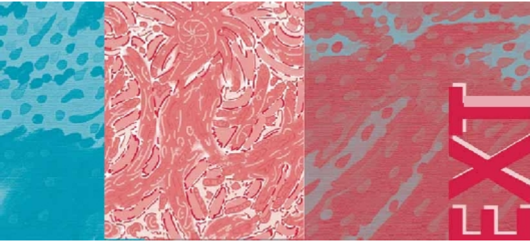
A. Gina
C. Seatzu

ANALYSIS OF DYNAMIC SYSTEMS



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Preface

The new educational system has necessitated a rapid adjustment of teaching programmes and undergraduate textbooks. The main novelty introduced by the new order is the fragmentation of the monolithic courses of the old degree into simpler courses spread over several years or even over several courses of study: bachelor*'s degree* and *master's degree*.

The classic texts that have formed the school of Automatica in Italy are not appropriate for the basic degree, not only because they presuppose a mathematical maturity that students may not yet have attained, but also because they present the various topics at a level of detail far beyond what the tight timeframe of the basic degree allows.

On the other hand, for the student who continues his or her studies until he or she obtains a master's degree, it is useful to have a single textbook intended as a guide and in-depth study of a discipline. The experience of An- glosian universities, where there has always been a basic (*bachelor's*) course followed by a specialized (*master's*) one, has taught us the usefulness of textbooks that can be used at multiple levels.

The text we present is devoted to the *analysis of continuous-time systems*. It is mainly devoted to the study of linear systems, but also contains some mention of nonlinear systems. Both *input-output models* and *models in state variables* are covered in it. The analysis techniques presented cover both the study in the *time domain, the Laplace variable domain* and the *frequency domain*. Although an attempt has been made to show the interconnections between all these analysis techniques, the various topics are treated in separate chapters and sections: in our intentions, this allows the text to be used as a teaching aid for teaching only a part of these topics.

The text covers the contents of:

a *systems analysis* (or *systems theory*) teaching devoted to the analysis of continuous-time linear systems for the undergraduate degree;

One or more teaching of *systems analysis complements* for the master's degree.

This necessitated a restructuring of the presentation to allow for two different reading paths.

First, special care has been taken to present each argument through a series of results that are first clearly stated and then demonstrated. On a first reading, it is always possible to skip the demonstration because one or more examples clarify how the result is to be applied. However, where-ever the reader wishes to explore the topic further, the demonstration is a useful complement: great care has been taken to present each demonstration in simple and intuitive terms, as far as possible.

Second, whole sections (and even an entire chapter, number 12) have been provided for in-depth topics. These sections are indi- cated with an asterisk and can be skipped without compromising the understanding of the remaining material.

Complementing the instructional materials presented in the text are a series of completed exercises and MATLAB programs that we believe will be useful to students at `http://www.diee.unica.it/ giua/ASD.`

We would like to thank our colleagues Maria Maddalena Pala and Elio Usai who read drafts of some chapters of this book and suggested useful changes. Additional thanks also go to all the students and tutors in the Systems Analysis course at the University of Cagliari, who in the years 2000-2005 read and corrected a number of notes and handouts from which this text later took shape.

Finally, special thanks go to our families who supported us by filling those gaps that the hard work involved in making this book inevitably generated.

Cagliari, September 2005

Alessandro Giua and *Carla Seatzu*

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1

Introduction

The objective of this chapter is to introduce the concepts underlying *Automatica*, the discipline of engineering to which this introductory text is devoted. The first section gives a brief definition of Automatica and the notion of a *system*, which is its main object of study. The second section briefly describes the problems that this discipline addresses and solves. A simple classification of the main approaches and models used is finally proposed in the third section.

1.1 Automatic and systems

Automatics or *Systems Engineering* is that discipline that studies the mathematical modeling of systems of different natures, analyzes their dynamic behavior, and makes appropriate control devices to make those systems behave as desired.

The underlying notion of Automatica is certainly that of a *system*. Numerous definitions of this entity have been proposed in the literature. At present, however, there is not one that can be considered universally recognized. The IEEE ma- nual, for example, defines a system as *a collection of elements that cooperate to perform a function that would otherwise be impossible for each of the individual components*. S. Battaglia's great dictionary of the Italian language defines a system as *a set, articulated complex of elements or instruments coordinated among themselves in view of a given function*.

In these definitions, however, an essential element that constitutes instead the main object of study of Automatica is not emphasized: the *dynamic behavior* of a system. According to the Automatica paradigm, in fact, a system is subject to external stresses that influence its evolution over time. In what follows, therefore, we will refer to the following definition according to which *a system is a physical entity, typically formed by several interacting components, that responds to external stresses by producing a certain behavior*.

Example 1.1 An electrical circuit consisting of components such as resistors, capacitors, inductors, diodes, current and voltage generators, etc., provides a simple example of a dynamic system. The behavior of the system can be described by the value of voltage and current signals in the branches of the circuit. The stresses acting on the system are the voltages and currents applied by the generators, which can be imposed from outside

Finally, it is important to note a peculiar aspect of Automatica: its indipendence from a particular technology. Many engineering disciplines are characterized by an interest in a particular application to which a particular technology corresponds: think of Electrotechnics which studies electrical circuits, Electronics which studies electronic devices, Computer Science which studies computing systems, etc. In contrast, Automatica is characterized by a formal *methodological approach* that is intended to be independent of a particular family of devices and is, therefore, potentially applicable in different application contexts.

1.2 Problems faced by Automatica

There are many activities covered by the interest of Automatica. Without claiming to be exhaustive, here we simply recall the main problems that this discipline allows to address and solve.

1.2.1 Modeling

In order to study a system, it is of fundamental importance to have a mathematical model- lo that describes its behavior in quantitative terms. Such a mo- del is usually constructed on the basis of knowledge of the devices that make up the system and the physical laws they obey.

