

## The $m(q, \Delta)$ estimator for fBM

Let  $X_t = \sigma B_t^H$ , and set  $SS_n^{(q)} := \frac{1}{n} \sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^q$ . Then

$$\mathbb{E}(SS_n^{(q)}) = \frac{1}{n} \mathbb{E}\left(\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^q\right) = \Delta^{qH} \frac{1}{n} \sum_{i=1}^n \mathbb{E}(|X_i - X_{i-1}|^q) = \sigma^q \mathbb{E}(|Z|^q) \Delta^{qH} = \sigma^q K_q \Delta^{qH} \quad (1)$$

where  $\Delta = \Delta_n = \frac{1}{n}$ ,  $K_q = \mathbb{E}(|Z|^q) = \frac{2^{q/2}}{\sqrt{\pi}} \Gamma(\frac{q+1}{2})$ , (for  $q > -1$ ) and  $Z \sim N(0, 1)$ .

This leads to a simple (and scale-independent) estimator for  $H$  given by

$$\hat{H}_n = -\frac{1}{2} \log_2 \frac{SS_n^{(2)}}{SS_{\frac{1}{2}n}^{(2)}} \quad (2)$$

which corresponds to just using two points for the linear regression method described in Eq (10) below, and one can show using a joint CLT for  $n^{2H} SS_n^{(2)} - 1$  and  $(\frac{1}{2}n)^{2H} SS_{\frac{1}{2}n}^{(2)} - 1$  that

$$\sqrt{n} (\hat{H}_n - H) \xrightarrow{d} \mathcal{N}(0, V_H)$$

where

$$V_H = \frac{1}{2(\ln 2)^2} \left( \sum_{k \in \mathbb{Z}} \rho_H(k)^2 + 2 \sum_{m \in \mathbb{Z}} r_W(m)^2 - 4 \sum_{m \in \mathbb{Z}} r_{ZW}(m)^2 \right) \quad (3)$$

(and we have numerically verified this), where

$$r_W(m) := 2^{-2H} (2\rho_H(2m) + \rho_H(2m-1) + \rho_H(2m+1)), \quad r_{ZW}(m) := 2^{-H} (\rho_H(2m) + \rho_H(2m-1)).$$

which allows us to compute a confidence interval for  $H$ .

However,  $\hat{H}_n$  is not **asymptotically optimal** since the lower bound for the variance for *any* consistent estimator for  $H$  (with  $\sigma$  unknown) is the (1,1) component of the inverse of the Fisher information matrix:

$$I(H, \sigma) = \frac{1}{2\pi} \begin{bmatrix} \int_0^\pi \left( \frac{\frac{\partial}{\partial H} f_H(\omega)}{f_H(\omega)} \right)^2 d\omega & \frac{2}{\sigma} \int_0^\pi \frac{\frac{\partial}{\partial H} f_H(\omega)}{f_H(\omega)} d\omega \\ \frac{2}{\sigma} \int_0^\pi \frac{\frac{\partial}{\partial H} f_H(\omega)}{f_H(\omega)} d\omega & \frac{4}{\sigma^2} \int_0^\pi d\omega \end{bmatrix}$$

where

$$f_H(\omega) = 2 \sin(\pi H) \Gamma(2H+1) (1 - \cos \lambda) \left( \sum_{k=1}^{\infty} \frac{1}{(\lambda + 2\pi k)^{2H+1}} + \sum_{k=-\infty}^{-1} \frac{1}{(-\lambda - 2\pi k)^{2H+1}} + \frac{1}{\lambda^{2H+1}} \right)$$

is the **spectral density** of fGN (see Figure 1 below for a plot of  $I(H, \sigma)_{1,1}^{-1}$  as a function of  $H$ ).

