Fast Smile Calibration in Discrete and Continuous Time Using Sinkhorn Algorithms

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Why Build an Arbitrage-Free, Fast Implied Vol Surface?

- **Arbitrage-free requirement**: Prevents mispricing and inconsistent hedging strategies.
- **Regulatory constraint**: implied vol surfaces must align with market prices.
- Crucial for market makers in equity, (exotic) options traders, structurers, and risk-managers (e.g., for stress-testing).
- Prerequisite for local and stochastic local vol models.
- Market models often struggle to fit the entire implied vol surface while remaining arbitrage-free.
- Inherent difficulty of interpolation/extrapolation (fixed number of strikes and maturities).
- Speed essential for most market players (real-time trading, risk assessment).

Existing Methods for Surface Construction

- **Parametric models**: SABR, SVI, SSVI, etc.
- **Spline interpolation or mixture models**: enforce no-arbitrage constraints on interpolated surfaces.
- **Discrete local volatility** from market quotes.
- **Optimal transport approach:**
  - Cast calibration as a divergence-minimization problem w.r.t. a prior model under market constraints.
  - Match market quotes $\Leftrightarrow$ Schrödinger problem.
  - Solved using an extension of the Sinkhorn algorithm.

Objectives

- (1) Discrete-Time Model: We build it by solving the Schrödinger system (SS) using the Sinkhorn algorithm. Mixed Newton-Sinkhorn and implied Newton methods are numerically shown to converge significantly faster than Sinkhorn.
- (2) Continuous-time extension: This enables pricing path-dependent options within a market-calibrated framework.

Objectives

- (1) is inspired by [DMHL19].
- (2) resembles the Bass local volatility of [BVBHK20] and [CHL22] (see also [AMP23]), but is fundamentally different:
  - Our purely forward Markov functional construction avoids solving a fixed-point problem $\Rightarrow$ much faster.
  - This is because we first build an arbitrage-free multimarginal discrete-time model consistent with market data.

Problem

- n maturities $0 < T_1 < \cdots < T_n$ and denote $S_i = S_{T_i}$.
- Risk-neutral distribution $\mu_i = \mu_{S_i}$ known (inferred from market prices).
- Let $\mathcal{M}(\{\mu_i\}_{i=1}^n)$ be the set of martingale probability measures with marginals μ_i:

$$\Big\{\mu \in \mathcal{P}(\mathbb{R}^n) : \ \forall i \in \{1, \ldots, n\},\ S_i \sim \mu_i, \\ \forall i \in \{2, \ldots, n\},\ \mathbb{E}^{\mu}[S_i | S_{i-1}, S_{i-2}, \ldots, S_1] = S_{i-1}\Big\}$$

- Let $\mathcal{M}_{\text{Markov}}(\{\mu_i\}_{i=1}^n)$ be the subset of $\mathcal{M}(\{\mu_i\}_{i=1}^n)$ made of Markov measures.

Theorem (Strassen). Let $\{\mu_i\}_{i=1}^n$ be probability measures on $\mathbb{R}$. Then, the following assertions are equivalent

- $\mathcal{M}(\{\mu_i\}_{i=1}^n) \neq \emptyset$,
- $\mathcal{M}_{\text{Markov}}(\{\mu_i\}_{i=1}^n) \neq \emptyset$,
- The sequence $\{\mu_i\}_{i=1}^n$ is increasing in convex order.

- No arbitrage implies $\mathcal{M}_{\mathrm{Markov}}(\{\mu_i\}_{i=1}^n) \neq \emptyset$. Sufficient to solve for $n = 2$, as concatenating solutions $\mu_{i,i+1} \in \mathcal{M}_{\mathrm{Markov}}(\mu_i, \mu_{i+1})$ produces a calibrated martingale measure which is also a Markov process.