Example 1.2 Suppose we have an electrical circuit consisting of two resistors and in series, as in Fig. 1.1. It is intended to describe how the current flowing through the circuit depends on the voltage.......Taking into account that both resistors satisfy Ohm's law, and taking into account how they are connected, we easily derive the model

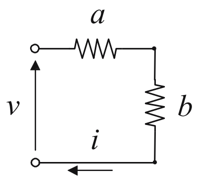


Fig. 1.1. Electrical system in Example 1.2

1.2.2 Identification

In some cases, knowledge of the devices that make up a system is not com- plete, and the model of the system can be constructed only on the basis of observation of its behavior. If it is known what and how many components there are but not all their parameters are known, one speaks of a *parametric identification* problem; in the more general case, however, one has no information about the constitution of the system and sometimes speaks of *black-box identification*.

Example 1.3 Suppose that in the circuit of the previous example the structure of the system is known but the value of the two resistors and are not known. In that case it is still possible to write the relation where it is an unknown parameter that needs to be identified. Based on the observation of the system- ma, several pairs of measurements are determined , for , represented on the graph in Fig. 1.2.a. Note that in general these points will not be perfectly aligned on a line of angular coefficient , basically due to two reasons. A first reason is due to the fact that the observations are always affected by inevitable *measurement errors*, which are more or less significant. A second reason is that *disturbances* act on the system that change its behavior: for example, a change in temperature between measurements can change the value of the resistance. One possible solution is to choose that value of *which* determines the straight line that best approximates the data, for example by interpolating in the least-squares sense as in Fig. 1.2.b

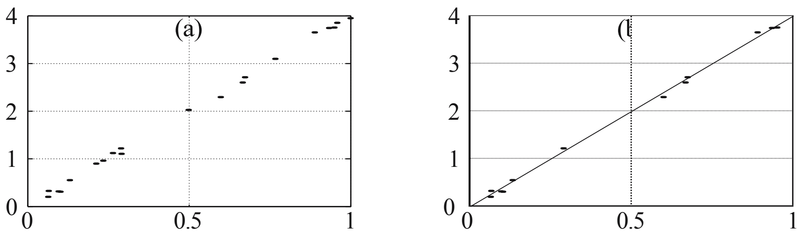


Fig. 1.2. Identification procedure in Example 1.3

1.2.3 Analysis

The fundamental problem of systems analysis is to predict the future behavior of a system based on the stresses to which it is subjected. To solve this problem in quantitative terms, it is essential to have a mathematical model of the system.

Example 1.4 The behavior of the marine ecosystem can be described by the e- volution over time of the population of fauna and flora, which is born, grows and dies. This behavior is influenced by climatic conditions, food preven- tion, human predators, pollutants in the water, etc. There has recently been a proposal to reduce the concentration of carbon dioxide in the earth's atmosphere by injecting the gases produced by industrial processes into the sea, where the carbon dioxide dissolves. An important analysis problem that has not yet been solved, partly due to the lack of an adequate model, is to determine what the behavior of the marine ecosystem would be under such stress.

1.2.4 Check

The goal of control is to impose a de- sidered behavior on a system. There are two main aspects related to this problem. First, it is necessary to define what is meant by desired behavior, through appropriate *specifications that* such behavior must satisfy. Second, one must design a device, called a *controller*, that by appropriately stressing the system is able to guide its evolution in the desired direction. The control problem is also called the *synthesis* problem, meaning the synthesis (or design) of the control device.

Example 1.5 In a water distribution network, it is desired to keep the pressure in the different branches constant. A pressure rating is given for each branch, and the specification requires that during network operation the instantaneous pressure value should not deviate from this by more than 10 percent. Two types of stresses act on this network modifying its behavior: the flows taken from the utilities and the pressures imposed by the pumps at some nodes of the network. The flows taken from the utilities are not variables that can be controlled and are to be considered as disturbances. The pressures imposed by the pumps, on the other hand, are variables that can be manipulated, and the purpose of the controller is precisely to determine how they must vary in order to meet the specification .

1.2.5 Optimization

The optimization problem can be seen as a special case of the control problem in which it is desired that the system achieve a given goal while optimizing *a* given *performance index*. This index, which measures the goodness of the system's behavior, typically takes into account multiple requirements.

Example 1.6 The suspension of a road vehicle is designed to contem- perate two different needs: to provide an adequate level of comfort for passengers and to ensure good road holding. Modern SUVs (sport utility vehicles) are equipped with semi-active suspension. In such devices, a controller oppor- tunately varies the suspension's damping coefficient to ensure the best compromise between these two needs depending on different driving c onditions (off-road or on road pavement). The performance index to be optimized takes into account the oscillations of the passenger compartment and the wheels .

1.2.6 Check

A verification procedure makes it possible, having a mathematical model with which to represent a system and a set of desired properties expressed in formal terms, to demonstrate through appropriate computational techniques that the model satisfies the desired properties.

This approach is particularly useful in the verification of a control device. In fact, it often happens that a control device is designed from the specification by semi-empirical methods: in such cases it is useful to verify that it meets the specification.

Example 1.7 An elevator is controlled in order to ensure that it responds to calls by serving the various floors. The control device is a programmable logic automaton (PLC: Programmable Logic Controller): to ensure that its pro- gram has no bugs and actually meets specifications, it may be useful to use formal verification techniques

1.2.7 Diagnosis of failure

A frequently occurring problem in dynamic systems is due to the occurrence of failures or malfunctions that change the nominal behavior of a si- stem. In such circumstances, it is necessary to have an approach to detect abnormal behavior that indicates the presence of a fault, identify the fault, and determine an appropriate corrective action that tends to re-establish nominal behavior.

Example 1.8 The human body is a complex system subject to a particular type of failure: disease. The presence of fever or other abnormal condition is a telltale symptom of the presence of disease. The physician, having identified the disease, treats the patient by prescribing appropriate therapy

1.3 Classification of systems

It has been said that Automatica is characterized by a methodological approach that is intended to be independent of a particular type of system. However, the great diversity of the systems one has an interest in studying and controlling has made it necessary to develop a substantial number of such approaches, each of which refers to a particular class of models and is applicable in particular contexts. It is then possible to give an initial classification of the methodologies and models under study in Au- tomatics as done in Fig. 1.3, where proceeding from top to bottom we go from a class to a subset of it.

