

# Homework 1

Let  $(W_t)_{t \geq 0}$  be a standard Brownian motion throughout.

1. Write down a formula for  $u(x, t) := \mathbb{E}(f(W_T) | W_t = x)$  for a general function  $f$ .

**Solution.** From the definition of BM, the conditional distribution of  $W_T$  given  $W_t = x$  is  $N(x, T - t)$ , so

$$\mathbb{E}(f(W_T) | W_t = x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} f(y) dy = (p_{T-t} * f)(x)$$

where  $p_t(\cdot)$  is the density of  $W_t$  and  $*$  denotes **convolution** (the convolution of two functions  $f$  and  $g$  is  $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$ . Note by setting  $x - y = u$  so  $y = x - u$  and  $dy = -du$ , we see that  $(f * g)(x) = -\int_{\infty}^{-\infty} f(x-u)g(u)du = \int_{-\infty}^{\infty} f(x-u)g(u)du = (g * f)(x)$ , i.e.  $f * g = g * f$ .

2. **Simulating random variables with a given distribution.** Let  $X$  be a random variable with a continuous strictly increasing distribution function  $F_X(x)$ . What is the distribution of  $F_X^{-1}(U)$ , where  $U$  is a standard Uniform random variable on  $[0, 1]$ ? Explain the significance of this result.

**Solution.**

$$\mathbb{P}(F_X^{-1}(U) \leq x) = \mathbb{P}(U \leq F_X(x)) = F_X(x) \quad (1)$$

since  $F_X(F_X^{-1}(x)) = x$ , i.e.  $F_X^{-1}(U) \sim X$ , where we have used that  $\mathbb{P}(U \leq x) = x$ . So the conclusion here is that  $F_X^{-1}(U)$  has the same distribution as  $X$ . This is how we typically generate a random variable with a given distribution in practice on a computer.

3. Let

$$B_t = (1-t)W_{\frac{t}{1-t}}$$

for  $0 \leq t < 1$ . Compute  $\mathbb{E}(B_s B_t)$  for  $0 < s < t < 1$ . What do you notice at  $t = 1$ ?

**Solution.**  $\mathbb{E}(B_s B_t) = (1-s)(1-t)\frac{s}{1-s} = s(1-t)$ . We note that  $\mathbb{E}(B_1^2) = 0$ , hence  $B_1 = 0$  a.s.  $B$  is known as the **Brownian bridge** (see Figure 1 below), which is Brownian motion conditioned to be at zero at time 1.

4. Let  $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$  (this is the famous **Black-Scholes model**). Compute  $\mathbb{E}(S_t^p)$  (hint: re-write  $S_t^p$  as  $S_0^p e^{pX_t}$  where  $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ ).

**Solution.**

$$\mathbb{E}(S_t^p) = S_0^p \mathbb{E}(e^{pX_t}) = S_0^p \mathbb{E}(e^{p((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)})$$

But  $(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \sim N(\mu_1, \sigma_1^2)$ , where  $\mu_1 = (\mu - \frac{1}{2}\sigma^2)t$  and  $\sigma_1^2 = \sigma^2 t$ . Thus

$$\mathbb{E}(S_t^p) = S_0^p e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2} = S_0^p e^{(\mu - \frac{1}{2}\sigma^2)pt + \frac{1}{2}\sigma^2 p^2 t}$$

where we have used that the mgf of a general Normal  $N(\mu_1, \sigma^2)$  random variable is  $e^{\mu_1 p + \frac{1}{2}\sigma^2 p^2}$  from Applied Probability Revision chapter. Note that  $\mathbb{E}(S_t^p) < \infty$  for all  $p \in \mathbb{R}$ , i.e. all moments of  $S_t$  are finite.

5. Let  $X_t = \sum_{i=1}^n (W_t^{(i)})^2$ , where  $W^{(i)}$  are  $n$  independent standard Brownian motions. Using that  $\mathbb{E}(e^{-\lambda Z^2}) = \frac{1}{(1+2\lambda)^{\frac{1}{2}}}$  for  $\lambda > 0$  where  $Z \sim N(0, 1)$ , compute  $\mathbb{E}(e^{-\lambda X_t})$  for  $\lambda > 0$ .  $X$  is known as a **Bessel squared process** of dimension  $n$ .

