A simplified unified propagator model for signed order flow, concave price impact and rough volatility

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Abstract

In this note, we simplify the setup in [MORS26] by exogenously modelling signed order flow as an integrated hyper-rough Heston process with Holder exponent in $(\frac{1}{2}, 1)$, which avoids the need for Hawkes processes, Mittag-Leffler functions and lengthy scaling limit arguments.

0.1 Positive order flow and the inversion formula

Similar to Theorem 3.1 in [MORS26][1], we consider a hyper-rough Heston process $(F_t^+)_{t\geq 0}$ for cumulative positive order flow which satisfies

$$F_t^+ \;=\; g(t) \,+\, \int_0^t f(t-s) B_{F_s^+} ds$$

where B is a standard Brownian motion with $B_0 = 0$ and $f(t) = e^{-\theta t} t^{H-\frac{1}{2}}$ is the usual Gamma kernel with $H \in (-\frac{1}{2}, \frac{1}{2})$, $\theta > 0$ so $f \in L^1$ and we assume $g \in C^1$. F_t^+ is a.s. $(2\alpha - \varepsilon) \wedge 1$-Hölder continuous where $\alpha = H + \frac{1}{2}$ (see e.g. Theorem 3.1 in [JM20] for details) [2] and F^+ is an increasing process which is non-differentiable if $H \leq 0$.

Lemma 0.1 $\mathbb{E}(F_t^+) < \infty$.

Proof. See Appendix A. ∎

Add proof of existence/uniqueness and that $B_{F_t^+}$ is a martingale.

If h is a function with $h * f \equiv 1$, then we have the inversion formula:

$$\int_0^t h(t-s)(F_s^+ - g(s))ds \;=\; \int_0^t h(t-s)\int_0^s f(s-u)B_{F_u^+}du\,ds \;=\; \int_0^t \int_u^t h(t-s)f(s-u)ds\,B_{F_u^+}du$$
$$= \int_0^t \int_0^{t-u} h(t-s-u)f(s)ds\,B_{F_u^+}du$$
$$= \int_0^t (h*f)(t-u)B_{F_u^+}du \;=\; \int_0^t B_{F_u^+}du$$

(where Fubini is justified since $F_t^+ < \infty$ a.s. for finite t and B is a.s. continuous), and hence

$$\frac{d}{dt}\int_0^t h(t-s)(F_s^+ - g(s))ds \;=\; B_{F_t^+} \tag{1}$$

Lebesgue a.e. Taking Laplace transforms, the condition $h * f \equiv 1$ becomes $\hat{f}(\lambda)\hat{h}(\lambda) = \frac{1}{\lambda}$, from which we find that

$$h(t) \;=\; \frac{\theta^\alpha}{\Gamma(\alpha)}(1 - \frac{\Gamma(-\alpha, t\theta)}{\Gamma(-\alpha)})$$

where $\Gamma(a, z) = \int_z^\infty s^{a-1}e^{-s}ds$ is the incomplete Gamma function, and in particular $h(t) = O(t^{-H-\frac{1}{2}})$ as $t \to 0$.

0.2 Persistent signed order flow

Again following [MORS26], we now assume the *signed* order flow for an asset is

$$V_t \;=\; F_t^+ - F_t^-$$

where $F_t^- = g(t) + \int_0^t f(t-s)W_{F_s^-}ds$ and W is another Brownian motion independent of B (so F^- is an i.i.d. copy of F^+). Then we see that

$$\frac{d}{dt}\int_0^t h(t-s)V_s ds \;=\; B_{F_t^+} - W_{F_t^-}$$

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[1]see also section 5.4 in [FGS21]
[2]we can simulate F^+ using the Monte Carlo scheme in [AA25] using Normal Inverse Gaussian variates

since the g terms cancel, and for $0 \le t \le u$ we have

$$\mathbb{E}(F_u^+|\mathcal{F}_{F_t^+}^B) \quad = \quad g(u) + \mathbb{E}(\int_0^u f(u-s)B_{F_s^+}ds \,|\, \mathcal{G}_t^+) \quad = \quad g(u) + \int_0^t f(u-s)B_{F_s^+}ds + B_{F_t^+}\int_t^u f(u-s)ds$$

(and similarly for $\mathbb{E}(F_u^-|\mathcal{F}_{F_t^-}^W)$), where $\mathcal{G}_t^+ = \mathcal{F}_{F_t^+}^B$, assuming $(B_{F_t^+})_{t \ge 0}$ is a $\mathcal{G}_t^+-$martingale (see e.g. Section 7 in [AJ21] for more on this).