By convention it is usual to denote these classes by the name of *systems* (e.g., hybrid systems, discrete event systems, etc.) but as mentioned it would be more correct to speak of *models* (e.g., hybrid models, discrete event models, etc.). In fact, one same system can often be described through one or the other of these models as will be seen in the examples presented in this section.

Finally, it should be noted that additional classifications are possible, which for the sake of brevity are not stated here. The sub-classes of interest in time-fasting systems, to which this text is devoted, are presented in Chapter 2.

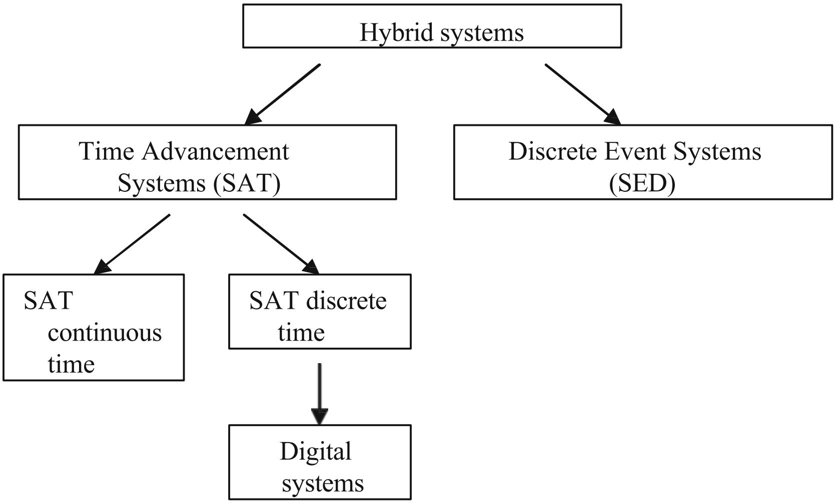


Fig. 1.3. Classification of systems under study by Automatica.

1.3.1 Advancement systems time

The systems that have so far been the main object of study of Auto- matics are the so-called *time advancing systems* (SAT). In such systems the behavior of the system is described by signals i.e., real functions of the independent time variable. If the time variable varies continuously we speak of *continuous-time* SAT, while if it takes values in a discrete set we speak of *discrete-time* SAT. In the particular case of discrete-time systems, it is possible to iden- tify the sub-class of *digital systems* in which the signals involved, and not only the time variable, also take discrete values.

The evolution of such systems arises from the passage of time. In the case of continuous-time SATs, the signals describing the behavior of the system satisfy a differential equation that specifies the instantaneous link between such signals and their derivatives. In the case of discrete-time SATs, the signals describing the behavior of the system satisfy a difference equation.

Example 1.9 (Continuous-time SAT) Consider the reservoir shown in Fig. 1.4. The volume of liquid in it [m] varies over time due to flow rates imposte da due pompe azionate dall'esterno. La portata entrante vale e quella uscente vale ; entrambe sono misurate in [m /s]. Supponendo che il serbatoio non si svuoti e non si riempia mai completamente, possiamo descrivere il comportamento di tale sistema mediante l'equazione

— (1.1)

Thus, it is a differential equation that binds together the continuous-time variables , and .

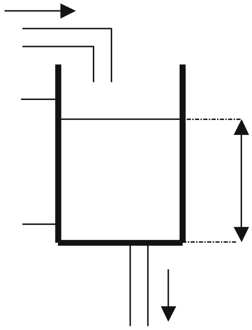


Fig. 1.4. A tank

Example 1.10 (discrete-time SAT) If in the reservoir shown in Fig. 1.4 the misure of volume and flow rate are not available continuously but only every unit of time, it is of interest to describe the behavior of the system only in the instants of time

Thus, discrete-time variables can be considered ,
and defined for .
Placed , approximating the derivative with the incremental ratio

and multiplying both members by , eq. (1.1) becomes

(1.2)

Thus, it is a difference equation that binds together discrete-time variables , and .

1.3.2 Event systems discrete

A discrete-event system can be defined as a dynamical system whose states take on different logical or symbolic, rather than numerical, values, and whose behavior is characterized by the occurrence of instantaneous *events* that occur with an irregular cadence that is not necessarily known. The behavior of such systems is described in terms, precisely, of states and events.

Example 1.11 (Discrete Event System) Consider a store of parts at- tending to be processed by a machine. Assume that the number of parts waiting cannot exceed two units and that the machine may be either machining or broken down.

The state of the overall system is given by the number of pending parts and the state of the machine. Thus, six states are possible:

: machine in process and empty storage, with one part or with two parts;
: broken machine and empty storage, with one part or with two parts.

The events that result in a change of state are:

: arrival of a new part in the repository;
: withdrawal by the machine of a part from the deposit;
: the machine breaks down;
: the machine is repaired.

The event can always occur as long as the filing does not contain two parts (in which case no new parts can arrive); this event changes the status from (i.e.) to (i.e.,). The event can occur only if the depot is not empty and the machine is being processed; this event changes the state from to Finally the events and determine, respectively, the transition from to and vice versa. This behavior can be formally described by means of the model in Fig. 1.5, which takes the form of a finite-state *a utomaton*.

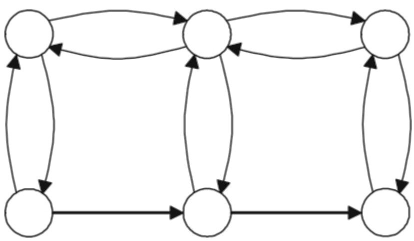


Fig. 1.5. Discrete event model of the repository in Example 1.11

There are inherently discrete-event systems such as the system described in the previous example. Many such systems are found in the fields of pro- ductivity, robotics, traffic, logistics (transportation and storage of products, organization and delivery of services), and computer and communications networks. At other times, given a system whose evolution could be de- scribed with a time advancing model, it is preferred to abstract and forgo a description of its behavior in terms of signals in order to highlight only the phenomena of interest. The following example presents such a case.

