions in the case of the at-the-money forward-starting straddle which of course requires an additional martingale condition (see next subsection), and [GMN17] for related problems). We implement this approach in Figure 2; specifically just use the five market bid-ask call prices on each cross-rate as inequality constraints and the forward prices as equality constraints.

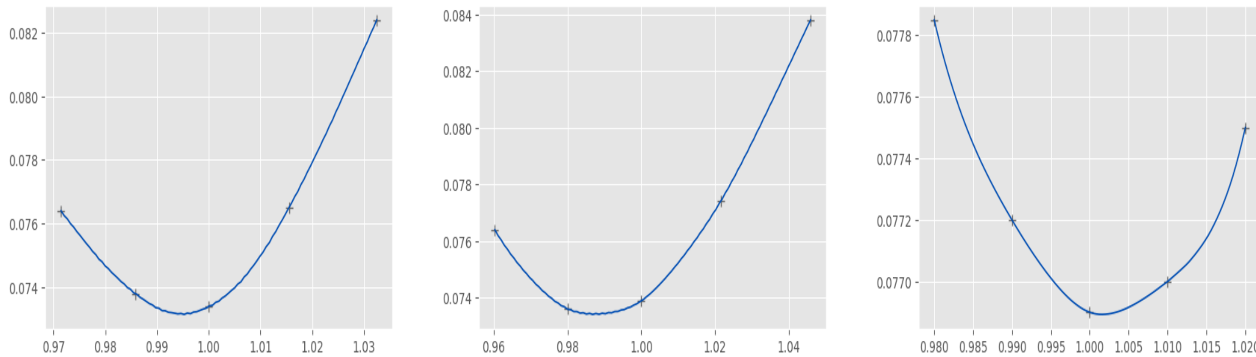


Figure 6: Fitting the 1M (left) and 2M (middle) EUR/USD smiles and a fictitious forward starter smile (right) on 5th Sept 2025 (since forward-starters are traded OTC and not liquidly traded on an exchanged) by solving the primal (entropy minimization) problem using cvxpy with MOSEK with the additional martingale condition (data taken from `https://www.investing.com/currencies/eur-usd-options`)

2.11 Calibration to forward-starter options

If instead $X = S_{T_1}$ and $Y = S_{T_2}$ for a martingale stock price process S with $0 < T_1 < T_2$ and μ_X and μ_Y are in convex order, then our third marginal constraint $\int_{[0,\infty)\times[0,\infty)}(x-Ky)^+\mu(dx,dy) = \int_{[0,\infty)}(z-K)^+\mu_3(dz)$ for all $K \geq 0$ now corresponds to having observed option prices for *forward-starter* options for all strikes rather than cross-rate options as above. In this case, we require the additional martingale constraint which here reads that $\mathbb{E}(\Delta(S_1)(S_2-S_1)) = 0$ i.e. $\mathbb{E}(\Delta(X)(Y-X)) = 0$ for all $\Delta \in C_b(\mathbb{R}^+)$ (using a denseness argument with Urysohn lemma to replace indicator functions with the C_b Δ functions here).

At this point we again make the following natural assumption:

Assumption 2.5 *There exists a $\mu^* \in \mathcal{P}(\mu_X,\mu_Y,\mu_Z)$ with $H(\mu^*|\bar{\mu}) < \infty$ which satisfies the margingale condition above.*

Then (assuming e.g. that $\bar{\mu}(x,y) = \mu_X(y)\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(\log\frac{y}{x}+\frac{1}{2}\sigma_2^2)^2}{2\sigma_2^2}}$ for some σ_2 so as to respect the martingale condition) the martingale condition $\mathbb{E}(Y-X|Y) = 0$ leads to a fourth Sinkhorn equation:

$$0 \;=\; \int_0^\infty (x-y)e^{u_{n+1}(x)+yw_{n+1}(\frac{x}{y})+\Delta_{n+1}(x)(y-x)}\frac{\bar{\mu}(x,y)}{\mu_Y(y)}dx \tag{24}$$

for Δ_{n+1} (see similar equation at top of page 9 in [Guy20]). Note we have divided out the $e^{v_{n+1}(y)}$ term here since we have zero on the right hand side and this terms do not depend on the variable of integration x, and the other three Sinkhorn equations take a similar form as before but with an additional hedging term in the exponent:

