PROBABILISTIC TITS ALTERNATIVE FOR CIRCLE DIFFEOMORPHISMS

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Abstract. Let μ_1, μ_2 be probability measures on $\mathrm{Diff}^1_+(S^1)$ satisfying a suitable moment condition and such that their supports genererate discrete groups acting proximally on S^1. Let $(f^n_\omega)_{n\geq 0}, (f^n_{\omega'})_{n\geq 0}$ be two independent realizations of the random walk driven by μ_1, μ_2 respectively. We show that almost surely there is an $N \in \mathbb{N}$ such that for all $n \geq N$ the elements $f^n_\omega, f^n_{\omega'}$ generate a non-abelian free group. The proof is inspired by the strategy by R. Aoun for linear groups. A weaker (and easier) statement holds for measures supported on $\mathrm{Homeo}_+(S^1)$ with no moment conditions.

1. Context and contributions

The Tits alternative is a celebrated theorem by J. Tits which asserts that finitely generated linear groups are either virtually solvable or contain a non-abelian free group [Tit72]. This alternative fails for groups of homeomorphisms of the circle, but a weaker alternative (sometimes called a *dynamical Tits alternative*, see [MM23]) still holds.

Theorem 1.1 (G. Margulis [Mar00], see also V. Antonov [Ant84]). *Let G be a subgroup of* $\mathrm{Homeo}(S^1)$. *Then either*

i. *G is* elementary*, that is, the action of G preserves a probability measure on S^1, or*

ii. *G contains a* ping-pong pair*, that is, two elements $f, g \in G$ such that there are pairwise disjoint open subsets U_1, U_2, V_1, V_2 of S^1 with $f(S^1 - U_1) \subseteq V_1$, $g(S^1 - U_2) \subseteq V_2$.*

The previous options are mutually exclusive and, by the ping-pong lemma, condition (ii) implies that f, g generate a non-abelian free group in G. We are interested in how generic these two elements are. As a first approximation, there is a dense G_δ subset W of $\mathrm{Homeo}(S^1) \times \mathrm{Homeo}(S^1)$ such that any pair of elements in W generate a non-abelian free group [Ghy01, Proposition 4.5] (see also [Tri14, Theorem 6.9]). Our viewpoint will be probabilistic instead of topological, inspired by the following result of R. Aoun for linear groups. We first fix some notation: a probability measure μ on a group G is said to be *non-degenerate* if the semigroup generated by its support is all G. Given non-degenerate probability measures μ_1, μ_2 on groups G_1, G_2 we let $(\Omega_i, \mathbb{P}_i)$, $i = 1, 2$ be the probability space $(G_i^{\mathbb{N}}, \mu_i^{\otimes\mathbb{N}})$, and we write $\omega = (f_{\omega_n})_{n\geq 0}$ for an element of Ω_1 and $\omega' = (f_{\omega'_n})_{n\geq 0}$ for an element of Ω_2. Also, denote f^n_ω for the right random walk $f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$ at time $n \in \mathbb{N}$.

Theorem 1.2 (R. Aoun [Aou11, Aou13]). *Let G be a real algebraic linear group that is semisimple and with no compact factors, and let G_1, G_2 be Zariski-dense subgroups of G. If μ_1, μ_2 are non-degenerate probability measures on G_1, G_2 respectively with finite exponential moment, then there exists $\rho \in (0, 1)$ such that*

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[(\omega, \omega') \in \Omega_1 \times \Omega_2 \text{ such that } f^n_\omega, f^n_{\omega'} \text{ are a ping-pong pair}\right] \geq 1 - \rho^n$$

for all sufficiently large $n \in \mathbb{N}$.

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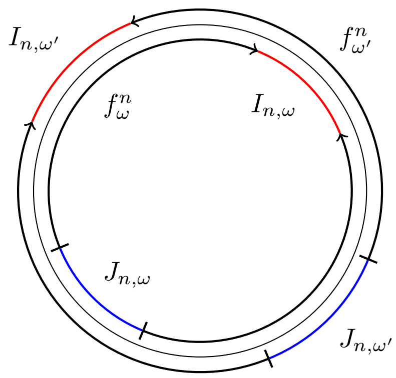
In the previous statement, two elements $f, g \in \mathrm{GL}_d(\mathbb{R}), d \in \mathbb{N}$ are said to be a *ping-pong pair* if the conditions in (ii) hold for their natural action on the projective space $\mathrm{P}\mathbb{R}^d$, and the measure μ is said to have *exponential moment* if $\int_G \|g\|^\delta \, \mathrm{d}\mu(g)$ is finite for some $\delta > 0$ (here $\|\cdot\|$ is any norm on $d \times d$ matrices).

When specialized to $\mathrm{PSL}_2(\mathbb{R})$ acting on S^1, the proof shows that the situation depicted in Figure 1 occurs with probability converging to 1 exponentially fast in $n \in \mathbb{N}$. That is, there exist disjoint intervals $I_{n,\omega}, I_{n,\omega'}, J_{n,\omega}, J_{n.\omega'} \subset S^1$ that testify that $f^n_\omega, f^n_{\omega'}$ are a ping-pong pair. The intervals $I_{n,\omega}, I_{n,\omega'}$ can be taken centered around $f^n_\omega(x), f^n_{\omega'}(y)$ where $x, y \in S^1$ are arbitrary and fixed beforehand. The intervals $J_{n,\omega}, J_{n,\omega'}$ converge as n increases to the *repellers* $\sigma(\omega), \sigma(\omega')$ of the random walks $f^n_\omega, f^n_{\omega'}$ (see the next section for the definition of σ). To control the probabilities that they intersect, their diameters decrease to 0 exponentially fast in n.



The main result of this paper shows that this situation remains typical for a pair of independent random walks on countable subgroups of $\mathrm{Diff}^1_+(S^1)$, the group of orientation-preserving diffeomorphisms of S^1, provided the action of the subgroups on S^1 is proximal. This condition is almost always fulfilled for a group acting on S^1 admitting no invariant measures on S^1, see Theorem 2.2 below for a precise statement. For a function $\phi\colon S^1 \to \mathbb{R}$, set

$$|\phi|_{\mathrm{Lip}} = \sup_{x \neq y \in S^1} \frac{|\phi(x) - \phi(y)|}{d(x,y)}.$$

Theorem A. *Let G_1, G_2 be countable subgroups of $\mathrm{Diff}^1_+(S^1)$ such that the actions of G_1 and of G_2 on S^1 are proximal. Let μ_1, μ_2 be non-degenerate probability measures on G_1, G_2 respectively such that there exists $\delta > 0$ so that the integral*

$$\int_{G_i} \max\left\{|g|_{\mathrm{Lip}}, \left|g^{-1}\right|_{\mathrm{Lip}}\right\}^\delta \mathrm{d}\mu(g)$$

is finite for $i = 1, 2$.

Then there exists $\rho \in (0,1)$ such that

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[(\omega, \omega') \in \Omega_1 \times \Omega_2 \text{ such that } f^n_\omega, f^n_{\omega'} \text{ are a ping-pong pair}\right] \geq 1 - \rho^n$$

for all sufficiently large $n \in \mathbb{N}$.

As with Theorem 1.2, the Borel-Cantelli lemma immediately implies the following.

Corollary B. *Let μ_1, μ_2 be probability measures on $\mathrm{Diff}^1_+(S^1)$ satisfying the same assumptions as in Theorem A. For $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost every (ω, ω') there exists $N \in \mathbb{N}$ such that $f^n_\omega, f^n_{\omega'}$ generate a non-abelian free group for all $n \geq N$.*

The conclusion of Theorem A is known to be true in other settings: if M is a proper hyperbolic space such that $\mathrm{Isom}(M)$ acts cocompactly on M and μ is a measure on $\mathrm{Isom}(M)$ generating a non-elementary group, then this is [AS22, Theorem 1.10]. The case of non-elementary hyperbolic groups acting on their Gromov boundary and finitely supported μ was treated previously in [GMO10].

We do not know if the statement in Corollary B is true for groups of bi-Lipschitz homeomorphisms of S^1.[1] To put this in perspective, we show that a weakening of Corollary B is true for groups of homeomorphisms of S^1, even without moment assumptions on the measures μ_i and relaxing the proximality assumption on the G_i to the absence of G_i-invariant probability measures on S^1. It is an application of results in [DKN07], but has not appeared previously in the literature up to our knowledge. Denote by $\mathrm{Homeo}_+(S^1)$ the group of orientation-preserving homeomorphisms of S^1.

