King's College London

UNIVERSITY OF LONDON

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MSc EXAMINATION

7CCMFM02 RISK NEUTRAL VALUATION MOCK QUESTIONS

JANUARY 2026

TIME ALLOWED: TWO HOURS

ALL QUESTIONS CARRY EQUAL MARKS. FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO ALL FOUR QUESTIONS.

WITHIN A GIVEN QUESTION, THE RELATIVE WEIGHTS OF THE DIFFERENT PARTS ARE INDICATED BY A PERCENTAGE FIGURE.

NO CALCULATORS ARE PERMITTED.

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TURN OVER WHEN INSTRUCTED

1. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Let

$$dX_t \;=\; \delta dt \,+\, 2\sqrt{X_t}dW_t$$

with $\delta = 2$. Derive the SDE for $Z_t = \log X_t$. **[30%]**

Solution. Let $f(x,t) = \log x$, so $f_x(x,t) = \frac{1}{x}$, $f_{xx}(x,t) = -\frac{1}{x^2}$ and $f_t(x,t) = 0$. From Ito's lemma

$$dZ_t \;=\; \frac{1}{X_t}(2dt \,+\, 2\sqrt{X_t}dW_t) - \frac{1}{2}\frac{1}{X_t^2}4X_t dt \;=\; \frac{2}{\sqrt{X_t}}dW_t \;=\; 2e^{-\frac{1}{2}Z_t}dW_t\,.$$

Note Z_t is driftless but it turns out that Z is not a martingale (not proved here).

b. Let $B_t^{(i)} = (1-t)W^{(i)}_{\frac{t}{1-t}}$ for $t \in [0,1)$, where $(W^{(i)})_{i=1..3}$ are three independent Brownian motions, and set $R_t = \sqrt{\sum_{i=1}^3 (B_t^{(i)})^2}$. Compute $\mathbb{E}(R_t^2)$. What can we say about the process R_t for $t \in [0,1]$? **[40%]**

Solution.

$$\mathbb{E}(R_t^2) \;=\; \mathbb{E}(\sum_{i=1}^3 (B_t^{(i)})^2) \;=\; \sum_{i=1}^3 \mathbb{E}((B_t^{(i)})^2) \;=\; 3(1-t)^2\frac{t}{1-t} \;=\; 3t(1-t)\,.$$

Hence R_t is a non-negative process on $[0,1)$ with $R_0 = 0$ and $\lim_{t\to 1}\mathbb{E}(R_t^2) = 0$, so in particular $R_t \to 0$ as $t \to 1$.

Remark: R_t is known as the Bessel-3 bridge or the Brownian excursion (see plot below).

c. Let M_t and m_t denote the max and min process of Brownian motion. Compute $\mathbb{P}(M_t = 0 \cup m_t = 0)$. **[30%]**

Solution. $\mathbb{P}(M_t = 0) = 0$ for $t > 0$ because M_t has a density, and hence so does m_t because $m_t \sim -M_t$. Then

$$\mathbb{P}(M_t = 0 \cup m_t = 0) \;\leq\; \mathbb{P}(M_t = 0) + \mathbb{P}(m_t = 0) \;=\; 0$$

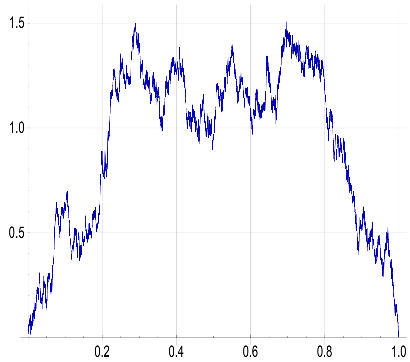

Figure 1: Monte Carlo simulation of R_t process in q1.

2. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. For the Black-Scholes model, write down the unique no-arbitrage price at time t of a contract paying $f(S_T)$ at time T, and the Delta of this contract at time t.

