



Figure 1: On the left and middle we see the exact P&L (blue) and the Taylor series approximation for P&L (grey) as a function of  $S$  for  $q_1$  when  $K = e^{0.1}$ ,  $T = 1$ ,  $\Delta t = .01$ ,  $\sigma = .1$ ,  $r = 0$  (the middle plot is just the first plot but over a wider range). In the third plot, we see the density of the two-sided maximum for standard Brownian motion (grey) for  $q_2$ .

## Homework 4

- 1.** Consider a trader who buys a European call option at  $t = 0$ , and  $\Delta$ -hedges the call option at  $t = 0$  (i.e. sells  $C_S(S_0, 0)$  units of stock at  $t = 0$  only), and assume  $r = 0$  for simplicity. Compute a Taylor series expansion for the Profit/Loss (P&L) of the trader over a small-time period  $\Delta t$ , and the expectation of this P&L under the risk-neutral measure  $\mathbb{Q}$ , and an exact expression for the P&L.

**Solution.** From the 2d version Taylor theorem:  $f(x+\Delta x, y+\Delta y) = f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y + \frac{1}{2}f_{xx}(x, y)(\Delta x)^2 + f_{xy}(x, y)\Delta x\Delta y + \frac{1}{2}f_{yy}(x, y)(\Delta y)^2 + h.o.t.$  applied to this problem, we see that

$$\text{P&L} \approx \frac{1}{2}C_{SS}(S_0, 0)(\Delta S)^2 + C_t(S_0, 0)\Delta t = \frac{1}{2}\Gamma(\Delta S)^2 + \Theta\Delta t$$

since the initial **total** delta of the position is zero by assumption so there is no  $O(\Delta S)$  term and  $C_{St}$  and  $C_{tt}$  are higher order terms (since  $\Delta S = O(\sqrt{\Delta t})$ ) so we ignore them here. Using a Simple Euler-scheme approximation we know that  $\Delta S \approx S_0\sigma\Delta W$ , so

$$\mathbb{E}^{\mathbb{Q}}(\text{P&L}) \approx \frac{1}{2}\Gamma S_0^2\sigma^2\mathbb{E}^{\mathbb{Q}}((\Delta W)^2) + \Theta\Delta t = 0 \quad (1)$$

because (from the Black-Scholes PDE)  $C_t(S_0, t) + \frac{1}{2}\sigma^2 S^2 C_{SS}(S_0, t) = \Theta + \frac{1}{2}\sigma^2 S_0^2\Gamma = 0$  when  $r = 0$ , and  $\mathbb{E}((\Delta W)^2) = \Delta t$ .

For the final part, the exact P&L is the difference between the portfolio value at time  $\Delta t$  and time zero:

$$C(S, K, \sigma, T - \Delta t) - C(S_0, K, \sigma, T) - (S - S_0)C_S(S_0, K, \sigma, T)$$

(see first two plots above).

- 2.** From Hwk3, we know that

$$\mathbb{P}(W_t \in dx, M_t < b, m_t > a) = \frac{2}{b-a} \sum_{n=1}^{\infty} e^{-\lambda_n t} \sin\left(\frac{n\pi(x-a)}{b-a}\right) \sin\left(\frac{n\pi(0-a)}{b-a}\right) dx \quad (2)$$

for  $a < x < b$ , where  $\lambda_n = \frac{n^2\pi^2}{2(b-a)^2}$ , and  $M_t$  and  $m_t$  are the running max and min processes of  $W$ . Use this to explicitly compute the cdf of the **two-sided maximum**  $R_t := \max_{0 \leq s \leq t} |W_s| = \max(M_t, -m_t)$  of  $W$  at time  $t$ . Is  $R_t = M_t - m_t$  i.e. the range of  $W$ ?

**Solution.** Setting  $b = r$  and  $a = -r$ , clearly  $b - a = 2r$  and we can re-write (2) as

$$\mathbb{P}(W_t \in dx, M_t < r, m_t > -r) = \mathbb{P}(W_t \in dx, R_t < r) = \frac{1}{r} \sum_{n=1}^{\infty} e^{-\lambda_n t} \sin\left(\frac{n\pi(x+r)}{2r}\right) \sin\left(\frac{n\pi(0+r)}{2r}\right) dx$$

for  $x \in (-r, r)$ , where now  $\lambda_n = \frac{n^2\pi^2}{8r^2}$ . Integrating each term here from  $x = -r$  to  $r$ , and assuming (without proof) that we can interchange the sum and integral, we obtain the cdf of  $R_t$ . Specifically (using that  $\int \sin x dx = -\cos x$ ) we see that