- Consider $0 < T_1 < T_2$. Let $\mathcal{M}(\mu_1, \mu_2)$ be the set of martingale probability measures with marginals μ_1 and μ_2:

$$\left\{\mu \in \mathcal{P}(\mathbb{R}^2) : S_1 \sim \mu_1, \quad S_2 \sim \mu_2, \quad \mathbb{E}^\mu[S_2|S_1] = S_1\right\}$$

- Goal: efficiently construct $\mu \in \mathcal{M}(\mu_1, \mu_2)$ minimizing KL divergence w.r.t. reference measure $\bar{\mu}$. Solve "measure problem":

$$D_{\bar{\mu}} = \inf_{\mu \in \mathcal{M}(\mu_1, \mu_2)} H(\mu \,\|\, \bar{\mu}) \tag{M}$$

(cf., [AFHS97, DMHL19])

If the minimum entropy problem is finite, there exists [Guy24] a unique minimizer $\mu^* \in \mathcal{M}(\mu_1, \mu_2)$ of the exponential-tilt form:

$$\mathrm{d}\mu^* = e_u(S_1, S_2)\,\mathrm{d}\bar{\mu}, \quad e_u(s_1, s_2) = \exp\Big\{u_1(s_1) + u_2(s_2) + \Delta_1(s_1)(s_2 - s_1)\Big\}$$

Here, $u = (u_1, u_2, \Delta_1)$ maximizers (Schrödinger potentials), if they exist, of the dual problem ("portfolio problem")

$$P_{\bar{\mu}} = \sup_{u \in \mathcal{U}} J_{\bar{\mu}}(u), \quad J_{\bar{\mu}}(u) = \mathbb{E}^{\mu_1}[u_1(S_1)] + \mathbb{E}^{\mu_2}[u_2(S_2)] - \mathbb{E}^{\bar{\mu}}[e_u(S_1, S_2)] + 1, \tag{P}$$

and $\mathcal{U}$ is the set of all measurable functions $u_1, u_2 : \mathbb{R}_{>0} \to \mathbb{R}$, $\Delta_1 : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying $u_i \in L^1(\mu_i)$ for $i \in \{1, 2\}$, and Δ_1 bounded.

- $D_{\bar{\mu}} = P_{\bar{\mu}}$. Both problems are dual to each other.
- (P) is an unconstrained concave maximization problem.

- In practice, only finitely many vanilla options traded; we restrict u_1 and u_2 to linear combinations of these, plus positions in the bond and S_1.
- For $i \in \{1, 2\}$, options with maturity T_i denoted as $\{P_j^{(i)}\}_{j=1}^{n_i}$ with strikes $\{K_i^{(1)} < \cdots < K_{n_i}^{(i)}\}$ and payoffs h_i.
- We then build a model of the form

$$\mathrm{d}\mu = e_\theta(S_1, S_2)\,\mathrm{d}\bar{\mu}$$

where $\theta = (c, \Delta_0, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \Delta_1)$ and

$$e_\theta(s_1, s_2) = \exp\Big\{c + \Delta_0 s_1 + \sum_{i=1}^{n_1} a_i^{(1)} h_1(s_1, K_i^{(1)}) + \sum_{j=1}^{n_2} a_j^{(2)} h_2(s_2, K_j^{(2)}) + \Delta_1(s_1)(s_2 - s_1)\Big\}$$

Schrödinger System

The measure μ is then a consistent, arbitrage-free model that calibrates to the market prices of SPX futures and options iff θ solves the SS

$$
\begin{cases}
\mathbb{E}^{\bar{\mu}}\left[e_\theta(S_1,S_2)\right] = 1, & \\
\mathbb{E}^{\bar{\mu}}\left[S_1\, e_\theta(S_1,S_2)\right] = S_0, & \\
\mathbb{E}^{\bar{\mu}}\left[h_1(S_1,K_i^{(1)})\, e_\theta(S_1,S_2)\right] = P_i^{(1)}, & \forall i \in \{1,\dots,n_1\}, \\
\mathbb{E}^{\bar{\mu}}\left[h_2(S_2,K_j^{(2)})\, e_\theta(S_1,S_2)\right] = P_j^{(2)}, & \forall j \in \{1,\dots,n_2\}, \\
\mathbb{E}^{\bar{\mu}}\left[(S_2-S_1)\ e_\theta(S_1,S_2) \middle| S_1 = s_1\right] = 0, & \forall s_1 > 0.
\end{cases}
$$

Choice of the Reference Measure $\bar{\mu}$

Assume the prior $\bar{\mu}$ is absolutely continuous w.r.t. Lebesgue measure and define:

$$\frac{\mathrm{d}\bar{\mu}(s_1, s_2)}{\mathrm{d}s_1\,\mathrm{d}s_2} = f_{\bar{\mu}}(s_1, s_2) = f_{\bar{\mu}_{2|1}}(s_2|s_1)\, f_{\bar{\mu}_1}(s_1)$$

Marginal $f_{\bar{\mu}_1}$: market risk-neutral density at T_1.

