LARGE-TIME ASYMPTOTICS FOR AN UNCORRELATED STOCHASTIC VOLATILITY MODEL

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Abstract. We derive a large-time large deviation principle with a convex, continuous rate function for the log stock price under an uncorrelated stochastic volatility model. For this we use a Donsker-Varadhan-type large deviation principle for the occupation measure of the Ornstein-Uhlenbeck process, combined with a simple application of the contraction principle and exponential tightness. From this we derive sharp large-time asymptotics for call options and implied volatility in the large-time, large-strike and large-time, fixed strike regimes.

1. The uncorrelated Ornstein-Uhlenbeck model with σ bounded

We work on a model $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ supporting two Brownian motions which satisfies the usual conditions.

Set $f(y)=\sigma^2(y)$, and assume that $0<f_{\min}\leq f\leq f_{\max}<\infty$. We consider an uncorrelated stochastic volatility model for a log stock price process $X_t=\log S_t$ defined by the following stochastic differential equations

$$\begin{cases} dX_t = -\frac{1}{2}\sigma(Y_t)^2dt+\sigma(Y_t)dW^1_t\,, \\ dY_t = -\alpha Y_t dt + dW^2_t\,, \end{cases} \tag{1}$$

for $\alpha>0$, $X_0=x_0, Y_0=y_0$, where W^1,W^2 are two independent standard Brownian motions and Y is an Ornstein-Uhlenbeck process. We set $S_0=1$ (i.e. $x_0=0$) without loss of generality, because X_t-x_0 is independent of x_0 as the SDEs have no dependence on x .

1.1. **Large deviations for the occupation measure of the OU process.** For each $t>0$ and $A\in\mathcal{B}(\mathbb{R})$, let

$$\mu_t(A)=\frac{1}{t}\int_0^t 1_A(Y_s)ds \tag{2}$$

denote the proportion of time up to t that the sample path of Y spends in A. For each $t>0$ and ω, $\mu_t(\omega,.)$ is a probability measure on $\mathbb{R}$. Let $\mathcal{P}(\mathbb{R})$ denote the space of probability measures on $\mathbb{R}$. Then $\mu_t(A)$ satisfies a large-time large deviation principle in the topology of weak convergence, with a good, convex, lower semicontinuous rate function given by

$$I_B(\mu)=\frac{1}{2}\int_{-\infty}^{\infty}|\partial_y\sqrt{\frac{d\mu}{d\mu_\infty}}(y)|^2\,\mu_\infty(dy) \tag{3}$$

for $\mu\in\mathcal{P}(\mathbb{R})$, where $\mu_\infty(y)=(\frac{\alpha}{\pi})^{\frac{1}{2}}\,e^{-\alpha y^2}$ is the unique stationary distribution for Y, i.e. $N(0,1/2\alpha)$ (see Donsker&Varadhan[2],[3], Stroock[8] and pages 367-8 in Feng&Kurtz[5]). If μ is not absolutely continuous with respect to μ_∞, then $I_B(\mu)=\infty$.

Remark 1. *Clearly $I_B(\mu)$ attains it minimum value of zero at $\mu=\mu_\infty$. Moreover, any measure which makes the rate function zero is a stationary distribution, and Y has a unique stationary distribution, so μ_∞ is the unique minimizer of $I_B(\mu)$.*

1.2. **The Prokhorov metric.** Given two measures μ and ν in $\mathcal{P}(\mathbb{R})$, the Prokhorov metric is defined by

$$d(\mu,\nu)=\inf\{\delta>0:\mu(C)\leq\nu(C^\delta)+\delta,\,\nu(C)\leq\mu(C^\delta)+\delta\quad\text{for all closed } C\in\mathcal{B}(\mathbb{R})\}\,.$$

$\mathcal{P}(\mathbb{R})$ then becomes a metric space (note that $d(\mu,\nu)\leq 1$), so the rate function $I_B(\mu)$ is *good*. Convergence of measures in the Prokhorov metric is equivalent to weak convergence of measures.

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1.3. **Contraction principle.** We note that $F:\mathcal{P}(\mathbb{R})\mapsto[f_{\min},f_{\max}]$ given by

$$F(\mu)=\langle f,\mu\rangle=\int_{-\infty}^{\infty}f(y)\mu(dy) \tag{4}$$

is a bounded, continuous functional. By the contraction principle from large deviations theory, the quantity

$$A_t=\frac{1}{t}\int_0^t f(Y_s)ds=\int_{-\infty}^{\infty}f(y)\mu_t(dy) \tag{5}$$

also satisfies the LDP, with good lower semicontinuous rate function given by

$$I_f(a)=\inf_{\mu\in\mathcal{P}(\mathbb{R}):\langle f,\mu\rangle=a}I_B(\mu)\quad,\quad a\in[f_{\min},f_{\max}]\,. \tag{6}$$

Remark 2. *$I_B(.)$ is non-negative and $I_B(\mu_\infty)=0$, so*

$$I_f(\bar{\sigma}^2)=0\,,$$

where

$$\bar{\sigma}^2=\langle f,\mu_\infty\rangle=\int_{-\infty}^{\infty}\sigma^2(y)\mu_\infty(y)dy\,. \tag{7}$$

Moreover, μ_∞ is the unique minimizer of I_B, so $\bar{\sigma}^2$ is the unique minimizer of I_f.

