We can re-write their u_t equation in operator form:

$$u_t \quad = \quad a_t + \int_0^t B(t,s)u_s ds \quad = \quad a \,+\, (Bu)(t)$$

so (formally at least) we can invert using the Neumann series:

$$u \quad = \quad (I-B)^{-1}(a) \quad = \quad (I + B + B^2 + ...)a$$

(see page 37 in their paper). Define the resolvent $R = R^B$ (of the second kind) of B via

$$I \quad = \quad (I-B)(I+R) \quad = \quad I - B + R - BR$$

so

$$R \quad = \quad B \,+\, BR\,.$$

Applying both sides here to an arbitrary function $f \in C([0,T])$ and assuming R is also of Volterra form $(Rf)(t) = \int_0^t r(t,u)f(u)du$, we find that

$$\begin{aligned}
\int_0^t r(t,u)f(u)du \quad &= \quad \int_0^t B(t,u)f(u)du \,+\, \int_0^t B(t,s)\int_0^s r(s,u)f(u)duds \\
\Rightarrow \quad \int_0^t r(t,u)f(u)du \quad &= \quad \int_0^t B(t,u)f(u)du \,+\, \int_0^t \int_u^t r(s,u)B(t,s)ds\, f(u)du \\
\Rightarrow \quad &\int_0^t (r(t,u) - B(t,u) - \int_u^t r(s,u)B(t,s)ds)f(u)du \quad = \quad 0\,.
\end{aligned}$$

This has to hold for all f, so

$$r(t,s) \quad = \quad B(t,s) \,+\, \int_s^t B(t,v)r(v,s)dv$$

for Leb a.e. $s \in [0,t]$. Note also the similar (but simpler) form of the optimal solution in Theorem 2.2 in my paper with Ben+Leandro where $u_t = \bar{u}(t) + \int_0^t k(v,t)dW_v$.