Conditional $f_{\bar{\mu}_{2|1}}(s_2|s_1)$: following [DMHL19], we set

$$f_{\bar{\mu}_{2|1}}(y|s_1) = \frac{\exp\left\{-\frac{1}{2}\left(\frac{y-s_1}{\sigma_1(s_1)\sqrt{T_2-T_1}}\right)^2\right\}}{\sigma_1(s_1)\sqrt{T_2-T_1}\sqrt{2\pi}}, \quad \forall y \in \mathbb{R}$$

That is, conditionally on $S_1 = s_1$, $S_2 \sim \mathcal{N}(s_1, \sigma_1(s_1)^2(T_2 - T_1))$.

We take $\sigma_1(s) = \alpha s^\beta$ (CEV-like), calibrated to minimize L_2 pricing error for T_2 options.

→ This prior choice speeds up computations significantly. Closed-form formulas for integrals $\int \cdots f_{\bar{\mu}_{2|1}}(y|s_1)\,\mathrm{d}y \Rightarrow$ one integral instead of two.

→ Contrast: independent prior $\bar{\mu} = \mu_1 \otimes \mu_2$ is standard in entropic OT, but not financially natural.

Solving the Schrödinger System

Approaches:

1. Sinkhorn algorithm (classical) – [Sin67]
2. Newton-Sinkhorn – faster
3. Implied Newton – even faster

- Method (1) iteratively solves each equation in the SS to converge to the optimizer θ^*.

- Popularized in ML [Cut13] for efficient computation of Wasserstein distances and entropic OT.

- Applied in quantitative finance to solve martingale optimal transport (MOT) and construct arbitrage-free implied vol surfaces [DM18, DMHL19].

- Extended in [Guy20, BG24] to handle martingale and VIX constraints, yielding fast joint calibration methods.

Newton-Sinkhorn

- Solving the SS is equivalent to setting the gradient of the concave function $J_{\bar{\mu}}$ to zero:

$$J_{\bar{\mu}}(\theta) = c + \Delta_0 S_0 + \sum_{i=1}^{n_1} a_i^{(1)} P_i^{(1)} + \sum_{j=1}^{n_2} a_j^{(2)} P_j^{(2)} - \mathbb{E}^{\bar{\mu}}[e_\theta(S_1, S_2)] + 1$$

- We solve the finite-payoff analogue of (P):

$$P_{\bar{\mu}} = \sup_{\theta \in \Theta} J_{\bar{\mu}}(\theta).$$

- We propose the Newton–Sinkhorn algorithm: each iteration performs a Newton step followed by a Sinkhorn step.

- To integrate with respect to S_1, we use Gauss–Legendre quadrature over an increasing grid $\mathcal{G}_1 = \{s_1^{(1)} \leq \cdots \leq s_1^{(N_1)}\}$ with N_1 nodes.

Newton–Sinkhorn algorithm

- Newton step. Start from an initial guess $\theta^{(0)}$. Then, solve for each iteration $n \in \mathbb{N}$, the portfolio problem

$$\theta^{-\Delta_1,(n+1)} = \underset{\theta^{-\Delta_1} \in \Theta^{-\Delta_1}}{\arg\max}\, J_{\bar{\mu}}(\theta^{-\Delta_1}, \Delta_1^{(n)}(s_1)), \qquad \forall s_1 \in \mathcal{G}_1,$$

  where $\theta^{-\Delta_1} = (c, \Delta_0, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$. The gradient and the Hessian are known in closed-form.

- Sinkhorn step for the martingale constraint.
  We find the solution $\Delta_1^{(n+1)}(s_1)$ of

$$\psi_{s_1}(\Delta_1^{(n+1)}(s_1), \boldsymbol{a}^{(2)}) = 0, \qquad \forall s_1 \in \mathcal{G}_1,$$

  where, for all $x \in \mathbb{R}$,

$$\psi_{s_1}(x, \boldsymbol{a}^{(2)}) = \int (s_2 - s_1) \exp\left\{ \sum_{j=1}^{n_2} a_j^{(2)} h_2(s_2, K_j^{(2)}) + x(s_2 - s_1) \right\} f_{\bar{\mu}_{2|1}}(s_2|s_1)\, \mathrm{d}s_2$$

