A simplified unified propagator model for signed order flow, concave price impact and rough volatility

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Abstract

In this note, we simplify the setup in [MORS26] by exogenously modelling signed order flow as the difference of two integrated hyper-rough Heston processes with Hölder exponent in $(\frac{1}{2},1)$, which endogenously determines the price process, the propagator kernel and the long-run permanent price impact. This obviates the need to use nearly unstable heavy-tailed Hawkes processes, and lengthy macroscopic scaling limit arguments with C-tightness.

1.1 Persistent order flow

Similar to Theorem 3.1 in [MORS26][1], we model positive order flow F^+_t for an asset as a weak solution the Stochastic Volterra Equation

$$F^+_t \quad = \quad g(t) + \int_0^t f(t-s)B_{F^+_s}ds \tag{1}$$

for a standard Brownian motion B with $B_0=0$, where $f(t)=e^{-\theta t}t^{H-\frac{1}{2}}$ with $H\in(-\frac{1}{2},\frac{1}{2})$, $\theta>0$ so $f\in L^1$ and $g(t)=V_0 t$ for some $V_0>0$ (note θ replaces the ergodicity parameter λ in Theorem 3.1 in [MORS26]). Weak existence and uniqueness for (1) is established in section 2 of [AJ21]), and F^+ is a non-decreasing, nonnegative, continuous and adapted process. F^+_t is also a.s. $(2\alpha-\varepsilon)\wedge 1$-Hölder continuous where $\alpha=H+\frac{1}{2}$ (see e.g. Theorem 3.1 in [JM20] for details) [2]. which is non-differentiable if $H\le 0$.

Lemma 1.1 $\mathbb{E}(F^+_t)<\infty$.

Proof. See Appendix A. ■

Corollary 1.2 $B_{F^+_t}$ *is an* $\mathcal{G}^+_t$*-martingale, where* $\mathcal{G}^+_t=\mathcal{F}^B_{F^+_t}$.

Proof. Let $S=F^+_s$ and $T=F^+_t$ for $0\le s\le t$, then T and S are $\mathcal{F}^B$-stopping times with $0\le S\le T$. To see this, consider two Brownian paths ω,ω' with $\omega(r)=\omega(r')$, then $F^+_t(\omega)\le u\Rightarrow F^+_t(\omega')\le u$. By the contrapositive, $F^+_t(\omega')>u\Rightarrow F^+_t(\omega)\ge u$, so $F^+_t(\omega)>u\Rightarrow F^+_t(\omega')\ge u$, hence $1_{F^+_t(\omega)\le u}=1_{F^+_t(\omega')\le u}$, so $\{F^+_t(\omega)\le u\}\in\mathcal{F}^B_u$.

Then $\mathbb{E}(B^2_{u\wedge T})=\mathbb{E}(u\wedge T)\le\mathbb{E}(T)=\mathbb{E}(F^+_t)<\infty$ (from the previous lemma), so $(B_{u\wedge T})$ is bounded in L^2 (and hence $M_u=B_{u\wedge T}$ is an $\mathcal{F}^B_u$-martingale with $\mathbb{E}(M^2_u)=\mathbb{E}(u\wedge T)\le\mathbb{E}(T)<\infty$ and hence is UI) so from the OST for U.I. martingales applied to M (e.g. Theorem 5.20 in [Eth18]), $\mathbb{E}(B_T|\mathcal{F}^B_S)=B_S$, and the martingale property follows. ■

1.2 The inversion formula

If h is a function with $h*f\equiv 1$, then we have the inversion formula:

$$\begin{aligned}\int_0^t h(t-s)(F^+_s-g(s))ds \quad &= \quad \int_0^t h(t-s)\int_0^s f(s-u)B_{F^+_u}duds \quad = \quad \int_0^t\int_u^t h(t-s)f(s-u)dsB_{F^+_u}du\\ &= \quad \int_0^t\int_0^{t-u} h(t-s-u)f(s)dsB_{F^+_u}du\\ &= \quad \int_0^t (h*f)(t-u)B_{F^+_u}du \quad = \quad \int_0^t B_{F^+_u}du\end{aligned}$$

