The conditional law of the Bacry-Muzy and Riemann-Liouville log correlated Gaussian fields and their GMC, via Gaussian Hilbert and fractional Sobolev spaces

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Abstract

We compute $\mathbb{E}(X_t|(X_s)_{0\leq s\leq L})$ for the standard Bacry-Muzy log-correlated Gaussian field X with covariance $\log^+ \frac{T}{|t-s|}$, which corrects the finite-horizon prediction formula in Vargas et al.[DRV12]. The problem can be viewed as a linear filtering problem, and we solve the problem by showing that the $L^2(\mathbb{P})$ closure of $\{\int_{[0,L]} \phi(s)X_s ds : \phi \in \mathcal{S}, \mathrm{supp}(\phi) \subseteq [0,L]\}$ is equal to $\{X(\phi) : \phi \in H^{-\frac{1}{2}}, \mathrm{supp}(\phi) \subseteq [0,L]\}$, where $X(\phi)$ is defined as a continuous linear extension of X acting on $\mathcal{S} \subset H^s$, H^s denotes the fractional Sobolev space of order s and $\mathbb{P}$ is the law of the field X on the space of tempered distributions. The explicit formula for the filter is obtained as the solution to a Fredholm integral equation of the first kind with logarithmic kernel. From this we characterize the conditional law of the Gaussian multiplicative chaos (GMC) M_γ generated by X, using that M_γ is measurable with respect to X. We also outline how one can adapt this result for the Riemann-Liouville GMC introduced in [FFGS19], which has a natural application to the Rough Bergomi volatility model in the $H \to 0$ limit.[1]

1 Introduction

Originally pioneered by Kahane[Kah85], Gaussian multiplicative chaos (GMC) is a random measure on a domain of $\mathbb{R}^d$ that can be formally written as

$$M_\gamma(dx) \quad = \quad e^{\gamma X_x - \frac{1}{2}\gamma^2 \mathbb{E}(X_x^2)} dx \tag{1}$$

where X is a Gaussian field with zero mean and covariance $K(x,y) := \mathbb{E}(X_x X_y) = \log^+ \frac{1}{|y-x|} + g(x,y)$ for some bounded continuous function g. X is not defined pointwise because there is a singularity in its covariance, rather X is a random tempered distribution, i.e. an element of the dual of the Schwartz space $\mathcal{S}$ under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of (1) requires a regularizing sequence X^ε of Gaussian processes (with the singularity removed, see e.g. [BBM13] and [BM03] for a description of such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular region, which is summarized in Section 2.3 in [FFGS19], or page 17 in [RV10] and section 3.4 in [Sha16] for a general method in $\mathbb{R}^d$ using a convolution to smooth X). In most of the literature on GMC, the choice of X^ε is a martingale in ε, from which we can then easily verify that $M_\gamma^\varepsilon(A) = \int_A e^{\gamma X_x^\varepsilon - \frac{1}{2}\gamma^2 \mathrm{Var}(X_x^\varepsilon)} dx$ is a martingale, and then obtain a.s. convergence of $M_\gamma^\varepsilon(A)$ using the martingale convergence to a random variable $M_\gamma(A)$ with $\mathbb{E}(M_\gamma(A)) = \mathrm{Leb}(A)$, and with a bit more work we can verify that $M_\gamma(.)$ defines a random measure (see the end of Section 4 on page 18 in [RV10]).

If $\gamma^2 < 2d$, $M_\gamma^\varepsilon(dx) = e^{\gamma X_x^\varepsilon - \frac{1}{2}\gamma^2 \mathbb{E}((X_x^\varepsilon)^2)} dx$ tends weakly to a multifractal random measure M_γ with full support a.s. which satisfies the multifractal property

$$\mathbb{E}(M_\gamma([0,t])^q) \quad = \quad c_q t^{\zeta(q)} \tag{2}$$

for $q \in (1, q^*)$ for some constant $c_q = \mathbb{E}(M_\gamma([0,1])^q)$, where $q^* = \frac{2}{\gamma^2}$ [2] and

$$\zeta(q) \quad = \quad q - \frac{1}{2}\gamma^2(q^2 - q)$$

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[2] see Lemma 3 in [BM03] to see why the critical q value is q^*

and $\mathbb{E}(M_\gamma([0,t])^q)=\infty$ if $q>\frac{2}{\gamma^2}$, see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). Moreover, we can show that the support of M_γ is a so-called γ-*thick points* of X, i.e. points such that $\lim_{\varepsilon\to 0}\frac{X^\varepsilon}{\log\frac{1}{\varepsilon}}=\gamma$ (see e.g. section 2 in [Aru17], [Ber17] and page 7 in [RV16] for more on this), and for $g\equiv 0$, explicit expressions are known for the Mellin transform of the law of $M_\gamma([0,1])$ (see e.g. [Ost09], [Ost13], [Ost18]), which show that $\log M_\gamma([0,1])$ has an infinitely divisible law, and an explicit formula for sampling the law of the total mass of the GMC on the interval is given in [RZ17].

M_γ is the zero measure for $\gamma^2=2d$ and $\gamma^2>2d$; in these cases a different re-normalization is required to obtain a non-trivial limit. Specifically, for $\gamma^2=2d$, we obtain a non-trivial limit by considering $\sqrt{\log\frac{1}{\varepsilon}}\cdot M_\varepsilon^{\gamma=2}$ as $\varepsilon\to 0$ or the "derivative measure" $\frac{d}{d\gamma}e^{\gamma X^\varepsilon-\frac{1}{2}\gamma^2\mathrm{Var}(X_\varepsilon)}|_{\gamma=\sqrt{2d}}$. [DRSV14] show that both these objects tend weakly to the same measure μ' as $\varepsilon\to 0$, and in 2d Aru et al.[APS19] have shown that $\frac{M_\gamma}{2-\gamma}\to 2\mu'$ in probability as γ tends to the critical value of 2, and the critical γ-value is particularly important in Liouville quantum gravity (again see [DRSV14] for further discussion). One can also construct a GMC for the super-critical phase, using an independent stable subordinator time-changed by a sub-critical GMC (see section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for γ-values greater than $\sqrt{2}$, which is closely related to the non-standard branch of gravity in conformal field theory.

In the sub-critical case, using a limiting argument it can be shown that M_γ satisfies the "master equations": $M(X+f,dz)=e^{\gamma f(z)}M(X,dz)$ and $\mathbb{E}(\int_D F(X,z)M_\gamma(dz))=\mathbb{E}(\int_D F(X+\gamma K(z,.),z)dz)$ for any measurable function F and any interval D, which comes from the Cameron-Martin theorem for Gaussian measures and the notion of *rooted measures* and the disintegration theorem (see section 2.1 in [Aru17] for a nice discussion on this). Moreover, either of these two can equations can be taken as the definition of GMC, and they uniquely determine M_γ as a measurable function of X, and hence also uniquely fix its law (see also Theorem 6 in Shamov[Sha16].