**Solution.** Using that  $B_t^{(i)} \sim \sqrt{t}Z$  and the independence of the  $\delta$ -Brownian motions, we see that

$$\mathbb{E}(e^{-\lambda X_t}) = \prod_{i=1}^{\delta} \mathbb{E}(e^{-\lambda (B_t^{(i)})^2}) = \prod_{i=1}^{\delta} \mathbb{E}(e^{-\lambda (\sqrt{t}Z)^2}) = (\mathbb{E}(e^{-\lambda t Z^2}))^{\delta} = \frac{1}{(1+2\lambda t)^{\frac{1}{2}\delta}}.$$

**6. Portfolio optimization, relevant for two of the summer projects.** Consider a financial market defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $d$  assets with **random payoffs**  $(\pi^1, \dots, \pi^d)$  at time  $T$  (which are **linearly independent**) with market prices  $p_i$  at  $t = 0$  (these assets can include **European call/put options**). Let  $Y_i = \pi_i - p_i$ , and assume a financial agent can only trade at time zero.

Derive the first order optimality condition for an agent to maximize their **expected utility**  $\mathbb{E}(U(b \cdot Y))$  over  $b \in \mathbb{R}^d$ , where  $U : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function with  $U''(x) < 0$  and  $b_i$  is the position in the  $i$ 'th asset and we assume that  $\mathbb{E}(U(b \cdot Y)) < \infty$  for all  $b \in \mathbb{R}^d$ .

**Solution.** As in first year calculus, we set compute derivatives wrt each  $b_i$  and then set the answer to zero:

$$\frac{\partial}{\partial b_i} \mathbb{E}(U(b \cdot Y)) = \mathbb{E}(Y_i U'(b \cdot Y)) = 0$$

for  $i = 1..d$ , i.e. we have  $d$  equations for the  $d$  unknowns  $b_1^*, \dots, b_d^*$  for the optimal portfolio allocation  $b^*$ . Note we can re-write this as

$$\mathbb{E}^{\mathbb{Q}}(Y_i) = 0 \quad (2)$$

where we define a **new probability measure** as  $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left(\frac{U'(b^* \cdot Y)}{\mathbb{E}^{\mathbb{P}}(U'(b^* \cdot Y))} 1_A\right)$  for events  $A \in \mathcal{F}$ , and (as a sanity check) we note that  $\mathbb{Q}(\Omega) = 1$ .

Under the moment condition stated in the question, it turns out that a unique solution  $b^*$  exists if the **no-arbitrage** condition is satisfied: if  $\mathbb{P}(b \cdot Y > 0) > 0$  then  $\mathbb{P}(b \cdot Y < 0) > 0$ . In this case, from (2), we see that under  $\mathbb{Q}$ , all contracts are priced according to the market, i.e.  $\mathbb{E}^{\mathbb{Q}}(\pi^i) = p_i$ .  $\mathbb{Q}$  is known as a **risk-neutral measure**. We can solve this maximization problem numerically using e.g. MOSEK convex optimization package in Python (used for summer project).

A common choice is the **exponential utility function**  $U(x) = -e^{-\lambda x}$ , in which case (for  $\lambda = 1$ ) we are computing  $\max_b (-\mathbb{E}(e^{-b \cdot Y})) = \max_b (-\mathbb{E}(e^{b \cdot Y})) = -\min_b \mathbb{E}(e^{b \cdot Y})$ , i.e. minus the **minimum of the mgf** of  $Y$ .