$$\begin{aligned}
u_{n+1}(x) &:= \log\mu_X(x) - \log\int_0^\infty e^{v_n(y)+yw_n(\frac{x}{y})+\Delta_n(y)(x-y)}\bar{\mu}(x,y)dy \\
v_{n+1}(y) &:= \log\mu_Y(y) - \log\int_0^\infty e^{u_{n+1}(x)+yw_n(\frac{x}{y})+\Delta_n(y)(x-y)}\bar{\mu}(x,y)dx \\
\mu_Z(z) &= \int_0^\infty e^{u_{n+1}(x)+v_{n+1}(\frac{x}{z})+\frac{x}{z}w_{n+1}(z)+\Delta_n(\frac{x}{z})(x-\frac{x}{z})}\frac{x^2}{z^3}\bar{\mu}(x,\frac{x}{z})dx\,.
\end{aligned} \tag{25}$$

For each fixed y, Eq (24) is just a one-dimensional root-finding exercise for $\Delta = \Delta(y)$.

A continuous-time martingale consistent with the three marginals here then takes the form

$$\begin{aligned}
S_t &= \mathbb{E}(F^{-1}_{S_{T_1}}(\Phi(\frac{W_{T_1}}{\sqrt{T_1}}))|\mathcal{F}^W_t) = \mathbb{E}(f(W_{T_1})|\mathcal{F}^W_t) \qquad (t\in[0,T_1]) \\
S_t &= \mathbb{E}(F^{-1}_{S_{T_2}|S_{T_1}}(\Phi(\frac{W_{T_2}-W_{T_1}}{\sqrt{T_2-T_1}}),S_{T_1})|\mathcal{F}^W_t) = \mathbb{E}(g(W_{T_2},S_{T_1})|\mathcal{F}^W_t) \qquad (T_1 < t \leq T_2)
\end{aligned}$$

for some function g, where $F_{S_{T_2}|S_{T_1}}(.;s_1)$ is the conditional distribution function of S_{T_2} given $S_{T_1} = s_1$), W is a standard Brownian motion, and S is continuous on $[0,T_2]$ (including at T_1). Note this is not a true Bass martingale because it is not of the "additive" form $S_t = \mathbb{E}(g(Z+W_{T_2}-W_{T_1})|\mathcal{F}^W_t)$ for some random variable Z which is independent of W as in [CL21] (see also Definition 1.2 in [BST23]).

EUR/JPY	156.71	159.59	162.06	164.15	165.93
Implied vol	8.368/9.824%	7.55/8.352%	6.63/7.265%	6.048/6.839%	5.54/6.977%
a, b, σ, ρ, m	0.0003044	0.004750	0.01382	-1.0000	0.005668
USD/JPY	143.94	146.87	149.35	151.50	153.40
Implied vol	9.546/10.496%	8.463/8.987%	7.46/7.875%	6.917/7.433%	6.616/7.551%
a, b, σ, ρ, m	0.0003764	0.005681	0.01445	-0.8632	0.005049
EUR/USD	1.0614	1.0731	1.0852	1.0971	1.1081
Implied vol	5.702/6.213%	5.589/5.871%	5.465/5.69%	5.404/5.686%	5.342/5.853%
a, b, σ, ρ, m	-0.00007481	0.005242	0.06315	-0.04437	0.006762

Corresponding table of (bid-ask) implied volatilities and SVI parameters calibrated to mid implied vols with 1-month maturity on 3rd Mar 2024 for the EUR-USD-JPY triangle. The forward prices here are 162.09, 149.39 and 1.0851.

2.12 Calibrating Markov stochastic volatility models to SPX and VIX options using HJB equations

We now consider a Markov stochastic volatility model of the form

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dB_t \\ dV_t &= \kappa(\theta - V_t)dt + \beta(S_t, V_t, t)dW_t \end{aligned}$$

where W and B are two Brownian motions with $dW_t dB_t = \rho dt$ with $\rho \in [-1, 0]$. The VIX index at time t is (theoretically) defined as $\mathrm{VIX}_t^2 := \frac{1}{\Delta}\mathbb{E}(\int_t^{t+\Delta} V_s ds|\mathcal{F}_t^{B,W})$ which in practice is inferred from option prices, where $\Delta = 1/12$ i.e. 1 month. For a V process with a drift of this form, we can easily show that $\mathrm{VIX}_t^2 = aV_t + b$, for two constants a and b that only depend on κ, θ and Δ, using the expression for $\mathbb{E}(V_t|V_s)$ for $s \le t$ (see e.g. section 1.6 in [FS23]). This also means we can observe V directly if the model is correct and we know κ and θ.