GMC also has a natural and important application in Liouville Quantum Field Theory; LQFT is is a 2d model of random surfaces, which (formally) we can view as a random metric in the context of quantum gravity, where we weight the classical free field action with an interaction term given by the exponential of a GMC and can be viewed as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time.

2 Construction of the standard Bacry-Muzy GMC on the line

Define the Gaussian process $\omega_\varepsilon(t)$ as in Eq 7 in [BBM13] with $\lambda=1$ (except here use ε instead of l), and set $X_t^\varepsilon:=\omega_\varepsilon(t)-\mathbb{E}(\omega_\varepsilon(t))$, so

$$X_t^\varepsilon \;=\; \int_{(u,s)\in\mathcal{A}_\varepsilon(t)} dW(u,s) \tag{3}$$

where $dW(u,s)$ is 2-dimensional Gaussian white noise with variance $s^{-2}duds$, and $\mathcal{A}_\varepsilon(t)$ is triangular region defined in Eq 8 (and Figure 1) in [BBM13]. Then

$$R_\varepsilon(s,t) \;:=\; \mathbb{E}(X_s^\varepsilon X_t^\varepsilon) \;=\; \begin{cases}\log\frac{T}{\tau} & \varepsilon\le\tau\le T\\ \log\frac{T}{\varepsilon}+1-\frac{\tau}{\varepsilon} & \tau\le\varepsilon\\ 0 & \tau>T\end{cases} \tag{4}$$

where $\tau=|t-s|$ (see Eq 10 in [BBM13]), and one can easily verify that

$$R_\varepsilon(s,t) \;\le\; \log\frac{T}{\tau} \;=\; R(s,t) \tag{5}$$

for $s,t\in[0,T]$ (see Eq 25 in [BM03]). Using (3), we also see that

$$\mathbb{E}(X_t^\varepsilon X_s^{\varepsilon'}) \;=\; \mathbb{E}(\int_{(u,v)\in\mathcal{A}_\varepsilon(t)} dW(u,v)\int_{(u,v)\in\mathcal{A}_{\varepsilon'}(s)} dW(u,v)) \;=\; \int_{\mathcal{A}_\varepsilon(t)\cap\mathcal{A}_\varepsilon(s)}\frac{1}{v^2}dudv \;=\; \mathbb{E}(X_s^\varepsilon X_t^\varepsilon)$$

for $0<\varepsilon'\le\varepsilon$ (i.e. the answer does not depend on ε'). We now define the measure

$$M_\gamma^\varepsilon(dt) \;=\; e^{\gamma X_t^\varepsilon-\frac{1}{2}\gamma^2\mathrm{Var}(X_t^\varepsilon)}dt\,.$$

One can easily verify that $M_\gamma^\varepsilon(A)$ is a backwards martingale with respect to the filtration $\mathcal{F}_\varepsilon:=\sigma(W(A,B):A\subset\mathbb{R}^+,B\subseteq[\varepsilon,\infty])$ (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and

$$\sup_{\varepsilon>0}\mathbb{E}(M_\gamma^\varepsilon(A)^q)<\infty \tag{6}$$

(Lemma 3 i) in [BM03]), so from the martingale convergence theorem, $M^{\varepsilon}_{\gamma}(A)$ converges to some random variable $M_{\gamma}(A)$ in L^q for $q \in (1, q^*)$, and from the reverse triangle inequality this implies that

$$\lim_{\varepsilon\to 0} \mathbb{E}((M^{\varepsilon}_{\gamma}(A))^q) = \mathbb{E}(M_{\gamma}(A)^q) \tag{7}$$

Moreover, one can show that $M_{\gamma}(.)$ defines measure (see e.g. end of Section 4 on page 18 in [RV10]), and since $M^{\varepsilon}_{\gamma}(A) \to M_{\gamma}(A)$ a.s. for any Borel set A this implies weak convergence of M^{ε}_{γ} to M_{γ} a.s. (from e.g. Theorem 3.1 parts a) and f) in Ethier&Kurtz[EK86]).

Moreover M_{γ} is multifractal, i.e. $\mathbb{E}(|M_{\gamma}([0,t])|^q) = c_{q,T}\, t^{\zeta(q)}$ (see e.g. Lemma 4 in [BM03]) for some finite constant $c_{q,T} > 0$, depending only on q and T. For integer $q \geq 1$, we also note that

$$\begin{aligned}
\mathbb{E}(M_{\gamma}(A)^q) &= \int_A \dots \int_A e^{\gamma^2 \sum_{1\leq i<j\leq q} \log \frac{T}{|u_i-u_j|}} du_i \dots du_q \\
&= \int_A \dots \int_A e^{\gamma^2 q(q-1)\log T + \sum_{1\leq i<j\leq q} \log \frac{1}{|u_i-u_j|}} du_i \dots du_q \\
&= T^{\gamma^2 q(q-1)} \int_A \dots \int_A e^{\sum_{1\leq i<j\leq q} \log \frac{1}{|u_i-u_j|}} du_i \dots du_q
\end{aligned} \tag{8}$$

so we see that

$$c_{q,T} = c_q T^{\gamma^2 q(q-1)}$$

where $c_q := c_{q,1}$, and this also holds for non-integer q (see e.g. Theorem 3.16 in [Koz06]).

3 The conditional law of the standard log correlated Gaussian field

Consider a standard log-correlated Gaussian field Z on $\mathbb{R}$ with covariance $R(s,t) = \log^+ \frac{T}{|t-s|}$. From the Minlos-Bochner theorem, we know that the law of Z is a Gaussian measure on the space $\mathcal{S}'$ of *tempered distributions* (see e.g. [DRSV17] and Appendix A in [FFGS19] for more details on tempered distributions) which is the dual of the Schwartz space $\mathcal{S}$ (see e.g. section 2.2 in [DRSV14] and Theorem 2.1 in [BDW17]). Moreover, $\mathcal{S}$ is a Montel space and thus is reflexive, i.e. $(\mathcal{S}')'$ is isomorphic to $\mathcal{S}$ using the canonical embedding of S into its bi-dual $(S')'$. From here on, we are only concerned with the restriction of Z to $[0,T]$ (on which the covariance of Z is just $\log \frac{T}{|t-s|}$, so we set Z equal to zero outside this interval for simplicity.

