The Riemann-Liouville field and its GMC as $H \to 0$, and skew flattening for the rough Bergomi model

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Abstract

We consider a re-scaled Riemann-Liouville (RL) process $Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, and using Lévy's continuity theorem for random fields we show that Z^H tends weakly to an almost log-correlated Gaussian field Z as $H \to 0$. Away from zero, this field differs from a standard Bacry-Muzy field by an a.s. Hölder continuous Gaussian process, and we show that $\xi_\gamma^H(dt) = e^{\gamma Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} dt$ tends to a Gaussian multiplicative chaos (GMC) random measure ξ_γ for $\gamma \in (0,1)$ as $H \to 0$. We also show convergence in law for ξ_γ^H as $H \to 0$ for $\gamma \in [0, \sqrt{2})$ using tightness arguments, and ξ_γ is non-atomic and locally multifractal away from zero. In the final section, we discuss applications to the Rough Bergomi model; specifically, using Jacod's stable convergence theorem, we prove the surprising result that (with an appropriate re-scaling) the martingale component X_t of the log stock price tends weakly to $B_{\xi_\gamma([0,t])}$ as $H \to 0$, where B is a Brownian motion independent of everything else. This implies that the implied volatility smile for the full rough Bergomi model with $\rho \leq 0$ is symmetric in the $H \to 0$ limit, and without re-scaling the model tends weakly to the Black-Scholes model as $H \to 0$. We also derive a closed-form expression for the conditional third moment $\mathbb{E}((X_{t+h} - X_t)^3 | \mathcal{F}_t)$ (for $H > 0$) given a finite history, and $\mathbb{E}(X_T^3)$ tends to zero (or blows up) exponentially fast as $H \to 0$ depending on whether γ is less than or greater than a critical $\gamma \approx 1.61711$ which is the root of $\frac{1}{4} + \frac{1}{2}\log\gamma - \frac{3}{16}\gamma^2$. We also briefly discuss the pros and cons of a $H = 0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable X_1 with non-zero skewness as $t \to 0$. [1]

1 Introduction

Gaussian multiplicative chaos (GMC) is a random measure on a domain of $\mathbb{R}^d$ that can be formally written as $M_\gamma(dx) = e^{\gamma X_x - \frac{1}{2}\gamma^2 \mathbb{E}(X_x^2)} dx$ where X is a Gaussian field with zero mean and covariance $K(x,y) := \mathbb{E}(X_x X_y) = \log^+ \frac{1}{|y-x|} + g(x,y)$ for some bounded continuous function g. X is not defined pointwise because there is a singularity in its covariance, rather X is a random tempered distribution, i.e. an element of the dual of the Schwartz space $\mathcal{S}$ under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of M_γ requires a regularizing sequence X^ϵ of Gaussian processes (with the singularity removed), (see e.g. [BBM13] and [BM03] and Section 2.2 here for such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular regions or page 17 in [RV10]. In most of the literature on GMC, the choice of X^ϵ is a martingale in ϵ, from which we can then easily verify that $M_\gamma^\epsilon(A) = \int_A e^{\gamma X_x^\epsilon - \frac{1}{2}\gamma^2 \mathrm{Var}(X_x^\epsilon)} dx$ is a martingale, and then obtain a.s. convergence of $M_\gamma^\epsilon(A)$ using the martingale convergence to a random variable $M_\gamma(A)$ with $\mathbb{E}(M_\gamma(A)) = \mathrm{Leb}(A)$, and with a bit more work we can verify that $M_\gamma(.)$ defines a random measure (see page 18 in [RV10]).

If $\gamma^2 < 2d$, $M_\gamma^\epsilon(dx) = e^{\gamma X_x^\epsilon - \frac{1}{2}\gamma^2 \mathbb{E}((X_x^\epsilon)^2)} dx$ tends weakly to a multifractal random measure M_γ with full support a.s. which satisfies the local multifractality property $\lim_{\delta \to 0} \frac{\log \mathbb{E}(M_\gamma([x, x+\delta]^d)^q)}{\log \delta}) = \zeta(q)$ for $q \in (1, q^*)$ (see Proposition 3.7 in [RV10]), where $\zeta(q^*) = 1$ [2] and

$$\zeta(q) \;=\; dq - \frac{1}{2}\gamma^2(q^2 - q)$$

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[2] see Lemma 3 in [BM03] to see why the critical q value is q^*

so $q^* = \frac{2}{\gamma^2}$ for $d=1$, and $\mathbb{E}(M_\gamma([0,t])^q) = \infty$ if $q > q^*$, see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). M_γ is the zero measure for $\gamma^2 = 2d$ and $\gamma^2 > 2d$; in these cases a different re-normalization is required to obtain a non-trivial limit.

In the sub-critical case, using a limiting argument it can be shown that M_γ satisfies

$$\mathbb{E}(\int_D F(X,z) M_\gamma(dz)) \quad = \quad \mathbb{E}(\int_D F(X+\gamma^2 K(z,.),z)dz) \tag{1}$$

for any measurable function F and any interval D, which comes from the Cameron-Martin theorem for Gaussian measures and the notion of *rooted measures* and the disintegration theorem (see [FS20]). (1) can be taken as the definition of GMC, and it uniquely determines M_γ as a measurable function of X, and hence also uniquely fix its law. GMC also has natural applications in Liouville Quantum Field Theory.

Continuing in the same vein as [NR18] (see also [HN20]), we consider a re-scaled Riemann-Liouville process $Z^H_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ in the $H \to 0$ limit. Using Lévy's continuity theorem for tempered distributions, we show that Z^H tends weakly to an almost log-correlated Gaussian field Z as $H \to 0$, which is a random tempered distribution, i.e. a random element of the dual of the Schwartz space $\mathcal{S}$. From Theorem A in [JSW19], we know this field differs from a standard Bacry-Muzy field by a Hölder continuous Gaussian process, and we show that $\xi^H_\gamma(dt) = e^{\gamma Z^H_t - \frac{1}{2}\gamma^2 \mathrm{Var}(Z^H_t)} dt$ tends to a Gaussian multiplicative chaos (GMC) random measure ξ_γ for $\gamma \in (0,1)$ as $H \searrow 0$. Unlike standard constructions of GMC, our approximating sequence Z^H_t is not a martingale so we cannot appeal to the martingale convergence theorem. We later address the more difficult "L^1-regime" where $\gamma \in [1,\sqrt{2})$ using standard tightness/weak convergence arguments and comparing ξ^H_γ to a sequence of GMCs ξ^H_φ constructed in using a Gaussian white noise integrated over curved regions in the upper half plane under the Haar measure.

These results have a natural application to the popular Rough Bergomi stochastic volatility model, since ξ^H_γ is the quadratic variation of the log stock price for this model and values of H as low as .03 have been reported in empirical studies of this model (see e.g. [FTW19]). In section 4, using our Riemann-Liouville GMC and Jacod's stable convergence theorem, the we prove the surprising result that the martingale component X_t of the log stock price for the Rough Bergomi model tends weakly to $B_{\xi_\gamma([0,t])}$ as $H \to 0$ where B is a Brownian motion independent of everything else, which means the smile for the rBergomi model with $\rho \le 0$ is symmetric in the $H \to 0$ limit for $\gamma \in (0,1)$, and we find that $\mathbb{E}(X_t^3)$ decays exponentially fast or blows up exponentially fast depending on whether γ is less than or greater than a critical $\gamma \approx 1.61711$ which solves $\frac{1}{4} + \frac{1}{2}\log\gamma - \frac{3}{16}\gamma^2 = 0$, and we also define a $H=0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable X_1 with non-zero skewness as $t \to 0$.

2 The Riemann-Liouville process and its GMC as $H \to 0$

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\ge 0}$ throughout, which satisfies the usual conditions. In this section we consider a re-scaled Riemann-Liouville process in the limit as $H \to 0$; To this end, let $(W_t)_{t \ge 0}$ denote a standard Brownian motion and consider the following family of re-scaled Riemann-Liouville processes:

$$Z^H_t \quad = \quad \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \tag{2}$$

for $H \in (0,\frac{1}{2})$, for which $R_H(s,t) := \mathbb{E}(Z^H_s Z^H_t) = \int_0^{s\wedge t} (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du$. The integrand here is dominated by

$$h(u,s,t) \quad = \quad ((s-u)^{-\frac{1}{2}} \vee 1) \cdot ((t-u)^{-\frac{1}{2}} \vee 1) \tag{3}$$

which is integrable for $s < t$, so using the dominated convergence theorem, we find that

$$R_H(s,t) \;\to\; R(s,t) \quad := \quad \int_0^{s\wedge t} (s-u)^{-\frac{1}{2}} (t-u)^{-\frac{1}{2}} du$$

for $s \ne t$ as $H \to 0$ and $R_H(s,t) \to \infty$ for $s = t > 0$. We note also that $R(0,0) = \lim_{n\to\infty} \int_0^0 n ds = 0$ (from the definition of Lebesgue integration) and we also note that $R_H(0,0) = 0$ so $\lim_{H\to 0} R_H(0,0) = R(0,0) = 0$. We can evaluate this integral to obtain

$$R(s,t) \quad := \quad 2\tanh^{-1}(\frac{\sqrt{s}}{\sqrt{t}}) \quad = \quad \log\frac{1+\frac{\sqrt{s}}{\sqrt{t}}}{1-\frac{\sqrt{s}}{\sqrt{t}}} \quad = \quad \log\frac{\sqrt{t}+\sqrt{s}}{\sqrt{t}-\sqrt{s}} \quad = \quad \log\frac{(\sqrt{t}+\sqrt{s})^2}{t-s} \quad = \quad \log\frac{1}{t-s} \;+\; g(s,t) \tag{4}$$

for $0 < s < t$, where

$$g(s,t) \quad = \quad 2\log(\sqrt{s}+\sqrt{t}) \tag{5}$$

and note that $R(s,t) \geq 0$ for all $s,t \geq 0$.

