Homework 6

1. From a famous result called the **reflection principle**, it is known that $\mathbb{P}(M_t > b) = 2\mathbb{P}(W_t > b)$, where $M_t = \max_{0\le s\le t} W_s$. Can Brownian motion remain non-negative over a non-zero interval $[0,\delta]$? Compute the cdf of $m_t = \min_{0\le s\le t} W_s$.

Solution. $\mathbb{P}(M_t > 0) = 1$, so (setting $t = \delta$) we see that W sets a new max over $[0,\delta]$. But by symmetry, $\mathbb{P}(m_t < 0) = 1$ as well, so W sets a new minimum as well a.s., so the claim is false. For the final part, let $b \le 0$. Then using that $m_t \sim -M_t$, we see that

$$\mathbb{P}(m_t \le b) \quad = \quad \mathbb{P}(-M_t \le b) \quad = \quad \mathbb{P}(M_t \ge -b) \quad = \quad 2\mathbb{P}(W_t \ge -b) \quad = \quad 2\Phi^c(\frac{|b|}{\sqrt{t}})\,.$$

2. Let $\theta_t = \max\{s : W_s = M_t\}$ denote the **last time** that W achieved its current maximum value M_t. The cdf of θ_t at s is given by

$$F(s) \quad = \quad \mathbb{P}(\theta_t \le s) \quad = \quad \frac{2}{\pi}\arcsin\frac{\sqrt{s}}{\sqrt{t}}$$

for t fixed (this is known as the **arcsine rule**), proof not asked for here. Explain how we can simulate θ_t.

Solution. We use the usual $F^{-1}(U)$ method from the first lecture where $U \sim U[0,1]$. In this case

$$F^{-1}(U) \quad = \quad (\sin(\frac{\pi U}{2})\sqrt{t})^2 \quad = \quad \sin^2(\frac{\pi U}{2})\,t$$

and note that $F^{-1} : [0,1] \to [0,t]$.

Remark 0.1 The density of θ_t is $\frac{d}{ds}F(s) = \frac{1}{\pi\sqrt{s(t-s)}}$ for $0 < s < t$, which is a special case of a beta distribution.

3. Explain how to simulate two random variables X and Y with joint density $f(x,y)$.

Solution. Simulate X in the usual way as $F_X^{-1}(U)$. Simulate Y using $F_{Y|X}^{-1}(U_2)$ where U_1, U_2 are i.i.d. $U[0,1]$, where $F_{Y|X}$ is the conditional cdf of Y given X. To compute $F_{Y|X}$, recall that conditional density of Y given X is

$$f(y|x) \quad = \quad \frac{f(x,y)}{p(x)}$$

where $p(x)$ is marginal density of X. The conditional cdf is then given by $F(y|x) = \int_{-\infty}^{y} f(u|x)du$.

4. Compute the asymptotic distribution of $\Pi_t^n = \sum_{i=0}^{n-1}(W_{\frac{(i+1)t}{n}} - W_{\frac{it}{n}})^2$ as $n \to \infty$.

Solution. Let $\Delta W_i = W_{i\Delta t} - W_{(i-1)\Delta t}$ where $\Delta t = \frac{t}{n}$. Then from the Brownian motion chapter we saw that

$$\Pi_t^n \quad = \quad \sum_{i=0}^{n-1}(W_{\frac{(i+1)t}{n}} - W_{\frac{it}{n}})^2 \quad = \quad \sum_{i=1}^{n}(\Delta W_i)^2 \quad \sim \quad \sum_{i=1}^{n}(\sqrt{\Delta t}\,Z_i)^2 \quad = \quad \Delta t\sum_{i=1}^{n} Z_i^2 \quad = \quad \frac{t}{n}\sum_{i=1}^{n} Z_i^2$$

where $Z_1, Z_2, ...$ is an i.i.d sequence of $N(0,1)$ random variables. Then from the Strong Law of Large Numbers (SLLN), $\Pi_1^t \to t$ a.s. as $n \to \infty$. Moreover, we also know that

