Signatures simplified

Let X_t be a semimartingale. Then the $(i_1, ..., i_n)$'th component of the order-n part of the signature $\hat{\mathbb{X}}_t$ of X_t is

$$\hat{\mathbb{X}}_t^{(i_1,\dots,i_n)} \quad = \quad \int_{u_n=0}^{t} \int_{u_{n-1}=0}^{u_n} .. \int_{u_1=0}^{u_2} dX_{u_1}^{i_1} \circ ... \circ dX_{u_n}^{i_n}$$

where $\int_0^t Y_s \circ dX_s := \int_0^t Y_s dX_s + \frac{1}{2}[X,Y]_t$ is the **Stratonovich** integral of Y with respect to X, and the final term is the quadratic covariation of X and Y. This ensures that $X_t Y_t = X_0 Y_0 + \int_0^t Y_s \circ dX_s + \int_0^t X_s \circ dY_s$ i.e. Stratonovich integration obeys the usual rules of calculus.

We will generally be interested in the case when $X_t = (t, W_t)$ (which we call *time-augmented* Brownian motion), in which case we denote $\hat{\mathbb{X}}$ by $\hat{\mathbb{W}}$, so $d[W,W]_t = dt$, $d[t,W]_t = d[W,t]_t = 0, d[t,t] = 0$.

Expectation of the signature of (t, W_t): the Fawcett formula

Let $i_k \in \{1, 2\}$ for $k = 1..n$ where 1 refers to the time dimension and 2 the spatial (W) dimension, and let x denote the number of time-dimension terms in $(i_1, .., i_n)$ (i.e. the number of $1's$). Then the $(i_1, .., i_n)$'th component of the level-n part of $\mathbb{E}(\hat{\mathbb{W}}_T)$ is

$$\frac{T^{\frac{1}{2}(n+x)}}{(\frac{1}{2}(n+x))!\ 2^{\frac{1}{2}(n-x)}} \tag{1}$$

if $x = n$ or (if $2's$ appear in $(i_1, .., i_n)$) the 2's appear as consecutive pairs [1], otherwise the expectation is zero. We list the non-zero components of $\mathbb{E}(\hat{\mathbb{W}}_T)$ here:

$$\begin{aligned}
\mathbb{E}(S^2) &= \frac{T^2}{2}(11) + \frac{T}{2}(22) \\
\mathbb{E}(S^3) &= \frac{T^3}{6}(111) + \frac{T^2}{4}((122), (221)) \\
\mathbb{E}(S^4) &= \frac{T^4}{24}(1111) + \frac{T^3}{12}((1122), (1221), (2211)) + \frac{T^2}{8}(2222)
\end{aligned} \tag{2}$$

(see Python code `https://colab.research.google.com/drive/1VgDaZ2zjx6aQvm7JDvV-qcTvOMHmrezy?usp=sharing` for a simple Monte Carlo test of these formula using the iisignature package with antithetic sampling).

Example: let $n = 3$. Then the $(1, 2, 2)$'th component of $\hat{\mathbb{W}}$ corresponds to $x = 1$ and has a pair of 2's, so the expectation computed using Eq (1) is $\frac{1}{4}T^2$. To check this manually from the definition of the signature, we compute

$$\int_0^T \int_0^t \int_0^s dW_u \circ dW_s\, dt \quad = \quad \int_0^T \int_0^t W_s \circ dW_s\, dt \quad = \quad \int_0^T \frac{1}{2} W_t^2 dt$$

which has expectation $\frac{1}{4}T^2$.

De-mystifying the shuffle product: expressing $\langle \ell_1, \hat{\mathbb{W}}_t \rangle \langle \ell_2, \hat{\mathbb{W}}_t \rangle$ as a linear functional of $\hat{\mathbb{W}}_t$

Consider the simple product of the linear functionals $\langle \ell_1, \hat{\mathbb{W}} \rangle = t$ and $\langle \ell_2, \hat{\mathbb{W}} \rangle = W_t$. Comparing this to the off-diagonal terms of the level-2 signature: $\int_0^t W_s ds$ and $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$ we see that their sum is equal to tW_t, and hence

$$\langle \ell_1, \hat{\mathbb{W}}_t \rangle \langle \ell_2, \hat{\mathbb{W}}_t \rangle \quad = \quad tW_t \quad = \quad \langle \ell_1 \sqcup\!\sqcup\, \ell_2, \hat{\mathbb{W}} \rangle$$

where in this case $\ell_1 \sqcup\!\sqcup\, \ell_2$ has components $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ at order 2 (and zero elsewhere), and this is still a linear functional of $\hat{\mathbb{W}}_t$ (of course $\hat{\mathbb{W}}_t$ itself contains non-linear terms). In this case we have "two decks of one card each", so the combinations for the shuffle product are just 12 and 21.

