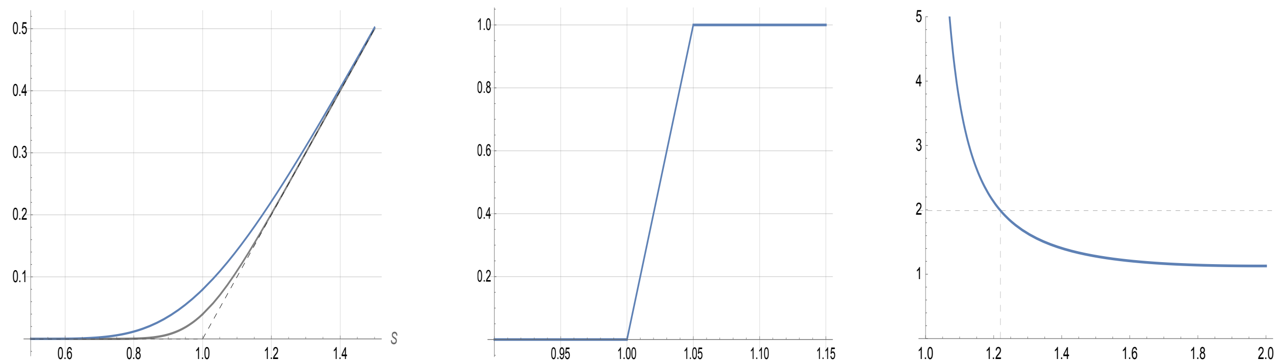

Figure 1: On the left we have plotted Call option values as function of the initial stock price S_0 for two maturities T_1 (grey) and T_2 (blue). In the middle we see the payoff profile of a call spread strategy in q2 with $K = 1$ and $\Delta = .05$. On the right we see a plot of $F(\alpha, t)$ in q3 as a function of α (for $t = 1$ fixed), which we invert to solve for $\hat{\alpha}$ from the realized value of $\bar{X}_t - \underline{X}_t$.

Homework 5

1. Plot the value of a European call option under the Black-Scholes model as a function of the initial stock price S_0 at $t = 0$ for two different maturities T_1 and T_2 (assume that $0 < T_1 < T_2$ and the interest rate $r = 0$). Derive the asymptotic behaviour of the call price as $S_0 \to \infty$.

Solution. From the Feynman-Kac formula with $r = 0$, the unique no-arbitrage call option price $C(S_0, 0)$ at $t = 0$ satisfies

$$\frac{C(S_0,0)}{S_0} \;=\; \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \;=\; \frac{1}{S_0}\mathbb{E}^{\mathbb{Q}}((S_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - K)^+) \;=\; \mathbb{E}^{\mathbb{Q}}((e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - \frac{K}{S_0})^+)$$

This tends to 1 as $S_0 \to \infty$ by the dominated convergence theorem, since the expression inside the expectation on the right hand side is $\leq \mathcal{E} := e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$ (and $\mathbb{E}^{\mathbb{Q}}(\mathcal{E}) = 1 < \infty$), and tends monotonically to $\mathcal{E}$ as $S_0 \to \infty$ a.s. since $\frac{K}{S_0} \to 0$. Hence we have shown that $C(S_0, 0) \sim S_0$ as $S_0 \to \infty$.

Remark 0.1 Note we can also make the more obvious statement that $C(S_0, 0) \nearrow \infty$ as $S_0 \nearrow \infty$, but the statement $C(S_0, 0) \sim S_0$ is a sharper statement which tells us exactly how C tends to ∞ (specifically that the slope of C tends to 1 as $S_0 \to \infty$), see first plot above. We also see from the plot that $C(S_0, 0) \sim S_0 - K$, which is the more useful asymptotic estimate in practice.

Remark 0.2 Note we also prove this without the Dominated Convergence theorem by using the Black-Scholes formula.

2. Consider a portfolio of buying $\frac{1}{\Delta}$ call options with strike K, and selling $\frac{1}{\Delta}$ call options of strike $K + \Delta$ for $\Delta > 0$. Plot the terminal payoff function $f(S)$ of this portfolio as a function of S. What does this payoff tend to as $\Delta \to 0$?

Solution. The payoff function $f(S) = 0$ for $S \leq K$, $f(S) = \frac{1}{\Delta}(S - K - (S - K - \Delta)) = 1$ for $S \geq K + \Delta$, and $f(S) = \frac{1}{\Delta}(S - K)$ for $S \in [K, K + \Delta]$, and in particular we see that $f(.)$ is continuous at $S = K$ and $K + \Delta$ (This is known as a **call spread** strategy.). From a picture, we see that this tends to a digital call payoff $1_{S_T \geq K}$ as $\Delta \to 0$.

