On differential equation driven by Brownian motion

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Background

Stochastic differential equations (SDEs) provide a mathematical framework for modelling systems that evolve in time under the influence of both deterministic forces and random fluctuations. Many real-world processes — such as stock prices, biological systems, climate models, and physical diffusion phenomena — exhibit inherent randomness that cannot be captured by deterministic ordinary differential equations (ODEs). SDEs extend the classical dynamical systems framework by incorporating stochastic terms driven by *Brownian motion*, a continuous-time random process with stationary, independent increments.

Itô and Stratonovich Integrals

To define SDEs rigorously, one must specify the meaning of integration with respect to Brownian motion B_t. Two main interpretations are used: the **Itô integral** and the **Stratonovich integral**.

The *Itô integral*, introduced by Kiyosi Itô in the 1940s, is defined in a non-anticipative (or *causal*) sense — the integrand at time t depends only on information available up to t. This makes Itô calculus particularly well-suited for modelling systems in finance and control theory, where future information cannot be used. However, the Itô integral does not satisfy the ordinary chain rule; instead, it follows the *Itô lemma*, which includes an additional correction term accounting for stochastic variance.

Theorem 1. *Let X_t satisfy the Itô stochastic differential equation driven by Brownian Motion B under the physical measure $\mathbb{P}$*

$$\mathrm{d}X_t = a(X_t,t)\,\mathrm{d}t + b(X_t,t)\,\mathrm{d}B_t^{\mathbb{P}},$$

and let $f(x,t)$ be a twice continuously differentiable function in x and once in t. Then the process $Y_t = f(X_t,t)$ evolves according to

$$\mathrm{d}f(X_t,t) = \left(\frac{\partial f}{\partial t}(X_t,t) + a(X_t,t)\frac{\partial f}{\partial x}(X_t,t) + \frac{1}{2}b^2(X_t,t)\frac{\partial^2 f}{\partial x^2}(X_t,t)\right)\mathrm{d}t + b(X_t,t)\frac{\partial f}{\partial x}(X_t,t)\,\mathrm{d}B_t^{\mathbb{P}}.$$

By contrast, the *Stratonovich integral* uses a symmetric (midpoint) approximation of the Brownian path and therefore preserves the standard rules of calculus, such as the chain rule. This makes the Stratonovich formulation more natural in physical and engineering contexts where stochastic perturbations arise as limits of smooth noise or when modelling continuous systems in contact with a fluctuating environment.

Itô Integral Definition.

$$\int_0^T X_t\,\mathrm{d}Y_t \;=\; \lim_{\|\Pi\|\to 0}\sum_{i=0}^{k-1} X_{t_i}\big(Y_{t_{i+1}} - Y_{t_i}\big), \tag{1}$$

where $0 = t_0 < t_1 < \cdots < t_k = T$ is a partition Π of $[0,T]$. By $\|\Pi\| \to 0$ we mean the maximum step size of the partition goes to zero.

In differential (SDE) notation, we may write:

$$\mathrm{d}X_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}Y_t \quad\Longleftrightarrow\quad X_t - X_0 = \int_0^t a_s\,\mathrm{d}s \;+\; \int_0^t b_s\,\mathrm{d}Y_s.$$

Stratonovich Integral Definition.

$$\int_0^T X_t \circ \mathrm{d}W_t \;=\; \lim_{\|\Pi\|\to 0} \sum_{i=0}^{k-1} \frac{X_{t_{i+1}} + X_{t_i}}{2} \left(W_{t_{i+1}} - W_{t_i}\right), \tag{2}$$

where $0 = t_0 < t_1 < \cdots < t_k = T$ again denotes a partition of the time interval with mesh $\|\Pi\| \to 0$.

In differential (SDE) notation, we may write:

$$\mathrm{d}X_t = a_t\,\mathrm{d}t + b_t \circ \mathrm{d}Y_t \quad\iff\quad X_T - X_0 = \int_0^t a_s\,\mathrm{d}s \;+\; \int_0^t b_s \circ \mathrm{d}Y_s.$$

Stochastic SINDy

Recently, *Stochastic Sparse Identification of Nonlinear Dynamics (Stochastic SINDy)* has emerged as a data-driven approach to discovering the governing equations of stochastic systems directly from trajectory data. It extends the deterministic SINDy framework—which represents the system's dynamics as a sparse linear combination of candidate nonlinear functions—by incorporating stochastic terms that capture random perturbations and diffusion effects.