$$\int_{[0,T]^2} R_H(s,t)dsdt \quad \leq \quad 2\int_{[0,T]^2}\int_0^t ((s-u)^{-\frac{1}{2}} \vee 1)\cdot((t-u)^{-\frac{1}{2}} \vee 1)dudsdt \quad < \quad \infty$$

so from the dominated convergence theorem, we have

$$\lim_{H\to 0}\int_{[0,T]^2} \phi_1(s)\phi_2(t)R_H(s,t)dsdt \quad = \quad \int_{[0,T]^2} \phi_1(s)\phi_2(t)R(s,t)dsdt \tag{6}$$

for any $\phi_1, \phi_2 \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz space. Similarly, for any sequence $\phi_k \in \mathcal{S}$ with $\|\phi_k\|_{m,j} \to 0$ for all $m, j \in \mathbb{N}_0^n$ for any $n \in \mathbb{N}$ (i.e. under the Schwartz space semi-norm defined in Eq 1 in e.g. [BDW18])

$$\lim_{k\to\infty}\int_{[0,T]^2} \phi_k(s)\phi_k(t)R(s,t)dsdt \quad = \quad 0 \tag{7}$$

since $\mu(A) = \int_A R(s,t)dsdt$ is a bounded non-negative measure (since $\int_0^T\int_0^t R(s,t)dsdt = \int_0^T 2tdt = T^2 < \infty$), and the convergence here implies in particular that ϕ_k tends to zero pointwise, so we can use the bounded convergence theorem. Thus if we define

$$\begin{aligned}\mathcal{L}_{Z^H}(f) \quad &:= \quad \mathbb{E}(e^{i(f,Z^H)}) \quad = \quad e^{-\frac{1}{2}\int_{[0,T]^2} f(s)f(t)R_H(s,t)dsdt} \\ \mathcal{L}(f) \quad &:= \quad e^{-\frac{1}{2}\int_{[0,T]^2} f(s)f(t)R(s,t)dsdt}\end{aligned}$$

for $f \in \mathcal{S}$, and note at the moment that we do not have a process or field as a subscript in $\mathcal{L}(f)$ since we have not yet shown that this is the characteristic functional of a random field. Then from (6) and (7) and Lévy's continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW18]), we see that $\mathcal{L}_{Z^H}(f)$ tends to $\mathcal{L}_Z(f)$ pointwise and $\mathcal{L}(.)$ is continuous at zero, then there exists a generalized random field Z (i.e. a random *tempered distribution*) such that $\mathcal{L}_Z = \mathcal{L}$ and Z^H tends to Z in distribution with respect to the strong and weak topology (see page 2 in [BDW18] for definition).Based on the right hand side of (4), we can say that Z is an *almost log-correlated Gaussian field* (LGF).

Remark 2.1 Since $g(s,t)$ is smooth away from $(0,0)$, from Theorem A in [JSW19], we know that Z differs from the standard Bacry-Muzy field on $(0,T]$ with covariance $\log\frac{1}{|t-s|}$ by some Gaussian process G_t which is a.s. Hölder continuous on $(0,T]$.

2.1 Constructing a Gaussian multiplicative chaos from Z^H as $H \to 0$

We now define the family of random measures : $\xi_\gamma^H(dt) := e^{\gamma Z_t^H - \frac{1}{2}\gamma^2\mathrm{Var}(Z_t^H)}dt$.

Theorem 2.1 *Let $H_n \searrow 0$. Then for any $A \in \mathcal{B}([0,T])$ and $\gamma \in (0,1)$, $\xi_\gamma^{H_n}(A)$ tends to some non-negative random variable $\xi_{\gamma,A}$ in L^2 (and hence also converges in probability), $\xi_\gamma([0,T])$ is a non-trivial random variable (i.e. has finite non-zero variance), and there exists a random measure ξ_γ on $[0,T]$ such that $\xi_\gamma(A) = \xi_{\gamma,A}$ a.s. for all $A \in \mathcal{B}([0,T])$. ξ_γ is the GMC associated with the family of process Z^H as $H \to 0$.*

Proof. We wish to show that $\mathbb{E}((\xi_\gamma^{H_n}[0,T]-\xi_\gamma^{H_m}[0,T]))^2 \to 0$, i.e. that $\xi_\gamma^{H_n}[0,T]$ is a Cauchy sequence in L^2. To this end, we first note that

$$\begin{aligned}\mathbb{E}(\xi_\gamma^{H_n}([0,T])\xi_\gamma^{H_m}([0,T])) \quad &= \quad \mathbb{E}(\int_{[0,T]^2} e^{\gamma(Z_t^{H_n}+Z_s^{H_m})-\frac{1}{2}\gamma^2\mathbb{E}((Z_t^{H_n})^2)-\frac{1}{2}\gamma^2\mathbb{E}((Z_s^{H_m})^2)}ds\,dt) \\ &= \quad \int_{[0,T]^2} \mathbb{E}(e^{\gamma(Z_t^{H_n}+Z_s^{H_m})-\frac{1}{2}\gamma^2\mathbb{E}((Z_t^{H_n})^2-\frac{1}{2}\gamma^2\mathbb{E}((Z_s^{H_m})^2})ds\,dt \\ &= \quad \int_{[0,T]^2} e^{\frac{1}{2}\gamma^2 R_{H_n}(t,t)+\frac{1}{2}\gamma^2 R_{H_m}(s,s)+\gamma^2\mathbb{E}(Z_t^{H_n}Z_s^{H_m})-\frac{1}{2}\gamma^2 R_{H_n}(t,t)-\frac{1}{2}\gamma^2 R_{H_m}(s,s)}ds\,dt \\ &= \quad \int_{[0,T]^2} e^{\gamma^2\mathbb{E}(Z_t^{H_n}Z_s^{H_m})}ds\,dt\,.\end{aligned}$$

The integrand here is bounded by $e^{\gamma^2\int_0^{s\wedge t}h(u,s,t)du}$ (where $h(u,s,t)$ is defined in (3)) and is integrable on $[0,T]^2$, and $\mathbb{E}(Z_t^{H_n}Z_s^{H_m})=\int_0^s(t-u)^{H_n-\frac{1}{2}}(s-u)^{H_m-\frac{1}{2}}du\to R(s,t)$ Lebesgue a.e. on $[0,T]^2$ as $n,m\to\infty$, so from the dominated convergence theorem we see that

$$\begin{aligned}\mathbb{E}(\xi_\gamma^{H_n}([0,T])\xi_\gamma^{H_m}([0,T])) &\to \int_{[0,T]^2}e^{\gamma^2R(s,t)}ds\,dt \qquad (n,m\to\infty)\\ &= 2\int_{[0,T]}\int_{[0,t]}e^{\gamma^2R(s,t)}ds\,dt\\ &= 2\int_{[0,T]}\int_{[0,t]}(\frac{\sqrt{t}+\sqrt{s}}{\sqrt{t}-\sqrt{s}})^{\gamma^2}dsdt\\ &= 2\int_{[0,T]}t\int_{[0,1]}(\frac{\sqrt{t}+\sqrt{tu}}{\sqrt{t}-\sqrt{tu}})^{\gamma^2}du\,dt\\ &= 2\int_{[0,T]}t\int_{[0,1]}(\frac{1+\sqrt{u}}{1-\sqrt{u}})^{\gamma^2}du\,dt \;=\; 2\int_0^T ta_\gamma dt \;=\; a_\gamma T^2 \;<\; \infty \quad (8)\end{aligned}$$

for $\gamma\in(0,1)$, where

$$a_\gamma \;:=\; \int_{[0,1]}(\frac{1+\sqrt{u}}{1-\sqrt{u}})^{\gamma^2}du \;=\; \frac{2\cdot{}_2F_1(2,-\gamma^2,3-\gamma^2,-1)}{(1-\gamma)(1+\gamma)(2-\gamma^2)} \tag{9}$$

where ${}_2F_1(z)$ is the hypergeometric function, and using that $1-\sqrt{u}\sim\frac{1}{2}(1-u)$ as $u\to1$, we can easily verify that $a_\gamma\to\infty$ as $\gamma\uparrow1$. Hence

$$\mathbb{E}((\xi_\gamma^{H_n}([0,T])-\xi_\gamma^{H_m}([0,T]))^2) \;=\; \mathbb{E}(\xi_\gamma^{H_n}([0,T])^2)\,-\,2\mathbb{E}(\xi_\gamma^{H_n}([0,T])\xi_\gamma^{H_m}([0,T]))\,+\,\mathbb{E}(\xi_\gamma^{H_m}([0,T])^2) \;\to\; 0$$

so $\xi_\gamma^{H_n}([0,T])$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$ to some a.s. non-negative random variable $\xi_{\gamma,[0,T]}$, and hence also converges in probability. Similarly, for any $A\in\mathcal{B}([0,T])$, we can trivially modify the argument above to show that

$$\mathbb{E}(\xi_\gamma^{H_n}(A)\xi_\gamma^{H_m}(A)) \;\to\; \int_A\int_A e^{\gamma^2R(s,t)}ds\,dt \;\le\; a_\gamma T^2 \;<\; \infty$$

so $\xi_\gamma^H(A)$ tends to some random variable $\xi_{\gamma,A}$ in L^2, and hence in probability.