Theorem C. *Let G_1, G_2 be countable subgroups of $\mathrm{Homeo}_+(S^1)$ such that the actions of G_1 and G_2 on S^1 do not admit any invariant probability measures, and let μ_1, μ_2 be non-degenerate probability measures on G_1, G_2 respectively. Then for $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost every $(\omega, \omega') \in \Omega_1 \times \Omega_2$, the set of $n \in \mathbb{N}$ such that f_ω^n, $f_{\omega'}^n$ are a ping-pong pair has density 1, that is,*

$$\lim_{N\to\infty} \frac{1}{N} |\{0 \leq n \leq N \mid f_\omega^n,\, f_{\omega'}^n \text{ are a ping-pong pair}\}| = 1.$$

The proof of Theorem C requires only the tools developped in [DKN07, Appendice] and general statements on contracting random dynamical systems from [Mal17]. In contrast, the proof of Theorem A requires more ingredients. For instance, to apply the strategy of [Aou11] in this context it is essential to know that exponential contractions occur in mean and that the stationary measure is Hölder continuous (see Theorems 4.2 and 2.6 below respectively). The first condition has been already proven in different situations in the literature by K. Gelfert and G. Salcedo [GS23], A. Gorodetski and V. Kleptsyn [GK21], and P. Barrientos and D. Malicet [BM24], all of which require (at least) that μ be supported on $\mathrm{Diff}^1_+(S^1)$. The second one is a very general theorem by A. Gorodetski, V. Kleptsyn and G. Monakov [GKM22]. One important difference with the linear setting lies in the dynamics of individual elements of $\mathrm{Homeo}_+(S^1)$: very "contracting" homeomorphisms of the circle do not have a canonically defined repeller or attractor. Proposition 4.4 below deals with this issue.

Remark. Theorem A and Corollary B are also true when the supports of the probability measures μ_i only generate *semigroup* (as opposed to a groups), with the same proofs. However, the proof of Theorem C uses the existence of maps $\pi_i \colon S^1 \to S^1$ intertwining the given actions of the G_i with minimal proximal actions. When G_i is only a semigroup with no invariant probability measure such a map may not exist, see [KKO18, Section 9.5].

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2. Preliminaries

We review some basic theory on random dynamical systems and groups acting on the circle, and introduce some notation. For more details on this material, see [Ghy01, DKN07, Mal17].

[1] The relevance of the bi-Lipschitz condition comes from the fact that any countable subgroup of $\mathrm{Homeo}_+(S^1)$ is conjugated to a group of bi-Lipschitz homeomorphisms, see [DKN07, Théorème D].

Notation. Given a metric space (X, d), $A \subseteq X$ and $\varepsilon > 0$, we write $A^\varepsilon = \{x \in X \mid d(x, A) \leq \varepsilon\}$. We also write 1_A for the indicator function on A. We will denote by d the metric on S^1 coming from an identification $S^1 = \mathbb{R}/\mathbb{Z}$, so that $\operatorname{diam}(S^1) = 1/2$. Probability measures on S^1 in this paper are always assumed to be Borel.

Random dynamical systems. A *random dynamical system* $(G, \mu) \curvearrowright X$ (or a *random walk on* X) is the data of a group G acting by homeomorphisms on a compact metric space (X, d) and of a probability measure μ on G. We always assume that μ is *non-degenerate*, that is, that the semigroup generated by μ is G. We remark that this notation does not coincide with that of [Mal17], where a measure on a semigroup is non-degenerate if its support generates the whole semigroup. Since we deal only with groups no confusion will arise.

Denote by $(\Omega, \mathbb{P})$ the probability space $(G^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$ and set $f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_0}$ when $n \in \mathbb{N}$, $\omega = (f_{\omega_k})_{k \geq 0} \in \Omega$. A *$\mu$-stationary measure* is a Borel probability measure on X such that $\mu * \nu = \nu$, where

$$\mu * \nu(A) = \int_G \nu(g^{-1}A) \, \mathrm{d}\mu(g)$$

for all Borel $A \subseteq X$.

We say that $(G, \mu) \curvearrowright X$ is *locally contracting* if for all $x \in X$, $\mathbb{P}$-almost surely there exists a neighborhood $B \subset X$ of x such that $\operatorname{diam}(f_\omega^n(B)) \xrightarrow[n \to \infty]{} 0$.

Proposition 2.1 ([Mal17, Propositions 4.8 and 4.9])**.** *Suppose $(G, \mu) \curvearrowright X$ is locally contracting. Then there are finitely many ergodic μ-stationary measures $\nu_1, \ldots, \nu_d$, and their supports are exactly the minimal G-invariant sets in X.*

Moreover, for every $x \in X$ and $\mathbb{P}$-almost every ω there exists a unique index $i = i(\omega, x) \in \{1, \ldots, d\}$ such that $f_\omega^n(x)$ equidistributes towards *ν_i, that is*

$$\frac{1}{N} \sum_{n=0}^{N} 1_{f_\omega^n(x)} \xrightarrow[N \to \infty]{} \nu_i \tag{2.1}$$

in the weak $*$-topology.

Groups acting on the circle. We say that a group action $G \curvearrowright S^1$ is *proximal* if for every strict subinterval $I \subset S^1$ and $\varepsilon > 0$ there exists $g \in G$ such that $\operatorname{diam}(g(I)) < \varepsilon$. The action is said to be *locally proximal* if every $x \in S^1$ is the endpoint of an interval $I \subset S^1$ such that for all $\varepsilon > 0$ there exists $g \in G$ with $\operatorname{diam}(g(I)) < \varepsilon$. In this context, we say that a group action $G \curvearrowright^{\phi} S^1$ is *semiconjugate* to $G \curvearrowright^{\psi} S^1$ if there exists a continuous surjection $\pi \colon S^1 \to S^1$ such that $\psi(g) \circ \pi = \pi \circ \phi(g)$ for all $g \in G$, and such that π is locally non-decreasing and has degree one (this means that any lift of π to $\widetilde{\pi} \colon \mathbb{R} \to \mathbb{R}$ is non-decreasing and satisfies $\widetilde{\pi}(x+1) = \widetilde{\pi}(x) + 1$ for all $x \in \mathbb{R}$). When such a π does not necessarily have degree 1, we call it a *factor map.*

The following theorem is essentially equivalent to Theorem 1.1 from the introduction: in cases (ii.b) and (ii.c) below G always contains a free group, and there exists an invariant probability measure for G in cases (i) (a mean of Dirac measures on a finite orbit on S^1) and (ii.a) (the image of Lebesgue measure under a conjugacy to a group of rotations).

Theorem 2.2 (see [Ghy01])**.** *Consider an action $G \curvearrowright^{\phi} S^1$ by orientation-preserving homeomorphisms. Then exactly one of the following options is satisfied.*

i. There exists a finite orbit.

ii. There exists a unique closed minimal set Λ, which is either S^1 or a Cantor set. In the latter case, by collapsing the countably many connected components of $S^1 - \Lambda$ we can semiconjugate ϕ to a minimal group action $G \curvearrowright S^1$.

Moreover, in the minimal case a further distinction exists: either *ii.a* *the action is free and thus conjugated to an action by rotations, or*

ii.b *the action is proximal, or*

ii.c *the action is locally proximal and not proximal, and there exists $d \in \mathbb{N}_+$, $d \geq 2$ and a continuous d-to-one covering $\pi\colon S^1 \to S^1$ that intertwines ϕ with a proximal action.*

Thus whenever $G \curvearrowright^{\phi} S^1$ does not preserve any probability measure on S^1, there exists $d \in \mathbb{N}_+$ and a factor map $\pi\colon S^1 \to S^1$ that is d-to-one on the minimal set of G (except for a countable number of points), and that intertwines ϕ with a proximal and minimal action. We will call this integer d the *degree of proximality* of ϕ, but this notation is not standard.

Random walks on S^1. In this subsection, fix a countable group G and a non-degenerate probability measure μ on G. The random walk on S^1 defined by a proximal group action $G \curvearrowright^{\phi} S^1$ has been well studied.

Theorem 2.3 ([DKN07, Appendice])**.** *Consider an action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1.*

i. *There exists a unique μ-stationary probability measure ν on S^1, which is atomless and is supported on the minimal set of G.*

ii. *If the action of G on S^1 is proximal, there exists a random variable $\omega \in \Omega \mapsto \sigma(\omega) \in S^1$ such that for $\mathbb{P}$-almost every ω we have*

$$(f_\omega^n)^{-1}\nu \xrightarrow[n\to\infty]{} 1_{\sigma(\omega)}$$

in the weak-$*$ topology.

We call the random variable $\sigma(\omega)$ from the previous theorem the *repeller* of the random walk $(f_\omega^n)_{n\geq 0}$. Its distribution is the unique $\overline{\mu}$-stationary measure on S^1 where $\overline{\mu} \in \mathrm{Prob}(G)$ is defined on $g \in G$ as $\mu(g^{-1})$, and is thus non-atomic.

Notice that the measure $(f_\omega^n)^{-1}\nu$ is given by $\nu(f_\omega^n(I))$ on every interval $I \subseteq S^1$, so the statement in (ii) says that $f_\omega^n(I)$ is contracted into a ν-null set unless I contains $\sigma(\omega)$, in which case it is expanded to the whole circle. As a consequence, for all $x, y \in S^1$ we have $\mathbb{P}$-almost surely that x and y are not $\sigma(\omega)$ since the law of $\sigma(\omega)$ is non-atomic, and hence $\lim_{n\to\infty} d(f_\omega^n(x), f_\omega^n(y)) = 0$. This conclusion is the subject of [KN04] (see also [Ant84]), and justifies the fact that we will use $f_\omega^n(0)$ (or $f_\omega^n(x)$ for some non-random $x \in S^1$) as an "attractor" for f_ω^n in the proofs below.