More generally, we can compute joint estimators  $(\hat{H}_n, \hat{\sigma}_n)$  for  $(H, \sigma)$  defined by

$$SS_n^{(q)} = \hat{\sigma}_n^q K_q \Delta^{q\hat{H}_n}$$

if we have computed  $SS_n^{(q)}$  for at least two  $\Delta$ -values. Taking logs we see that

$$\log SS_n^{(q)} = q \log \hat{\sigma}_n + \log K_q + q \hat{H}_n \log \Delta \quad (4)$$

so we can perform **linear regression** on  $\log SS_n^{(q)}$  vs  $\log \Delta = \log \frac{1}{n}$  for a range of  $\Delta$ -values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal  $q\hat{H}_n$  ( $q$  is chosen by you, e.g.  $q = 1, 2, 2.5, 3$  etc), and the **intercept** at  $\log \Delta = 0$  is  $q \log \hat{\sigma}_n + \log K_q$ , from which we can compute  $\hat{\sigma}_n$  since  $K_q$  has an explicit formula above. This is the  $m(q, \Delta)$  estimator discussed in [GJR18]. One can then also compute the  **$R^2$ -statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of  $\hat{H}_n$  and  $\hat{\sigma}_n$ .

To approximate the effect of using **realized variance** with  $m$  subwindows to estimate  $V_t$  (as in Part 2 of the project), we can use the Central Limit Theorem approximation from FM02:

$$V_{i\Delta} = V_0 e^{\sigma B_{i\Delta}^H} \left( 1 + \sqrt{\frac{2}{m}} \varepsilon_i \right)$$

where the  $\varepsilon_i$ 's here are i.i.d.  $N(0, 1)$  (and independent of  $B^H$ ), so (using that  $\log(1+x) = x + O(x^2)$ ), we see that  $\log V_{i\Delta} = \log V_0 + \sigma B_{i\Delta}^H + \sqrt{\frac{2}{m}} \varepsilon_i + O(\frac{1}{m}) = \log V_0 + X_{i\Delta} + \sqrt{\frac{2}{m}} \varepsilon_i + O(\frac{1}{m})$ .

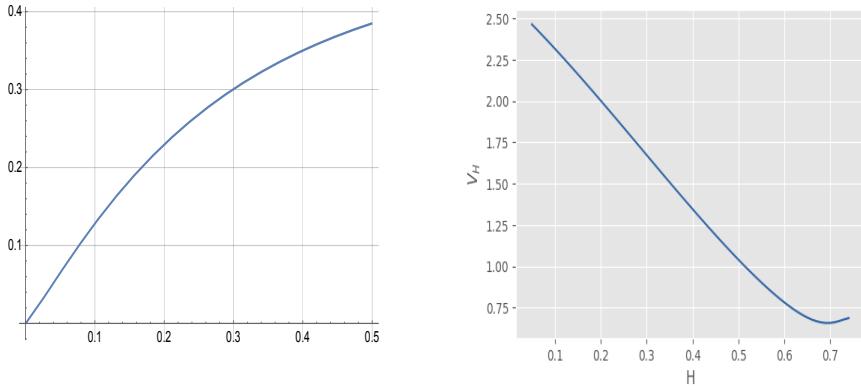


Figure 1: On the left we see the lower bound for variance of  $\sqrt{n}(\hat{H}_n - H)$  for any consistent estimator for  $H$  for  $\sigma B^H$  with  $\sigma$  unknown. On the right we have plotted  $V_H$  in Eq (3).

For convenience we now define  $\tilde{X}_{i\Delta} = X_{i\Delta} + \sqrt{\frac{2}{m}}\varepsilon_i$ . Then  $\tilde{X}_{i\Delta} - \tilde{X}_{(i-1)\Delta} = X_{i\Delta} - X_{(i-1)\Delta} + \sqrt{\frac{2}{m}}(\varepsilon_i - \varepsilon_{(i-1)}) \sim N(0, \sigma^2\Delta^{2H} + \frac{4}{m})$ , and setting  $\widetilde{SS}_n^{(2)} := \frac{1}{n} \sum_{i=1}^n |\tilde{X}_{i\Delta} - \tilde{X}_{(i-1)\Delta}|^2$ , adding the effect of  $\varepsilon$  into the computation above we find that

$$\mathbb{E}(\widetilde{SS}_n^{(2)}) = \sigma^2\Delta^{2H} + \frac{4}{m}$$

(since  $K_q = 1$  for  $q = 2$ ) so we now regress  $\log(\widetilde{SS}_n^{(2)} - \frac{4}{m})$  vs  $\log \Delta$ , which provides a smart adjustment to  $\hat{H}_n$ . (this adjustment is made in