Implied Newton

- Inspired by [DM18], we observe

$$\theta^* = \arg\max_{\theta \in \Theta} J_{\bar{\mu}}(\theta) = \arg\max_{\theta^{-\Delta_1} \in \Theta^{-\Delta_1}} J_{\bar{\mu}}\big(\theta^{-\Delta_1}, \Delta_1^*(\cdot, \boldsymbol{a}^{(2)})\big)$$

  where $\Delta_1^*(\cdot, \boldsymbol{a}^{(2)})$ solves

$$\psi_{s_1}(\Delta_1^*(\cdot, \boldsymbol{a}^{(2)}), \boldsymbol{a}^{(2)}) = 0.$$

- The modified objective $\tilde{J}_{\bar{\mu}}$ shares the same gradient and Hessian as $J_{\bar{\mu}}$, except for the terms involving differentiation with respect to $\boldsymbol{a}^{(2)}$. They remain explicit.

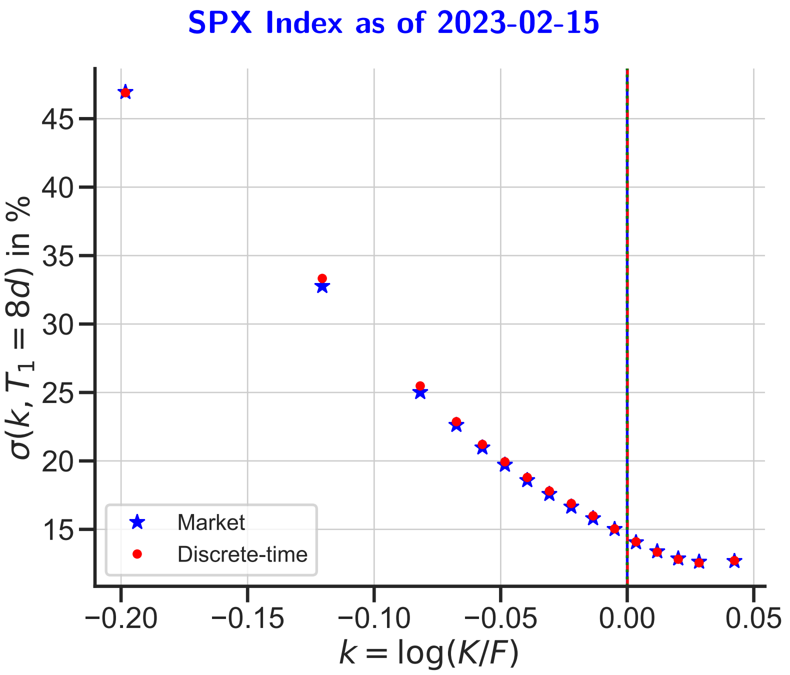
Updated Conditional Density

Let μ be the optimal joint measure associated to some parameter θ. Then:

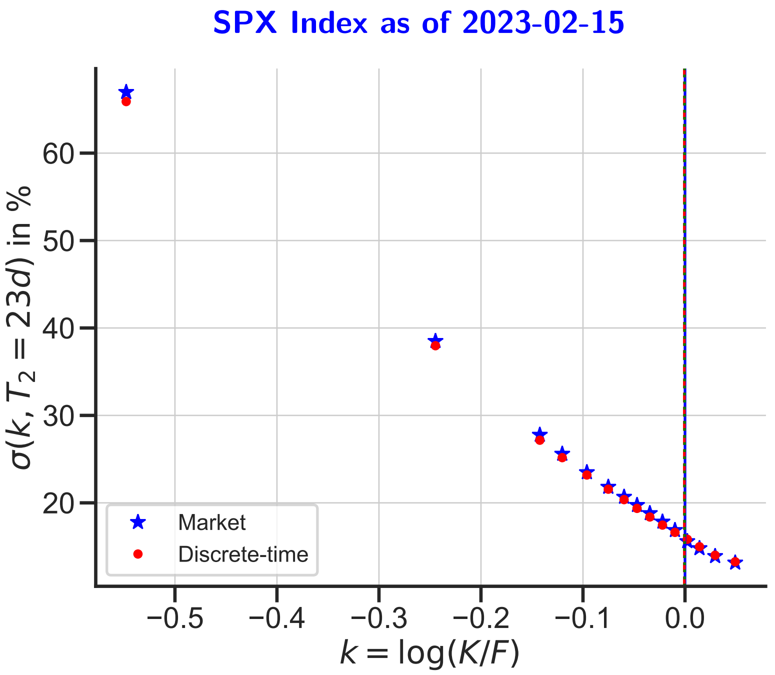
$$\frac{\mathrm{d}\mu(s_1, s_2)}{\mathrm{d}s_1\,\mathrm{d}s_2} = f_\mu(s_1, s_2) = e_\theta(s_1, s_2)\, f_{\bar{\mu}_{2|1}}(s_2|s_1)\, f_{\bar{\mu}_1}(s_1).$$

