# Example questions

**1**. (Estimating volatility). Let $X_t = \mu t + \sigma W_t$. Show that $\hat{\sigma}_n^2 = \sum_{i=0}^{n-1}(X_{(i+1)/n} - X_{i/n})^2$ is a consistent estimator for $\sigma^2$ (i.e. that $\hat{\sigma}_n^2 \to \sigma^2$ as $n \to \infty$ in some sense). Is $\hat{\sigma}_n^2$ an unbiased estimator?

**Solution**.

$$\begin{aligned}
\hat{\sigma}_n^2 \quad = \quad \sum_{i=0}^{n-1}(X_{(i+1)/n} - X_{i/n})^2 \quad = \quad \sum_{i=0}^{n-1}(\frac{\mu}{n} + \sigma(W_{(i+1)/n} - W_{i/n}))^2 \quad &\sim \quad \sum_{i=1}^{n}(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}Z_i)^2 \\
&= \quad \frac{\mu^2}{n} + \frac{2\mu\sigma}{n}\sum_{i=1}^{n}\frac{Z_i}{\sqrt{n}} + \frac{\sigma^2}{n}\sum_{i=1}^{n}Z_i^2 \\
&= \quad \frac{\mu^2}{n} + \frac{2\mu\sigma}{\sqrt{n}} \cdot \frac{1}{n}\sum_{i=1}^{n}Z_i + \frac{\sigma^2}{n}\sum_{i=1}^{n}Z_i^2
\end{aligned}$$

where the $Z_i$'s are i.i.d. $N(0,1)$, and we have used that $W_{(i+1)/n} - W_{i/n} \sim \frac{1}{\sqrt{n}}Z_i$ (from the third property of Brownian motion). The $\frac{\mu^2}{n}$ term in the final line trivially tends to zero, and the second term also tends to zero a.s. because $\frac{1}{n}\sum_{i=1}^{n}Z_i$ tends to $\mathbb{E}(Z_i) = 0$ from the SLLN. Hence $\hat{\sigma}_n^2$ tends to the constant $\sigma^2$ in distribution by applying the SLLN to the final term (which also implies convergence in probability), so $\hat{\sigma}_n^2$ is a consistent estimator for $\sigma^2$.

Note this applies to the log stock price $X_t = \log S_t$ for the Black-Scholes model if we just replace $\mu$ here with $\mu - \frac{1}{2}\sigma^2$, and the final limit does not depend on $\mu$.

For the second part, for $n$ finite, we see that $\mathbb{E}(\hat{\sigma}_n^2) = \frac{\mu^2}{n} + \sigma^2$; hence $\hat{\sigma}_n^2$ is only unbiased when $\mu = 0$.

**2**. Let $R_t := \bar{X}_t - \underline{X}_t$ denote the **range** of $dX_t = \sigma dW_t$ over $[0,t]$, where $\bar{X}_t = \max_{0 \le s \le t} X_s$ and $\underline{X}_t = \min_{0 \le s \le t} X_s$. Using that $\mathbb{E}(R_t) = 2\sigma\sqrt{\frac{2t}{\pi}}$, derive an **unbiased estimator** $\hat{\sigma}$ for $\sigma$ using an observed value for $R_t$ (hint: your answer should not contain an expectation). Using that $\mathbb{E}(R_t^2) = 4\log 2 \cdot \sigma^2\, t$, compute the variance of $\hat{\sigma}$.

**Solution**. $\hat{\sigma} = \frac{R_t}{2\sqrt{\frac{2t}{\pi}}}$ is an unbiased estimator. Then

$$\mathrm{Var}(\hat{\sigma}) \quad = \quad \frac{\mathrm{Var}(R_t)}{4\frac{2t}{\pi}} \quad = \quad \frac{\mathbb{E}(R_t^2) - \mathbb{E}(R_t)^2}{\frac{8t}{\pi}} \quad = \quad \frac{4\log 2 \cdot \sigma^2\, t - (2\sigma\sqrt{\frac{2t}{\pi}})^2}{\frac{8t}{\pi}} \quad = \quad \sigma^2\frac{4\log 2 - \frac{8}{\pi}}{\frac{8}{\pi}} \quad \approx \quad 0.0888\sigma^2\,.$$