We wish to find a $\beta(S, V, t)$ so this model calibrates to n European call options and m VIX options at a fixed maturity T with market prices $(c_i)_{i=1}^n$ and $(c_j^v)_{j=1}^m$ respectively. From standard PDE theory we know that

$$u(S, v, t) = \mathbb{E}(\sum_{i=1}^n w_i(S_T - K_i)^+ + \sum_{j=1}^m w_i^v(\mathrm{VIX}_T - K_j^v)^+ + \alpha\int_0^T(\beta(S_t, V_t, t) - \nu V_t^p)^2 dt \,|\, S_t = S, V_t = v) \tag{26}$$

(for $p \in (0, 1]$) satisfies the backward Kolmogorov equation

$$\mathcal{L}u \;:=\; u_t + \frac{1}{2}S^2 V u_{SS} + \kappa(\theta - v)u_v + \frac{1}{2}\beta^2 u_{vv} + \rho S\sqrt{v}\beta u_{Sv} + \alpha(\beta - \nu v^p)^2) \;=\; 0$$

with $u(S, v, T) = \sum_{i=1}^n w_i(S - K_i)^+ + \sum_{j=1}^m w_i^v(\sqrt{aV + b} - K_j^v)^+$. We can consider a more general model where

$$dV_t = \kappa(\theta - V_t)dt + \beta_t dW_t$$

where β_t is any $\mathcal{F}_t^{W,B}$-adapted process, and then ask how do we optimally choose β so as to minimize $\sum_{i=1}^n w_i\mathbb{E}((X_T - K_i)^+) + \ldots + \alpha\mathbb{E}(\int_0^T(\beta_t - \nu V_t^p)^2 dt)$.

This approach penalizes the "distance" from a reference stochastic volatility model, which is the Hull-White model when $p = 1$ and Heston when $p = \frac{1}{2}$. From standard stochastic control theory, formally at least we know the solution satisfies the HJB eq: $\min_\beta \mathcal{L}u = 0$. Then (assuming we can interchange the sup and inf)

$$\begin{aligned} &\sup_w \inf_{\beta\in\mathcal{A}}(-\sum_i \tilde{w}_i\tilde{c}_i + \sum_i \tilde{w}_i\mathbb{E}^\beta(f_i(X_T, V_T)) + \alpha\mathbb{E}^\beta(\int_0^T(\beta_t - \nu V_t^p)^2 dt) \\ = &\inf_{\beta\in\mathcal{A}}\sup_{\tilde{w}_i}(\ldots) \\ = &\inf_{\beta\in\mathcal{A}\,:\,\mathbb{E}^\beta((X_T - K_i)^+) = c_i,\, \mathbb{E}^\beta((\sqrt{aV_T + b} - K_j^v)^+) = c_j^v} \alpha\mathbb{E}^\beta(\int_0^T(\beta_t - \nu V_t^p)^2 dt) \end{aligned} \tag{27}$$

(where we have aggregated the payoffs, weights and market prices of the European and VIX payoffs into single vectors f, $\tilde{w}$ and $\tilde{c}$ in the first line here to ease notation), and the third line line follows because the inner sup in the middle line is $+\infty$ if for a particular $a \in \mathcal{A}$ the options are not correctly calibrated, since e.g. we can choose w_i to be arbitrarily large if $\sum_{i=1}^n w_i\mathbb{E}^\beta((X_T - K_i)^+) - \sum_{i=1}^n w_i c_i > 0$, and vice versa, and similarly for the VIX options.

The main contributions on this problem are due to Guo,Loeper,Obłój&Wang et al.(see [GLW22],[GLOW22]), and conceptually similar ideas for a specific case are considered in Henry-Labordére[HL19] and [Guy22] using an entropic penalty function which only allows Girsanov perturbations from the reference model. These articles do not appear to have noticed/used the simple relation $\mathrm{VIX}_t^2 = aV_t + b$ to simplify this problem.