In Stochastic SINDy, one typically assumes a model of the form:

$$dX_t = f(X_t)\,dt + g(X_t)\,dB_t^{\mathbb{P}},$$

where both the drift $f(X_t)$ and diffusion $g(X_t)$ are inferred from observed data using sparse regression techniques. This allows researchers to recover interpretable models of complex stochastic systems, even when the underlying equations are not known *a priori*.

1 Literature review

Example solution. An ideal project should address most of the bullets more or less:

Now sure how much we want to prompt the students. I am more tempted to give this to the solution file so that supervisors can direct the students in the correct direction.

• Describe the motivation for studying stochastic differential equations (SDEs) — why randomness is essential in modelling real-world systems.

• Outline the historical development of stochastic calculus: the emergence of Brownian motion and Wiener processes. In particular, the contributions of Kiyosi Itô and Ruslan Stratonovich.

• Compare the Itô and Stratonovich approaches: Definitions, intuition, and differences in interpretation. Typical use cases for each (e.g. finance vs. physics). This should include key references such as Itô (1944), Stratonovich (1966), and standard SDE textbooks (e.g. Øksendal, Karatzas & Shreve).

• Discuss Numerical Simulation Methods (including accuracy, convergence, and computational cost), including (but not limited to) Euler–Maruyama method. Some other interesting methods are the Milstein scheme, stochastic Runge–Kutta methods. Highlight challenges in simulating Stratonovich vs Itô equations.

• Summarise literature on system identification for stochastic systems. Introduce Sparse Identification of Nonlinear Dynamics (SINDy) and its stochastic extensions (Stochastic SINDy, Neural SINDy, etc.). Discuss how stochastic SINDy relates to classical estimation techniques (e.g. Maximum Likelihood, Kalman filtering). Highlight current research gaps (e.g. robustness to noise, interpretability, or scalability).

• Discuss assumptions made by different authors (e.g. stationarity, ergodicity, Gaussian noise). Point out limitations in current models or simulation techniques. Reflect on how these gaps motivate the present study or project. □

2 Part 2: Numerical computations

This part is for us, will be removed in the final version: The general theme is– in Q1, the students learns about uniqueness and existence of SDE (might have some overlap with FM04, but not necessary for students to have prior knowledge). This leads to an integral representation, which can be written in either Itô or in Stratonovich sense. This brings us to Q2, where they learn how to calculate Stratonovich integrals and Itô integrals separately. From this, they can verify the correction term in the Itô formula. Then, in Q3, in the absence of noise, they learn how to numerically compute the functional form of μ (this is a 3-dim case). In Q4, the students will learn how to identify the drift and diffusion functions from data.

Q1 **Itô-Stratonovich correction:** Consider the stochastic differential equation:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{3}$$

where B is a Brownian motion. Write down a sufficient condition for $\mu, \sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the above SDE has existence and uniqueness for the solution of the form

$$X_t = X_s + \int_s^t \mu(X_u)du + \int_s^t \sigma(X_u)dB_u^{\mathbb{P}}. \tag{4}$$

Example solution. One such condition can be global Lipschitz + linear growth. They need to define the conditions properly in the project. □

Under the assumption of existence and uniqueness of the solution of the form (4) and differentiability of σ, write down the corresponding Stratonovich form differential equation.

Example solution. We have σ that is differentiable. Let X solve

$$X_t = X_0 + \int_0^t \mu(u, X_u)du + \int_0^t \sigma(u, X_u)dB_u.$$

Then X solves the Stratonovich equation

$$X_t = X_0 + \int_0^t \left(\mu(u, X_u) - \frac{1}{2}\sigma(u, X_u)\partial_x\sigma(u, X_u)\right) du + \int_0^t \sigma(u, X_u) \circ dB_u. \tag{5}$$

□

Consider now the ODE

$$y'(s) = \sigma(y(s)) \tag{6}$$

and assume that σ is smooth and a solution $y = y(s)$ is defined for all $s \in \mathbb{R}$. Derive the Itô SDE satisfied by

$$X_t = y(B_t).$$

Derive also the corresponding Stratonovich SDE for X. What do you observe?