3. A **symmetric α-stable process** X with parameters $\alpha \in (0, 2]$, $\sigma > 0$ is a generalization of Brownian motion, which has independent stationary increments like Brownian motion but now $\mathbb{E}(e^{iu(X_t - X_s)}|X_s) = e^{-(t-s)\sigma^\alpha |u|^\alpha}$ for $u \in \mathbb{R}$ and $0 \leq s \leq t$, so X is only a (multiple of) BM if $\alpha = 2$, but for $\alpha < 2$ the increments of X are not normally distributed. If $\bar{X}_t = \max_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \min_{0 \leq s \leq t} X_s$ and $\sigma = 1$, it can be shown that

$$\mathbb{E}(\bar{X}_t - \underline{X}_t) \;=\; F(\alpha, t) \;:=\; \frac{2\alpha\Gamma(1 - \frac{1}{\alpha})}{\pi} t^{\frac{1}{\alpha}}$$

for $\alpha \in (1, 2]$. Use this identity to define a statistical estimator $\hat{\alpha}$ for α from observed values for $\bar{X}_t$ and $\underline{X}_t$. Is $\hat{\alpha}$ biased? (you may use that $\frac{\partial^2}{\partial \alpha^2} F(\alpha, t) > 0$ and that $F(\alpha, t)$ is a decreasing function of α for t fixed).

Solution. We just solve $\bar{X}_t - \underline{X}_t = F(\hat{\alpha}, t)$ for $\hat{\alpha}$ to get $\hat{\alpha}$ (see plot of $F(\alpha, t)$ on the right above for $t = 1$).

For the 2nd part, we see that

$$F(\alpha, t) \quad = \quad \mathbb{E}(\bar{X}_t - \underline{X}_t) \quad = \quad \mathbb{E}(F(\hat{\alpha}, t)) \quad \geq \quad F(\mathbb{E}(\hat{\alpha}), t)$$

where the final inequality follows from **Jensen's inequality** from last lecture. Assuming the $\geq$ is actually a $>$ here, we can apply F^{-1} to both sides (with t fixed) and (since F^{-1} with t fixed is decreasing, see plot above), we see that $\alpha < \mathbb{E}(\hat{\alpha})$, so $\hat{\alpha}$ is biased.

4. Let $M_t^{(n)} = \max_{0\le k\le n} W_{kt/n}$ denote the **discretely sampled** maximum of W. Is $M_t^{(n)}$ more or less than $M_t = \max_{0\le s\le t} W_s$? It is known that

$$\mathbb{E}(M_t^{(n)}) \;=\; \sqrt{\frac{t}{2\pi n}}\, H_n^{(\frac{1}{2})}$$

where $H_n^{(\frac{1}{2})} = \sum_{k=1}^n k^{-\frac{1}{2}}$ is known as the nth Harmonic number of order $\frac{1}{2}$. Using that $H_n^{\frac{1}{2}} = 2\sqrt{n} + O(1)$ as $n\to\infty$, show this formula is consistent with the formula $\mathbb{E}(R_t) = 2\sqrt{\frac{2t}{\pi}}$ in Hwk4, q5, where R_t is the range of W.

Solution. Clearly $M_t^{(n)} \le M_t$ [1]. Using the given asymptotic relation, we see that

$$\mathbb{E}(M_t^{(n)}) \;=\; \sqrt{\frac{t}{2\pi n}}\,(2\sqrt{n}+O(1)) \;\to\; \sqrt{\frac{2t}{\pi}}$$

as $n\to\infty$, and we see this is half $\mathbb{E}(R_t)$ as expected, since $\mathbb{E}(R_t) = \mathbb{E}(M_t) - \mathbb{E}(m_t) = 2\mathbb{E}(M_t)$ since $m_t \sim -M_t$, where $m_t = \min_{0\le s\le t} W_s$.

5. Consider the famous **GARCH(1,1)** discrete-time stochastic volatility model where

$$\begin{aligned} r_t &= \sqrt{V_t}\,\varepsilon_t \\ V_t &= \omega + \alpha r_{t-1}^2 + \beta V_{t-1} \;=\; V_{t-1} - (1-\beta)V_{t-1} + \omega + \alpha r_{t-1}^2 \end{aligned}$$

for $t\in\mathbb{Z}$, $\omega,\alpha,\beta>0$, where $r_t = (S_t - S_{t-1})/S_{t-1}$ is the return on a stock price process S at time t so $S_t = S_0\prod_{i=1}^t(1+r_i)$, and ε_t is an i.i.d. sequence of random variables with $\mathbb{E}(\varepsilon_t) = 0$ and $\mathrm{Var}(\varepsilon_t) = 1$. From the final equation we see that $1-\beta$ is the **mean-reversion speed**. Express V_t in terms of the ε_t sequence only, and compute $\mathbb{E}(V_t)$ and give a condition on the model parameters for the formula to make sense.