The final line in (27) is the model-independent lower bound for a contract which pays $\alpha\int_0^T(\beta_t-\nu V_t^p)^2dt$ at time T, subject to matching the market prices of the n given European options and m VIX options. If duality holds i.e. $\inf\sup=\sup\inf$, this is also the maximum subhedging cost of this contract, using just cash, dynamic trading in X and a static position in the stock and VIX options. Note when we minimize here we are also including rough models since we do not assume a priori that β_t is Markov, but the optimal model is Markovian by the usual "rules" of how the HJB eq works, because the reference model is Markovian. One can generalize this methodology to use a rough reference model, but one ends up with an intractable non-standard FBSDE so we omit the details.

If we have an additional SPX options to fit at a later maturity $T_2>T$, then we need a nested PDE scheme to solve this problem, i.e. we solve the PDE on $[T,T_2]$ with the boundary condition for the weights for the additional European options expiring at T_2. This then gives us a (non-trivial/non-explicit) boundary condition at time T and we then add-on the usual boundary condition for the SPX and VIX options expiring at the earlier maturity T. One can use explicit an FD scheme or semi-explicit ADI scheme, e.g. the Douglas-Rachford scheme to solve the resulting PDE.

3 Robust price bounds for the cross options given full marginals μ_X and μ_Y

We first note the triangle-type inequality:

$$(x-y)^+ \quad\le\quad (x-x_1)^+ \,+\, (x_1-y)^+ \tag{28}$$

for all $x,y,x_1\in\mathbb{R}$ (we just have to check all cases to verify this identity). Now let $y=p(x):=F_{\mu_Y}^{-1}(1-F_{\mu_X}(x))$ where $F_\mu(x):=\mu([0,x])$ so $Y=p(X)\sim\mu_Y$ if $X\sim\mu_X$ i.e. the Fréchet-Hoeffding lower bound coupling, and note that p is strictly decreasing if we assume μ_X and μ_Y have strictly positive densities. We further assume there is a unique root x_* of $p(x)=x$, with $p(x)>x_*$ for $x<x_*$ and vice versa (see first plot in Figure 1 below). Then setting $x_1=x^*$ and $y=p(x)$ and assuming $x>x_*$, (28) becomes

$$x-p(x) \quad\le\quad x-x_* \,+\, x_*-p(x)$$

(since $p(x)<x$) i.e. an equality. Conversely if $y=p(x)$ and $x<x_*$, $p(x)>x_*>x$ so both sides of (28) are zero. Hence $(x-x_*)^+ + (x_*-y)^+$ (i.e. a call option on X plus a put option on Y, both with strike x^*) is a superhedge for $c(x,y)=(x-y)^+$, and equality is obtained for the coupling where $Y=p(X)$, so this coupling is optimal for the max problem, i.e.

$$\int_0^1(F_{\mu_X}^{-1}(u)-F_{\mu_Y}^{-1}(1-u))^+du \quad=\quad \sup_{\mu\in\Pi(\mu_X,\mu_Y)}\mathbb{E}^\mu((X-Y)^+)$$

(see also [HLW05],[HLW05b] who look at this problem in the context of basket options).

For the general case when $c(x,y)=(x-Ky)^+$, we just regard KY as the new Y variable, then x_* is now the root of $p(x)=Ky$ in the proof above, which is otherwise unchanged except now x_* depends on K.[6]

For the lower bound, we first note that

$$(x-y)^+ \quad\ge\quad -(x_1-x)^+ \,+\, (x_1-y)^+ \tag{29}$$

for all $x,y,x_1\in\mathbb{R}$. Now let $y=p(x)=F_{\mu_Y}^{-1}(F_{\mu_X}(x))$ so $Y=p(X)\sim\mu_Y$ if $X\sim\mu_X$ i.e. the Fréchet-Hoeffding upper bound coupling, and note that p is strictly increasing if we assume μ_X and μ_Y have strictly positive densities. We again assume there is a unique root x^* of $p(x)=x$, but now with $p(x)>x^*$ for $x>x^*$ and vice versa (see 3rd plot in Figure 1). Then again setting $x_1=x^*$ and $y=p(x)$ and assuming $x\le x^*$, (29) becomes

$$x-p(x) \quad\ge\quad -(x^*-x)\,+\,(x^*-p(x)) \quad=\quad x-p(x)$$

so we have equality. Conversely, if $x>x^*$, both sides of (29) vanish. Hence (repeating the same arguments as above), the coupling with $Y=p(X)$ is optimal for the min problem, i.e.