<https://colab.research.google.com/drive/1jJGf4bVWETJqWRIMZ6STjsJ9jINgEaPd#scrollTo=e9fn6ABHNf52>

### The $m(q, \Delta)$ estimator for the RL process

If  $X_t = \sigma Z_t^H$  where  $Z_t^H = \sqrt{2H} \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$  is an RL process, then for  $q = 2$  and  $H \in (0, \frac{1}{2})$  one can show that

$$\mathbb{E}(SS_n^{(2)}) = \frac{1}{n} \mathbb{E}\left(\sum_{i=1}^n |X_{i\Delta} - X_{(i-1)\Delta}|^2\right) = \sigma^2 \Delta^{2H} \frac{1}{n} \sum_{i=1}^n \mathbb{E}(|Z_i^H - Z_{i-1}^H|^2) \sim \sigma^2 c_H \Delta^{2H}$$

as  $n \rightarrow \infty$  where  $c_H = -\frac{4H\Gamma(\frac{1}{2}+H)\Gamma(-2H)}{\Gamma(\frac{1}{2}-H)} > 0$ . Hence for the regression we now consider

$$\log(SS_n^{(2)}) = 2 \log \hat{\sigma}_n + \log c_{\hat{H}} + 2\hat{H}_n \log \Delta$$

so now the intercept is  $2 \log \hat{\sigma}_n + \log c_{\hat{H}}$ , which leads to an adjusted estimate for  $\hat{\sigma}_n$ . This can also be used for driftless rough Heston model below since (modulo an unimportant constant) both processes have the same covariance.

### Convergence of $\hat{H}_n$ to $H$ for the first task in Part 2, and asymptotic normality

Above we considered sums of the form

$$SS_n^{(q)} := \frac{1}{n} \sum_{j=1}^n |B_j^H - B_{j-1}^H|^q. \quad (5)$$

We cannot apply the SLLN to this since the RVs in the sum here are not i.i.d. However, since the increments process  $X_j = B_j^H - B_{j-1}^H \sim N(0, 1)$  is stationary (in particular  $X_j \sim N(0, 1)$  for all  $j$ ) and moreover is known to be **ergodic**, which means that for any measurable function  $g$  with  $\mathbb{E}(|g(X_1)|) < \infty$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n g(X_j) = \mathbb{E}(g(X_1)) \quad \text{a.s.} \quad (6)$$

In general, a stationary Gaussian process  $Y$  is ergodic if its covariance function  $R(k) \rightarrow 0$  as  $k \rightarrow \infty$ , which we verified in the fBM.pdf document. Hence we can apply the general ergodic property in (6) to show that  $SS_n^{(q)}$

in (5) tends to the **non-random** limit  $K_q := \mathbb{E}(|Z|^q)$  a.s. (where  $Z \sim N(0, 1)$ , see Homework 3); hence from the self-similarity of fBM

$$SS_n^{(q)} \xrightarrow{w} \frac{1}{n} n^{qH} \sum_{j=1}^n |B_{j/n}^H - B_{(j-1)/n}^H|^q$$

i.e. the right hand side tends weakly to the constant  $K_q$  as  $n \rightarrow \infty$ , which also implies convergence in probability.

Recall this also suggests the estimator of  $\hat{H} = \hat{H}_n$  for  $H$  defined by the relation

$$K_q = \frac{1}{n} n^{q\hat{H}} \sum_{j=1}^n |B_{j/n}^H - B_{(j-1)/n}^H|^q = \frac{1}{n} n^{qH} \sum_{j=1}^n |B_{j/n}^H - B_{(j-1)/n}^H|^q \cdot n^{q(\hat{H}-H)}.$$

But from the discussion immediately before this, we know that  $\frac{1}{n} n^{qH} \sum_{j=1}^n |B_{j/n}^H - B_{(j-1)/n}^H|^q$  tends to  $K_q$  in probability as  $n \rightarrow \infty$ , hence we must have that  $n^{q(\hat{H}-H)} \rightarrow 1$  in probability as well, which implies that

$$q(\hat{H}_n - H) \log n \rightarrow 0$$

in probability (where we are now emphasizing the dependence of  $\hat{H}$  on  $n$ ), which can only be true if  $\hat{H}_n - H \rightarrow 0$  in probability (recall that  $\hat{H}$  depends on  $n$ ).

