Synthesis of Control Laws and State Observers

Feedback Control

Using measurements of the system's output (or state) to adjust the input in order to achieve the desired behavior

- Helps stabilize the system
- Reduces the effect of external disturbances
- Ensures the system follows desired performance criteria
- Can compensate for modeling uncertainties

Different Types

- Static state feedback control

  Uses full state vector to compute control input:

  $$u(t) = -Kx(t)$$

  K is state feedback gain matrix

  goal is to modify dynamics by shifting eigs of $A-BK$

  $\div$ Requires full state information

- Output feedback control

  If only system output is available, the control law uses

  $$u(t) = -Ky(t)$$

  Less powerfull since it uses limited information

- Dynamic output feedback control

When full state is not available, can use an observer to estimate the state. Control law

$$u(t) = -K\hat{x}(t)$$

where $\hat{x}(t)$ is the estimated state from the observer

Regulation Control Problem

Focuses on keeping the system at a desired constant state or output despite disturbances.

Tracking Control Problem

Making the output follow a time-varying reference trajectory

Static Linear State Feedback Control Design

Designing control law where control input is directly computed as a linear function of the system's state

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$u(t) = -Kx(t)$$

Goal is to modify closed-loop system dynamics and obtain stability

Putting control law $u(t)$ in the state equation:

$$\dot{x}(t) = (A - BK)\,x(t)$$

The matrix $A-BK$ determines the behavior

Pole Allocation Theorem

Method for designing feedback matrix K

A system $\dot{x}(t) = Ax(t) + Bu(t)$

is completely state controllable iff it is possible to place all poles of the closed-loop system arbitrarily using K

The system must be controllable

R must be full rank

Stabilizability

A system is stabilizable if it is possible to place all unstable poles in the left half-plane using a K.

A system may not be fully controllable but can still be stabilizable if all unstable modes can be controlled

Stabilizability is a weaker condition than controllability but still ensures that the system can be stabilized with feedback

Use the PBH test:

If for every eig λ with $Re(\lambda) \geq 0$

$$\text{rank}[\lambda I - A \quad B] = n$$

the system is stabilizable.

Dead-Beat Control (Only D.T.)

Drive the system to zero in a finite number of steps (n steps for an n-dimensional system)

The system must be completely controllable

Static Output Feedback

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$u(t) = -Ky(t)$$

$$\Rightarrow \quad \dot{x}(t) = (A - BKC)\, x(t) \qquad \text{closed-loop}$$

Asymptotical Observers (Luenberger)

System designed to estimate the full state vector of a system using measurements of the output. Provides an asymptotically accurate estimate of the state as time progresses.

System

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \ x(0)$$
$$y(t) = Cx(t) + Du(t)$$

The Luenberger observer estimates the state using the measured output and a correction term

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

where

$\hat{x}(t)$ is the estimated state vector

L is the observer gain matrix

Error Dynamics

Define the state estimation error as

$$e(t) = x(t) - \hat{x}(t)$$
$$\Rightarrow \quad \dot{e}(t) = (A - LC)\, e(t)$$

The observer is asymptotically stable if all eigenvalues of $A-LC$ lie in the left half-plane

Observer Allocation Theorem

If the system is observable, then there exists a gain matrix L s.t. the eigenvalues of $A-LC$ can be placed arbitrarily in the complex plane.

Given the desired poles $\lambda_{des} = \{\lambda_1^0, \lambda_2^0, \dots, \lambda_n^0\} \in \mathbb{C}$

there exists an L s.t.:

$$\det(\lambda I - (A - LC)) = P_{des}(\lambda) = (\lambda - \lambda_1^o)(\lambda - \lambda_2^o)\cdots(\lambda - \lambda_n^o)$$

iff (A, C) is observable

Detectability

weaker condition than observability, ensuring that only the unstable modes can be observed

Def:

A system is detectable if all unstable modes of the system can be observed from the output

A system is detectable if $\forall$ eigs λ with $Re(\lambda) \geq 0$:

$$\text{rank}[\lambda I - A^T \quad C^T] = n$$

Detectability is sufficient for designing asymptotic observers because only unstable modes need correction

The Ackermann Formula

$$L^T = [0, 0, \cdots, 1]\,\Theta^{-1} P_{des}(A^T)$$

Where

$P_{des}(A^T)$ is the desired characteristic polynomial evaluated at A^T

Θ is the observability matrix

Dead-Beat Observer (D.T. Only)

Goal is to estimate the state in finite number of steps
A dead-beat observer drives the estimation error to zero in exactly n steps for an n-dimensional system.