$$\int_{-r}^r \sin\left(\frac{n\pi(x+r)}{2r}\right) dx = -\frac{2r \cos\left(\frac{n\pi(x+r)}{2r}\right)}{n\pi} \Big|_{x=-r}^{x=r} = \frac{2r(1 - \cos(n\pi))}{n\pi} = \frac{4r}{n\pi}$$

for  $n$  odd, and **zero** for  $n$  even. Using this, and setting  $n = 2k - 1$  for  $k \in \mathbb{N}$  and that  $\sin(\frac{n\pi(0+r)}{2r}) = \sin(\frac{1}{2}n\pi)$ , we get

$$\mathbb{P}(R_t < r) = \sum_{k=1}^{\infty} e^{-\lambda_{2k-1}t} \frac{4}{(2k-1)\pi} (-1)^{k-1}$$

since  $\sin(\frac{1}{2}n\pi) = \sin(\frac{1}{2}(2k-1)\pi) = (-1)^{k-1}$  (and the  $\frac{1}{r}$  and  $r$  terms have cancelled) and note that each  $\lambda_{2k-1} = \frac{(2k-1)^2\pi^2}{8r^2}$  term here depends on  $r$  (see **right plot above**).

For the final part,  $R$  is not the range of  $W$ , since if e.g.  $M_t = -m_t$ , then  $M_t - m_t = 2R_t$ .

**3.** Compute the asymptotic price of a European call option as the maturity  $T$  tends to infinity (assuming  $r \geq 0$ ).

**Solution.** Recall from the notes that

$$C^{BS}(S, K, \sigma, T-t, r) = S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

where  $\tau = T-t$  is the time-to-maturity and

$$d_1 = \frac{\log \frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau} = \frac{\log \frac{S}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Then we see that  $d_1 \rightarrow +\infty$  and  $Ke^{-r\tau}\Phi(d_2) \rightarrow 0$  as  $T \rightarrow \infty$  (since  $e^{-rT} \rightarrow 0$  if  $r > 0$  and  $d_2 \rightarrow -\infty$  if  $r = 0$ ), so  $C^{BS}(S, K, \sigma, T-t, r) \rightarrow S$  as  $T \rightarrow \infty$ .

For an alternate proof for  $r > 0$  that does not require the Black-Scholes formula, we note that

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \geq e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T - K) = e^{-rT}S_0e^{rT} - Ke^{-rT} = S_0 - Ke^{-rT}.$$

But we also know that

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \leq e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T) = S_0.$$

Hence the call price is sandwiched as follows:

$$S_0 - Ke^{-rT} \leq e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T - K)^+) \leq S_0$$

and both left and right hand sides tends to  $S_0$  as  $T \rightarrow \infty$ , and hence so does the middle expression.

For  $r = 0$ , for European put options we have the limit:  $\lim_{T \rightarrow \infty} \mathbb{E}^{\mathbb{Q}}((K - S_T)^+) = K$  (by the bounded convergence theorem), since  $S_t \rightarrow 0$  a.s. as  $t \rightarrow \infty$  (see Hwk 3), so (by the put-call parity)  $C + K = P + S_0$ , we see that  $\lim_{T \rightarrow \infty} \mathbb{E}^{\mathbb{Q}}((S_T - K)^+) = S_0$ .

**4.** For the SDE  $dX_t = X_t^{\beta} dW_t$  with  $X_0 > 0$  (CEV process), it can be shown that

$$\mathbb{E}(X_t | X_0 = x) = \begin{cases} x(1 - 2\Phi(-\frac{1}{x\sqrt{t}})) & (\beta = 2) \\ x(1 - e^{-\frac{2}{xt}}) & (\beta = \frac{3}{2}) \end{cases}$$

for  $t \geq 0$ , where  $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$  (proof of these formulae not required). Is  $X$  a martingale in either of these cases? (explain your answer).

**Solution.**  $\mathbb{E}(X_t | X_0 = x) < x$  in both cases, so  $X$  cannot be a martingale (note that  $X$  is a true martingale when  $\beta \leq 1$ ).

