The $m(q,\Delta)$ estimator for fBM

Let $X_t=\sigma B^H_t$, and set $SS^{(q)}_n:=\frac{1}{n}\sum_{i=1}^n|X_{i\Delta}-X_{(i-1)\Delta}|^q$. Then

$$\mathbb{E}(SS^{(q)}_n) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^n|X_{i\Delta}-X_{(i-1)\Delta}|^q) \;=\; \Delta^{qH}\frac{1}{n}\sum_{i=1}^n\mathbb{E}(|X_i-X_{i-1}|^q) \;=\; \sigma^q\mathbb{E}(|Z|^q)\Delta^{qH} \;=\; \sigma^q K_q\Delta^{qH} \quad (1)$$

where $\Delta=\frac{1}{n}$, $K_q=\mathbb{E}(|Z|^q)=\frac{2^{q/2}}{\sqrt{\pi}}\Gamma(\frac{q+1}{2})$, (for $q>-1$) and $Z\sim N(0,1)$. This naturally leads to the estimates $(\hat{H}_n,\hat{\sigma}_n)$ for (H,σ) defined by

$$SS^{(q)}_n \;=\; \hat{\sigma}^q_n\, K_q\Delta^{q\hat{H}_n}$$

if we have computed $SS^{(q)}_n$ for at least two Δ-values. Taking logs we see that

$$\log SS^{(q)}_n \;=\; q\log\hat{\sigma}_n+\log K_q \;+\; q\hat{H}_n\log\Delta$$

so we can perform **linear regression** on $\log SS^{(q)}_n$ vs $\log\Delta$ for a range of Δ-values (i.e. using a log-log plot, see plot overleaf). Then for the line of best fit, the **slope** will equal $q\hat{H}_n$ (q is chosen by you, e.g. $q=1,2,2.5,3$ etc), and the **intercept** at $\log\Delta=0$ is $q\log\hat{\sigma}_n+\log K_q$, from which we can compute $\hat{\sigma}_n$ since K_q has an explicit formula above. This is the $m(q,\Delta)$ estimator discussed in [GJR18]. One can then also compute the R^2**-statistic** for the regression (which measures how close the data is to the line of best fit), and try to estimate the **sample variance** of $\hat{H}_n$ and $\hat{\sigma}_n$.

To approximate the effect of using **realized variance** with m subwindows to estimate V_t (as in Part 2 of the project), we can use the Central Limit Theorem approximation from FM02:

$$V_{i\Delta} \;=\; V_0e^{\sigma B^H_{i\Delta}}(1+\sqrt{\frac{2}{m}}\,\varepsilon_i)$$

where the ε_i's are i.i.d. $N(0,1)$ (and independent of B^H), so (using that $\log(1+x)=x+O(x^2)$), we see that $\log V_{i\Delta}=\log V_0+\sigma B^H_{i\Delta}+\sqrt{\frac{2}{m}}\varepsilon_i+O(\frac{1}{m})=\log V_0+X_{i\Delta}+\sqrt{\frac{2}{m}}\varepsilon_i+O(\frac{1}{m})$.

For convenience we now define $\tilde{X}_{i\Delta}=X_{i\Delta}+\sqrt{\frac{2}{m}}\,\varepsilon_i$. Then $\tilde{X}_{i\Delta}-\tilde{X}_{(i-1)\Delta}=X_{i\Delta}-X_{(i-1)\Delta}+\sqrt{\frac{2}{m}}(\varepsilon_i-\varepsilon_{(i-1)})\sim N(0,\sigma^2\Delta^{2H}+\frac{4}{m})$, and setting $\widetilde{SS}^{(2)}_n:=\frac{1}{n}\sum_{i=1}^n|\tilde{X}_{i\Delta}-\tilde{X}_{(i-1)\Delta}|^2$, adding the effect of ε into the computation above we find that

$$\mathbb{E}(\widetilde{SS}^{(2)}_n) \;=\; \sigma^2\Delta^{2H}+\frac{4}{m}$$

(since $K_q=1$ for $q=2$) so we now regress $\log(\widetilde{SS}^{(2)}_n-\frac{4}{m})$ vs $\log\Delta$, which provides a smart adjustment to $\hat{H}_n$ (this adjustment is made in
`https://colab.research.google.com/drive/1jJGf4bVWETJqWRIMZ6STjsJ9jINgEaPd#scrollTo=e9fn6ABHNf52`