Moreover, it can be shown that  $\sqrt{n} \log n (\hat{H}_n - H)$  is asymptotically  $N(0, 2\gamma)$  as  $n \rightarrow \infty$ , where

$$\gamma = \sum_{r=-\infty}^{\infty} (|r+1|^{2H} - 2|r|^{2H} + |r-1|^{2H})$$

(see code and histogram numerically verifying this at <https://colab.research.google.com/drive/18ISGhNSN9ybbcpJ1P0-0je>)

If a function  $X_t$  is  $\frac{1}{q}$  Hölder continuous on  $[0, T]$  i.e.  $|X_t - X_s| \leq c|t - s|^{1/q}$  for some constant  $c$ , then it has finite  $q$ -variation. To see this, we just note that

$$\sup_{\mathcal{P}} \sum_{t_k} |X_{t_k} - X_{t_{k-1}}|^q \leq \sum_{t_k} |c(t_k - t_{k-1})^{\frac{1}{q}}|^q = c^q \sum_{t_k} |t_k - t_{k-1}| = c^q T < \infty \quad (7)$$

where  $\mathcal{P}$  is the set of partitions of  $[0, T]$ .

## The Han-Schied [HS21] estimator for fBM

Let  $X_t = \sigma B_t^H$  and let

$$\theta_{m,k} = 2^{\frac{m}{2}} (2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}) = -2^{\frac{m}{2}} (X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} + X_{\frac{2k}{2^{m+1}}})$$

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for  $R(s, t) = \mathbb{E}(B_s^H B_t^H)$ , one can check that

$$\mathbb{E}(\theta_{m,k}^2) = \sigma^2 2^{m-2H(1+m)} (4 - 4^H). \quad (8)$$

Then setting  $s_n^2 = \sum_{m=0}^{n-1} \sum_{k=0}^{2^m-1} \theta_{m,k}^2$ , we see that  $\mathbb{E}(s_n^2) = \sum_{m=0}^{n-1} 2^m \mathbb{E}(\theta_{m,k}^2)$  (since (8) does not depend on  $k$ ) which simplifies to

$$\mathbb{E}(s_n^2) = \sigma^2 (4^{n(1-H)} - 1) \sim \sigma^2 4^{n(1-H)} = \sigma^2 2^{2n(1-H)} \quad (9)$$

as  $n \rightarrow \infty$ , which suggests an estimator  $\hat{H}_n$  defined by  $s_n = \hat{\sigma}_n 2^{n(1-\hat{H}_n)}$  which (assuming  $\hat{\sigma}_n = O(1)$  as  $n \rightarrow \infty$ ) we can re-arrange as

$$\hat{H}_n = 1 - \frac{1}{n} \log_2 \left( \frac{s_n}{\hat{\sigma}_n} \right) = 1 - \frac{1}{n} \log_2 s_n + O\left(\frac{1}{n}\right)$$

where  $\log_2$  denotes the base-2 logarithm, so (ignoring the  $O(\frac{1}{n})$  remainder term), we recover the Han-Schied[HS21] estimator  $\hat{H}_n = 1 - \frac{1}{n} \log_2 s_n$ . Then

$$\begin{aligned} \mathbb{E}(\hat{H}_n) &= 1 - \frac{1}{n} \mathbb{E}(\log_2(s_n)) = 1 - \frac{1}{2n} \mathbb{E}(\log_2(s_n^2)) \geq 1 - \frac{1}{2n} \log_2 \mathbb{E}(s_n^2) = 1 - \frac{1}{2n} \log_2 (\sigma^2 (4^{n(1-H)} - 1)) \\ &\geq 1 - \frac{1}{2n} \log_2 (\sigma^2 (4^{n(1-H)})) \\ &= 1 - \frac{1}{2n} \log_2 (\sigma^2) - \frac{1}{2n} \log_2 (4^{n(1-H)}) \\ &= H - \frac{1}{n} \log_2 \sigma \end{aligned}$$

and the final line is  $> H$  if  $\sigma < 1$ , so  $\mathbb{E}(\hat{H}_n) > H$  if  $\sigma < 1$ . Note this argument does not show that  $\mathbb{E}(\hat{H}_n) < H$  if  $\sigma > 1$ .