We also know that $\mathbb{E}(\xi_\gamma^{H_n}([0,T]))=T$ for all n and we have already established L^2-convergence for $\xi_\gamma^{H_n}(A)$ as $n\to\infty$ which implies L^1 convergence, so (by Scheffe's lemma) $\mathbb{E}(\xi_{\gamma,[0,T]})=T$, which further implies that $\mathbb{P}(\xi_{\gamma,[0,T]}>0)>0$ and (from the reverse triangle inequality)

$$|\mathbb{E}(\xi_{\gamma,[0,T]}^2)^{\frac{1}{2}}-\mathbb{E}((\xi_{\gamma,[0,T]}^H)^2)^{\frac{1}{2}}| \;\le\; \mathbb{E}((\xi_\gamma([0,T])-\xi_\gamma^H([0,T]))^2) \;\to\; 0$$

so

$$\mathbb{E}(\xi_{\gamma,[0,T]}^2) \;=\; \lim_{H\to0}\mathbb{E}((\xi_{\gamma,[0,T]}^H)^2) \;=\; a_\gamma T^2$$

so in particular ξ_γ is not multifractal at zero, since the power is 2 here and not $\zeta(2)$. The L^2-convergence also means that $\xi_\gamma^H[0,T]\to\xi_{\gamma,[0,T]}$ in L^q as $H\to0$ for all $q\in[1,2]$ which (again from the reverse triangle inequality) implies that

$$\lim_{H\to0}\mathbb{E}(\xi_\gamma^H([0,T])^q) \;=\; \mathbb{E}(\xi_{\gamma,[0,T]}^q)\,. \tag{10}$$

Given that $\mathbb{E}(\xi_{\gamma,[0,T]})=T$ and $\mathrm{Var}(\xi_{\gamma,[0,T]})=\int_{[0,T]^2}e^{\gamma^2R(s,t)}ds\,dt-T^2>0$ since $a_\gamma>1$ for $\gamma\in(0,1)$, we see that $\xi_{\gamma,[0,T]}$ is a non-trivial random variable.

For $A,B\in\mathcal{B}([0,T])$ disjoint, $\xi_{\gamma,A\cup B}^H=\xi_{\gamma,A}^H+\xi_{\gamma,B}^H$ a.s. since ξ_γ^H is a measure, and we know that both sides tend to $\xi_{\gamma,A\cup B}$ and $\xi_{\gamma,A}+\xi_{\gamma,B}$ in probability. But by a standard result, if $X_n\xrightarrow{p}X$ and $X_n\xrightarrow{p}Y$, then $X=Y$ a.s., hence

$$\xi_{\gamma,A\cup B} \;=\; \xi_{\gamma,A}\,+\,\xi_{\gamma,B} \tag{11}$$

a.s.

Similarly for any sequence $A_n\downarrow\emptyset$ with $A_n\in\mathcal{B}([0,T])$, $\mathbb{E}(\xi_{\gamma,A_n})=\mathrm{Leb}(A_n)$, so by Markov's inequality $\mathbb{P}(\xi_\gamma(A_n)>\delta)\le\frac{\mathrm{Leb}(A_n)}{\delta}$, so $\xi_\gamma(A_n)$ tends to zero in probability, and from (11), we know that $\xi_\gamma(A_n)$ is decreasing, and hence also tends to some random variable Y a.s. (and hence also in probability). Thus by the same standard result discussed above, $Y=0$ a.s. Thus by Theorem 9.1.XV in [DV07] (see also the end of Section 4 on page 18 in [RV10]), there exists a random measure ξ_γ on $[0,T]$ such that $\xi_\gamma(A)=\xi_{\gamma,A}$ a.s. for all $A\in\mathcal{B}([0,T])$. ■

Remark 2.2 If we replace the definition of Z^H with the usual Riemann-Liouville process $Z_t^H = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, then adapting the arguments above, we see that

$$\mathbb{E}((\int_A e^{\gamma^2 Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} dt)^2) \quad \to \quad \mathrm{Leb}(A)^2$$

as $H \to 0$, for all $A \in \mathcal{B}([0,T])$. But we know that the first moment of $\int_A e^{\gamma^2 Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} dt$ is $\mathrm{Leb}(A)$ as well, hence $\int_A e^{\gamma^2 Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} dt \to \mathrm{Leb}(A)$ in L^2.

Remark 2.3 For $c \in (0,1]$, $(W_c, \xi_\gamma([0,c]) \sim (\sqrt{c}W_1, c\xi_\gamma[0,1])$, so in particular, $\xi_\gamma([0,(.)])$ is a self-similar process, and we can easily verify $\xi_\gamma([0,c])$ is monofractal at zero, i.e. $\mathbb{E}(\xi_\gamma([0,c])^q) = c^q\,\mathbb{E}(\xi_\gamma([0,1])^q)$.

2.2 Construction and properties of the usual Bacry-Muzy multifractal random measure (MRM) via Gaussian white noise on triangles

In this subsection we briefly describe the family of (stationary) Gaussian process used in [BM03]; the Bacry-Muzy multifractal random measure (MRM) is then the GMC associated with this family of processes as the l parameter tends to zero. Define $\omega_l(t)$ as in Eq 7 in [BBM13] with $\lambda = 1$ and $T = 1$, and set $\bar{\omega}_l(t) := \omega_l(t) - \mathbb{E}(\omega_l(t))$, so $\bar{\omega}_l(t) = \int_{(u,s)\in\mathcal{A}_l(t)} dW(u,s)$ where (in this subsection alone) $dW(u,s)$ is a two-dimensional Gaussian white noise with variance $s^{-2}duds$, and $\mathcal{A}_l(t) = \{(u,s) : |u-t| \le (\frac{1}{2}s) \wedge T, s \ge l\}$ is the cone-like region defined in Eq 11 in [BM03] (for the special case when $f(l) = f^{(e)}(t)$ in their notation, see Eqs 12 and 15 in [BM03]). Then

$$K_l^T(s,t) \quad := \quad \mathbb{E}(\bar{\omega}_l(t)\bar{\omega}_l(s)) \quad = \quad \begin{cases} \log\frac{T}{\tau} & l \le \tau \le T \\ \log\frac{T}{l} + 1 - \frac{\tau}{l} & \tau \le l \\ 0 & \tau > T \end{cases} \tag{12}$$

where $\tau = |t-s|$, and one can easily verify that $K_l^T(s,t) \le \log\frac{T}{\tau}$ (see Eq 25 in [BM03]). From a picture, we also see that $\mathbb{E}(\bar{\omega}_l(t)\,\bar{\omega}_{l'}(s)) = K_l(s,t)$ for $l > l'$ (i.e. the answer does not depend on l'), and $K_l^T(s,t) \nearrow \log\frac{T}{|t-s|}$ as $l \to 0$. We now define the measure

$$M_\gamma^{T,l}(dt) \quad = \quad e^{\gamma\bar{\omega}_l(t) - \frac{1}{2}\gamma^2 \mathrm{Var}(\bar{\omega}_l(t))} dt \tag{13}$$

and we use $M_\gamma^l(dt)$ as shorthand for $M_\gamma^{1,l}(dt)$. One can easily verify that $M_\gamma^l(A)$ is a martingale with respect to the filtration $\mathcal{F}_l := \sigma(W(A,B) : A \subset \mathbb{R}^+, B \subseteq [l,\infty])$ (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and $\sup_l \mathbb{E}(M_\gamma^l(A)^q) < \infty$ (Lemma 3 i) in [BM03]), so from the martingale convergence theorem, $M_\gamma^{T,l}(A)$ converges to $M_\gamma^T(A)$ in L^q for $q \in (1, q^*)$, and from the reverse triangle inequality this implies that

$$\lim_{l\to 0} \mathbb{E}((M_\gamma^{T,l}(A))^q) \quad = \quad \mathbb{E}((M_\gamma^T(A))^q) \tag{14}$$

and M^T is perfectly multifractal, i.e. $\mathbb{E}(|M_\gamma^T([0,t])|^q) = c_{q,T}\, t^{\zeta(q)}$ (see e.g. Lemma 4 in [BM03]) for some finite constant $c_{q,T} > 0$, depending only on q and T. For integer $q \ge 1$, we also note that

$$\begin{aligned} \mathbb{E}(M_\gamma^T(A)^q) \quad &= \quad \int_A \ldots \int_A e^{\gamma^2 \sum_{1\le i<j\le q} \log\frac{T}{|u_i - u_j|}} du_i \ldots du_q \\ &= \quad \int_A \ldots \int_A e^{\gamma^2 q(q-1)\log T + \sum_{1\le i<j\le q} \log\frac{1}{|u_i-u_j|}} du_i \ldots du_q \quad = \quad T^{\gamma^2 q(q-1)}\mathbb{E}(M_\gamma(A)^q) \end{aligned} \tag{15}$$

so we see that

$$c_{q,T} \quad = \quad c_q T^{\gamma^2 q(q-1)} \tag{16}$$

where $c_q = c_{q,1}$, and this also holds for non-integer q (see e.g. Theorem 3.16 in [Koz06]).

3 ξ_γ for the full sub-critical range $\gamma \in (0, \sqrt{2})$

3.1 The Sandwich lemma

We now look to extend the definition of ξ_γ to $\gamma \in (0, \sqrt{2})$. We will use the following standard result:

Theorem 3.1 (Kahane's Inequality) *(see e.g. Appendix of [RV10]). Let I be a bounded subinterval of $\mathbb{R}$ and $(X(u))_{u\in I}$, $(Y(u))_{u\in I}$ be two centred continuous Gaussian processes with $\mathbb{E}[X(u)X(u')] \leq \mathbb{E}[Y(u)Y(u')]$ for all u, u'. Then, for all convex functions $F : \mathbb{R} \to \mathbb{R}$, we have:*

$$\mathbb{E}[F(\int_I e^{X(u)-\frac{1}{2}\mathbb{E}(X(u)^2)}du)] \quad \leq \quad \mathbb{E}[F(\int_I e^{Y(u)-\frac{1}{2}\mathbb{E}(Y(u)^2)}du)]\,.$$

Lemma 3.2 *(The Sandwich lemma). Fix any τ and δ such that $0 < \tau < \tau + \delta < 1$. Then for $\tau \leq s \leq t \leq t + \delta$ and $H > 0$ sufficiently small, we can sandwich $R_H(s,t)$ as follows:*

$$K^{4\tau}_{l_*(H,\tau)}(k) \quad \leq \quad R_H(s,t) \quad \leq \quad K^4_{l^*(H)}(k) \tag{17}$$

for $k = |t-s| < \delta$ for $0 < s < t < 1$, where $l_*(H,\tau) = \frac{1}{F'_H(k^*)} > 0$ and $l^*(H) := 4e^{-\frac{1}{2H}} > 0$ (which both tend to zero as $H \to 0$), and $F_H(k) := R_H(\tau, \tau + k)$. Note the upper bound trivially holds for $s = 0$ as well, since $R_H(0,k) = 0$ and $K^T_l(k) \geq 0$. We also remind the reader that if $0 = s < t$, $R(s,t) = 0$ not $\log\frac{1}{t-0} + g(0,t) = \infty$.