Example solution. From the classical chain rule we know that $y''(s) = \sigma'(y(s))y'(s) = \sigma'(y(s))\sigma(y(s))$. Then from Itô's lemma

$$\begin{aligned} dX_t = y'(B_t)dB_t + \frac{1}{2}y''(B_t)dt &= \sigma(y(B_t))dB_t + \frac{1}{2}\sigma'(y(B_t))\sigma(y(B_t))dt \\ &= \sigma(X_t)dB_t + \frac{1}{2}\sigma'(X_t)\sigma(X_t)dt. \end{aligned}$$

To compute the Stratonovich counterpart, by means of the conversion formula above, we obtain

$$dX_t = \sigma(X_t) \circ dB_t,$$

as one would get by "blindly" applying the classical chain rule to X_t. □

Simulate S_T for $N = 100,000$ price paths where the path follows the dynamics with time grid $t_i = \frac{i}{500}$ (do not use $\log S_t$ process to simulate)

$$dS_t = \kappa(\theta - S_t)\,dt + \sigma S_t \circ dW_t,$$

where, $S_0 = 100, T = 1, \kappa = 2, \theta = 100, \sigma = 0.6$. Calculate the European call option price for strike price $K = 100$ and maturity $T = 1$ when the risk-free interest rate is 5%.

Now simulate another $N = 100,000$ many price path S'_T which comes from the dynamics

$$dS'_t = \kappa(\theta - S_t)S'_t\,dt + \sigma S'_t dW_t,$$

where, $S'_0 = 100, T = 1, \kappa =?, \theta =?, \sigma = 0.2$ as before. Calculate again the European call option price for strike price $K = 100$ and Maturity $T = 1$ from the simulated S'_T, when the risk-free interest rate is 5%. Plot the histogram of S_T and S'_T in a single graph. What is your observation?

Example solution. For the Stratonovich sense $dS_t = \mu S_t\,dt + \sigma S_t \circ dW_t$ first the students need to convert to Stratonovich form by adding the Itô correction term to the drift

$$dS_t = \kappa(\theta - S_t)\,dt + \frac{1}{2}\sigma^2 S_t\,dt + \sigma S_t dW_t. \tag{7}$$

One can now use the Euler–Maruyama scheme on (7). i.e. the asset price S_t is discretized as:

$$S_{t+\Delta t} = S_t + \kappa(\theta - S_t)\Delta t + \frac{1}{2}\sigma^2 S_t \Delta t + \sigma S_t \Delta W_t.$$

Then they compute the payoff for a European call option from this formula:

$$\text{payoff}_i = \max(S_T^{(i)} - K, 0), \tag{8}$$

where $i = 1, \ldots, N$. The discounted call option price is then estimated as

$$C_0 \approx e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\text{payoff}_i, \tag{9}$$

with $r = 0.05$.
Need the values for solution. □

Q2 **Computing iterated integrals:** Iterated integrals capture how different parts of a path interact over time, and together they form something called 'signature' (these are nothing but a collection of (Stratnovich type) iterated integrals) of that path, which encodes its essential shape and behaviour. In finance, signatures can help describe and analyse how prices evolve, making them useful for modelling, forecasting, and understanding market dynamics. For $k \in \mathbb{N}$, the k-th order signature of a path/process X is defined as follows.

$$S^{(k)}(X)_{s,t} = \int_s^t \int_s^{u_k} \cdots \int_s^{u_2} \circ dX_{u_1} \circ dX_{u_2} \cdots \circ dX_{u_k}.$$

Calculate the $S^{(1)}(X)_{s,t}$, $S^{(2)}(X)_{s,t}$ and $S^{(3)}(X)_{s,t}$ for $X_t = a + bt$.

Example solution. Let $X_t = a + bt$, where $a, b \in \mathbb{R}$ and $t \geq 0$. Then $dX_t = b\,dt$.
Since $dX_{u_i} = b\,du_i$, each integral is an ordinary (Riemann) integral, so

$$S^{(k)}(X)_{s,t} = b^k \int_s^t \int_s^{u_k} \cdots \int_s^{u_2} du_1\,du_2 \cdots du_k.$$

The iterated integral of 1 over the simplex $\{s < u_1 < \cdots < u_k < t\}$ equals $\frac{(t-s)^k}{k!}$, hence (the students can simply calculate by hand for the first 3, no need to do this general approach)

$$S^{(k)}(X)_{s,t} = \frac{b^k(t-s)^k}{k!}.$$

In particular,

$$S^{(1)}(X)_{s,t} = b(t-s), \qquad S^{(2)}(X)_{s,t} = \frac{b^2(t-s)^2}{2}, \qquad S^{(3)}(X)_{s,t} = \frac{b^3(t-s)^3}{6}.$$

□

Now consider $X_t = B_t$, the standard Brownian motion. Calculate $S^{(1)}(X)_{s,t}$ and $S^{(2)}(X)_{s,t}$.

