Homework 2

1. Let B^α denote a Brownian motion with $B^\alpha_0 \sim \alpha$ (i.e. a **random initial starting point** with density $\alpha(x)$, and assume the process $B^\alpha_t - B^\alpha_0$ for $t > 0$ is independent of B^α_0). Write down an integral expression for the density of B^α_t as a convolution.

Solution. If we condition on $B^\alpha_0 = x$, then the conditional distribution of B^α_t is $N(x,t)$ with density $p_t(y-x)$. Hence (integrating over the density of B^α_0) we see that

$$\int_{-\infty}^{\infty} p_t(y-x)\alpha(x)dx \;=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \alpha(x)dx \;=\; (p_t * \alpha)(y)$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ denotes the density of Brownian motion.

2. Write an expression for $M_t = \mathbb{E}(F(B^\alpha_1)|B^\alpha_t)$ for $t \in (0,1]$, where B^α is defined as above.

Solution. From Hwk 1 q1, we know that

$$\mathbb{E}(F(B^\alpha_1)|B^\alpha_t = x) \;=\; (p_{1-t} * F)(x)\,.$$

Then replacing x with the random B^α_t, we see that

$$M_t \;=\; (p_{1-t} * F)(B^\alpha_t)\,.$$

3. (relevant for the new Project). Let $X_t = \mu t + \sigma W_t$ and assume we have observations of X at equidistant times on the interval $[0,T]$, and assume σ is known. Show that the variance of any **unbiased estimator** for μ is $\geq \frac{\sigma^2}{T}$.

Solution. Let $\Delta X_i = X_{\frac{i}{n}T} - X_{\frac{(i-1)}{n}T}$ for $i = 1,...,n$ denote the increments of X. Then the ΔX_i's are i.i.d. $N(\mu\Delta t, \sigma^2\Delta t)$ random variables with $\Delta t = \frac{T}{n}$, so their joint density is just the product

$$\frac{1}{(2\pi\sigma^2\Delta t)^{\frac{1}{2}n}}\, e^{-\sum_{i=1}^n \frac{(\Delta X_i - \mu\Delta t)^2}{2\sigma^2\Delta t}}\,.$$

Taking the log of this we obtain

$$\ell_n(\mu) \;=\; (...) - \sum_{i=1}^n \frac{(\Delta X_i - \mu\Delta t)^2}{2\sigma^2\Delta t}$$

$$\Rightarrow \qquad \frac{\partial}{\partial\mu}\ell_n(\mu) \;=\; \sum_{i=1}^n \frac{(\Delta X_i - \mu\Delta t)}{\sigma^2}$$

$\ell_n(\mu)$ is known as the **score**, and in general it can be easily shown that $\mathbb{E}(\ell_n(\mu)) = 0$. Then the **Fisher information**: $I(\mu) := \mathrm{Var}(\frac{\partial\ell_n}{\partial\mu}) = \mathbb{E}((\frac{\partial\ell_n}{\partial\mu})^2) = \sum_{i=1}^n \frac{\sigma^2\Delta t}{\sigma^4} = \frac{T}{\sigma^2}$, so (by the **Cramer-Rao bound**) from undergrad Statistics, the variance of any unbiased estimator $\hat{\mu}$ for μ satisfies $\mathrm{Var}(\hat{\mu}) \geq \frac{1}{I(\mu)} = \frac{\sigma^2}{T}$. Hence we need T large to get a good estimator for μ. Note this bound is attained by the obvious unbiased estimator $\hat{\mu} = X_T/T$.

4. Consider the **Bessel process** which satisfies

$$dR_t \;=\; \frac{2\delta - 1}{R_t}dt + dW_t$$

for $\delta \geq 0, R_0 > 0$. Using Ito's lemma, compute the SDE satisfied by $Z_t = R_t^2$.

Solution.

$$dZ_t \;=\; 2R_t dR_t + \frac{1}{2}\cdot 2dt \;=\; 2[(2\delta-1)dt + R_t dW_t] + dt \;=\; (4\delta - 1)dt + 2\sqrt{Z_t}dW_t\,.$$

5. Consider a process X_t satisfying the SDE

$$dX_t \;=\; X_t^2 dW_t\,.$$

Compute the SDE for $R_t = 1/X_t$ in terms of R_t.

Solution:

$$dR_t \;=\; -\frac{1}{X_t^2}dX_t + \frac{1}{2}\frac{2}{X_t^3}X_t^4 dt \;=\; -\frac{1}{X_t^2}X_t^2 dW_t + \frac{1}{2}\frac{2}{X_t^3}X_t^4 dt \;=\; -dW_t + \frac{1}{R_t}dt\,.$$

X is an example of a process which is driftless but it is not an $\mathcal{F}^W$-martingale (in fact it can be shown that $\mathbb{E}(X_t|X_s) < X_s$, see FM04 for details).

6. Apply Ito's lemma to $(1-t/T)W_t$, and integrate the resulting equation from $t=0$ to $t=T$. Use this to compute the distribution of $\frac{1}{T}\int_0^T W_t dt$.