Proposition 3.1 *X^{ε} tends to X in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definitions), where X has the same law as the field Z defined above.*

Proof. $0 \leq R_{\varepsilon}(s,t) \leq R(s,t)$ for $s,t \in [0,T]$ (see (5)), so from the dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0} \int_{[0,T]^2} \phi_1(s)\phi_2(t) R_{\varepsilon}(s,t)\, ds dt = \int_{[0,T]^2} \phi_1(s)\phi_2(t) R(s,t)\, ds dt \tag{9}$$

for any $\phi_1, \phi_2 \in \mathcal{S}$, where $R_{\varepsilon}(s,t)$ is defined as in (4). Similarly, for any sequence $\phi_k \in \mathcal{S}$ with $\|\phi_k\|_{m,j} \to 0$ for all $m, j \in \mathbb{N}^n_0$ for any $n \in \mathbb{N}$ (i.e. under the Schwartz space semi-norm defined in Eq 1 in [BDW17])

$$\lim_{k\to\infty} \int_{[0,T]^2} \phi_k(s)\phi_k(t) R(s,t)\, ds dt = 0 \tag{10}$$

since $\nu(A) := \int_A R(s,t) ds dt$ is a bounded non-negative measure (since $\int_0^T \int_0^t R(s,t) ds dt < \infty$), and the convergence here implies in particular that ϕ_k tends to ϕ pointwise, so we can use the bounded convergence theorem. Thus if we define

$$\begin{aligned}
\mathcal{L}_{X^{\varepsilon}}(f) &:= \mathbb{E}(e^{i(f,X^{\varepsilon})}) = e^{-\frac{1}{2}\int_{[0,T]^2} f(s)f(t) R_{\varepsilon}(s,t) ds dt} \\
\mathcal{L}(f) &:= e^{-\frac{1}{2}\int_{[0,T]^2} f(s)f(t) R(s,t) ds dt}
\end{aligned}$$

for $f \in \mathcal{S}$, then from (9) and (10) and Lévy's continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW17]), we see that $\mathcal{L}_{X^{\varepsilon}}(f)$ tends to $\mathcal{L}(f)$ pointwise and $\mathcal{L}(.)$ is continuous at zero, then there exists a generalized random field

X (i.e. a random tempered distribution, such that $L_X = L$ and X^ε tends to X in distribution with respect to the strong and weak topology (see page 2 in [BDW17] for definition). ■

In general, the conditional expectation of a random variable is equal to its projection onto the Gaussian Hilbert space (sub-Hilbert space of $L^2(\Omega, \mathcal{F}, \mathbb{P})$) generated by the variables on which we are conditioning. To this end, we let $\bar{F}$ denote the Hilbert space given by the $L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$ closure of

$$F \quad = \quad \{X(\phi) : \phi \in \mathcal{S},\, \mathrm{supp}(\phi) \subseteq [0, L]\}$$

where $\mathcal{F}_L = \sigma((X_u)_{0 \le u \le L})$. The closure here is necessary because the notion of orthogonal projection requires the Hilbert space structure, and there is no guarantee that the conditional expectation $\mathbb{E}(X(\psi)|\mathcal{F}_L)$ will be a random variable of the form $\int_{[0,L]} X_s \phi(s) ds$ with $\phi \in \mathcal{S}$.

In order to characterize $\bar{F}$, we first note that

$$\mathbb{E}[(\int X_s \phi(s) ds)^2] \quad = \quad \int \int R(s,t) \phi(s) \phi(t) ds dt\,.$$

From Eqs 2.1 in [DRV12], we also know that

$$c\|\phi\|^2_{H^{-\frac{1}{2}}} \quad \le \quad \int \int R(s,t)\phi(s)\phi(t) dsdt \quad \le \quad C\|\phi\|^2_{H^{-\frac{1}{2}}} \tag{11}$$

where $0 < c < C < \infty$. Let H^s denotes the fractional Sobolev space of order s (see e.g. Section 2.2 in [JSW18] for definitions). Then we can put two inner products on the linear space $\mathcal{S}$ of Schwarz functions:

1. $\langle \phi, \psi \rangle_{H^{-\frac{1}{2}}} := \int_{-\infty}^{\infty} (1 + |k|^2)^{-\frac{1}{2}} \hat{\phi}(k) \bar{\hat{\psi}}(k) dk$ (i.e. the standard inner product on $H^{-\frac{1}{2}}$)
2. $\langle \phi, \psi \rangle := \mathbb{E}[X(\phi)X(\psi)] = \int \int \phi(s)\psi(t)R(s,t)dsdt$

Eq 2.2 in [DRV12] shows that these two inner products are equivalent and thus generate the same topologies on $\mathcal{S}$.

We now make the following observations:

- Let $\phi \in H^{-\frac{1}{2}}$, with $\mathrm{supp}(\phi) \subseteq [0, L]$. $\mathcal{S}$ is dense in $H^{-\frac{1}{2}}$, so there exists a sequence $\phi_n \in \mathcal{S}$ with $\mathrm{supp}(\phi_n) \subseteq [0, L]$ such that $\|\phi_n - \phi\|_{H^{-\frac{1}{2}}} \to 0$, and ϕ is a Cauchy sequence in $H^{-\frac{1}{2}}$ so (by the equivalence of norms) $X(\phi_n)$ is a Cauchy sequence in $\bar{F}$, and thus converges to some Y in $\bar{F}$. This defines $X(\phi) := Y$ as a continuous linear extension of X from $\mathcal{S}$ to the larger space $H^{-\frac{1}{2}}$, which we will also often write as $\int \phi(t) X_t dt$. To check that $X(\phi)$ is uniquely specified, consider two such sequences ϕ_n and ϕ'_n. Then from the triangle inequality

$$\|\phi_n - \phi'_n\|_{H^{-\frac{1}{2}}} \quad \le \quad \|\phi_n - \phi\|_{H^{-\frac{1}{2}}} + \|\phi - \phi'_n\|_{H^{-\frac{1}{2}}} \quad \to \quad 0$$

  and thus (by the equivalence of norms) we have $\|X(\phi_n) - X(\phi'_n)\|_{L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})} = \|X(\phi_n) - X(\phi'_n)\|_{\bar{F}} \to 0\,.$

- Conversely, for any $Z \in \bar{F}$, there exists a sequence $\phi_n \in \mathcal{S}$ such that $X(\phi_n)$ converges to $Z \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$, so ϕ_n is a Cauchy sequence with respect to the second norm defined above, and hence also a Cauchy sequence with respect to the $H^{-\frac{1}{2}}$ norm (by the equivalence of the two norms). $H^{-\frac{1}{2}}$ is a Hilbert space so Cauchy sequences in $H^{-\frac{1}{2}}$ converge i.e. there exists a ϕ in $H^{-\frac{1}{2}}$ such that $\phi_n \to \phi \in H^{-\frac{1}{2}}$.

Thus we have shown that

$$\bar{F} \quad = \quad \{X(\phi) : \phi \in H^{-\frac{1}{2}}, \mathrm{supp}(\phi) \subseteq [0, L]\}$$

where we are using the extension of X to $H^{-\frac{1}{2}}$ on the right hand side here as defined in the first bullet point above.

Moreover (since $\mathbb{E}(X(\psi)|\mathcal{F}_L) \in \bar{F}$) we see that for any $\psi \in \mathcal{S}$

$$\mathbb{E}(X(\psi)|\mathcal{F}_L) \quad = \quad \int_{[0,L]} X_s k_\psi(s) ds \quad := \quad X(k_\psi)$$

for some $k_\psi(s) \in H^{-\frac{1}{2}}([0, L])$, where $X(.)$ in the final expression is the linear extension we have just defined. This analysis shows that $\bar{F}$ is isometrically isomorphic to the set of functions in $H^{-\frac{1}{2}}$ with support in $[0, L]$.