(where Fubini is justified since $F^+_t<\infty$ a.s. for finite t and B is a.s. continuous), and hence

$$B_{F^+_t} \quad = \quad \frac{d}{dt}\int_0^t h(t-s)(F^+_s-g(s))ds \quad = \quad \frac{d}{dt}\int_0^t h(t-s)F^+_s ds-\frac{d}{dt}\int_0^t h(t-s)g(s)ds \tag{2}$$

[1] see also section 5.4 in [FGS21]
[2] we can simulate F^+ using the Monte Carlo scheme in [AA25] using Normal Inverse Gaussian variates

Lebesgue a.e., where the final equality holds since $\int_0^t h(t-s)(F_s^+ - g(s))ds$ and $\int_0^t h(t-s)g(s)ds$ are both absolutely continuous, and hence so is their sum. Moreover, setting $H(t)=\int_0^t h(s)ds$, integrating by parts we see that

$$\int_0^t H(t-s)dF_s^+ \;=\; F_s^+ H(t-s)_{s=0}^{s=t} + \int_0^t h(t-s)F_s^+ ds \;=\; \int_0^t h(t-s)F_s^+ ds\,.$$

Taking Laplace transforms, the condition $h*f\equiv 1$ becomes $\hat{f}(\lambda)\hat{h}(\lambda)=\frac{1}{\lambda}$, from which we find that

$$h(t) \;=\; \frac{\theta^\alpha}{\Gamma(\alpha)}\Big(1-\frac{\Gamma(-\alpha,t\theta)}{\Gamma(-\alpha)}\Big)$$

where $\Gamma(a,z)=\int_z^\infty s^{a-1}e^{-s}ds$ is the incomplete Gamma function, and $h(t)=O(t^{-H-\frac{1}{2}})$ as $t\to 0$.

1.3 Signed order flow

Again following [MORS26], we now assume the *signed* order flow for an asset is

$$V_t \;=\; F_t^+ - F_t^-$$

where $F_t^- = g(t)+\int_0^t f(t-s)W_{F_s^-}ds$ and W is another Brownian motion independent of B (so F^- is an i.i.d. copy of F^+). Then we see that

$$\frac{d}{dt}\int_0^t h(t-s)V_s ds \;=\; \frac{d}{dt}\int_0^t H(t-s)dV_s \;=\; \int_0^t h(t-s)dV_s \;=\; B_{F_t^+}-W_{F_t^-} \tag{3}$$

since the g terms cancel (and the second equality follows from the Stieltjes-Leibniz rule using the known Hölder continuity of $F^\pm$ and that $|h(u)|\le Cu^{-\alpha}$ so $\int_0^t |h(t-s)||dV_s|<\infty$), and for $0\le t\le u$ we have

$$\mathbb{E}(F_u^+|\mathcal{F}^B_{F_t^+}) \;=\; g(u)+\mathbb{E}\Big(\int_0^u f(u-s)B_{F_s^+}ds\,|\,\mathcal{G}_t^+\Big) \;=\; g(u)+\int_0^t f(u-s)B_{F_s^+}ds + B_{F_t^+}\int_t^u f(u-s)ds$$

(and similarly for $\mathbb{E}(F_u^-|\mathcal{F}^W_{F_t^-})$), where $\mathcal{G}_t^+=\mathcal{F}^B_{F_t^+}$, since $(B_{F_t^+})_{t\ge 0}$ is a $\mathcal{G}_t^+-$martingale by Corollary 1.2.

1.4 Price dynamics and the propagator model

Making the usual assumption that the asset price $P_t=\kappa\lim_{u\to\infty}\mathbb{E}(F_u|\mathcal{G}_t)$ where $F_t=F_t^+-F_t^-$ and $\mathcal{G}_t=\sigma(\mathcal{G}_t^+,\mathcal{G}_t^-)$ with $\mathcal{G}_t^-=\mathcal{F}^W_{F_t^-}$ (for some $\kappa>0$) [3], we find that $f(u-s)\to 0$ as $u\to\infty$ since $\theta>0$, but $\lim_{u\to\infty}\int_t^u f(u-s)ds=c_{H,\theta}$ where $c_{H,\theta}=\theta^{-\frac{1}{2}-H}\Gamma(\frac{1}{2}+H)$, so

