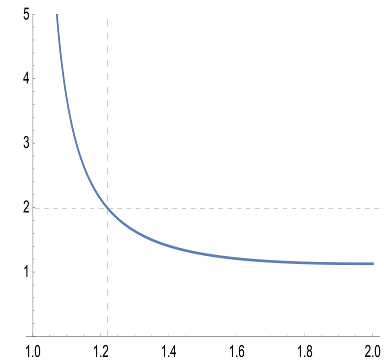

Figure 1: Plot of $F(\alpha, t)$ in q3 as a function of α (for $t = 1$ fixed).

Homework 5

1. Plot the value of a European call option as a function of the initial stock price at $t = 0$ for two different maturities T_1 and T_2 (assume that $0 < T_1 < T_2$ and the interest rate $r = 0$). Derive the asymptotic behaviour of the call price as $S_0 \to \infty$.

Solution. The unique no-arbitrage call option price $C(S_0, 0)$ at $t = 0$ satisfies

$$\frac{C(S_0,0)}{S_0} \quad = \quad \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \quad = \quad \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}((S_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - K)^+) \quad = \quad \mathbb{E}^{\mathbb{Q}}((e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - \frac{K}{S_0})^+)\,.$$

This tends to 1 as $S_0 \to \infty$ by the dominated convergence theorem, since the expression inside the expectation is less than or equal to $\mathcal{E} = e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$ (and tends monotonically to $\mathcal{E}$ as $S_0 \to \infty$ a.s. since $\frac{K}{S_0} \to 0$), and $\mathbb{E}^{\mathbb{Q}}(\mathcal{E}) = 1 < \infty$. Hence we have shown that $C(S_0, 0) \sim S_0$ as $S_0 \to \infty$.

2. Consider a portfolio of buying $\frac{1}{\Delta}$ call options with strike K, and selling $\frac{1}{\Delta}$ call options of strike $K + \Delta$ for $\Delta > 0$. Plot the terminal payoff function $f(S)$ of this portfolio as a function of S. What does this payoff tend to as $\Delta \to 0$?

Solution. The payoff function $f(S) = 0$ for $S \leq K$, $f(S) = \frac{1}{\Delta}(S - K - (S - K - \Delta)) = 1$ for $S \geq K + \Delta$, and $f(S) = \frac{1}{\Delta}(S - K)$ for $S \in [K, K + \Delta]$, and in particular we see that $f(.)$ is continuous at $S = K$ and $K + \Delta$ (This is known as a **call spread** strategy.). From a picture, we see that this tends to a digital call payoff $1_{S_T \geq K}$ as $\Delta \to 0$.

3. A **symmetric α-stable process** X with parameters $\alpha \in (0, 2]$, $\sigma > 0$ is a generalization of Brownian motion, which has independent stationary increments like Brownian motion but now $\mathbb{E}(e^{iu(X_t - X_s)}|X_s) = e^{-(t-s)\sigma^\alpha |u|^\alpha}$ for $u \in \mathbb{R}$ and $0 \leq s \leq t$, so X is only a (multiple of) BM if $\alpha = 2$, but for $\alpha < 2$ the increments of X are not normally distributed. If $\bar{X}_t = \max_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \min_{0 \leq s \leq t} X_s$, it can be shown that

$$\mathbb{E}(\bar{X}_t - \underline{X}_t) \quad = \quad F(\alpha, t) \quad := \quad \frac{2\alpha\Gamma(1 - \frac{1}{\alpha})}{\pi} t^{\frac{1}{\alpha}}$$

for $\alpha \in (1, 2]$. Use this identity to define a statistical estimator $\hat{\alpha}$ for α (assuming $\sigma = 1$) from observed values for $\bar{X}_t$ and $\underline{X}_t$. Is $\hat{\alpha}$ biased? (you may use that $\frac{\partial^2}{\partial \alpha^2} F(\alpha, t) > 0$).