0.3   Price dynamics and the propagator model

Making the usual assumption that the asset price $P_t = \kappa \lim_{u \to \infty} \mathbb{E}(F_u|\mathcal{G}_t)$ where $F_t = F_t^+ - F_t^-$ and $\mathcal{G}_t = \sigma(\mathcal{G}_t^+, \mathcal{G}_t^-)$ with $\mathcal{G}_t^- = \mathcal{F}_{F_t^-}^W$ (for some $\kappa > 0$)[3], we find that $f(u-s) \to 0$ as $u \to \infty$ since $\theta > 0$, but $\lim_{u \to \infty} \int_t^u f(u-s)ds = c_{H,\theta}$ where $c_{H,\theta} = \theta^{-\frac{1}{2}-H}\Gamma(\frac{1}{2}+H)$, so

$$P_t \quad = \quad \kappa \lim_{u \to \infty} \mathbb{E}(F_u^+ - F_u^- \,|\mathcal{G}_t) \quad = \quad \kappa\, c_{H,\theta}(B_{F_t^+} - W_{F_t^-})$$

i.e. P is the difference of two i.i.d. hyper-rough Heston models (each with correlation $\rho = 1$ since each process only has one driving Brownian motion), and

$$P_t \quad = \quad \kappa\, c_{H,\theta} \frac{d}{dt}\int_0^t h(t-s)V_s ds$$

which is a weak formulation of the usual Propagator model, and the *unsigned* order flow is $U_t = F_t^+ + F_t^-$.

0.4   Power-law price impact

In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 up to time t_0 is given by

$$MI(t) \quad = \quad \frac{d}{dt}\int_0^t h(t-s)s ds \quad = \quad \frac{\theta^{-\alpha_-}(t\theta\Gamma(-\alpha) - t\theta\Gamma(-\alpha, t\theta) - \Gamma(\alpha_-, 0) + \Gamma(\alpha_-, t\theta))}{\Gamma(-\alpha)\Gamma(\alpha)}$$

for $0 \le t < t_0$ (where $\alpha_- = \frac{1}{2} - H$) for $0 \le t \le t_0$ which is $O(t^{\frac{1}{2}-H})$ as $t \to 0$ and globally concave in t for $H \in (-\frac{1}{2}, \frac{1}{2})$, consistent with empirical evidence (note the usual square-root impact law corresponds to $H = 0$ here), and we can show that the asymptotic (i.e. permanent) impact as $t \to \infty$ (given no trading after t_0) is

$$MI(\infty) \quad = \quad \frac{t_0\theta^\alpha}{\Gamma(\frac{1}{2}+H)}$$

(this is the asymptotic grey line in the plot below). For $H = \frac{1}{2}$, $h(t) = \theta$ which corresponds to pure permanent price impact; conversely $\lim_{\theta \to 0} h(t) = const. \times t^{-\frac{1}{2}-H}$. Recall from above that F_t^+ has Hölder regularity $(2\alpha - \varepsilon) \wedge 1$, and empirical evidence in [MORS26] suggests that $2\alpha \in (\frac{1}{2}, 1)$ which corresponds to $H \in (-\frac{1}{4}, 0)$.

0.5   Extensions

If we augment f to a function of the form $f(t) = e^{-\theta t}t^{H-\frac{1}{2}} + e^{-\theta_2 t}t^{H_2-\frac{1}{2}}$ with $H_2 \in (0, \frac{1}{2})$, $\theta_2 > 0$, then we find that

$$h(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\sum_{n=0}^\infty (-c)^n \sum_{k=0}^\infty \frac{(\beta n)_k}{k!}\,\delta^k\,\theta^{-(\mu_n+k)}\frac{\gamma(\mu_n+k, \theta t)}{\Gamma(\mu_n+k)}$$

where $\beta = \frac{1}{2} + H_2$, $c = \Gamma(\beta)/\Gamma(\alpha)$, $\delta = \theta - \theta_2$, $\mu_n = \beta n - \alpha(n+1)$, $\gamma(a, z) = \int_0^z s^{a-1}e^{-s}ds$ is the lower incomplete Gamma function and $(a)_k$ denotes the Pochammer symbol. We can then use two (possibly different) f functions to drive $F^\pm$, and the presence of the positive H_2 parameter means the price dynamics will now have a (non-hyper) rough component, which is arguably more realistic. We can also assume W and B correlated which gives greater flexibility in capturing skew for the price process P_t.