Example solution. In this case, $dX_t = dB_t$. So,

$$S^{(1)}(X)_{s,t} = \int_s^t \circ dB_u = \lim_{N\to\infty} \sum_{i=0}^{N-1} 1 \times (B_{t_{i+1}} - B_{t_i}) = B_t - B_s.$$

Second order:

$$\begin{aligned}
S^{(2)}(X)_{s,t} &= \int_s^t \int_s^{u_2} \circ dB_{u_1} \circ dB_{u_2} = \int_s^t (B_{u_2} - B_s) \circ dB_{u_2}. \\
&= \int_s^t B_{u_2} \circ dB_{u_2} - B_s \int_s^t \circ dB_{u_2} \\
&= \left(\lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{B_{t_i} + B_{t_{i+1}}}{2} \times (B_{t_{i+1}} - B_{t_i}) \right) - B_s(B_t - B_s) \\
&= \cdots = \frac{B_t - B_s}{2}.
\end{aligned}$$

□

Calculate the Itô iterated integrals $\int_s^t dX_u$ and $\int_s^t \int_s^{u_2} dX_{u_1} dX_{u_2}$ for the above two cases of X. What is the Itô-Stratonovich correction term for both cases?

Example solution. For the linear case, it is the same as above (as a result of X_t being smooth):

$$S^{(1)}(X)_{s,t} = b(t-s), \qquad S^{(2)}(X)_{s,t} = \frac{b^2(t-s)^2}{2}.$$

For the case $X_t = B_t$, since BM is not smooth (has non-zero quadratic variation), the Itô integral is different from the Stratonovich integral, and is as follows:

$$\int_s^t dB_u = \lim_{N\to\infty} \sum_{i=0}^{N-1} 1 \times (B_{t_{i+1}} - B_{t_i}) = B_t - B_s.$$

$$\begin{aligned}
\int_s^t \int_s^{u_2} dB_{u_1} dB_{u_2} &= \int_s^t (B_{u_2} - B_s) dB_{u_2}. \\
&= \int_s^t B_u dB_u - B_s(B_t - B_s) \\
&=^{\text{Itô formula}} \frac{B_t^2 - B_s^2}{2} - \frac{t-s}{2} - B_s(B_t - B_s) \\
&= \frac{(B_t - B_s)^2 - (t-s)}{2}.
\end{aligned}$$

□

Now plot the histogram of $\frac{2S^{(2)}(B)_{s,t}}{t-s}$ for, $t = 10$, $s = 5$ by generating $M = 10,000$ sample paths. Calculate the mean, variance and empirical 95% confidence band. Identify the distribution.

Example solution. The distribution should be $\mathcal{X}^2$ with 1 degree of freedom.
Need the values to provide in solution □

Maybe we can also ask them to return the p-value of the KS statistic. However, I want the students to find $\mathcal{X}^2$ on their own. So unsure how to formulate the question

Happy to change the question to calculating signatures of $\begin{pmatrix} t \\ B_t \end{pmatrix}$ which is used for signature models in finance, but I feel the two dimensional case might be too difficult for weak students, open to feedbacks.

Q3 **ODE setup** Assume now that we are in a deterministic setting, and we consider a physical system described by an ODE. Viz.

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)). \tag{10}$$

The vector $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^\top \in \mathbb{R}^n$ denotes the state of the system at time t, and the function $\mathbf{f}(\mathbf{x}(t)) = (f_1(x(t)), \ldots, f_n(x(t)))^\top$ represents the dynamic constraints that define the equations of motion of the system.