Moreover, we can now extend the inner product to $H^{-\frac{1}{2}}$ as

$$\langle \phi, \psi \rangle \;=\; \lim_{n\to\infty} \mathbb{E}[X(\phi_n)X(\psi_n)] \;=\; \lim_{n\to\infty} \int\int \phi_n(s)\psi_n(t)R(s,t)dsdt$$

where $\phi_n, \psi_n \in \mathcal{S}$ and $\phi_n \to \phi$ in $H^{-\frac{1}{2}}$ and $\psi_n \to \psi$ in $H^{-\frac{1}{2}}$.

Proposition 3.2 *$X \in H^{-\frac{1}{2}-\delta}$ a.s. for any $\delta > 0$.*

Proof. The proof is almost identical to Proposition 2.1 in [FFGS19], but since some of its arguments are needed for the next Proposition as well, we have put a proof in Appendix A. ■

Remark 3.1 One can actually show the stronger result that $X \in H^{-\delta} \subset H^{-\frac{1}{2}}$ a.s. for any $\delta > 0$, but we will not need this here (see also [BDW17]).

Proposition 3.3 *$X^\varepsilon \to X$ in $H^{-\frac{1}{2}-\delta}$ in probability for any $\delta > 0$, where X^ε is defined as in* (3).

Proof. See Appendix B. ■

We know that for any $\psi \in \mathcal{S}$ with $\mathrm{supp}(\psi) \subseteq [L,T]$, the conditional expectation $\mathbb{E}(X(\psi)|\mathcal{F}_L) = X(k_\psi)$ minimizes

$$\mathbb{E}((X(\psi) - Y)^2)$$

over all $Y \in L^2(\mathcal{S}, \mathcal{F}_L, \mathbb{P})$, and $\mathbb{E}((\int_{[L,T]} X_t\psi(t)dt - \mathbb{E}(\int_{[L,T]} X_t\psi(t)dt \,|\, \mathcal{F}_L))Z) = 0$ for all $Z \in \mathcal{F}_L$, so in particular setting $Z = \int_{[0,L]} \psi_2(s)X_s ds$ for $\psi_2 \in \mathcal{S}$ with $\mathrm{supp}(\psi_2) \subseteq [0,L]$, we see that

$$\begin{aligned} 0 &= \mathbb{E}((X(\psi) - X(k_\psi))X(\psi_2)) \\ &= \mathbb{E}((\int_{[L,T]} \psi(t)X_t dt - \int_{[0,L]} k_\psi(u)X_u du) \int_{[0,L]} \psi_2(s)X_s ds) \\ &= \int_{[L,T]}\int_{[0,L]} \psi(t)\psi_2(s)R(t-s)dsdt \;-\; \int_{[L,T]}\int_{[0,L]} R(s-u)k_\psi(u)\psi_2(s)duds\,. \end{aligned} \tag{12}$$

In (14) below we construct an explicit solution $k_t(.)$ to

$$0 \;=\; \mathbb{E}((X_t - \int_{[0,L]} k(u)X_u du)X_s) \;=\; R(s,t) \;-\; \int_{[0,L]} R(u,s)k_t(u)du \tag{13}$$

for $s \in [0,L]$, with $k_t \in \mathrm{supp}(\psi) \subseteq [t,T]$, which implies that (12) holds if we set $k_\psi(u) = \int_{[L,T]} \psi(t)k_t(u)dt$.

Proposition 3.4 *The covariance operator $R\phi = \int_0^T R(s,t)\phi(s)ds$ acting on $H^{-\frac{1}{2}}$ is positive definite, and $\int_0^T R(s,t)\phi(s)ds = 0$ if and only if $\phi \equiv 0$ Lebesgue a.e.*

Proof. From the discussion on page 4, we know that bilinear form R is (up to an equivalence) the inner product on $H^{-\frac{1}{2}}$ so it has to be positive definite (from the definition of a norm), and thus $\int_0^T R(s,t)\phi(s)ds \neq 0$ if $\phi \neq 0$, since otherwise $R(\phi,\phi) = \int_0^T\int_0^T R(s,t)\phi(s)ds\phi(t)dt = 0$. ■

The integral equation in (13) (with t fixed) is the well known *Wiener-Hopf equation*. We refer the reader to [Poor94] for more details on the Wiener-Hopf equation in the context of ordinary Gaussian processes.

Corollary 3.5 *Proposition 3.4 shows that the Wiener-Hopf equation in* (13) *has a unique solution.*

If $t \leq T$ (so we can replace $\log^+$ with log), we can re-write (12) as

$$\int_{[0,L]} k_t(u)\log\frac{T}{|s-u|}du \;=\; f(s) \;:=\; \log\frac{T}{t-s}$$

and we see that this is now a Fredholm integral equation of the 1st kind with logarithmic kernel, which can be solved explicitly by a minor extension of page 299 in [EK00] (who consider $T = 1$) to give

$$k_t(u) \;=\; \frac{1}{\pi^2}\int_0^L \frac{\sqrt{v(L-v)}}{\sqrt{u(L-u)}}\frac{f'(v)}{u-v}dv + \frac{c_t}{\pi\sqrt{u(L-u)}} \;=\; \frac{(c_t-1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}} \tag{14}$$

where the integral in the second expression is understood in the principal value sense, and

$$c_t \quad = \quad \int_0^L k_t(u)du \quad = \quad \frac{1}{\pi(\log(\frac{1}{4}L) - \log T)} \int_0^L \frac{\log\frac{t-v}{T}}{\sqrt{v(L-v)}}dv \quad < \quad \infty\,.$$

We now verify that $k_\psi(u) \in H^{-\frac{1}{2}}$. To this end, we first note that

$$\pi\log\frac{L}{4} - \pi\log T \quad = \quad \int_0^L \frac{\log\frac{L-v}{T}}{\sqrt{v(L-v)}}dv \quad \leq \quad \int_0^L \frac{\log\frac{t-v}{T}}{\sqrt{v(L-v)}}dv \quad \leq \quad \int_0^L \frac{\log\frac{t}{T}}{\sqrt{v(L-v)}}dv \quad \leq \quad \pi\log t - \pi\log T\,.$$

$$\frac{(c_t-1)u + t - c_t t - \sqrt{t(t-L)}}{\pi(u-t)\sqrt{u(L-u)}}1_{u\in[0,L]}1_{t\in[L,T]} \quad \leq \quad h(u,t) \quad = \quad \frac{c_1}{(t-u)\sqrt{u(L-u)}}1_{u\in[0,L]}1_{t\in[L,T]}$$