Now consider $\langle \ell_1, \hat{\mathbb{W}}_t \rangle \langle \ell_2, \hat{\mathbb{W}}_t \rangle = \int_0^t W_s ds \cdot W_t$. We wish to re-write this as a linear functional of order-3 signature terms. We first re-write as

$$\int_0^t W_u du \cdot W_t \quad = \quad \int_0^t \int_0^u dW_r du \cdot \int_0^t dW_s \quad = \quad \int_0^t \int_0^t \int_0^u dW_r dW_s du$$

[1] e.g. 122 and 221 but not 212

We can break this up as

$$\int_0^t \int_0^u \int_0^u dW_r dW_s du + \int_0^t \int_u^t \int_0^u dW_r dW_s du \tag{3}$$

and then re-write the 1st integral in (3) as

$$\int_0^t \int_0^u \int_0^s dW_r dW_s du + \int_0^t \int_0^u \int_s^u dW_r dW_s du \tag{4}$$

and then further re-write the final integral here as

$$\int_0^t \int_0^u \int_0^r dW_s dW_r du\,.$$

And we can re-write the 2nd integral in (3) as

$$\int_0^t \int_0^s \int_0^u dW_r du dW_s$$

so adding the three blue items, we have components $2 \cdot (2,2,1)$, plus $(2,1,2)$ for the shuffle product.

Similarly the functional $\langle \ell_1, \hat{\mathbb{W}}_t\rangle\langle \ell_2, \hat{\mathbb{W}}_t\rangle = \int_0^t u dW_u \cdot W_t$ corresponds to $(1,2)$ and 2, so the correct combinations for the shuffle product are $2 \cdot (1,2,2)$, and $(2,1,2)$.

The proof of the Shuffle formula is given in Lemma 22.2 of the 1994 article of Gaines: "The algebra of iterated stochastic integrals".

Gaussian Volterra processes as a linear combination of signature elements

(see e.g. section 4.3 of [AGH24]). For a Gaussian Volterra process $Z_t = \int_0^t K(t-s)dW_s$ with $K \in L^2$ and smooth away from zero, we can Taylor expand K around t to get

$$Z_t \;=\; \langle \ell, \hat{\mathbb{W}}_t\rangle \;=\; \int_0^t \sum_{n=0}^{\infty} K^{(n)}(t)\frac{(-s)^n}{n!} dW_s \;=\; \sum_{n=0}^{\infty} K^{(n)}(t)\int_0^t \frac{(-s)^n}{n!} dW_s$$

which is an infinite linear combination of signature terms of $\hat{\mathbb{W}}_t$ (more specifically, the nth term in the series is a just a multiple of the order-$(n+1)$ signature term $\int_{s=0}^t \int_{u_n=0}^s \int_{u_{n-1}=0}^{u_n} .. \int_{u_1=0}^{u_2} du_1...du_n dW_s$). In particular, for the Riemann-Liouville process $Z_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, this simplifies to

$$Z_t \;=\; t^{H-\frac{1}{2}} \sum_{n=0}^{\infty} t^{-n} (\frac{1}{2} - H)^{\bar{n}}\; \frac{1}{n!}\int_0^t u^n dW_u\,.$$

We can numerically check this by computing the covariance of both sides (the covariance of the RHS involves a doubly infinite sum). Note a *finite* number of linear signature elements is still a semi-martingale since individual terms of $\hat{\mathbb{W}}_t$ are semimartingales, but in this case clearly the infinite sum is not for $H < \frac{1}{2}$ because Z is not a semi-martingale for $H < \frac{1}{2}$ since it has infinite quadratic variation