When $G \curvearrowright S^1$ is not necessarily proximal a similar statement holds, and even more is true: the rate of contraction of $f_\omega^n(I)$ when $\sigma(\omega) \notin I$ is exponential.

Theorem 2.4. *Consider an action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1. Let $d \in \mathbb{N}_+$ be the degree of proximality of $G \curvearrowright S^1$.*

There exist measurable functions $\sigma_1, \ldots, \sigma_d\colon \Omega \to S^1$ such that the following hold.

i. [Mal17, Theorem D] *There exists $\lambda > 0$ such that for $\mathbb{P}$-almost every ω and every closed interval $I \subset S^1 \setminus \{\sigma_1(\omega), \ldots, \sigma_d(\omega)\}$ we have*

$$\mathrm{diam}(f_\omega^n(I)) \leq e^{-\lambda n}$$

for all sufficiently large $n \in \mathbb{N}$.

ii. [Mal17, Proposition 4.3] *$\mathbb{P}$-almost surely, the set $\{\sigma_1(\omega), \ldots, \sigma_d(\omega)\}$ has size d.*

The random set $\{\sigma_1, \ldots, \sigma_d\}$ from the previous theorem is called the *repelling set* of the random walk $(f_\omega^n)_{n\geq 0}$. In this setting, let $\pi\colon S^1 \to S^1$ be a factor map to a minimal and proximal action and $\Lambda \subseteq S^1$ be the minimal set of $G \curvearrowright S^1$. Define $E \subset S^1$ as the countable set of images of connected components of $S^1 \setminus \Lambda$, so for all $x \in S^1 \setminus E$, the fiber $\pi^{-1}(x)$ has size d. Denote by $\sigma(\omega)$

the repeller of the random walk in the image of π. If $\sigma(\omega) \in S^1 \setminus E$ (which happens $\mathbb{P}$-almost surely, since the distribution of σ is non-atomic), then $\pi^{-1}(\sigma(\omega)) = \{\sigma_1(\omega), \dots . \sigma_d(\omega)\}$ by the defining properties of $\sigma(\omega)$ and $\{\sigma_1(\omega), \dots, \sigma_d(\omega)\}$. We record this as a proposition.

Proposition 2.5. *Consider an action $G \curvearrowright S^1$ by orientation-preserving homeomorphisms with no invariant probability measure on S^1. Denote by $\pi\colon S^1 \to S^1$ a factor map to a minimal and proximal action, and by $\omega \mapsto \sigma(\omega)$ the repeller of the random walk induced in the image of π.*

Then $\mathbb{P}$-almost surely, the repelling set $F(\omega) = \{\sigma_1(\omega), \dots, \sigma_d(\omega)\}$ of $(f_\omega^n)_{n\geq 0}$ is such that $\pi^{-1}(\sigma(\omega)) = F(\omega)$.

We finish with the Hölder regularity of the unique μ-stationary measure of a proximal random dynamical system $(G, \mu) \curvearrowright S^1$. The original statement in [GKM22, Theorem 2.3] is written for $G \leq \mathrm{Diff}^1(M)$ for any compact smooth manifold M, but [GKM22, Remark 2.10] shows that differentiability of the maps in G is not essential: what is truly needed is that all maps of G be bi-Lipschitz.

Theorem 2.6 ([GKM22, Theorem 2.3])**.** *Consider an action $G \curvearrowright S^1$ by orientation-preserving diffeomorphisms of class C^1 with no invariant probability measure on S^1, and assume that for some $\delta > 0$ the integral*

$$\int_G \max\left\{|g|_{\mathrm{Lip}}, \left|g^{-1}\right|_{\mathrm{Lip}}\right\}^\delta \mathrm{d}\mu(g)$$

is finite.

*Then there exist $C, \alpha > 0$ such that any μ-stationary probability measure ν on S^1 is (C,α)-*Hölder continuous*, that is, $\nu(B(x,r)) \leq Cr^\alpha$ for all $x \in S^1$ and $r > 0$.*

3. Probabilistic Tits alternative in $\mathrm{Homeo}_+(S^1)$

Proof of Theorem C. Fix μ_1, μ_2 two non-degenerate probability measures on countable subgroups G_1, G_2 of $\mathrm{Homeo}_+(S^1)$ that do not preserve any probability measure on S^1. For $i = 1, 2$, let

- ν_i be the unique μ_i-stationary measure on S^1 and $\Lambda_i \subseteq S^1$ the minimal set of G_i,
- $d_i \in \mathbb{N}_+$ be the degree of proximality of $G_i \curvearrowright S^1$, and
- $\pi_i\colon S^1 \to S^1$ be a factor map of $G_i \curvearrowright S^1$ to a minimal proximal action of G_i such that π_i is d_i-to-one ν_i-almost everywhere.

Recall that the degree of proximality of $G_i \curvearrowright S^1$ is the unique integer such that the map π_i exists, see the discussion after Theorem 2.2.

For $\omega \in \Omega_1$ we write

- $(g_\omega^n)_{n\geq 0}$ for the random walk driven by μ_1 acting on the image of π_1 and $\sigma(\omega) \in S^1$ for its repelling point, and
- $F(\omega) \subset S^1$ for the repelling set of the random walk $(f_\omega^n)_{n\geq 0}$.

Recall that π_1 intertwines the action $G_1 \curvearrowright S^1$ with a minimal proximal action of G_1, so the random walk $(g_\omega^n)_{n\geq 0}$ verifies the conclusion of Theorem 2.3, (ii). When $\omega' \in \Omega_2$ we denote by $g_{\omega'}^n$, $\sigma(\omega')$ and $F(\omega')$ the same objects associated to μ_2.

Fix once and for all $x \in S^1$, $y \in S^1$ such that $\pi_1^{-1}(x)$ and $\pi_2^{-1}(y)$ have size d_1, d_2 respectively and are disjoint.

Claim. *The following properties are true for $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost every $(\omega, \omega') \in \Omega_1 \times \Omega_2$.*

i. The sequence $\{(f_\omega^n(a), f_{\omega'}^n(b))\}_{n\geq 0} \subset S^1 \times S^1$ equidistributes with respect to $\nu_1 \otimes \nu_2$ for every $a \in \pi_1^{-1}(x)$ and $b \in \pi_2^{-1}(y)$.

ii. The equalities $F(\omega) = \pi_1^{-1}(\sigma(\omega))$ and $F(\omega') = \pi_2^{-1}(\sigma(\omega'))$ hold.

iii. The sets $F(\omega)$, $F(\omega')$, $\pi_1^{-1}(x)$ and $\pi_2^{-1}(y)$ are pairwise disjoint.

Proof of the claim: (i) The random dynamical system $(G_1 \times G_2, \mu_1 \otimes \mu_2) \curvearrowright S^1 \times S^1$ is locally contracting since $(G_1, \mu_1) \curvearrowright S^1$, $(G_2, \mu_2) \curvearrowright S^1$ are locally contracting. Moreover, for any pair $(x, y) \in S^1 \times S^1$ the orbit $\mathrm{Orb}_{G_1 \times G_2}((x, y))$ accumulates on $\Lambda_1 \times \Lambda_2$, so $\Lambda_1 \times \Lambda_2$ is the unique $G_1 \times G_2$-minimal set and Proposition 2.1 shows that $S^1 \times S^1$ has a unique $\mu_1 \otimes \mu_2$-stationary measure, namely $\nu_1 \otimes \nu_2$. Again Proposition 2.1 gives equidistribution $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost surely.
(ii) This is Proposition 2.5.
(iii) By independence it suffices to show that $\mathbb{P}_1 [z \in F(\omega)] = \mathbb{P}_2 [z \in F(\omega')] = 0$ for any fixed $z \in S^1$, but this follows from $\mathbb{P}_1 [z \in F(\omega)] = \mathbb{P}_1 [\pi_1(z) = \sigma(\omega)]$ and the fact that the distribution of $\sigma(\omega)$ is non-atomic. $\square$

We will assume in what follows that the pair $(\omega, \omega') \in \Omega_1 \times \Omega_2$ satisfies the previous properties. Fix $\varepsilon > 0$ and pick $\delta > 0$ such that any interval $I \subset S^1$ with $|I| \leq \delta$ has $\nu_1(I), \nu_2(I) \leq \varepsilon$ and also $\nu_1 \otimes \nu_2(\triangle^\delta) \leq \varepsilon$ where $\triangle \subset S^1 \times S^1$ is the diagonal (here $S^1 \times S^1$ is equipped with the l^∞-metric). Choose $\chi = \chi(\omega, \omega') \in (0, \delta/2)$ such that $F(\omega)^\chi$ and $F(\omega')^\chi$ are disjoint.