Solution. From the Feynman-Kac formula

$$P(S,t) \;=\; e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(f(S_T)|\mathcal{F}_t^W) \;=\; e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(f(S_T)|S_t)$$

and the Delta at time t is $P_S(S_t,t)$.

b. Let

$$S_t \;=\; S_0 + \int_0^t Y_u du + \sigma W_t$$

where Y is an OU process $dY_t = -\theta Y_t dt + dB_t$ (independent of W) for which $\mathbb{E}(Y_t) = 0$ and $\mathbb{E}(Y_s Y_t) = \frac{1}{2\theta}(e^{-\theta|t-s|} - e^{-\theta|t+s|})$ with $\theta > 0$. Compute the covariance of S_s and S_t.

Solution. $\mathbb{E}(S_t) = S_0$ because $\mathbb{E}(\int_0^t Y_u du) = \int_0^t \mathbb{E}(Y_u)du = 0$ and $\mathbb{E}(\sigma W_t) = 0$, so the required covariance is

$$\begin{aligned}\mathbb{E}((S_s - S_0)(S_t - S_0)) \;&=\; \mathbb{E}((\int_0^s Y_u du + \sigma W_s)(\int_0^t Y_v dv + \sigma W_t)) \\ &=\; \mathbb{E}(\int_0^s Y_u du \int_0^t Y_v dv) + \sigma^2 \mathbb{E}(W_s W_t)\end{aligned}$$

and the cross terms have vanished since Y and W are independent and both have zero mean, and

$$\begin{aligned}\mathbb{E}(\int_0^s Y_u du \int_0^t Y_v dv) \;&=\; \int_0^s \int_0^t \frac{1}{2\theta}(e^{-\theta|u-v|} - e^{-\theta|u+v|})dudv \\ \mathbb{E}(W_s W_t) \;&=\; \min(s,t)\end{aligned}$$

Note we have not assumed that $0 \leq s \leq t$.

Remark. S is a Gaussian process, and this type of process with an independent Gaussian drift process is often used in optimal trade execution problems, see e.g. FM20 module after Xmas.

c. Consider the CEV process $dS_t = S_t^\beta dW_t$ with $S_0 > 0$ and $\beta > 1$. Using that

$$\mathbb{E}(\frac{S_t}{S_0}) \quad = \quad 1 - \frac{\Gamma(a,b)}{\Gamma(a,0)}$$

where $a = \frac{1}{2(\beta-1)}$, $b = \frac{2a^2 S_0^{-1/a}}{t}$, and $\Gamma(a,b) = \int_b^\infty s^{a-1}e^{-s}ds$, is S a martingale with respect to $\mathcal{F}^W$? Using that $\mathbb{P}(S_t \geq K) \leq \frac{1}{K}\mathbb{E}(S_t)$ for $K > 0$, describe the behaviour of S_t as $t \to \infty$. **[40%]**

Solution. S is not a martingale since $\mathbb{E}(S_t) < S_0$ for $t > 0$ unless $b = \infty$, which only happens when $t = 0$. For the second part, we first note that $\lim_{t\to\infty} \mathbb{E}(S_t) = 0$ because $b \to 0$ as $t \to \infty$. Then for $K > 0$, since $S_t \geq 0$, we see that

$$\mathbb{P}(|S_t - 0| \geq K) \quad = \quad \mathbb{P}(S_t \geq K) \quad \leq \quad \frac{1}{K}\mathbb{E}(S_t) \quad \to \quad 0$$

as $t \to \infty$ (since $1_{S_t \geq K} \leq \frac{1}{K}S_t$ so $\mathbb{E}(1_{S_t\geq K}) = \mathbb{P}(S_t \geq K) \leq \frac{1}{K}\mathbb{E}(S_t)$, and $|S_t - 0| = S_t$), so S_t tends to zero in probability as $t \to \infty$.

d. Consider a jump process X_t with i.i.d. increments for which $M(p) = \mathbb{E}(e^{p(X_t - X_s)}|X_s) = e^{c(t-s)((G-p)^Y + (G+p)^Y - 2G^Y)}$ for $p \in [-G, G]$ with $Y \in (1,2)$, and $c > 0$. Using basic properties of mgfs, compute

$$\mathbb{E}(\sum_{i=0}^{n-1}(X_{(i+1)t/n} - X_{it/n})^2)\,.$$

You may use that $M''(0) = a(t-s)$, where $a = 2cG^{Y-2}(Y-1)Y$.