**5.** Let  $R_t := \bar{X}_t - \underline{X}_t$  denote the **range** of  $dX_t = \sigma dW_t$  over  $[0, t]$ , where  $\bar{X}_t = \max_{0 \leq s \leq t} X_s$  and  $\underline{X}_t = \min_{0 \leq s \leq t} X_s$ . Using that  $\mathbb{E}(R_t) = 2\sigma\sqrt{\frac{2t}{\pi}}$ , derive an **unbiased estimator**  $\hat{\sigma}$  for  $\sigma$  using an observed value for  $R_t$  (hint: your answer should not contain an expectation). Using that  $\mathbb{E}(R_t^2) = 4\log 2 \cdot \sigma^2 t$ , compute the variance of  $\hat{\sigma}$ .

**Solution.**  $\hat{\sigma} = \frac{R_t}{2\sqrt{\frac{2t}{\pi}}}$  is an unbiased estimator. Then

$$\text{Var}(\hat{\sigma}) = \frac{\text{Var}(R_t)}{4\frac{2t}{\pi}} = \frac{\mathbb{E}(R_t^2) - \mathbb{E}(R_t)^2}{\frac{8t}{\pi}} = \frac{4\log 2 \cdot \sigma^2 t - (2\sigma\sqrt{\frac{2t}{\pi}})^2}{\frac{8t}{\pi}} = \sigma^2 \frac{4\log 2 - \frac{8}{\pi}}{\frac{8}{\pi}} \approx 0.0888\sigma^2.$$

**6.** Following on from q5, it can be show that the density of the range  $R_t = M_t - m_t$  of Brownian motion is

$$p(r) = \sum_{k=1}^{\infty} a_k(r) \quad \text{where} \quad a_k(r) = \frac{8}{\sqrt{t}} (-1)^{k-1} k^2 \phi\left(\frac{kr}{\sqrt{t}}\right)$$

where  $\phi(z) = \Phi'(z)$  is the standard Normal density. Using that  $\int_0^\infty r^2 a_k(r) dr = -\frac{4(-1)^k t}{k}$  (\*), verify the formula for  $\mathbb{E}(R_t^2)$  in q5 when  $\sigma = 1$ . Hint: use the Taylor series  $\log(1 + x) = -\sum_{k=1}^\infty \frac{(-1)^k x^k}{k}$ .

**Solution.** Applying \*, to each term of the series, we see that

$$\mathbb{E}(R_t^2) = \int_0^\infty r^2 p(r) dr = \sum_{k=1}^\infty \left( \int_0^\infty r^2 a_k(r) dr \right) = -4t \sum_{k=1}^\infty \frac{(-1)^k}{k}.$$

Then using the hint with  $x = 1$ , we obtain that  $-\sum_{k=1}^\infty \frac{(-1)^k}{k} = \log 2$ , so  $\mathbb{E}(R_t^2) = 4t \log 2$ .

### 7. Constructing Brownian motion using Fourier series.

Show that

$$B_t = tZ_0 + \sum_{n=1}^\infty Z_n \frac{\sqrt{2} \sin(n\pi t)}{n\pi}$$

is a Brownian motion for  $t \in [0, 1]$ , where  $(Z_n)_{n=0}^\infty$  is a sequence of i.i.d.  $N(0, 1)$  random variables. You may use that  $\sum_{n=1}^\infty \frac{2}{n^2 \pi^2} \sin(n\pi s) \sin(n\pi t) = s(1-t)$  for  $0 \leq s \leq t$ .

**Solution.** Since  $\mathbb{E}(Z_n Z_m) = 1$  if  $m = n$  and zero otherwise, we see that

$$\mathbb{E}(\hat{B}_s \hat{B}_t) = \mathbb{E}\left(\sum_{m=1}^\infty Z_m \frac{\sqrt{2} \sin(m\pi s)}{m\pi} \cdot \sum_{n=1}^\infty Z_n \frac{\sqrt{2} \sin(n\pi t)}{n\pi}\right) = \sum_{n=1}^\infty \frac{2}{n^2 \pi^2} \sin(n\pi s) \sin(n\pi t) = s(1-t)$$

for  $0 \leq s \leq t$  where  $\hat{B}_t = \sum_{n=1}^\infty Z_n \frac{\sqrt{2} \sin(n\pi t)}{n\pi}$ , using the hint in the question. From this we see that

$$\mathbb{E}(B_s B_t) = \mathbb{E}((sZ_0 + \hat{B}_s)(tZ_0 + \hat{B}_t)) = st + s(1-t) = s$$

for  $0 \leq s \leq t$ , which is the covariance function of Brownian motion, where we have used that  $\mathbb{E}(Z_0^2) = 1$  and the first term in  $B_t$  is independent of the second term in  $B_t$ .