**3.** (**Estimating volatility**). Let $X_t = \mu t + \sigma W_t$ and let $\bar{X}_t = \max_{0 \le s \le t} X_s$ and $\underline{X}_t = \min_{0 \le s \le t} X_s$, and assume the unit of time here is in days. Using that

$$\mathbb{E}^{\mathbb{P}}(\bar{X}_t(\bar{X}_t - X_t) + \underline{X}_t(\underline{X}_t - X_t)) \quad = \quad \sigma^2 t \tag{1}$$

derive an **unbiased estimate** for $\sigma^2$ using the **daily returns** $r_i := X_i - X_{(i-1)}$, **daily (relative) highs**: $H_i = \max_{s \in [(i-1),i]}(X_s - X_{(i-1)})$, and **daily (relative) lows**: $L_i = \min_{s \in [(i-1),i]}(X_s - X_{(i-1)})$ for $i \in 1, 2, ..., n$.

**Solution**. $X$ has i.i.d. increments and the $r_i$'s are the increments of $X$ with time increment 1 so $r_i \sim X_1$ for all $i$.

Moreover, for each $i$, the *process* $X_s - X_{(i-1)}$ for $s \in [(i-1), i]$ is independent (and distributed the same) as the process $X_s - X_{(j-1)}$ for $s \in [(j-1), j]$ for $j \neq i$, so (in particular) the $H_i(H_i - r_i)$'s are i.i.d. and so are the $L_i(L_i - r_i)$ (this doesn't mean that $H_i(H_i - r_i)$ and $L_i(L_i - r_i)$ are independent of each other, but we don't require that).

Hence from this i.i.d. property, we see that

$$\mathbb{E}^{\mathbb{P}}(\frac{1}{n}\sum_{i=1}^{n}(H_i(H_i - r_i) + L_i(L_i - r_i))) \quad = \quad \mathbb{E}^{\mathbb{P}}(H_i(H_i - r_i) + L_i(L_i - r_i)) \quad = \quad \sigma^2$$

so $\hat{\sigma}^2 := \frac{1}{n}\sum_{i=1}^{n}(H_i(H_i - r_i) + L_i(L_i - r_i))$ has expectation $\sigma^2$, and hence is an unbiased estimate for $\sigma^2$, which is robust to unknown $\mu$.

**4**. Let $X_t = \mu t + \sigma W_t$ and assume we have observations of $X$ at equidistant times on the interval $[0,T]$, and assume $\sigma$ is known. Show that the variance of any **unbiased estimator** for $\mu$ is $\geq \frac{\sigma^2}{T}$.

**Solution**. Let $\Delta X_i = X_{\frac{i}{n}T} - X_{\frac{(i-1)}{n}T}$ for $i = 1,...,n$ denote the increments of $X$. Then the $\Delta X_i$'s are i.i.d. $N(\mu\Delta t, \sigma^2\Delta t)$ random variables with $\Delta t = \frac{T}{n}$, so their joint density is just the product

$$\frac{1}{(2\pi\sigma^2\Delta t)^{\frac{1}{2}n}} e^{-\sum_{i=1}^n \frac{(\Delta X_i - \mu\Delta t)^2}{2\sigma^2\Delta t}} .$$

Taking the log of this we obtain

$$\begin{aligned} \ell_n(\mu) &= (...) - \sum_{i=1}^n \frac{(\Delta X_i - \mu\Delta t)^2}{2\sigma^2\Delta t} \\ \Rightarrow \quad \frac{\partial}{\partial\mu}\ell_n(\mu) &= \sum_{i=1}^n \frac{(\Delta X_i - \mu\Delta t)}{\sigma^2} \end{aligned}$$

$\ell_n(\mu)$ is known as the **score**, and in general it can be easily shown that $\mathbb{E}(\ell_n(\mu)) = 0$. Then the **Fisher information**: $I(\mu) := \mathrm{Var}(\frac{\partial\ell_n}{\partial\mu}) = \mathbb{E}((\frac{\partial\ell_n}{\partial\mu})^2) = \sum_{i=1}^n \frac{\sigma^2\Delta t}{\sigma^4} = \frac{T}{\sigma^2}$, so (by the **Cramer-Rao bound**) from undergrad Statistics, the variance of any unbiased estimator $\hat{\mu}$ for $\mu$ satisfies $\mathrm{Var}(\hat{\mu}) \geq \frac{1}{I(\mu)} = \frac{\sigma^2}{T}$. Hence we need $T$ large to get a good estimator for $\mu$. Note this bound is attained by the obvious unbiased estimator $\hat{\mu} = X_T/T$.