Solution. M is an mgf, so $M''(0) = \mathbb{E}((X_t - X_s)^2|X_s)$ (see AppliedProbabilityRevision chapter). Then the answer is

$$\mathbb{E}(\sum_{i=0}^{n-1}(X_{(i+1)t/n} - X_{it/n})^2) \quad = \quad n\mathbb{E}((X_{(i+1)t/n} - X_{it/n})^2) \quad = \quad na\Delta t \quad = \quad at$$

where $\Delta t = t/n$.

3. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Let $M_t = \max_{0\leq s\leq t} W_s$. Write down an integral expression for the cdf of $2M_t - W_t$ in terms of the joint density $f(x,b)$ of (W_t, M_t).

Solution.

$$\mathbb{P}(2M_t - W_t \leq \rho) \quad = \quad \mathbb{E}(1_{2M_t - W_t \leq \rho}) \quad = \quad \int_0^\infty \int_{-\infty}^b 1_{\{2b-x\leq\rho\}} f(x,b) dx db\,. \qquad (1)$$

Let $u = 2b - x$ so $du = -dx$. Then for b fixed, when x runs from $-\infty$ to b:

$$x = -\infty \;\Rightarrow\; u = 2b - (-\infty) \quad = \quad +\infty, \qquad x = b \;\Rightarrow\; u = 2b - b = b.$$

Hence we can re-write the inner integral in Eq (1) as

$$\int_\infty^b 1_{\{u\leq\rho\}} f(2b-u,b)\,(-du) \quad = \quad \int_b^\infty 1_{\{u\leq\rho\}} f(2b-u,b)\,du \quad = \quad \int_b^\rho f(2b-u,b)\,du\,.$$

Not asked for here, but if we evaluate the double integral in Eq (1) using the explicit expression for $f(x,b)$ in the Reflection Principle chapter and then differentiate wrt ρ, we find that the density of ρ is $\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\rho^2}{t^{\frac{3}{2}}}e^{-\frac{\rho^2}{2t}}$.

b. From e.g. Hwk4 q2, we know that

$$\mathbb{P}(W_t \in dx, M_t < b, m_t > a) = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\lambda_n t}\sin(\frac{n\pi(x-a)}{b-a})\sin(\frac{n\pi(0-a)}{b-a})dx$$

for $a < x < b$, where $\lambda_n = \frac{n^2\pi^2}{2(b-a)^2}$, and M_t and m_t are the running max and min processes of W. Using that $\int_a^b \sin(\frac{n\pi(x-a)}{b-a})dx = \frac{b-a}{n\pi}(1-(-1)^n)$, explain how we can jointly simulate M_t and m_t.

Solution. (see also Hwk4 q2 and quiz 2, q12).

$$\begin{aligned} P(a,b) = \mathbb{P}(M_t < b, m_t > a) &= \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-\lambda_n t}(1-(-1)^n)\sin(-\frac{n\pi a}{b-a}) \\ &= -\sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-\lambda_n t}(1-(-1)^n)\sin(\frac{n\pi a}{b-a}) \\ &= -\sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} e^{-\lambda_{2k-1} t}\sin(\frac{(2k-1)\pi a}{b-a}). \end{aligned}$$

Moreover, $\frac{\partial}{\partial b}P(a,b)db = \mathbb{P}(M_t \in db, m_t > a)$, so (from Bayes rule) we see that

$$\mathbb{P}(m_t > a | M_t = b) = \frac{\mathbb{P}(M_t \in db, m_t > a)}{\mathbb{P}(M_t \in db)} = \frac{\frac{\partial}{\partial b}P(a,b)}{p(b)} \qquad (b \geq 0)$$

where $p(b) = \frac{2}{\sqrt{2\pi t}}e^{-\frac{b^2}{2t}}$ is the (marginal) density of M_t (see HittingTimes+ReflectionPrinciple chapter), and note that two db's cancelled out in the final term. Then the conditional cdf of m_t is $F_{m_t|M_t}(a;b) = \mathbb{P}(m_t \leq a | M_t = b) = 1 - \frac{\frac{\partial}{\partial b}P_b(a,b)}{p(b)}$.