Example 1.12 (Discrete Event System) You want to control the tank stu- dered in Examples 1.9 and 1.10 to keep its level within a range

To do this, it is decided to use a supervisory device that turns off the pump associated with the incoming flow rate when the level is reached and turns off the pump associated with the outgoing flow rate when the level is ....reache dAi fini
of supervision, it is sufficient to describe the behavior of the system by means of a discrete event model such as the one represented by the automaton in Fig. 1.6. Such an automaton has three states (High, Medium, Low) and the corresponding events, which indicate the rag- reaching of levels and , can be detected by two simple level sensors placed in the tank

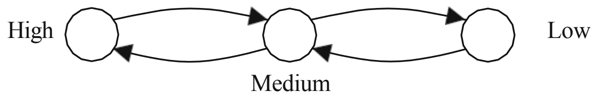


Fig. 1.6. Discrete event model of the reservoir in Fig. 1.4.

1.3.3 Systems hybrids

In common parlance, a hybrid is defined as a system consisting of components of different na- ture. Within Automatica, this term is used with a specific signi- fication: a *hybrid* system is a system whose behavior is described by means of a model that combines time-advancing dynamics with discrete-event dynamics. Because of their characteristics, hybrid systems can be considered as the most general class of dynamical systems, containing SAT and SED as subclasses, as shown in Fig. 1.3.

Example 1.13 (Hybrid system) Consider a Finnish sauna whose temperature is regulated by a thermostat. We can distinguish two main components in such a system.

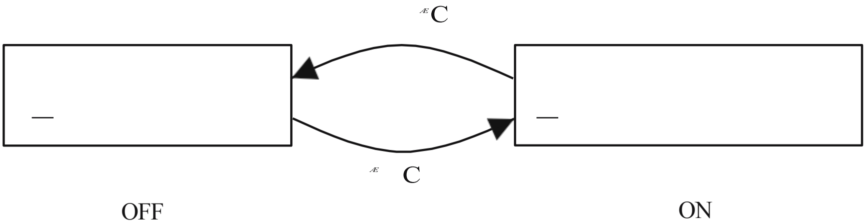


Fig. 1.7. Hybrid model of a Finnish sauna with thermostat in Example 1.13

A first component is the thermostat, whose behavior can well be described by an event-driven system: in the ON state it puts the heater on, and in the OFF state it keeps it off. Since it is desired to maintain the temperature between C and C, the events that cause it to switch from ON to OFF and vice versa are related to the attainment of these temperature levels.

A second component is the sauna cabin, whose state can be represented by its temperature , which is a continuous-time signal. When the thermostat is in the OFF state, the temperature decreases because the cabin loses heat to the external environment that is at temperature , and the behavior of the system is described at the generic instant by the equation

—

where is an appropriate coefficient that accounts for heat transfer. Conversely, when the thermostat is in the ON state, the temperature increases with the law

—

where represents the temperature increase in the unit time due to the heat produced by the heating device.

The status of such a system

thus has two components: the logical variable is called the *location* and represents the *discrete state*; the temperature signal represents the *continuous state*.

Finally, we can give the hybrid model shown in Fig. 1.7, where each rectangle represents a location, the arrows describe the event-driven behavior, and within each location a differential equation describes the time-driven behavior .

2

Systems, models and their classification

The objective of this chapter is to provide some fundamental concepts in the study of time advancing dynamical systems, that is, those systems whose evolution, as seen in Chapter 1, arises from the passage of time. In particular, with reference to time-advancing and continuous-time systems, which constitute the class of systems that will be examined in this text, the two main descriptions that can be given of a system, depending on the quantities or variables of interest, are introduced. The first is the *input-output* (UI) description, the second is the *description in state variables* (VS). Depending on the type of de- scription chosen, it is then necessary to formulate different types of mathematical model, namely the *UI model* or the *model in VS*. The derivation of both types of ma- tematic models is illustrated within the chapter through some simple physical examples, such as hydraulic, electrical, mechanical, and thermal systems.

An important classification of such models is finally proposed, based on al- cationary properties that systems may enjoy. In particular, in the following, systems will be classified as, *dynamic* or *instantaneous*, *linear* or *nonlinear*, *stationary or non-stationary*, *proper* or *improper*, with *concentrated* or *distributed parameters*, *with* or *without delay elements*.

2.1 Description of system

The first fundamental step in being able to apply formal techniques to the study of systems is, of course, to describe the behavior of the system by means of *quantities* (or *variables*, or *signals*) that evolve over time. In the case of the time-evolving systems to which this text is devoted, there are two possible descriptions: the first known as *input-output* (IU) description, the second known as description in terms of *state variables* (VS).

2.1.1 Description input-output

The quantities underlying a UI description are *causes external to* the system and *effects*. External causes are quantities that are generated outside the system; their evolution influences the behavior of the system but does not depend on it. Effects, on the other hand, are quantities whose evolution depends on causes external to the system and the nature of the system itself. We usually use the convention of defining external causes as *inputs* to the system, and effects as *outputs*. In general more than one input may act on a system as well as more than one may be the output quantities. The classic graphical representation of a system for which inputs and outputs have been identified is that shown in Fig. 2.1 where it can be considered as an operator that assigns a specific trend to the output quantities corresponding to each possible trend of the inputs.

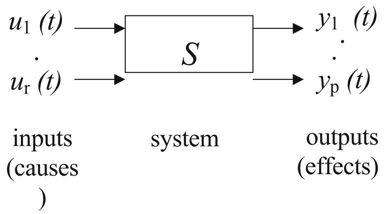


Fig. 2.1. Input-output description

We usually use the convention of denoting by

the vector of inputs, and with

The vector of outputs.

