

The problem is to estimate  $\mu(\cdot)$  and  $\sigma(\cdot)^2$  for a 1d diffusion  $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$  from a single path. As a warm up example, consider estimating  $\theta = (\mu, \sigma)$  for arithmetic Brownian motion  $X_t = \mu t + \sigma W_t$  with  $n$  observations at equidistant intervals  $\delta_n$  with  $\delta_n \rightarrow 0$ . Then the log likelihood of the increments  $\Delta X_i$  of  $X$  is

$$\ell_n(\mu, \sigma) = \text{const.} - n \log \sigma - \sum_{i=1}^n \frac{(\Delta X_i - \mu \delta_n)^2}{2\sigma^2 \delta_n}$$

where *const.* is independent of  $(\mu, \sigma)$ . Then the **Fisher information**  $-\mathbb{E}(\frac{\partial^2 \ell_n}{\partial \theta_i \partial \theta_j})$  is given by

$$I_n(\mu, \sigma) = \begin{bmatrix} \frac{T}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$

where  $T = n\delta_n$ , so (by **Cramer-Rao**) the covariance of any unbiased estimator for  $\theta = (\mu, \sigma)$  is  $\geq I_n(\mu, \sigma)^{-1}$ . In particular, the variance of any unbiased estimator for  $\mu$  is  $\geq I_{1,1}(\mu, \sigma)^{-1} = \frac{\sigma^2}{T}$ . Hence we need  $T \rightarrow \infty$  (i.e. the observation window tending to  $\infty$ ) to get a consistent estimator for  $\mu$ , as also discussed earlier this week (note the condition that  $\delta_n \rightarrow 0$  and  $T \rightarrow \infty$  also appears in **Theorem 4.1** in [WM24]).<sup>1</sup>

More generally, if  $\sigma$  is known and  $\hat{\mu}$  is *any* estimator for  $\mu$  with **bias**  $b(\mu)$ , then

$$\text{MSE}_{\hat{\mu}}(\hat{\mu}) := \mathbb{E}_{\mu}((\hat{\mu} - \mu)^2) \geq \frac{(1 + b'(\mu))^2}{I(\mu)} + b(\mu)^2 = \frac{\sigma^2}{T} (1 + b'(\mu))^2 + b(\mu)^2.$$

so for  $T$  fixed, we can't drive the right hand side down to zero if  $b(\mu) \neq 0$ .

Note **Theorem 4.1** in the cited article [WM24] also requires  $X$  to be **ergodic** (see Eq 3.2 in [WM24] for definition) which is obviously not the case for the process in **q4** of the draft project since  $X$  there is not mean-reverting.

## SINDy method discussed in the draft and [WM24]

The simplest case is when the “dictionary” of functions only consists of the constant function 1. Then (for an ergodic 1d diffusion) Eq 4.1 in [WM24] is  $\sum_{m=0}^{N-1} (\frac{\Delta X_m}{\Delta t} - v)^2$ . Minimizing in  $v$  leads to the first-order optimality condition:

$$2 \sum_{m=0}^{N-1} (\frac{\Delta X_m}{\Delta t} - \hat{v}) = 0$$

so we recover the obvious unbiased estimate:  $\hat{\mu} = \frac{1}{N} \sum_{m=0}^{N-1} \frac{\Delta X_m}{\Delta t} = (X_T - X_0)/T$ . Similarly, to estimate the diffusion coefficient, we consider

$$\frac{d}{dv} \sum_{m=0}^{N-1} (\frac{(\Delta X_m)^2}{\Delta t} - v)^2 \big|_{v=\hat{v}} = -2 \sum_{m=0}^{N-1} (\frac{(\Delta X_m)^2}{\Delta t} - \hat{v}) = 0$$

which leads to  $\hat{\sigma}^2 N \Delta t = \sum_{m=0}^{N-1} (\Delta X_m)^2$ , i.e. usual estimation method for  $\hat{\sigma}^2$  using realized variance (see also more involved analysis on page 7 in [WM24]).

Again if  $X$  is arithmetic Brownian motion:  $X_t = \mu t + \sigma W_t$ , the  $\frac{\Delta X_i}{\Delta t}$ 's are i.i.d. If  $\Delta t = 1$ , then  $\mathbb{E}(\hat{v}) = \mu$ ,  $\text{Var}(\hat{v}) = \frac{\sigma^2}{N}$  and (from the SLLN)  $\hat{v} \rightarrow \mu$  as  $N \rightarrow \infty$ .

But if  $\Delta t = T/N$  for some  $T$  **fixed**,  $v = (X_T - X_0)/T$ , which has variance  $\frac{\sigma^2}{T}$  which clearly does not vanish as  $N \rightarrow \infty$  (recall Theorem 4.1 in [WM24] requires  $T \rightarrow \infty$  and  $\Delta t \rightarrow 0$ ).

If we now include the next term  $v_1$ , the task becomes more or less the same as the first part of Project 3: just performing linear regression on

$$\frac{\Delta X_t}{\Delta t} \quad \text{vs} \quad v_0 + v_1 X_t$$

(in P3 the equation the corresponding linear regression is

$$\Delta \xi_t \quad \text{vs} \quad -a(\xi_{t-1} - \bar{\xi})$$

where they are fitting the parameters for a discrete-time OU process (AR(1) process), see page 74 in [Kut04] for the case of a general AR( $d$ ) process. For P3, their unit of time is days with  $\Delta t = 1$  day, and their  $\xi$  process is ergodic and  $T$  is very large ( $\approx 35\text{yrs} \times 252 = 8820$  days).

