

# Hawkes processes

(updated Sun 10th Aug 4.53pm). Consider a time-inhomogenous Poisson process  $(N_t)_{t \geq 0} \in \mathbb{N}$  whose intensity is itself a random process  $\lambda_t$  which evolves as

$$\lambda_t = \mu + \int_{[0,t]} \phi(t-s) dN_s$$

i.e.  $\lambda$  depends on the history of  $N$  itself, where  $\mu$  is a positive constant and  $\phi$  a positive function. For this reason we say that  $N$  is **self-exciting**, and this is a special type of **Stochastic Volterra equation** with no Brownian motion. The meaning of the  $\lambda_t$  is that  $\lim_{h \rightarrow 0} \frac{1}{h} \mathbb{P}(N_{t+h} - N_t > 0 | \mathcal{F}_t^{\lambda, N}) = \lambda_t$ , and note we can re-write  $\lambda$  as

$$\lambda_t = \lambda_0 + \sum_{0 \leq s_i \leq t} \phi(t-s_i)$$

where  $s_1, s_2, \dots$  are the random **jump times** of  $N$ , which we can also take as the definition of  $\lambda$ .

If we let  $M_t = N_t - \int_0^t \lambda_u du$ , then  $M$  is a martingale and we can re-write the  $\lambda$  equation as

$$\lambda_t = \mu + \int_{[0,t]} \phi(t-s)(dM_t + \lambda_t dt) = \mu + (\phi * dM)_t + (\phi * \lambda)_t.$$

For any  $f \in L^1(0, \infty)$ , we can easily check that  $\|\phi * f\| = \|\phi\| \|f\|$ , where all norms will refer to the  $L^1(0, \infty)$  norm, so the operator  $\phi*$  is a contraction on  $L^1(0, \infty)$  if  $\|\phi\| = \int_0^\infty \phi(u) du < 1$ , so the inverse  $(I - \phi*)^{-1} = I + (\phi*) + (\phi*)^2 + \dots$  is well defined. We can re-write the eq for  $\lambda_t$  as

$$(I - \phi*)\lambda = \mu + \phi * dM.$$

To make further sense of  $(I - \phi*)^{-1}$ , we look for a function  $\psi$  such that

$$(I - \phi*)^{-1}f = (I + \psi*)f$$

for any test function  $f \in L^1(0, \infty)$ , so

$$\begin{aligned} f &= (I - \phi*)(I + \psi*)f = (I - \phi*)(f + \psi * f) \\ &= f - \phi * f + \psi * f - \phi * \psi * f \end{aligned}$$

which we can re-write in operator form (i.e. without the  $f$ ) as  $\phi * \psi = \psi - \phi$ .  $\psi$  is known as the **resolvent** of  $\phi$ , note definition here is opposite way round to chap 3 in FM14. Applying this to our Hawkes process i.e. setting  $f(t) = \lambda_t$ , we see that

$$\begin{aligned} \lambda &= (I - \phi*)^{-1}(\mu + \phi * dM) = (1 + \psi) * \mu + (I + \psi) * (\phi * dM) \\ &= \mu + \psi * \mu + (\phi * + (\phi*)^2 + \dots) * dM \\ &= \mu + \psi * \mu + \psi * dM \end{aligned}$$

where  $\psi = \sum_{k=1}^\infty (\phi*)^k$ , which is shorthand for

$$\lambda_t = \mu + \mu \int_{[0,t]} \psi(t-s) ds + \int_{[0,t]} \psi(t-s) dM_s \quad (1)$$