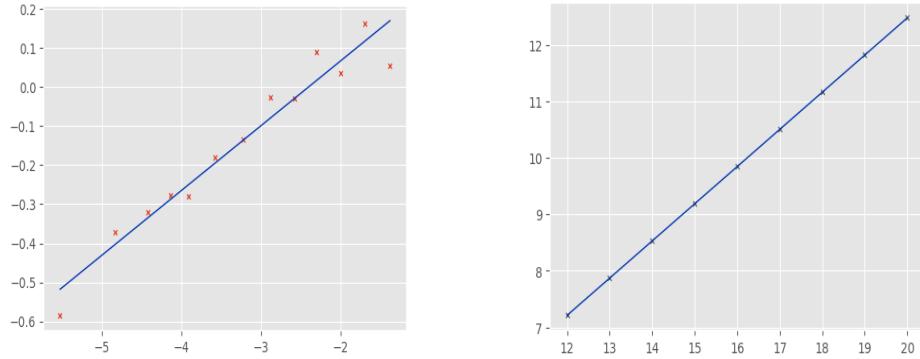


Figure 2: On the left, we see estimates of  $H$  for the SPX using the  $m(q, \Delta)$  method for the SPX from 3rdJan22-15thJul24 for  $q = 2$  for which  $\hat{H} = 0.0830$  and  $\hat{\sigma} = 1.221$  (see similar plots in [GJR18]). On the right we see the linear regression in (10) for the Han-Schied method ( $\log s_n$  vs  $n$ ) for a true fBM path with  $2^{20}$  time points, for which  $\hat{H} = 0.0508$ , and  $\hat{\sigma} = 1.010$ .

### Jointly estimating $H$ and $\sigma$ using [HS21]

If  $X_t = \sigma B_t^H$ , we can jointly estimate  $H$  and  $\sigma$  by performing linear regression since

$$\log s_n = \log \hat{\sigma}_n + n(1 - \hat{H}_n) \log 2 \quad (10)$$

but we now have to compute  $\log s_n$  for a range of different  $n$ -values to get a line of best fit, for which the slope is  $(1 - \hat{H}_n) \log 2$  and the intercept is  $\log \hat{\sigma}_n$ .

### Proof of scale-invariance of the [HS21] SSE

Recall Definition 8.1 in [HS21]:

$$\lambda_n^s(x) := \operatorname{argmin}_{\lambda > 0} \sum_{k=n-m}^n \alpha_{n-k} (\hat{R}_k(\lambda x) - \hat{R}_{k-1}(\lambda x))^2 \quad (11)$$

<sup>1</sup> where  $\hat{R}_n(x)$  is the basic [HS21] estimator at order  $n$ . Then we see that  $\lambda_n^s(cx) = \frac{\lambda_n^s(x)}{c}$ , so

$$\hat{R}_n(\lambda_n^s(cx) cx) = \hat{R}_n(\lambda_n^s(x) x).$$

But the right hand side here is the SSE  $R_n^s(x) := \hat{R}_n(\lambda_n^s(x)x)$  applied to the path  $x$ , and the left hand side is the SSE applied to the scaled path  $cx$ . Hence the SSE is **scale-invariant**.

Corollary 8.4 in [HS21] shows if  $X = \sigma B^H$ , then all of their three SSEs ( $R_n^s$ ,  $R_n^t$  and  $R_n^r$ ) have error

$$|R_n - H| = O(2^{-n/2} \sqrt{\log n}) = O\left(\frac{1}{\sqrt{N}} \sqrt{\log_2 \log_2 N}\right) \quad (12)$$

since  $N = 2^n$  so  $2^{-n/2} = 2^{-\log_2 N/2} = \frac{1}{\sqrt{N}}$ . Note this is slightly slower convergence than  $O(\frac{1}{\sqrt{N}})$  because  $\sqrt{\log_2 \log_2 N}$  tends to infinity (very slowly). The intuition behind the SSE is that we are trying to find the  $\lambda$ -value that minimizes the difference between consecutive iterates:  $|\hat{R}_k(\lambda x) - \hat{R}_{k-1}(\lambda x)|$ , since (loosely speaking) this difference being smaller implies that the sequence  $R_k(\lambda x)$  is converging quicker. Note (12) does not say anything about the asymptotic variance of  $|R_n - H|$ .