**5.** A **symmetric $\alpha$-stable process** $X$ with parameters $\alpha \in (0,2]$, $\sigma > 0$ is a generalization of Brownian motion, which has independent stationary increments like Brownian motion but now $\mathbb{E}(e^{iu(X_t - X_s)}|X_s) = e^{-(t-s)\sigma^\alpha|u|^\alpha}$ for $u \in \mathbb{R}$ and $0 \leq s \leq t$, so $X$ is only a (multiple of) BM if $\alpha = 2$, but for $\alpha < 2$ the increments of $X$ are not normally distributed. If $\bar{X}_t = \max_{0\leq s\leq t} X_s$ and $\underline{X}_t = \min_{0\leq s\leq t} X_s$ and $\sigma = 1$, it can be shown that

$$\mathbb{E}(\bar{X}_t - \underline{X}_t) = F(\alpha,t) := \frac{2\alpha\Gamma(1-\frac{1}{\alpha})}{\pi} t^{\frac{1}{\alpha}}$$

for $\alpha \in (1,2]$. Use this identity to define a statistical estimator $\hat{\alpha}$ for $\alpha$ from observed values for $\bar{X}_t$ and $\underline{X}_t$. Is $\hat{\alpha}$ biased? (you may use that $\frac{\partial^2}{\partial\alpha^2}F(\alpha,t) > 0$ and that $F(\alpha,t)$ is a decreasing function of $\alpha$ for $t$ fixed).

**Solution**. We just solve $\bar{X}_t - \underline{X}_t = F(\hat{\alpha},t)$ for $\hat{\alpha}$ to get $\hat{\alpha}$ (see plot of $F(\alpha,t)$ on the right above for $t=1$).
For the 2nd part, we see that

$$F(\alpha,t) = \mathbb{E}(\bar{X}_t - \underline{X}_t) = \mathbb{E}(F(\hat{\alpha},t)) \geq F(\mathbb{E}(\hat{\alpha}),t)$$

where the final inequality follows from **Jensen's inequality** from last lecture. Assuming the $\geq$ is actually a $>$ here, we can apply $F^{-1}$ to both sides (with $t$ fixed) and (since $F^{-1}$ with $t$ fixed is decreasing, see plot above), we see that $\alpha < \mathbb{E}(\hat{\alpha})$, so $\hat{\alpha}$ is biased.

**6**. For a symmetric $\alpha$-stable process $(X_t)$ with parameters $(\alpha,\sigma)$, it is known that

$$\mathbb{E}(\bar{X}_1(\bar{X}_1 - X_1)) = \frac{\sigma^2}{\Gamma(1+\frac{2}{\alpha})\sin(\frac{\pi}{\alpha})^2} \quad , \quad \mathbb{E}(\bar{X}_1 - \underline{X}_1) = \sigma\frac{2\alpha\Gamma(1-\frac{1}{\alpha})}{\pi}$$

for $\alpha \in (1,2)$ [1]. Use this to construct an estimator $\hat{\alpha}$ for $\alpha$. Show that $\hat{\alpha} \to \alpha$ as $n \to \infty$.

**Remark 0.1** Note $\mathbb{E}(\bar{X}_1(\bar{X}_1 - X_1))$ is finite even though $\mathbb{E}(X_1^2)$ and $\mathbb{E}(\bar{X}_1^2)$ are not.

**Solution**. From the i.i.d. property of $X$, we first note that

$$\mathbb{E}(\bar{X}_1(\bar{X}_1 - X_1)) = \mathbb{E}(\frac{1}{n}\sum_{i=1}^n H_i(H_i - r_i)) \quad , \quad \mathbb{E}(\bar{X}_1 - \underline{X}_1) = \mathbb{E}(\frac{1}{n}\sum_{i=1}^n (H_i - L_i))$$

where $H_i = \max_{s\in[i-1,i]}(X_s - X_{i-1})$, $L_i = \min_{s\in[i-1,i]}(X_s - X_{i-1})$ denote the **daily (relative) highs** and **lows**. As always for these type of questions, we then remove the expectation and replace the true parameters $(\alpha,\sigma)$ with their estimates $(\hat{\alpha},\hat{\sigma})$, and now wish to solve:

$$\frac{1}{n}\sum_{i=1}^n H_i(H_i - r_i) = \frac{\hat{\sigma}^2}{\sin(\frac{\pi}{\hat{\alpha}})^2\Gamma(1+\frac{2}{\hat{\alpha}})} \quad , \quad \frac{1}{n}\sum_{i=1}^n (H_i - L_i) = \hat{\sigma}\frac{2\hat{\alpha}\Gamma(1-\frac{1}{\hat{\alpha}})}{\pi} .$$