Below we have tabulated estimates of 2α (i.e. the Hölder exponent of F_t) using signed order flow for major US tech stocks using the model-independent $\hat{H}_{L,K}^\pi(X)$ estimator in Eq 7 in Cont&Das[CD24] which allows for irregularly spaced data (with their $L/K = [\sqrt{N}]$ where N is the total data size). For this we have used the free data at `https://data.lobsterdata.com/info/DataSamples.php`: [4]

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[3]note we are assuming $P_0 = 0$ without loss of generality since we can easily add a P_0 term

[4]only using columns 4 and 5 from the lobsterdata.csv file which correspond to actual realized trades.

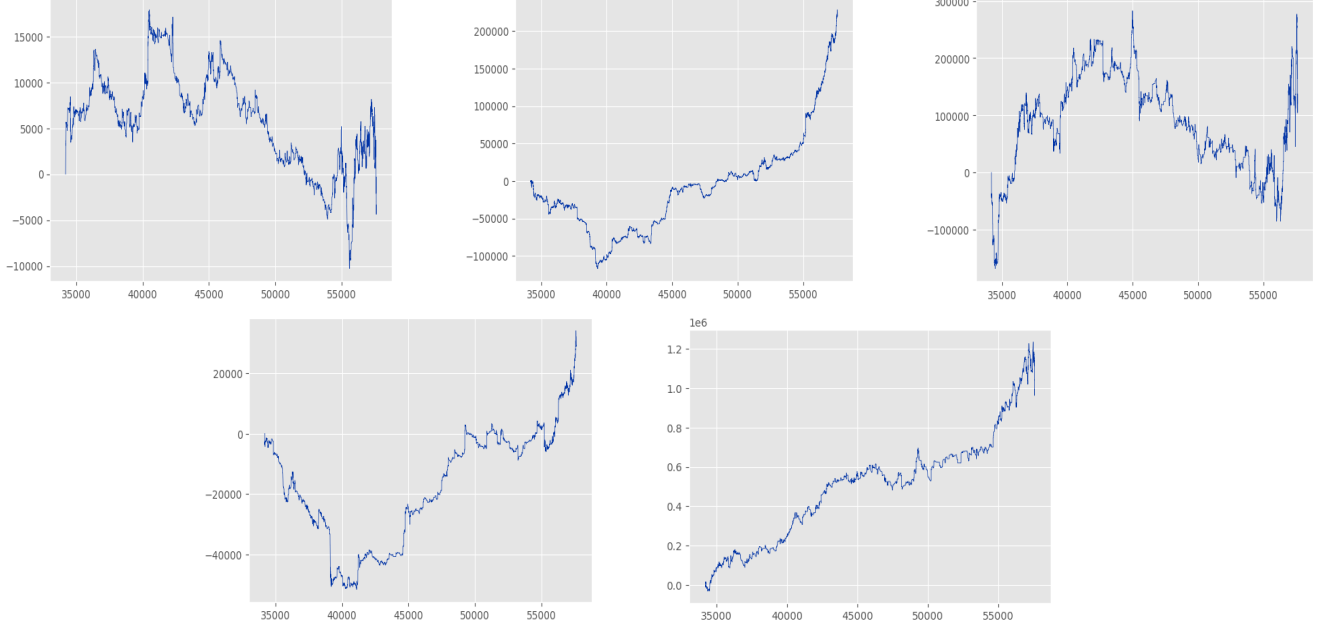|	GOOG	AAPL	MSFT	AMZN	INTC
N	49284	118326	410881	57360	404496
$2\hat{\alpha}$	0.692	0.720	0.847	0.735	0.890
$\hat{H}$	-0.154	-0.140	-.0767	-0.133	-0.0553

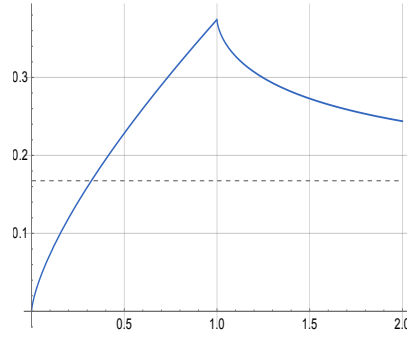Figure 1: Signed order flow for GOOG, AAPL, MSFT, AMZN and INTC

Figure 2: Concave price impact of an exogenous metaorder executed at constant trading speed 1 over $[0,1]$ with $H = -0.2$, $\theta = 0.1$ and $\kappa = 1$. The blue line asymptotes to the constant level $MI(\infty)$ (grey line) as $t \to \infty$ which represents the asymptotic permanent price impact.