$$P_t \;=\; \kappa\lim_{u\to\infty}\mathbb{E}(F_u^+-F_u^-\,|\mathcal{G}_t) \;=\; \kappa\, c_{H,\theta}(B_{F_t^+}-W_{F_t^-})$$

i.e. P is the difference of two i.i.d. hyper-rough Heston models (each with correlation $\rho=1$ since each process only has one driving Brownian motion), and we recover the Propagator equation

$$P_t \;=\; \int_0^t G(t-s)dV_s$$

where $G(t)=\kappa c_{H,\theta}h(t)$, and the *unsigned* order flow is $U_t=F_t^++F_t^-$.

1.5 Power-law price impact

In particular, the *market impact function* of an exogenous metaorder executed at constant trading speed 1 up to time t_0 is given by

$$MI(t) \;=\; \frac{d}{dt}\int_0^t h(t-s)sds \;=\; \frac{\theta^{-\alpha_-}(t\theta\Gamma(-\alpha)-t\theta\Gamma(-\alpha,t\theta)-\Gamma(\alpha_-,0)+\Gamma(\alpha_-,t\theta))}{\Gamma(-\alpha)\Gamma(\alpha)}$$

for $0\le t<t_0$ (where $\alpha_-=\frac{1}{2}-H$) which is $O(t^{\frac{1}{2}-H})$ as $t\to 0$ (for $H\ne\frac{1}{2}$) and globally concave in t for $H\in(-\frac{1}{2},\frac{1}{2})$, consistent with empirical evidence (note the usual square-root impact law corresponds to $H=0$ here), and we can show that the asymptotic (i.e. permanent) impact as $t\to\infty$ (given no trading after t_0) is

$$MI(\infty) \;=\; \frac{t_0\theta^\alpha}{\Gamma(\frac{1}{2}+H)}$$

[3] note we are assuming $P_0=0$ without loss of generality to ease notation, but we can easily add a positive P_0 term

(this is the asymptotic grey line in the plot below).

For the special case $H = \frac{1}{2}$ that we have previously excluded, $h(t) = \theta + \delta(t)$, which corresponds to linear permanent and temporary price impact; conversely $\lim_{\theta \to 0} h(t) = const. \times t^{-\frac{1}{2}-H}$. Recall from above that F_t^+ has Hölder regularity $(2\alpha - \varepsilon) \wedge 1$, and empirical evidence in [MORS26] and below suggests that $2\alpha \in (\frac{1}{2}, 1)$ which corresponds to $H \in (-\frac{1}{4}, 0)$.

1.6 Controlling liquidity

In the setup above, we have no control over the depth of liquidity, since the kernel $G(t)$ which controls the amount of transient price impact is endogenously determined via the f function in (1). To resolve this inflexibility, we can make the natural assumption that $P_t = \theta M_t + (1-\theta)\int_0^t G(t-s)dV_s$ for $\theta \in [0,1]$ where M is an additional martingale, so θ close to 1 implies greater liquidity and vice versa.

1.7 Other extensions

If we augment f to a function of the form $f(t) = e^{-\theta t} t^{H-\frac{1}{2}} + e^{-\theta_2 t} t^{H_2 - \frac{1}{2}}$, then we find that

$$h(t) = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} (-c)^n \sum_{k=0}^{\infty} \frac{(\beta n)_k}{k!} \delta^k \, \theta^{-(\mu_n + k)} \frac{\gamma(\mu_n + k, \theta t)}{\Gamma(\mu_n + k)}$$

where $\beta = \frac{1}{2} + H_2$, $c = \Gamma(\beta)/\Gamma(\alpha)$, $\delta = \theta - \theta_2$, $\mu_n = \beta n - \alpha(n+1)$, $\gamma(a,z) = \int_0^z s^{a-1} e^{-s} ds$ is the lower incomplete Gamma function and $(a)_k$ denotes the Pochhammer symbol. We can then use two (possibly different) f functions to drive $F^{\pm}$, and if $H_2 \in (0, \frac{1}{2})$, P_t will have a conventional (i.e. non-hyper) rough component, which may be more realistic. We can also assume W and B are correlated to give greater control over the skewness of P_t.