Remark 3.1 The lower bound of the Sandwich lemma will only be used to prove the local multifractality of ξ_γ, and is not needed for everything else in the article.

Proof. We define $G_H(k) := R_H(\tau+\delta-k, \tau+\delta)$, and at this point we refer the reader to Appendix A for some basic properties of $G_H(k)$. Then choosing $l^* = l^*(H)$ such that $G_H(0) = \frac{(\tau+\delta)^{2H}}{2H} \leq \frac{1}{2H} = \log(\frac{4}{l^*})$, we see that

$$l^*(H) \quad = \quad 4e^{-\frac{1}{2H}} \downarrow 0 \qquad \text{as} \qquad H \to 0\,.$$

(A-1) implies that $G_H(k) \leq \log\frac{4}{k}$, and for $k \in [l^*, 4]$, $K^4_{l^*}(k) = \log\frac{4}{k}$ (see Eq 12 for definiton of $K^T(.)$), so in this case $G_H(k) \leq K^4_{l^*}(k)$. For $k \in (0, l^*)$, $K^4_{l^*}(k) = \log(\frac{4}{l^*}) + 1 - \frac{k}{l^*} > \log\frac{4}{l^*} \geq G_H(0) > G_H(k)$. Hence for both cases, we have the following upper bound:

$$G_H(k) \quad = \quad R_H(\tau+\delta-k, \tau+\delta) \quad \leq \quad K^4_{l^*(H)}(k)\,.$$

From Appendix A, we recall that

$$R_H(s, k+s) \quad = \quad \int_0^s (u(k+u))^{H-\frac{1}{2}}du$$

and if we restrict attention to $A_\delta := \{(s,t) : t-s = k \text{ and } (s,t) \in [\tau, \tau+\delta]^2\}$ for $0 < \tau < \tau+\delta < 1$ with $k \in [0,\delta]$, then from Appendix A we know that $R_H(s,t)$ is maximized at $s = \tau + \delta - k$ and minimized at $s = \tau$ (see Figure 2). Thus

$$R_H(s,t) \quad \leq \quad G_H(k) \quad \leq \quad K^4_{l^*(H)}(k) \tag{18}$$

for $(s,t) \in [\tau, \tau+\delta]^2$ where $k = |t-s|$.

From the second part of Appendix A, we know that $F_0(k) := \log\frac{1}{k} + 2\log(\sqrt{\tau} + \sqrt{\tau+k}) > \log\frac{4\tau}{k}$ but we also know that $F_H(k) \uparrow F_0(k)$ uniformly on compact intervals away from zero, and $F_H(0) < \infty$ and $\log(\frac{4\tau}{k}) \to \infty$ as $k \to 0$, so from the aforementioned uniform convergence, we see that for $H > 0$ sufficiently small there exists a $k^* = k^*(H,\tau) > 0$ such that

$$F_H(k^*) \quad = \quad \log\frac{4\tau}{k^*} \tag{19}$$

(see middle plot in Figure 2) with

$$F_H(k) \geq \log\frac{4\tau}{k} \quad \text{for} \quad k \in [k^*, 4\tau] \quad , \quad F_H(k) \leq \log\frac{4\tau}{k} \quad \text{for} \quad k \leq k^*\,. \tag{20}$$

Now set $l_* = l_*(H,\tau)$ such that $|F'_H(k^*)| = \frac{1}{l_*}$. $l_* \in [\tau, \tau+\delta]$ for H sufficiently small, and $l_* \geq k^*$ since

$$\frac{1}{k^*} \quad = \quad |\frac{d}{dk}\log\frac{4\tau}{k}|_{k=k^*}| \quad > \quad |F'_H(k^*)| \tag{21}$$

(see Figure 2 middle plot). We now note the following:

- In the region $[k^*, l_*]$, $F_H(k) > \log(4\tau/k)$ so $F_H(k) > \log(4\tau/l_*) + 1 - k/l_*$ (since the latter is just the tangent line to $\log(4\tau/k)$ at $k = l_*$), see Figure 2 middle plot.

- At $k = k_*$, F_H is greater than said tangent and by construction has the same gradient as the tangent, i.e. $\frac{1}{l_*}$. Then as k decreases to zero, the gradient of F_H increases in absolute value (due to the convexity of F_H) so F_H is greater than the tangent line.

Thus $K^{4\tau}_{l_*}(k) = \log \frac{4\tau}{l_*} + 1 - \frac{k}{l_*} < F_H(k)$ for $k \in (0, l_*)$. We also see that $l_* \downarrow 0$ as $H \downarrow 0$, since $k^* \to 0$ as $H \to 0$. Thus, to sum up, we have shown that

$$G_H(k) \quad = \quad R_H(\tau+\delta-k, \tau+\delta) \quad \leq \quad K^4_{l^*(H)}(k)$$

and

$$K^{4\tau}_{l_*(H,\tau)}(k) \quad \leq \quad F_H(k) \quad = \quad R_H(\tau, \tau+k)$$

for $k \in [0, 4\tau]$. From Appendix A, we recall that $R_H(s, k+s) = \int_0^s (u(k+u))^{H-\frac{1}{2}} du$ and if we restrict attention to $A_\delta := \{(s,t) : t-s = k, (s,t) \in [\tau, \tau+\delta]\}$ for $0 < \tau < \tau+\delta < 1$ with $k \in [0, \delta]$, then $R_H(s,t)$ is maximized at $s = \tau+\delta-k$ and minimized at $s = \tau$. Thus

$$K^{4\tau}_{l_*(H,\tau)}(k) \quad \leq \quad F_H(k) \quad \leq \quad R_H(s,t) \quad \leq \quad G_H(k) \quad \leq \quad K^4_{l^*(H)}(k) \tag{22}$$

for $(s,t) \in [\tau, \tau+\delta]^2$ where $k = |t-s|$. ■

3.2 Existence of a limiting law for ξ_γ for $\gamma \in (0, \sqrt{2})$

Let P be an independently scattered infinitely divisible random measure (see [BM03] for details) with

$$\mathbb{E}(e^{iqP(A)}) \quad = \quad e^{\varphi(q)\mu(A)}$$

for $q \in \mathbb{R}$ where $\mu(du, dw) = \frac{1}{w^2} dw du$ denotes the *Haar measure*. Here we restrict attention to the special case where $\varphi(q) = \frac{1}{2}\gamma^2 q^2$, in which case $P(du, dw)$ is just γ times a Gaussian white noise with variance $\frac{1}{w^2} du dw$ (similar to Section 2.2). Let $A^H_t := \{0 \leq u \leq t,\, w \geq g_H(u,t)\}$ for a family of functions which satisfy the following condition:

Condition 1 *$g_H(.,t) \geq 0$ with $g_H(u,t)$ increasing in t and H.*

We now define the process $\omega^H_t = P(A^H_t)$ for $t \geq 0$ with filtration

$$\mathcal{F}_H \quad := \quad \sigma(P(A \times B) : B \subseteq [H, \infty], A, B \in \mathcal{B}(\mathbb{R})) \tag{23}$$

(compare to a similar filtration on page 17 in [RV10]), and ω^H_t is a Gaussian process since $\varphi(q)$ is the characteristic function of a Gaussian with covariance

$$\mathbb{E}(\omega^H_s \omega^H_t) \quad = \quad \int_0^s \int_{g_H(u,t)}^\infty \frac{1}{w^2} dw du \quad = \quad \int_0^s \frac{1}{g_H(u,t)} du$$

for $0 \leq s \leq t$, and differentiating with respect to s, we see that if g satisfies $\frac{1}{g_H(s,t)} = R^H_s(s,t)$ then (for H fixed) the Gaussian process ω^H has the same covariance as our process Z^H, and the explicit formula for g_H is given as

$$g_H(s,t) \quad = \quad \frac{1}{\gamma} \frac{2 s^{\frac{1}{2}-H} t^{\frac{3}{2}-H}}{\Gamma(\frac{1}{2}+H)(t(1+2H)\,{}_2F_1(1, \frac{1}{2}-H, \frac{3}{2}+H, \frac{s}{t}) + s(1-2H)\,{}_2F_1(2, \frac{3}{2}-H, \frac{5}{2}+H, \frac{s}{t}))}$$

where ${}_2F_1(a,b,c,z)$ is the regularized hypergeometric function[3] (and in Appendix B we verify that Condition 1 above is satisfied. For $H = 0$ we have $g_0(s,t) = \frac{\sqrt{s}\,(t-s)}{\sqrt{t}}$. For $H_2 < H_1$, $\omega^{H_2}_t - \omega^{H_1}_t = P(A^{H_2}_t \setminus A^{H_1}_t)$ and $\omega^H_t = P(A^H_t)$ are independent for any $H \geq H_1$, so ω^H_t is an $\mathcal{F}_H$-martingale (see (23) for definition of $\mathcal{F}_H$, and we refer to this as a backward martingale since the martingale evolves as H goes smaller not larger and we start the martingale at some $H > 0$), and from this one can easily verify that $\xi^H_\varphi(I)$ is also an $\mathcal{F}_H$-backward martingale for any Borel set I.