$$\mathrm{Var}(\sum_{i=1}^{n} Z_i^2) \quad = \quad \sum_{i=1}^{n}\mathrm{Var}(Z_i^2) \quad = \quad 2n$$

since $\mathbb{E}((Z_i^2)^2) = \mathbb{E}(Z_i^4) = 3$ and $\mathbb{E}(Z_i^2) = 1$, so $\mathrm{Var}(Z_i^2) = 3 - 1 = 2$. Then

$$\mathrm{Var}(\sum_{i=1}^{n}(\Delta W_i)^2) \quad = \quad \mathrm{Var}(\sum_{i=1}^{n}(\sqrt{\Delta t}\,Z_i)^2) \quad = \quad n\mathrm{Var}(\Delta t\,Z_i^2) \quad = \quad n(\Delta t)^2 \cdot 2 \quad = \quad 2n(\frac{t}{n})^2 \quad = \quad \frac{2t^2}{n}\,.$$

Moreover

$$\sqrt{n}\,(\sum_{i=1}^{n}(\Delta W_i)^2 - t) \quad \sim \quad \sqrt{n}\,(\sum_{i=1}^{n}\Delta t Z_i^2 - t) \quad = \quad t\sqrt{n}\,(\frac{1}{n}\sum_{i=1}^{n} Z_i^2 - 1) \qquad (1)$$

and from the **Central Limit Theorem** the right hand side tends weakly to a $N(0, 2t^2)$ random variable. Hence for n large we can approximately say that $\sum_{i=1}^{n}(\Delta W_i)^2 - t \sim \frac{1}{\sqrt{n}}N(0, 2t^2)$

5. Recall from Hwk 5 that a symmetric α-stable process X with parameters $\alpha \in (0,2]$, $\sigma > 0$ is a generalization of Brownian motion, which has i.i.d. increments like Brownian motion and $\mathbb{E}(e^{iu(X_t-X_s)}|X_s) = \mathbb{E}(e^{iu(X_t-X_s)}) = e^{-(t-s)\sigma^\alpha|u|^\alpha}$ for $u \in \mathbb{R}$, $0 \le s \le t$, and we assume $X_0 = 0$. Explain how to estimate (α, σ) using **linear regression** if we have observations of X_1, X_2,..., X_n.

Solution. Let $\Delta X_i = X_i - X_{i-1}$ for $i = 1...n$; then the ΔX_i's are i.i.d. and $\mathbb{E}(e^{iu\Delta X_i}) = |\mathbb{E}(e^{iu\Delta X_i})| = e^{-\sigma^\alpha|u|^\alpha}$ for $u \in \mathbb{R}$. Moreover if we define $\hat{\phi}_n(u) = \frac{1}{n}\sum_{i=1}^n e^{iu\Delta X_i}$, then $|\mathbb{E}(\hat{\phi}_n(u))| = |\mathbb{E}(e^{iu\Delta X_1})| = e^{-\sigma^\alpha|u|^\alpha}$ for all $i = 1...n$, and taking logs we see that

$$\log(-\log|\mathbb{E}(\hat{\phi}_n(u))|) \quad = \quad \alpha(\log\sigma + \log|u|)\,.$$

Then to estimate (α, σ), we remove the expectation from the LHS and perform linear regression on $f(u) := \log(-\log|\hat{\phi}_n(u)|)$ versus $\log|u|$ on a grid of u-values $(u_j)_{j=1...m}$, which minimizes $\sum_{j=1}^m (y_j - a - bx_j)^2$ with $x_j = \log|u_j|$ and $y_j = f(u_j)$. Then the slope will then be $b = \hat{\alpha}$ and the intercept will be $a = \hat{\alpha}\log\hat{\sigma}$.