1.8 Empirical results

Below we have tabulated estimates of 2α (i.e. the Hölder exponent of $(F_t)_{t \geq 0}$) using signed order flow for major US tech stocks using the model-independent $\hat{H}^{\pi}_{L,K}(X)$ estimator in Eq 7 in Cont&Das[CD24] which is consistent and allows for irregularly spaced data (with their $L/K = [\sqrt{N}]$ where N is the total sample size). For this we have used the free data at `https://data.lobsterdata.com/info/DataSamples.php`: [4]

	GOOG	AAPL	MSFT	AMZN	INTC
N	49284	118326	410881	57360	404496
$2\hat{\alpha}$	0.692	0.720	0.847	0.735	0.890
$\hat{H}$	-0.154	-0.140	-.0767	-0.133	-0.0553

[4]only using columns 4 and 5 from the lobsterdata.csv files which correspond to actual realized trades.

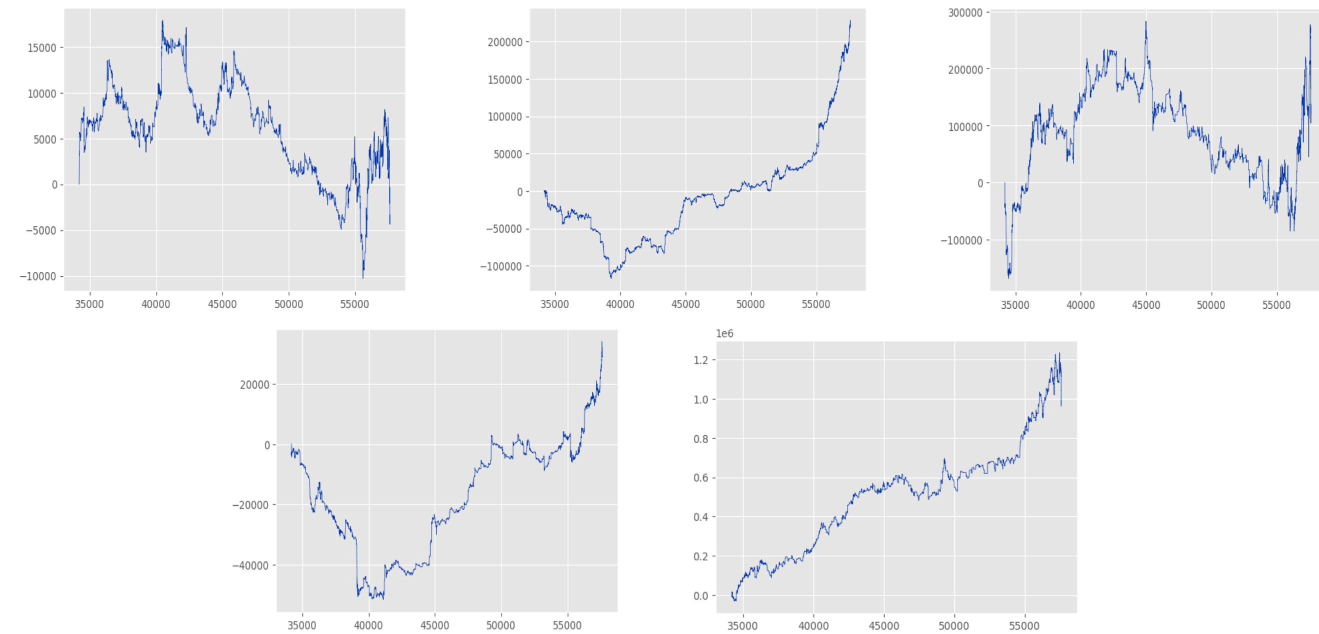

Figure 1: Signed order flow for GOOG, AAPL, MSFT, AMZN and INTC

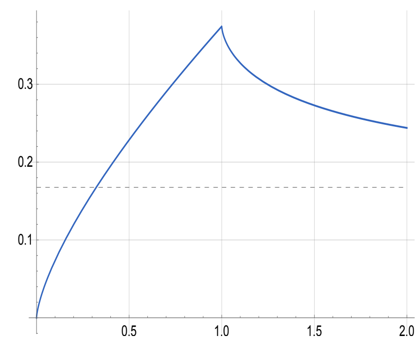

Figure 2: Concave price impact of an exogenous metaorder executed at constant trading speed 1 over $[0,1]$ with $H=-0.2$, $\theta=0.1$ and $\kappa=1$. The blue line asymptotes to the constant level $MI(\infty)$ (grey line) as $t\to\infty$ which represents the asymptotic permanent price impact.