Lemma 1. *$I_f(a)$ is convex in a on $[f_{\min},f_{\max}]$.*

Proof. $0\le\frac{1}{t}\int_0^t f(Y_s)ds\le f_{\max}<\infty$, so by Varadhan's lemma we know that

$$\Lambda(p)=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}(e^{p\int_0^t f(Y_s)ds})=\sup_{a\in[f_{\min},f_{\max}]}[pa-I_f(a)]\,.$$

If we extend the domain of I_f to $\mathbb{R}$ by setting $I_f(a)=+\infty$ for $a\notin[f_{\min},f_{\max}]$, then $I_f(a)$ is lower semicontinuous on $\mathbb{R}$, and $\Lambda(p)=\sup_{a\in\mathbb{R}}[pa-I_f(a)]$ is the Fenchel-Legendre transform of $I_f(a)$. Using Hölder's inequality, we can show that $\Lambda_t(p)=\frac{1}{t}\log\mathbb{E}(e^{p\int_0^t f(Y_s)ds})$ is convex in p for $t<\infty$, so the limit $\Lambda(p)$ is convex. $\Lambda(p)<pf_{\max}<\infty$ for $p\in\mathbb{R}$, so $\Lambda(p)$ is also continuous. By Lemma 2.3.9 in Dembo&Zeitouni[1], the Fenchel-Legendre transform $\Lambda^*(a)=\sup_{p\in\mathbb{R}}[pa-\Lambda(p)]=I_f^{**}(a)$ is a good, convex rate function. Thus I_f^{**} is convex and lower semicontinuous, so $I_f^{**}=I_f$.

□

2. A joint large deviation principle for $(X_t/t,A_t)$

Proposition 1. *$(X_t/t,A_t)$ satisfies a joint LDP as $t\to\infty$ with good rate function $I(x,a)=\frac{(x+\frac{1}{2}a)^2}{2a}+I_f(a)$.*

Proof. See Appendix. □

From this we obtain the following proposition:

Proposition 2. *(X_t/t) satisfies the LDP as $t\to\infty$ with a good rate function given by*

$$I(x)=\inf_{a\in[f_{\min},f_{\max}]}[\frac{(x+\frac{1}{2}a)^2}{2a}+I_f(a)]\le\frac{(x+\frac{1}{2}\bar{\sigma}^2)^2}{2\bar{\sigma}^2}$$

and $x=-\frac{1}{2}\bar{\sigma}^2$ is the unique minimizer of I.

Proof. The LDP with a good rate function just follows from the contraction principle. Setting $a=\bar{\sigma}^2$ defined in (7) and using that $I_f(\bar{\sigma}^2)=0$, we see that $I(-\frac{1}{2}\bar{\sigma}^2)=0$. Moreover, for any $x\neq-\frac{1}{2}\bar{\sigma}^2$, we cannot find an $a\in[f_{\min},f_{\max}]$ which simultaneously makes $\frac{(x+\frac{1}{2}a)^2}{2a}$ and $I_f(a)$ vanish, so $x=-\frac{1}{2}\bar{\sigma}^2$ is the unique minimizer. □

Lemma 2. *$I(x)$ is convex and continuous on $\mathbb{R}$.*

Proof. Define $G:\mathbb{R}\times\mathbb{R}^+\mapsto\mathbb{R}^+$ by $G(x,a)=\frac{(x+\frac{1}{2}a)^2}{2a}$. Then the determinant of the Hessian D^2G is zero everywhere, so the Hessian is positive semi-definite; thus G is convex in (x,a). $I_f(a)$ is convex in a (and (x,a)), so $I(x,a)=G(x,a)+I_f(a)$ is also convex in (x,a). The interval $C=[f_{\min},f_{\max}]$ is a convex set, so $I(x)=\inf_{a\in C}I(x,a)$ is also convex. From the previous proposition, we know that $I(x)$ is finite for all $x\in\mathbb{R}$, hence $I(x)$ is also continuous on any open interval. □

