Homework 1

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion throughout.

1. Write down a formula for $u(x,t) := \mathbb{E}(f(W_T)|W_t = x)$ for a general function f.

Solution. From the definition of BM, the conditional distribution of W_T given $W_t = x$ is $N(x, T-t)$, so

$$\mathbb{E}(f(W_T)|W_t = x) \quad = \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} f(y)dy \quad = \quad (p_{T-t} * f)(x)$$

where $p_t(.)$ is the density of W_t and $*$ denotes **convolution** (the convolution of two functions f and g is $(f*g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$. Note by setting $x-y=u$ so $y=x-u$ and $dy=-du$, we see that $(f*g)(x) = -\int_{\infty}^{-\infty} f(x-u)g(u)du = \int_{-\infty}^{\infty} f(x-u)g(u)du = (g*f)(x)$, i.e. $f*g = g*f$.

2. **Simulating random variables with a given distribution**. Let X be a random variable with a continuous **strictly increasing distribution function** $F_X(x)$. What is the distribution of $F_X^{-1}(U)$, where U is a standard Uniform random variable on $[0,1]$? Explain the significance of this result.

Solution.

$$\mathbb{P}(F_X^{-1}(U) \leq x) \quad = \quad \mathbb{P}(U \leq F_X(x)) \quad = \quad F_X(x) \tag{1}$$

since $F_X(F_X^{-1}(x)) = x$, i.e. $F_X^{-1}(U) \sim X$, where we have used that $\mathbb{P}(U \leq x) = x$. So the conclusion here is that $F_X^{-1}(U)$ has the same distribution as X. This is how we typically generate a random variable with a given distribution in practice on a computer.

3. Let

$$B_t \quad = \quad (1-t)W_{\frac{t}{1-t}}$$

for $0 \leq t < 1$. Compute $\mathbb{E}(B_sB_t)$ for $0 < s < t < 1$. What do you notice at $t=1$?

Solution. $\mathbb{E}(B_sB_t) = (1-s)(1-t)\frac{s}{1-s} = s(1-t)$. We note that $\mathbb{E}(B_1^2) = 0$, hence $B_1 = 0$ a.s. B is known as the **Brownian bridge** (see Figure 1 below), which is Brownian motion conditioned to be at zero at time 1.

4. Let $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ (this is the famous **Black-Scholes model**). Compute $\mathbb{E}(S_t^p)$ (hint: re-write S_t^p as $S_0^p e^{pX_t}$ where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$).

Solution.

$$\mathbb{E}(S_t^p) \quad = \quad S_0^p\,\mathbb{E}(e^{pX_t}) \quad = \quad S_0^p\mathbb{E}(e^{p((\mu-\frac{1}{2}\sigma^2)t+\sigma W_t)})\,.$$

But $(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \sim N(\mu_1, \sigma_1^2)$, where $\mu_1 = (\mu - \frac{1}{2}\sigma^2)t$ and $\sigma_1^2 = \sigma^2 t$. Thus

$$\mathbb{E}(S_t^p) \quad = \quad S_0^p e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2} \quad = \quad S_0^p e^{(\mu-\frac{1}{2}\sigma^2)pt + \frac{1}{2}\sigma^2 p^2 t}$$

where we have used that the mgf of a general Normal $N(\mu_1, \sigma^2)$ random variable is $e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2}$ from Applied Probability Revision chapter. Note that $\mathbb{E}(S_t^p) < \infty$ for all $p \in \mathbb{R}$, i.e. all moments of S_t are finite.

5. Let $X_t = \sum_{i=1}^n (W_t^{(i)})^2$, where $W^{(i)}$ are n independent standard Brownian motions. Using that $\mathbb{E}(e^{-\lambda Z^2}) = \frac{1}{(1+2\lambda)^{\frac{1}{2}}}$ for $\lambda > 0$ where $Z \sim N(0,1)$, compute $\mathbb{E}(e^{-\lambda X_t})$ for $\lambda > 0$. X is known as a **Bessel squared process** of dimension n.

Solution. Using that $B_t^{(i)} \sim \sqrt{t}Z$ and the independence of the δ-Brownian motions, we see that

$$\mathbb{E}(e^{-\lambda X_t}) \quad = \quad \prod_{i=1}^{\delta} \mathbb{E}(e^{-\lambda (B_t^{(i)})^2}) \quad = \quad \prod_{i=1}^{\delta} \mathbb{E}(e^{-\lambda(\sqrt{t}Z)^2}) \quad = \quad (\mathbb{E}(e^{-\lambda t Z^2}))^{\delta} \quad = \quad \frac{1}{(1+2\lambda t)^{\frac{1}{2}\delta}}\,.$$

6. **Portfolio optimization, relevant for two of the summer projects**. Consider a financial market defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with d assets with **random payoffs** $(\pi^1, ..., \pi^d)$ at time T (which are **linearly independent**) with market prices p_i at $t = 0$. Let $Y_i = \pi_i - p_i$, and assume a financial agent can only trade at time zero.

Derive the first order optimality condition for an agent to maximize their **expected utility** $\mathbb{E}(U(b \cdot Y))$ over $b \in \mathbb{R}^d$, where $U : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $U''(x) < 0$ and b_i is the position in the i'th asset and we assume that $\mathbb{E}(U(b \cdot Y)) < \infty$ for all $b \in \mathbb{R}^d$.

