

Consider an integrated variance process $A_t = \int_0^t V_s ds$ which satisfies an eq of the form:

$$A_t = G_0(t) + \int_0^t \kappa(t-s) W_{A_s} ds$$

for some $\kappa \in L^1$. Consider a discrete-time Monte Carlo scheme for A :

$$A_{j\Delta t} = G_0(j\Delta t) + \sum_{k=1}^j b_{j,k} W_{A_{k\Delta t}} \quad (1)$$

where $b_{j,k} = \int_{t_{k-1}}^{t_k} K(t_j - s) ds$ so in particular $b_{j,j} = \int_{t_{j-1}}^{t_j} K(t_j - s) ds$, and $t_k = k\Delta t$. Setting $G_0(t) = V_0 t$, we have

$$\begin{aligned} \Delta A_{j\Delta t} &= V_0 \Delta t + \sum_{k=1}^{j-1} (b_{j,k} - b_{j-1,k}) W_{A_{k\Delta t}} + b_{j,j} W_{A_{j\Delta t}} \\ &= V_0 \Delta t + Z_j + b_{j,j} (W_{A_{j\Delta t}} - W_{A_{(j-1)\Delta t}}) \\ &= \alpha_j + b_{j,j} (W_{A_{j\Delta t}} - W_{A_{(j-1)\Delta t}}) \end{aligned}$$

for $j = 1, 2, \dots$, where $\tilde{W}_t = W_{A_{j\Delta t}} - W_{A_{(j-1)\Delta t}}$, where $Z_j = \sum_{k=1}^{j-1} (b_{j,k} - b_{j-1,k}) W_{A_{k\Delta t}} + b_{j,j} W_{A_{(j-1)\Delta t}}$.

If $H = \frac{1}{2}$ then

$$A_{j\Delta t} = V_0 j\Delta t + \nu \sum_{k=1}^j W_{A_{k\Delta t}} \Delta t$$

so

$$\Delta A_{j\Delta t} = V_0 \Delta t + \nu W_{A_{j\Delta t}} \Delta t = V_0 \Delta t + \nu W_{A_{(j-1)\Delta t}} \Delta t + \nu (W_{A_{j\Delta t}} \Delta t - W_{A_{(j-1)\Delta t}} \Delta t)$$

Then using the Appendix E argument below, ΔA_j has an Inverse Gaussian distribution.

Appendix E

$$A_t = G_0(t) + \sigma W_{A_t}. \quad (2)$$

Now let

$$Y_t = -t + \sigma W_t \quad (3)$$

and set $\tilde{A}_t = H_{-G_0(t)}$, where $H_b^Y = \inf\{t : Y_t = b\}$. Then setting $t \mapsto \tilde{A}_t$ in (3), we see that

$$\begin{aligned} -G_0(t) &= -\tilde{A}_t + \sigma W_{\tilde{A}_t} \\ &= -\tilde{A}_t + Y_{\tilde{A}_t} + \tilde{A}_t = Y_{\tilde{A}_t} \end{aligned}$$

i.e. \tilde{A} satisfies the same equation as A_t in (2), and H_b has an Inverse Gaussian distribution (first hitting time to a barrier for a Brownian motion with drift), and note in particular that

$$\sigma W_{\tilde{A}_t} = Y_{\tilde{A}_t} + \tilde{A}_t = -G_0(t) + \tilde{A}_t$$

For us $\sigma = b_{j,j}$ for each j .