6. **Fundamental Theorem of Asset Pricing**. Consider a general stock price process S of the form

$$dS_t \quad = \quad \mu_t dt \,+\, \sigma_t dW_t$$

with $r = 0$ where μ_t and σ_t are $\mathcal{F}_t^W$-adapted and $\int_0^T(|\mu_u|+\sigma_u^2)du < \infty$ a.s. Assume there exists another probability measure $\mathbb{Q}$ **equivalent to** $\mathbb{P}$ such that S is a martingale under $\mathbb{Q}$. Show there is no-arbitrage in the model under this general condition.

Solution. Let $V_t = \int_0^t \phi_u dS_u$ with $\mathbb{E}^{\mathbb{Q}}(\int_0^T \phi_u^2\sigma_u^2 du) < \infty$. Assume the strategy is self-financing and starts from zero initial wealth, so $V_0 = 0$. There is an arbitrage if there exists an $\mathcal{F}_t$-adapted process ϕ such that $\mathbb{P}(V_T \ge 0) = 1$ and $\mathbb{P}(V_T > 0) > 0$. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent we also have $\mathbb{Q}(V_T \ge 0) = 1$ and $\mathbb{Q}(V_T > 0) > 0$, which implies that $\mathbb{E}^{\mathbb{Q}}(V_T) > 0$.

But S is a $\mathbb{Q}$-martingale, and (for an admissible trading strategy ϕ) $V_t = \int_0^t \phi_u\, dS_u$ is also a $\mathbb{Q}$-martingale with $V_0 = 0$. Hence $\mathbb{E}^{\mathbb{Q}}(V_T) = V_0 = 0$, so $V_T = 0$ $\mathbb{Q}$-a.s. This contradicts $\mathbb{Q}(V_T > 0) > 0$, so no such arbitrage strategy can exist.

Remark 0.2 The density $p(x)$ of $\chi = \sum_{i=1}^n Z_i^2$ is χ^2 with n degrees of freedom, and has the explicit formula $p(x) = \frac{1}{2^{n/2}\Gamma(\frac{1}{2}n)}x^{\frac{1}{2}n-1}e^{-\frac{1}{2}x}$ for $x > 0$.

7. A Generalized Brownian bridge $\hat{B}$ on $[0,T]$ is a Brownian motion B conditioned to end at b at time T. Compute the density of $\hat{B}_t$ for $t \in [0,1)$.

Solution. Let $p_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ denote the density of Brownian motion at time t. Then from Bayes rule for conditional densities

$$\mathbb{P}(\hat{B}_t \in dx) \quad = \quad \frac{\mathbb{P}(B_t \in dx, B_T \in db)}{\mathbb{P}(B_T \in db)} \quad = \quad \frac{p_t(x)\,p_{T-t}(b-x)}{p_T(b)}dx\,.$$

If we tidy this expression up further, we recover a Normal density for $\hat{B}_t$.

8. From e.g. Hwk4 q2, we know that

$$\mathbb{P}(W_t \in dx, M_t < b, m_t > a) \quad = \quad \frac{2}{b-a}\sum_{n=1}^\infty e^{-\lambda_n t}\sin(\frac{n\pi(x-a)}{b-a})\sin(\frac{n\pi(0-a)}{b-a})dx \tag{2}$$

for $a < x < b$, where $\lambda_n = \frac{n^2\pi^2}{2(b-a)^2}$, and M_t and m_t are the running max and min processes of W. Use this to explicitly compute the **conditional** cdf of the **two-sided maximum** $R_t := \max_{0\le s\le t}|W_s|$ of W at time t, **given that** $W_t = 0$.