Finally we sample M_t using $F^{-1}_{M_t}(U)$ (see Applied Probability Revision chapter) and m_t as $F^{-1}_{m_t|M_t}(U_2; M_t)$ (see q3 Hwk 6), where U and U_2 are i.i.d. $U[0,1]$ random variables.

c. For a 1d diffusion $dX_t = \sigma(X_t)dW_t$ with $0 < \underline{\sigma} \le \sigma \le \bar{\sigma} < \infty$, we have

$$p_t(x,y)dy \;=\; \mathbb{P}(X_t \in dy, \bar{X}_t < b, \underline{X}_t > a | X_0 = x) \;=\; m(y)\sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_{\lambda_n}(x)\psi_{\lambda_n}(y)dy$$

for $a < x < b$ for some sequence $0 < \lambda_1 < \lambda_2 < ...$, where $m(y) = 1/\sigma(y)^2$, and ψ_λ satisfies $\frac{1}{2}\sigma(x)^2\psi''_\lambda(x) + \lambda\psi_\lambda(x) = 0$ with $\psi_\lambda(a) = \psi_\lambda(b) = 0$, and $\int_a^b \psi_{\lambda_m}(y)\psi_{\lambda_n}(y)m(y)dy = 1_{\{m=n\}}$.

Use this expansion to compute the price P of a double barrier option which pays $\max(X_T - K, 0)1_{\bar{X}_T < b, \underline{X}_T > a}$ at time T given that $X_0 = x$ for $K < b$ (you may assume interest rates are zero).

Solution. We just integrate $p_T(x,y)$ like a standard 1d density to get

$$P \;=\; \int_a^b (y-K)^+ p_T(x,y)dy$$

d. We know that the density of the maximum M_t of Brownian motion is $p_t(b) = \frac{2}{\sqrt{2\pi t}}e^{-\frac{b^2}{2t}}$, and one can also show that

$$\int_0^\infty \lambda e^{-\lambda t}p_t(b)dt \;=\; \sqrt{2\lambda}\, e^{-b\sqrt{2\lambda}}.$$

What does this tell us?

Solution. It tells us that $M_{T_\lambda} \sim \text{Exp}(\sqrt{2\lambda})$, if T_λ is an $\text{Exp}(\lambda)$ random variable which is independent of W, i.e. the maximum of W evaluated at an independent exponential time is also an exponential random variable.

e. Compute the mgf of M_{T_λ} and m_{T_λ}, where T_λ is defined in the previous question.

Solution.

The mgf of an $\text{Exp}(\lambda)$ random variable is $\frac{\lambda}{\lambda - p}$ for $p < \lambda$ (and $+\infty$ otherwise), so $\mathbb{E}(e^{pM_{T_\lambda}}) = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda}-p}$. By symmetry $m_{T_\lambda} \sim -M_{T_\lambda}$, so $\mathbb{E}(e^{pm_{T_\lambda}}) = \mathbb{E}(e^{-pM_{T_\lambda}}) = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda}+p}$ for $p > -\sqrt{2\lambda}$, and $+\infty$ otherwise.

4. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Construct a function G so that $G(W_T)$ has a given density μ (you may assume μ is strictly positive for simplicity and that $\int_{-\infty}^{\infty} |x|\mu(x)dx < \infty$). Use this to construct a martingale X such that $X_T \sim \mu$

Solution. Let $F_\mu(x) = \int_{-\infty}^{x} \mu(u)du$ denote the cdf associated with μ. W_T has cdf $\mathbb{P}(W_T \leq x) = \Phi(\frac{x}{\sqrt{T}})$, so (from Eq 8 in the AppliedProbabilityRevision chapter), we know that $U = \Phi(\frac{W_T}{\sqrt{T}}) \sim U[0,1]$, since $Z = \frac{W_T}{\sqrt{T}} \sim N(0,1)$ and Φ is the cdf of Z.

