For $q = 2$

$$\mathbb{E}(SS_n^{(q)}) \quad = \quad \frac{1}{n}\mathbb{E}(\sum_{i=1}^n |Z_{i\Delta}^H - Z_{(i-1)\Delta}^H|^q) \quad = \quad \Delta^{qH}\frac{1}{n}\sum_{i=1}^n \mathbb{E}(|Z_i^H - Z_{i-1}^H|^q) \quad \sim \quad c_H\Delta^{qH}$$

as $n \to \infty$, for some constant $c_H > 0$ with $c_H \to 1$ as $H \nearrow \frac{1}{2}$.

Alternate estimators for H

The $m(q,\Delta)$ estimator from [GJR18]

Process with history no longer self-similar...

For the first task, let $SS_n^{(q)} := \frac{1}{n}\sum_{i=1}^n |B_{i\Delta}^H - B_{(i-1)\Delta}^H|^q$. Then

$$\mathbb{E}(SS_n^{(q)}) \quad = \quad \frac{1}{n}\mathbb{E}(\sum_{i=1}^n |B_{i\Delta}^H - B_{(i-1)\Delta}^H|^q) \quad = \quad \Delta^{qH}\frac{1}{n}\sum_{i=1}^n \mathbb{E}(|B_i^H - B_{i-1}^H|^q) \quad = \quad \mathbb{E}(|Z|^q)\Delta^{qH} \quad = \quad K_q\Delta^{qH} \tag{1}$$

where $K_q = \mathbb{E}(|Z|^q) = \frac{2^{q/2}}{\sqrt{\pi}}\Gamma(\frac{q+1}{2})$, with $q > -1$ and $Z \sim N(0,1)$. From this we can then derive the estimator for the first task. Alternatively, if let $X = \sigma B_t^H$ and now define $SS_n^{(q)} = \frac{1}{n}\sum_{i=1}^n |X_{i\Delta}^H - X_{(i-1)\Delta}^H|^q$, and assume H and σ are unknown, then (1) changes to

$$\mathbb{E}(SS_n^{(q)}) \quad = \quad \sigma^q K_q\Delta^{qH}$$

which leads to the estimates $(\hat{H}_n, \hat{\sigma}_n)$ for (H,σ) defined by

$$SS_n^{(q)} \quad = \quad \hat{\sigma}_n^q\, K_q\Delta^{q\hat{H}_n}$$

if we have computed $SS_n^{(q)}$ for at least two Δ-values. Taking logs we see that

$$\log SS_n^{(q)} \quad = \quad q\log\hat{\sigma}_n + \log K_q \; + \; q\hat{H}_n\log\Delta$$

so we can perform **linear regression** on $\log SS_n^{(q)}$ vs $\log\Delta$ for a range of Δ-values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal $q\hat{H}_n$ (q is chosen by you, e.g. $q = 1, 2, 2.5, 3$ etc), and the **intercept** at $\log\Delta = 0$ is $q\log\hat{\sigma}_n + \log K_q$, from which we can compute $\hat{\sigma}_n$ since K_q has an explicit formula above. This is the $m(q,\Delta)$ estimator discussed in [GJR18]. One can then also compute the R^2**-statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of $\hat{H}_n$ and $\hat{\sigma}_n$.

The Han-Schied [HS21] estimator

Let $X_t = \sigma B_t^H$ and let

$$\theta_{m,k} \quad = \quad 2^{\frac{m}{2}}(2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}) \quad = \quad -2^{\frac{m}{2}}(X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{2k}{2^{m+1}}})$$

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for $R(s,t) = \mathbb{E}(B_s^H B_t^H)$, one can check that

$$\mathbb{E}(\theta_{m,k}^2) \quad = \quad \sigma^2 2^{m-2H(1+m)}(4 - 4^H)\,. \tag{2}$$

Then setting $s_n^2 = \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\theta_{m,k}^2$, we see that $\mathbb{E}(s_n^2) = \sum_{m=0}^{n-1} 2^m\mathbb{E}(\theta_{m,k}^2)$ (since (2) does not depend on k) which simplifies to

$$\mathbb{E}(s_n^2) \quad = \quad \sigma^2(4^{n(1-H)} - 1) \quad \sim \quad \sigma^2 4^{n(1-H)} \quad = \quad \sigma^2 2^{2n(1-H)} \tag{3}$$

as $n \to \infty$, which suggests an estimator $\hat{H}_n$ defined by $s_n = \hat{\sigma}_n\, 2^{n(1-\hat{H}_n)}$ which (assuming $\hat{\sigma}_n = O(1)$ as $n \to \infty$) we can re-arrange as

$$\hat{H}_n \quad = \quad 1 - \frac{1}{n}\log_2(\frac{s_n}{\hat{\sigma}_n}) \quad = \quad 1 - \frac{1}{n}\log_2 s_n \; + \; O(\frac{1}{n})$$