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A Appendix A

Proof. Set $G_t := \sup_{0\le r\le t} |g(r)|$, $K_t := \int_0^t |f(r)|dr > 0$. For $n \ge 1$ define the level hitting time $\tau_n := \inf\{s \ge 0 : F_s^+ \ge n\}$.

Since F^+ is continuous and nondecreasing, on $\{\tau_n \le t\}$ we have $F_{\tau_n}^+ = n$ and $F_r^+ \le n$ for all $0 \le r \le \tau_n$. Fix $n \ge 1$ and work on the event $\{\tau_n \le t\}$. For $0 \le r \le \tau_n$ we have $|B_{F_r^+}| \le \sup_{0\le u\le n} |B_u|$. Setting $s = \tau_n$ and taking absolute values we see that

$$
\begin{aligned}
n \;=\; g(\tau_n) + \int_0^{\tau_n} f(\tau_n - r)\, B_{F_r^+}\, dr \;&\le\; |g(\tau_n)| + \int_0^{\tau_n} |f(\tau_n - r)|\, |B_{F_r^+}|\, dr \\
&\le\; G_t + \Big(\int_0^{\tau_n} |f(\tau_n - r)|\, dr\Big) \sup_{0\le u\le n} |B_u| \\
&=\; G_t + \Big(\int_0^{\tau_n} |f(s)|\, ds\Big) \sup_{0\le u\le n} |B_u| \;\le\; G_t + K_t \sup_{0\le u\le n} |B_u|.
\end{aligned}
$$

Hence for $n > G_t$, on $\{\tau_n \le t\}$ we have $\{\sup_{0\le u\le n} |B_u| \ge \frac{n-G_t}{K_t}\}$. Since $F_t^+ \ge n$ iff $\tau_n \le t$, we obtain

$$
\mathbb{P}(F_t^+ \ge n) \;\le\; \mathbb{P}\Big(\sup_{0\le u\le n} |B_u| \ge \frac{n - G_t}{K_t}\Big), \qquad n > G_t.
$$

By the standard reflection principle bound $\mathbb{P}(\sup_{0\le u\le n} |B_u| \ge a) \le 4\exp(-\frac{a^2}{2n})$, for $a > 0$, so for $n > G_t$,

$$
\mathbb{P}(F_t^+ \ge n) \;\le\; 4\exp\Big(-\frac{(n - G_t)^2}{2nK_t^2}\Big).
$$

In particular, if $n \ge 4G_t$ then $(n - G_t)^2/n \ge \frac{1}{2}n$, hence

$$
\mathbb{P}(F_t^+ \ge n) \;\le\; 4e^{-\frac{n}{4K_t^2}}, \qquad n \ge 4G_t. \tag{A-1}
$$

Since $F_t^+ \ge 0$, we also know that $\mathbb{E}(F_t^+) = \int_0^\infty \mathbb{P}(F_t^+ \ge x)\, dx$. Now set $m := \lceil 4G_t \rceil$. Then

$$
\mathbb{E}(F_t^+) \;\le\; m + \int_m^\infty \mathbb{P}(F_t^+ \ge x)\, dx \;=\; m + \sum_{n=m}^\infty \int_n^{n+1} \mathbb{P}(F_t^+ \ge x)\, dx \;\le\; m + \sum_{n=m}^\infty \mathbb{P}(F_t^+ \ge n). \tag{A-2}
$$

Using (A-1) for all $n \ge m$,

$$
\sum_{n=m}^\infty \mathbb{P}(F_t^+ \ge n) \;\le\; \sum_{n=m}^\infty 4\exp\Big(-\frac{n}{4K_t^2}\Big) \;<\; \infty. \tag{A-3}
$$

Combining (A-2) and (A-3) yields $\mathbb{E}(F_t^+) < \infty$. ■