Theorem 3.3 *Let ξ^H_φ denote the GMC of $\gamma\omega^H$ on $[0,1]$. Then for any $q \in (1, q^*)$ and any interval $I \subseteq [0,1]$, $\xi^H_\varphi(I)$ tends to some non-negative random variable $\xi_{\varphi,I}$ as $H \to 0$ a.s. and in L^q, and $\mathbb{E}(\xi^H_\varphi(I)^q) \to \mathbb{E}(\xi^q_{\varphi,I})$.*

[3] we are using Mathematica's definition here

Proof. From the upper bound in the Sandwich Lemma $R_H(s,t) \le K^{\theta}_{l^*(H)}(s,t)$ for $0<s<t<1$, where $\theta = 4 \cdot \sup(I)$ and $K^T_l(s,t)$ is the covariance of the model in [BM03], and $l^*(H) \downarrow 0$ as $H \downarrow 0$. Then from Kahane's inequality we have that

$$\mathbb{E}(\xi^H_{\varphi}(I)^q) \quad \le \quad \mathbb{E}(M^{\theta}_{l^*(H)}(I)^q) \tag{24}$$

where M^T_l is defined as in Section 2.2. Moreover, from Lemma 3 in [BM03] we know that $\sup_{l>0} \mathbb{E}(M^{\theta}_l(I)^q) < \infty$ for $q \in [1,q^*)$, so we have the uniform bound $\sup_{H>0} \mathbb{E}(\xi^H_{\varphi}(I)^q) < \infty$.

From above we know that $\xi^H_{\varphi}(I)$ is a $\mathcal{F}^H$-backwards martingale. Then (by Doob's martingale convergence theorem for continuous martingales) $\xi^H_{\varphi}(I)$ tends to some random variable (which we call $\xi_{\varphi,I}$) as $H \to 0$ a.s. and in L^q for $q \in [1,q^*)$. Moreover, from the reverse triangle inequality, the aforementioned L^q-convergence implies that

$$\mathbb{E}((\xi^H_{\varphi}(I))^q) \quad \to \quad \mathbb{E}(\xi^q_{\varphi,I}) \tag{25}$$

as $H \to 0$, for $q \in [1,q^*)$. ■

Theorem 3.4 *The laws of $\xi^H_{\gamma}([0,.])$ on $C_0([0,1])$ converge weakly as $H \to 0$ to the law of a non decreasing process on $C_0([0,1])$ which induces a non-atomic measure ξ_{γ} on $[0,T]$ with $\mathbb{E}(\xi_{\gamma}(A)) = \mathrm{Leb}(A)$.*

Remark 3.2 In a previous version, we gave a slightly stronger result involving L^1-convergence using Theorem 25 in [Sha16]) via generalized randomized shifts, but in practice we are really just interested in simulating ξ^H for some single small H-value, and seeing whether the law of ξ^H is close to some limiting law.

Proof. Note that although $\mathbb{E}(\omega^H_s \omega^H_t) = \mathbb{E}(Z^H_s Z^H_t)$ this does not imply that $\mathbb{E}(\omega^H_s \omega^{H_2}_t) = \mathbb{E}(Z^H_s Z^{H_2}_t)$ for $H \ne H_2$. However (crucially) ξ^H_{φ} (defined in Theorem 3.3) has the same law as our original ξ^H_{γ} measure for all $H>0$, and the non-decreasing process $\xi^H_{\varphi}([0,(.))$ and $\xi^H_{\gamma}([0,(.))$ have the same finite-dimensional distributions, so it suffices to prove weak convergence in law of the sequence $\xi^H_{\varphi}([0,(.))$. Thus from the a.s. convergence in Theorem 3.3 and the bounded convergence theorem, we see that for n distinct time values $t_1,...t_n \in [0,1]$ and $u_1,..u_n \in \mathbb{R}$

$$\lim_{H\to 0} \mathbb{E}(e^{\sum_{k=1}^n \; iu_k \xi^H_{\varphi}([0,t_k))}) \quad = \quad \mathbb{E}(e^{\sum_{k=1}^n \xi_{\gamma,[0,t_k]}})\,.$$

So we have convergence of the finite-dimensional distributions of the process $\xi^H_{\gamma}([0,.]))$. Moreover, from the upper bound for the Sandwich lemma, for $0<s<t<1$ we have

$$\mathbb{E}(\xi^H_{\gamma}([s,t])^q) \quad \le \quad \mathbb{E}((M^{4,l^*(H)}_{\gamma}([s,t]))^q) \;\nearrow\; \mathbb{E}((M^4_{\gamma}([s,t]))^q) \quad = \quad c_{q,4}|t-s|^{\zeta(q)}\,.$$

Moreover, $\zeta(q) = 1 + (1-\frac{1}{2}\gamma^2)(q-1) + O((q-1)^2)$, and hence $\zeta(q)>1$ for $q>1$ sufficiently small for $\gamma \in (0,\sqrt{2})$. Hence by Problem 2.4.11 in [KS91] (or Theorem 1.8 in chapter XIII in [RY99]) with $X^m_t := \xi^H_{\gamma}([0,t])$ and $H = 1/m$, the probability measures $\mathbb{Q}^H = \mathbb{P} \circ (X^m)^{-1}$ induced by the sequence of processes $\xi^H_{\gamma}([0,.])$ on $C_0([0,1])$ are tight under the usual sup norm topology. Thus by Proposition 2.4.15 in [KS91] (see also Theorem B.1.3 in [FH05] and page 1 in [BM16]), the sequence $\mathbb{Q}^H$ converges weakly to a probability measure $\mathbb{Q}$ on $C_0([0,1])$. Moreover, since

$$\xi^H_{\varphi}([0,s]) \quad \le \quad \xi^H_{\varphi}([0,t])$$

for $0<s<t$, and we have a.s. convergence of both sides, so $\xi_{\varphi}([0,s])) \le \xi_{\varphi}([0,t]))$ and hence $\mathbb{Q}$ is the law of a non-decreasing continuous process, which induces a measure on $[0,1]$ which we call ξ_{γ}, with no atoms. We know that $\mathbb{E}(\xi_{\gamma,A}) = \mathrm{Leb}(A)$, so $\mathbb{E}(\xi_{\gamma}(A)) = \mathrm{Leb}(A)$. ■

3.2.1 Local multifractality

Proposition 3.5 *For $\gamma \in (0,\sqrt{2})$, ξ_{γ} has the following locally multifractal behaviour away from zero:*

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}(\xi_{\gamma}([t,t+\delta])^q)}{\log \delta} \quad = \quad \zeta(q) \tag{26}$$

for $t \in (0,1)$ and $q \in (0,q^*)$.

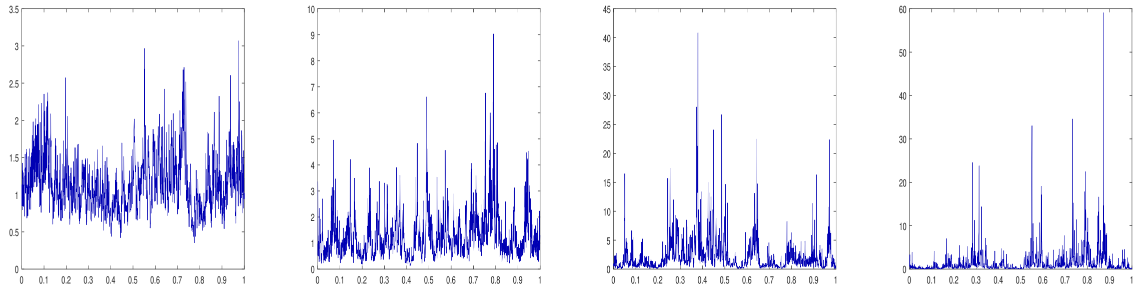

Figure 1: Here we see simulations of ξ_γ using a spectral expansion for (from left to right) $\gamma = 0.125$, 0.25, 0.375 and 0.5 with $n = 1000$ eigenfunctions, 1000 time points, $H = 0$ and we have used Gauss-Legendre quadrature. For this range of γ-values, the first four raw sample moments are in very close agreement with the theoretical values for $H = 0$.

Proof. Applying Kahane's inequality and Sandwich Lemma for $q \in (1, q^*)$ we have

$$\mathbb{E}[(M_\gamma^{4\tau, l_*(H,\tau)}([\tau, \tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(\xi_\gamma^H([\tau, \tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(M_\gamma^{4, l^*(H)}([\tau, \tau+\delta]))^q] \tag{27}$$

where $M_\gamma^{T,l}$ is defined as in Section 2.2. Using the L^q convergence of $M_\gamma^{T,l}(A)$ in (14) and (25), we see that

$$\mathbb{E}[(M_\gamma^{4\tau}([\tau, \tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(\xi_\gamma([\tau, \tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(M_\gamma^{4}([\tau, \tau+\delta]))^q]\,.$$

Then using the multifractality property of M_γ^T we see that:

$$c_{q,4\tau}\delta^{\zeta(q)} \quad = \quad c_{q,1}(4\tau)^{\gamma^2 q(q-1)}\delta^{\zeta(q)} \quad \leq \quad \mathbb{E}[(\xi_\gamma([\tau, \tau+\delta]))^q] \quad \leq \quad c_{q,4}\delta^{\zeta(q)} \quad = \quad c_{q,1}4^{\gamma^2 q(q-1)}\delta^{\zeta(q)}$$

where we have used (16) in the final line. Taking the logarithm of the above inequality, dividing by $\log \delta$ and taking limits yields the local multifractality property for ξ_γ (recall that we are assuming that $\tau > 0$ here). ■

4 Application to the Rough Bergomi model - skew flattening/blowup as $H \to 0$

We consider the standard Rough Bergomi model for a stock price process X_t^H:

$$\begin{cases} dX_t^H = -\frac{1}{2}\sqrt{V_t^H}dt + \sqrt{V_t^H}dW_t\,, \\ V_t^H = e^{\gamma Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} \\ Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}}(\rho dW_s + \bar{\rho} dW_t^\perp) \end{cases} \tag{28}$$

where $\gamma \in (0,1)$, $|\rho| \leq 1$ and W, $W^\perp$ are independent Brownian motions, and (without loss of generality) we set $\tilde{X}_0^H = 0$. We let $\tilde{X}_t^H = \int_0^t \sqrt{V_s^H}dW_s$ denote the martingale part of X^H.