A system that has only one input ( ) and only one output ( ) is called *SISO* (single-input single-output). In contrast, a system that has multiple inputs and/or multiple outputs is called *MIMO* (multiple-inputs multiple-outputs).

By convention, it is assumed that both inputs and outputs are all *measurable* gran- deities, that is, quantities whose magnitude can be detected by appropriate measuring instruments.

With regard to inputs, an important distinction is also made whether or not they are *manipulatable* quantities. More precisely, if the inputs are manipulatable quantities, they constitute precisely the quantities by which an attempt is made to impose the desired behavior on the system; conversely, if they are nonmanipulatable quantities, their action on the system constitutes a disturbance that can alter the desired behavior of the system itself. This is the reason for where in the latter case such quantities are called *disturbances* in the input to the system. For the purposes of Systems Analysis, however, this distinction is not important, since the goal of this discipline is to understand how the system evolves in response to certain causes external to the system, whether or not these are manipulable.

Example 2.1 Suppose the system under study is an automobile. Let po- siation and speed be the output quantities, both measurable. Steering position and throttle position can be taken as input variables (see Fig. 2.2), both of which can be both measured and manipulated. In fact, acting on these magnitudes causes a change in the output quantities, to an extent that depends on the particular system under study, i.e., the particular dynamics of the automobile. As an additional input quantity to the system, let us assume wind thrust, which obviously influences the position and speed of the vehicle, but on which the driver cannot act, i.e., it is not a manipulable quantity.

This is a simple example of a MIMO system, being e......................................

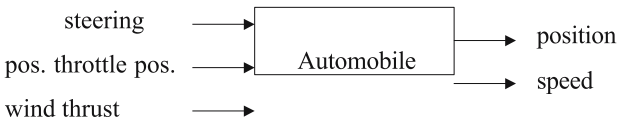


Fig. 2.2. System related to Example 2.1

Example 2.2 Consider the system depicted in Fig. 2.3.a given by two cylindrical tanks of base m . On the first reservoir acts the inlet flow rate m s and the outlet flow rate m s ; on the other hand, on the second reservoir acts the in- gress flow rate and the outlet flow rate m s , where the outlet flow rate from the first reservoir coincides with the inlet flow rate to the second reservoir. Finally, let m and m be the liquid levels in the two tanks.

Assume that the desired value can be imposed on and by oppor- tinently operating pumps, while the flow rate is a linear function of the liquid level in the reservoir, i.e. , where m s is an appropriate proportionality coefficient. In this case, the flow rates and can be considered as inputs external to the system (measurable and manipulat- able) that influence the trend of the liquid level in the two reservoirs. Finally, assume as an output variable , i.e., the difference between the level of the first reservoir and the level of the second reservoir. This quantity is, of course, measurable but not manipulable: in fact, its value can be changed only indirectly, that is, by appropriately acting on the inputs.

Because of what was said before, this is still an example of a MIMO system being 2 input quantities. The schematic representation of such a system in terms of UI variables is given in Fig
2.3.b.

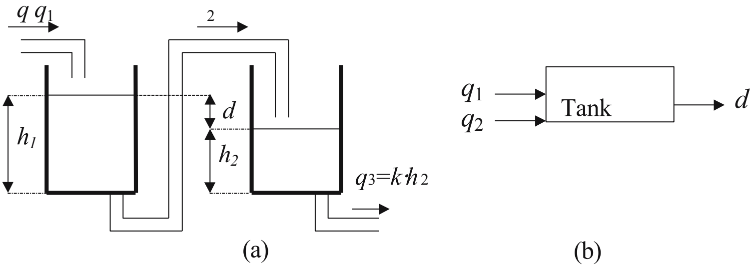


Fig. 2.3. System related to Example 2.2

2.1.2 Description in variables of status

With reference to Fig. 2.1 it has been said that, given a specific trend of the ingress, through results identified a definite trend of the output quantities. However, it is easy to realize that in general the output of a system at a certain instant of time does not depend only on the input at time , but also depends on the previous evolution of the system.

Example 2.3 Consider again the system in Fig. 2.3. Let be *t h e* value of the output at time instant , where and represent the liquid levels in the two reservoirs at time instant . Assume further that at all input quantities are zero, i.e. .

It is clear that the output at the generic time instant depends on the value assumed by the flow rates and during the entire time interval .

This fact can be taken into account by introducing an interme- dary quantity between inputs and outputs, called the system *state.* The system state enjoys the property of concentrating in itself information about the past and present of the system.

Just as the input and output quantities, state is also in general a vector quantity and is indicated by a state vector

where the number of components of the state vector is denoted by and is called the *order of* the system. The vector is also called the *state vector of* the system, and the following formal definition applies to it.

Definition 2.4. *The* state *of a system at the time instant is the quantity that contains the information necessary to uniquely determine the u- course , for each , based on knowledge of the input course*
, for and precisely of the state in .

The representative scheme of a system described in terms of state variables is of the type shown in Fig. 2.4.

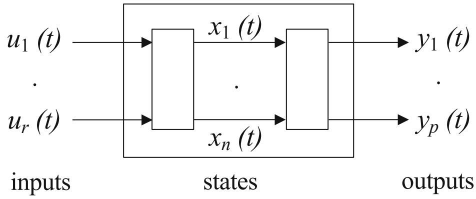


Fig. 2.4. Description in state variables

Example 2.5 Consider again the system consisting of the two reservoirs in Fig. 2.3. Assume as state variables the volumes of fluid in the two reservoirs, which we denote as and , respectively. In this case, as shown in detail in the following Example 2.10, the value of the output at time can be evaluated based on knowledge of the initial state of the system (and) and based on knowledge of the input vector during the time interval .

In general, several physical quantities related to a given system can be chosen as state variables, so the state vector is not uniquely determined. The most natural and most common choice, however, is to assume as state variables the quantities that immediately characterize the system energetically.

Example 2.6 Consider the following elementary physical systems.