Suppose that $I \subset S^1$ has diameter at most 2χ and consider $a \in \pi_1^{-1}(x)$, $b \in \pi_2^{-1}(y)$. Equidistribution implies that the quantities

$$\limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N \mid f_\omega^n(a) \in I\}|, \quad \limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N \mid f_{\omega'}^n(b) \in I\}|$$
$$\text{and} \qquad \limsup_{N\to\infty} \frac{1}{N} |\{0 \leq n < N \mid (f_\omega^n(a), f_{\omega'}^n(b)) \in \triangle^\chi\}|$$

are all smaller than ε. By considering all the combinations in which the intervals in $f_\omega^n(\pi_1^{-1}(x))^\chi$, $f_{\omega'}^n(\pi_2^{-1}(y))^\chi$, $F(\omega)^\chi$ and $F(\omega')^\chi$ can intersect, we conclude that

$$\limsup_{N\to\infty} \frac{1}{N} \big|\{0 \leq n < N \mid f_\omega^n(\pi_1^{-1}(x))^\chi,\ f_{\omega'}^n(\pi_2^{-1}(y))^\chi,\ F(\omega)^\chi \text{ and } F(\omega')^\chi \text{ are not pairwise disjoint}\}\big| \tag{3.1}$$

is at most $(d_1^2 + d_2^2 + 3d_1d_2)\varepsilon$. More explicitly, (3.1) is at most

$$\begin{aligned}
\sum_{\substack{a\in\pi_1^{-1}(x), b\in\pi_2^{-1}(y)\\ f\in F(\omega), f'\in F(\omega')}} & \limsup_{N\to\infty} \frac{1}{N} \Big(|\{0 \leq n < N \mid d(f_\omega^n(a), f_{\omega'}^n(b)) \leq 2\chi\}| \\
& + |\{0 \leq n < N \mid d(f_\omega^n(a), f) \leq 2\chi\}| + |\{0 \leq n < N \mid d(f_{\omega'}^n(b), f') \leq 2\chi\}| \\
& + |\{0 \leq n < N \mid d(f_\omega^n(a), f') \leq 2\chi\}| + |\{0 \leq n < N \mid d(f_{\omega'}^n(b), f) \leq 2\chi\}| \Big) \\
& \leq d_1d_2\varepsilon + d_1^2\varepsilon + d_2^2\varepsilon + d_1d_2\varepsilon + d_2d_2\varepsilon,
\end{aligned}$$

as desired.

Take $\bar{\chi} > 0$ such that for $i = 1, 2$, the connected components of $\pi_i^{-1}(I)$ have diameter at most χ if $I \subset S^1$ has diameter at most $\bar{\chi}$. By Theorem 2.3, $\mathbb{P}_1 \otimes \mathbb{P}_2$-almost surely we can find $n_0 = n_0(\omega, \omega') \in \mathbb{N}$ such that for all $n \geq n_0$ the inclusions

$$g_\omega^n(S^1 - \sigma(\omega)^{\bar{\chi}}) \subseteq g_\omega^n(x)^{\bar{\chi}} \quad \text{and} \quad g_{\omega'}^n(S^1 - \sigma(\omega')^{\bar{\chi}}) \subseteq g_{\omega'}^n(y)^{\bar{\chi}}$$

hold, so $F(\omega) = \pi_1^{-1}(\sigma(\omega))$, $F(\omega') = \pi_2^{-1}(\sigma(\omega'))$ shows that

$$f_\omega^n(S^1 - F(\omega)^\chi) \subseteq f_\omega^n(\pi_1^{-1}(x))^\chi \quad \text{and} \quad f_{\omega'}^n(S^1 - F(\omega')^\chi) \subseteq f_{\omega'}^n(\pi_2^{-1}(y))^\chi. \tag{3.2}$$

The inclusions (3.2) imply that every n in the set

$$\mathcal{N} = \{n \geq n_0 \mid f_\omega^n(\pi_1^{-1}(x))^\chi,\ f_{\omega'}^n(\pi_2^{-1}(y))^\chi,\ F(\omega)^\chi \text{ and } F(\omega')^\chi \text{ are pairwise disjoint}\}$$

is such that $f_\omega^n, f_{\omega'}^n$ are a ping-pong pair. Thus

$$\begin{aligned}\varliminf_{N\to\infty} \frac{1}{N}|\{0 \le n < N \mid f_\omega^n,\, f_{\omega'}^n \text{ are a ping-pong pair}\}| &\ge \varliminf_{N\to\infty} \frac{1}{N}|\mathcal{N} \cap [0, N]| \\ &\ge 1 - (d_1^2 + d_2^2 + 3d_1d_2)\varepsilon,\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary the conclusion follows. □

4. Probabilistic Tits alternative in $\mathrm{Diff}^1_+(S^1)$

Preliminary statements. In this subsection G is a countable subgroup of $\mathrm{Diff}^1_+(S^1)$ that acts proximally on S^1 and μ is a non-degenerate probability measure on G such that

$$\text{there exists } \delta > 0 \text{ so that } \int_G \max\left\{|g|_{\mathrm{Lip}}, |g^{-1}|_{\mathrm{Lip}}\right\}^\delta \mathrm{d}\mu(g) \text{ is finite.} \tag{M}$$

We now state and prove Theorem 4.2, which gives (uniform) exponential contractions in mean in our context. It is a variation on similar statements that have appeared independently in [GK21, Proposition 4.18] and [GS23, Theorem 1.3], assuming that μ has finite support in $\mathrm{Diff}^1_+(S^1)$. The proof follows [GK21, Proposition 4.18] closely, along with additional input from [BM24, Proposition 4.5]. This is the only point in the proof where we use that μ is supported in $\mathrm{Diff}^1_+(S^1)$, namely to obtain inequality (4.1) below.

Theorem 4.1 ([BM24, Proposition 4.5])**.** *If μ satisfies the condition* (M)*, there exist constants $r, \lambda > 0$, $s_0 \in (0, 1]$ and $k \in \mathbb{N}_+$ such that*

$$\mathbb{E}\left[d(f_\omega^{k_1}(x), f_\omega^{k_1}(y))^s\right] \le e^{-\lambda} d(x, y)^s \tag{4.1}$$

for all $x, y \in S^1$ such that $d(x, y) \le r$ and all $s \in (0, s_0]$.

Theorem 4.2. *There exist $\lambda_+ > 0$, $s \in (0, 1]$ and $N \in \mathbb{N}$ such that for all $n \ge N$ we have*

$$\sup_{x \ne y \in S^1} \mathbb{E}\left[\frac{d(f_\omega^n(x), f_\omega^n(y))^s}{d(x, y)^s}\right] \le e^{-\lambda_+ n}.$$

In particular, for all $n \ge N$ we have

$$\sup_{x, y \in S^1} \mathbb{E}\left[d(f_\omega^n(x), f_\omega^n(y))\right] \le e^{-\lambda_+ n}.$$

Proof. Take $r, \lambda > 0$, $s_0 \in (0, 1]$ and $k_1 = k \in \mathbb{N}_+$ given by Theorem 4.1, so that (4.1) holds for all $x, y \in S^1$ with $d(x, y) \le r$ and all $s \in (0, s_0]$.

Claim. *For every $\varepsilon_1, \varepsilon_2 > 0$ there exists $k_2 \in \mathbb{N}_+$ such that*

$$\mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < \varepsilon_1\right] > 1 - \varepsilon_2$$

for all $x, y \in S^1$.

Proof of the claim. This is [GK21, Lemma 4.23], but we give the proof for completeness.

Let $l \in \mathbb{N}_+$ be large enough so that the points $x_j = j/l \in S^1$, $0 \le j \le l - 1$ satisfy

$$\mathbb{P}\left[\sigma(\omega) \in (x_j, x_{j+1})\right] \le \varepsilon_2/4$$

for each j. When $0 \le j \le l - 1$ denote by I_j the open interval with endpoints x_j, x_{j+1} and length $1 - 1/l$. By the defining property of $\sigma(\omega)$, we may choose $k_2 \in \mathbb{N}$ large enough so that for every $0 \le j \le l - 1$, we have

$$\mathbb{P}\left[\mathrm{diam}(f_\omega^{k_2}(I_j)) \le \varepsilon_1 \mid \sigma(\omega) \in (x_j, x_{j+1})\right] \ge 1 - \varepsilon_2/2.$$

If $x, y \in S^1$, then $x, y \in I_j$ for all indices $0 \leq j \leq l-1$ except at most two values j_1, j_2. Thus

$$\begin{aligned}\mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < \varepsilon_1\right] &\geq \sum_{\substack{j=0 \\ j \neq j_1, j_2}}^{l-1} \mathbb{P}\left[\operatorname{diam}(f_\omega^{k_2}(I_j)) \leq \varepsilon_1 \mid \sigma(\omega) \in (x_j, x_{j+1})\right] \mathbb{P}\left[\sigma(\omega) \in (x_j, x_{j+1})\right] \\ &\geq (1 - \varepsilon_2/2)(1 - 2\varepsilon_2/4) = (1 - \varepsilon_2/2)^2 \geq 1 - \varepsilon_2. \qquad \square\end{aligned}$$