Solution. Setting $b = r$ and $a = -r$ we did in Hwk2 q2, $b - a = 2r$ and we can re-write (2) as

$$\mathbb{P}(W_t \in dx, M_t < r, m_t > -r) \quad = \quad \mathbb{P}(W_t \in dx, R_t < r) \quad = \quad \frac{1}{r}\sum_{n=1}^\infty e^{-\lambda_n t}\sin(\frac{n\pi(x+r)}{2r})\sin(\frac{n\pi(0+r)}{2r})dx$$

for $x \in (-r,r)$, where $\lambda_n = \frac{n^2\pi^2}{8r^2}$. Then since $\{M_t < r\} \cup \{m_t > -r\} = \{R_t < r\}$, using Bayes' formula and setting $x = 0$ in (3), we see that

$$\mathbb{P}(R_t < r|W_t = 0) \quad = \quad \frac{\mathbb{P}(R_t < r, W_t \in dx)}{\mathbb{P}(W_t \in dx)}|_{x=0} \quad = \quad \frac{\frac{1}{r}\sum_{n=1}^\infty e^{-\lambda_n t}\sin(\frac{1}{2}n\pi)^2}{\frac{1}{\sqrt{2\pi t}}} \quad = \quad \frac{\sqrt{2\pi t}}{r}\sum_{k=1}^\infty e^{-\lambda_{2k-1}t} \tag{3}$$

where we have set $n = 2k-1$ for $k \in \mathbb{N}$, and used that $\sin(\frac{1}{2}n\pi)$ vanishes for n even (see final plot below).

9. Let $X_1, ..., X_n$ denote a sequence of i.i.d. random variables with common cdf F, and let $F_n(x) = \frac{1}{n}\sum_{i=1}^n 1_{X_i \leq x}$ denote the **empirical cdf**.

It can be shown that

$$\hat{B}_t^{(n)} \quad = \quad \sqrt{n}\,(F_n(F^{-1}(t)) - t)$$

(defined for $t \in [0,1]$) tends to a (zero-mean) Gaussian process $\hat{B}$ with covariance function $R(s,t) = \mathbb{E}(\hat{B}_s\hat{B}_t) = s(1-t)$ (for $0 \leq s \leq t \leq 1$) as $n \to \infty$. Recall we have seen the Gaussian process with this covariance function before in Hwk 1 which can be realized as $\hat{B}_t = (1-t)W_{\frac{t}{1-t}}$ where W is a standard BM. We don't prove it here, but one can show that $\hat{B}$ can also be realized as a Brownian motion conditioned to be zero at time 1 (i.e. the same process considered in q7+8 when the final time $T = 1$).

If the observed value of $D_n := \max_{t\in[0,1]}|\hat{B}_t^{(n)}| = R$ for a sequence of observations $X_1, ..., X_n$ with n large, explain how we can test the null hypothesis H_0 that the sequence (X_i) is i.i.d. with cdf F, at the 5% significance level.

Solution. If H_0 is true, then from the stated asymptotic result, D_n has approximately the same distribution as $\max_{t\in[0,1]}|\hat{B}_t|$, which is the same as the two-sided maximum R_t in Eq (3) in q8 if we set $t = 1$ there. Hence if $\mathbb{P}(R_1 > R) < .05$, we reject H_0. This is known as the **Kolmogorov–Smirnov** goodness-of-fit test, which is a famous statistical test, which can be computed in Python using `scipy.stats.kstest(Z, 'norm')`, or `stats.ks_1samp(Z1, stats.norm.cdf, method='exact')` which uses the exact distribution of D_n).

10. Let $X_i \sim N(0,1)$ be i.i.d, and $\bar{m}_{3,n} = \frac{1}{n}\sum_{i=1}^n X_i^3$, $\bar{m}_{4,n} = \frac{1}{n}\sum_{i=1}^n(X_i^4 - 3)$. Then by the (multivariate) CLT,

$$\sqrt{n}\begin{bmatrix}\bar{m}_{3,n} \\ \bar{m}_{4,n}\end{bmatrix} \quad \xrightarrow{d} \quad N(0,\Sigma),$$

where

$$\Sigma \quad = \quad \begin{bmatrix}\mathbb{E}[X^6] & \mathbb{E}[X^3(X^4-3)] \\ \mathbb{E}[X^3(X^4-3)] & \mathbb{E}[(X^4-3)^2]\end{bmatrix} \quad = \quad \begin{bmatrix}15 & 0 \\ 0 & 96\end{bmatrix}.$$

Use this to devise a χ^2-test statistic to assess whether a sequence of observations $(X_1, .., X_n)$ are i.i.d. standard Normals.