$$\int_0^1(F_{\mu_X}^{-1}(u)-F_{\mu_Y}^{-1}(u))^+du \quad=\quad \inf_{\mu\in\Pi(\mu_X,\mu_Y)}\mathbb{E}^\mu((X-Y)^+)$$

and again we can extend this argument to the general case when $c(x,y)=(x-Ky)^+$. Note our assumption that $p(x)=x$ has a unique root fails for our EUR-USD-GBP triangle example in the third plot in Fig 4 where $p(x)=x$ at three distinct x-values (see also Corollary 1.2 in [BJ16], although their result requires strict convexity condition on $c(x,y)=h(y-x)$).

[6] Note if we set $p(x)=V'(x)$, then $dy=V''(x)dx$, and since $\mu_Y(y)dy=\mu_X(x)dx$, $\mu_X(x)=V''(x)\mu_Y(V'(x))=\nabla^2V\,\mu_Y(\nabla V(x))$ which is the *Monge-Ampére* equation for V in one dimension (see e.g. [Wu21]).

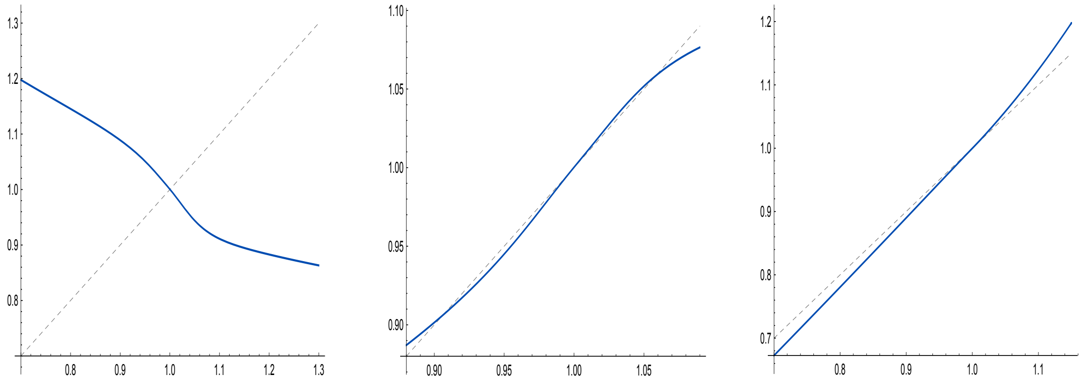


Figure 7: On the left we see the optimal transport map $y = p(x)$ (in blue) from Subsection 7 using the lower Frechet-Hoeffding bound verses the $y = x$ line (dashed). In the 2nd plot we see the $p(x)$ map for the minimal price for EUR/GBP options which comes from the upper Frechet-Hoeffding bound (and we see there are three points of equality which is a problem case), and the final plot shows the same $p(x)$ function for EUR/USD options (given the EUR/JPY and USD/JPY smiles in the later data set) which only has 1 point of equality (non-problem case) as assumed in the proof in subsection .

Remark 3.1 For a single cross-rate option paying $(x - Ky)^+$, we can construct a "model" consistent with P and the given marginals for X and Y if and only if P lies within the upper and lower price bounds computed above, in which case the model can just be chosen as a weighted coin toss between the two extremal models with the weight chosen to match P. It is an interesting open problem as to whether we can construct a consistent model if we have prices for two (or n) such cross-rate options when the market prices for each these options lie between their respective upper and lower price bounds [7]

[7] We thank David Hobson for pointing this out.

References

[APR18] Armstrong, A., T.Pennanen, and U.Rakwongwan, "Pricing Index Options by Static Hedging under Finite Liquidity," *International Journal of Theoretical and Applied Finance*, vol.21, no.6, 2018.

[ACD15] Atar, R., K. Chowdhary, and P. Dupuis, Robust bounds on risk-sensitive functionals via Rényi divergence. *SIAM Journal on Uncertainty Quantification*, 3(1):1–30, 2015.