Theorem 4.1 *For $\gamma \in (0,1)$, $\tilde{X}^H$ tends to $B^\perp_{\xi_\gamma([0,(.)])}$ stably (and hence weakly) in law on any finite interval $[0,T]$, where $B^\perp$ is a Brownian motion independent of everything else.*

Corollary 4.2 *From the weak convergence of $\xi_\gamma^H([0,T)$ and the previous result we see that*

$$\lim_{H\to 0} \mathbb{E}(e^{ikX_t^H}) \quad = \quad \lim_{H\to 0} \mathbb{E}(e^{-\frac{1}{2}(ik+k^2)\xi_\gamma^H([0,t])}) \quad = \quad \mathbb{E}(e^{-\frac{1}{2}(ik+k^2)\xi_\gamma([0,t])}) \quad = \quad \mathbb{E}(e^{ik(-\frac{1}{2}\xi_\gamma([0,t]) + B_{\xi_\gamma([0,t])})})$$

which (by a well known result in Renault&Touzi[RT96]) implies that implied volatility smile for the true Rough Bergomi model in (28) is symmetric in the log-moneyness $k = \log \frac{K}{S_0}$.

Remark 4.1 We call this the *skew flattening phenomenon*, so in particular $\tilde{X}_t^H$ (for a single fixed t) tends weakly to a symmetric distribution μ.

Proof. From Theorem 2.1, we know that $\langle\tilde{X}^H\rangle_t$ tends to a random variable $\xi_\gamma([0,t])$ in L^2 (and hence in probability), and $\langle\tilde{X}^H,W\rangle_t=\rho\int_0^t\sqrt{V_u^H}du$. But

$$\begin{aligned}\mathbb{E}((V_t^H)^{\frac{1}{2}}) &= \mathbb{E}(e^{\frac{1}{2}(\gamma Z_t^H-\frac{1}{2}\gamma^2\frac{1}{2H}t^{2H})})\\ &= \mathbb{E}(e^{\frac{1}{2}\gamma Z_t^H-\frac{1}{2}\cdot\frac{1}{4}\gamma^2\cdot\frac{1}{2H}+\frac{1}{2}\cdot\frac{1}{4}\gamma^2\cdot\frac{1}{2H}-\frac{1}{2}\gamma^2\frac{1}{4H}t^{2H}}) \;=\; e^{-\frac{1}{16H}\gamma^2t^{2H}} \;\to\; 0\end{aligned}$$

as $H\to0$, so (by Markov's inequality) $\mathbb{P}(\sqrt{V_t^H}>\delta)\le\frac{1}{\delta}\mathbb{E}(\sqrt{V_t^H})\to0$, so $\sqrt{V_t^H}$ tends to zero in probability, and hence

$$G_t \;:=\; \langle\tilde{X}^H,W\rangle_t \;\overset{p}{\to}\; 0. \tag{29}$$

Moreover, for any bounded martingale N orthogonal to W

$$\langle\tilde{X}^H,N\rangle_t \;=\; 0. \tag{30}$$

Thus setting $Z_t=W_t$ and applying Theorem IX.7.3 in Jacod&Shiryaev[JS03] (see also Proposition II.7.5 and Definition II.7.8 in [JS03]), we can construct an extension $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t),\tilde{\mathbb{P}})$ of our original filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ and a continuous Z-biased $\mathcal{F}$-progressive conditional PII martingale $\tilde{X}$ on this extension (see Definition 7.4 in chapter II in [JS03] for definition), such that $\tilde{X}^H$ converges stably (and hence weakly) to $\tilde{X}$ (see Definition 5.28 in chapter XIII in [JS03] for definition of stable convergence) for which

$$\begin{aligned}\langle\tilde{X}\rangle_t &= \xi_\gamma([0,t])\\ \langle\tilde{X},M\rangle_t &= 0\end{aligned}$$

for all continuous (bounded) martingales M with respect to the original filtration $\mathcal{F}_t$. From Proposition 7.5 and Definition 7.8 in Chapter 2 in [JS03], this means that $\tilde{X}_t=X'_t+\int_0^t u_s dW_s$ where X' is an $\tilde{\mathcal{F}}_t$-local martingale and u is a predictable process on the original space $(\Omega,\mathcal{F},\mathbb{P})$. One such M is $M_t=W_{t\wedge\tau_b\wedge\tau_{-b}}$, where $\tau_b=\inf\{t:W_t=b\}$, so we have a pair of continuous local martingales (M,X) with $\langle\tilde{X},M\rangle_t=\langle\tilde{X},W\rangle_t=\int_0^t u_s ds \;=\; 0$ for $t\le\tau_b\wedge\tau_{-b}$, so in fact $u_t\equiv0$. Then applying F.Knight's Theorem 3.4.13 in [KS91] with $M^{(1)}=X$ and $M^{(2)}=W$, if $T_t=\inf\{s\ge0:\langle X\rangle_s>t\}$, then X_{T_t} is a Brownian motion independent of W. Hence X has the same law as $B^\perp_{\xi_\gamma([0,t])}$ for any Brownian motion $B^\perp$ independent of W. ■

4.1 $H\to0$ behaviour for the usual rough Bergomi model

If we replace the definition of Z^H with the usual RL process $Z_t^H=\sqrt{2H}\int_0^t(t-s)^{H-\frac{1}{2}}ds$ (as is usually done), then from Remark 2.4, we know that $\xi_\gamma^H(A)$ tends Leb(A) in L^2 for any Borel set $A\subseteq[0,1]$, so adapting Theorem 4.1 for this case, we see that $\tilde{X}^H$ tends weakly to a standard Brownian motion, which means the rough Bergomi model tends weakly to the Black-Scholes model in the $H\to0$ limit.

4.2 A closed-form expression for $\mathbb{E}((\tilde{X}_t^H)^3)$

In this subsection we compute an explicit expression for the skewness of $\tilde{X}_t^H$ (conditioned on its history), which (as a by-product) gives a more "hands-on" proof as to why the skew tends to zero as $H\to0$, and also allows us to see how fast the skew decays.

We first note that (trivially) $\tilde{X}^H$ has the same law as $\tilde{X}^H$ defined by

$$\begin{cases} d\tilde{X}_t^H = \sqrt{V_t^H}(\rho dB_t+\bar{\rho}dW_t)\,,\\ V_t^H \;=\; e^{\gamma Z_t^H-\frac{1}{2}\gamma^2\mathrm{Var}(Z_t^H)}\\ Z_t^H \;=\; \int_0^t(t-s)^{H-\frac{1}{2}}dB_s\end{cases} \tag{31}$$

where B is independent of W, and this is the version of the model we use in this subsection. We henceforth use $\mathbb{E}_t((.))$ as shorthand for the conditional expectation $\mathbb{E}((.)|\mathcal{F}_t^{B,W})$, and we now replace the constant ρ with a time-dependent $\rho(t)$, and replace our original V_t^H process with

$$V_t^H \;=\; \xi_0(t)\,e^{\gamma Z_t^H-\frac{1}{2}\gamma^2\mathrm{Var}(Z_t^H)}$$

to incorporate a non-flat initial variance term structure.

Proposition 4.3

$$\mathbb{E}_{t_0}((\tilde{X}^H_T - \tilde{X}^H_{t_0})^3) \quad = \quad 3\gamma\int_{t_0}^T\int_0^t \rho(s)\,\xi^{\frac{1}{2}}_{t_0}(s)\xi_{t_0}(t)\,e^{\frac{1}{2}\gamma^2\mathrm{Cov}_{t_0}(Z^H_s Z^H_t)-\frac{1}{8}\gamma^2\mathrm{Var}_{t_0}(Z^H_s)}(t-s)^{H-\frac{1}{2}}dsdt \quad (32)$$

where $\xi_{t_0}(t) = \xi_0(t)e^{\gamma\int_0^{t_0}(t-u)^{H-\frac{1}{2}}dB_u - \frac{\gamma^2}{4H}[t^{2H}-(t-t_0)^{2H}]}$. *This simplifies to*

$$\mathbb{E}((\tilde{X}^H_T)^3) \quad = \quad 3\rho\gamma V_0^{\frac{3}{2}}\int_0^T\int_0^t e^{\frac{1}{2}\gamma^2(R_H(s,t)-\frac{s^{2H}}{8H})}(t-s)^{H-\frac{1}{2}}dsdt \quad < \quad \infty \quad (33)$$

if $t_0 = 0$*,* ρ *is constant and* $\xi_0(t) = V_0$ *for all* t *(i.e. flat initial variance term structure).*

Proof. See Appendix C. ■

Remark 4.2 Using that $R_H(s,t) \to R^{\mathrm{fBM}}(s,t)$ as $s,t \to 0$ (for $H > 0$ fixed), where $R^{\mathrm{fBM}}(s,t) = \frac{1}{2H}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ is the covariance function of $\frac{1}{\sqrt{2H}}W^H$ where W^H is a standard (one or two-sided) fractional Brownian motion, we find that the exponent in (33) behaves like $\frac{1}{16H}(s^{2H} + 2t^{2H} - 2(t-s))^{2H})$ for $s < t$ as $s,t \to 0$, and thus can effectively be ignored, so (for ρ constant)

$$\mathbb{E}((\tilde{X}^H_T)^3) \quad \sim \quad 3\rho\gamma V_0^{\frac{3}{2}}\int_0^T\int_0^t (t-s)^{H-\frac{1}{2}}dsdt \quad = \quad \frac{3\rho\gamma V_0^{\frac{3}{2}}}{(H+\frac{1}{2})(H+\frac{3}{2})}T^{H+\frac{3}{2}} \quad (T \to 0)\,.$$

Remark 4.3 Note that $\tilde{X}^H$ is driftless so (31) is only a toy model at the moment, but we easily adapt Proposition 4.3 and the two remarks above to incorporate the additional $-\frac{1}{2}\langle\tilde{X}^H\rangle_t$ drift term required to make $S_t = e^{\tilde{X}^H_t}$ a martingale. However, the relative contribution from this drift will disappear in the small-time limit, so we omit the tedious details, since rough stochastic volatility models are generally used (and considered more realistic) over small time horizons.

4.3 Convergence of the skew to zero

Corollary 4.4 *For* $\gamma \in (0,1)$ *and* $0 \le t \le T \le 1$, $\mathbb{E}_{t_0}((\tilde{X}^H_T - \tilde{X}^H_{t_0})^3) \to 0$ *a.s. as* $H \to 0$.