Solution. If we normalize by the appropriate constants $Z_{1,n} = \frac{\sqrt{n}\,\bar{m}_{3,n}}{\sqrt{15}}$, $Z_{2,n} = \frac{\sqrt{n}\,\bar{m}_{4,n}}{\sqrt{96}}$, then

$$(Z_{1,n}, Z_{2,n}) \xrightarrow{d} (Z_1, Z_2), \qquad (Z_1, Z_2) \sim N(0, I)$$

where $I = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$, so $J_n = Z_{1,n}^2 + Z_{2,n}^2$ tends to a χ^2-random variable with $\nu = 2$ df.

Remark 0.3 This is a simpler version of the well known **Jarque-Bera test** for normality, which uses $J = \frac{n}{6}(S^2 + \frac{1}{4}(K-3)^2)$ as the test statistic, where S is the sample skewness and K is the sample kurtosis. This test can be computed in Python using `stats.jarque_bera(Z1)`.

11. Consider a general stochastic volatility model $dS_t = S_t\sqrt{V_t}dB_t$. Then the VIX volatility index at time t satisfies

$$\mathrm{VIX}_t^2 \quad = \quad \frac{1}{\Delta}\mathbb{E}(\int_t^{t+\Delta} V_u du|\mathcal{F}_t^W) \quad = \quad \frac{1}{\Delta}\int_t^{t+\Delta}\mathbb{E}(V_u|\mathcal{F}_t^W)du\,.$$

Now assume V satisfies the rough CEV-Volterra model:

$$V_t \quad = \quad V_0 + \nu\int_0^t (t-s)^{H-\frac{1}{2}}V_s^p dW_s$$

with $H \in (0,\frac{1}{2})$, $\nu, V_0, p > 0$ and $\mathbb{E}(W_tB_t) = \rho t$ with $\rho \in [-1,0]$. Compute an explicit expression for VIX_t^2 under this model. **Solution**.

$$\begin{aligned}
\xi_t(u) \quad &:= \quad \mathbb{E}(V_u|\mathcal{F}_t^W) \quad = \quad V_0 + \nu\int_0^t (u-s)^{H-\frac{1}{2}}V_s^p dW_s \\
\Rightarrow \quad \mathrm{VIX}_t^2 \quad &= \quad V_0 + \frac{\nu}{\Delta}\int_t^{t+\Delta}\int_0^t (u-s)^{H-\frac{1}{2}}V_s^p dW_s du \quad = \quad V_0 + \frac{\nu}{\Delta}\int_0^t(\int_t^{t+\Delta}(u-s)^{H-\frac{1}{2}}du)V_s^p dW_s \\
&= \quad V_0 + \frac{\nu}{\Delta(\frac{1}{2}+H)}\int_0^t((t+\Delta-s)^{\frac{1}{2}+H} - (t-s)^{\frac{1}{2}+H})\,V_s^p dW_s
\end{aligned}$$