[1] we have seen the second identity in Hwk 5 q3, and the first identity in Hwk 1 for the case $\alpha = 2$ so $X_t = \sqrt{2}W_t$ where $W_t$ is BM

Dividing the first term by the square of the 2nd terms, the $\hat{\sigma}^2$ terms cancel, and we see that

$$\frac{\frac{1}{n}\sum_{i=1}^n H_i(H_i - r_i)}{(\frac{1}{n}\sum_{i=1}^n (H_i - L_i))^2} \quad = \quad \frac{1}{\sin(\frac{\pi}{\hat{\alpha}})^2 \Gamma(1+\frac{2}{\hat{\alpha}})} \cdot \frac{1}{(\frac{2\hat{\alpha}\Gamma(1-\frac{1}{\hat{\alpha}})}{\pi})^2} \quad = \quad g(\hat{\alpha})\,. \tag{2}$$

We then just solve numerically for $\hat{\alpha}$ from the observed test statistic on the left hand side (see plot of $g(\alpha)$ below). Note if we replace $X$ for $\lambda X$ (with $\lambda > 0$) then the left hand side becomes $\frac{\frac{1}{n}\sum_{i=1}^n \lambda H_i(\lambda H_i - \lambda r_i)}{(\frac{1}{n}\sum_{i=1}^n (\lambda H_i - \lambda L_i))^2}$ but all $\lambda$ terms cancel, so we say $\hat{\alpha}$ is **scale-invariant**.

We can improve the estimate by replacing $\frac{1}{n}\sum_{i=1}^n H_i(H_i - r_i)$ on the left hand with the **antithetic** version: $\frac{1}{2n}\sum_{i=1}^n H_i(H_i - r_i) + \frac{1}{2n}\sum_{i=1}^n L_i(L_i - r_i)$, since both these expressions here have the same expectation but the latter has smaller variance.

Finally, since the $H_i(H_i - r_i)$'s are i.i.d. and the $(H_i - L_i)^2$'s are i.i.d., by the SLLN, the numerator and denominator of (2) tend to $\mathbb{E}(H_1(H_1 - r_1))$ and $\mathbb{E}((H_1 - L_1))^2$ respectively as $n \to \infty$, whose ratio is $g(\alpha)$; hence $g(\hat{\alpha}) \to g(\alpha)$ as $n \to \infty$, and $g^{-1}$ is well defined and continuous so $\hat{\alpha} \to \alpha$, so $\alpha$ is what we call a **consistent estimator** for $\alpha$.

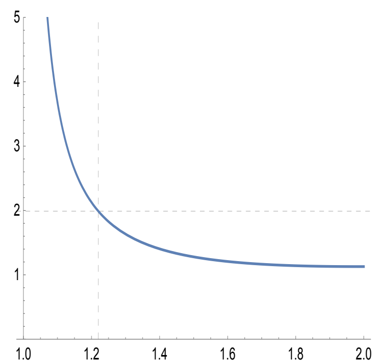


Figure 1: Here we see a plot of $F(\alpha, t)$ in q5 as a function of $\alpha$ (for $t = 1$ fixed), which we invert to solve for $\hat{\alpha}$ from the realized value of $\bar{X}_t - \underline{X}_t$.

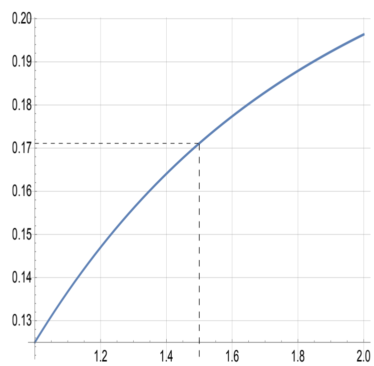


Figure 2: Histogram of $\hat{\alpha}$ and $\hat{\sigma}$ for q6 using 200 samples, where we have replaced $\bar{X}_t - \underline{X}_t$ with a sample mean of 4000 i.i.d. realizations per sample and the true parameter values are $\alpha = 1.7$ and $\sigma = 1$. The mean values are $(1.6526, .9644)$.