Solution. We just solve $\bar{X}_t - \underline{X}_t = F(\hat{\alpha}, t)$ for $\hat{\alpha}$ to get $\hat{\alpha}$ (see plot of $F(\alpha, t)$ above for $t = 1$). For the 2nd part, we see that

$$F(\alpha, t) \quad = \quad \mathbb{E}(\bar{X}_t - \underline{X}_t) \quad = \quad \mathbb{E}(F(\hat{\alpha}, t)) \quad \geq \quad F(\mathbb{E}(\hat{\alpha}), t)$$

where the final inequality follows from **Jensen's inequality** from last lecture. Assuming the $\geq$ is actually a $>$ here, we can apply F^{-1} to both sides and (since F^{-1} is decreasing), we see that $\alpha < \mathbb{E}(\hat{\alpha})$, so $\hat{\alpha}$ is biased.

4. Let $M_t^{(n)} = \max_{0\le k\le n} W_{kt/n}$ denote the **discretely sampled** maximum of W. Is $M_t^{(n)}$ more or less than $M_t = \max_{0\le s\le t} W_s$? It is known that

$$\mathbb{E}(M_t^{(n)}) \;=\; \sqrt{\frac{t}{2\pi n}}\, H_n^{(\frac{1}{2})}$$

where $H_n^{(\frac{1}{2})} = \sum_{k=1}^n k^{-\frac{1}{2}}$ is known as the nth Harmonic number of order $\frac{1}{2}$. Using that $H_n^{\frac{1}{2}} \sim 2\sqrt{n}$ as $n\to\infty$, show this formula is consistent with the formula $\mathbb{E}(R_t) = 2\sqrt{\frac{2t}{\pi}}$ in Hwk4, q5, where R_t is the range of W.

Solution. $M_t^{(n)} \le M_t$. Using the given asymptotic relation, we see that

$$\mathbb{E}(M_t^{(n)}) \;\sim\; \sqrt{\frac{t}{2\pi n}}\, 2\sqrt{n} \;=\; \sqrt{\frac{2t}{\pi}}$$

and we see this is half $\mathbb{E}(R_t)$ as expected, since $\mathbb{E}(R_t) = \mathbb{E}(M_t) - \mathbb{E}(m_t) = 2\mathbb{E}(M_t)$ since $m_t \sim -M_t$, where $m_t = \min_{0\le s\le t} W_s$.

5. Consider the famous **GARCH(1,1)** discrete-time stochastic volatility model where

$$\begin{aligned} r_t &= \sqrt{V_t}\,\varepsilon_t \\ V_t &= \omega + \alpha r_{t-1}^2 + \beta V_{t-1} \end{aligned}$$

for $t\in\mathbb{Z}$ and $\omega,\alpha,\beta>0$, where $r_t = (S_t - S_{t-1})/S_{t-1}$ is the return on a stock price process S at time t, and ε_t is an i.i.d. sequence of random variables with $\mathbb{E}(\varepsilon_t)=0$ and $\mathrm{Var}(\varepsilon_t)=1$. Express V_t in terms of the ε_t sequence only, and compute $\mathbb{E}(V_t)$ and give a condition on the model parameters for the formula to make sense.

Solution. We first note that

$$V_t \;=\; \omega + \alpha r_{t-1}^2 + \beta V_{t-1} \;=\; \omega + \alpha V_{t-1}\varepsilon_{t-1}^2 + \beta V_{t-1} \;=\; \omega + A_t V_{t-1} \tag{1}$$

where $A_t = \alpha\varepsilon_{t-1}^2 + \beta$. "Unrolling" this expression, we see that

$$\begin{aligned} V_t \;=\; \omega + A_t(\omega + A_{t-1}V_{t-2}) &= \omega + A_t(\omega + A_{t-1}(\omega + A_{t-2}V_{t-3})) \\ &= \omega + A_t\omega + A_tA_{t-1}\omega + A_tA_{t-1}A_{t-2}V_{t-3} \\ &= \omega(1 + A_t + A_tA_{t-1} + A_tA_{t-1}A_{t-2} + \ldots)\,. \end{aligned}$$

We also note that $\mathbb{E}(A_t) = \alpha+\beta$ for all t, so

$$\mathbb{E}(V_t) \;=\; \omega(1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots) \;=\; \frac{\omega}{1-\alpha-\beta}$$

(using the formula for the sum of a **Geometric series**), so the model only makes sense when $\alpha+\beta<1$, which is known as the **stationarity condition** which ensures that V_t has the same distribution for all t (see notes on my webpage for more details on the model, many students have looked at this model in their projects and we often use the student t-distribution for the ε_t's).