

Bid-ask spreads/finite liquidity, and the dual portfolio optimization problem

Let C^X, C^Y, C^Z denote random vectors with components $C_j^X = (X - K_j^X)^+$, $C_j^Y = (Y - K_j^Y)^+$, $C_j^Z = (Z - K_j^Z)^+$ for $j = 0, \dots, m$ for X and Y and $j = 1, \dots, m$ for Z , with corresponding market price vectors P^X, P^Y, P^Z , and assume that the zeroth strikes are zero, i.e. $K_0^X = 0, K_0^Y = 0$, which correspond to the tradeable forward contracts (note we don't need an additional zero strike contract on Z since it has the same payoff as the zero strike contract on X).

Assume $X, Y \in [0, R]$ for simplicity. To allow for finite liquidity, we modify the original problem to

$$\inf_{\mu \in \mathcal{P}([0, R]^2)} \sup_{u, v \in \mathbb{R}^{m+1}, w \in \mathbb{R}^m, q_b^X \leq u \leq q_a^X, q_b^Y \leq v \leq q_a^Y, q_b^Z \leq w \leq q_a^Z} \left(\frac{1}{\alpha} H(\mu | \bar{\mu}) + \mathbb{E}^\mu(u \cdot C^X + v \cdot C^Y + w \cdot C^Z) + \chi(u, v, w) \right) \quad (1)$$

where the inequalities in the sup are understood componentwise, and $\chi(u, v, w) = -P_a^X \cdot u^+ + P_b^X \cdot u^- - P_a^Y \cdot v^+ + P_b^Y \cdot v^- - P_a^Z \cdot w^+ + P_b^Z \cdot w^-$, and $\alpha > 0$ is a risk-aversion parameter (the reason the primal takes this form will become clear below when we compute the dual). Using e.g. Sion minimax theorem to interchange the inf and sup, and then the inner inf and the optimal μ can be computed explicitly, so we can further re-write the sup inf as a portfolio optimization problem with liquidity constraints:

$$\sup_{u, v \in \mathbb{R}^{m+1}, w \in \mathbb{R}^m, q_b^X \leq u \leq q_a^X, q_b^Y \leq v \leq q_a^Y, q_b^Z \leq w \leq q_a^Z} \left(-\frac{1}{\alpha} \log \mathbb{E}^\mu(e^{-\alpha(u \cdot C^X + v \cdot C^Y + w \cdot C^Z)}) + \chi(u, v, w) \right) \quad (2)$$

where the optimal μ is still $\mu(x, y) = \frac{e^{-\alpha(u \cdot C^X + v \cdot C^Y + w \cdot C^Z)}}{\mathbb{E}^\mu(e^{-\alpha(u \cdot C^X + v \cdot C^Y + w \cdot C^Z)})} \bar{\mu}(x, y)$ as before. $H(\mu | \bar{\mu})$ does not depend on u, v, w in (1), so we can easily evaluate the inner sup to re-write (1) more explicitly as

$$\begin{aligned} \inf_{\mu \in \mathcal{P}([0, R]^2)} & \left[\frac{1}{\alpha} H(\mu | \bar{\mu}) + q_a^X \cdot (\mathbb{E}^\mu(C^X) - P_a^X)^+ + |q_b^X| \cdot (P_b^X - \mathbb{E}^\mu(C^X))^+ \right. \\ & + q_a^Y \cdot (\mathbb{E}^\mu(C^Y) - P_a^Y)^+ + |q_b^Y| \cdot (P_b^Y - \mathbb{E}^\mu(C^Y))^+ \\ & \left. + q_a^Z \cdot (\mathbb{E}^\mu(C^Z) - P_a^Z)^+ + |q_b^Z| \cdot (P_b^Z - \mathbb{E}^\mu(C^Z))^+ \right] \end{aligned} \quad (3)$$

(where $|q_b^X|$ etc refers to component-wise absolute values) i.e. we minimize entropy over models which fall within the bid-offer spread, *and* models which don't (but the latter incur an additional finite penalty for each option which falls outside which is proportional to the available liquidity). Note we could also extend this setup to include a full limit order book.

