The left curtain martingale coupling

sub/super hedge

We conjecture that if $X \le m$, then $Y = X$, otherwise $Y = T_u(X)$ with probability $q(X)$ or $T_d(X)$ with probability $q(X)$, with $T_d(x) \le x \le T_u(x)$, with $q(x) = \frac{x - T_d(x)}{T_u(x) - T_d(x)}$ to respect the martingale condition, and we guess that $T_d : (m, \infty) \to (-\infty, m)$ is decreasing and $T_u : (m, \infty) \to (m, \infty)$ increasing for the left curtain coupling.

- For $y > m$ with m in the support of ν, then $y = T_u(x)$ for some $x > m$. Then

  $$1 - F_\nu(y) \quad = \quad \mathbb{P}(Y > y) \quad = \quad \mathbb{P}(\mathbb{P}(Y > y|X)) \quad = \quad \mathbb{E}(q(X)1_{T_u(X)>y})$$

  so

  $$1 - F_\nu(T_u(x)) \quad = \quad \mathbb{E}(q(X)1_{T_u(X)>T_u(x)}) \quad = \quad \mathbb{E}(q(X)1_{X>x})$$

  and differentiating we find that

  $$\frac{d}{dx}F_\nu(T_u(x)) \quad = \quad T_u'(x)F_\nu'(T_u(x)) \quad = \quad q(x)F_\mu'(x)$$

  for $x > T_u^{-1}(m) = m$, so we see that $T_u'(x) > 0$ as required for the left curtain coupling.

- Conversely for $y \le m$, then $y = T_d(x)$ for some $x > m$, but there are two ways for Y to get to y: either we came from $X = y$, or $y = T_d(x)$ with $x > m$, so

  $$\begin{aligned} F_\nu(y) \quad = \quad \mathbb{P}(\mathbb{P}(Y \le y|X)) \quad &= \quad \mathbb{P}(X \le y) + \mathbb{E}((1 - q(X))1_{T_d(X)\le y}) \\ &= \quad F_\mu(y) + \mathbb{E}((1 - q(X))1_{T_d(X)\le y}) \end{aligned}$$

  so

  $$F_\nu(T_d(x)) - F_\mu(T_d(x)) \quad = \quad \mathbb{E}((1 - q(X))1_{T_d(X)\le T_d(x)}) \quad = \quad \mathbb{E}((1 - q(X))1_{X\ge x})$$

  (note the change of inequality in the final indicator since T_d is decreasing), then differentiating we find that

  $$\frac{d}{dx}(F_\nu(T_d(x)) - F_\mu(T_d(x))) \quad = \quad T_d'(x)(F_\nu'(T_d(x)) - F_\mu'(T_d(x))) \quad = \quad -(1 - q(x))F_\mu'(x) \quad (1)$$

  for $x > T_u^{-1}(m) = m$, so we have recovered Eqs 3.12 and 3.13 in [HT15], and we see that $T_d'(x) > 0$ in the region where $F_\nu' < F_\mu'$ (which is $[-1, 1]$ for the simple uniform example below). T_d is increasing?

Example

For the case when μ is $U[-1, 1]$ and ν is $U[-2, 2]$, turns out we don't need an m, so there is no F_μ' in (1), in which case $T_d'(x) < 0$, and we seemingly find that $T_u(x) = \frac{3}{2}(x + 1) - 1$, and $T_d(x) = -\frac{1}{2}(x + 1) - 1$.