Fix $s \in (0, s_0]$ and find $k_2 \in \mathbb{N}_+$ such that

$$\mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < r/4^{1/s}\right] > 1 - r^s/4$$

for all $x, y \in S^1$. If we take $x, y \in S^1$ such that $d(x, y) \geq r$, then by conditioning on whether $d(f_\omega^{k_2}(x), f_\omega^{k_2}(y))$ is smaller or larger than $r/4$ we obtain

$$\begin{aligned}\mathbb{E}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y))^s\right] &\leq 1 \cdot \mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) > r/4^{1/s}\right] + \left(\frac{r}{4^{1/s}}\right)^s \mathbb{P}\left[d(f_\omega^{k_2}(x), f_\omega^{k_2}(y)) < r/4^{1/s}\right] \\ &\leq \frac{r^s}{4} + \frac{r^s}{4} \leq \frac{1}{2} d(x, y)^s. \qquad (4.2)\end{aligned}$$

For every $k \in \mathbb{N}_+$ and $x, y \in S^1$, define a random variable $K_k(\omega) \in \mathbb{N}_+$ (which depends on x, y) as follows: if $d(x, y) \leq r$ (resp. $d(x, y) > r$) apply k_1 (resp. k_2) random iterations of $\omega = (f_{\omega_n})_{n \geq 0}$ to the pair x, y. Repeat the process on the pair $f_\omega^{k_1}(x), f_\omega^{k_1}(y)$ (resp. $f_\omega^{k_2}(x), f_\omega^{k_2}(y)$) applying iterations of $(f_{\omega_{n+k_1}})_{n \geq 0}$ (resp. $(f_{\omega_{n+k_2}})_{n \geq 0}$) until the total number of iterations exceeds k for the first time. By definition we have $k \leq K_k \leq k + \max\{k_1, k_2\}$.

Claim. *The inequality*

$$\mathbb{E}\left[d\left(f_\omega^{K_k(\omega)}(x), f_\omega^{K_k(\omega)}(y)\right)^s\right] \leq \max\left\{\left(\frac{1}{2}\right)^{\frac{k}{k_2}}, e^{-\frac{k}{k_1}\lambda}\right\} d(x, y)^s. \qquad (4.3)$$

holds.

Proof. Define $r_1(\omega)$ as the (random) number of times where k_1 elements of the $(f_{\omega_i})_{i \geq 0}$ were applied in the definition of $K_k(\omega)$ and define $r_2(\omega)$ similarly, so $r_1(\omega)k_1 + r_2(\omega)k_2 = K_k(\omega)$. The Markov property and inequalities (4.1), (4.2) show that

$$\mathbb{E}\left[d\left(f_\omega^{K_k(\omega)}(x), f_\omega^{K_k(\omega)}(y)\right)^s\right] \leq \mathbb{E}\left[\left(\frac{1}{2}\right)^{r_2(\omega)} e^{-\lambda r_1(\omega)}\right] d(x, y). \qquad (4.4)$$

Set $r = r_1(\omega)k_1 + r_2(\omega)k_2$ and $c_1 = e^{-\lambda}, c_2 = 1/2$, so

$$r_1(\omega) \ln c_1 + r_2(\omega) \ln c_2 = r\left(\frac{r_1(\omega)k_1}{r} \frac{\ln c_1}{k_1} + \frac{r_2(\omega)k_2}{r} \frac{\ln c_2}{k_2}\right) \leq r \max\left\{\frac{\ln c_1}{k_1}, \frac{\ln c_2}{k_2}\right\}. \qquad (4.5)$$

The right-hand side of (4.5) is at most $k \max\left\{\frac{\ln c_1}{k_1}, \frac{\ln c_2}{k_2}\right\}$ since the c_i belong to $(0, 1)$ and $r \geq k$. By exponentiating we obtain

$$\left(\frac{1}{2}\right)^{r_2(\omega)} e^{-\lambda r_1(\omega)} \leq \max\left\{\left(\frac{1}{2}\right)^{\frac{k}{k_2}}, e^{-\frac{k}{k_1}\lambda}\right\},$$

which together with (4.4) proves the desired conclusion. $\square$

The f_{ω_i} are independent and distributed along μ, and hence

$$
\begin{aligned}
\mathbb{E}\left[\left|\left(f_{\omega_{K_k}}\circ\cdots\circ f_{\omega_k}\right)^{-1}\right|_{\mathrm{Lip}}^{s}\right] &\le \mathbb{E}\left[\left|f_{\omega_{K_k}}^{-1}\right|_{\mathrm{Lip}}^{s}\cdots\left|f_{\omega_k}^{-1}\right|_{\mathrm{Lip}}^{s}\right]\\
&\le \mathbb{E}\left[\left|f_{\omega_{k+\max\{k_1,k_2\}}}^{-1}\right|_{\mathrm{Lip}}^{s}\cdots\left|f_{\omega_k}^{-1}\right|_{\mathrm{Lip}}^{s}\right]\\
&= \int_G |g^{-1}|_{\mathrm{Lip}}^{s\max\{k_1,k_2\}}\,\mathrm{d}\mu(g). \qquad (4.6)
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\mathbb{E}\left[d(f_\omega^k(x),f_\omega^k(y))^{s/2}\right] &\le \mathbb{E}\left[\left|\left(f_{\omega_{K_k}}\circ\cdots\circ f_{\omega_k}\right)^{-1}\right|_{\mathrm{Lip}}^{s/2} d\left(f_\omega^{K_k(\omega)}(x),f_\omega^{K_k(\omega)}(y)\right)^{s/2}\right]\\
&\le \mathbb{E}\left[\left|\left(f_{\omega_{K_k}}\circ\cdots\circ f_{\omega_k}\right)^{-1}\right|_{\mathrm{Lip}}^{s}\right]^{1/2}\mathbb{E}\left[d\left(f_\omega^{K_k(\omega)}(x),f_\omega^{K_k(\omega)}(y)\right)^{s}\right]^{1/2}\\
&\le \left(\int_G |g^{-1}|_{\mathrm{Lip}}^{s\max\{k_1,k_2\}}\,\mathrm{d}\mu(g)\right)^{1/2}\max\left\{\left(\frac{1}{2}\right)^{\frac{k}{2k_2}}, e^{-\frac{k}{2k_1}\lambda}\right\} d(x,y)^{s/2},
\end{aligned}
$$

where we have used (4.3) and (4.6) in the last inequality. If $s\le \delta/\max\{k_1,k_2\}$, where $\delta>0$ is provided by the condition (M), the term $\int_G|g^{-1}|_{\mathrm{Lip}}^{s\max\{k_1,k_2\}}\,\mathrm{d}\mu(g)$ is also finite. But the term $\max\left\{\left(\frac{1}{2}\right)^{\frac{k}{2k_2}}, e^{-\frac{k}{2k_1}\lambda}\right\}$ converges to 0 as $k\to\infty$, and hence there exists $k\in\mathbb{N}_+$ and $\lambda>0$ such that

$$
\mathbb{E}\left[d(f_\omega^k(x),f_\omega^k(y))^{s}\right]\le e^{-\lambda}d(x,y)^s
$$

for all $x,y\in S^1$ and $0<s\le\min\{s_0/2,\delta/(2k_1),\delta/(2k_2)\}$. By the Markov property, for all $n\in\mathbb{N}$ we have

$$
\mathbb{E}\left[d(f_\omega^n(x),f_\omega^n(y))^{s}\right]\le e^{-\lambda\lfloor n/k\rfloor}d(x,y)^s\le e^{\lambda}e^{-\lambda n/k}d(x,y)^s.
$$

The conclusion follows by setting $\lambda_+=\lambda/(2k)$ and choosing $N\in\mathbb{N}$ so that $e^{\lambda}e^{-\lambda N/(2k)}<1$. $\square$

Remark. The previous theorem says that $(G,\mu)\curvearrowright S^1$ is *μ-contracting*, according to the terminology of Benoist-Quint in [BQ16, Section 11.1]. As a consequence, all of the limit laws available for cocycles in this setting (that is, the central limit theorem, the law of the iterated logarithm and large deviations estimates, see [BQ16, Section 12.1]) hold in this setting. In particular the Lyapunov cocycle $(g,x)\mapsto\log g'(x)$ satisfies these limit laws. This recovers [GS24, Theorem 1.14], for instance. However, [BQ16, Theorem 12.1] requires the cocycle to be Lipschitz with integrable Lipschitz constant, but the proofs go through without relevant changes if the Lipschitz condition is replaced by a τ-Hölder one.

Denote by $(\overline{f}_\omega^n)_{n\ge0}$ the *left (or inverse) random walk* $\overline{f}_\omega^n=f_{\omega_0}\circ f_{\omega_1}\circ\cdots f_{\omega_n}$. Define the random variable $T(\omega)\in S^1$ as the repeller of the random walk $\left(f_{\omega_n}^{-1}\circ f_{\omega_{n-1}}^{-1}\circ\cdots\circ f_{\omega_0}^{-1}\right)_{n\ge0}$. The following is an analogue of [Aou11, Theorem 4.16].

