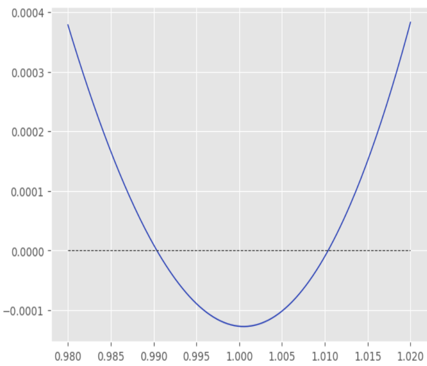

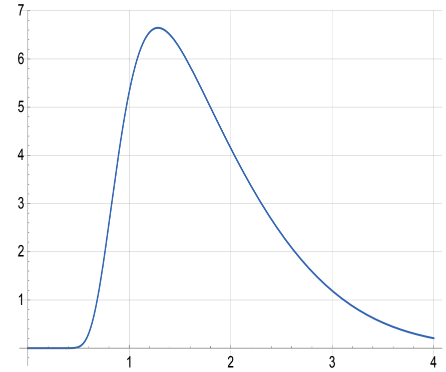

Figure 1: On the left we see the exact P&L as a function of S for q1 when $K = e^{0.1}$, $T = 1$, $\Delta t = .01$, $\sigma = .1$, $r = 0$. In the second plot, we see the density of the two-sided maximum for standard Brownian motion (grey) for q2.

Homework 4

1. Consider a trader who buys a European call option at $t = 0$, and Δ-hedges the call option at $t = 0$ (i.e. sells $C_S(S_0, 0)$ units of stock at $t = 0$ only), and assume $r = 0$ for simplicity. Compute a Taylor series expansion for the Profit/Loss (P&L) of the trader over a small-time period Δt, and the expectation of this P&L under the risk-neutral measure $\mathbb{Q}$.

Solution.

$$\text{P\&L} \;\approx\; \frac{1}{2}C_{SS}(S_0,0)(\Delta S)^2 \,+\, C_t(S_0,0)\Delta t \;=\; \frac{1}{2}\Gamma(\Delta S)^2 \,+\, \Theta\Delta t$$

since the initial total delta of the position is zero by assumption, and $\Gamma > 0$ and $\Theta < 0$ are the initial Gamma and Theta respectively. Note that $\Delta S \approx S_0\sigma\Delta W$, so

$$\mathbb{E}^{\mathbb{Q}}(\text{P\&L}) \;\approx\; \frac{1}{2}\Gamma S_0^2\sigma^2\mathbb{E}^{\mathbb{Q}}((\Delta W)^2) \,+\, \Theta\Delta t \;=\; 0 \tag{1}$$

because (from the Black-Scholes PDE) we know that $C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} = \Theta + \frac{1}{2}\sigma^2 S^2\Gamma = 0$ when $r = 0$, and $\mathbb{E}((\Delta W)^2) = \Delta t$.

2. From Hwk3, we know that

$$\mathbb{P}(W_t \in dx, M_t < b, m_t > a) \;=\; \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\lambda_n t}\sin\Big(\frac{n\pi(x-a)}{b-a}\Big)\sin\Big(\frac{n\pi(0-a)}{b-a}\Big)dx \tag{2}$$

for $a < x < b$, where $\lambda_n = \frac{n^2\pi^2}{2(b-a)^2}$, and M_t and m_t are the running max and min processes of W. Use this to explicitly compute the cdf of the **two-sided maximum** $R_t := \max_{0\le s\le t}|W_s| = \max(M_t, -m_t)$ of W at time t. Is $R_t = M_t - m_t$ i.e. the range of W?

