Homework 2

1. Consider the **Bessel process** which satisfies

$$dR_t \;=\; \frac{2\delta-1}{R_t}dt \,+\, dW_t$$

for $\delta \geq 0, R_0 > 0$. Using Ito's lemma, compute the SDE satisfied by $Z_t = R_t^2$.

Solution.

$$\begin{aligned} dZ_t &= 2R_t dR_t + \frac{1}{2}\cdot 2dt \\ &= 2[(2\delta-1)dt + R_t dW_t] + dt \\ &= (4\delta-1)dt + 2\sqrt{Z_t}dW_t\,. \end{aligned}$$

2. Consider a process X_t satisfying the SDE

$$dX_t \;=\; X_t^2 dW_t\,.$$

Compute the SDE for $R_t = 1/X_t$ in terms of R_t. X is a rare example of a process which is driftless but it not an $\mathcal{F}^W$-martingale (in fact it can be shown that $\mathbb{E}(X_t|X_s) < X_s$, see FM04 for details).

Solution:

$$dR_t \;=\; -\frac{1}{X_t^2}dX_t \,+\, \frac{1}{2}\frac{2}{X_t^3}X_t^4 dt \;=\; -\frac{1}{X_t^2}X_t^2 dW_t \,+\, \frac{1}{2}\frac{2}{X_t^3}X_t^4 dt \;=\; -dW_t \,+\, \frac{1}{R_t}dt$$

3. Apply Ito's lemma to $(1-t/T)W_t$, and integrate the resulting equation from $t=0$ to $t=T$. Use this to compute the distribution of $\frac{1}{T}\int_0^T W_t dt$.

Solution. Let $f(x,t) = (1-t/T)x$. Then applying Ito's lemma to $f(W_t,t)$, we see that

$$df(W_t,t) \;=\; -\frac{1}{T}W_t dt \,+\, (1-\frac{t}{T})dW_t\,.$$

Integrating from 0 to T, we see that

$$f(W_T,T) - f(W_0,0) \;=\; 0 \;=\; -\frac{1}{T}\int_0^T W_t dt \,+\, \int_0^T (1-\frac{t}{T})dW_t$$

so we see that the average of W over the interval $[0,T]$ is given by $\int_0^T(1-\frac{t}{T})dW_t$. Moreover, since this is a stochastic integral of the form $\int_0^T \phi(t)dW_t$, where ϕ is non-random, $\int_0^T(1-\frac{t}{T})dW_t \sim N(0, \int_0^T \phi(t)^2 dt)$ (see part of lecture notes on the Ornstein-Uhlenbeck process) and when you evaluate the integral here, one finds that $\int_0^T \phi(t)^2 dt = \frac{1}{3}T$. This means that $\mathrm{Var}(\frac{1}{T}\int_{t=0}^T W_t dt)$, i.e. the variance of the average of W over $[0,T]$ is one-third the variance of W_T itself (which we know is T).

4. Consider the following SDE

$$dR_t \;=\; (\frac{1}{R_t} - \frac{R_t}{1-t})dt \,+\, dW_t$$

for $t<1$ with $R_0 > 0$ (you may assume that $R_t > 0$ for $t<1$). Compute an SDE for $Y_t = R_t^2$. R is known as the **Brownian excursion process**, which is Brownian motion conditioned to return to zero for the first time at time 1.

Solution. Let $Y_t = R_t^2$. Then

$$\begin{aligned} dY_t \;=\; 2R_t dR_t \,+\, \frac{1}{2}\cdot 2dt &= 2R_t((\frac{1}{R_t} - \frac{R_t}{1-t})dt \,+\, dW_t) \,+\, dt \\ &= 2R_t dW_t \,+\, 3dt - \frac{2R_t^2}{1-t}dt \\ &= (3 - \frac{2Y_t}{1-t})dt \,+\, 2\sqrt{Y_t}dW_t\,. \end{aligned}$$

5. Consider the following SDE

$$dS_t \quad = \quad \delta(\beta S_t + 1 - \beta)dW_t\,.$$

for $\delta > 0$. Derive the SDE satisfied by $X_t = \beta S_t + 1 - \beta$. S is known as a **displaced-diffusion** process.

Solution.

$$dX_t \quad = \quad d(\beta S_t + 1 - \beta) \quad = \quad \delta\beta X_t dW_t\,.$$

and we note that X is Geometric Brownian motion. S is often used to approximate the **CEV process** $dS_t = \delta S_t^\beta dW_t$ for $\beta \in (0,1)$ when $S_0 = 1$, since S^β and $\beta S + 1 - \beta$ have the same slope and value at $S = 1$.