Dynamic Regulator (Compensator)

Use the estimated state $\hat{x}(t)$ together with the feedback controller instead of $x(t)$.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0)$$

$$y(t) = Cx(t) + Du(t)$$

$$\dot{\hat{x}}(t) = (A-LC)\hat{x}(t) + [(B-LD), \quad L]\begin{bmatrix} u \\ y \end{bmatrix}$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t)$$

Control law:

$$u(t) = K\hat{x}(t) + v(t)$$

$$\dot{\hat{x}}(t) = (A-LC)\hat{x}(t) + LCx(t) + Bu(t)$$

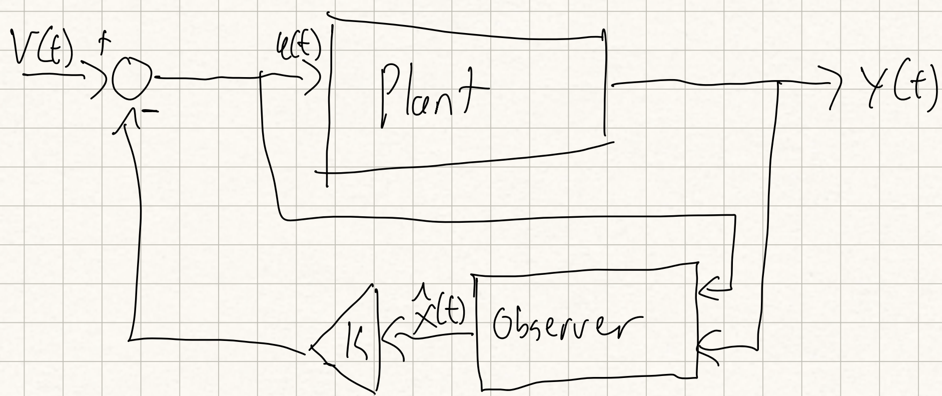
Compensated system:

$$\dot{x}(t) = Ax(t) + BK\hat{x}(t) + Bv(t)$$

$$\dot{\hat{x}}(t) = LCx(t) + (A-LC+BK)\hat{x}(t) + Bv(t)$$

$$y(t) = Cx(t) + DK\hat{x}(t) + Dv(t)$$

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ LC & A-LC+BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v \\ y = \begin{bmatrix} C, & DK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + Dv \end{cases}$$



Conditions for existence

Eigs of $A+BK$ arbitrarily requires (A,B) reachable

Eigs of $A+LC$ arbitrarily requires (A,C) observable

If only detectable and stabilizable: compensator can exist but cannot assign eigs arbitrarily

Polynomial Approach

Design compensator based on polynomials instead of state-space model

Consider single-input, single-output system described by transfer function:

$$G(s) = \frac{N(s)}{D(s)}$$

where

$N(s)$, $D(s)$ are the numerator and denominator polynomials of the system

s is the Laplace variable

Compensator can be represented as a transfer function:

$$C(s) = \frac{Q(s)}{P(s)}$$

where

$Q(s)$, $P(s)$ are polynomials chosen to shape the closed-loop response

Closed-loop transfer function:

$$T(s) = \frac{N(s)\,Q(s)}{D(s)\,P(s) + N(s)\,Q(s)}$$

goal is to design $P(s)$, $Q(s)$ s.t.:

- Stability: denominator roots must be stable
- Performance: response should meet the performance criteria

Polynomial Diophantine Equations

General form:

$$A(s)\,X(s) + B(s)\,Y(s) = C(s)$$

where

$A(s)$, $B(s)$, $C(s)$ are known

$X(s)$, $Y(s)$ are to be determined

In compensator design:

Desired cl-polynomial:

$$D_{cl}(s) = P(s)D(s) + Q(s)N(s)$$