Solution. Setting $b = r$ and $a = -r$, clearly $b - a = 2r$ and we can re-write (2) as

$$\mathbb{P}(W_t \in dx, M_t < r, m_t > -r) \;=\; \mathbb{P}(W_t \in dx, R_t < r) \;=\; \frac{1}{r}\sum_{n=1}^{\infty} e^{-\lambda_n t}\sin\Big(\frac{n\pi(x+r)}{2r}\Big)\sin\Big(\frac{n\pi(0+r)}{2r}\Big)dx$$

for $x \in (-r, r)$, where $\lambda_n = \frac{n^2\pi^2}{8r^2}$. Integrating each term here from $x = -r$ to r, and assuming we can apply Fubini's theorem to interchange the sum and integral, we obtain the cdf of R_t. For this we just need that

$$\int_{-r}^{r}\sin\Big(\frac{n\pi(x+r)}{2r}\Big)dx \;=\; -\frac{2r\cos(\frac{n\pi(x+r)}{2r})}{n\pi}\Big|_{x=-r}^{x=r} \;=\; \frac{2r(1-\cos(n\pi))}{n\pi} \;=\; \frac{4r}{n\pi}$$

for n odd, and **zero** for n even. Using this + setting $n = 2k-1$ for $k \in \mathbb{N}$ and that $\sin(\frac{n\pi(0+r)}{2r}) = \sin(\frac{1}{2}n\pi)$, we get

$$\mathbb{P}(R_t < r) \;=\; \sum_{k=1}^{\infty} e^{-\lambda_{2k-1}t}\frac{4}{(2k-1)\pi}(-1)^{k-1}$$

since $\sin(\frac{1}{2}n\pi) = \sin(\frac{1}{2}(2k-1)\pi) = (-1)^{k-1}$ (where the $\frac{1}{r}$ and r terms have cancelled) and note that each $\lambda_{2k-1} = \frac{(2k-1)^2\pi^2}{8r^2}$ term here depends on r (**see right plot above**). R is not the range of W, since if e.g. $\bar{W}_t = -\underline{W}_t$, then $M_t - m_t = 2R_t$.

3. Compute the asymptotic price of a European call option as the maturity T tends to infinity (assuming $r \geq 0$).

Solution. Recall from the notes that

$$C^{BS}(S, K, \sigma, T-t, r) \quad = \quad S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where $\tau = T - t$ is the time-to-maturity and

$$d_1 \quad = \quad \frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad , \quad d_2 \quad = \quad d_1 - \sigma\sqrt{\tau} \quad = \quad \frac{\log \frac{S}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \, .$$

Then we see that $d_1 \to +\infty$ and $Ke^{-r\tau}\Phi(d_2) \to 0$ as $T \to \infty$ (since $e^{-rT} \to 0$ if $r > 0$ and $d_2 \to -\infty$ if $r = 0$), so $C^{BS}(S, K, \sigma, T-t, r) \to S$ as $T \to \infty$.

For an alternate proof for $r > 0$ that does not require the Black-Scholes formula, we note that

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \;\; \geq \;\; e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T - K) \;\; = \;\; e^{-rT}S_0 e^{rT} \; - \; Ke^{-rT} \;\; = \;\; S_0 - Ke^{-rT} \, .$$

But we also know that

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \;\; \leq \;\; e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T) \;\; = \;\; S_0 \, .$$

Hence the call price is sandwiched as follows:

$$S_0 - Ke^{-rT} \;\; \leq \;\; e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \;\; \leq \;\; S_0$$

and both left and right hand sides tends to S_0 as $T \to \infty$, and hence so does the middle expression.

For $r = 0$, for European put options we have the limit: $\lim_{T\to\infty} \mathbb{E}^{\mathbb{Q}}((K - S_T)^+) = K$ (by the bounded convergence theorem), since $S_t \to 0$ a.s. as $t \to \infty$ (see Hwk 3), so (by the put-call parity) $C + K = P + S_0$, we see that $\lim_{T\to\infty} \mathbb{E}^{\mathbb{Q}}((S_T - K)^+) = S_0$.

4. For the SDE $dX_t = X_t^{\beta} dW_t$ with $X_0 > 0$, it can be shown that

$$\mathbb{E}(X_t | X_0 = x) \quad = \quad \begin{cases} x(1 - 2\Phi(-\frac{1}{x\sqrt{t}})) & (\beta = 2) \\ x(1 - e^{-\frac{2}{xt}}) & (\beta = \frac{3}{2}) \end{cases}$$

for $t \geq 0$, where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$ (proof of these formulae not required). Is X a martingale in either of these cases? (explain your answer).

Solution. $\mathbb{E}(X_t | X_0 = x) < x$ in both cases, so X cannot be a martingale.