## The propagator model - concave price impact from Hawkes order flow

Consider two independent Hawkes processes  $N_t^\pm$  with associated intensities  $\lambda_t^\pm$  which evolve as with

$$\lambda_t^\pm = \mu + \mu \int_0^t \psi(t-s) ds + \int_0^t \psi(t-s) dM_s^\pm$$

where  $dM_t^\pm = dN_t^\pm - \lambda_t^\pm dt$ . Then

$$\begin{aligned} \mathbb{E}_t(N_u^+) &= g(t) + \mathbb{E}_t\left(\int_0^u \int_0^s \psi(s-v) dM_v^+ ds\right) = g(t) + \mathbb{E}_t\left(\int_0^u \int_v^u \psi(s-v) ds dM_v^+\right) \\ &= g(t) + \int_0^t \int_v^u \psi(s-v) ds dM_v^+ \end{aligned}$$

for some function  $g(t)$ , and note that  $\int_v^\infty \psi(s-v)ds = \int_0^\infty \psi(s)ds = \|\psi\|_1$ . Then if we assume the current price  $P_t = \kappa \lim_{u \rightarrow \infty} \mathbb{E}(N_u^+ - N_u^- | \mathcal{F}_t)$  for some constant  $\kappa > 0$ , then

$$\frac{1}{\kappa} P_t = \lim_{u \rightarrow \infty} \mathbb{E}(N_u^+ - N_u^- | \mathcal{F}_t) = \int_0^t \sigma(dM_v^+ - dM_v^-)$$

where  $\sigma = \|\psi\|$ . Then

$$\begin{aligned} \frac{1}{\sigma} P_t &= N_t^+ - N_t^- - \int_0^t \lambda_s^+ ds + \int_0^t \lambda_s^- ds = N_t^+ - N_t^- - \int_0^t \int_u^t \phi(s-u) ds (dN_u^+ - dN_u^-) \\ &= \int_0^t (1 - \int_0^{t-u} \phi(s) ds) (dN_u^+ - dN_u^-) \end{aligned}$$

so

$$P_t = \int_0^t \zeta(t-u) (dN_u^+ - dN_u^-) \quad (2)$$

where  $\zeta(t) = \kappa \sigma (1 - \int_0^t \phi(s) ds)$ , so

$$\zeta'(t) = -\kappa \sigma \phi(t) = -\zeta(0) \phi(t)$$

and since  $\zeta(\infty) > 0$  under our assumption that  $\int_0^\infty \phi(t) dt < \infty$ , we interpret  $\zeta(\infty)$  as the (non-transient) permanent price impact component. Hence from the Volterra form in (2), we see that  $P$  is a **propagator model** (e.g. like transient price impact) but also retains the martingale property. Differentiating the expression in Prop 3.2 in [Jai15] we get

$$\zeta'(t) = \kappa (-\psi(t) - \int_0^t \psi(t-s) \phi(s) ds)$$

Can re-write as

$$\zeta(t) = \frac{\|\phi\|}{1 - \|\phi\|} (1 - \int_0^t \phi(s) ds)$$

so  $\zeta(t) \rightarrow \|\phi\|$ .

## Impact of a metaorder executed at constant rate before and after completion

Consider the additional contribution from an additional agent who buys at a fixed rate  $v$  for duration  $\tau$ . Then impact the cumulative impact time  $t$  is

$$P_t = \int_0^t \zeta(t-u) (dN_u^+ - dN_u^-) + v \int_0^{t \wedge \tau} \zeta(t-s) ds$$

whose expectation is  $MI(t) = v \int_0^{t \wedge \tau} \zeta(t-s) ds$ , which we call the **market impact function**, see plot below where we see **concave price impact** up to  $\tau$ , and then decay thereafter (which is broadly consistent with empirical findings where  $MI(t)$  is often found to be  $const. \times t^{\frac{1}{2}}$  for  $t \leq \tau$  (the so-called **square root impact law**.)