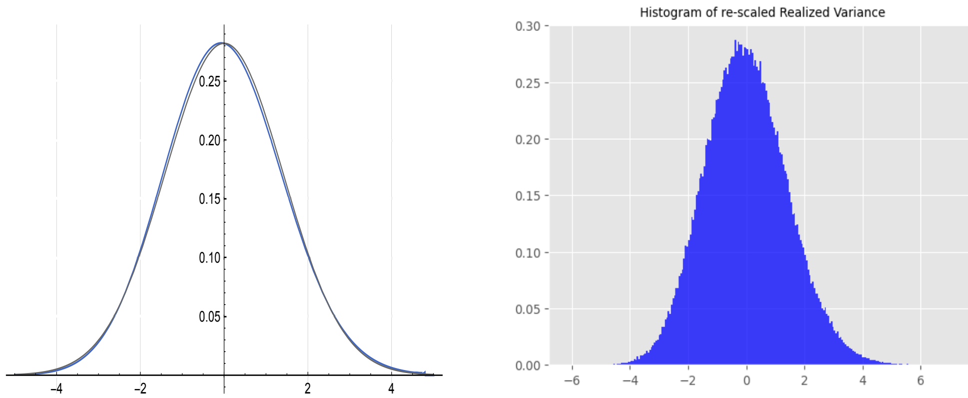

Figure 1: On the left we have plotted the exact density of $\sqrt{n}\,(\sum_{i=1}^{n}(\Delta W_i)^2 - t)$ (in blue, computed using a χ^2 density) with $t = 1$ and $n = 800$, verses the density of an $N(0, 2)$ random variable (grey), and we see they are almost identical, as the CLT shows above. On the right we have plotted a histogram of $\sqrt{n}\,(\sum_{i=1}^{n}(\Delta W_i)^2 - t)$ with 250,000 Brownian paths and $n = 200$ time steps which again tends to an $N(0, 2)$ distribution as $n \to \infty$, and the sample variance here came out to 2.0111, i.e. close to 2.

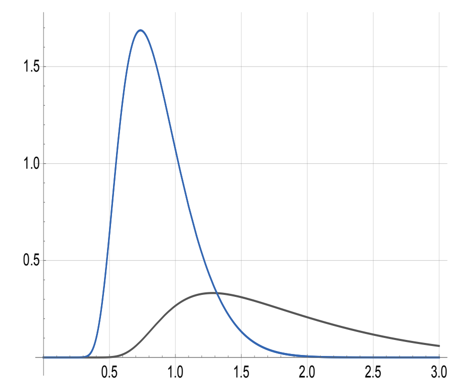

Figure 2: Density of the two-sided maximum for standard Brownian motion from Hwk 4 q2 (grey) and the Brownian bridge in q8 here (blue) on $[0, 1]$.

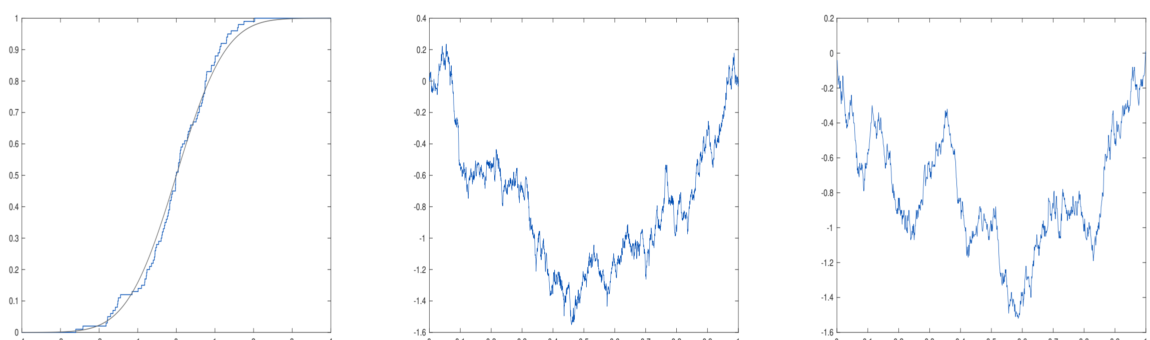

Figure 3: On the left we see the empirical cdf of 100 i.i.d standard Normals (in blue) and the exact cdf of a standard Normal (in grey). In the middle we see an exact Monte Carlo simulation of a Brownian bridge. On the right we see a simulation of $\sqrt{n}(F_n(F^{-1}(u)) - u)$ for $n = 10000$ from $u = .001$ to $.999$, which is close to a true Brownian bridge by the result discussed above

.