

# The conditional law of the Bacry-Muzy and Riemann-Liouville log correlated Gaussian fields and their GMC, via Gaussian Hilbert and fractional Sobolev spaces

Martin Forde

Benjamin Smith\*

November 18, 2025

## Abstract

We compute  $\mathbb{E}(X_t|(X_s)_{0 \leq s \leq L})$  for the standard Bacry-Muzy log-correlated Gaussian field  $X$  with covariance  $\log^+ \frac{T}{|t-s|}$ , which corrects the finite-horizon prediction formula in Vargas et al.[DRV12]. The problem can be viewed as a linear filtering problem, and we solve the problem by showing that the  $L^2(\mathbb{P})$  closure of  $\{\int_{[0,L]} \phi(s)X_s ds : \phi \in \mathcal{S}, \text{supp}(\phi) \subseteq [0, L]\}$  is equal to  $\{X(\phi) : \phi \in H^{-\frac{1}{2}}, \text{supp}(\phi) \subseteq [0, L]\}$ , where  $X(\phi)$  is defined as a continuous linear extension of  $X$  acting on  $\mathcal{S} \subset H^s$ ,  $H^s$  denotes the fractional Sobolev space of order  $s$  and  $\mathbb{P}$  is the law of the field  $X$  on the space of tempered distributions. The explicit formula for the filter is obtained as the solution to a Fredholm integral equation of the first kind with logarithmic kernel. From this we characterize the conditional law of the Gaussian multiplicative chaos (GMC)  $M_\gamma$  generated by  $X$ , using that  $M_\gamma$  is measurable with respect to  $X$ . We also outline how one can adapt this result for the Riemann-Liouville GMC introduced in [FFGS19], which has a natural application to the Rough Bergomi volatility model in the  $H \rightarrow 0$  limit.<sup>1</sup>

## 1 Introduction

Originally pioneered by Kahane[Kah85], Gaussian multiplicative chaos (GMC) is a random measure on a domain of  $\mathbb{R}^d$  that can be formally written as

$$M_\gamma(dx) = e^{\gamma X_x - \frac{1}{2}\gamma^2 \mathbb{E}(X_x^2)} dx \quad (1)$$

where  $X$  is a Gaussian field with zero mean and covariance  $K(x, y) := \mathbb{E}(X_x X_y) = \log^+ \frac{1}{|y-x|} + g(x, y)$  for some bounded continuous function  $g$ .  $X$  is not defined pointwise because there is a singularity in its covariance, rather  $X$  is a random tempered distribution, i.e. an element of the dual of the Schwartz space  $\mathcal{S}$  under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of (1) requires a regularizing sequence  $X^\varepsilon$  of Gaussian processes (with the singularity removed, see e.g. [BBM13] and [BM03] for a description of such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular region, which is summarized in Section 2.3 in [FFGS19], or page 17 in [RV10] and section 3.4 in [Sha16] for a general method in  $\mathbb{R}^d$  using a convolution to smooth  $X$ ). In most of the literature on GMC, the choice of  $X^\varepsilon$  is a martingale in  $\varepsilon$ , from which we can then easily verify that  $M_\gamma^\varepsilon(A) = \int_A e^{\gamma X_x^\varepsilon - \frac{1}{2}\gamma^2 \text{Var}(X_x^\varepsilon)} dx$  is a martingale, and then obtain a.s. convergence of  $M_\gamma^\varepsilon(A)$  using the martingale convergence to a random variable  $M_\gamma(A)$  with  $\mathbb{E}(M_\gamma(A)) = \text{Leb}(A)$ , and with a bit more work we can verify that  $M_\gamma(\cdot)$  defines a random measure (see the end of Section 4 on page 18 in [RV10]).

If  $\gamma^2 < 2d$ ,  $M_\gamma^\varepsilon(dx) = e^{\gamma X_x^\varepsilon - \frac{1}{2}\gamma^2 \mathbb{E}((X_x^\varepsilon)^2)} dx$  tends weakly to a multifractal random measure  $M_\gamma$  with full support a.s. which satisfies the multifractal property

$$\mathbb{E}(M_\gamma([0, t])^q) = c_q t^{\zeta(q)} \quad (2)$$

for  $q \in (1, q^*)$  for some constant  $c_q = \mathbb{E}(M_\gamma([0, 1])^q)$ , where  $q^* = \frac{2}{\gamma^2 - 2}$  and

$$\zeta(q) = q - \frac{1}{2}\gamma^2(q^2 - q)$$

\*Dept. Mathematics, King's College London, Strand, London, WC2R 2LS (Benjamin.Smith@kcl.ac.uk)

<sup>1</sup>We would like to thank Juhan Aru, Janne Junnila, Vincent Vargas and Lauri Viitasaari for helpful discussions.

<sup>2</sup>see Lemma 3 in [BM03] to see why the critical  $q$  value is  $q^*$

and  $\mathbb{E}(M_\gamma([0, t])^q) = \infty$  if  $q > \frac{2}{\gamma^2}$ , see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). Moreover, we can show that the support of  $M_\gamma$  is a so-called  $\gamma$ -*thick points* of  $X$ , i.e. points such that  $\lim_{\varepsilon \rightarrow 0} \frac{X_\varepsilon}{\log \frac{1}{\varepsilon}} = \gamma$  (see e.g. section 2 in [Aru17], [Ber17] and page 7 in [RV16] for more on this), and for  $g \equiv 0$ , explicit expressions are known for the Mellin transform of the law of  $M_\gamma([0, 1])$  (see e.g. [Ost09], [Ost13], [Ost18]), which show that  $\log M_\gamma([0, 1])$  has an infinitely divisible law, and an explicit formula for sampling the law of the total mass of the GMC on the interval is given in [RZ17].

$M_\gamma$  is the zero measure for  $\gamma^2 = 2d$  and  $\gamma^2 > 2d$ ; in these cases a different re-normalization is required to obtain a non-trivial limit. Specifically, for  $\gamma^2 = 2d$ , we obtain a non-trivial limit by considering  $\sqrt{\log \frac{1}{\varepsilon}} \cdot M_\varepsilon^{\gamma=2}$  as  $\varepsilon \rightarrow 0$  or the “derivative measure”  $\frac{d}{d\gamma} e^{\gamma X_\varepsilon - \frac{1}{2} \gamma^2 \text{Var}(X_\varepsilon)}|_{\gamma=\sqrt{2d}}$ . [DRSV14] show that both these objects tend weakly to the same measure  $\mu'$  as  $\varepsilon \rightarrow 0$ , and in 2d Aru et al. [APS19] have shown that  $\frac{M_\gamma}{2-\gamma} \rightarrow 2\mu'$  in probability as  $\gamma$  tends to the critical value of 2, and the critical  $\gamma$ -value is particularly important in Liouville quantum gravity (again see [DRSV14] for further discussion). One can also construct a GMC for the super-critical phase, using an independent stable subordinator time-changed by a sub-critical GMC (see section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for  $\gamma$ -values greater than  $\sqrt{2}$ , which is closely related to the non-standard branch of gravity in conformal field theory.

In the sub-critical case, using a limiting argument it can be shown that  $M_\gamma$  satisfies the “master equations”:  $M(X + f, dz) = e^{\gamma \int f(z)} M(X, dz)$  and  $\mathbb{E}(\int_D F(X, z) M_\gamma(dz)) = \mathbb{E}(\int_D F(X + \gamma K(z, \cdot), z) dz)$  for any measurable function  $F$  and any interval  $D$ , which comes from the Cameron-Martin theorem for Gaussian measures and the notion of *rooted measures* and the disintegration theorem (see section 2.1 in [Aru17] for a nice discussion on this). Moreover, either of these two can equations can be taken as the definition of GMC, and they uniquely determine  $M_\gamma$  as a measurable function of  $X$ , and hence also uniquely fix its law (see also Theorem 6 in Shamov [Sha16]).