Solution. Let $f(x,t) = (1-t/T)x$. Then applying Ito's lemma to $f(W_t,t)$, we see that

$$df(W_t,t) \;=\; -\frac{1}{T}W_t dt + (1-\frac{t}{T})dW_t\,.$$

Integrating from 0 to T, we see that

$$f(W_T,T) - f(W_0,0) \;=\; 0 \;=\; -\frac{1}{T}\int_0^T W_t dt + \int_0^T (1-\frac{t}{T})dW_t$$

so we see that the average of W over the interval $[0,T]$ is given by $\int_0^T (1-\frac{t}{T})dW_t$. Moreover, since this is a stochastic integral of the form $\int_0^T \phi(t)dW_t$, where ϕ is non-random, $\int_0^T (1-\frac{t}{T})dW_t \sim N(0, \int_0^T \phi(t)^2 dt)$ (see part of lecture notes on the Ornstein-Uhlenbeck process) and when you evaluate the integral here, one finds that $\int_0^T \phi(t)^2 dt = \frac{1}{3}T$. This means that $\mathrm{Var}(\frac{1}{T}\int_{t=0}^T W_t dt)$, i.e. the variance of the average of W over $[0,T]$ is one-third the variance of W_T itself (which we know is T).

7. Consider the following SDE

$$dR_t \;=\; (\frac{1}{R_t} - \frac{R_t}{1-t})dt + dW_t$$

for $t<1$ with $R_0>0$ (you may assume that $R_t>0$ for $t<1$). Compute an SDE for $Y_t = R_t^2$. R is known as the **Brownian excursion process**, which is BM conditioned to return to zero for the first time at time 1.

Solution. Let $Y_t = R_t^2$. Then from Ito's lemma

$$\begin{aligned} dY_t \;&=\; 2R_t dR_t + \frac{1}{2}\cdot 2dt \;=\; 2R_t((\frac{1}{R_t} - \frac{R_t}{1-t})dt + dW_t) + dt \\ &=\; 2R_t dW_t + 3dt - \frac{2R_t^2}{1-t}dt \\ &=\; (3 - \frac{2Y_t}{1-t})dt + 2\sqrt{Y_t}dW_t\,. \end{aligned}$$

8. Consider the following SDE

$$dS_t \;=\; \delta(\beta S_t + 1 - \beta)dW_t$$

for $\delta>0$. Derive the SDE satisfied by $X_t = \beta S_t + 1 - \beta$. S is known as a **displaced-diffusion** process.

Solution.

$$dX_t \;=\; d(\beta S_t + 1 - \beta) \;=\; \delta\beta X_t dW_t$$

and we note that X is Geometric Brownian motion. S is often used to approximate the **CEV process** $dS_t = \delta S_t^\beta dW_t$ for $\beta\in(0,1)$ when $S_0=1$, since S^β and $\beta S + 1 - \beta$ have the same slope and value at $S=1$.

9. Consider the ODE

$$y'(t) \;=\; \sigma(y(t))\,.$$

Derive the SDE satisfied by $X_t = y(W_t)$.

Solution. From the chain rule we know that $y''(t) = \sigma'(y(t))y'(t) = \sigma'(y(t))\sigma(y(t))$. Then from Ito's lemma

$$\begin{aligned} dX_t \;=\; y'(W_t)dW_t + \frac{1}{2}y''(W_t)dt \;&=\; \sigma(y(W_t))dW_t + \frac{1}{2}\sigma'(y(W_t))\sigma(y(W_t))dt \\ &=\; \sigma(X_t)dW_t + \frac{1}{2}\sigma'(X_t)\sigma(X_t)dt\,. \end{aligned}$$

The final drift term above is known as the **Stratonovich correction**.

10. Consider an i.i.d. sequence $X_1, X_2, ..$ of Cauchy random variables with density $p(x) = \frac{1}{\pi(1+x^2)}$. Does the sample mean $\bar{X}_n = \frac{1}{n}\sum_{j=1}^n X_j$ tend to zero as $n \to \infty$? You may use that $\mathbb{E}(e^{iuX_j}) = e^{-|u|}$ for $u \in \mathbb{R}$.

Solution.

$$\mathbb{E}(e^{iu\bar{X}_n}) \;=\; \mathbb{E}(e^{iu\frac{1}{n}\sum_{j=1}^n X_j}) \;=\; \mathbb{E}(\prod_{j=1}^n e^{i\frac{u}{n}X_j}) \;=\; \prod_{j=1}^n \mathbb{E}(e^{i\frac{u}{n}X_j}) \;=\; \mathbb{E}(e^{i\frac{u}{n}X_j})^n \;=\; e^{-|\frac{u}{n}|n} \;=\; e^{-|u|}$$

where the 2nd equality follows since the X_j's are independent, and the third equality follows since the X_j's are identically distributed. Hence we see that $\bar{X}_n \sim X_1$ because $\bar{X}_n$ has the same (complex) mgf as X_1, so the answer is no. This is a case where the SLLN fails because $\mathbb{E}(|X_j|) = \infty$.