For the 1d case  $d = 1$ , if  $\pi_1 = f(S)$  with market price  $p$  and  $S$  has density  $p(S)$ , then we can re-write expected utility as

$$-\int_0^\infty e^{-\lambda b(f(S)-p)} p(S) dS.$$

We can evaluate this integral explicitly in certain cases, e.g. if  $S \sim \text{Exp}(1)$  and  $f(S) = S$ , we see that

$$\mathbb{E}(U(b(S-p))) = \mathbb{E}(-e^{-\lambda b(S-p)}) = \int_0^\infty -e^{-\lambda b(S-p)} e^{-S} dS = -\frac{e^{pb\lambda}}{1+b\lambda}$$

if  $\lambda b + 1 > 0$ , and  $-\infty$  otherwise. Differentiating this expression wrt  $b$  and setting the answer to zero, we find that

$$b^* = \frac{1-p}{p\lambda} \quad (3)$$

and the risk-neutral density  $\mathbb{Q}$  corresponding to  $b^*$  is the density of a  $\text{Exp}(\frac{1}{p})$  random variable, under which  $\mathbb{E}^{\mathbb{Q}}(S) = p$  as claimed.

Note the “fair price” of the stock is  $\int_0^\infty S e^{-S} dS = \mathbb{E}(S) = 1$ , so (3) says that we **buy** stock when the stock is **underpriced**, and **sell** when the stock is **overpriced**, and  $|b^*|$  is smaller when  $\lambda$  is larger i.e. when the trader is more risk-averse.

We can also modify the problem to account for interest rates, **bid-ask spreads**, **finite liquidity** or a **limit order book** structure.

**7.** Compute the **conditional distribution** of  $W_t$  given the known value of  $W_s$ , for  $s, t > 0$  (you do not have to assume that  $s < t$ ). You may use that for two correlated Normal random variables  $X \sim N(\mu_X, \sigma_X^2)$  and  $Y \sim N(\mu_Y, \sigma_Y^2)$  with  $\text{Corr}(X, Y) = \rho$ ,

$$Y|X \sim N(\mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (X - \mu_X), (1 - \rho^2) \sigma_Y^2)$$

and recall that the correlation of two random variables  $X$  and  $Y$  is defined as  $\text{Corr}(X, Y) = \frac{\mathbb{E}((X - \mu_X)(Y - \mu_Y))}{\sigma_X \sigma_Y}$ .

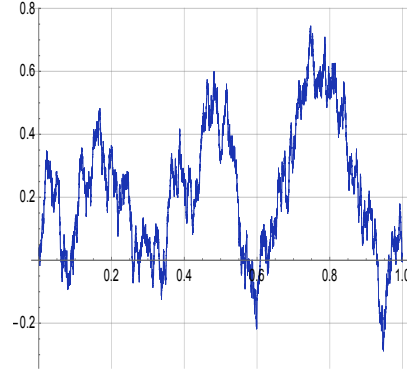


Figure 1: Simulation of a Brownian bridge on  $[0, 1]$ .

**Solution.** For our case here,  $X = W_s$ ,  $Y = W_t$ ,  $\mu_X = 0$ ,  $\mu_Y = 0$ ,  $\sigma_X = \sqrt{s}$ ,  $\sigma_Y = \sqrt{t}$  and  $\rho = \frac{\min(s,t)}{\sqrt{st}}$ , and recall that we have shown in the lecture notes that  $\mathbb{E}(W_s W_t) = \min(s, t)$ . Thus

$$W_t | W_s \sim N\left(\rho \frac{\sqrt{t}}{\sqrt{s}} W_s, (1 - \rho^2)t\right).$$

8. Compute  $\mathbb{E}(W_t^3 | W_s = x)$  for  $0 \leq s \leq t$ .

**Solution.**  $W_t - W_s \sim N(0, t - s)$ , so

$$\begin{aligned} \mathbb{E}((x + W_t - W_s)^3 | W_s = x) &= \mathbb{E}(x^3 + 3x^2(W_t - W_s) + 3x(W_t - W_s)^2 + (W_t - W_s)^3 | W_s = x) \\ &= x^3 + 3x(t - s). \end{aligned}$$

We can generalize this computation to compute  $\mathbb{E}(W_t^n | W_s = x)$  for any  $n \in \mathbb{N}$ , since (from the **binomial theorem**) we know that  $(x + W_t - W_s)^n = \sum_{i=0}^n x^{n-i} (W_t - W_s)^i \binom{n}{i}$ , and we also know that all odd moments of  $W_t - W_s$  are zero.