The $m(q,\Delta)$ estimator for the RL process

If $X_t=\sigma Z^H_t$ where $Z^H_t=\sqrt{2H}\int_0^t(t-s)^{H-\frac{1}{2}}dW_s$ is an RL process, then for $q=2$ and $H\in(0,\frac{1}{2})$ one can show that

$$\mathbb{E}(SS^{(2)}_n) \;=\; \frac{1}{n}\mathbb{E}(\sum_{i=1}^n|X_{i\Delta}-X_{(i-1)\Delta}|^2) \;=\; \sigma^q\Delta^{2H}\frac{1}{n}\sum_{i=1}^n\mathbb{E}(|Z^H_i-Z^H_{i-1}|^2) \;\sim\; \sigma^2c_H\Delta^{2H}$$

as $n\to\infty$ where $c_H=-\frac{4H\Gamma(\frac{1}{2}+H)\Gamma(-2H)}{\Gamma(\frac{1}{2}-H)}>0$. Hence for the regression we now consider

$$\log(SS^{(2)}_n) \;=\; 2\log\hat{\sigma}_n\;+\;\log c_{\hat{H}}\;+\;2\hat{H}_n\log\Delta$$

so now the intercept is $2\log\hat{\sigma}_n\;+\;\log c_{\hat{H}}$, which leads to an adjusted estimate for $\hat{\sigma}_n$. This can also be used for driftless rough Heston model below since (modulo an unimportant constant) both processes have the same covariance.

Convergence of $\hat{H}_n$ to H for the first task in Part 2, and asymptotic normality

Above we considered sums of the form

$$SS^{(q)}_n \;:=\; \frac{1}{n}\sum_{j=1}^n|B^H_j-B^H_{j-1}|^q\,. \quad (2)$$

We cannot apply the SLLN to this since the RVs in the sum here are not i.i.d. However, since the increments process $X_j = B^H_j - B^H_{j-1} \sim N(0,1)$ is stationary (in particular $X_j \sim N(0,1)$ for all j) and moreover is known to be **ergodic**, which means that for any measurable function g with $\mathbb{E}(|g(X_1)|) < \infty$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n g(X_j) \quad = \quad \mathbb{E}(g(X_1)) \qquad \text{a.s.} \tag{3}$$

In general, a stationary Gaussian process Y is ergodic if its covariance function $R(k) \to 0$ as $k \to \infty$, which we verified in the fBM.pdf document. Hence we can apply the general ergodic property in (3) to show that $SS^{(q)}_n$ in (2) tends to the **non-random** limit $K_q := \mathbb{E}(|Z|^q)$ a.s. (where $Z \sim N(0,1)$, see Homework 3); hence from the self-similarity of fBM

$$SS^{(q)}_n \quad \overset{w}{\to} \quad \frac{1}{n} n^{qH} \sum_{j=1}^n |B^H_{j/n} - B^H_{(j-1)/n}|^q$$

i.e. the right hand side tends weakly to the constant K_q as $n \to \infty$, which also implies convergence in probability.