for some constant c_1. We know that

$$\int_{[L,T]}(\int_{[0,L]}|\psi(t)h(u,t)|^p)du)^{\frac{1}{p}}dt \quad \leq \quad \|\psi\|_{L^\infty}\int_{[L,T]}(\int_{[0,L]}|h(u,t)|^p)du)^{\frac{1}{p}}dt \tag{15}$$

and setting $p = \frac{3}{2}$ we find that

$$\int_{[0,L]}|h(u,t)|^p)du \quad = \quad G(t) \quad := \quad const. \times \frac{2t-L}{t(t-L)^{\frac{5}{4}}}$$

which implies that

$$\int_{[L,T]} G(t)^{\frac{1}{p}}dt \quad < \quad \infty$$

so for $p = \frac{3}{2}$ the double integral in (15) is finite, so (from the Minkowski integral inequality) $\int_{[L,T]} h(.,t)dt$ and thus $\int_{[L,T]}\psi(t)k_t(.)dt \in L^p$, and hence its Fourier transform is in $L^q = L^3$ where $1/p + 1/q = 1$, and thus is $O(|\xi|^{-\frac{1}{3}-\varepsilon})$ for $\xi \gg 1$ and $O(|\xi|^{-\frac{1}{3}+\varepsilon})$ for $\xi \ll 1$.

Hence

$$\begin{aligned}\|k_\psi\|_{H^{-\frac{1}{2}}} \quad &= \quad \int_{-\infty}^{\infty}(1+|\xi^2|)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{i\xi u}\int_{[L,T]}\psi(t)k_t(u)1_{u\in[0,L]}dt\,du\,d\xi \\ &= \quad \int_{-\infty}^{\infty}(1+|\xi^2|)^{-\frac{1}{2}}\int_{[0,L]}e^{i\xi u}\int_{[L,T]}\psi(t)k_t(u)dt\,du\,d\xi \quad < \quad \infty\end{aligned}$$

which verifies the validity of our explicit solution for $k_u(t)$.

Remark 3.2 Corollary 3.3 in [DRV12] gives the following nice prediction formula for a log-correlated Gaussian field X with covariance $\log\frac{T}{|t-s|}$: [3]

$$\mathbb{E}(X_t|(X_s)_{s\leq 0}) \quad = \quad \frac{1}{\pi}\int_{-\infty}^0 \frac{\sqrt{t}}{(t-s)\sqrt{-s}}X_s ds$$

which we can verify satisfies the associated Wiener-Hopf equation (and is also very similar to the predicition formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the $H \to 0$ limit). However the prediction formula for the finite history case stated in Theorem 3.5 in [DRV12] appears to be wrong since numerical tests confirm that it does not satisfy the Wiener-Hopf equation. Our linear filter $\int_{[0,L]} k_t(u)X_u du$ corrects this formula for the case when $L + t \leq T$.

Remark 3.3 Clearly if $t - L > T$, the history of X over $[0,L]$ is of no use for prediction since in this case $\mathbb{E}(X_sX_t) = 0$ for $s \in [0,L]$, and the conditioned process then has the same law as the unconditioned process.

[3] i.e. log not $\log^+$

3.1 The conditional covariance

We use $\mathbb{E}_L(.)$ as shorthand for $\mathbb{E}(.|(X_u)_{0\leq u\leq L})$. Then from the tower property we see that

$$\begin{aligned}
&\mathbb{E}((X_t-\mathbb{E}_L(X_t))(X_s-\mathbb{E}_L(X_s)))\\
=\;&\mathbb{E}(\mathbb{E}_L((X_t-\mathbb{E}_L(X_t))(X_s-\mathbb{E}_L(X_s))))\\
=\;&\mathbb{E}_L((X_t-\mathbb{E}_L(X_t))(X_s-\mathbb{E}_L(X_s)))
\end{aligned}$$

and the final equality follows since the conditional covariance of a Gaussian process or field is deterministic, and does not depend on its history. Given $k_t(u)$, we can now compute the conditional covariance in the final line explicitly (for $s,t\in[L,T]$) as

$$\begin{aligned}
R_L(s,t) &:= \mathbb{E}_L((X_t-\mathbb{E}_L(X_t))(X_s-\mathbb{E}_L(X_s)))\\
&= \mathbb{E}((X_t-\mathbb{E}_L(X_t))(X_s-\mathbb{E}_L(X_s))\\
&= \mathbb{E}((X_t-\int_{[0,L]}k_t(u)X_u du)(X_s-\int_{[0,L]}k_s(v)X_v dv))\\
&= R(s,t)\;-\;\int_{[0,L]}k_t(u)R(u,s)du\;-\;\int_{[0,L]}k_s(v)R(v,t)dv\;+\;\int_{[0,L]}\int_{[0,L]}k_t(u)k_s(v)R(u,v)dudv\,.
\end{aligned}$$

4 Application to Gaussian multiplicative chaos

4.1 Rooted measures

Proposition 4.1 *(see also Lemma 2.1 in [Aru17] and Theorems 4 and 17 in [Sha16]). We have the following "master equation" for any bounded continuous function F on $H^{-\frac{1}{2}-\delta}\times[0,T]$ (under the product topology induced by the Hilbert space norm on $H^{-\frac{1}{2}-\delta}$ and the usual Euclidean metric on $[0,T]$):*

$$\frac{1}{T}\mathbb{E}(\int_0^T F(X,t)M_\gamma(dt)) \;=\; \frac{1}{T}\mathbb{E}(\int_0^T F(X+\gamma R(t,.),t)dt\,. \tag{16}$$

Proof. See Appendix C. [4] ■

Corollary 4.2 *M_γ is measurable with respect to X.*

Proof. $\mathcal{H}=H^{-\frac{1}{2}-\delta}\times[0,T]$ is a metric space, so if μ and ν are two finite Borel measures on $\mathcal{H}$ then $\int f d\mu=\int f d\nu$ for all $f\in C_b(\mathcal{H})$ means that $\mu=\nu$, so the left hand hand side of (16) uniquely defines a measure $\mathbb{P}^*$ on $\mathcal{H}\times[0,T]$ which satisfies

$$\frac{1}{T}\mathbb{E}(\int_0^T F(X,t)M_\gamma(dt)) \;=\; \int\int F(\omega,t)\mathbb{P}^*(d\omega,dt)$$

where

$$\begin{aligned}
\mathbb{P}^*(d\omega,dt) \;:=\; \frac{1}{T}\mathbb{E}(1_{X\in d\omega}M_\gamma(\omega,ds)) \;=\; \frac{1}{T}\mathbb{Q}^X(d\omega)M_\gamma(\omega,dt) \;&=\; \frac{1}{T}\mathbb{E}(1_{X+\gamma R(t,.)\in d\omega})dt\\
&=\; \mathbb{P}(X+\gamma R(t,.)\in d\omega)\frac{1}{T}dt
\end{aligned}$$

where $\mathbb{Q}^X$ denotes the law of X on $H^{-\frac{1}{2}-\delta}$.