[AMP25] Acciaio, B., A.Marini, G.Pammer, "Calibration of the Bass Local Volatility model", SIAM Journal on Financial Mathematics, 16/3, 803-833, 2025.

[Aus11] Austing, P., "Repricing the Cross Smile", `risk.net`, 2011

[BBHK20] Backhoff-Veraguas, J., M.Beiglbock, M.Huesmann and S.Källblad, "Martingale Benamou–Brenier: A Probabilistic Perspective", *The Annals of Probability*, Vol. 48, No. 5, 2258–2289, 2020

[BST23] Backhoff-Veraguas, J., W.Schachermayer, and B.Tschiderer, "The Bass functional of martingale transport", preprint, 2023

[Bass93] Bass, R.F., "Skorokhod imbedding via stochastic integrals", in J.Azéma and M.Yor, editors, *Semin. Probab. XVII* 1981/82 — Proceedings, volume 986 of Lecture Notes in Math., pages 221–224. Springer, Berlin, Heidelberg, 1983

[BJ16] Beiglböck, M. and N.Juillet, "On a problem of optimal transport under marginal martingale constraints", *The Annals of Probability*, 2016, Vol. 44, No. 1, 42–106

[Bert15] Bertseka, D., "Convex Optimization Algorithms", Athena Scientific, 2015.

[BCH18] Beiglböck, M., A.M.G.Cox, and M.Huesmann, "Optimal Transport and Skorokhod Embedding", *Invent. Math.*, 2018

[BG24] Bourgey, F. and J.Guyon, "Fast Exact Joint S&P 500/VIX Smile Calibration in Discrete and Continuous Time". `risk.net`, Feb 2024

[Bour25] "Fast Smile Calibration in Discrete and Continuous Time Using Sinkhorn Algorithms", SIAM conference, Miami, July 2025.

[Burke] Burke, J., Lecture Notes, `https://sites.math.washington.edu//~burke/crs/407/notes/section4.pdf`

[Carr09] Carr, P., "Local Variance Gamma Option Pricing Model", presentation Bloomberg/Courant Institute, April 28, 2009.

[CN17] Carr, P. and S.Nadtochiy, "Local Variance Gamma and Explicit Calibration to Option Prices", *Mathematical Finance*, 27(1):151–193, 2017

[CL21] Conze, A. and P.Henry-Labordère, "Bass Construction with Multi-Marginals: Lightspeed Computation in a New Local Volatility Model", preprint, 2021.

[Csi75] Csiszár, I. "I-divergence geometry of probability distributions and minimization problems", *The Annals of Probability*, 3(1):146–158, 1975.

[dMHL15] de Marco, S. and P.Henry-Labordère, "Linking Vanillas and VIX options: A constrained martingale optimal transport problem", *SIAM J. Finan. Math*, 4(54), pp. 1171–1194, 2015

[DZ10] Dembo, A. and O.Zeitouni, "Large deviations techniques and applications", Jones and Bartlet publishers, Boston, second edition, 2010.

[F19] Forde, M., "Pathwise superhedging for time-dependent barrier options on cadlag paths - finite or infinite tradeable European, One-Touch, lookback or forward starting options", *Stoch. Proc. Appl.*, 129(3), 799-821, 2019

[F24] Forde, M., "Rough Bergomi revisited - exact and minimal-variance hedging for VIX and European options, and exact calibration to multiple smiles", preprint, 2024.

[FS23] Forde, M. and B.Smith, "Markovian stochastic volatility with stochastic correlation - joint calibration and consistency of SPX/VIX short-maturity smiles", *Int. J. Theor. Appl. Finance*, 26, 2-3, 2023

[GLOW22] Guo, I., G.Loeper, J.Obłój, S.Wang, "Optimal transport for model calibration", `risk.net` , `risk.net`, Jan 2022

[GLW22] Guo, I., G.Loeper, S.Wang, "Calibration of Local-Stochastic Volatility Models by Optimal Transport", *Mathematical Finance*, 32(1), 46-77, 2022.