Given a capacitor of capacitance , the energy stored in it at time è pari a dove è la tensione ai capi del condensatore at the instant of time . As a state variable it is therefore natural to assume . Given an inductor of inductance , the energy stored in it at time is pari a dove è la corrente che lo attraversa al tempo . As a state variable it is then natural to assume .

Given a spring of elastic constant , the energy stored in it at the instant of time is equal to where is the deformation of the mol- la from the equilibrium condition. The most natural choice, therefore, is to assume the deformation of the spring as the state variable.

Given a mass in motion at a velocity in a plane, the (kinetic) energy possessed by the mass is equal ....toIn this case the state of the system is equal to the velocity of the mass.

Si consideri un serbatoio cilindrico di sezione costante e sia il livello del liquido al suo interno al tempo . L'energia (potenziale) che tale sistema possiede al tempo è pari a dove è la densità del liquido nel serbatoio, è l'accelerazione di gravità e è il volume del liquido nel serbatoio. In questo caso una scelta naturale consiste nell'assumere lo stato del sistema pari al volume Si noti che una scelta altrettanto naturale consiste In assuming the state equal to the liquid level in the tank.

Example 2.7 Consider the system in Fig. 2.3. In each reservoir it is possible to imma- gine potential energy that depends on the volume (or equivalently the level) of the liquid in the reservoirs. The order of the system is therefore equal to .

Note that if there is stored energy in the system (i.e., if its state is not null) the system can evolve even in the absence of external inputs. This means that *the state of a system must also be seen as a possible cause of evolution* (internal and not external to the system).

Example 2.8 Consider a circuit consisting of a charged capacitor with a resistor in parallel. Current circulates in the resistor although there is no voltage ge- nerator until the capacitor is completely discharged.

2.2 Mathematical model of a system

The goal of *Systems Analysis* is to study the link between the inputs and outputs of a system and/or between the states, inputs and outputs of the system. In other words, solving an analysis problem means understanding, given certain input signals to the system, how the states and outputs of that system will evolve.

This makes it necessary to define a *mathematical model that quantitatively* describes the behavior of the system under study, that is, provides an exact mathematical description of the link between inputs (states) and outputs.

Depending on the type of description one wants to give to the system (UI or VS), two different types of models need to be formulated.

The *input-output* (UI*) model* describes the link between the output *(and* its derivatives) and the input *(*and its derivatives) in the form of a differential equation.

The *model in state variables* (VS) describes how:

1. the evolution of the state depends on the state and from the input *(equation of state*),
2. output depends on the state and input *(output transformation*).

2.2.1 Model input-output

The UI model for a SISO system, that is, a system with only one input and one output, is expressed by means of *a* differential equation of the type :[1]

(2.1)

uscita ingress

wher
e

, , ,

is a function of multiple parameters that depends on the particular system under study,
is the maximum degree of derivation of the output and coincides with the order of the system,
Is the maximum degree of derivation of the input.

Example **2.9** An example of a model in the form (2.1) is given by the diffe- rential equation

where ....edIn particular, it can be seen that in this case the function binds , , according to a relationship that depends explicitly on time due to the presence of the coefficient .

Instead, the UI model for a MIMO system with outputs and inputs is expressed by differential equations of the type:

exit entrance entrance

uscita ingress o ingres o

. .

uscita ingress o ingres o

(2.2)

where

[1]Note that in reality this statement is true only if the system is parameter con- centrated, that is, as will be seen more fully below (see § 2.4.5) when the only indepen- dent variable is time. In what follows we will always assume that the systems we are talking about are parameter con- centrated.

, , are functions of multiple parameters that depend on the particular system under study,

is the maximum degree of derivation of the -th component of the output ,

Is the maximum degree of derivation of the -th component of the input

.

2.2.2 Model in state variables

The model in VS for a SISO system instead of considering differential equations of order , binds the derivative of each state variable with the different state variables and with the input, by a relationship known as the *equation of state*; in addition, this model binds the output variable to the state components and to the input by a relationship known as the *output transformation*:

. .
. .

where , e are functions of multiple parameters that depend on the dynamics of the particular system under study.

Now, if we denote by

— :

the vector whose components are equal to the first derivatives of the state components, the model in VS of a SISO system can be rewritten in a more compact form as

(2.3)

where it is a vector function whose -ma component is equal to .

The model in VS for a MIMO system with inputs and outputs, on the other hand, has a structure such as.

. .

(2.4)

. .

which rewritten in matrix form becomes

(2.5)

The *equation of state* is therefore a system of first-order differential equations, regardless of whether the system is SISO or MIMO. In contrast, the *output transformation* is an algebraic equation, scalar or vector depending on the number of output variables.

Therefore, the schematic representation that can be given of a model in VS is as shown in Fig. 2.5.

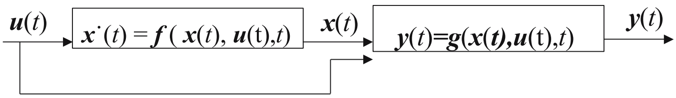


Fig. 2.5. Schematic representation of a model in VS.

2.3 Formulation of the model mathematical

Let us now illustrate through some simple physical examples how to proceed in deriving the mathematical model of a system. In particular, in the following we will present examples of hydraulic, electrical, mechanical and thermal systems.

2.3.1 Systems hydraulic

Example 2.10 Consider again the system in Fig. 2.3 and let , , be the input variables; the output variable; and the state variables. Note that there are 2 state variables being 2 elements capable of storing energy in the system (see Example 2.7).

We deduce for such a system the UI model and the model in VS.

Let us first observe that by virtue of the *law of conservation of mass*
For an incompressible fluid, the following applies[2]

(2.6)

---

[2]Note that in fact such differential equations have a limited range of validity. This is defined by the nonnegativity constraints and constraints limiting the maximum value of these volumes, which of course cannot exceed the capacity of the reservoirs.