Proof. For $T \le 1$, using that $R_H(s,t) \uparrow R(s,t)$ and $(t-s)^{H-\frac{1}{2}} \uparrow (t-s)^{-\frac{1}{2}}$ we see that

$$\begin{aligned}
|\mathbb{E}_{t_0}((\tilde{X}^H_T - \tilde{X}^H_{t_0})^3)| \quad &\le \quad 3|\rho|\gamma\int_{t_0}^T\int_0^t \xi^{\frac{1}{2}}_{t_0}(s)\xi_{t_0}(t)e^{\frac{1}{2}\gamma^2(R_{t_0}(s,t)-\frac{s^{2H}}{8H})}(t-s)^{-\frac{1}{2}}dsdt \\
&\le \quad 3|\rho|\gamma\int_{t_0}^T\int_0^t \xi^{\frac{1}{2}}_{t_0}(s)\xi_{t_0}(t)e^{\frac{1}{2}\gamma^2(R(s,t)-\frac{s^{2H}}{8H})-\frac{1}{2}\log(t-s)}dsdt \\
&\le \quad 3\bar{\xi}^{\frac{1}{2}}_{t_0}(s)\bar{\xi}_{t_0}(t)\,|\rho|\gamma\int_{t_0}^T\int_0^t e^{\frac{1}{2}(1+\gamma^2)\log\frac{1}{t-s}+\frac{1}{2}\gamma^2\bar{g}}dsdt \;\le\; \mathit{const.}\times\mathbb{E}(M_{\sqrt{\frac{1}{2}(1+\gamma^2)}}([0,T])^2) \;<\; \infty
\end{aligned}$$

for $\gamma \in (0,1)$ where $M_\gamma(dt)$ is the usual [BM03] GMC, and $R_0(s,t) = \mathbb{E}_{t_0}(Z_s Z_s) = \int_{t_0}^s(s-u)^{-\frac{1}{2}}(t-u)^{-\frac{1}{2}}duds$, $\bar{g} = 2\log(2\sqrt{2})$, $\bar{\xi}_t = \sup_{0\le s\le t}\xi_s$. The result follows from dominated convergence theorem. ■

4.4 Speed of convergence of the skew to zero

Proposition 4.5 *(see [Ger20]). Let* $\rho(.)$ *be continuous and bounded away from zero with constant sign for* t *sufficiently small. Then*

$$-\lim_{H\to 0} H\log[\mathrm{sgn}(\rho)\mathbb{E}((\tilde{X}^H_T)^3)] \quad = \quad \hat{r}(\gamma) \quad = \quad \begin{cases} \frac{1}{16}\gamma^2 & 0 \le \gamma \le 1\,, \\ \frac{1}{4}+\frac{1}{2}\log\gamma - \frac{3}{16}\gamma^2 & \gamma \ge 1 \end{cases} \quad (34)$$

$\hat{r}(\gamma)$ is negative for γ larger than the root of $\frac{1}{4}+\frac{1}{2}\log\gamma-\frac{3}{16}\gamma^2$ at ≈ 1.61711, which makes the integral explode as $H \to 0$ for such values of γ.

4.5 A $H = 0$ model - pros and cons

Returning to Section 4.1, we can circumvent the problem of vanishing skew, by considering a toy model of the form

$$X_t = \sigma(\rho W_t + \bar{\rho} B^{\perp}_{\xi_\gamma([0,t])}) \tag{35}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$, W and $\xi_\gamma([0,t])$ are defined as in Section 2.1 with $\gamma \in (0,1)$, and $B^{\perp}$ is a Brownian motion independent of W. Then (setting $\alpha = \sigma\rho$ and $\beta = \sigma\bar{\rho}$), from the tower property we see that

$$\mathbb{E}(e^{ikX_t}) = \mathbb{E}(\mathbb{E}(e^{ik(\alpha W_t + \beta B_{\xi_\gamma([0,t])})}|W)) = \mathbb{E}(e^{ik\alpha W_t - \frac{1}{2}k^2\beta^2\xi_\gamma([0,t])})$$

and (from Remark 2.3) we know that $\xi_\gamma([0,t]) \sim t\xi_\gamma([0,1])$ (i.e. self-similarity), so

$$\mathbb{E}(e^{\frac{ik}{\sqrt{t}}X_t}) = \mathbb{E}(e^{ik\alpha W_t/\sqrt{t} - \frac{1}{2}k^2\beta^2\xi_\gamma([0,t])/t}) = \mathbb{E}(e^{ik\alpha W_1 - \frac{1}{2}k^2\beta^2\xi_\gamma([0,1])})$$

so X is self-similar: $X_t/\sqrt{t} \sim X_1$ for all $t > 0$, and X_1 (and hence X_t) has non-zero skewness for $\alpha \neq 0$; more specifically

$$\mathbb{E}((\frac{X_t}{\sqrt{t}})^3) = 4\sigma^3\rho(1-\rho^2)\gamma \tag{36}$$

and $\mathbb{E}(X_1^2) = \sigma^2$, and we can derive a similar (slightly more involved) expression for $\mathbb{E}(X_1^4)$. The ρ component achieves the goal of a $H = 0$ model with non-zero skewness, and one can establish the following small-time behaviour for European put options in the Edgeworth Central Limit Theorem regime:

$$\frac{1}{\sqrt{t}}\mathbb{E}((e^{x\sqrt{t}} - e^{X_t})^+) \sim e^{x\sqrt{t}}\mathbb{E}((x - \frac{X_t}{\sqrt{t}})^+) \sim \mathbb{E}((x - \frac{X_t}{\sqrt{t}})^+) \sim \mathbb{E}((x - \bar{X}_1)^+)$$

and $\lim_{t\to 0}\hat{\sigma}_t(x\sqrt{t}, t) = C_B(x, .)^{-1}(C(x))$ for $x > 0$, where $\hat{\sigma}_t(x,t)$ denotes the implied volatility of a European call option with strike $e^{x\sqrt{t}}$ maturity t and $S_0 = 1$ ($C_B(x,\sigma)$ is the Bachelier model call price formula). Hence we see the full smile effect in the small-time FX options Edgeworth regime unlike the $H > 0$ case where the leading order term is just Black-Scholes, followed by a next order skew term, followed by an even higher order convexity term.

We can go from a toy model to a real model adding back the usual $-\frac{1}{2}\langle X\rangle_t$ drift term for the log stock price X so $S_t = e^{X_t}$ is a martingale, and in this case we lose self-similarity for X but $X_t/\sqrt{t}$ still tends weakly to a non-Gaussian random variable, and in particular $\lim_{t\to 0}\mathbb{E}((\frac{X_t}{\sqrt{t}})^3) = 4\sigma^3\rho\bar{\rho}^2\gamma$. [4]. This model overcomes two of the main drawbacks of the original Bacry et al. multifractal random walk, namely zero skewness and unrealistic small-time behaviour. However, the property in (36) does not appear to be time-consistent, since if we define $\eta^h_t := \mathbb{E}((\frac{X_{t+h}-X_t}{\sqrt{h}})^3|\mathcal{F}_t)$ for $t > 0$, then $\mathbb{E}((\eta^h_t)^2) = O(h^{-\gamma^2})$ (and not $O(1)$ as we would want), so we do not pursue this model further at the present time.

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A Definition and properties of $F_H(k)$ and $G_H(k)$ for the Sandwich lemma

$R_H(s,t) = \int_0^{s\wedge t}(s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du = \int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{1}{2}}du$ for $0 \le s \le t$, and note that the integrand is non-negative. Going forward we set $k = t - s$. We restrict $R_H(s,t)$ to $A_\delta := \{(s,t) : t-s = k, (s,t) \in [\tau, \tau+\delta]^2)$ with $k \in (0,\delta)$ and $\delta \in (0, 1-\tau)$, i.e. $R_H(s, k+s) = \int_0^s (u(k+u))^{H-\frac{1}{2}}du$. This expression is maximized at $s = \tau + \delta - k$ and minimized at $s = \tau$ for constant k (see Figure 2). Recall that $G_H(k) := R_H(\tau+\delta-k, \tau+\delta)$, we will now establish some basic properties of $G_H(k)$. From the analysis above: $G_H(k) = \int_0^{\tau+\delta-k}(u(k+u))^{H-\frac{1}{2}}du$. Taking the derivative with respect to k and using the Leibniz rule, we see that

$$G'_H(k) \quad = \quad -(\tau+\delta-k)^{H-\frac{1}{2}}(\tau+\delta)^{H-\frac{1}{2}} + (H-\frac{1}{2})\int_0^{\tau+\delta-k} u^{H-\frac{1}{2}}(k+u)^{H-\frac{3}{2}}du$$

which is negative (since $H < \frac{1}{2}$), so $G_H(k)$ is decreasing in k. The integral term in the previous equation explodes as $k \downarrow 0$:

$$\int_0^{\tau+\delta-k} u^{H-\frac{1}{2}}(k+u)^{H-\frac{3}{2}}du \quad \ge \quad \int_0^{\tau+\delta-k}(k+u)^{2H-2}du \quad = \quad \frac{(\tau+\delta)^{2H-1}}{2H-1} - \frac{k^{2H-1}}{2H-1} \quad \uparrow \quad \infty\,.$$

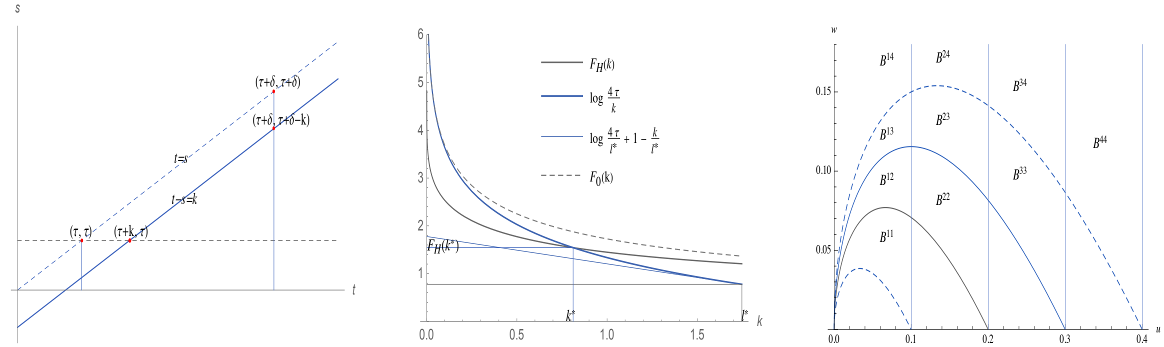

Figure 2: Left plot: $R(s,t)$ is maximized at $s = \tau + \delta - k$, and minimized at $s = \tau$. In the middle, we have plotted the various quantities appearing in the lower bound part of the proof of the Sandwich Lemma with $H = .1$, $\tau = .95$ (of course in practice we care about much lower H-values but it is clearer to see what is going on here for a larger H-value so the curves are not so close to each other). Note the blue dashed line is tangential to the grey line at $k = k^*$, and the blue line has steeper slope than the grey line at this point. On the right we we have plotted $g_H(s,t)$ for different t values for the RL process/field with $H = 0$ (left).