[Guy13] Guyon, J., "A New Class of Local Correlation Models", Quantitative Research, Bloomberg LP, 2013

[Guy20] Guyon, J., "The Joint S&P 500/Vix Smile Calibration Puzzle Solved", `risk.net`, April 2020

[Guy22] Guyon, J., "Dispersion-Constrained Martingale Schrödinger Bridges: Joint Entropic Calibration of Stochastic Volatility Models to S&P 500 and VIX Smiles", preprint, 2022

[Guy24] Guyon, J., "Dispersion-Constrained Martingale Schrödinger Problems and the Exact Joint S&P 500/VIX Smile Calibration Puzzle", *Finance and Stochastics*, 28(1):27-79, 2024

[Guy24b] Guyon, J., "Fast exact joint S&P 500/VIX smile calibration in discrete and continuous time", London Mathematical Finance Seminar, Feb 2024

[GMN17] Guyon, J., R.Menegaux, and M.Nutz, "Bounds for VIX Futures given S&P 500 Smiles", *Finance&Stochastics*, Vol. 21, No. 3, pp. 593-630, 2017

[HN12] Hobson, D. and A.Neuberger, "Robust bounds for forward start options", *Mathematical Finance*, 22, 31-56, 2012

[HK15] Hobson, D. and M.Klimmek, "Robust price bounds for the forward starting straddle", *Finance and Stochastics*, Vol 19, Issue 1, 189-214, 2015

[HL19] Henry-Labordère, P., "From (Martingale) Schrodinger bridges to a new class of Stochastic Volatility Models", preprint, 2019.

[HLW05] Hobson, D., P.Laurence, T-H.Wang,"Static-arbitrage upper bounds for the prices of basket options", *Quantitative Finance*, 5 (4) 329-342, 2005

[HLW05b] Hobson, D., P.Laurence, T-H.Wang, "Static-arbitrage optimal subreplicating strategies for basket options." *Insurance: Mathematics and Economics*, 37(3):553–572, 2005.

[HLT16] Henry-Labordere, P. and N.Touzi, "An explicit martingale version of the one-dimensional Brenier theorem", *Finance and Stochastics*, 20(3):635–668, 2016 (version from 10th April 2015 cited here)

[JLO23] Joseph, B., G.Loeper and Jan Obłój, "The Measure Preserving Martingale Sinkhorn Algorithm", preprint

[Nutz21] Nutz, M., "Entropic Optimal Transport -Geometry and convergence", Talk, March 2021

[Nutz22] Nutz, M., "Introduction to Entropic Optimal Transport", Lecture Notes, 2022

[Nutz24] Nutz, M., "Quadratically Regularized Optimal Transport: Existence and Multiplicity of Potentials", preprint

[Tho18] Thorpe, M., "Introduction to Optimal Transport", Notes, Cambridge University, March 2018

[Wu21] Wu, L., "Entropical Optimal Transport, Schrödinger's System and Algorithms", *Acta Mathematica Scientia*, 2021, 41B(6): 2183–2197

Appendix A

For any random variable X, let $\frac{d\bar{\mu}^n_X}{d\bar{\mu}} = \frac{e^{X\wedge n}}{\mathbb{E}^{\bar{\mu}}(e^{X\wedge n})}$ (assuming the denominator is finite). Then

$$\begin{aligned}\inf_{\mu\in\mathcal{P}}\{H(\mu|\bar{\mu}) - \mathbb{E}^{\mu}(X)\} \;=\; \inf_{\mu\in\mathcal{P}}\mathbb{E}^{\mu}(\log\frac{d\mu}{d\bar{\mu}} - X) &= \inf_{\mu\in\mathcal{P}}\mathbb{E}^{\mu}(\log\frac{d\mu}{d\bar{\mu}^n_X} + \log\frac{d\bar{\mu}^n_X}{d\bar{\mu}} - X)\\ &= \inf_{\mu\in\mathcal{P}}\mathbb{E}^{\mu}(\log\frac{d\mu}{d\bar{\mu}^n_X} + \log\frac{e^{X\wedge n}}{\mathbb{E}^{\bar{\mu}}(e^{X\wedge n})} - X)\\ &= \inf_{\mu\in\mathcal{P}} H(\mu|\bar{\mu}^n_X) - \log\mathbb{E}^{\bar{\mu}}(e^{X\wedge n}) + \mathbb{E}^{\mu}(X\wedge n - X)\\ &= \inf_{\mu\in\mathcal{P}}[H(\mu|\bar{\mu}^n_X) - \mathbb{E}^{\mu}(X1_{X>n})] - \log\mathbb{E}^{\bar{\mu}}(e^{X\wedge n})\\ &\leq -\mathbb{E}^{\bar{\mu}^n_X}(X1_{X>n})] - \log\mathbb{E}^{\bar{\mu}}(e^{X\wedge n}) \qquad \text{(A-1)}\end{aligned}$$