## Example $\phi$ and $\psi$ functions

One can easily check that  $\|\phi\| = \frac{\|\psi\|}{1 + \|\psi\|}$  so  $\|\psi\| = \frac{\|\phi\|}{1 - \|\phi\|}$ . A common choice for  $\phi$  is

$$\phi(t) = \nu t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$$

where  $E_{\alpha,\alpha}$  is the **Mittag-Leffler** function (which is **heavy-tailed** since  $\phi(t) \sim \frac{const.}{t^{1+\alpha}}$  as  $t \rightarrow \infty$ ), and  $\int_0^\infty \phi(t) dt = \frac{\nu}{\lambda}$  so we choose  $\nu < \lambda$ . For this choice of  $\phi$ , the resolvent is

$$\psi(t) = \nu t^{\alpha-1} E_{\alpha,\alpha}(-(\lambda - \nu)t^\alpha)$$

and as  $\nu \nearrow \lambda$ ,  $\|\phi\| \nearrow 1$  and  $\|\psi\| \nearrow \infty$  (see also table below).

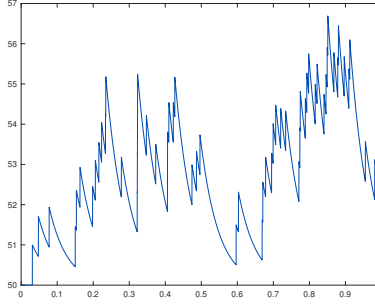


Figure 1: Here we we have simulated the intensity process of the form  $\lambda_t = \lambda_0 + \int_0^t k(t-s)dN_s$  for  $k(t) = e^{-10t}$  and  $\lambda_0 = 50$ .

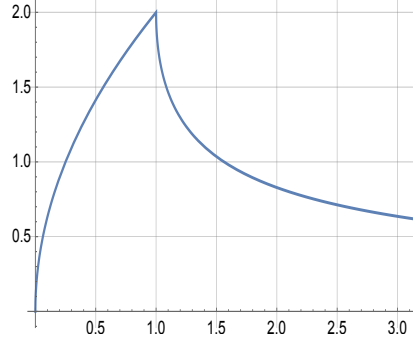


Figure 2: Here we see a typical concave market impact function  $MI(t) = \int_0^{t \wedge \tau} \zeta(t-s)ds$  with  $\tau = 1$ .

	$\psi(t)$	$\phi(t)$
Constant	$c$	$c e^{-ct}$
Fractional	$\frac{c t^{\alpha-1}}{\Gamma(\alpha)}$	$c t^{\alpha-1} E_{\alpha,\alpha}(-c t^\alpha)$
Exponential	$c e^{-\lambda t}$	$c e^{-(\lambda+c)t}$
Gamma	$\frac{c e^{-\lambda t} t^{\alpha-1}}{\Gamma(\alpha)}$	$c e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}(-c t^\alpha)$

## Interpretation of a Hawkes process in terms of population dynamics

Let us define a population model: At time zero, there are no individuals. Some individuals (migrants) arrive as a uniform Poisson process with intensity  $\mu$ . If a migrant arrives at time  $s$ , the birth dates of its children form a Poisson process of intensity  $\phi(t-s)$  at time  $t$ , with  $\int_0^\infty \phi(t)dt < 1$ . In the same way, if a child is born at  $s'$ , the birth dates of its children form a Poisson process of intensity  $\phi(\cdot - s')$ . Let  $N_t$  be the number of individuals who were born or migrated until time  $t$ . Then  $N$  is a Poisson-type process with intensity

$$\lambda_t = \mu + \int_0^t \phi(t-s) dN_s \quad (3)$$

i.e.  $N$  is a Hawkes process. This captures the notion of the process being self-exciting.

## References

- [FL04] J. Doyn Farmer and F. Lillo, “A theory for long-memory in supply and demand”, *Quantitative Finance*, 2004.
- [Jai15] T. Jaisson, “No-arbitrage implies power-law market impact and rough volatility”, Working paper, 2015.
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[DRS20] B. Durin, M. Rosenbaum, and G. Szymanski, “The two square root laws of market impact and the role of sophisticated market participants”, Working paper, 2020.