GMC also has a natural and important application in Liouville Quantum Field Theory; LQFT is a 2d model of random surfaces, which (formally) we can view as a random metric in the context of quantum gravity, where we weight the classical free field action with an interaction term given by the exponential of a GMC and can be viewed as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time.

## 2 Construction of the standard Bacry-Muzy GMC on the line

Define the Gaussian process  $\omega_\varepsilon(t)$  as in Eq 7 in [BBM13] with  $\lambda = 1$  (except here use  $\varepsilon$  instead of  $l$ ), and set  $X_t^\varepsilon := \omega_\varepsilon(t) - \mathbb{E}(\omega_\varepsilon(t))$ , so

$$X_t^\varepsilon = \int_{(u,s) \in \mathcal{A}_\varepsilon(t)} dW(u, s) \quad (3)$$

where  $dW(u, s)$  is 2-dimensional Gaussian white noise with variance  $s^{-2} du ds$ , and  $\mathcal{A}_\varepsilon(t)$  is triangular region defined in Eq 8 (and Figure 1) in [BBM13]. Then

$$R_\varepsilon(s, t) := \mathbb{E}(X_s^\varepsilon X_t^\varepsilon) = \begin{cases} \log \frac{T}{\tau} & \varepsilon \leq \tau \leq T \\ \log \frac{T}{\varepsilon} + 1 - \frac{\tau}{\varepsilon} & \tau \leq \varepsilon \\ 0 & \tau > T \end{cases} \quad (4)$$

where  $\tau = |t - s|$  (see Eq 10 in [BBM13]), and one can easily verify that

$$R_\varepsilon(s, t) \leq \log \frac{T}{\tau} = R(s, t) \quad (5)$$

for  $s, t \in [0, T]$  (see Eq 25 in [BM03]). Using (3), we also see that

$$\mathbb{E}(X_t^\varepsilon X_s^{\varepsilon'}) = \mathbb{E}\left(\int_{(u,v) \in \mathcal{A}_\varepsilon(t)} dW(u, v) \int_{(u,v) \in \mathcal{A}_{\varepsilon'}(s)} dW(u, v)\right) = \int_{\mathcal{A}_\varepsilon(t) \cap \mathcal{A}_{\varepsilon'}(s)} \frac{1}{v^2} du dv = \mathbb{E}(X_s^\varepsilon X_t^\varepsilon)$$

for  $0 < \varepsilon' \leq \varepsilon$  (i.e. the answer does not depend on  $\varepsilon'$ ). We now define the measure

$$M_\gamma^\varepsilon(dt) = e^{\gamma X_t^\varepsilon - \frac{1}{2} \gamma^2 \text{Var}(X_t^\varepsilon)} dt.$$

One can easily verify that  $M_\gamma^\varepsilon(A)$  is a backwards martingale with respect to the filtration  $\mathcal{F}_\varepsilon := \sigma(W(A, B) : A \subset \mathbb{R}^+, B \subseteq [\varepsilon, \infty])$  (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and

$$\sup_{\varepsilon > 0} \mathbb{E}(M_\gamma^\varepsilon(A)^q) < \infty \quad (6)$$

(Lemma 3 i) in [BM03]), so from the martingale convergence theorem,  $M_\gamma^\varepsilon(A)$  converges to some random variable  $M_\gamma(A)$  in  $L^q$  for  $q \in (1, q^*)$ , and from the reverse triangle inequality this implies that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}((M_\gamma^\varepsilon(A))^q) = \mathbb{E}(M_\gamma(A)^q) \quad (7)$$

Moreover, one can show that  $M_\gamma(\cdot)$  defines measure (see e.g. end of Section 4 on page 18 in [RV10]), and since  $M_\gamma^\varepsilon(A) \rightarrow M_\gamma(A)$  a.s. for any Borel set  $A$  this implies weak convergence of  $M_\gamma^\varepsilon$  to  $M_\gamma$  a.s. (from e.g. Theorem 3.1 parts a) and f) in Ethier&Kurtz[EK86]).

Moreover  $M_\gamma$  is multifractal, i.e.  $\mathbb{E}(|M_\gamma([0, t])|^q) = c_{q,T} t^{\zeta(q)}$  (see e.g. Lemma 4 in [BM03]) for some finite constant  $c_{q,T} > 0$ , depending only on  $q$  and  $T$ . For integer  $q \geq 1$ , we also note that

$$\begin{aligned} \mathbb{E}(M_\gamma(A)^q) &= \int_A \dots \int_A e^{\gamma^2 \sum_{1 \leq i < j \leq q} \log \frac{T}{|u_i - u_j|}} du_i \dots du_q \\ &= \int_A \dots \int_A e^{\gamma^2 q(q-1) \log T + \sum_{1 \leq i < j \leq q} \log \frac{1}{|u_i - u_j|}} du_i \dots du_q \\ &= T^{\gamma^2 q(q-1)} \int_A \dots \int_A e^{\sum_{1 \leq i < j \leq q} \log \frac{1}{|u_i - u_j|}} du_i \dots du_q \end{aligned} \quad (8)$$

so we see that

$$c_{q,T} = c_q T^{\gamma^2 q(q-1)}$$

where  $c_q := c_{q,1}$ , and this also holds for non-integer  $q$  (see e.g. Theorem 3.16 in [Koz06]).

### 3 The conditional law of the standard log correlated Gaussian field

Consider a standard log-correlated Gaussian field  $Z$  on  $\mathbb{R}$  with covariance  $R(s, t) = \log^+ \frac{T}{|t-s|}$ . From the Minlos-Bochner theorem, we know that the law of  $Z$  is a Gaussian measure on the space  $\mathcal{S}'$  of *tempered distributions* (see e.g. [DRSV17] and Appendix A in [FFGS19] for more details on tempered distributions) which is the dual of the Schwartz space  $\mathcal{S}$  (see e.g. section 2.2 in [DRSV14] and Theorem 2.1 in [BDW17]). Moreover,  $\mathcal{S}$  is a Montel space and thus is reflexive, i.e.  $(\mathcal{S}')'$  is isomorphic to  $\mathcal{S}$  using the canonical embedding of  $\mathcal{S}$  into its bi-dual  $(\mathcal{S}')'$ . From here on, we are only concerned with the restriction of  $Z$  to  $[0, T]$  (on which the covariance of  $Z$  is just  $\log \frac{T}{|t-s|}$ , so we set  $Z$  equal to zero outside this interval for simplicity).

**Proposition 3.1**  *$X^\varepsilon$  tends to  $X$  in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definitions), where  $X$  has the same law as the field  $Z$  defined above.*

**Proof.**  $0 \leq R_\varepsilon(s, t) \leq R(s, t)$  for  $s, t \in [0, T]$  (see (5)), so from the dominated convergence theorem, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{[0, T]^2} \phi_1(s) \phi_2(t) R_\varepsilon(s, t) ds dt = \int_{[0, T]^2} \phi_1(s) \phi_2(t) R(s, t) ds dt \quad (9)$$

for any  $\phi_1, \phi_2 \in \mathcal{S}$ , where  $R_\varepsilon(s, t)$  is defined as in (4). Similarly, for any sequence  $\phi_k \in \mathcal{S}$  with  $\|\phi_k\|_{m,j} \rightarrow 0$  for all  $m, j \in \mathbb{N}_0^n$  for any  $n \in \mathbb{N}$  (i.e. under the Schwartz space semi-norm defined in Eq 1 in [BDW17])

$$\lim_{k \rightarrow \infty} \int_{[0, T]^2} \phi_k(s) \phi_k(t) R(s, t) ds dt = 0 \quad (10)$$

since  $\nu(A) := \int_A R(s, t) ds dt$  is a bounded non-negative measure (since  $\int_0^T \int_0^t R(s, t) ds dt < \infty$ ), and the convergence here implies in particular that  $\phi_k$  tends to  $\phi$  pointwise, so we can use the bounded convergence theorem. Thus if we define

$$\begin{aligned} \mathcal{L}_{X^\varepsilon}(f) &:= \mathbb{E}(e^{i(f, X^\varepsilon)}) = e^{-\frac{1}{2} \int_{[0, T]^2} f(s) f(t) R_\varepsilon(s, t) ds dt} \\ \mathcal{L}(f) &:= e^{-\frac{1}{2} \int_{[0, T]^2} f(s) f(t) R(s, t) ds dt} \end{aligned}$$

for  $f \in \mathcal{S}$ , then from (9) and (10) and Lévy's continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW17]), we see that  $\mathcal{L}_{X^\varepsilon}(f)$  tends to  $\mathcal{L}(f)$  pointwise and  $\mathcal{L}(\cdot)$  is continuous at zero, then there exists a generalized random field