Solution. We first note that

$$V_t \;=\; \omega + \alpha r_{t-1}^2 + \beta V_{t-1} \;=\; \omega + \alpha V_{t-1}\varepsilon_{t-1}^2 + \beta V_{t-1} \;=\; \omega + A_t V_{t-1} \tag{1}$$

where $A_t = \alpha\varepsilon_{t-1}^2 + \beta$. "Unrolling" this expression, we see that

$$\begin{aligned} V_t \;=\; \omega + A_t(\omega + A_{t-1}V_{t-2}) &= \omega + A_t(\omega + A_{t-1}(\omega + A_{t-2}V_{t-3})) \\ &= \omega + A_t\omega + A_tA_{t-1}\omega + A_tA_{t-1}A_{t-2}V_{t-3} \\ &= \omega(1 + A_t + A_tA_{t-1} + A_tA_{t-1}A_{t-2} + \ldots)\,. \end{aligned} \tag{2}$$

We also note that $\mathbb{E}(A_t) = \alpha+\beta$ for all t, so

$$\mathbb{E}(V_t) \;=\; \omega(1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots) \;=\; \frac{\omega}{1-\alpha-\beta}$$

(using the formula for the sum of a **Geometric series**), so we need $\alpha+\beta<1$ for the series to converge, which is known as the **weak stationarity condition**. Moreover from (2) we see that

$$V_{t+k} \;=\; \omega(1 + A_{t+k} + A_{t+k}A_{t+k-1} + A_{t+k}A_{t+k-1}A_{t+k-2} + \ldots)\,.$$

but $(A_t)_{t\in\mathbb{Z}}$ and the *shifted* sequence $(A_{t+k})_{t\in\mathbb{Z}}$ for all $k\in\mathbb{Z}$ both have the same joint distributions, so we see that $V_{t+k} \overset{(d)}{=} V_t$, i.e. the process is stationary (see notes on my webpage for more details on the model, many students have looked at this model in their projects and we often use the student t-distribution for the ε_t's).

6. Explain why we cannot estimate μ perfectly even with continuous observation of S on $[0,T]$ under the Black-Scholes model. for $T<\infty$.

Solution. Let $\mathbb{P}^{\mu}_{S_0}$ denote the probability measure induced on $C_{S_0}([0,T])$ by S; if we had a perfect estimator $\hat{\mu}_T$ then

$$\mathbb{P}^{\mu_0}_{S_0}(\hat{\mu}_T = \mu_0) \;=\; 1\,.$$

But since $\mathbb{P}^{\mu_0}_{S_0}$ and $\mathbb{P}^{\mu}_{S_0}$ are **equivalent** (by Girsanov), $\mathbb{P}^{\mu}_{S_0}(\hat{\mu}_T = \mu_0) = 1$ as well, so $\hat{\mu}_T$ gives the wrong estimate in this case.

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[1] in fact $M_t^{(n)} \nearrow M_t$ a.s. since W is continuous and hence $\lim_{n\to\infty}\mathbb{E}(M_t^{(n)}) = \mathbb{E}(M_t)$ from the monotone convergence theorem, but you are not asked to prove this here

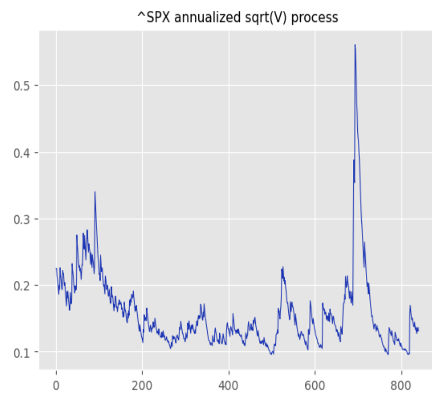

Figure 2: Here we have fitted a GARCH(1,1) model to the last 3 years of SPX data (ending on 10th Nov 2025) using maximum likelihood estimates using a (skewed) t-distribution for the residuals ε_t, and plotted the V process implied by the data (see my website for Python code for this)

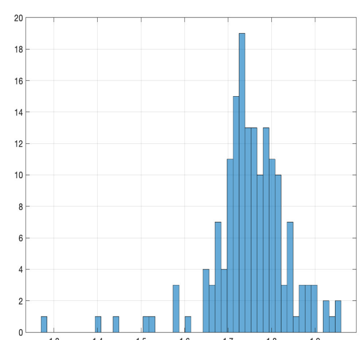

Figure 3: Histogram of $\hat{\alpha}$ for q3 using 200 samples, where we have replaced $\bar{X}_t - \underline{X}_t$ with a sample mean of 1000 i.i.d. realizations per sample and the true value of $\alpha = 1.7$.