<sup>1</sup>One can also replace  $W$  by an fBM and compute a 3x3 matrix  $I_n(\mu, \sigma, H)$  although one runs into issues with a singular matrix in the high frequency regime because the increment sizes are of order  $\sigma^2 \Delta^{2H}$ , so taking logs gives  $\log(\sigma^2) + 2H \log(\Delta)$  so  $H$  dominates when  $\Delta \ll 1$ , see [Kaw13].

**Remark 0.1** ChatGPT generated this code in Python:

<https://colab.research.google.com/drive/19fwbrMJu4WOMNNG7DuFuoGVTb8qbjXVn?usp=sharing> for the SINDy problem for a standard OU process with a dictionary of constant and linear functions, which seems to work well with a large  $T$  and small  $\delta_n$ , but as expected (from the arguments above) it doesn't work well with a fixed time horizon  $T$ . Note I didn't use any **regularization** so not sure if this is **sparse** in the sense you mean, and this only took about 5 minutes to write and clean up and the code is rather short, so this task maybe needs to be more substantial in the project.

## Continuous observation - the Trajectory Fitting Estimator (TFE)

For a discrete-time autoregressive (AR) process, the TFE method is essentially the first task of Project 3 as discussed above.

We now discuss the analog of this approach for diffusion processes with continuous observation. Following pages 5 and 74 in [Kut04], let  $X$  be an ergodic 1d diffusion:

$$X_t = X_0 + \int_0^t S(\theta, X_s) ds + \int_0^t \sigma(X_s) dW_s$$

for  $\theta \in \Theta \subset \mathbb{R}^d$  as in [WM24], and assume  $\sigma(\cdot)$  is known now since it can easily be estimated with realized variance if  $X$  is observed continuously. Set

$$\hat{X}_t(\theta) = X_0 + \int_0^t S(\theta, X_s) ds \Rightarrow \nabla_\theta \hat{X}_t(\theta) = \int_0^t \nabla_\theta S(\theta, X_s) ds. \quad (1)$$

Then the TFE estimator for  $\theta$  is  $\theta_T^* = \arg \inf_{\theta \in \Theta} \int_0^T (X_t - \hat{X}_t(\theta))^2 dt$ , which leads to the first order conditions:

$$\nabla_\theta \int_0^T (X_t - \hat{X}_t(\theta))^2 dt |_{\theta=\theta_T^*} = 2 \int_0^T (X_t - \hat{X}_t(\theta_T^*)) \nabla_\theta \hat{X}_t(\theta_T^*) dt = 0 \quad (2)$$

which gives us  $n$  non-linear equations for the  $n$  unknowns  $\theta_T^*$  if  $\Theta$  has dimension  $n$ .

**Remark 0.2** See discussion on the **identifiability condition** for the TFE on page 75 in [Kut04], although there may be a typo in that formula.

### TFE Explicit Example: the OU process

**Example 1.4.3 in [Kut04]:** For the OU case  $dX_t = (b - aX_t)dt + \sigma dW_t$  with  $b, \sigma$  known,  $\nabla_\theta S(\theta, x) = \frac{\partial}{\partial a} S(a, x) = -x$ , so (1) becomes

$$\partial_a \hat{X}_t(a) = - \int_0^t X_s ds = -Y_t.$$

Hence (2) simplifies to

$$\int_0^T (X_t - X_0 - bt + aY_t) \cdot -Y_t dt = 0$$

where  $Y_t = \int_0^t X_s ds$ , which we then re-arrange to obtain the TFE estimate for  $a$  as

$$a_T^* = - \frac{\int_0^T (X_t - X_0 - bt) Y_t dt}{\int_0^T Y_t^2 dt}$$

(the **MLE** for  $a$  which is obtained by maximizing the Girsanov factor with respect to a reference  $\theta_0$  involves a similar formula). Page 76 in [Kut04] shows that  $a_T^*$  is consistent and  $\sqrt{T}(a_T^* - a)$  is **asymptotically Normal** as  $T \rightarrow \infty$ , although the MLE for  $a$  has lower asymptotic variance, so in general  $\hat{a}_T$  is not **asymptotically efficient**.

**Remark 0.3** See page 5 in [Kut04] for discussion on what happens when the observed diffusion process does not belong to the prescribed parametric family.

## References

- [Kaw13] Kawai, R., “Fisher Information for Fractional Brownian Motion Under High-Frequency Discrete Sampling”, *Communications in Statistics – Theory and Methods*, vol. 42, no. 9, pp. 1628–1636, 2013.
- [Kut04] Y.A. Kutoyants, *Statistical Inference for Ergodic Diffusion Processes*. Springer, 2004,  
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- [WM24] M. Wanner and I. Mezic, “On higher order drift and diffusion estimates for stochastic SINDy”, *SIAM Journal on Applied Dynamical Systems*, 23 (2024), pp. 1504–1539