Let us now say we observe the state of the system $\mathbf{x}(t)$ and its derivative $\dot{\mathbf{x}}(t)$ at m points in time $(t_1, t_2, \cdots, t_m)$, and let us arrange these data into two matrices:

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}^\top(t_1) \\ \mathbf{x}^\top(t_2) \\ \vdots \\ \mathbf{x}^\top(t_m) \end{pmatrix} = \begin{pmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_n(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_n(t_2) \\ & \vdots \; \vdots \; \vdots & & \\ x_1(t_m) & x_2(t_m) & \cdots & x_n(t_m) \end{pmatrix}$$

and

$$\dot{\mathbf{X}} = \begin{pmatrix} \dot{\mathbf{x}}^\top(t_1) \\ \dot{\mathbf{x}}^\top(t_2) \\ \vdots \\ \dot{\mathbf{x}}^\top(t_m) \end{pmatrix} = \begin{pmatrix} \dot{x}_1(t_1) & \dot{x}_2(t_1) & \cdots & \dot{x}_n(t_1) \\ \dot{x}_1(t_2) & \dot{x}_2(t_2) & \cdots & \dot{x}_n(t_2) \\ & \vdots \; \vdots \; \vdots & & \\ \dot{x}_1(t_m) & \dot{x}_2(t_m) & \cdots & \dot{x}_n(t_m) \end{pmatrix}$$

The goal is to learn $\mathbf{f}$ from the data. The steps are summarised as follows (Feel free to refer to e.g. https://en.wikipedia.org/wiki/Sparse_identification_of_non-linear_dynamics):

A library $\mathbf{\Theta}(\mathbf{X})$ of nonlinear candidate functions of the columns of $\mathbf{X}$ is constructed, which may be constant, polynomial, or more exotic functions (like trigonometric and rational terms, and so on):

$$\mathbf{\Theta}(\mathbf{X}) = \begin{bmatrix} | & | & | & | & & | & | & \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \cdots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \cdots \\ | & | & | & | & & | & | & \end{bmatrix}$$

Here, higher polynomials are denoted as $\mathbf{X}^{P_2}, \mathbf{X}^{P_3}, \ldots$ e.g.

$$\mathbf{X}^{P_2} = \begin{pmatrix} x_1^2(t_1) & x_1(t_1)x_2(t_1) & \cdots & x_2^2(t_1) & \cdots & x_n^2(t_1) \\ x_1^2(t_2) & x_1(t_2)x_2(t_2) & \cdots & x_2^2(t_2) & \cdots & x_n^2(t_2) \\ & & \vdots \; \vdots \; \vdots & & & \\ x_1^2(t_m) & x_1(t_m)x_2(t_m) & \cdots & x_2^2(t_m) & \cdots & x_n^2(t_m) \end{pmatrix}$$

and so forth.

Each column of $\mathbf{\Theta}(\mathbf{X})$ represents a candidate function for the right-hand side of (10). Because only a few terms are expected to be active at each point in time, an assumption is made that $\mathbf{f}(\mathbf{x}(t))$ admits a sparse representation in $\mathbf{\Theta}(\mathbf{X})$.

We set up a sparse regression problem to determine the sparse vectors of coefficients $\mathbf{\Xi} = [\xi_{\mathbf{1}} \; \xi_{\mathbf{2}} \; \cdots \; \xi_{\mathbf{n}}]$ determining the active terms in $\mathbf{f}(\mathbf{x})$:

$$\dot{\mathbf{X}} = \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi}.$$

This then becomes an optimisation problem in finding a sparse $\mathbf{\Xi}$ which optimally embeds $\dot{\mathbf{X}}$. In other words, a parsimonious model is obtained by performing least squares regression on the system (10) as follows.

$$\xi_{\mathbf{k}} = \arg\min_{\xi'_{\mathbf{k}}} \left|\left|\dot{\mathbf{X}}_{\mathbf{k}} - \mathbf{\Theta}(\mathbf{X})\xi'_{\mathbf{k}}\right|\right|_{\mathbf{2}} + \lambda \left|\left|\xi'_{\mathbf{k}}\right|\right|_{\mathbf{1}},$$

where λ is a regularisation parameter. Finally, the sparse set of $\xi_{\mathbf{k}}$ can be used to reconstruct the dynamical system:

$$\dot{x}_k = f_k(\mathbf{x}) = \mathbf{\Theta}(\mathbf{x}^\top)\xi_k.$$

Notice that $\mathbf{\Theta}(\mathbf{x}^\top)$ is a vector of symbolic functions of elements of $\mathbf{x}$, as opposed to $\mathbf{\Theta}(\mathbf{X})$, which is a data matrix. Thus, in conclusion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \mathbf{\Xi}^\top(\mathbf{\Theta}(\mathbf{x}^\top))^\top.$$