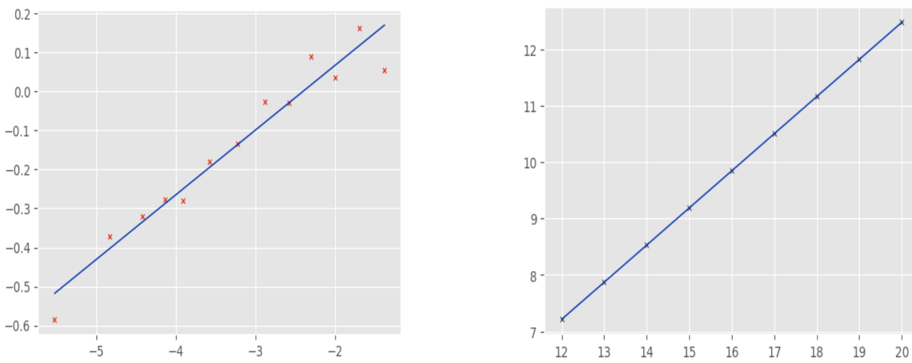

Figure 1: On the left, we see estimates of H for the SPX using the $m(q,\Delta)$ method for the SPX from 3rdJan22-15thJul24 for $q=2$ for which $\hat{H}=0.0830$ and $\hat{\sigma}=1.221$ (see similar plots in [GJR18]. On the right we see the linear regression in (4) for the Han-Schied method ($\log s_n$ vs n) for a true fBM path with 2^{20} time points, for which $\hat{H}=0.0508$, and $\hat{\sigma}=1.010$.

where $\log_2$ denotes the base-2 logarithm, so (ignoring the $O(\frac{1}{n})$ remainder term), we recover the Han-Schied[HS21] estimator $\hat{H}_n = 1-\frac{1}{n}\log_2 s_n$. Then

$$\begin{aligned}\mathbb{E}(\hat{H}_n) \;=\; 1-\frac{1}{n}\mathbb{E}(\log_2(s_n)) \;=\; 1-\frac{1}{2n}\mathbb{E}(\log_2(s_n^2)) \;\geq\; 1-\frac{1}{2n}\log_2\mathbb{E}(s_n^2) &= 1-\frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)}-1))\\ &\geq 1-\frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)}))\\ &= 1-\frac{1}{2n}\log_2(\sigma^2) - \frac{1}{2n}\log_2(4^{n(1-H)}))\\ &= H-\frac{1}{2n}\log_2(\sigma^2)\end{aligned}$$

and the final line is $>H$ if $\sigma<1$, so $\mathbb{E}(\hat{H}_n)>H$ if $\sigma<1$. See also discussion on optimal scaling factors in [HS21].

For more a general process X, (under certain conditions) [HS21] show that $\hat{H}_n \to R$ as $n\to\infty$, where $R=1/q$, where q is the critical p-value at which $\lim_{n\to\infty}\sum_{i=1}^{n}|X_{i\Delta}-X^H_{(i-1)\Delta}|^p$ switches from being zero (for $p>q$) to $+\infty$ (for $p<q$) (see Eq 2.2 in [HS21]).

Or we can jointly estimate H and σ by performing linear regression since

$$\log s_n \;=\; \log\hat{\sigma}_n \;+\; n(1-\hat{H}_n)\log 2 \tag{4}$$

but we now have to compute $\log s_n$ for a range of different n-values to get a line of best fit, for which the slope is $(1-\hat{H}_n)\log 2$ and the intercept is $\log\hat{\sigma}_n$.

You can then draw histograms of $\hat{H}_n$ if you simulate M fBM paths and compute the sample variance for $\hat{H}_n$ (or a confidence interval), or compute $\hat{H}_n$ for real data, e.g. using the SPX data file or data from yahoo finance.

The rough Heston model

The driftless rough Heston model satisfies

$$V_t \;=\; V_0 + \nu \int_0^t (t-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \,.$$

Then $\mathbb{E}(V_t) = V_0$, and V has covariance function:

$$\begin{aligned}
\mathbb{E}((V_s - V_0)(V_t - V_0)) &= \nu^2 \mathbb{E}(\int_0^s (s-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \cdot \int_0^t (t-r)^{H-\frac{1}{2}} \sqrt{V_r} dW_r) \\
&= \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \mathbb{E}(V_u) du \\
&= V_0 \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du \;=\; V_0 \nu^2 \bar{R}(s,t)
\end{aligned}$$

for $0 \le s \le t$, where $\bar{R}(s,t)$ is the covariance function for the Riemann-Liouville (RL) process $Z_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u$ used for the rough Bergomi model (note Z is a Gaussian process but V is not), but the explicit formula for $\bar{R}(s,t)$ is more complicated than the $R(s,t)$ formula for fBM.

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H \;=\; c_H (\int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}) dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s) \;=\; c_H (A_t + Z_t)$$

for $t \ge 0$, in terms of an RL process Z (and note that A_t is known at time zero for all $t \ge 0$), and $c_H = (\frac{2H\,\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\,\Gamma(2-2H)})^{\frac{1}{2}}$. Note also that A_t and Z_t are independent.

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