Stability

Not stability of a system, but stability of a nominal solution $\hat{x}(t)$, $\hat{u}(t)$

Internal stability: study stability of $\hat{x}(t)$, $\hat{y}(t)$ when initial condition is perturbated (input remains the same)

External stability: study stability of $\hat{x}(t)$, $\hat{y}(t)$ when only the input is changed $\hat{u}(t) + \tilde{u}(t)$

Internal Stability

Equilibrium points

For a system $\dot{x}(t) = f(x)$, an equilibrium point x_e satisfies

$$f(x_e) = 0$$

The system remains unchanged if undisturbed

We linearize nonlinear systems around such points

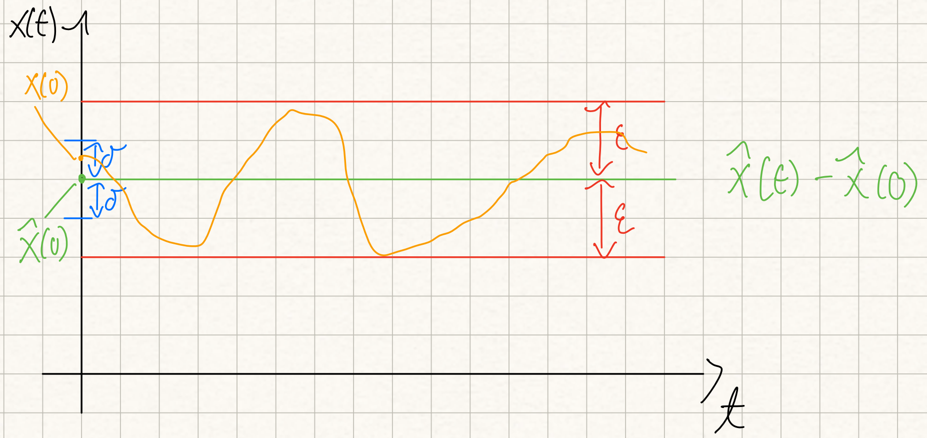
Local stability

Describes behavior near an equilibrium point.

A nominal solution $\hat{x}(t)$ is locally stable if for every small neighborhood around $\hat{x}(t)$ (ball of radius ε) there exists a smaller neighborhood such that all trajectories starting within that neighboorhood remain inside it for all time.

$\forall \varepsilon > 0, \exists \delta > 0$, s.t.

$$\|x(0) - \hat{x}(0)\| < \delta \Rightarrow \|x(t) - \hat{x}(t)\| < \varepsilon, \forall t \geq 0$$



Imagine a marble inside a bowl. If slightly nudged, the marble oscillates inside the bowl but doesn't escape. This is stable behavior.

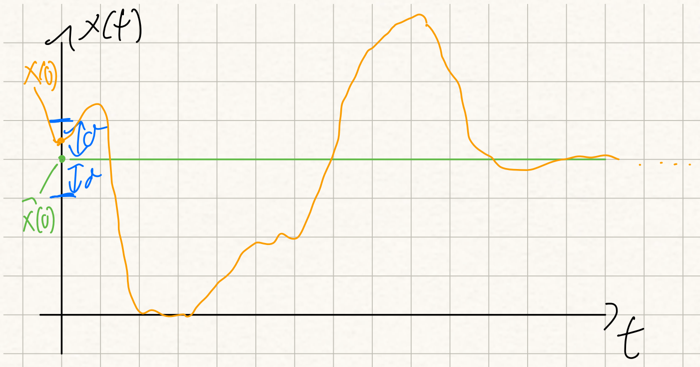
Local Attractivity

Describes whether the system converges back to the equilibrium point over time

A nominal solution $\hat{x}(t)$ is locally attractive if all trajectories starting sufficiently close to the nominal solution eventually converge to it

$\exists \delta > 0$ s.t.

$$\|x(0) - \hat{x}(0)\| < \delta \Rightarrow \lim_{t \to \infty} \|x(t) - \hat{x}(t)\| = 0$$



Asymptotic stability

A nominal solution $\hat{x}(t)$ is locally asymptotically stable if it is both locally stable and locally attractive.

Lyapunov's Criteria for Local Stability

Direct way to analyze stability without solving the diff. equations.
Construct a Lyapunov function - a scalar energy-like function that decreases over time

Nonlinear system

$$\dot{x}(t) = f(x)$$

A Lyapunov function is a scalar function $V(x): \mathbb{R}^n \to \mathbb{R}$ with the properties:

1) $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$ (positive definite)
2) The time derivative along trajectories is $\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$

- Local stability:

  Equilibrium at $x=0$ is locally stable if there exists a continously differentiable Lyapunov function s.t.:

- $V(x)$ is positive definite
- $\dot{V}(x) \leq 0$ (negative semi-definite)

- Local Asymptotic Stability:
  Equilibrium at $x=0$ is locally asymptotically stable if:
  - $V(x)$ is positive definite
  - $\dot{V}(x) < 0$ (negative definite)

Krasovski's Criteria for Local Asymptotic Stability

A modification of Lyapunov's Criteria that's sometimes easier to apply.