Remark 3. *For non-zero correlation and/or unbounded σ, the approach outlined here will not work. However, we can transform the problem to a small-noise, fast mean-reverting regime, which is the same scaling used in the recent paper by Feng et al.* [4]*, aside from the fact that the drift of the log Stock price process is not small in this case. This problem then falls into the class of homogenization and averaging problems for nonlinear HJB type equations, where the fast volatility variable lives on a non-compact space. The Feng et al. argument based on viscosity solutions can be easily adapted to the large-time regime, using Bryc's lemma combined with exponential tightness to prove a large deviation principle. The leading order term is the unique viscosity solution to a HJB equation where the Hamiltonian is given in terms of the limiting log mgf for the integrated variance; this will be dealt with in a sequel article. For the well known SABR model with $\beta = 1$, we can derive large-time asymptotics for the correlated case using the Willard mixing formula, see Forde*[6].

2.1. **Large-time behaviour of the distribution function.** $I(-\frac{1}{2}\bar{\sigma}^2) = 0$ so from the convexity and continuity of $I(x)$, $I(x)$ is non-decreasing for $x > -\frac{1}{2}\bar{\sigma}^2$ and non-increasing for $x < -\frac{1}{2}\bar{\sigma}^2$. From this we obtain the following:

Corollary 1.

$$\begin{aligned} -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}(X_t > xt) &= I(x) \qquad (x > -\frac{1}{2}\bar{\sigma}^2)\,, \\ -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}(X_t > x) &= I(0) \qquad (x \in \mathbb{R})\,. \end{aligned}$$

Proof. The first result just follows from the LDP and the continuity of I. For the second result, we first assume $x > 0$. By Proposition 2, we know that for all $\epsilon, \delta > 0$, there exists a $t^* = t^*(\delta, \epsilon, x)$ such that for all $t > t^*$ we have

$$e^{-(I(\delta)+\epsilon)t} \le \mathbb{P}(\frac{X_t}{t} > \delta) \le \mathbb{P}(X_t > x) = \mathbb{P}(\frac{X_t}{t} > \frac{x}{t}) \le \mathbb{P}(\frac{X_t}{t} > 0) \le e^{-(I(0)-\epsilon)t}.$$

(recall that $X_0 = 0$). Taking the limit as $t \to \infty$, we obtain $-I(\delta) \le \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{P}(X_t > x) \le \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(X_t > x) \le I(0)$, and we then just let $\delta \to 0$ and use the continuity of $I(x)$ at $x = 0$. We proceed similarly for $x < 0$. □

3. Call option prices

Let $\mathbb{P}^*(A) = \frac{1}{S_0}\mathbb{E}(S_t 1_A)$ for $A \in \mathcal{F}_t$ denote the *Share measure.*

Lemma 3. *(X_t/t) satisfies the large deviation principle under $\mathbb{P}^*$ with rate function $I(-x)$.*

Proof. From a simple Girsanov change of measure, we see that $-X_t$ satisfies the same SDEs under $\mathbb{P}^*$ as X_t does under $\mathbb{P}$. The result then follows from Proposition 2. □

Proposition 3. *For the OU model defined in* (1)*, we have the following large-time asymptotic behaviour for put/call options*

$$\begin{aligned} -\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}(S_t - S_0e^{xt})^+ &= I(-x) \qquad (x > \frac{1}{2}\bar{\sigma}^2)\,, \\ -\lim_{t\to\infty}\frac{1}{t}\log(S_0 - \mathbb{E}(S_t - S_0e^{xt})^+) &= I(-x) \qquad (|x| < \frac{1}{2}\bar{\sigma}^2)\,, \\ -\lim_{t\to\infty}\frac{1}{t}\log(\mathbb{E}(S_0e^{xt} - S_t)^+) &= I(-x) \qquad (x < -\frac{1}{2}\bar{\sigma}^2)\,, \\ -\lim_{t\to\infty}\frac{1}{t}\log(S_0 - \mathbb{E}(S_t - S_0e^{x})^+) &= I(0) \qquad (x \in \mathbb{R})\,. \end{aligned} \tag{8}$$

Proof. We first assume $x > \frac{1}{2}\bar{\sigma}^2$, and note that $I(-x)$ is non-decreasing for $x > \frac{1}{2}\bar{\sigma}^2$. From Lemma 3, we know that for all $\epsilon > 0$ there exists a $t^* = t^*(\epsilon)$ such that for all $t > t^*$ we have

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0e^{xt}) \;=\; \mathbb{P}^*(X_t > xt) - e^{xt}\mathbb{P}(X_t > xt) \;\le\; \mathbb{P}^*(X_t > xt) \;\le\; e^{-(I(-x)-\epsilon)t}$$

which gives the upper bound for the call price. For the lower bound we have

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0e^{xt})^+ = \mathbb{E}^{\mathbb{P}^*}(1 - e^{xt}e^{-X_t})^+ = e^{xt}\mathbb{E}^{\mathbb{P}^*}(e^{-xt} - e^{-X_t})^+\,. \tag{9}$$