Application to stochastic volatility models - sampling the VIX

Following slide 27 in [Ger25], let

$$\begin{aligned} dS_t &= S_t \Sigma_t dB_t \\ \Sigma_t &= \langle \sigma, \hat{\mathbb{W}}_t\rangle \end{aligned}$$

where $B_t = \rho W_t + \bar{\rho} W_t^{\perp}$, and W and $W^{\perp}$ are independent Brownians. Then (from shuffle formula above) $V_t := \Sigma_t^2 = \langle \sigma \sqcup\!\sqcup\, \sigma, \hat{\mathbb{W}}_t\rangle$, so $\mathbb{E}(V_t) = \langle \sigma \sqcup\!\sqcup\, \sigma, \mathbb{E}(\hat{\mathbb{W}}_t)\rangle$, and we know $\mathbb{E}(\hat{\mathbb{W}}_t)$ from (1). Then

$$\mathrm{VIX}_0^2 \;=\; \frac{1}{\Delta}\int_0^{\Delta} \mathbb{E}(V_u) du \;=\; \frac{1}{\Delta}\int_0^{\Delta} \langle \sigma \sqcup\!\sqcup\, \sigma, \mathbb{E}(\hat{\mathbb{W}}_u)\rangle du\,.$$

The **Quintic** model of Abi-Jaber&Li model uses $n=5$ with W replaced by an OU process Y (but only uses trivial polynomial terms Y_t^n for n odd from $\hat{\mathbb{Y}}_t$, (i.e. the last element of $\hat{\mathbb{Y}}$ for odd values of n).

Computing VIX_t^2 using conditional expectations of $\hat{\mathbb{W}}_t$

To compute $\mathrm{VIX}_t^2 = \frac{1}{\Delta}\mathbb{E}(\int_t^{t+\Delta} V_u du|\mathcal{F}_t^W)$ for $t > 0$, we need to be able to compute conditional expectations of signature elements. To this end, we first note that

$$\begin{aligned} d\hat{\mathbb{W}}_t^{(i_1,\dots,i_{n-1},1)} &= \hat{\mathbb{W}}_t^{(i_1,\dots,i_{n-1})}dt \\ d\hat{\mathbb{W}}_t^{(i_1,\dots,i_{n-1},2)} = \hat{\mathbb{W}}_t^{(i_1,\dots,i_{n-1})} \circ dW_t &= \hat{\mathbb{W}}_t^{(i_1,\dots,i_{n-1})}dW_t + \frac{1}{2}d\langle \hat{\mathbb{W}}^{(i_1,\dots,i_{n-1})}, W\rangle_t \\ &= \hat{\mathbb{W}}_t^{(i_1,\dots,i_{n-1})}dW_t + \frac{1}{2}\hat{\mathbb{W}}^{(i_1,\dots,i_{n-2})}dt\, 1_{i_{n-1}=2} \end{aligned}$$

so

$$\frac{\partial}{\partial u}\mathbb{E}(\hat{\mathbb{W}}_u^{(i_1,\dots,i_{n-1},i_n)}|\mathcal{F}_t) = \mathbb{E}(\hat{\mathbb{W}}_u^{i_1,\dots,i_{n-1}}|\mathcal{F}_t)1_{i_n=1} + \frac{1}{2}\mathbb{E}(\hat{\mathbb{W}}_u^{i_1,\dots,i_{n-2}}|\mathcal{F}_t)1_{i_{n-1}=2,i_n=2}$$

which gives a recursive ODE for $\mathbb{E}(\hat{\mathbb{W}}_u^{(i_1,\dots,i_{n-1},i_n)}|\mathcal{F}_t)$, which (when $t=0$) is consistent with the Fawcett formula above.

Here is a specific example:

$$\begin{aligned} d(\int_{s=0}^t\int_{u=0}^s\int_{v=0}^u dv dW_u \circ dW_s) &= \int_{u=0}^t\int_{v=0}^u dv\, dW_u \circ dW_t \\ &= \int_{u=0}^t u\, dW_u \circ dW_t = (\int_{u=0}^t u\, dW_u)dW_t + \frac{1}{2}tdt \end{aligned}$$

so integrating we see that

$$\mathbb{E}(\int_{s=0}^t\int_{u=0}^s\int_{v=0}^u dv dW_u \circ dW_s) = \frac{1}{4}t^2\,.$$

(which agrees with Eq (2)).

References

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[Ger25] Gérard, L.Gérard and Y.Huang, "Signatures for Volatility: Pricing and Hedging", SIAM conference talk, Miami, 2025