Solution. As in first year calculus, we set compute derivatives wrt each b_i and then set the answer to zero:

$$\frac{\partial}{\partial b_i}\mathbb{E}(U(b \cdot Y)) \quad = \quad \mathbb{E}(Y_i U'(b \cdot Y)) \quad = \quad 0$$

for $i = 1..d$, i.e. we have d equations for the d unknowns $b_1^*, ..., b_d^*$ for the optimal portfolio allocation b^*. Note we can re-write this as

$$\mathbb{E}^{\mathbb{Q}}(Y_i) \quad = \quad 0 \tag{2}$$

where we define a **new probability measure** as $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(\frac{U'(b^* \cdot Y)}{\mathbb{E}^{\mathbb{P}}(U'(b^* \cdot Y))} 1_A)$ for events $A \in \mathcal{F}$, and (as a sanity check) we note that $\mathbb{Q}(\Omega) = 1$.

Under the moment condition stated in the question, it turns out that a unique solution b^* exists if the **no-arbitrage** condition is satisfied: if $\mathbb{P}(b \cdot Y > 0) > 0$ then $\mathbb{P}(b \cdot Y < 0) > 0$. In this case, from (2), we see that under $\mathbb{Q}$, all contracts are priced according to the market, i.e. $\mathbb{E}^{\mathbb{Q}}(\pi^i) = p_i$. $\mathbb{Q}$ is known as a **risk-neutral measure**. We can solve this maximization problem numerically using e.g. MOSEK convex optimization package in Python (used for summer project).

A common choice is the **exponential utility function** $U(x) = -e^{-\lambda x}$, in which case (for $\lambda = 1$) we are computing $\max_b(-\mathbb{E}(e^{-b \cdot Y})) = \max_b(-\mathbb{E}(e^{b \cdot Y})) = -\min_b \mathbb{E}(e^{b \cdot Y})$, i.e. minus the **minimum of the mgf** of Y.

For the 1d case $d = 1$, if $\pi_1 = f(S)$ with market price p and S has density $p(S)$, then we can re-write expected utility as

$$-\int_0^\infty e^{-\lambda b(f(S)-p)} p(S) dS\,.$$

We can evaluate this integral explicitly in certain cases, e.g. if $S \sim \text{Exp}(1)$ and $f(S) = S$, we see that

$$\mathbb{E}(U(b(S-p))) \quad = \quad \mathbb{E}(-e^{-\lambda b(S-p)}) \quad = \quad \int_0^\infty -e^{-\lambda b(S-p)} e^{-S} dS \quad = \quad -\frac{e^{pb\lambda}}{1+b\lambda}$$

if $\lambda b + 1 > 0$, and $-\infty$ otherwise. Differentiating this expression wrt b and setting the answer to zero, we find that

$$b^* \quad = \quad \frac{1-p}{p\lambda} \tag{3}$$

and the risk-neutral density $\mathbb{Q}$ corresponding to b^* is the density of a $\text{Exp}(\frac{1}{p})$ random variable, under which $\mathbb{E}^{\mathbb{Q}}(S) = p$ as claimed.

Note the "fair price" of the stock is $\int_0^\infty S e^{-S} dS = \mathbb{E}(S) = 1$, so (3) says that we buy stock when the stock is underpriced, and sell when the stock is overpriced, and $|b^*|$ is smaller when λ is larger i.e. when the trader is more risk-averse.

We can also modify the problem to account for interest rates, **bid-ask spreads**, **finite liquidity** or a **limit order book** structure.

7. Compute the **conditional distribution** of W_t given the known value of W_s, for $s, t > 0$ (you do not have to assume that $s < t$). You may use that for two correlated Normal random variables $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ with $\text{Corr}(X, Y) = \rho$,

$$Y|X \quad \sim \quad N(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X), (1-\rho^2)\sigma_Y^2)$$

and recall that the correlation of two random variables X and Y is defined as $\text{Corr}(X, Y) = \frac{\mathbb{E}((X-\mu_X)(Y-\mu_Y)}{\sigma_X \sigma_Y}$.

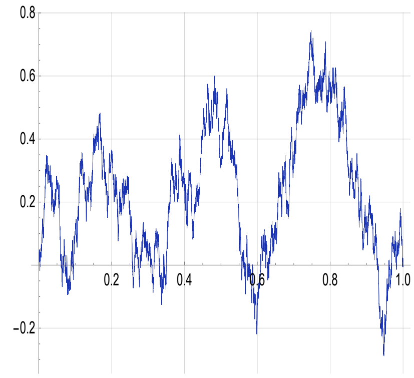

Figure 1: Simulation of a Brownian bridge on $[0,1]$.

Solution. For our case here, $X = W_s$, $Y = W_t$, $\mu_X = 0$, $\mu_Y = 0$, $\sigma_X = \sqrt{s}$, $\sigma_Y = \sqrt{t}$ and $\rho = \frac{\min(s,t)}{\sqrt{st}}$, and recall that we have shown in the lecture notes that $\mathbb{E}(W_s W_t) = \min(s,t)$. Thus

$$W_t | W_s \quad \sim \quad N(\rho \frac{\sqrt{t}}{\sqrt{s}} W_s, (1-\rho^2)t)\,.$$

8. Compute $\mathbb{E}(W_t^3 | W_s = x)$ for $0 \leq s \leq t$.

Solution. $W_t - W_s \sim N(0, t-s)$, so

$$\begin{aligned}
\mathbb{E}((x + W_t - W_s)^3 | W_s = x) \quad &= \quad \mathbb{E}(x^3 + 3x^2(W_t - W_s) + 3x(W_t - W_s)^2 + (W_t - W_s)^3) | W_s = x) \\
&= \quad x^3 + 3x(t-s)\,.
\end{aligned}$$

We can generalize this computation to compute $\mathbb{E}(W_t^n | W_s = x)$ for any $n \in \mathbb{N}$, since (from the **binomial theorem**) we know that $(x + W_t - W_s)^n = \sum_{i=0}^{n} x^{n-i}(W_t - W_s)^i \binom{n}{i}$, and we also know that all odd moments of $W_t - W_s$ are zero.