From a picture, we see that for any random variable X on $\mathbb{R}$, $\mathbb{E}(e^k - e^X)^+ \ge (e^k - e^{k-\delta})\mathbb{P}(X < k - \delta)$. In this case we set $X = -X_t$ and $k = -xt$ to obtain

$$\mathbb{E}^{\mathbb{P}^*}(e^{-xt} - e^{-X_t})^+ \;\ge\; (e^{-xt} - e^{-xt-\delta})\mathbb{P}^*(-X_t < -xt - \delta)\,.$$

Combining this with (9) we have

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0 e^{xt})^+ \geq e^{xt}(e^{-xt} - e^{-xt-\delta})\mathbb{P}^*(-X_t < -xt - \delta) = (1-e^{-\delta})\mathbb{P}^*(-X_t < -xt-\delta) \geq e^{-(I(-x-\delta)+\epsilon)t},$$

and the first result follows from the continuity of $I(x)$. The other cases follow similarly. □

Remark 4. *Using the same proofs as in Corollary 1.7 and Corollary 2.17 in Forde&Jacquier[7] for the Heston model, we have the following large-time asymptotic behaviour for the implied volatility $\hat{\sigma}_t(x)$ of a put/call option with strike $S_0 e^x$*

$$\lim_{t\to\infty} \hat{\sigma}_t^2(xt) = \begin{cases} 2(2I(x) - x - 2\sqrt{I(x)^2 - I(x)x}) & |x| > \frac{1}{2}\bar{\sigma}^2, \\ 2(2I(x) - x + 2\sqrt{I(x)^2 - I(x)x}) & (|x| < \frac{1}{2}\bar{\sigma}^2), \end{cases}$$

$$\lim_{t\to\infty} \hat{\sigma}_t^2(x) = 8I(0).$$

We omit the details for the sake of brevity.

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Appendix A. Proof of Proposition 1

Let $Z_t = X_t/t$. We first note that $(Z_t, A_t) \stackrel{d}{=} (\frac{1}{t}W^1_{tA_t} - \frac{1}{2}A_t, A_t)$. We first assume $x + \frac{1}{2}a > 0$. Now choose δ so that $0 < \delta < x + \frac{1}{2}a$. Then

$$\mathbb{P}(|Z_t - x| < \frac{\delta}{\sqrt{2}}, |A_t - a| < \frac{\delta}{\sqrt{2}}) \leq \mathbb{P}(\|(Z_t, A_t) - (x,a)\| < \delta) \leq \mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta).$$

From the Gärtner-Ellis theorem, we can easily verify that $\frac{W^1_{at} - \frac{1}{2}at}{t}$ satisfies the LDP as $t \to \infty$ with convex rate function $\frac{(x+\frac{1}{2}a)^2}{2a}$. Then for any $\epsilon > 0$, conditioning on A_t and using the LDP for A_t and the LDP for $\frac{W^1_{at} - \frac{1}{2}at}{t}$, we see that there exists a $t = t^*(\epsilon, \delta)$ such that

$$\mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \leq e^{-t[-\epsilon + \inf_{y \in B_\delta(x)} \frac{(y+\frac{1}{2}(a-\delta))^2}{2(a+\delta)}]} e^{-t[-\epsilon + \inf_{a_1 \in \bar{B}_a(\delta)} I_f(a_1)]}.$$

Then

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \leq -\inf_{y\in B_\delta(x)} \frac{(y+\frac{1}{2}(a-\delta))^2}{2(a+\delta)} - \inf_{a_1 \in \bar{B}_a(\delta)} I_f(a_1),$$

and by the lower semicontinuity of $I_f(a)$ we have

$$\lim_{\delta\to 0}\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \leq -[\frac{(x+\frac{1}{2}a)^2}{2a} + I_f(a)].$$

Using a similar argument for the lower bound, we replace the limsup here by a genuine limit, so (Z_t, A_t) satisfies the weak LDP with rate function $\frac{(x+\frac{1}{2}a)^2}{2a} + I_f(a)$. The rate function $I_f(a)$ is good, so (A_t) is exponentially tight; hence for all $R > 0, a > 0$, there exists a compact set $K_a \subset \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}((Z_t, A_t) \in [-R,R] \times K_a^c) \leq \limsup_{t\to\infty} \frac{1}{t}\log \mathbb{P}(A_t \in K_a^c) \leq -a,$$

so (Z_t, A_t) is exponentially tight, hence (Z_t, A_t) satisfies the full LDP and the rate function is good. We proceed similarly for $x + \frac{1}{2}a \leq 0$.

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