Then (from Eq 9 in the AppliedProbabilityRevision chapter) we know that $F_\mu^{-1}(U)$ has density μ. Hence the required function here is

$$G(x) \;=\; F_\mu^{-1}(\Phi(\frac{x}{\sqrt{T}}))\,.$$

For the final part, we just set $X_t = \mathbb{E}(G(W_T)|\mathcal{F}_t^W) = \mathbb{E}(G(W_T)|W_t) = g(W_t,t)$ for some $C^{2,1}$ function $g(x,t)$, which is an $\mathcal{F}^W$-martingale (see q10 in Revision quiz 2 for proof) [1].

b. Apply Ito's lemma to the process X_t, and simplify your expression for dX_t.

Solution.

$$dX_t \;=\; dg(W_t,t) \;=\; g_x(W_t,t)dW_t + (\frac{1}{2}g_{xx}(W_t,t) + g_t(W_t,t))dt\,.$$

But X is a martingale so the drift must be zero, hence $dX_t = g_x(W_t,t)dW_t$. If we want, we can further write this as $dX_t = g_x(g^{-1}(X_t,t),t)dW_t$, so we see X is a diffusion process of the form $dX_t = \sigma(X_t,t)dW_t$.

[1] note we require the technical condition that $\mathbb{E}(|G(W_T)|) < \infty$ but $\mathbb{E}(|G(W_T)|) = \int_{-\infty}^{\infty} |x|\mu(x)dx$ because $G(W_T) \sim \mu$ by construction, and the integral here is finite by assumption

5. Throughout this question, we let $W = (W_t)_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

a. Find the SDE satisfied by $X_t = \sin W_t$.

Solution. From Ito's lemma

$$dX_t = \cos(W_t)dW_t - \frac{1}{2}\sin(W_t)dt = \sqrt{1-X_t^2}dW_t - \frac{1}{2}X_t dt$$

where the second equality only holds for $t < \min\{s : |W_s| \geq \frac{1}{2}\pi\}$.

b. Let

$$dY_t = \frac{b - Y_t}{T - t}dt + dW_t\,. \tag{2}$$

Verify that $Y_t = a(1-\frac{t}{T}) + b\frac{t}{T} + (T-t)\int_0^t \frac{dW_u}{T-u}$ satisfies this eq.

Solution.

$$dY_t = -\frac{a}{T}dt + \frac{b}{T}dt - \int_0^t \frac{1}{T-u}dW_u dt + dW_t \tag{3}$$

But $\frac{b-Y_t}{T-t} = \frac{b}{T} - \frac{a}{T} - \int_0^t \frac{dW_u}{T-u}$, so Eq 3 simplifies to Eq 2.

c. Let (Y_i) be a sequence of i.i.d. random variables with density $\nu(x)$ and mgf $M(p) = \mathbb{E}(e^{pY_i}) = e^{V(p)}$, and $(N_t)_{t\geq 0}$ be a Poisson process for which $\mathbb{E}(e^{pN_t}) = e^{\lambda t(e^p - 1)}$ (and recall that $N_t \in \mathbb{N}$), and assume N_t is independent of (Y_i). Compute the mgf of $X_t = \sum_{i=1}^{N_t} Y_i$.

Solution. From the tower property of conditional expectation

$$\mathbb{E}(e^{p\sum_{i=1}^{N_t} Y_i}) = \mathbb{E}(\mathbb{E}(e^{p\sum_{i=1}^{N_t} Y_i}|N_t)) = \mathbb{E}(M(p)^{N_t}) = \mathbb{E}(e^{V(p)N_t}) = e^{\lambda t(e^{V(p)}-1)}$$

where the second equality follows since the Y_i's are i.i.d. We can further re-write the final line as

$$\mathbb{E}(e^{pX_t}) = e^{\lambda t(\int_{-\infty}^{\infty} e^{px}\nu(x)dx - 1)} = e^{\lambda t\int_{-\infty}^{\infty}(e^{px}-1)\nu(x)dx}\,. \tag{4}$$

Note we can also consider $\tilde{X}_t = \mu t + \sigma W_t + X_t$ as a log stock price process (with W and X independent), which is known as a **jump-diffusion** model, which is a special case of a Lévy process. Eq 4 is known as the Lévy-Khintchine formula.