In addition to the two conditions for Lyapunov's criteria, we have:

3) The set $\mathcal{V} = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ contains no complete trajectories except $\hat{x}(t)$

Lyapunov's Criteria for Global Stability

Extends Lyapunov's local criteria to entire state space

Equilibrium point $x_e = 0$ is globally stable if there exists a Lyapunov function $V(x)$ s.t.:

1) $V(x)$ positive definite $\forall x \neq x_e$
2) $\dot{V}(x) \leq 0 \quad \forall x \neq x_e$
3) $V(x)$ radially unbounded: $\lim_{\|x\| \to \infty} V(x) = \infty$

Global Asymptotic Stability:
Condition 2) above becomes:

2) $\dot{V}(x) < 0 \quad \forall x \neq x_e$

Krasowski's Criteria for Global Stability

Add another condition to the local stability criteria:

4) $V(x)$ is radially unbounded, $\lim_{\|x\| \to \infty} V(x) = \infty$

Reduced Lyapunov Criteria for Local Stability

Simplify standard Lyapunov criteria by focusing on the linearized dynamics near the equilibrium

For a linear system

$$\dot{x} = f(x) \quad , \quad O_x = f(O_x)$$

$$A = \frac{\partial f(x)}{\partial x}\bigg|_{x=O_x} \qquad \dot{\tilde{x}} = A\tilde{x} \text{ linearized system around } O_x$$

1) A has all eigenvalues asymptotically stable
   C.T.: $Re(\lambda_i) < 0 \ , \ \forall i = 1, \ldots, n$
   D.T.: $|\lambda_i| < 1 \ , \ \forall i = 1, \ldots, n$
   then O_x is locally asymptotically stable
2) If there exists at least one eigenvalue that is "unstable"
   C.T.: $Re(\lambda) > 0$
   D.T.: $|\lambda| > 1$
   then O_x is unstable

3) In all other situation we cannot conclude anything

Instability criteria

Lyapunov's Instability criterion

For a system $\dot{x} = f(x)$ with equilibrium point x_e, if there exists a Lyapunov function $V(x) \in C^1$ s.t.:

1) $V(x) = 0$
2) $\dot{V}(x) > 0 \quad \forall x \neq x_e$ in a neighbhoorhood of x_e
3) In every neighboorhood of x_e, there exists a point x_1 where $V(x_1) > 0$

   $\forall \varepsilon > 0 \quad \exists x \in B_\varepsilon(0_x)$ s.t. $V(x) > 0$

then x_e is unstable

Chetaev's Instability Criteria

If there exists a Lyapunov function $V(x) \in C^1$ and an open set A that contains x_e in its boundary s.t.:

1) $V(x) > 0, \; \dot{V}(x) > 0, \; \forall x \in B_r(x_e) \cap A \setminus \{x_e\}$
2) $V(x) = 0, \; \forall x \in B_r(x_e) \cap \partial A / \{x_e\}$

then x_e is unstable

Stability for LTI Systems

$$\dot{x}(t) = Ax(t)$$
$$x_{k+1} = Ax_k$$

Equilibrium points:

C.T.: $0_x = Ax_e$, x_e are all eq. points $x_e \in \ker(A)$

D.T.: $0_x = Ax_e$, x_e are all eq. points $x_e \in \ker(I-A)$

C.T.:

A is full rank $\Leftrightarrow$ $x_e = 0_x$ is the only eq. point

$\Rightarrow$ 0_x could be asymptotically stable or unstable

A is not full rank $\Leftrightarrow$ x_e are all vectors in $\ker(A)$

$\Rightarrow$ 0_x could be stable or unstable

D.T.:

$(I-A)$ is full rank $\Leftrightarrow$ $x_e = 0_x$ is the only eq. point

$\Rightarrow$ 0_x could be asymptotically stable or unstable

$(I-A)$ is not full rank $\Leftrightarrow$ x_e are all vectors in $\ker(I-A)$

$\Rightarrow$ 0_x could be stable or unstable

Lyapunov Criteria LTI Systems

Lyapunov function

$$V(x) = x^T P x \quad , \quad P \text{ symmetric, positive definite}$$

$$\dot{V}(x) = x^T(A^TP + PA)x$$

$$A^TP + PA = -Q \quad , \quad Q = Q^T > 0 \; \left(\begin{matrix}\text{symmetric} \\ \text{positive definite}\end{matrix}\right)$$

Theorem

$\dot{x} = Ax$ is asymptotically stable iff for any $Q = Q^T > 0$, the Lyapunov equation

$$A^TP + PA = -Q$$

has a unique solution $P = P^T > 0$

Theorem

For any choice of $Q = Q^T > 0$, the Lyapunov equation admits a solution $P = P^T > 0$ iff the eigs of A have all $Re(\lambda_i) < 0$, $i = 1, \dots, n$

External Stability

Describes relationship between the input and output of a system

External stability: if bounded inputs produce bounded outputs over time

BIBO stability $\Longleftrightarrow$ All poles of $W_{yu}(s)$ have $Re(s) < 0$