Recall this also suggests the estimator of $\hat{H} = \hat{H}_n$ for H defined by the relation

$$K_q \quad = \quad \frac{1}{n} n^{q\hat{H}} \sum_{j=1}^n |B^H_{j/n} - B^H_{(j-1)/n}|^q \quad = \quad \frac{1}{n} n^{qH} \sum_{j=1}^n |B^H_{j/n} - B^H_{(j-1)/n}|^q \cdot n^{q(\hat{H}-H)}\,.$$

But from the discussion immediately before this, we know that $\frac{1}{n} n^{qH} \sum_{j=1}^n |B^H_{j/n} - B^H_{(j-1)/n}|^q$ tends to K_q in probability as $n \to \infty$, hence we must have that $n^{q(\hat{H}-H)} \to 1$ in probability as well, which implies that

$$q(\hat{H}_n - H)\log n \quad \to \quad 0$$

in probability (where we are now emphasizing the dependence of $\hat{H}$ on n), which can only be true if $\hat{H}_n - H \to 0$ in probability (recall that $\hat{H}$ depends on n).

Moreover, it can be shown that $\sqrt{n}\log n\,(\hat{H}_n - H)$ is asymptotically $N(0, 2\gamma)$, where

$$\gamma \quad = \quad \sum_{r=-\infty}^{\infty} (|r+1|^{2H} - 2|r|^{2H} + |r-1|^{2H})$$

(see code to verify this at `https://colab.research.google.com/drive/18ISGhNSN9ybbcpJlP0-OjeBNOz96f7kc#scrollTo=vde`

The Han-Schied [HS21] estimator

Let $X_t = \sigma B^H_t$ and let

$$\theta_{m,k} \quad = \quad 2^{\frac{m}{2}}(2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{k}{2^m}} - X_{\frac{k+1}{2^m}}) \quad = \quad -2^{\frac{m}{2}}(X_{\frac{2(k+1)}{2^{m+1}}} - 2X_{\frac{2k+1}{2^{m+1}}} - X_{\frac{2k}{2^{m+1}}})$$

(note the similarity of the second expression to a 2nd order finite difference estimate). Then (with some tedious algebra) using the formula for $R(s,t) = \mathbb{E}(B^H_s B^H_t)$, one can check that

$$\mathbb{E}(\theta^2_{m,k}) \quad = \quad \sigma^2 2^{m-2H(1+m)}(4 - 4^H)\,. \tag{4}$$

Then setting $s^2_n = \sum_{m=0}^{n-1}\sum_{k=0}^{2^m-1}\theta^2_{m,k}$, we see that $\mathbb{E}(s^2_n) = \sum_{m=0}^{n-1} 2^m \mathbb{E}(\theta^2_{m,k})$ (since (4) does not depend on k) which simplifies to

$$\mathbb{E}(s^2_n) \quad = \quad \sigma^2(4^{n(1-H)} - 1) \quad \sim \quad \sigma^2 4^{n(1-H)} \quad = \quad \sigma^2 2^{2n(1-H)} \tag{5}$$

as $n \to \infty$, which suggests an estimator $\hat{H}_n$ defined by $s_n = \hat{\sigma}_n\, 2^{n(1-\hat{H}_n)}$ which (assuming $\hat{\sigma}_n = O(1)$ as $n \to \infty$) we can re-arrange as

$$\hat{H}_n \quad = \quad 1 - \frac{1}{n}\log_2(\frac{s_n}{\hat{\sigma}_n}) \quad = \quad 1 - \frac{1}{n}\log_2 s_n \,+\, O(\frac{1}{n})$$

where $\log_2$ denotes the base-2 logarithm, so (ignoring the $O(\frac{1}{n})$ remainder term), we recover the Han-Schied[HS21] estimator $\hat{H}_n = 1 - \frac{1}{n}\log_2 s_n$. Then

$$\begin{aligned}
\mathbb{E}(\hat{H}_n) \quad = \quad 1 - \frac{1}{n}\mathbb{E}(\log_2(s_n)) \quad = \quad 1 - \frac{1}{2n}\mathbb{E}(\log_2(s^2_n)) \quad \geq \quad 1 - \frac{1}{2n}\log_2 \mathbb{E}(s^2_n) \quad &= \quad 1 - \frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)} - 1)) \\
&\geq \quad 1 - \frac{1}{2n}\log_2(\sigma^2(4^{n(1-H)})) \\
&= \quad 1 - \frac{1}{2n}\log_2(\sigma^2) \; - \; \frac{1}{2n}\log_2(4^{n(1-H)})) \\
&= \quad H - \frac{1}{2n}\log_2(\sigma^2)
\end{aligned}$$