Now consider the dataset $(x_{t_i}, y_{t_i}, z_{t_i})$ for $i = 1, 2, \cdots 50,000$ provided in 'Lorenz.csv' and consider a polynomial library of x, y, z up to order 4. Use the 'pysindy' package (Code available at `https://github.com/dynamicslab/pysindy`) for two regularisation parameters of λ (say 0.001 and 0.5) to learn the function f_1, f_2, f_3 such that the following system holds

@Martin change them if needed, these are ad-hoc

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}.$$

Example solution. **Chaotic Lorenz System** The data matrix should come from

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned} \tag{11}$$

where $\sigma, \beta, \rho > 0$. Let $\sigma = 1, \rho = 2, \beta = 3$. This question is inspired by [1], where they provide a link to the code. . Need the final estimates for the solution file □

@Martin change them if needed

Q4 **Learning drift and diffusion** Consider the EUR-USD time series (10-year history, excluding weekends) provided in 'data.csv'. Assume this data follows an SDE of the form (3). If we denote the EUR-USD time series as $X_{t_1}, \cdots, X_{t_N}$, then the drift vector $(\mu_{t_1}, \cdots, \mu_{t_{N-1}})$ and diffusion vector $(\sigma_{t_1}, \cdots, \sigma_{t_{N-1}})$ can be estimated as

$$\mu_{X_{t_i}} = \frac{X_{t_{i+1}} - X_{t_i}}{t_{i+1} - t_i}, \text{ and, } \sigma_{X_{t_i}} = \frac{(X_{t_{i+1}} - X_{t_i})^2}{2(t_{i+1} - t_i)}. \tag{12}$$

Now use the Stochastic SINDy method (you might want to look at [2]) with a polynomial library of order 5, to estimate the drift and diffusion function. You can use the 'pysindy' package in Python. Plot the drift vector estimate from (12) and the drift function estimate from stochastic SINDy in the same graph. Repeat this for diffusion vector/function.

There is a better estimator of μ in the same paper Eq (6.2) in [2]. I intentionally chose the simpler one as this is the one the students can use in part 3

Example solution. Need the plots in the solution file □

The stochastic SINDy above provides a functional form of $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$. Plot both μ and σ. Using this functional form, you can simulate new sample paths of X_t from (3). Now use Monte Carlo with antithetic variables (see FM06), tabulate and plot the implied volatility smile for this EUR/USD path.

Example solution. Need the plot/table in the solution file □

Last exercise (simulated setup):

(a) Take $N = 50,000$ and define $t_i = \frac{i}{N}$ for all $i \in \{0, 1, \cdots N\}$. Simulate $(X_{t_i})_{i=1}^N$ from the following generalized OU process

$$dX_t = \kappa(\theta - X_t)dt + \sigma X_t dW_t,$$

where $\kappa = ?, \sigma = ?, \theta = ?$.

(b) Use the Stochastic SINDy algorithm as described above with a polynomial library up to order 5 to get $\hat{\mu}$ and $\hat{\sigma}$.

(c) Repeat (a) and (b) $M = 20,000$ times and get

$$\hat{\mu}_1, \ldots, \hat{\mu}_M, \quad \hat{\sigma}_1, \ldots, \hat{\sigma}_M : [0, 1] \to \mathbb{R}$$

(d) Define $Y_i = \int_0^1 \hat{\mu}_i(u) du$. Compute the average, std deviation and confidence interval for Y_i.

this integral exists... as Θ is just a polynomial library, so both $\hat{\mu}$ and $\hat{\sigma}$ are polynomials up to order 5 only

(e) Do the same for $\hat{\sigma}$.

References

[1] S. L. Brunton, J. L. Proctor, and J. N. Kutz, *Discovering governing equations from data by sparse identification of nonlinear dynamical systems*, Proceedings of the national academy of sciences, 113 (2016), pp. 3932–3937.

[2] M. Wanner and I. Mezić, *On higher order drift and diffusion estimates for stochastic sindy*, SIAM Journal on Applied Dynamical Systems, 23 (2024), pp. 1504–1539.