$X$  (i.e. a random tempered distribution, such that  $L_X = L$  and  $X^\varepsilon$  tends to  $X$  in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definition). ■

In general, the conditional expectation of a random variable is equal to its projection onto the Gaussian Hilbert space (sub-Hilbert space of  $L^2(\Omega, \mathcal{F}, \mathbb{P})$ ) generated by the variables on which we are conditioning. To this end, we let  $\bar{F}$  denote the Hilbert space given by the  $L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$  closure of

$$F = \{X(\phi) : \phi \in \mathcal{S}, \text{supp}(\phi) \subseteq [0, L]\}$$

where  $\mathcal{F}_L = \sigma((X_u)_{0 \leq u \leq L})$ . The closure here is necessary because the notion of orthogonal projection requires the Hilbert space structure, and there is no guarantee that the conditional expectation  $\mathbb{E}(X(\psi)|\mathcal{F}_L)$  will be a random variable of the form  $\int_{[0, L]} X_s \phi(s) ds$  with  $\phi \in \mathcal{S}$ .

In order to characterize  $\bar{F}$ , we first note that

$$\mathbb{E}[(\int X_s \phi(s) ds)^2] = \int \int R(s, t) \phi(s) \phi(t) ds dt.$$

From Eqs 2.1 in [DRV12], we also know that

$$c \|\phi\|_{H^{-\frac{1}{2}}}^2 \leq \int \int R(s, t) \phi(s) \phi(t) ds dt \leq C \|\phi\|_{H^{-\frac{1}{2}}}^2 \quad (11)$$

where  $0 < c < C < \infty$ . Let  $H^s$  denotes the fractional Sobolev space of order  $s$  (see e.g. Section 2.2 in [JSW18] for definitions). Then we can put two inner products on the linear space  $\mathcal{S}$  of Schwarz functions:

1.  $\langle \phi, \psi \rangle_{H^{-\frac{1}{2}}} := \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}} \hat{\phi}(k) \bar{\hat{\psi}}(k) dk$  (i.e. the standard inner product on  $H^{-\frac{1}{2}}$ )
2.  $\langle \phi, \psi \rangle := \mathbb{E}[X(\phi)X(\psi)] = \int \int \phi(s) \psi(t) R(s, t) ds dt$

Eq 2.2 in [DRV12] shows that these two inner products are equivalent and thus generate the same topologies on  $\mathcal{S}$ .

We now make the following observations:

- Let  $\phi \in H^{-\frac{1}{2}}$ , with  $\text{supp}(\phi) \subseteq [0, L]$ .  $\mathcal{S}$  is dense in  $H^{-\frac{1}{2}}$ , so there exists a sequence  $\phi_n \in \mathcal{S}$  with  $\text{supp}(\phi_n) \subseteq [0, L]$  such that  $\|\phi_n - \phi\|_{H^{-\frac{1}{2}}} \rightarrow 0$ , and  $\phi$  is a Cauchy sequence in  $H^{-\frac{1}{2}}$  so (by the equivalence of norms)  $X(\phi_n)$  is a Cauchy sequence in  $\bar{F}$ , and thus converges to some  $Y$  in  $\bar{F}$ . This defines  $X(\phi) := Y$  as a continuous linear extension of  $X$  from  $\mathcal{S}$  to the larger space  $H^{-\frac{1}{2}}$ , which we will also often write as  $\int \phi(t) X_t dt$ . To check that  $X(\phi)$  is uniquely specified, consider two such sequences  $\phi_n$  and  $\phi'_n$ . Then from the triangle inequality

$$\|\phi_n - \phi'_n\|_{H^{-\frac{1}{2}}} \leq \|\phi_n - \phi\|_{H^{-\frac{1}{2}}} + \|\phi - \phi'_n\|_{H^{-\frac{1}{2}}} \rightarrow 0$$

and thus (by the equivalence of norms) we have  $\|X(\phi_n) - X(\phi'_n)\|_{L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})} = \|X(\phi_n) - X(\phi'_n)\|_{\bar{F}} \rightarrow 0$ .

- Conversely, for any  $Z \in \bar{F}$ , there exists a sequence  $\phi_n \in \mathcal{S}$  such that  $X(\phi_n)$  converges to  $Z \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$ , so  $\phi_n$  is a Cauchy sequence with respect to the second norm defined above, and hence also a Cauchy sequence with respect to the  $H^{-\frac{1}{2}}$  norm (by the equivalence of the two norms).  $H^{-\frac{1}{2}}$  is a Hilbert space so Cauchy sequences in  $H^{-\frac{1}{2}}$  converge i.e. there exists a  $\phi \in H^{-\frac{1}{2}}$  such that  $\phi_n \rightarrow \phi \in H^{-\frac{1}{2}}$ .

Thus we have shown that

$$\bar{F} = \{X(\phi) : \phi \in H^{-\frac{1}{2}}, \text{supp}(\phi) \subseteq [0, L]\}$$

where we are using the extension of  $X$  to  $H^{-\frac{1}{2}}$  on the right hand side here as defined in the first bullet point above.

Moreover (since  $\mathbb{E}(X(\psi)|\mathcal{F}_L) \in \bar{F}$ ) we see that for any  $\psi \in \mathcal{S}$

$$\mathbb{E}(X(\psi)|\mathcal{F}_L) = \int_{[0, L]} X_s k_\psi(s) ds := X(k_\psi)$$

for some  $k_\psi(s) \in H^{-\frac{1}{2}}([0, L])$ , where  $X(\cdot)$  in the final expression is the linear extension we have just defined. This analysis shows that  $\bar{F}$  is isometrically isomorphic to the set of functions in  $H^{-\frac{1}{2}}$  with support in  $[0, L]$ .

Moreover, we can now extend the inner product to  $H^{-\frac{1}{2}}$  as

$$\langle \phi, \psi \rangle = \lim_{n \rightarrow \infty} \mathbb{E}[X(\phi_n)X(\psi_n)] = \lim_{n \rightarrow \infty} \int \int \phi_n(s)\psi_n(t)R(s, t)dsdt$$

where  $\phi_n, \psi_n \in \mathcal{S}$  and  $\phi_n \rightarrow \phi$  in  $H^{-\frac{1}{2}}$  and  $\psi_n \rightarrow \psi$  in  $H^{-\frac{1}{2}}$ .

**Proposition 3.2**  $X \in H^{-\frac{1}{2}-\delta}$  a.s. for any  $\delta > 0$ .

**Proof.** The proof is almost identical to Proposition 2.1 in [FFGS19], but since some of its arguments are needed for the next Proposition as well, we have put a proof in Appendix A. ■

**Remark 3.1** One can actually show the stronger result that  $X \in H^{-\delta} \subset H^{-\frac{1}{2}}$  a.s. for any  $\delta > 0$ , but we will not need this here (see also [BDW17]).