Note if  $m = 0$  and  $\alpha_0 = 1$ , then (11) simplifies to

$$\lambda_n^s(x) := \operatorname{argmin}_{\lambda > 0} (\hat{R}_k(\lambda x) - \hat{R}_{k-1}(\lambda x))^2$$

and the SSE simplifies to:

$$R_n = 1 - \frac{1}{2} \log_2 \left( \frac{s_n^2}{s_{n-1}^2} \right). \quad (13)$$

(this is conceptually similar to (2), since in both cases we are just using a ratio of the same statistic at the finest and next finest resolution). Note  $H$ -Hölder continuity alone is not enough to guarantee convergence of  $R_n$  here, since if e.g.  $x$  is linear then  $s_n = 0$  for all  $n$ , so the right hand side is undefined.

<sup>1</sup>one can check the minimizer is unique here since the objective function is strictly convex

## The driftless rough Heston model as a non-Gaussian process with the same covariance as the RL process

The driftless rough Heston model satisfies

$$V_t = V_0 + \nu \int_0^t (t-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u.$$

Then  $\mathbb{E}(V_t) = V_0$ , and  $V$  has covariance function:

$$\begin{aligned} \mathbb{E}((V_s - V_0)(V_t - V_0)) &= \nu^2 \mathbb{E}\left(\int_0^s (s-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \cdot \int_0^t (t-r)^{H-\frac{1}{2}} \sqrt{V_r} dW_r\right) \\ &= \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \mathbb{E}(V_u) du \\ &= V_0 \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du = V_0 \nu^2 \bar{R}(s, t) \end{aligned}$$

for  $0 \leq s \leq t$ , where  $\bar{R}(s, t)$  is the covariance function for the Riemann-Liouville (RL) process  $Z_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u$  used for the rough Bergomi model (note  $Z$  is a Gaussian process but  $V$  is not), but the explicit formula for  $\bar{R}(s, t)$  is more complicated than the  $R(s, t)$  formula for fBM.

## The Mandelbrot-van Ness representation for fBM and relation to the RL process

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H = c_H \left( \int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}) dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \right) = c_H (A_t + Z_t)$$

for  $t \geq 0$ , in terms of an RL process  $Z$  (and note that  $A_t$  is known at time zero for all  $t \geq 0$ ), and  $c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}$ . Note also that  $A_t$  and  $Z_t$  are independent.

## References

- [BFG16] Bayer, C., P.K.Friz, and J.Gatheral, “Pricing under rough volatility”, *Quantitative Finance*, 16(6), pp. 887–904, 2016.
- [Bib20] Bibinger, M., “Cusum tests for changes in the Hurst exponent and volatility of fractional Brownian motion”, 2020
- [Cha14] Chang, Y-C., “Efficiently Implementing the Maximum Likelihood Estimator for Hurst Exponent”, *Mathematical Problems in Engineering*, vol. 2014
- [CD22] Cont, R. and P.Das, “Rough Volatility: Fact or Artefact?”, preprint, 2022
- [F23] Forde, M., “Statistical issues and calibration problems under rough and Markov volatility”, <https://martinfoerde.github.io/Talk.pdf>
- [GJR18] Gatheral, J., T.Jaisson and M.Rosenbaum, “Volatility is rough”, *Quantitative Finance*, 18(6), 2018
- [HS21] Han, X. and A.Schied, “The Hurst roughness exponent and its model-free estimation”, preprint, 2021
- [HS24] Han, X. and A.Schied, “Estimating the roughness exponent of stochastic volatility from discrete observations of the integrated variance”, preprint, 2024