Now, since and , from eq. (2.6) it follows that

Furthermore, being by definition , it is worth

therefore

The UI model of the system under consideration is then given by the following ordinary differential equation

Note that this equation is in the form (2.2) where , , ,

.

Finally, to deduce the model in VS, we observe that the equation of state is given precisely by (2.6) where we pose , while the output transformation is defined as

The model in VS is then

Which is in the form (2.4).

It is important to remember that state choice is generally not unique.

In the case of the hydraulic system under consideration, we could have assumed the liquid levels in the reservoirs as state variables, i.e., pose e In this case it is immediate to verify that the model in VS would have been

2.3.2 Systems electrical

We now present two simple examples of electrical circuits.

Example 2.11 (Purely resistive circuit**)** Consider the circuit in Fig. 2.6 consisting of a resistor placed in parallel with a voltage generator
V .

Input:	$u(t)=v(t)$
Status:	$x(t)=?$
Output:	$y(t)=i(t)$

Fig. 2.6. Resistive circuit related to Example 2.11

We assume voltage as the input variable and current A as the output variable, i.e., we place

Regarding the choice of state, we immediately observe that the system has no elec-tives capable of storing energy. This means that the order of the system is i.e., the state does not exist.

To derive a model that can describe the behavior of such a system, we write *the laws of the components* (in this case, the resistance alone) and the *laws of the connections* (in this case, the *mesh equation*):

e

from which we get

—

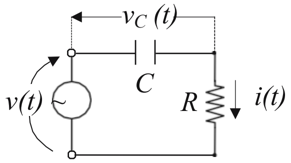
This equation can be considered at the same time:

a UI model in which the order of derivation is *(* thus the differential equation is reduced to an algebraic equation),

a model in VS of order that includes only the output transformation (the equation of state does not appear because the state does not exist).

Example 2.12 (RC Circuit) Consider the electrical circuit in Fig. 2.7 consisting of a resistor , a capacitor of capacitance F and a voltage generator V .

We denote by *[A*] the current in the circuit and by V the voltage at the ends of the capacitor.



Input:	$u(t)=v(t)$
Status:	$x(t)=v_C\,(t)$
Output:	$y(t)=i(t)$

Fig. 2.7. RC circuit related to Example 2.12

We assume voltage as the input variable , voltage at the ends of the capacitor as the state variable and current as the output , i.e., we place

Note that in this case there is only one state variable since there is only one element (the capacitor) in the system that can store energy.

To deduce a mathematical model that describes the dynamics of this system, we first write down *the component* laws, that is, the laws that describe the dynamics of each component. The first is *Ohm's law*:

(2.7)

The second is the law governing capacitor dynamics:

— (2.8)

It is also necessary to take into account how these components are connected to each other. This is equivalent to writing the *equation of the mesh*:

(2.9)

Now, from (2.9) we get , which substituted into (2.7) leads to

(2.10)

Finally deriving from (2.10) there remains

(a)

(2.11)

(b)

or

(a)

(2.12)

(b)

To determine the UI model, the state must be eliminated. For this purpose we *derive* from (2.12.b), we derive and substitute in (2.12.a). The UI model results defined by the differential equation:

(2.13)

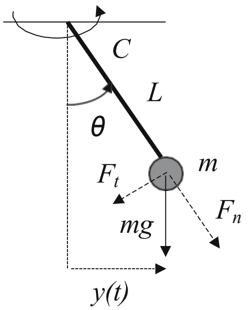
Instead, to determine the model in VS, one must eliminate the output from the equation of state. For this purpose, we substitute (2.12.b) into (2.12.a) and obtain

2.3.3 Systems mechanical

We now present two mechanical systems, the first given by a pendulum and the second by a mass-spring system.

Example 2.13 (Pendulum) Consider the pendulum in Fig. 2.8 consisting of a mass Kg placed at the end of a rod of length m and negligible mass. The position of the mass is identified by the angle rad that the rod forms with the vertical, where the direction of is assumed positive when directed counterclockwise, as shown in Fig. 2.8.

The pendulum moves in the vertical plane under the action of the weight force whose tangential component is worth , where it is equal to the acceleration of gravity, and under the effect of an external mechanical torque N m . Finally, there is a frictional force opposing the motion, which we assume to be proportional to the velocity of the mass through a coefficient of friction .



Input : $u(t) = C(t)$
Status : $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$
Output : $y(t) = L \sin\theta$

Fig. 2.8. Pendulum

From the second principle of rotational dynamics, we know that the total motor momentum

is equal to the sum of the motor moment due to the weight force , plus the friction force, plus the motor moment due to the external torque , i.e.

(2.14)

If , and the external torque is assumed as the output variable , given Eq. (2.14) it is immediate to verify that the UI model holds true:

(2.15)

Moreover, if we assume the following as state variables

the model in VS of such a system is worth

(2.16)

Note that both the IU and VS models of such a system can be simplified under the assumption that the oscillations to which the system is subjected are very small. In fact, in such a case it is permissible to assume

(2.17)

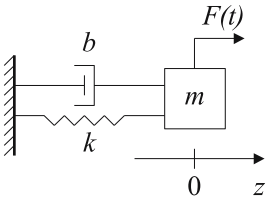
Under this assumption, the UI model is worth

(2.18)

while the model in VS is equal to

$$(2.19)$$

Example 2.14 (Mass-spring system) Consider the system in Fig. 2.9 given by a mass Kg connected to a fixed reference by a spring of ela- stic constant N m and a damper with viscous friction coefficient N s m placed in parallel. Let N be the external force acting on the mass (positive if directed to the right) and m be the position of the mass with respect to a reference whose origin coincides with the equilibrium position of the system.



Input:	$u(t) = F(t)$
Status:	$x_1(t) = z(t)$
	$x_2(t) = \dot{z}(t)$
Output:	$y(t) = z(t)$

Fig. 2.9. Mass-spring system related to Example 2.14

We assume as input the force applied to the mass, i.e., we *place* and as the output the position of the mass with respect to the chosen reference, i.e.