Proposition 4.3. *Let $\lambda_+>0$ be the constant provided by Theorem 4.2. There exist $\lambda_->0$, $N\in\mathbb{N}$ such that*

$$
\sup_{x\in S^1}\mathbb{E}\left[d\left((f_\omega^n)^{-1}(x),\sigma(\omega)\right)\right]\le e^{-\lambda_- n} \qquad (4.7)
$$

and

$$
\sup_{x\in S^1}\mathbb{E}\left[d\left(\overline{f}_\omega^n(x),T(\omega)\right)\right]\le e^{-\lambda_+ n} \qquad (4.8)
$$

hold for $n\ge N$. Moreover, there exist $C_-,\alpha_->0$ such that the distribution of T is (C_-,α_-)-Hölder continuous.

Proof. By applying (4.7) to the random walk on $\mathrm{Diff}^1_+(S^1)$ driven by $\overline{\mu}$ where $\overline{\mu}(g) = \mu(g^{-1})$ for all $g \in \mathrm{Diff}^1_+(S^1)$ we conclude that (4.8) holds. The Hölder continuity of the distribution of T also follows from Theorem 2.6 since this distribution is μ-stationary. It suffices then to prove (4.7).

Take $n, k \in \mathbb{N}$ with $0 < n < k$ and fix $x, y \in S^1$. We have that

$$\mathbb{E}\left[d\left((f^n_\omega)^{-1}(x), \sigma(\omega)\right)\right] \leq \mathbb{E}\left[d\left((f^n_\omega)^{-1}(x), (f^k_\omega)^{-1}(y)\right)\right] + \mathbb{E}\left[d\left((f^k_\omega)^{-1}(y), \sigma(\omega)\right)\right].$$

Theorem 4.2 applied to the random walk driven by $\overline{\mu}$ gives $\lambda_- > 0$ such that

$$\sup_{u,v \in S^1} \mathbb{E}\left[d\left(f^{-1}_{\omega_n} \circ \cdots \circ f^{-1}_{\omega_0}(u), f^{-1}_{\omega_n} \circ \cdots \circ f^{-1}_{\omega_0}(v)\right)\right] \leq e^{-\lambda_- n}$$

for all sufficiently large $n \in \mathbb{N}$. In particular we deduce that

$$\begin{aligned}
\mathbb{E}\left[d\left((f^n_\omega)^{-1}(x), (f^k_\omega)^{-1}(y)\right)\right] &= \int \mathbb{E}\left[d\left((f^n_\omega)^{-1}(x), (f^n_\omega)^{-1} \circ \gamma^{-1}(y)\right)\right] \mathrm{d}\mu^{*(k-n)}(\gamma) \\
&\leq \sup_{u,v \in S^1} \mathbb{E}\left[d\left((f^n_\omega)^{-1}(u), (f^n_\omega)^{-1}(v)\right)\right] \\
&= \sup_{u,v \in S^1} \mathbb{E}\left[d\left(f^{-1}_{\omega_n} \circ \cdots \circ f^{-1}_{\omega_0}(u), f^{-1}_{\omega_n} \circ \cdots \circ f^{-1}_{\omega_0}(v)\right)\right] \leq e^{-\lambda_- n}
\end{aligned}$$

where we have used that the f_{ω_j} are independent and identically distributed in the last equality. Hence the inequality

$$\sup_{x \in S^1} \mathbb{E}\left[d\left((f^n_\omega)^{-1}(x), \sigma(\omega)\right)\right] \leq e^{-\lambda_- n} + \mathbb{E}\left[d\left((f^k_\omega)^{-1}(y), \sigma(\omega)\right)\right] \tag{4.9}$$

holds, and by integrating (4.9) in $\mathrm{d}\nu(y)$ we conclude that

$$\begin{aligned}
\sup_{x \in S^1} \mathbb{E}\left[d\left((f^n_\omega)^{-1}(x), \sigma(\omega)\right)\right] &\leq e^{-\lambda_- n} + \mathbb{E}\left[\int_{S^1} d\left((f^k_\omega)^{-1}(y), \sigma(\omega)\right) \mathrm{d}\nu(y)\right] \\
&= e^{-\lambda_- n} + \mathbb{E}\left[\int_{S^1} d(y, \sigma(\omega))\, \mathrm{d}(f^k_\omega)^{-1}\nu(y)\right].
\end{aligned}$$

The dominated convergence theorem and Theorem 2.3, (ii) imply that

$$\mathbb{E}\left[\int_{S^1} d(y, \sigma(\omega))\, \mathrm{d}(f^k_\omega)^{-1}\nu(y)\right] \xrightarrow[k \to \infty]{} \mathbb{E}\left[\int_{S^1} d(y, \sigma(\omega))\, \mathrm{d}1_{\sigma(\omega)}(y)\right] = 0,$$

so (4.7) holds. ☐

Recall that the proof of Theorem C (when the subgroups of $\mathrm{Homeo}_+(S^1)$ act proximally) involves trying to find for a given $n \in \mathbb{N}$ small disjoint open intervals $U, V \subset S^1$ containing $\sigma(\omega)$ and $f^n_\omega(0)$ respectively such that $f^n_\omega(S^1 \setminus U) \subseteq V$. Here, the diameters of U, V depend on ω but not on n. In this sense, $\sigma(\omega)$ and $f^n_\omega(0)$ are a repeller-attractor pair for f^n_ω in a weak sense that is sufficient for the proof of the qualitative statement in Theorem C. On the other hand, to show Theorem A we need to show that $\mathbb{P}\left[f^n_\omega(S^1 \setminus U_n) \subset V_n\right]$ is exponentially close to 1 as n increases, where U_n and V_n are disjoint intervals centered around $\sigma(\omega)$ and $f^n_\omega(0)$ respectively such that $\mathrm{diam}(U_n)$, $\mathrm{diam}(V_n)$ are exponentially small in n. The fact that this contraction takes place does not follow from the definition of $\sigma(\omega)$, since $\sigma(\omega)$ is defined by an asymptotic condition saying that $\mathbb{P}$-almost surely any fixed closed interval inside $S^1 \setminus \{\sigma(\omega)\}$ is eventually contracted by f^n_ω. Nevertheless, the following proposition shows that $\sigma(\omega)$ and $f^n_\omega(0)$ are a repeller-attractor pair for f^n_ω in a strong sense suitable for our purposes. This is one of the main points where the strategy deviates from the linear case.

Proposition 4.4. *There exists $\varepsilon \in (0,1)$ such that*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left[f^n_\omega\left(S^1 \setminus \sigma(\omega)^{\varepsilon^n}\right) \textit{ is not contained in } f^n_\omega(0)^{\varepsilon^n}\right] < 0.$$

Proof. Let $\lambda > 0$, $N \in \mathbb{N}$ be the constants given by Theorem 4.2, so

$$\sup_{x,y\in S^1} \mathbb{E}\left[d\left(f_\omega^n(x), f_\omega^n(y)\right)\right] \le e^{-\lambda n} \tag{4.10}$$

for all $n \ge N$.

For $n \ge N$ define $K_n = \lfloor e^{\lambda n/3} \rfloor$, the grid $x_k^n = k/K_n \in S^1$, $0 \le k \le K_n - 1$ and the event

$$V_n = \{\omega \in \Omega \mid d(f_\omega^n(x_k^n), f_\omega^n\left(x_{k+1}^n\right)) \le e^{-\lambda n/2} \text{ for all } 0 \le k \le K_n - 1\},$$

so we have

$$\mathbb{P}\left[V_n^c\right] \le \sum_{k=1}^{K_n} \mathbb{P}\left[d\left(f_\omega^n(x_k^n), f_\omega^n(x_{k+1}^n)\right) \ge e^{-\lambda n/2}\right] \le e^{-\lambda n/2} K_n \le e^{-\lambda n/6},$$

where we have used the Markov inequality and (4.10).

Notice that if $\omega \in V_n$, then there exists a unique interval $J_{n,\omega} \subset S^1$ of the form $[x_j^n, x_{j+1}^n)$ such that $\operatorname{diam}\left(f_\omega^n(J_{n,\omega})\right) \ge 1 - e^{-\lambda n/2}$.