References

[AJ21] Abi Jaber, E., "Weak existence and uniqueness for affine stochastic Volterra equations with L^1-kernels", Bernoulli 27(3), 2021, 1583–1615

[AA25] Abi Jaber, E. and E.Attal, "Simulating integrated Volterra square-root processes and Volterra Heston models via Inverse Gaussian", preprint, 2025

[CD24] Cont, R. and P. Das, "Rough Volatility: Fact or Artefact?", Sankhya: The Indian Journal of Statistics, Series B, 86(1), 191-223, 2024.

[FGS21] Forde, M., S.Gerhold and B.Smith, "Small-time, large-time and $H \to 0$ asymptotics for the rough Heston model", *Mathematical Finance, 31*(1), 203-241, 2021.

[JM20] Jusselin, P. and M.Rosenbaum, "No-arbitrage implies power-law market impact and rough volatility", *Mathematical Finance*, Volume 30, Issue 4 pp. 1309-1336

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A   Appendix A

Proof. Set $G_t := \sup_{0 \le r \le t} |g(r)|$, $K_t := \int_0^t |f(r)| dr > 0$. For $n \ge 1$ define the level hitting time $\tau_n := \inf\{s \ge 0 : F_s^+ \ge n\}$.

Since F^+ is continuous and nondecreasing, on $\{\tau_n \le t\}$ we have $F_{\tau_n}^+ = n$ and $F_r^+ \le n$ for all $0 \le r \le \tau_n$. Fix $n \ge 1$ and work on the event $\{\tau_n \le t\}$. For $0 \le r \le \tau_n$ we have $|B_{F_r^+}| \le \sup_{0 \le u \le n} |B_u|$. Setting $s = \tau_n$ and taking absolute values we see that

$$
\begin{aligned}
n = g(\tau_n) + \int_0^{\tau_n} f(\tau_n - r) B_{F_r^+} \, dr &\le |g(\tau_n)| + \int_0^{\tau_n} |f(\tau_n - r)| \, |B_{F_r^+}| \, dr \\
&\le G_t + (\int_0^{\tau_n} |f(\tau_n - r)| \, dr) \sup_{0 \le u \le n} |B_u| \\
&= G_t + (\int_0^{\tau_n} |f(s)| \, ds) \sup_{0 \le u \le n} |B_u| \le G_t + K_t \sup_{0 \le u \le n} |B_u|.
\end{aligned}
$$

Hence for $n > G_t$, on $\{\tau_n \le t\}$ we have $\{\sup_{0 \le u \le n} |B_u| \ge \frac{n - G_t}{K_t}\}$. Since $F_t^+ \ge n$ iff $\tau_n \le t$, we obtain

$$
\mathbb{P}(F_t^+ \ge n) \le \mathbb{P}(\sup_{0 \le u \le n} |B_u| \ge \frac{n - G_t}{K_t}), \qquad n > G_t.
$$

By the standard reflection principle bound $\mathbb{P}(\sup_{0 \le u \le n} |B_u| \ge a) \le 4\exp(-\frac{a^2}{2n})$, for $a > 0$, so for $n > G_t$,

$$
\mathbb{P}(F_t^+ \ge n) \le 4\exp(-\frac{(n - G_t)^2}{2nK_t^2}). \tag{A-1}
$$

In particular, if $n \ge 4G_t$ then $(n - G_t)^2/n \ge \frac{1}{2}n$, hence

$$
\mathbb{P}(F_t^+ \ge n) \le 4e^{-\frac{n}{4K_t^2}}, \qquad n \ge 4G_t. \tag{A-2}
$$

Since $F_t^+ \ge 0$, we also know that $\mathbb{E}(F_t^+) = \int_0^\infty \mathbb{P}(F_t^+ \ge x) \, dx$. Now set $m := \lceil 4G_t \rceil$. Then

$$
\mathbb{E}(F_t^+) \le m + \int_m^\infty \mathbb{P}(F_t^+ \ge x) \, dx = m + \sum_{n=m}^\infty \int_n^{n+1} \mathbb{P}(F_t^+ \ge x) \, dx \le m + \sum_{n=m}^\infty \mathbb{P}(F_t^+ \ge n). \tag{A-3}
$$

Using (A-2) for all $n \ge m$,

$$
\sum_{n=m}^\infty \mathbb{P}(F_t^+ \ge n) \le \sum_{n=m}^\infty 4\exp(-\frac{n}{4K_t^2}) < \infty. \tag{A-4}
$$

Combining (A-3) and (A-4) yields $\mathbb{E}(F_t^+) < \infty$. ∎