**Proposition 3.3**  $X^\varepsilon \rightarrow X$  in  $H^{-\frac{1}{2}-\delta}$  in probability for any  $\delta > 0$ , where  $X^\varepsilon$  is defined as in (3).

**Proof.** See Appendix B. ■

We know that for any  $\psi \in \mathcal{S}$  with  $\text{supp}(\psi) \subseteq [L, T]$ , the conditional expectation  $\mathbb{E}(X(\psi)|\mathcal{F}_L) = X(k_\psi)$  minimizes

$$\mathbb{E}((X(\psi) - Y)^2)$$

over all  $Y \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$ , and  $\mathbb{E}((\int_{[L, T]} X_t \psi(t) dt - \mathbb{E}(\int_{[L, T]} X_t \psi(t) dt | \mathcal{F}_L))Z) = 0$  for all  $Z \in \mathcal{F}_L$ , so in particular setting  $Z = \int_{[0, L]} \psi_2(s)X_s ds$  for  $\psi_2 \in \mathcal{S}$  with  $\text{supp}(\psi_2) \subseteq [0, L]$ , we see that

$$\begin{aligned} 0 &= \mathbb{E}((X(\psi) - X(k_\psi))X(\psi_2)) \\ &= \mathbb{E}((\int_{[L, T]} \psi(t)X_t dt - \int_{[0, L]} k_\psi(u)X_u du) \int_{[0, L]} \psi_2(s)X_s ds) \\ &= \int_{[L, T]} \int_{[0, L]} \psi(t)\psi_2(s)R(t-s)dsdt - \int_{[L, T]} \int_{[0, L]} R(s-u)k_\psi(u)\psi_2(s)duds. \end{aligned} \quad (12)$$

In (14) below we construct an explicit solution  $k_t(\cdot)$  to

$$0 = \mathbb{E}((X_t - \int_{[0, L]} k(u)X_u du)X_s) = R(s, t) - \int_{[0, L]} R(u, s)k_t(u)du \quad (13)$$

for  $s \in [0, L]$ , with  $k_t \in \text{supp}(\psi) \subseteq [t, T]$ , which implies that (12) holds if we set  $k_\psi(u) = \int_{[L, T]} \psi(t)k_t(u)dt$ .

**Proposition 3.4** The covariance operator  $R\phi = \int_0^T R(s, t)\phi(s)ds$  acting on  $H^{-\frac{1}{2}}$  is positive definite, and  $\int_0^T R(s, t)\phi(s)ds = 0$  if and only if  $\phi \equiv 0$  Lebesgue a.e.

**Proof.** From the discussion on page 4, we know that bilinear form  $R$  is (up to an equivalence) the inner product on  $H^{-\frac{1}{2}}$  so it has to be positive definite (from the definition of a norm), and thus  $\int_0^T R(s, t)\phi(s)ds \neq 0$  if  $\phi \neq 0$ , since otherwise  $R(\phi, \phi) = \int_0^T \int_0^T R(s, t)\phi(s)ds\phi(t)dt = 0$ . ■

The integral equation in (13) (with  $t$  fixed) is the well known *Wiener-Hopf equation*. We refer the reader to [Poor94] for more details on the Wiener-Hopf equation in the context of ordinary Gaussian processes.

**Corollary 3.5** Proposition 3.4 shows that the Wiener-Hopf equation in (13) has a unique solution.

If  $t \leq T$  (so we can replace  $\log^+$  with  $\log$ ), we can re-write (12) as

$$\int_{[0, L]} k_t(u) \log \frac{T}{|s-u|} du = f(s) := \log \frac{T}{t-s}$$

and we see that this is now a Fredholm integral equation of the 1st kind with logarithmic kernel, which can be solved explicitly by a minor extension of page 299 in [EK00] (who consider  $T = 1$ ) to give

$$k_t(u) = \frac{1}{\pi^2} \int_0^L \frac{\sqrt{v(L-v)}}{\sqrt{u(L-u)}} \frac{f'(v)}{u-v} dv + \frac{c_t}{\pi \sqrt{u(L-u)}} = \frac{(c_t-1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}} \quad (14)$$

where the integral in the second expression is understood in the principal value sense, and

$$c_t = \int_0^L k_t(u) du = \frac{1}{\pi(\log(\frac{1}{4}L) - \log T)} \int_0^L \frac{\log \frac{t-v}{T}}{\sqrt{v(L-v)}} dv < \infty.$$

We now verify that  $k_\psi(u) \in H^{-\frac{1}{2}}$ . To this end, we first note that

$$\pi \log \frac{L}{4} - \pi \log T = \int_0^L \frac{\log \frac{L-v}{T}}{\sqrt{v(L-v)}} dv \leq \int_0^L \frac{\log \frac{t-v}{T}}{\sqrt{v(L-v)}} dv \leq \int_0^L \frac{\log \frac{t}{T}}{\sqrt{v(L-v)}} dv \leq \pi \log t - \pi \log T.$$

$$\frac{(c_t - 1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}} 1_{u \in [0, L]} 1_{t \in [L, T]} \leq h(u, t) = \frac{c_1}{(t-u)\sqrt{u(L-u)}} 1_{u \in [0, L]} 1_{t \in [L, T]}$$

for some constant  $c_1$ . We know that

$$\int_{[L, T]} \left( \int_{[0, L]} |\psi(t)h(u, t)|^p du \right)^{\frac{1}{p}} dt \leq \|\psi\|_{L^\infty} \int_{[L, T]} \left( \int_{[0, L]} |h(u, t)|^p du \right)^{\frac{1}{p}} dt \quad (15)$$

and setting  $p = \frac{3}{2}$  we find that

$$\int_{[0, L]} |h(u, t)|^p du = G(t) := \text{const.} \times \frac{2t - L}{t(t-L)^{\frac{5}{4}}}$$

which implies that

$$\int_{[L, T]} G(t)^{\frac{1}{p}} dt < \infty$$

so for  $p = \frac{3}{2}$  the double integral in (15) is finite, so (from the Minkowski integral inequality)  $\int_{[L, T]} h(\cdot, t) dt$  and thus  $\int_{[L, T]} \psi(t) k_t(\cdot) dt \in L^p$ , and hence its Fourier transform is in  $L^q = L^3$  where  $1/p + 1/q = 1$ , and thus is  $O(|\xi|^{-\frac{1}{3}-\varepsilon})$  for  $\xi \gg 1$  and  $O(|\xi|^{-\frac{1}{3}+\varepsilon})$  for  $\xi \ll 1$ .

Hence

$$\begin{aligned} \|k_\psi\|_{H^{-\frac{1}{2}}} &= \int_{-\infty}^{\infty} (1 + |\xi^2|)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\xi u} \int_{[L, T]} \psi(t) k_t(u) 1_{u \in [0, L]} dt du d\xi \\ &= \int_{-\infty}^{\infty} (1 + |\xi^2|)^{-\frac{1}{2}} \int_{[0, L]} e^{i\xi u} \int_{[L, T]} \psi(t) k_t(u) dt du d\xi < \infty \end{aligned}$$

which verifies the validity of our explicit solution for  $k_u(t)$ .

**Remark 3.2** Corollary 3.3 in [DRV12] gives the following nice prediction formula for a log-correlated Gaussian field  $X$  with covariance  $\log \frac{T}{|t-s|}$ :<sup>3</sup>

$$\mathbb{E}(X_t | (X_s)_{s \leq 0}) = \frac{1}{\pi} \int_{-\infty}^0 \frac{\sqrt{t}}{(t-s)\sqrt{-s}} X_s ds$$

which we can verify satisfies the associated Wiener-Hopf equation (and is also very similar to the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the  $H \rightarrow 0$  limit). However the prediction formula for the finite history case stated in Theorem 3.5 in [DRV12] appears to be wrong since numerical tests confirm that it does not satisfy the Wiener-Hopf equation. Our linear filter  $\int_{[0, L]} k_t(u) X_u du$  corrects this formula for the case when  $L + t \leq T$ .