The updated conditional density is:

$$\begin{aligned}
f_{\mu_{2|1}}(s_2|s_1) &= \frac{f_\mu(s_1, s_2)}{f_{\mu_1}(s_1)} \\
&= \frac{e_\theta(s_1, s_2)\, f_{\bar{\mu}}(s_1, s_2)}{\int e_\theta(s_1, y)\, f_{\bar{\mu}}(s_1, y)\, dy} \\
&= \frac{\exp\left\{\sum_{j=1}^{n_2} a_j^{(2)} h_2(s_2, K_j^{(2)}) + \Delta_1(s_1)(s_2 - s_1)\right\} f_{\bar{\mu}_{2|1}}(s_2|s_1)}{\int \exp\left\{\sum_{j=1}^{n_2} a_j^{(2)} h_2(y, K_j^{(2)}) + \Delta_1(s_1)(y - s_1)\right\} f_{\bar{\mu}_{2|1}}(y|s_1)\, \mathrm{d}y}
\end{aligned}$$



SPX Index as of 2023-02-15
$\sigma(k, T_1 = 8d)$ in %
$k = \log(K/F)$
Market
Discrete-time
45
40
35
30
25
20
15
−0.20
−0.15
−0.10
−0.05
0.00
0.05



SPX Index as of 2023-02-15
σ(k, T₂ = 23d) in %
k = log(K/F)
60
50
40
30
20
−0.5
−0.4
−0.3
−0.2
−0.1
0.0
Market
Discrete-time

Continuous-Time Extension: Step 1

- Assume we have solved the discrete-time problem and denote the resulting measure by μ.
- We seek a continuous-time model $(S_t)_{t\in[0,T_2]}$ with law $\mathbb{P}$ on $C([0,T_2],\mathbb{R})$.

Goal: Construct $\mathbb{P}$ such that $(S_t)_{t\in[0,T_1]}$ is a $\mathbb{P}$-martingale and $S_1 \overset{\mathbb{P}}{\sim} \mu_1$.

Construction: Use a Markov functional model:

$$S_t = u(t, W_t), \quad \text{where } W \text{ is a } \mathbb{P}\text{-Brownian motion},$$

and u solves the backward heat equation:

$$\partial_t u + \frac{1}{2}\partial_x^2 u = 0, \quad u(T_1, x) = g(x) = F_{\mu_1}^{-1}\left(\Phi\left(\frac{x}{\sqrt{T_1}}\right)\right)$$

Then,

$$u(t,x) = \mathbb{E}[g(W_{T_1}) \mid W_t = x] = (g * \phi_{0,\sqrt{T_1 - t}})(x)$$

with the heat kernel

$$\phi_{0,\sqrt{T}}(x) = \frac{1}{\sqrt{2\pi T}} e^{-x^2/(2T)}$$

Continuous-Time Extension: Step 2

Goal: Extend the dynamics to $t \in [T_1, T_2]$, conditionally on $\mathcal{F}_{T_1}$.

Conditional dynamics: Given $S_{T_1} = s$, define for $t \in [T_1, T_2]$,

$$S_t = u_s(t, W_t - W_{T_1})$$

where

$$u_s(t,x) = \mathbb{E}[g_s(W_{T_2} - W_{T_1}) \mid W_t - W_{T_1} = x] = (g_s * \phi_{0,\sqrt{T_2 - t}})(x)$$

and

$$g_s(x) = \left(F_{\mu_{2|1}}(\cdot|s)\right)^{-1}\left(\Phi\left(\frac{x}{\sqrt{T_2 - T_1}}\right)\right)$$