Claim. *There exists $C > 0$ such that $\mathbb{E}\left[d(J_{n,\omega}, \sigma(\omega)) \mid V_n\right] \le Ce^{-\lambda_1 n}$ for all sufficiently large $n \in \mathbb{N}$, where $\lambda_1 = \max\{\lambda/6, \lambda_-\}$ and λ_- is given by Proposition 4.3.*

Proof of the claim. Given $\omega \in V_n$, define $j_{n,\omega} \in S^1$ by

$$j_{n,\omega} = \begin{cases} (f_\omega^n)^{-1}(0) & \text{if } 0 \in f_\omega^n(J_{n,\omega}) \\ (f_\omega^n)^{-1}(1/2) & \text{otherwise.} \end{cases}$$

Since $\operatorname{diam}\left(f_\omega^n(J_{n,\omega})\right) \ge 1 - e^{-\lambda n/2}$, for all $n > 2\log 2/\lambda$ we have that $f_\omega^n(J_{n,\omega})$ contains 0 or 1/2, so $j_{n,\omega} \in J_{n,\omega}$. Thus

$$\begin{aligned}
\mathbb{E}\left[d(J_{n,\omega}, \sigma(\omega)) \mid V_n\right] &\le \mathbb{E}\left[\operatorname{diam}(J_{n,\omega}) + d(j_{n,\omega}, \sigma(\omega)) \mid V_n\right] \\
&\le \frac{1}{K_n} + \mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \sigma(\omega)\right) \mid V_n\right] + \mathbb{E}\left[d\left((f_\omega^n)^{-1}(1/2), \sigma(\omega)\right) \mid V_n\right] \\
&\le \frac{1}{K_n} + \mathbb{P}\left[V_n\right]^{-1}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \sigma(\omega)\right)\right] + \mathbb{E}\left[d\left((f_\omega^n)^{-1}(1/2), \sigma(\omega)\right)\right]\right)
\end{aligned}$$

for $n > \max\{2\log 2/\lambda, N\}$. From the bound (4.7) and the fact that $\mathbb{P}[V_n]$ is bounded away from 0 we obtain the conclusion. □

Let $C, \lambda_1 > 0$ be the constants given by the previous claim and take $\varepsilon \in (e^{-\lambda_1}, 1)$, so that

$$\begin{aligned}
\mathbb{P}\left[f_\omega^n\left(S^1 \setminus \sigma(\omega)^{\varepsilon^n}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n}\right] &\le \mathbb{P}\left[f_\omega^n\left(S^1 \setminus \sigma(\omega)^{\varepsilon^n}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] + \mathbb{P}\left[V_n^c\right] \\
&\le \mathbb{P}\left[J_{n,\omega} \not\subseteq \sigma(\omega)^{\varepsilon^n} \text{ or } f_\omega^n\left(S^1 \setminus J_{n,\omega}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] + \mathbb{P}\left[V_n^c\right].
\end{aligned} \tag{4.11}$$

for all $n \in \mathbb{N}$ large enough. Since $\varepsilon > e^{-\lambda/3}$ and $K_n = \lfloor e^{\lambda n/3} \rfloor$, there exists a constant $C' > 0$ such that the inequalities

$$\begin{aligned}
\mathbb{P}\left[J_{n,\omega} \not\subseteq \sigma(\omega)^{\varepsilon^n} \,\middle|\, V_n\right] &\le \mathbb{P}\left[d(J_{n,\omega}, \sigma(\omega)) \ge \varepsilon^n - \operatorname{diam}(J_{n,\omega}) \,|\, V_n\right] \\
&\le \frac{\mathbb{E}\left[d(J_{n,\omega}, \sigma(\omega) \,|\, V_n\right]}{\varepsilon^n - 1/K_n} \\
&\le C\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n \left(\frac{1}{1 - 1/(\varepsilon^n K_n)}\right) \le C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n
\end{aligned} \tag{4.12}$$

hold, so the right-hand side of (4.12) decreases with exponential speed towards 0 by the choice of ε. Here, the first inequality follows from the fact that whenever $\operatorname{diam}(J_{n,\omega}) + d(J_{n,\omega}, \sigma(\omega)) \le \varepsilon^n$ then necessarily $J_{n,\omega}$ is included in $\sigma(\omega)^{\varepsilon^n}$.

Similarly, since $\varepsilon^n \geq e^{-\lambda n/6} \geq \operatorname{diam}\left(f_\omega^n\left(S^1 \setminus J_{n,\omega}\right)\right)$ for all sufficiently large $n \in \mathbb{N}$, we see that

$$\begin{aligned}\mathbb{P}\left[f_\omega^n\left(S^1 \setminus J_{n,\omega}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] &\leq \mathbb{P}\left[0 \in J_{n,\omega} \,|\, V_n\right] \\ &\leq \mathbb{P}\left[0 \in J_{n,\omega} \text{ and } J_{n,\omega} \subseteq \sigma(\omega)^{\varepsilon^n} \,\middle|\, V_n\right] + C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n \\ &\leq \mathbb{P}\left[d(0,\sigma(\omega)) \leq \varepsilon^n \,|\, V_n\right] + C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n,\end{aligned}$$

where we have used (4.12) in the second inequality. Moreover, Theorem 2.6 applied to the distribution of σ (which is $\overline{\mu}$-stationary, where $\overline{\mu}(g) = \mu(g^{-1})$ for $g \in G$) provides $C'', \alpha > 0$ such that

$$\mathbb{P}\left[d(0,\sigma(\omega)) \leq \varepsilon^n\right] \leq C''\varepsilon^{\alpha n},$$

and from (4.11) we conclude that

$$\mathbb{P}\left[f_\omega^n\left(S^1 \setminus J_{n,\omega}\right) \not\subseteq f_\omega^n(0)^{\varepsilon^n} \,\middle|\, V_n\right] \leq \mathbb{P}\left[V_n\right]^{-1} C''\varepsilon^{\alpha n} + C'\left(\frac{e^{-\lambda_1}}{\varepsilon}\right)^n. \tag{4.13}$$

The bounds (4.13) and (4.12) show that the right-hand side of (4.11) is exponentially small in n. This finishes the proof of the proposition. ☐

From now on, the rest of the proof of Theorem A follows the strategy of [Aou11].

Proposition 4.5. *For every $t \in (0,1)$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[d(f_\omega^n(0), \sigma(\omega)) \leq t^n\right] < 0.$$

Proof. We start with a version of [Aou11, Theorem 4.35] (which in turn is inspired by [Gui90, Lemme 8]), which states that the variables $f_\omega^n(0)$ and $\sigma(\omega)$ become asymptotically independent with exponential speed as $n \to \infty$.

Claim. *There exists a random variable $S(\omega) \in S^1$ independent of σ and constants $C, \lambda > 0$ such that for any Lipschitz function $\psi\colon S^1 \times S^1 \to \mathbb{R}$ we have*

$$\left|\mathbb{E}\left[\psi\left(f_\omega^n(0), \sigma(\omega)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), S(\omega)\right)\right]\right| \leq Ce^{-\lambda n}|\psi|_{\mathrm{Lip}}$$

for sufficiently large $n \in \mathbb{N}$, where

$$|\psi|_{\mathrm{Lip}} = \sup_{\substack{x,y,u,v \in S^1 \\ x \neq y \text{ or } u \neq v}} \frac{|\psi(x,u) - \psi(y,v)|}{d(x,y) + d(u,v)}.$$

Proof of the claim. Let $\lambda_-, \lambda_+ > 0$ be the constants given by Proposition 4.3. Consider an independent copy $\omega' = (f_{\omega'_n})_{n\geq 0}$ of the process ω (that is, a coupling of $\mathbb{P}$ with itself). Define $S(\omega')$ as the repeller of the random walk $\left(f_{\omega'_n}^{-1} \circ f_{\omega'_{n-1}}^{-1} \circ \cdots \circ f_{\omega'_0}^{-1}\right)_{n\geq 0}$, so that

$$\sup_{x\in S^1} \mathbb{E}\left[d\left(\overline{f}_{\omega'}^n(x), S(\omega')\right)\right] \leq e^{-\lambda_+ n}$$

holds for all large $n \in \mathbb{N}$.

Decompose

$$\left|\mathbb{E}\left[\psi\left(f_\omega^n(0), \sigma(\omega)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), S(\omega')\right)\right]\right| \leq \triangle_1 + \triangle_2 + \triangle_3 + \triangle_4$$

where

- $\triangle_1 = \left|\mathbb{E}\left[\psi\left(\sigma(\omega), f_\omega^n(0)\right)\right] - \mathbb{E}\left[\psi\left((f_\omega^n)^{-1}(0), f_\omega^n(0)\right)\right]\right|$,
- $\triangle_2 = \left|\mathbb{E}\left[\psi\left((f_\omega^n)^{-1}(0), f_\omega^n(0)\right)\right] - \mathbb{E}\left[\psi\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), f_{\omega_n} \circ \cdots \circ f_{\omega_{\lceil n/2\rceil+1}}(0)\right)\right]\right|$,

- $\triangle_3 = \left|\mathbb{E}\left[\psi\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), f_{\omega_n}\circ\cdots\circ f_{\omega_{\lceil n/2\rceil+1}}(0)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right]\right|$
$= \left|\mathbb{E}\left[\psi\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right]\right|$, and
- $\triangle_4 = \left|\mathbb{E}\left[\psi\left(\sigma(\omega), \overline{f}_{\omega'}^{\lfloor n/2\rfloor}(0)\right)\right] - \mathbb{E}\left[\psi\left(\sigma(\omega), S(\omega')\right)\right]\right|$.