**Remark 3.3** Clearly if  $t - L > T$ , the history of  $X$  over  $[0, L]$  is of no use for prediction since in this case  $\mathbb{E}(X_s X_t) = 0$  for  $s \in [0, L]$ , and the conditioned process then has the same law as the unconditioned process.

---

<sup>3</sup>i.e.  $\log$  not  $\log^+$

### 3.1 The conditional covariance

We use  $\mathbb{E}_L(\cdot)$  as shorthand for  $\mathbb{E}(\cdot | (X_u)_{0 \leq u \leq L})$ . Then from the tower property we see that

$$\begin{aligned} & \mathbb{E}((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \\ &= \mathbb{E}(\mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s)))) \\ &= \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \end{aligned}$$

and the final equality follows since the conditional covariance of a Gaussian process or field is deterministic, and does not depend on its history. Given  $k_t(u)$ , we can now compute the conditional covariance in the final line explicitly (for  $s, t \in [L, T]$ ) as

$$\begin{aligned} R_L(s, t) &:= \mathbb{E}_L((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \\ &= \mathbb{E}((X_t - \mathbb{E}_L(X_t))(X_s - \mathbb{E}_L(X_s))) \\ &= \mathbb{E}((X_t - \int_{[0, L]} k_t(u) X_u du)(X_s - \int_{[0, L]} k_s(v) X_v dv)) \\ &= R(s, t) - \int_{[0, L]} k_t(u) R(u, s) du - \int_{[0, L]} k_s(v) R(v, t) dv + \int_{[0, L]} \int_{[0, L]} k_t(u) k_s(v) R(u, v) dudv. \end{aligned}$$

## 4 Application to Gaussian multiplicative chaos

### 4.1 Rooted measures

**Proposition 4.1** (see also Lemma 2.1 in [Aru17] and Theorems 4 and 17 in [Sha16]). We have the following “master equation” for any bounded continuous function  $F$  on  $H^{-\frac{1}{2}-\delta} \times [0, T]$  (under the product topology induced by the Hilbert space norm on  $H^{-\frac{1}{2}-\delta}$  and the usual Euclidean metric on  $[0, T]$ ):

$$\frac{1}{T} \mathbb{E}(\int_0^T F(X, t) M_\gamma(dt)) = \frac{1}{T} \mathbb{E}(\int_0^T F(X + \gamma R(t, \cdot), t) dt). \quad (16)$$

**Proof.** See Appendix C. <sup>4</sup> ■

**Corollary 4.2**  $M_\gamma$  is measurable with respect to  $X$ .

**Proof.**  $\mathcal{H} = H^{-\frac{1}{2}-\delta} \times [0, T]$  is a metric space, so if  $\mu$  and  $\nu$  are two finite Borel measures on  $\mathcal{H}$  then  $\int f d\mu = \int f d\nu$  for all  $f \in C_b(\mathcal{H})$  means that  $\mu = \nu$ , so the left hand side of (16) uniquely defines a measure  $\mathbb{P}^*$  on  $\mathcal{H} \times [0, T]$  which satisfies

$$\frac{1}{T} \mathbb{E}(\int_0^T F(X, t) M_\gamma(dt)) = \int \int F(\omega, t) \mathbb{P}^*(d\omega, dt)$$

where

$$\begin{aligned} \mathbb{P}^*(d\omega, dt) &:= \frac{1}{T} \mathbb{E}(1_{X \in d\omega} M_\gamma(\omega, ds)) = \frac{1}{T} \mathbb{Q}^X(d\omega) M_\gamma(\omega, dt) = \frac{1}{T} \mathbb{E}(1_{X + \gamma R(t, \cdot) \in d\omega}) dt \\ &= \mathbb{P}(X + \gamma R(t, \cdot) \in d\omega) \frac{1}{T} dt \end{aligned}$$

where  $\mathbb{Q}^X$  denotes the law of  $X$  on  $H^{-\frac{1}{2}-\delta}$ .

Moreover, if  $F \equiv 1$ ,  $\frac{1}{T} \mathbb{E}(\int_0^T F(X, t) M_\gamma(dt)) = 1$ , so  $\mathbb{P}^*(d\omega, dt)$  is a probability measure, known as a *rooted* or *Peyri re measure* (see [Aru17] and [Sha16] for more on this). Moreover, using a similar argument to the third bullet point in Appendix C, we know that the conditional law of  $\mathbb{P}^*$  given  $X$  is  $M_\gamma(dt)/M_\gamma([0, T])$  and from the disintegration theorem, we know that this (probability) measure is a measurable with respect to  $X$ . Then using a similar argument to the second bullet point in Appendix C, if we take the sample space  $\Omega$  to be  $H^{-\frac{1}{2}-\delta}$  with  $\sigma$ -algebra  $\sigma(H^{-\frac{1}{2}-\delta})$ , then the “tilted” probability measure  $\mathbb{Q}_\gamma^X(d\omega) := \frac{1}{T} M_\gamma([0, T]) \mathbb{Q}^X(d\omega)$  on  $(\Omega, \mathcal{F})$  is the marginal law of  $\mathbb{P}^*$  on  $H^{-\frac{1}{2}-\delta}$  (where  $\mathbb{Q}^X$  is the law of  $X$  on  $H^{-\frac{1}{2}-\delta}$ ) and  $\mathbb{Q}_\gamma^X \ll \mathbb{Q}^X$ , so  $\frac{1}{T} M_\gamma([0, T])(\omega)$  is the (a.s.) unique Radon-Nikodym derivative of  $\mathbb{Q}_\gamma^X$  with respect to  $\mathbb{Q}^X$ , which is a measurable function of  $\omega$ . Thus we have shown that  $M_\gamma(dt)/M_\gamma([0, T])$  and  $M_\gamma([0, T])$  are measurable wrt  $X$  and thus so is  $M_\gamma$ . ■

<sup>4</sup>We thank Juhan Aru for his help with multiple parts of this proof.

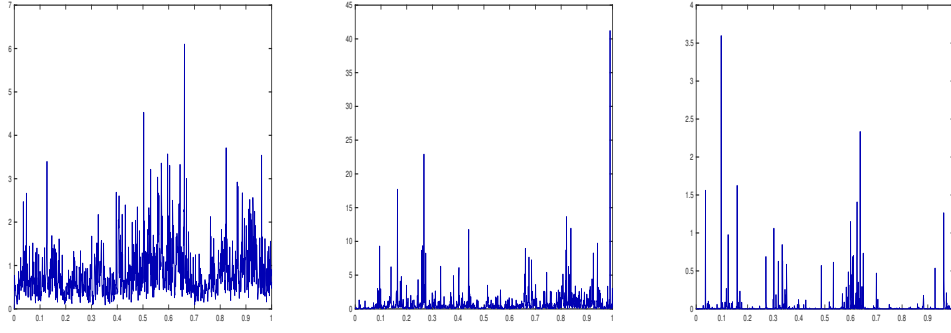


Figure 1: Here we have plotted a Monte Carlo simulation of the multifractal random measure  $M_\gamma(dt)$  on  $[0, 1]$  with  $\gamma = 0.20, 0.45$  and  $1$  using the regularized autocovariance  $\log^+ \frac{T}{|t|+\varepsilon}$  for  $\varepsilon = .000001$ , and we see greater intermittency as  $\gamma$  increases.