Moreover, if $F\equiv 1$, $\frac{1}{T}\mathbb{E}(\int_0^T F(X,t)M_\gamma(dt))=1$, so $\mathbb{P}^*(d\omega,dt)$ is a probability measure, known as a *rooted* or *Peyriére measure* (see [Aru17] and [Sha16] for more on this). Moreover, using a similar argument to the third bullet point in Appendix C, we know that the conditional law of $\mathbb{P}^*$ given X is $M_\gamma(dt)/M_\gamma([0,T])$ and from the disintegration theorem, we know that this (probability) measure is a measurable with respect to X. Then using a similar argument to the second bullet point in Appendix C, if we take the sample space Ω to be $H^{-\frac{1}{2}-\delta}$ with σ-algebra $\sigma(H^{-\frac{1}{2}-\delta})$, then the "tilted" probability measure $\mathbb{Q}^X_\gamma(d\omega):=\frac{1}{T}M_\gamma([0,T])\mathbb{Q}^X(d\omega)$ on $(\Omega,\mathcal{F})$ is the marginal law of $\mathbb{P}^*$ on $H^{-\frac{1}{2}-\delta}$ (where $\mathbb{Q}^X$ is the law of X on $H^{-\frac{1}{2}-\delta}$) and $\mathbb{Q}^X_\gamma\ll\mathbb{Q}^X$, so $\frac{1}{T}M_\gamma([0,T])(\omega)$ is the (a.s.) unique Radon-Nikodym derivative of $\mathbb{Q}^X_\gamma$ with respect to $\mathbb{Q}^X$, which is a measurable function of ω. Thus we have shown that $M_\gamma(dt)/M_\gamma([0,T])$ and $M_\gamma([0,T])$ are measurable wrt X and thus so is M_γ. ■

[4] We thank Juhan Aru for his help with multiple parts of this proof.

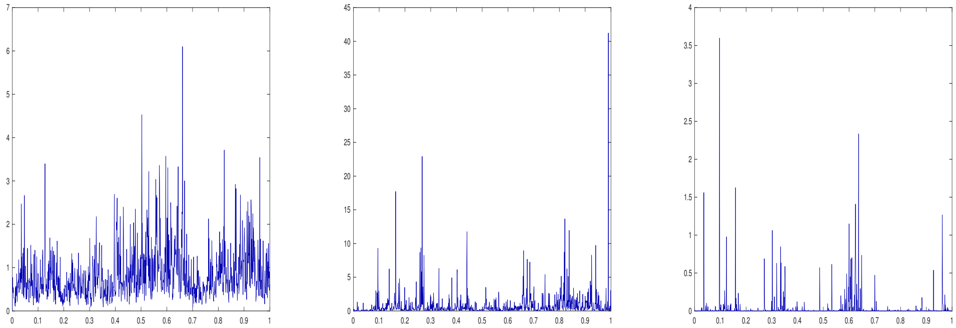

Figure 1: Here we have plotted a Monte Carlo simulation of the multifractal random measure $M_\gamma(dt)$ on $[0,1]$ with $\gamma = 0.20$, 0.45 and 1 using the regularized autocovariance $\log^+ \frac{T}{|t|+\varepsilon}$ for $\varepsilon = .000001$, and we see greater intermittency as γ increases.

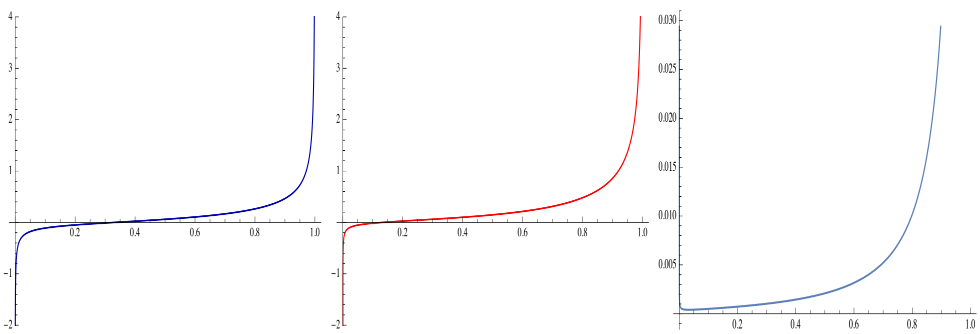

Figure 2: In the first three graphs we have plotted the optimal linear filter $k(u)$ in (14) associated with the multifractal random walk with $L = 1$, $T = 2$ for $t = 2, 1.5$ and 1.00001 respectively, and the numerics confirm that the Wiener-Hopf equation is satisfied (Mathematica code available on request), and $k(u)$ is U-shaped and strictly positive for all $u \in [0, L]$ for t sufficiently small

.

4.2 The conditional law of M_γ

From the Corollary above, $M_\gamma(dt)$ is a measurable wrt X, so M_γ given $(X)_{0\leq s\leq L}$ is just obtained as

$$M_\gamma((X)_{0\leq s\leq L} \oplus X', dt) \tag{17}$$

where $\oplus$ denotes concatenation, and X' is a Gaussian field (which is also a random element of $\mathcal{S}'$) on $[L, T]$ with mean $\mathbb{E}_L(X_t)$ and covariance $R_L(s,t)$. This then uniquely specifies the law of M_γ conditioned on its history over $[0, L]$.

4.3 Conditional law of the Riemann-Liouville field

Formally letting $H \to 0$ in the prediction formula for the Riemann-Liouville process in Proposition 2.9 in [FSV19] in the $H \to 0$ limit, we obtain the following conditional law for the Riemann-Liouville field Z defined in section 2 in [FFGS19]:

Proposition 4.3 *Z has conditional mean and covariance given by*

$$\begin{aligned} \mathbb{E}(Z_u|(Z_v)_{0\leq v\leq t}) &= \int_0^t \bar{k}(s) Z_s ds \\ \mathrm{Cov}(Z_s, Z_u|(Z_v)_{0\leq v\leq t}) &= \int_t^{s\wedge u} (u-v)^{-\frac{1}{2}}(s-v)^{-\frac{1}{2}} dv \end{aligned} \tag{18}$$

for $u \geq t$, where $\bar{k}(s) = \frac{1}{\pi}(\frac{u-t}{t-s})^{\frac{1}{2}} \frac{1}{u-s}$.