Proposition 4.3 shows that $\triangle_1 \le |\psi|_{\mathrm{Lip}} e^{-\lambda_- n}$, $\triangle_3 \le |\psi|_{\mathrm{Lip}} e^{-\lambda_- n/2}$ and $\triangle_4 \le |\psi|_{\mathrm{Lip}} e^{-\lambda_+ n/2}$ for all large $n \in \mathbb{N}$. Moreover

$$\begin{aligned}
\triangle_2 &\le |\psi|_{\mathrm{Lip}}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0)\right)\right] + \mathbb{E}\left[d\left(f_\omega^n(0), f_{\omega_n}\circ\cdots\circ f_{\omega_{\lceil n/2\rceil+1}}(0)\right)\right]\right)\\
&= |\psi|_{\mathrm{Lip}}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0)\right)\right] + \mathbb{E}\left[d\left(\overline{f}_\omega^n(0), \overline{f}_\omega^{\lfloor n/2\rfloor}(0)\right)\right]\right)\\
&\le |\psi|_{\mathrm{Lip}}\left(\mathbb{E}\left[d\left((f_\omega^n)^{-1}(0), \sigma(\omega)\right)\right] + \mathbb{E}\left[d\left(\left(f_\omega^{\lceil n/2\rceil}\right)^{-1}(0), \sigma(\omega)\right)\right]\right.\\
&\qquad \left. + \mathbb{E}\left[d\left(\overline{f}_\omega^n(0), T(\omega)\right)\right] + \mathbb{E}\left[d\left(\overline{f}_\omega^{\lfloor n/2\rfloor}(0), T(\omega)\right)\right]\right)
\end{aligned}$$

which is at most

$$|\psi|_{\mathrm{Lip}}\left(e^{-\lambda_- n} + e^{-\lambda_- n/2} + e^{-\lambda_+ n} + e^{-\lambda_+ n/2}\right)$$

by Proposition 4.3 again. The claim follows. □

For any $\varepsilon \in (0, 1/2)$, take a $1/\varepsilon$-Lipschitz function $\phi_\varepsilon\colon [0,1] \to [0,1]$ such that $\phi\big|_{[0,\varepsilon]} = 1$ and $\phi\big|_{[2\varepsilon,1]} = 0$, so

$$1_{[0,\varepsilon]} \le \phi_\varepsilon \le 1_{[0,2\varepsilon]}$$

holds and $\psi_\varepsilon \doteq \phi_\varepsilon \circ d\colon S^1 \times S^1 \to [0,1]$ is also $1/\varepsilon$-Lipschitz. Let $C, \lambda > 0$ be the constants given by the previous claim.

Now for all $n \in \mathbb{N}$ large enough we have

$$\begin{aligned}
\mathbb{P}\left[d\left(f_\omega^n(0), \sigma(\omega)\right) \le t^n\right] \le \mathbb{E}\left[\psi_{t^n}\left(f_\omega^n(0), \sigma(\omega)\right)\right] &\le \mathbb{E}\left[\psi_{t^n}\left(\sigma(\omega), S(\omega)\right)\right] + Ce^{-\lambda n}|\psi_{t^n}|_{\mathrm{Lip}}\\
&\le \mathbb{P}\left[d\left(\sigma(\omega), S(\omega)\right) \le 2t^n\right] + Ce^{-\lambda n}|\psi_{t^n}|_{\mathrm{Lip}}\\
&\le \sup_{x \in S^1} \mathbb{P}\left[d(\sigma(\omega), x) \le 2t^n\right] + Ce^{-\lambda n}|\psi_{t^n}|_{\mathrm{Lip}},
\end{aligned}$$

where we have used the independence of σ and S in the last inequality. The first term

$$\sup_{x \in S^1} \mathbb{P}\left[d(\sigma(\omega), x) \le 2t^n\right]$$

is exponentially small in n by Theorem 2.6, and the second term

$$Ce^{-\lambda n/2}|\psi_{t^n}|_{\mathrm{Lip}} = C\left(\frac{e^{-\lambda/2}}{t}\right)^n$$

is also exponentially small in n whenever $t > e^{-\lambda/2}$. The proposition is thus proven in this case, and is also true for $t \le e^{-\lambda/2}$ as a consequence. □

Remark. In the proof of the previous theorem we have abused notation by writing $S(\omega)$: the random variable S is not a function of ω, and is defined on a larger probability space. We have done so as to not to weigh down the notation with distinctions between this larger probability space and its quotient $(\Omega, \mathbb{P})$, since all relevant means and measures of sets coincide with those of the measure $\mathbb{P}$.

Proof of Theorem A. Fix two non-degenerate probability measures μ_1, μ_2 on countable subgroups G_1, G_2 of $\mathrm{Diff}_+^1(S^1)$ acting proximally on S^1 such that μ_1, μ_2 satisfy the moment condition (M).

Claim. *For every $n \in \mathbb{N}$ let $\omega' \in \Omega_2 \mapsto l_n(\omega') \in S^1$ be a measurable map. For every $t \in (0,1)$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(f^n_\omega(0), l_n(\omega')) \le t^n\right] < 0 \tag{4.14}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(\sigma(\omega), l_n(\omega')) \le t^n\right] < 0. \tag{4.15}$$

Proof of the claim. By independence, we have

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(f^n_\omega(0), l_n(\omega')) \le t^n\right] \le \sup_{x\in S^1} \mathbb{P}_1 \left[d(f^n_\omega(0), x) \le t^n\right] = \sup_{x\in S^1} \mathbb{P}_1 \left[d(\overline{f}^n_\omega(0), x) \le t^n\right].$$

Proposition 4.3 and the Markov inequality imply that

$$\mathbb{P}_1 \left[d\left(\overline{f}^n_\omega(0), T(\omega)\right) \ge e^{-\lambda_+ n/2}\right] \le e^{-\lambda_+ n/2},$$

for some $\lambda_+ > 0$ and all large $n \in \mathbb{N}$, and thus

$$\sup_{x\in S^1} \mathbb{P}_1 \left[d\left(\overline{f}^n_\omega(0), x\right) \le t^n\right] \le \sup_{x\in S^1} \mathbb{P}_1 \left[d\left(T(\omega), x\right) \le t^n + e^{-\lambda_+ n/2}\right] + e^{-\lambda_+ n/2}.$$

Take $C, \alpha > 0$ such that the distribution of T is (C,α)-Hölder, so

$$\sup_{x\in S^1} \mathbb{P}_1 \left[d(T(\omega), x) \le t^n + e^{-\lambda_+ n/2}\right] \le C(t^n + e^{-\lambda_+ n/2})^\alpha$$

and

$$\mathbb{P}_1 \otimes \mathbb{P}_2 \left[d(f^n_\omega(0), l_n(\omega')) \le t^n\right] \le C(t^n + e^{-\lambda_+ n/2})^\alpha + e^{-\lambda_+ n/2}$$

is exponentially small in n. This gives (4.14), and (4.15) follows in the same way. □

Take $\varepsilon \in (0,1)$ so that the conclusion of Proposition 4.4 is verified for $\mathbb{P}_1 = \mu_1^{\otimes\mathbb{N}}$ and $\mathbb{P}_2 = \mu_2^{\otimes\mathbb{N}}$. Given $\omega \in \Omega_1$, $\omega' \in \Omega_2$ and $n \in \mathbb{N}$ we say that f^n_ω *and* $f^n_{\omega'}$ *are in ε-transverse position at time n* if the intervals $f^n_\omega(0)^{\varepsilon^n}$, $f^n_{\omega'}(0)^{\varepsilon^n}$, $\sigma(\omega)^{\varepsilon^n}$ and $\sigma(\omega')^{\varepsilon^n}$ are pairwise disjoint. This is exactly the situation in Figure 1 for

$$I_{n,\omega} = f^n_\omega(0)^{\varepsilon^n}, \quad I_{n,\omega'} = f^n_{\omega'}(0)^{\varepsilon^n}, \quad J_{n,\omega} = \sigma(\omega)^{\varepsilon^n} \text{ and } J_{n,\omega'} = \sigma(\omega')^{\varepsilon^n}.$$

Proposition 4.5 shows that the probability that the pair of sets $(f^n_\omega(0)^{\varepsilon^n}, \sigma(\omega)^{\varepsilon^n})$ intersect or that the pair $(f^n_{\omega'}(0)^{\varepsilon^n}, \sigma(\omega')^{\varepsilon^n})$ intersect is exponentially small in n. The previous claim shows that the probability that the remaining pairs

$$(f^n_\omega(0)^{\varepsilon^n}, \sigma(\omega')^{\varepsilon^n}), \quad (f^n_{\omega'}(0)^{\varepsilon^n}, \sigma(\omega)^{\varepsilon^n}), \quad (f^n_\omega(0)^{\varepsilon^n}, f^n_{\omega'}(0)^{\varepsilon^n}) \text{ or } (\sigma(\omega)^{\varepsilon^n}, \sigma(\omega')^{\varepsilon^n})$$

intersect is exponentially small in n. We conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_1 \otimes \mathbb{P}_2 \left[f^n_\omega \text{ and } f^n_{\omega'} \text{ are not in } \varepsilon\text{-transverse position}\right] < 0. \qquad \square$$

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