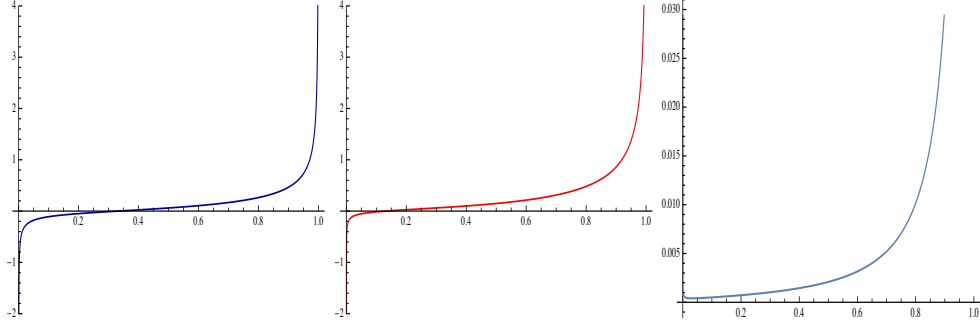


Figure 2: In the first three graphs we have plotted the optimal linear filter  $k(u)$  in (14) associated with the multifractal random walk with  $L = 1$ ,  $T = 2$  for  $t = 2, 1.5$  and  $1.00001$  respectively, and the numerics confirm that the Wiener-Hopf equation is satisfied (Mathematica code available on request), and  $k(u)$  is U-shaped and strictly positive for all  $u \in [0, L]$  for  $t$  sufficiently small

## 4.2 The conditional law of $M_\gamma$

From the Corollary above,  $M_\gamma(dt)$  is a measurable wrt  $X$ , so  $M_\gamma$  given  $(X)_{0 \leq s \leq L}$  is just obtained as

$$M_\gamma((X)_{0 \leq s \leq L} \oplus X', dt) \quad (17)$$

where  $\oplus$  denotes concatenation, and  $X'$  is a Gaussian field (which is also a random element of  $\mathcal{S}'$ ) on  $[L, T]$  with mean  $\mathbb{E}_L(X_t)$  and covariance  $R_L(s, t)$ . This then uniquely specifies the law of  $M_\gamma$  conditioned on its history over  $[0, L]$ .

## 4.3 Conditional law of the Riemann-Liouville field

Formally letting  $H \rightarrow 0$  in the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the  $H \rightarrow 0$  limit, we obtain the following conditional law for the Riemann-Liouville field  $Z$  defined in section 2 in [FFGS19]:

**Proposition 4.3**  *$Z$  has conditional mean and covariance given by*

$$\begin{aligned} \mathbb{E}(Z_u | (Z_v)_{0 \leq v \leq t}) &= \int_0^t \bar{k}(s) Z_s ds \\ \text{Cov}(Z_s, Z_u | (Z_v)_{0 \leq v \leq t}) &= \int_t^{s \wedge u} (u-v)^{-\frac{1}{2}} (s-v)^{-\frac{1}{2}} dv \end{aligned} \quad (18)$$

for  $u \geq t$ , where  $\bar{k}(s) = \frac{1}{\pi} \left( \frac{u-t}{t-s} \right)^{\frac{1}{2}} \frac{1}{u-s}$ .

**Remark 4.1** This is essentially the same type of linear filter that we have obtained in section 3 for the Bacry-Muzy field. To make this rigorous, we can consider  $Y_t = e^{Z_t}$ ; then one can verify that



$Y$  is a strictly stationary Gaussian field with covariance  $R_Y(s, t) = R(\tau) := 2 \tanh^{-1}(e^{-\frac{1}{2}|\tau|})$  where  $\tau = t - s$ , and from Parseval's theorem (similar to Eq 2.1 in [DRV12]) we obtain

$$\int \int \phi(t) \phi(s) R_Y(s, t) ds dt = \int \hat{R}(k) |\hat{\phi}(k)|^2 dk$$

where  $\hat{R}(k) = \frac{-iH_{-\frac{1}{2}-ik} + iH_{-\frac{1}{2}+ik} + 2\pi \tanh(k\pi)}{k\sqrt{2\pi}}$  and  $H_n$  denotes the  $n$ th harmonic number. Then  $\hat{R}(|k|)$  is continuous, strictly positive and decreasing with  $\hat{R}(0) < \infty$  and  $\hat{R}(|k|) \sim \frac{\sqrt{\pi}}{|k|\sqrt{2}} \sim \text{const.} \times (1 + |k|^2)^{-\frac{1}{2}}$  as  $|k| \rightarrow \infty$ . Hence (11) still holds with  $R$  replaced by  $R_Y$  and we can then repeat our previous arguments to make (18) rigorous (after transforming back from  $Y$  to  $Z$ ). In [FFGS19] we define the GMC associated with  $Z$  (which we call  $\xi_\gamma$ ) and one can show that  $\xi_\gamma$  is also measurable with respect to  $Z$  so (17) still holds with  $M_\gamma$  replaced by  $\xi_\gamma$  and  $X$  replaced by  $Z$ .

## References

- [APS19] Aru, J., E.Powell and A.Sepulveda, “Critical Liouville Measure as a limit of Sub-critical measures”, *Electron. Commun. Probab.*, Volume 24 (2019), paper no. 18, 16 pp.
- [Aru17] Aru, J., “Gaussian Multiplicative Chaos through the lens of the 2D Gaussian Free Field”, to appear in IRS 2017 special issue in *Markov. Process Relat.*
- [BDM01] Bacry, E., J.Delour, and J.Muzy, “Multifractal Random Walks”, *Phys. Rev. E*, 64, 026103-026106, 2001.
- [BDM01b] Bacry, E., J.Delour, and J.Muzy, “Modelling Financial time series using multifractal random walks”, *Physica A*, 299, 84-92, 2001.
- [BBM13] Bacry, E., Rachel Baïle, and J.Muzy, “Random cascade model in the limit of infinite integral scale as the exponential of a nonstationary  $1/f$  noise: Application to volatility fluctuations in stock markets”, *PHYSICAL REVIEW E* 87, 042813, 2013.
- [BM03] Bacry, E., and J.Muzy, “Log-Infinitely Divisible Multifractal Process”, *Commun. Math. Phys.*, 236, 449-475, 2003.
- [Ber17] Berestycki, N., “Introduction to the Gaussian Free Field and Liouville Quantum Gravity”, draft lecture notes, updated version, Dec 2017.
- [BJRV14] Barral, J., Jin, X., R.Rhodes, SV.Vargas, “Gaussian multiplicative chaos and KPZ duality”, *Communications in Mathematical Physics*, 2014.
- [BDW17] Bierme, H., O.Durieu, and Y.Wang, “Generalized Random Fields and Lévy’s continuity Theorem on the space of Tempered Distributions”, preprint, 2017.
- [DRV12] Duchon, J., R.Robert and V.Vargas, “Forecasting Volatility With The Multifractal Random Walk Model”, *Mathematical Finance*, 22,1, 83-108.
- [DRSV14] Duplantier, D., R.Rhodes, S.Sheffield, and V.Vargas, “Renormalization of Critical Gaussian Multiplicative Chaos and KPZ Relation”, *Communications in Mathematical Physics*, August 2014, Volume 330, Issue 1, pp 283-330.
- [DRSV17] Duplantier, D., R.Rhodes, S.Sheffield, and V.Vargas, “Log-correlated Gaussian Fields: An Overview”, *Geometry, Analysis and Probability*, August pp 191-216, 2017.
- [DS99] Decreusefond, L. and A.S. Ustunel, “Stochastic analysis of the fractional Brownian motion”, *Potential Anal.*, 10: 177-214, 1999.
- [EK00] Estrada R., Kanwal R.P., “Integral Equations with Logarithmic Kernels”, in *Singular Integral Equations*, Birkhauser, Boston, MA, 2000.
- [EK86] Ethier, S.N. and T.G.Kurtz, “Markov Processes: characterization and convergence”, John Wiley and Sons, New York, 1986.
- [FFGS19] Forde, M., M.Fukasawa, S.Gerhold and B.Smith, “Sub and super-critical Gaussia multiplicative chaos for the the Riemann-Liouville process as  $H \rightarrow 0$ , and skew flattening/blow up for the Rough Bergomi model”, preprint, 2019.
- [FSV19] Forde, M., B.Smith and L.Viitasaari, “Rough volatility and CGMY jumps with a finite history and the Rough Heston model - small-time asymptotics in the  $k\sqrt{t}$  regime,”, preprint, 2019.