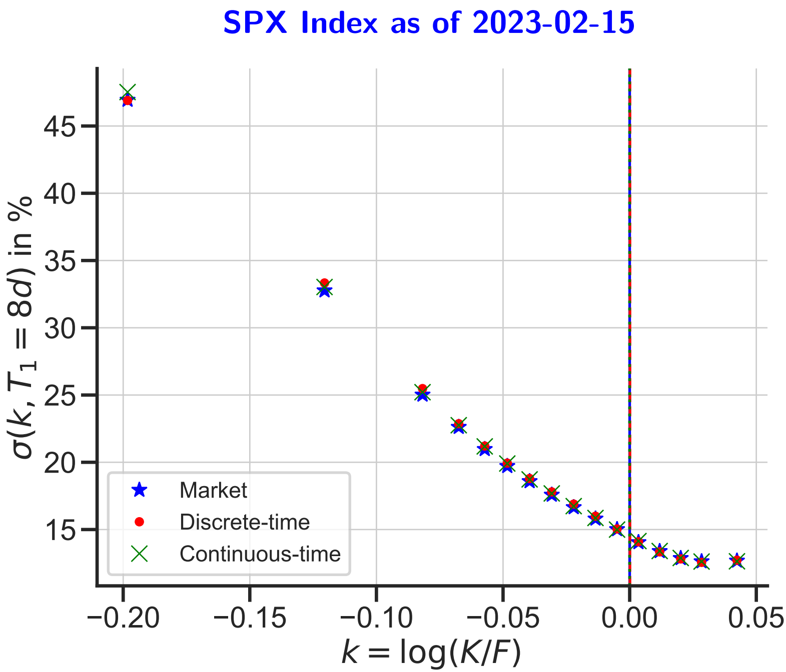
The conditional CDF is given by

$$F_{\mu_{2|1}}(x|s) = \mathbb{P}^{\mu}(S_2 \leq x | S_1 = s)$$

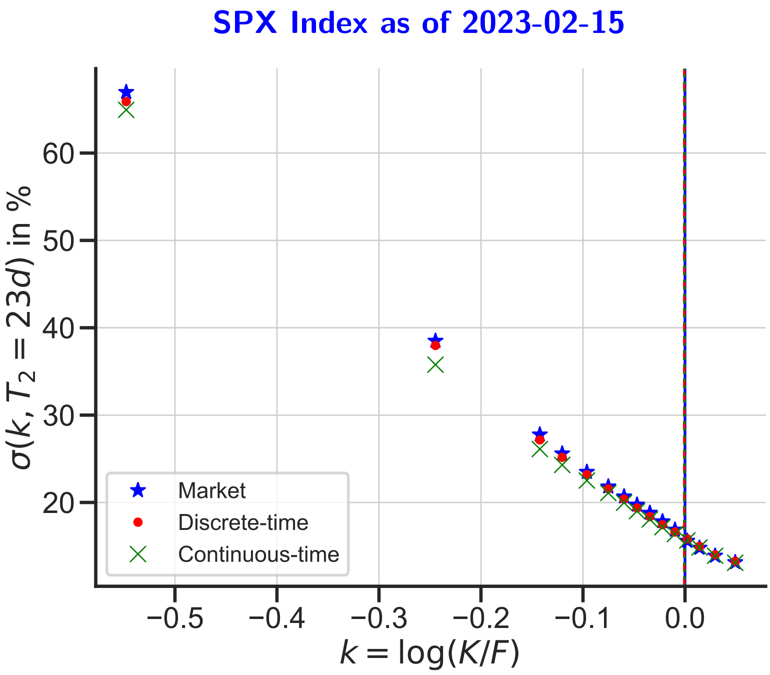
with conditional density

$$f_{\mu_{2|1}}(y|s) = \frac{e^{\sum_j a_j^{(2)} h_2(y, K_j^{(2)}) + \Delta_1(s)(y-s)} f_{\bar{\mu}_{2|1}}(y|s)}{\int e^{\sum_j a_j^{(2)} h_2(z, K_j^{(2)}) + \Delta_1(s)(z-s)} f_{\bar{\mu}_{2|1}}(z|s) \, \mathrm{d}z}$$

Closed-form expressions for $F_{\mu_{2|1}}$ and its inverse exist.



SPX Index as of 2023-02-15
$\sigma(k, T_1 = 8d)$ in %
$k = \log(K/F)$
45
40
35
30
25
20
15
−0.20
−0.15
−0.10
−0.05
0.00
0.05
Market
Discrete-time
Continuous-time



SPX Index as of 2023-02-15
$\sigma(k, T_2 = 23d)$ in %
$k = \log(K/F)$
Market
Discrete-time
Continuous-time
20
30
40
50
60
−0.5
−0.4
−0.3
−0.2
−0.1
0.0

Thank you for your attention!