where the final line follows from considering $\mu = \bar{\mu}^n_X$ for which $H(\bar{\mu}^n_X, \bar{\mu}^n_X) = 0$. For the awkward case when $\mathbb{E}^{\bar{\mu}}(e^X) = \infty$, the second term in (A-1) tends to $-\infty$ as $n \to \infty$ by the monotone convergence theorem, so $\inf_{\mu\in\mathcal{P}} H(\mu|\bar{\mu}) = -\infty$.

Appendix B

Proof. We break the proof into subparts and let $C = \mathcal{P}(\mu_X, \mu_Y, \mu_Z)$ and $K_c = \mathcal{P}_c(\mathbb{R}^2_+)$ to ease notation.

1. Closedness of the feasible set C

Fix $K \geq 0$. For $m \in \mathbb{N}$, define the truncations

$$f_{K,m}(x,y) \quad := \quad \big((x-Ky)^+\big) \wedge m, \qquad g_{K,m}(z) := (z-K)^+ \wedge m.$$

Each $f_{K,m} \in C_b(\mathbb{R}^2_+)$, and $f_{K,m} \uparrow (x-Ky)^+$ as $m \to \infty$. Suppose $\mu_n \Rightarrow \mu$ weakly with $\mu_n \in C$. Then

$$\int f_{K,m}(x,y)\,\mu_n(dx,dy) \quad = \quad \int g_{K,m}(z)\,\mu_Z(dz).$$

Then letting $n \to \infty$ we see that

$$\int f_{K,m}(x,y)\,\mu(dx,dy) \quad = \quad \int g_{K,m}(z)\,\mu_Z(dz).$$

Now let $m \uparrow \infty$. Using monotone convergence on both sides, we see that

$$\int (x-Ky)^+\,\mu(dx,dy) \quad = \quad \int (z-K)^+\,\mu_Z(dz).$$

Existence. By Assumption 2.4, there exists some $\mu^* \in C$ with $H(\mu^*|\mu) < \infty$. Pick $c > H(\mu^* \mid \bar{\mu})$. Then $C \cap K_c$ is nonempty and the intersection of a compact set K_c with a closed set C, hence also compact. Since $H(\cdot|\mu)$ is l.s.c., it attains its minimum on $C \cap K_c$, hence on C. So a minimizer μ^* exists.

2. Strict convexity. For $\mu_1, \mu_2 \ll \bar{\mu}$ with densities f_1, f_2 with respect to $\bar{\mu}$, and $0 < \lambda < 1$, we have

$$H(\lambda\mu_1 + (1-\lambda)\mu_2|\bar{\mu}) \quad = \quad \int \big(\lambda f_1 + (1-\lambda)f_2\big)\log\big(\lambda f_1 + (1-\lambda)f_2\big)\,d\bar{\mu}.$$

Since $z \mapsto z\log z$ is strictly convex on $[0,\infty)$, we see that

$$H(\lambda\mu_1 + (1-\lambda)\mu_2|\mu) \quad < \quad \lambda H(\mu_1|\mu) + (1-\lambda)H(\mu_2|\mu)$$

as long as $\mu_1 \neq \mu_2$ on a set of positive $\bar{\mu}$-measure. Thus $H(\cdot|\mu)$ is strictly convex on $\{\mu \ll \bar{\mu}\}$.

Step 3. Uniqueness. Suppose $\mu_1, \mu_2 \in C$ are distinct minimizers with finite entropy. Then $\lambda\mu_1 + (1-\lambda)\mu_2 \in C$ by convexity of C, and Step 2 gives

$$H(\lambda\mu_1 + (1-\lambda)\mu_2|\mu) \quad < \quad \lambda H(\mu_1|\mu) + (1-\lambda)H(\mu_2|\mu)$$

contradicting optimality. Hence the minimizer is unique. ∎