Note if we remove the liquidity constraints and set $\alpha = 1$, we can re-write (2) as

$$\sup_{u, v \in \mathbb{R}^{m+1}, w \in \mathbb{R}^m} (-\log \mathbb{E}^\mu(e^{-(u \cdot C^X + v \cdot C^Y + w \cdot C^Z)}) + \chi(u, v, w)) = \sup_{u, v \in \mathbb{R}^{m+1}, w \in \mathbb{R}^m} (-\log \mathbb{E}^\mu(e^{u \cdot C^X + v \cdot C^Y + w \cdot C^Z}) + \chi(-u, -v, -w))$$

which reduces to the concave maximization problem in [Guy20], but with bid/ask spreads, and note that the portfolio weights have flipped sign for the finite liquidity problem due to the additional $-\alpha$ term in the exponent.

Below we have tabulated our finite-liquidity numerical results using cvxpy/MOSEK to (separately) solve the primal (Eq 3) and dual (Eq 1) problems above using a Gaussian copula for $\bar{\mu}$ with $\rho = 0.6$ using same smile data from Table 1, with risk-aversion parameter $\alpha = \frac{1}{2}$ and available liquidity $q_{a/b}^X = q_{a/b}^Y = q_{a/b}^Z = 10$ for all three cross-rates using a 200-point Gauss-Legendre quadrature scheme, and we see that the primal and dual values are in very close agreement.

== Primal objective: 0.03390034, == Dual objective: 0.03390021

EUR-USD, GBP-USD and EUR-GBP call prices for primal:

| | | | | |
|----------------------|----------------------|----------------------|-----------------------|---------------------|
| 0.02205613972935358 | 0.013276907389933082 | 0.006354530059493843 | 0.001705926574171197 | 0.0007786524890174 |
| 0.024965160638936012 | 0.01494277916755357 | 0.007141103617805143 | 0.004185912852411529 | 0.00088154999401955 |
| 0.015218198662611626 | 0.009968285435874235 | 0.005678920176476505 | 0.0027151487847044643 | 0.00111447686241345 |

Distance outside bid-ask (primal):

| | | | | |
|-----------------------|-----------------------|----------------------|-----------------------|---------------------|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 8.113538660370345e-11 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0005551918798219913 | 0.000847418669037898 | 0.0006065420360578068 | 0.00015251103408142 |

EUR-USD, GBP-USD and EUR-GBP call prices for dual:

| | | | | |
|----------------------|----------------------|----------------------|-----------------------|---------------------|
| 0.02205616356722562 | 0.013276937250783662 | 0.006354565832117385 | 0.0017059597771090077 | 0.00077868069881520 |
| 0.02496516007345911 | 0.014942786980476514 | 0.007141120657613855 | 0.004185933323148571 | 0.0008815682735120 |
| 0.015218197850083473 | 0.009968284348058323 | 0.005678920877665079 | 0.002715150828526764 | 0.0011144802319625 |

Distance outside bid-ask (dual):

| | | | | |
|-----|-----------------------|-----------------------|-----------------------|---------------------|
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.0 | 0.0005551907920060793 | 0.0008474193702264722 | 0.0006065440798801065 | 0.00015251440363048 |

Optimal weights for dual (including forwards as first component)

| | | | | | |
|-------------|-----------------|-----------------|----------------|----------------|----------------|
| -7.80981828 | -1.58080857e-10 | -1.38417944e-10 | 4.33745760e-11 | 2.57709274e-09 | 8.89171392e-10 |
| 10 | 4.99965677 | 1.01404430e-09 | 7.74568465e-10 | 3.52643321e-10 | 1.19819712e-10 |
| -7.80356366 | 2.7979127 | 10 | 10 | 10 | 10 |

Note the q_a constraints are binding here for the Y -forward and the four options on Z with the smallest strikes.