- [Jan09] Jost, C., “Gaussian Hilbert Spaces”, Cambridge University Press, 2009.
- [Jost06] Jost, C., “Transformation formulas for fractional Brownian motion”, *Stochastic Processes and their Applications*, 116, 1341-1357, 2006.
- [JSW18] Junnila, J., E.Saksman, C.Webb, “Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model”, preprint.
- [Kah85] Kahane, J.-P., “Sur le chaos multiplicatif”, *Ann. Sci. Math.*, Québec, 9 (2), 105-150, 1985.
- [Koz06] Kozhemyak, A., “Modélisation de séries financières à l’aide de processus invariants d’échelle. Application à la prédiction du risque.”, thesis, Ecole Polytechnique, 2006,
- [LSSW16] Lodhia, A., S.Sheffield, X.Sun and S.S. Watson, “Fractional Gaussian fields: A survey”, *Probability Surveys*, Vol. 13,1-56, 2016.
- [Ost09] Ostrovsky, D., “Mellin Transform of the Limit Lognormal Distribution Dmitry Ostrovsky”, *Commun. Math. Phys.*, 288, 287-310.
- [Ost13] Ostrovsky, D., “Selberg Integral as a Meromorphic Function”, *International Mathematics Research Notices*, Vol. 2013, No. 17, pp. 3988–4028, 2013.
- [Ost18] Ostrovsky, D., “A review of conjectured laws of total mass of Bacry–Muzy GMC measures on the interval and circle and their applications”, *Reviews in Mathematical Physics*, Vol. 30, No. 10, 1830003, 2018.
- [Poor94] Poor, H.V., “An Introduction to Signal Estimation and Detection”, 1994, Springer.
- [RV10] Robert, R. and V.Vargas, “Gaussian Multiplicative Chaos Revisited”, *Annals of Probability*, 38, 2, 605-631, 2010.
- [RV14] Rhodes, R. and V.Vargas, “Gaussian multiplicative chaos and applications: a review”, *Probab. Surveys*, 11, 315-392, 2014.
- [RV16] Rhodes, R. and V.Vargas, “Lecture notes on Gaussian multiplicative chaos and Liouville Quantum Gravity”, preprint.
- [RZ17] Remy, G. and T.Zhu, “The distribution of Gaussian multiplicative chaos on the unit interval”,
- [Sha16] Shamov, A., “On Gaussian multiplicative chaos”, *Journal of Functional Analysis*, 270(9):3224-3261, 2016.
- [Var17] Vargas, V., “Lecture notes on Liouville theory and the DOZZ formula”, 2017.
- [Wil91] Williams, D., “Probability with Martingales”, *Cambridge Mathematical Textbooks*, 1991.

## A Proof of Proposition 3.2

$$\begin{aligned}
\mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) &= \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} |\hat{X}_k|^2 dk\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \hat{X}_k \bar{\hat{X}}_k dk\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T e^{ikt} X_t dt \int_0^T e^{-iks} X_s ds dk\right) \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T \int_0^T e^{ik(t-s)} X_s X_t ds dt dk\right) \\
&= \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T \int_0^T e^{ik(t-s)} R(s, t) ds dt dk
\end{aligned}$$

Using that  $R \in L^1([0, T]^2)$ , we see that  $\int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} \int_0^T \int_0^T \mathbb{E}(X_s X_t) ds dt dk = \int_0^T \int_0^T R(s, t) ds dt \cdot \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}-\delta} dk < \infty$  iff  $\delta > 0$ , so by Fubini we have

$$\begin{aligned}
\mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) &= \mathbb{E}\left(\int_0^T \int_0^T R(s, t) \int_{-\infty}^{\infty} e^{ik(t-s)} (1 + |k|^2)^{-\frac{1}{2}-\delta} dk ds dt\right) \\
&= 2c_\delta \int_0^T \int_0^t R(s, t) (t-s)^\delta \text{BesselK}(\delta, t-s) ds dt \\
&\leq c_\delta \int_{[0, T]^2} R(s, t) ds dt < \infty
\end{aligned} \tag{A-1}$$

where we have used that the Fourier transform of  $\hat{f}(k) := (1 + |k|^2)^{-\frac{1}{2}-\delta}$  is  $f(t) = c_\delta |t|^\delta \text{BesselK}(\delta, |t|)$  for some real constant  $c_\delta$ , and that  $t^\delta \text{BesselK}(\delta, t)$  is bounded on  $[0, T]$  if  $\delta > 0$ . For  $\delta \leq 0$ , the integrand in the triple integral in the first line is not absolutely integrable.

## B Proof of Proposition 3.3

Using that

$$\begin{aligned} \chi(s, t, \varepsilon, \varepsilon_2) &:= \mathbb{E}((X_t^{\varepsilon_2} - X_t^\varepsilon)(X_s^{\varepsilon_2} - X_s^\varepsilon)) = R_{\varepsilon_2}(s, t) - \mathbb{E}(X_s^{\varepsilon_2} X_t^\varepsilon) - \mathbb{E}(X_s^\varepsilon X_t^{\varepsilon_2}) + R_\varepsilon(s, t) \rightarrow 0 \\ &= R_{\varepsilon_2}(s, t) - \mathbb{E}(X_s^{\varepsilon \vee \varepsilon_2} X_t^{\varepsilon \vee \varepsilon_2}) - \mathbb{E}(X_s^{\varepsilon \vee \varepsilon_2} X_t^{\varepsilon \vee \varepsilon_2}) + R_\varepsilon(s, t) \end{aligned}$$

as  $\varepsilon, \varepsilon_2 \rightarrow 0$  and that  $|\chi(s, t, \varepsilon, \varepsilon_2)| \leq 4R(s, t)$ , we can use a similar argument to (A-1) and the dominated convergence theorem to show that

$$\mathbb{E}(\|X^{\varepsilon_2} - X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}}^2) \leq c_\varepsilon \int_{[0, T]^2} \chi(s, t, \varepsilon, \varepsilon_2) ds dt \rightarrow 0 \quad (\text{B-1})$$

as  $\varepsilon, \varepsilon_2 \rightarrow 0$ , so  $X^\varepsilon$  is a Cauchy sequence in the Hilbert space  $L^2(\Omega, \mathcal{F}, \mathbb{P}; H^{-\frac{1}{2}-\delta})$  of  $H^{-\frac{1}{2}-\delta}$ -valued random variables  $X$  with  $\mathbb{E}(\|X\|_{H^{-\frac{1}{2}-\delta}}^2) < \infty$ , and thus converges in this space. Using that

$$\mathbb{P}(\|X^{\varepsilon_2} - X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}} > k) \leq \frac{1}{k^2} \mathbb{E}(\|X^{\varepsilon_2} - X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}}^2)$$

the claim is proved.

## C Proof of Proposition 4.1

Similar to the analysis before Lemma 2.1 in [Aru17] with rooted measures, we let  $D = [0, T]$  and we can define a sequence of approximate “rooted” probability measures  $\mathbb{P}_\varepsilon^*$  on  $D \times H^{-\frac{1}{2}-\delta}$  as

$$\mathbb{P}_\varepsilon^*(dt, d\omega) = \frac{dt}{\text{Leb}(D)} e^{\gamma\omega(t) - \frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)} \mathbb{Q}^{X^\varepsilon}(d\omega)$$

where  $\mathbb{Q}^{X^\varepsilon}$  denotes the law of  $X^\varepsilon$  on  $H^{-\frac{1}{2}-\delta}$ , and  $X^\varepsilon$  is defined as in (3). Then

- The marginal law on  $D$  is

$$\frac{dt}{\text{Leb}(D)} \mathbb{E}^{\mathbb{Q}^{X^\varepsilon}}(e^{\gamma\omega(t) - \frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}) = \frac{dt}{\text{Leb}(D)}$$

i.e. the uniform probability measure on  $D$ .