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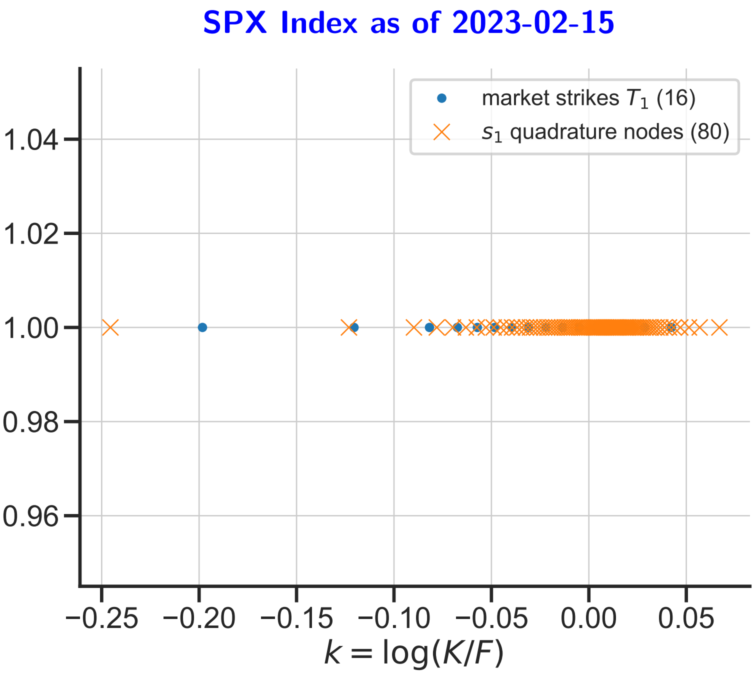
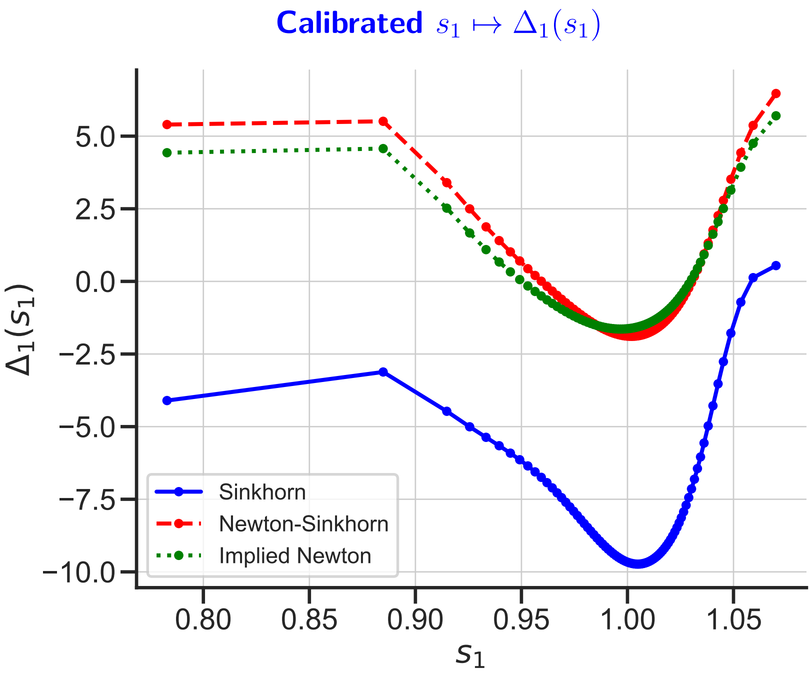


SPX Index as of 2023-02-15
market strikes T_1 (16)
s_1 quadrature nodes (80)
1.04
1.02
1.00
0.98
0.96
−0.25
−0.20
−0.15
−0.10
−0.05
0.00
0.05
$k = \log(K/F)$



Calibrated $s_1 \mapsto \Delta_1(s_1)$
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
$\Delta_1(s_1)$
0.80
0.85
0.90
0.95
1.00
1.05
s_1
Sinkhorn
Newton-Sinkhorn
Implied Newton