Hence $G'_H(k) \to -\infty$ as $k \searrow 0$. Conversely, if we fix k and let $H \to 0$, we find that

$$\begin{aligned} G_H(k) \quad &\uparrow \quad G_0(k) \;=\; \log\frac{1}{k} + 2\log(\sqrt{\tau+\delta-k} + \sqrt{\tau+\delta}) \qquad (H \to 0) \\ &\le \quad g(k) \;:=\; \log\frac{1}{k} + 2\log(2\sqrt{\tau+\delta}) \;=\; \log\frac{1}{k} + \log(4(\tau+\delta)) \end{aligned}$$

with equality at $k = 0$ in the sense that both sides of the inequality are infinite. Thus

$$G_H(k) \quad \le \quad G_0(k) \quad \le \quad g(k) \quad \le \quad \log\frac{4}{k} \tag{A-1}$$

since $\tau + \delta < 1$ by assumption.

Similarly, we recall that $F_H(k) := R_H(\tau, \tau+k) = \int_0^\tau (\tau-u)^{H-\frac{1}{2}}(\tau+k-u)^{H-\frac{1}{2}}du$, so

$$\begin{aligned} F'_H(k) &= (H-\frac{1}{2})\int_0^\tau (\tau-u)^{H-\frac{1}{2}}(\tau+k-u)^{H-\frac{3}{2}}du \quad \ge \quad (H-\frac{1}{2})\int_0^\tau (\tau-u)^{2H-2}du \\ F''_H(k) &= (H-\frac{1}{2})(H-\frac{3}{2})\int_0^\tau (\tau-u)^{H-\frac{1}{2}}(\tau+k-u)^{H-\frac{5}{2}}du \end{aligned}$$

so $F_H(k)$ is decreasing and convex in k, and $F'_H(k) \searrow -\infty$ as $k \searrow 0$. $F_H(k)$ increases pointwise as $H \downarrow 0$ to $F_0(k) := \log\frac{1}{k} + 2\log(\sqrt{\tau} + \sqrt{\tau+k})$. The second term is minimized at $k = 0$, so we define: $f(k) := \log\frac{4\tau}{k}$ and note that $f(k) < F_0(k)$.

B Monotonicity properties of $g_H(s,t)$

The covariance of the RL process for $s < t < 1$ is $R(s,t) = \int_0^s (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du = \int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{1}{2}}du$. Differentiating this expression using the Leibniz rule we see that $R_s(s,t) = s^{H-\frac{1}{2}}t^{H-\frac{1}{2}} + (\frac{1}{2}-H)\int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{3}{2}}du$ and recall that $g_H(s,t) = \frac{1}{R_s(s,t)}$. Then we can infer monotonicity properties of g from R_s:

- By inspection R_s is a decreasing function of t, so g is increasing in t.
- For $0 < s < t$, $(t-s+u)^{H-\frac{1}{2}}$ is a smooth function of u on $[0,s]$ so the integral term in our expression for R_s is finite $\forall t > 0$. Thus $R_s(s,t)$ tends to $+\infty$ as $s \to 0$ so $g_H(0,t) = 0$ for $t > 0$.
- For $s = t > 0$ the first term in (3) is finite but the integral diverges, so we also have $g_H(t,t) = 0$.
- For $s,t \in (0,1]^2$, $(st)^{H-\frac{1}{2}}$, $\frac{1}{2} - H$ and $u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{3}{2}}$ are non-negative and decreasing in H, so $g_H(s,t)$ is increasing in H.
- By inspection, $g_H(s,t)$ is continuous for $s \in [0,t]$, and performing a Taylor series expansion of $\frac{\partial}{\partial s}g_H(s,t)(s,t)$ we can show that $\frac{\partial}{\partial s}g_H(s,t) \to -\infty$ as $s \searrow 0$ and $s \nearrow t$.

These properties can be seen in the right plot in Figure 2.

C Proof of Proposition 4.3

We first recall that for any continuous martingale M, using Ito's lemma and integrating by parts we know that $\mathbb{E}(M_t^3) = 3\mathbb{E}(\int_0^t M_s d\langle M\rangle_s) = 3\mathbb{E}(M_t\langle M\rangle_t)$. Thus we see that

$$\begin{aligned}
& \mathbb{E}_{t_0}((\tilde{X}_T^H - \tilde{X}_{t_0}^H)^3) \\
= \; & 3\mathbb{E}_{t_0}((\tilde{X}_T^H - \tilde{X}_{t_0}^H)(\langle \tilde{X}_T^H\rangle - \langle \tilde{X}_{t_0}^H\rangle)) \\
= \; & 3\mathbb{E}_{t_0}(\int_{t_0}^T \rho(s)\sqrt{V_s^H}\, dB_s \cdot \int_{t_0}^T V_t^H dt) \\
= \; & 3\mathbb{E}_{t_0}(\int_{t_0}^T \rho(s)\xi_{t_0}^{\frac{1}{2}}(s)\, e^{\frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}}dB_u - \frac{1}{2}\cdot\frac{1}{2}\gamma^2\int_{t_0}^s (s-u)^{2H-1}du}\, dB_s \cdot \int_{t_0}^T \xi_{t_0}(t)\, e^{\gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}}dB_u - \frac{1}{2}\gamma^2\int_{t_0}^t (t-u)^{2H-1}du} dt)\,.
\end{aligned}$$

So we (formally) need to compute

$$\begin{aligned}
\delta I \;&=\; \mathbb{E}_{t_0}(e^{\frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}}dB_u - \frac{1}{2}\cdot\frac{1}{2}\gamma^2\int_{t_0}^s (s-u)^{2H-1}du} dB_s \cdot e^{\gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}}dB_u - \frac{1}{2}\gamma^2\int_{t_0}^t (t-u)^{2H-1}du}) \\
&=\; \mathbb{E}_{t_0}(e^{\gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}}dB_u + \frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}}dB_u - (\ldots)} dB_s)
\end{aligned}$$

where (...) refers to the non-random terms. To this end, let $X = \gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}}dB_u + \frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}}dB_u$ and $Y = dB_s$. Then $\mathbb{E}(XY) = \gamma(t-s)^{H-\frac{1}{2}}ds\, 1_{s<t}$ (since formally $\mathbb{E}(\frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}}dB_u \cdot dB_s) = 0$, see end of proof for discussion on how to make this argument rigorous) and

$$\begin{aligned}
\mathbb{E}(Ye^X) \;&=\; e^{\frac{1}{2}\mathbb{E}(X^2)}\mathbb{E}(XY) \;=\; e^{\frac{1}{2}V_H(s,t)}\gamma(t-s)^{H-\frac{1}{2}}ds\, 1_{s<t} \\
\Rightarrow \quad \delta I \;&=\; e^{-\frac{1}{2}\gamma^2\int_{t_0}^t (t-u)^{2H-1}du - \frac{1}{2}\cdot\frac{1}{2}\gamma^2\int_{t_0}^s (s-u)^{2H-1}du}\, e^{\frac{1}{2}V_H(s,t)}\gamma(t-s)^{H-\frac{1}{2}}ds\, 1_{s<t}
\end{aligned}$$

where $V_H(s,t) = \gamma^2\int_{t_0}^t [(t-u)^{H-\frac{1}{2}} + \frac{1}{2}(s-u)^{H-\frac{1}{2}}1_{s<t}]^2 du$. Cancelling terms in the exponent, we see that δI simplifies to

$$\begin{aligned}
\delta I \;&=\; e^{\frac{1}{2}\gamma^2\int_{t_0}^s (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du - \frac{1}{8}\gamma^2\int_{t_0}^s (s-u)^{2H-1}du)}(t-s)^{H-\frac{1}{2}}ds\,\gamma\, 1_{s<t} \\
&=\; e^{\frac{1}{2}\gamma^2\mathrm{Cov}_{t_0}(Z_s^H Z_t^H) - \frac{1}{8}\gamma^2\mathrm{Var}_{t_0}(Z_s^H))}\gamma(t-s)^{H-\frac{1}{2}}ds\, 1_{s<t}\,.
\end{aligned}$$

Then

$$\mathbb{E}_{t_0}((\tilde{X}_T^H - \tilde{X}_{t_0}^H)^3) \;=\; 3\mathbb{E}_{t_0}\int_{t_0}^T\int_{t_0}^T \rho(s)\xi_{t_0}^{\frac{1}{2}}(s)\,\xi_{t_0}(t)\delta I dt$$

and (32) and (33) follow. Finally we recall that a general stochastic integral $\int_0^t \phi_s dM_s$ with respect to a continuous martingale M is defined as an L^2- limit of $\int_0^t \phi_{\frac{1}{n}[ns]}dM_s$; using this construction we can rigourize the formal argument above with δI (we omit the tedious details for the sake of brevity).