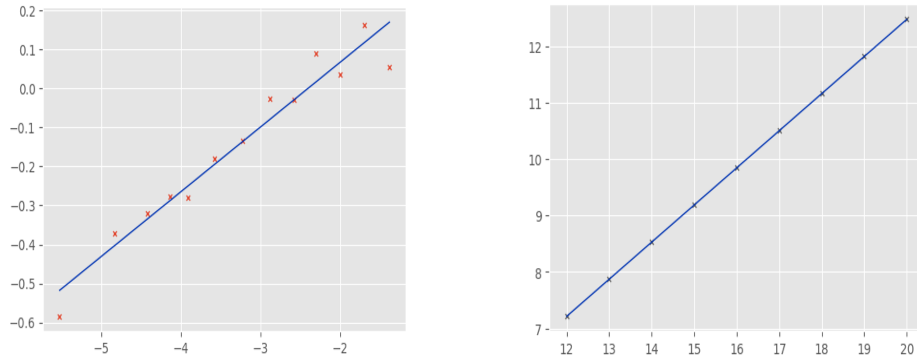

Figure 1: On the left, we see estimates of H for the SPX using the $m(q, \Delta)$ method for the SPX from 3rdJan22-15thJul24 for $q = 2$ for which $\hat{H} = 0.0830$ and $\hat{\sigma} = 1.221$ (see similar plots in [GJR18]. On the right we see the linear regression in (6) for the Han-Schied method ($\log s_n$ vs n) for a true fBM path with 2^{20} time points, for which $\hat{H} = 0.0508$, and $\hat{\sigma} = 1.010$.

and the final line is $> H$ if $\sigma < 1$, so $\mathbb{E}(\hat{H}_n) > H$ if $\sigma < 1$. See also the sequential scale independent estimator in [HS21].

Or we can jointly estimate H and σ by performing linear regression since

$$\log s_n \quad = \quad \log \hat{\sigma}_n \; + \; n(1 - \hat{H}_n) \log 2 \tag{6}$$

but we now have to compute $\log s_n$ for a range of different n-values to get a line of best fit, for which the slope is $(1 - \hat{H}_n) \log 2$ and the intercept is $\log \hat{\sigma}_n$.

The rough Heston model

The driftless rough Heston model satisfies

$$V_t = V_0 + \nu \int_0^t (t-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \,.$$

Then $\mathbb{E}(V_t) = V_0$, and V has covariance function:

$$\begin{aligned}
\mathbb{E}((V_s - V_0)(V_t - V_0)) &= \nu^2 \mathbb{E}(\int_0^s (s-u)^{H-\frac{1}{2}} \sqrt{V_u} dW_u \cdot \int_0^t (t-r)^{H-\frac{1}{2}} \sqrt{V_r} dW_r) \\
&= \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} \mathbb{E}(V_u) du \\
&= V_0 \nu^2 \int_0^s (s-u)^{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} du \;=\; V_0 \nu^2 \bar{R}(s,t)
\end{aligned}$$

for $0 \leq s \leq t$, where $\bar{R}(s,t)$ is the covariance function for the Riemann-Liouville (RL) process $Z_t = \int_0^t (t-u)^{H-\frac{1}{2}} dW_u$ used for the rough Bergomi model (note Z is a Gaussian process but V is not), but the explicit formula for $\bar{R}(s,t)$ is more complicated than the $R(s,t)$ formula for fBM.

We also have the **Mandelbrot-van Ness** representation for fBM:

$$W_t^H = c_H (\int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}) dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s) = c_H (A_t + Z_t)$$

for $t \geq 0$, in terms of an RL process Z (and note that A_t is known at time zero for all $t \geq 0$), and $c_H = (\frac{2H\,\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\,\Gamma(2-2H)})^{\frac{1}{2}}$. Note also that A_t and Z_t are independent.

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