5. Let $R_t := \bar{X}_t - \underline{X}_t$ denote the **range** of $dX_t = \sigma dW_t$ over $[0, t]$, where $\bar{X}_t = \max_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \min_{0 \leq s \leq t} X_s$. Using that $\mathbb{E}(R_t) = 2\sigma\sqrt{\frac{2t}{\pi}}$, derive an **unbiased estimator** $\hat{\sigma}$ for σ using an observed value for R_t (hint: your answer should not contain an expectation). Using that $\mathbb{E}(R_t^2) = 4 \log 2 \cdot \sigma^2 \, t$, compute the variance of $\hat{\sigma}$.

Solution. $\hat{\sigma} = \frac{R_t}{2\sqrt{\frac{2t}{\pi}}}$ is an unbiased estimator. Then

$$\mathrm{Var}(\hat{\sigma}) \quad = \quad \frac{\mathrm{Var}(R_t)}{4\frac{2t}{\pi}} \quad = \quad \frac{\mathbb{E}(R_t^2) - \mathbb{E}(R_t)^2}{\frac{8t}{\pi}} \quad = \quad \frac{4\log 2 \cdot \sigma^2 \, t - (2\sigma\sqrt{\frac{2t}{\pi}})^2}{\frac{8t}{\pi}} \quad = \quad \sigma^2 \frac{4\log 2 - \frac{8}{\pi}}{\frac{8}{\pi}} \quad \approx \quad 0.0888\sigma^2 \, .$$

6. Following on from q5, it can be show that the density of the range R_t of Brownian motion is

$$p(r) \quad = \quad \sum_{k=1}^{\infty} a_k(r) \qquad \text{where} \qquad a_k(r) = \frac{8}{\sqrt{t}}(-1)^{k-1}k^2\phi(\frac{kr}{\sqrt{t}})$$

where $\phi(z) = \Phi'(z)$ is the standard Normal density. Using that $\int_0^{\infty} r^2 a_k(r) dr = -\frac{4(-1)^k t}{k}$ $(*)$, verify the formula for $\mathbb{E}(R_t^2)$ in q5 when $\sigma = 1$. Hint: use the Taylor series $\log(1 + x) = -\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$.

Solution. Applying $*$, to each term of the series, we see that

$$\mathbb{E}(R_t^2) \quad = \quad \int_0^{\infty} r^2 p(r) dr \quad = \quad \sum_{k=1}^{\infty} (\int_0^{\infty} r^2 a_k(r) dr) \quad = \quad -4t \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \, .$$

Then using the hint with $x = 1$, we obtain that $-\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = \log 2$, so $\mathbb{E}(R_t^2) = 4t \log 2$.

7. Constructing Brownian motion using Fourier series. Show that

$$B_t = tZ_0 + \sum_{n=1}^{\infty} Z_n \frac{\sqrt{2}\sin(n\pi t)}{n\pi}$$

is a Brownian motion for $t \in [0,1]$, where $(Z_n)_{n=0}^{\infty}$ is a sequence of i.i.d. $N(0,1)$ random variables. You may use that $\sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \sin(n\pi s)\sin(n\pi t) = s(1-t)$ for $0 \le s \le t$.

Solution. Since $\mathbb{E}(Z_n Z_m) = 1$ if $m = n$ and zero otherwise, we see that

$$\begin{aligned}\mathbb{E}(\hat{B}_s \hat{B}_t) &= \mathbb{E}\Big(\sum_{m=1}^{\infty} Z_m \frac{\sqrt{2}\sin(m\pi s)}{m\pi} \cdot \sum_{n=1}^{\infty} Z_n \frac{\sqrt{2}\sin(n\pi t)}{n\pi}\Big) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \sin(n\pi s)\sin(n\pi t) \\ &= s(1-t)\end{aligned}$$

for $0 \le s \le t$, using the hint in the question. From this we see that

$$\mathbb{E}(B_s B_t) = \mathbb{E}((sZ_0 + \hat{B}_s)(tZ_0 + \hat{B}_t)) = st + s(1-t) = s$$

for $0 \le s \le t$, which is the covariance function of Brownian motion.