- The marginal law on  $H^{-\frac{1}{2}-\delta}$  is  $\frac{\int_D e^{\gamma\omega(t) - \frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)} dt}{\text{Leb}(D)} \mathbb{Q}^{X^\varepsilon}(d\omega) = \frac{M_\gamma^\varepsilon(D)}{\text{Leb}(D)} \mathbb{Q}^{X^\varepsilon}(d\omega)$ , i.e. the law of  $X^\varepsilon$  tilted by  $M_\gamma^\varepsilon(D)/\text{Leb}(D)$ .
- The conditional law on  $D$  given  $\omega$  is the probability measure:  $\frac{e^{\gamma\omega(t) - \frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}}{M_\gamma^\varepsilon(D)} dt = \frac{M_\gamma^\varepsilon(dt)}{M_\gamma^\varepsilon(D)}$ .
- The conditional law on  $H^{-\frac{1}{2}-\delta}$  given  $t$  is  $e^{\gamma\omega(t) - \frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)} \mathbb{Q}^{X^\varepsilon}(d\omega)$ . From Girsanov’s theorem (see e.g. section 6.1 in [Var17]), we can re-write this as

$$\mathbb{Q}(X^\varepsilon + \gamma R_\varepsilon(., t) \in d\omega) \quad (\text{C-1})$$

Thus we can sample from  $\mathbb{P}_\varepsilon^*$  by either (i) sampling from  $\frac{M_\gamma^\varepsilon(D)}{\text{Leb}(D)} \mathbb{Q}^{X^\varepsilon}(d\omega)$  and then sampling a point according to  $M_\gamma^\varepsilon(dt)/M_\gamma^\varepsilon(D)$ , or ii) sampling  $t$  from the uniform measure on  $[0, T]$ , and then sampling  $X^\varepsilon + \gamma R_\varepsilon(., t)$ , with  $X^\varepsilon$  independent of  $t$ . Combining these two prescriptions, we see that

$$\mathbb{E}\left(\frac{M_\gamma^\varepsilon(D)}{\text{Leb}(D)} \int_0^T F(X^\varepsilon, t) \frac{M_\gamma^\varepsilon(dt)}{M_\gamma^\varepsilon(D)}\right) = \frac{1}{\text{Leb}(D)} \mathbb{E}\left(\int_0^T F(X^\varepsilon + \gamma R_\varepsilon(t, .), t) dt\right)$$

which we can re-write as

$$\mathbb{E}\left(\int_0^T F(X^\varepsilon, t) M_\gamma^\varepsilon(dt)\right) = \mathbb{E}\left(\int_0^T F(X^\varepsilon + \gamma R_\varepsilon(t, .), t) dt\right). \quad (\text{C-2})$$

We first consider the left hand side of this expression as  $\varepsilon \rightarrow 0$ . To begin with, we note that

$$\begin{aligned} & |\mathbb{E}(\int_0^T F(X^\varepsilon, t) M_\gamma^\varepsilon(dt) - \int_0^T F(X, t) M_\gamma(dt))| \\ & \leq |\mathbb{E}(\int_0^T (F(X^\varepsilon, t) - F(X, t)) M_\gamma^\varepsilon(dt))| + |\mathbb{E}(\int_0^T F(X, t) (M_\gamma^\varepsilon(dt) - M_\gamma(dt)))| \end{aligned} \quad (\text{C-3})$$

and we can bound the first term in the final expression using Hölder's inequality as

$$\begin{aligned} \mathbb{E}(\int_0^T (F(X^\varepsilon, t) - F(X, t)) M_\gamma^\varepsilon(dt)) & \leq \mathbb{E}(\sup_{t \in [0, T]} |F(X^\varepsilon, t) - F(X, t)| \cdot M_\gamma^\varepsilon([0, T])) \\ & \leq \mathbb{E}((\sup_{t \in [0, T]} |F(X^\varepsilon, t) - F(X, t)|)^p)^{\frac{1}{p}} \cdot \mathbb{E}((M_\gamma^\varepsilon([0, T]))^q)^{\frac{1}{q}} \end{aligned} \quad (\text{C-4})$$

for  $1/p + 1/q = 1$ , and from (2) we know that

$$\mathbb{E}((M_\gamma^\varepsilon([0, T]))^q) = c_q T^{\zeta(q)} < \infty$$

for any  $q \in (1, q^*) = \frac{2}{\gamma^2}$ .

We claim that  $\sup_{t \in [0, T]} |F(X^\varepsilon, t) - F(X, t)| \rightarrow 0$  a.s. Indeed, suppose to the contrary. Let  $f_\varepsilon(t) := F(X^\varepsilon, t)$  and  $f(t) := F(X, t)$ . If the claim is false,  $f_\varepsilon$  does not tend to  $f$  uniformly on  $[0, T]$ , so there exists a sequence  $\varepsilon_n \rightarrow 0$ , a  $\delta > 0$  and a sequence  $t_n \in [0, T]$  such that

$$|f_{\varepsilon_n}(t_n) - f(t_n)| \geq \delta \quad (\text{C-5})$$

for all  $n \in \mathbb{N}$ . But by Bolzano-Weierstrass, we can choose a convergent subsequence  $(t_{n_k})$  of  $(t_n)$  with  $t_{n_k} \rightarrow t_\infty \in [0, T]$ . Then  $f_{\varepsilon_{n_k}}(t_{n_k}) = F(X^{\varepsilon_{n_k}}, t_{n_k})$  and  $f(t_{n_k}) = F(X, t_{n_k})$ . From Proposition 3.3 we know that  $X^\varepsilon$  tends to  $X$  in  $H^{-\frac{1}{2}-\delta}$  in probability, and thus almost surely along a further subsequence  $\varepsilon_{n_{k_j}}$ , thus (by continuity of  $F$  in both arguments)  $F(X^{\varepsilon_{n_{k_j}}}, t_{n_{k_j}}) \rightarrow F(X, t_\infty)$  a.s. and hence

$$|F(X^{\varepsilon_{n_{k_j}}}, t_{n_{k_j}}) - F(X, t_{n_{k_j}})| = |f_{\varepsilon_{n_{k_j}}}(t_{n_{k_j}}) - f(t_{n_{k_j}})| \rightarrow 0 \quad (\text{C-6})$$

a.s., which violates (C-5). Hence the right hand side of (C-4) tends to zero (along *any* subsequence) for  $q \in (1, q^*)$

The term  $\int_0^T F(X, t) (M_\gamma^\varepsilon(dt) - M_\gamma(dt))$  inside the expectation on the right hand side of (C-3) converges to zero a.s. since  $M_\gamma^\varepsilon$  tends weakly to  $M_\gamma$  a.s. (see top of page 3 for details) and the random  $F(X, t)$  is continuous in  $t$  for each  $\omega$ . Moreover

$$\int_0^T F(X, t) (M_\gamma^\varepsilon(dt) - M_\gamma(dt)) \leq \|F\|_\infty (M_\gamma^\varepsilon([0, T]) + M_\gamma([0, T])).$$

From (6) we also know that  $M_\gamma^\varepsilon([0, T])$  is uniformly integrable, so by e.g. the Theorem in section 13.7 in [Wil91], the rightmost term of (C-3) tends to zero

Finally, the right hand side of (C-2) converges by the a.s. convergence of  $X^\varepsilon$  to  $X$  in Proposition 3.3 and the bounded convergence theorem.