Remark 4.1 This is essentially the same type of linear filter that we have obtained in section 3 for the Bacry-Muzy field. To make this rigorous, we can consider $Y_t = e^{Z_t}$; then one can verify that Y

is a strictly stationary field with covariance $R_Y(s,t) = R(\tau) := 2\tanh^{-1}(e^{-\frac{1}{2}|\tau|})$ where $\tau = t - s$, and from Parseval's theorem (similar to Eq 2.1 in [DRV12]) we obtain

$$\int\int \phi(t)\phi(s)R_Y(s,t)dsdt \quad = \quad \int \hat{R}(k)\,|\hat{\phi}(k)|^2 dk$$

where $\hat{R}(k) = \frac{-iH_{-\frac{1}{2}-ik}+iH_{-\frac{1}{2}+ik}+2\pi\tanh(k\pi)}{k\sqrt{2\pi}}$ and H_n denotes the nth harmonic number. Then $\hat{R}(|k|)$ is continuous, strictly positive and decreasing with $\hat{R}(0) < \infty$ and $\hat{R}(|k|) \sim \frac{\sqrt{\pi}}{|k|\sqrt{2}} \sim const. \times (1+|k|^2)^{-\frac{1}{2}}$ as $|k| \to \infty$. Hence (11) still holds with R replaced by R_Y and we can then repeat our previous arguments to make (18) rigorous (after transforming back from Y to Z). In [FFGS19] we define the GMC associated with Z (which we call ξ_γ) and one can show that ξ_γ is also measurable with respect to Z so (17) still holds with M_γ replaced by ξ_γ and X replaced by Z.

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A Proof of Proposition 3.2

$$\begin{aligned}
\mathbb{E}(\|X\|^2_{H^{-\frac{1}{2}-\delta}}) &= \mathbb{E}(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}|\hat{X}_k|^2dk)\\
&= \mathbb{E}(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\hat{X}_k\bar{\hat{X}}_kdk)\\
&= \mathbb{E}(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T e^{ikt}X_tdt\int_0^T e^{-iks}X_sdsdk)\\
&= \mathbb{E}(\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T\int_0^T e^{ik(t-s)}X_sX_t\,dsdtdk)\\
&= \int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T\int_0^T e^{ik(t-s)}R(s,t)\,dsdtdk
\end{aligned}$$

Using that $R\in L^1([0,T]^2)$, we see that $\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}\int_0^T\int_0^T\mathbb{E}(X_sX_t)dsdtdk=\int_0^T\int_0^T R(s,t)dsdt\cdot\int_{-\infty}^{\infty}(1+|k|^2)^{-\frac{1}{2}-\delta}dk<\infty$ iff $\delta>0$, so by Fubini we have

$$\begin{aligned}
\mathbb{E}(\|X\|^2_{H^{-\frac{1}{2}-\delta}}) &= \mathbb{E}(\int_0^T\int_0^T R(s,t)\int_{-\infty}^{\infty}e^{ik(t-s)}(1+|k|^2)^{-\frac{1}{2}-\delta}dkdsdt)\\
&= 2c_\delta\int_0^T\int_0^t R(s,t)(t-s)^\delta\mathrm{BesselK}(\delta,t-s)dsdt\\
&\le c_\delta\int_{[0,T]^2}R(s,t)dsdt \quad < \quad \infty
\end{aligned} \tag{A-1}$$

where we have used that the Fourier transform of $\hat{f}(k) := (1+|k|^2)^{-\frac{1}{2}-\delta}$ is $f(t) = c_\delta |t|^\delta \mathrm{BesselK}(\delta,|t|)$ for some real constant c_δ, and that $t^\delta \mathrm{BesselK}(\delta,t)$ is bounded on $[0,T]$ if $\delta > 0$. For $\delta \le 0$, the integrand in the triple integral in the first line is not absolutely integrable.

B Proof of Proposition 3.3

Using that

$$\begin{aligned} \chi(s,t,\varepsilon,\varepsilon_2) \;:\; = \mathbb{E}((X_t^{\varepsilon_2}-X_t^\varepsilon)(X_s^{\varepsilon_2}-X_s^\varepsilon)) &= R_{\varepsilon_2}(s,t) - \mathbb{E}(X_s^{\varepsilon_2}X_t^\varepsilon) - \mathbb{E}(X_s^\varepsilon X_t^{\varepsilon_2}) + R_\varepsilon(s,t) \;\to\; 0 \\ &= R_{\varepsilon_2}(s,t) - \mathbb{E}(X_s^{\varepsilon\vee\varepsilon_2}X_t^{\varepsilon\vee\varepsilon_2}) - \mathbb{E}(X_s^{\varepsilon\vee\varepsilon_2}X_t^{\varepsilon\vee\varepsilon_2}) + R_\varepsilon(s,t) \end{aligned}$$

as $\varepsilon,\varepsilon_2 \to 0$ and that $|\chi(s,t,\varepsilon,\varepsilon_2)| \le 4R(s,t)$, we can use a similar argument to (A-1) and the dominated convergence theorem to show that

$$\mathbb{E}(\|X^{\varepsilon_2}-X^\varepsilon\|^2_{H^{-\frac{1}{2}-\delta}}) \;\le\; c_\varepsilon \int_{[0,T]^2} \chi(s,t,\varepsilon,\varepsilon_2)\,ds\,dt \;\to\; 0 \tag{B-1}$$

as $\varepsilon,\varepsilon_2 \to 0$, so X^ε is a Cauchy sequence in the Hilbert space $L^2(\Omega,\mathcal{F},\mathbb{P};H^{-\frac{1}{2}-\delta})$ of $H^{-\frac{1}{2}-\delta}$-valued random variables X with $\mathbb{E}(\|X\|^2_{H^{-\frac{1}{2}-\delta}}) < \infty$, and thus converges in this space. Using that

$$\mathbb{P}(\|X^{\varepsilon_2}-X^\varepsilon\|_{H^{-\frac{1}{2}-\delta}} > k) \;\le\; \frac{1}{k^2}\mathbb{E}(\|X^{\varepsilon_2}-X^\varepsilon\|^2_{H^{-\frac{1}{2}-\delta}})$$

the claim is proved.

C Proof of Proposition 4.1

Similar to the analysis before Lemma 2.1 in [Aru17] with rooted measures, we let $D=[0,T]$ and we can define a sequence of approximate "rooted" probability measures $\mathbb{P}_\varepsilon^*$ on $D \times H^{-\frac{1}{2}-\delta}$ as

$$\mathbb{P}_\varepsilon^*(dt,d\omega) \;=\; \frac{dt}{\mathrm{Leb}(D)}\, e^{\gamma\omega(t)-\frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}\mathbb{Q}^{X^\varepsilon}(d\omega)$$

where $\mathbb{Q}^{X^\varepsilon}$ denotes the law of X^ε on $H^{-\frac{1}{2}-\delta}$, and X^ε is defined as in (3). Then

- The marginal law on D is

  $$\frac{dt}{\mathrm{Leb}(D)}\,\mathbb{E}^{\mathbb{Q}^{X^\varepsilon}}(e^{\gamma\omega(t)-\frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}) \;=\; \frac{dt}{\mathrm{Leb}(D)}$$

  i.e. the uniform probability measure on D.