Useful formulas

Denote $z_x = \frac{x-m}{\sigma} - C\sigma$ and $\widetilde{m} = m + C\sigma^2$. For any $m, A, B, C, D, E \in \mathbb{R}$ and $\sigma > 0$, we have

$$\mathcal{I}_{m,\sigma}(A,B,C) := \int_A^B e^{Cx}\phi_{m,\sigma}(x)\,\mathrm{d}x = e^{Cm+\frac{1}{2}C^2\sigma^2}\Big(\Phi(z_B) - \Phi(z_A)\Big),$$

and

$$\begin{aligned}\mathcal{J}_{m,\sigma}(A,B,C,D) &:= \int_A^B (x-D)e^{Cx}\phi_{m,\sigma}(x)\,\mathrm{d}x \\ &= (\widetilde{m} - D)\mathcal{I}_{m,\sigma}(A,B,C) + \sigma e^{Cm+\frac{1}{2}C^2\sigma^2}\Big(\phi(z_A) - \phi(z_B)\Big),\end{aligned}$$

$$\mathcal{K}_{m,\sigma}(A,B,C,D,E) := \int_A^B (x-D)(x-E)e^{Cx}\phi_{m,\sigma}(x)\,\mathrm{d}x$$

equals

$$\begin{aligned}\Big(\sigma^2 + \widetilde{m}^2 + DE - (D+E)\widetilde{m}\Big)\,&\mathcal{I}_{m,\sigma}(A,B,C)\\ &+ \sigma e^{Cm+\frac{1}{2}C^2\sigma^2}\Big((A+\widetilde{m}-(D+E))\phi(z_A) - (B+\widetilde{m}-(D+E))\phi(z_B)\Big),\end{aligned}$$

and

$$\mathcal{L}_{m,\sigma}(A,B,C,D,E,F) := \int_A^B (x-D)(x-E)(x-F)e^{Cx}\phi_{m,\sigma}(x)\,\mathrm{d}x$$

equals

$$\begin{aligned}\Big[(\widetilde{m}-D)(\widetilde{m}-E)(\widetilde{m}-F) + \sigma^2\Big(3\widetilde{m} - (D+E+F)\Big)\Big]&\mathcal{I}_{m,\sigma}(A,B,C)\\ &+ \sigma\, e^{Cm+\frac{1}{2}C^2\sigma^2}\Big(\Lambda(A)\,\phi(z_A) - \Lambda(B)\,\phi(z_B)\Big),\end{aligned}$$

where

$$\begin{aligned}\Lambda(x) = (x-\widetilde{m})^2 + \Big(3\widetilde{m} - D - E - F\Big)&(x-\widetilde{m})\\ &+ (\widetilde{m}-D)(\widetilde{m}-E) + (\widetilde{m}-D)(\widetilde{m}-F) + (\widetilde{m}-E)(\widetilde{m}-F) + 2\sigma^2.\end{aligned}$$

Closed-form integrals

Let $n \in \mathbb{N}^*$, $m, m' \in \{1, \dots, n\}$, $K_0 = -\infty < K_1 < \cdots < K_n < K_{n+1} = +\infty$, $(a_i)_{1 \le i \le n} \in \mathbb{R}$, and $\Delta : \mathbb{R} \to \mathbb{R}$. Assume that $f_{\bar{\mu}_{2|1}}(s_2|s_1) = \phi_{s_1,\sigma(s_1)}(s_2)$. Empty sums are understood to be zero. Define

$$I_0(s_1) := \int \exp\left\{ \sum_{i=1}^{n} a_i h(s_2, K_i) + \Delta(s_1) s_2 \right\} f_{\bar{\mu}_{2|1}}(s_2|s_1) \, \mathrm{d}s_2,$$

$$I_1(s_1) := \int (s_2 - s_1) \exp\left\{ \sum_{i=1}^{n} a_i h(s_2, K_i) + \Delta(s_1) s_2 \right\} f_{\bar{\mu}_{2|1}}(s_2|s_1) \, \mathrm{d}s_2,$$

$$I_2(s_1, m) = \int h(s_2, K_m) \exp\left\{ \sum_{i=1}^{n} a_i h(s_2, K_i) + \Delta(s_1) s_2 \right\} f_{\bar{\mu}_{2|1}}(s_2|s_1) \, \mathrm{d}s_2,$$

$$I_3(s_1, m, m') = \int h(s_2, K_m) \, h(s_2, K_{m'}) \exp\left\{ \sum_{i=1}^{n} a_i h(s_2, K_i) + \Delta(s_1) s_2 \right\} f_{\bar{\mu}_{2|1}}(s_2|s_1) \, \mathrm{d}s_2$$