- The marginal law on $H^{-\frac{1}{2}-\delta}$ is $\frac{\int_D e^{\gamma\omega(t)-\frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}dt}{\mathrm{Leb}(D)}\mathbb{Q}^{X^\varepsilon}(d\omega) = \frac{M_\gamma^\varepsilon(D)}{\mathrm{Leb}(D)}\mathbb{Q}^{X^\varepsilon}(d\omega)$, i.e. the law of X^ε tilted by $M_\gamma^\varepsilon(D)/\mathrm{Leb}(D)$.

- The conditional law on D given ω is the probability measure: $\frac{e^{\gamma\omega(t)-\frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}}{M_\gamma^\varepsilon(D)}\,dt = \frac{M_\gamma^\varepsilon(dt)}{M_\gamma^\varepsilon(D)}$.

- The conditional law on $H^{-\frac{1}{2}-\delta}$ given t is $e^{\gamma\omega(t)-\frac{1}{2}\gamma^2\mathbb{E}(X_\varepsilon^2)}\mathbb{Q}^{X^\varepsilon}(d\omega)$. From Girsanov's theorem (see e.g. section 6.1 in [Var17]), we can re-write this as

  $$\mathbb{Q}(X^\varepsilon + \gamma R_\varepsilon(.,t) \in d\omega) \tag{C-1}$$

Thus we can sample from $\mathbb{P}_\varepsilon^*$ by either (i) sampling from $\frac{M_\gamma^\varepsilon(D)}{\mathrm{Leb}(D)}\mathbb{Q}^{X^\varepsilon}(d\omega)$ and then sampling a point according to $M_\gamma^\varepsilon(dt)/M_\gamma^\varepsilon(D)$, or ii) sampling t from the uniform measure on $[0,T]$, and then sampling $X^\varepsilon + \gamma R_\varepsilon(.,t)$, with X^ε independent of t. Combining these two prescriptions, we see that

$$\mathbb{E}(\frac{M_\gamma^\varepsilon(D)}{\mathrm{Leb}(D)}\int_0^T F(X^\varepsilon,t)\frac{M_\gamma^\varepsilon(dt)}{M_\gamma^\varepsilon(D)}) \;=\; \frac{1}{\mathrm{Leb}(D)}\mathbb{E}(\int_0^T F(X^\varepsilon + \gamma R_\varepsilon(t,.),t)dt)$$

which we can re-write as

$$\mathbb{E}(\int_0^T F(X^\varepsilon,t)\,M_\gamma^\varepsilon(dt)) \;=\; \mathbb{E}(\int_0^T F(X^\varepsilon + \gamma R_\varepsilon(t,.),t)dt)\,. \tag{C-2}$$

We first consider the left hand side of this expression as $\varepsilon \to 0$. To begin with, we note that

$$\begin{aligned}
& |\mathbb{E}(\int_0^T F(X^\varepsilon,t)M_\gamma^\varepsilon(dt) - \int_0^T F(X,t)M_\gamma(dt))| \\
\leq \quad & |\mathbb{E}(\int_0^T (F(X^\varepsilon,t) - F(X,t))M_\gamma^\varepsilon(dt))| + |\mathbb{E}(\int_0^T F(X,t)(M_\gamma^\varepsilon(dt) - M_\gamma(dt)))| \qquad \text{(C-3)}
\end{aligned}$$

and we can bound the first term in the final expression using Hölder's inequality as

$$\begin{aligned}
\mathbb{E}(\int_0^T (F(X^\varepsilon,t) - F(X,t))M_\gamma^\varepsilon(dt)) \quad & \leq \quad \mathbb{E}(\sup_{t\in[0,T]} |F(X^\varepsilon,t) - F(X,t)| \cdot M_\gamma^\varepsilon([0,T])) \\
& \leq \quad \mathbb{E}((\sup_{t\in[0,T]} |F(X^\varepsilon,t) - F(X,t)|)^p)^{\frac{1}{p}} \cdot \mathbb{E}((M_\gamma^\varepsilon([0,T])^q)^{\frac{1}{q}} \quad \text{(C-4)}
\end{aligned}$$

for $1/p + 1/q = 1$, and from (2) we know that

$$\mathbb{E}((M_\gamma^\varepsilon([0,T])^q) \quad = \quad c_q T^{\zeta(q)} \quad < \quad \infty$$

for any $q \in (1, q^*) = \frac{2}{\gamma^2}$.

We claim that $\sup_{t\in[0,T]} |F(X^\varepsilon,t) - F(X,t)| \to 0$ a.s. Indeed, suppose to the contrary. Let $f_\varepsilon(t) := F(X^\varepsilon,t)$ and $f(t) := F(X,t)$. If the claim is false, f_ε does not tend to f uniformly on $[0,T]$, so there exists a sequence $\varepsilon_n \to 0$, a $\delta > 0$ and a sequence $t_n \in [0,T]$ such that

$$|f_{\varepsilon_n}(t_n) - f(t_n)| \quad \geq \quad \delta \qquad \text{(C-5)}$$

for all $n \in \mathbb{N}$. But by Bolzano-Weierstrass, we can choose a convergent subsequence (t_{n_k}) of (t_n) with $t_{n_k} \to t_\infty \in [0,T]$. Then $f_{\varepsilon_{n_k}}(t_{n_k}) = F(X^{\varepsilon_{n_k}}, t_{n_k})$ and $f(t_{n_k}) = F(X, t_{n_k})$. From Proposition 3.3 we know that X^ε tends to X in $H^{-\frac{1}{2}-\delta}$ in probability, and thus almost surely along a further subsequence $\varepsilon_{n_{k_j}}$, thus (by continuity of F in both arguments) $F(X^{\varepsilon_{n_{k_j}}}, t_{n_{k_j}}) \to F(X, t_\infty)$ a.s. and hence

$$|F(X^{\varepsilon_{n_{k_j}}}, t_{n_{k_j}}) - F(X, t_{n_{k_j}})| \quad = \quad |f_{\varepsilon_{n_{k_j}}}(t_{n_{k_j}}) - f(t_{n_{k_j}})| \to 0 \qquad \text{(C-6)}$$

a.s., which violates (C-5). Hence the right hand side of (C-4) tends to zero (along *any* subsequence) for $q \in (1, q^*)$

The term $\int_0^T F(X,t)(M_\gamma^\varepsilon(dt) - M_\gamma(dt))$ inside the expectation on the right hand side of (C-3) converges to zero a.s. since M_γ^ε tends weakly to M_γ a.s. (see top of page 3 for details) and the random $F(X,t)$ is continuous in t for each ω. Moreover

$$\int_0^T F(X,t)(M_\gamma^\varepsilon(dt) - M_\gamma(dt)) \quad \leq \quad \|F\|_\infty (M_\gamma^\varepsilon([0,T]) + M_\gamma([0,T]))\,.$$

From (6) we also know that $M_\gamma^\varepsilon([0,T])$ is uniformly integrable, so by e.g. the Theorem in section 13.7 in [Wil91], the rightermost term of (C-3) tends to zero

Finally, the right hand side of (C-2) converges by the a.s. convergence of X^ε to X in Proposition 3.3 and the bounded convergence theorem.