.

The system certainly has order 2 being 2 components capable of imma- gaging energy, namely mass and spring (see Example 2.6). Let us assume as state variables and .

We first write the *laws of components*, that is, the laws governing the dynamics of the spring, damper and mass:

$$(2.20)$$

where the magnitudes at the first member represent the forces acting on the spring, damper and mass, respectively, assumed positive when directed to the right.

It also requires a relationship that takes into account how these components are connected to each other, i.e., the *law of connections*:

$$(2.21)$$

Substituting (2.20) into (2.21) gives:

(2.22)

i.e.

— — —

that is, for the choice of variables made, we obtain the UI model

— — —

Furthermore, according to the definition of , and it applies:

(2.23)

Finally from (2.22) follows.

— — —

which replaced in (2.23) provides the model in VS of the system

— — —

2.3.4 Systems thermal

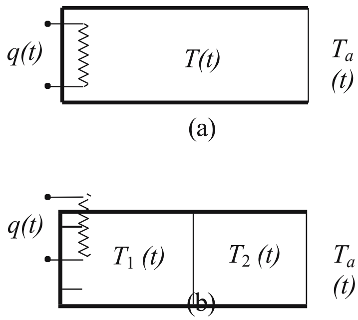
Example 2.15 Consider the furnace shown in Fig. 2.10.a, which exchanges ca- lor with the external environment through the right wall that, unlike the al- three, is not adiabatic. Through a resistor it is possible to supply the furnace with a cer- tain power J s . The temperature of the external environment is K while the temperature inside the furnace, assumed to be uniform, is K .

The heat capacity of the furnace is worth J K and finally the heat transfer coefficient through the nonadiabatic wall is assumed to be J K s . Thus, the following *law of conservation of energy* applies.

(2.24)

Assume as inputs and , as *outputs* and as a state variable .

From the law of conservation of energy, by introducing the input and output variables, the UI model is obtained:



Input: $u_1(t) = q(t)$, $u_2(t) = T_a(t)$
Status: $x(t) = T(t)$
Output: $y(t) = T(t)$

Input: $u_1(t) = q(t)$, $u_2(t) = T_a(t)$
Status: $x_1(t) = T_1(t)$, $x_2(t) = T_2(t)$
Output: $y(t) = (T_1(t)+T_2(t))/2$

Fig. 2.10. A furnace with a non-adiabatic wall. **(a)** Schematic of a first-order model (uniform internal temperature); **(b)** Schematic of a second-order model (non-uniform internal temperature).

Again from the energy conservation equation by introducing the state variable and input variables, the equation of state is obtained:

In addition, oven temperature was assumed as the output variable, so that

Therefore, the system is described by the following model in VS

Suppose now that we want to use a more detailed model that takes into account the fact that the temperature inside the furnace is not uniform. Specifically, as shown in Fig.2.10.b, consider the furnace divided into two areas of the same dimen- sion, the first of temperature and the second of ....temperatur eThe capacity
thermal of each of the two areas is worth while assuming that the heat transfer coefficient between the two areas is worth J K s Assuming as output variable
the average temperature between the two areas

we want to determine the new model in terms of VS.

The first oven area receives the power supplied by the heating element and exchanges heat with the second area according to the equation while the second oven area exchanges heat with the first area and the external environment according to the equation

It is then immediate to verify that the model in VS of the system is worth

2.4 Properties of systems

In Chapter 1 we saw a classification of the systems under study in the Automatica, of which time-advancing systems (SATs) are a part. In the following we will present a number of elementary *properties* that SATs can enjoy and that can be used to classify them.

For example, we will classify SATs into linear and *nonlinear* depending on whether or not they enjoy the property of linearity.

However, it generally makes more sense to talk about the properties by referring them to *the patterns*
rather than to *systems*.

In fact, models provide a description of system behavior but are always based on a number of simplifying assumptions. For example, a large class of systems can be described by linear models. In practice, however, a linear system is a pure abstraction that does not exist in nature. The same applies to all other properties.

In what follows we will define elementary properties in general terms by referring them to systems. We will also see that these properties are *structural* in that they depend on the particular structure of the model, be it a UI model or a model in VS.

2.4.1 Dynamic systems or instantaneous

The first important distinction that can be made is between instantaneous systems and dynamic systems.

Definition 2.16. *A system is said to be* instantaneous*: if the value taken by the output at time depends only on the value assumed from entry to time ;*

Dynamic*: Otherwise.*

Let us now see how it is possible, based solely on observation of the structure of the model, to determine whether a system is instantaneous or dynamic.

Let us first consider a UI model and assume for simplicity that the system is SISO.

Proposition 2.17 (IU model, SISO system) A *necessary and sufficient condition for a SISO system to be* instantaneous *is that the IU bond be expressed by an equation of the form:*

By virtue of this proposition we can therefore conclude that *if* a SISO system is *instantaneous* the UI bond reduces to an *algebraic equation*, that is, the order of the derivatives of and is .

In contrast, if the UI bond related to a given SISO system is described by a differential equation then the system is *dynamic*.

It is important to note that the fact that the IU bound of a SISO system is expressed by an algebraic equation is a necessary but not suf- ficient condition for a SISO system to be instantaneous. Indeed, consider a SISO system whose IU model is defined by the algebraic equation

Such a system, known as *the delay element*, is clearly dynamic in that the u- lution at time does not depend on the value of the input at time , but depends on the value that the input took at a previous instant. In this regard, see also
§ 2.4.6.

The above can easily be extended to the case of a MIMO system.

Proposition 2.18 (UI model, MIMO system) *A necessary and suffi- cient condition for a MIMO system with inputs and outputs to be* instantaneous *is that the UI bond be expressed by a set of equations of the form:*

:

This implies that if a MIMO system is instantaneous the following conditions are verified:

the order of the derivatives of is , for each ,
the order of the derivatives of is for each , ,
the UI bond reduces to a set of *algebraic equations*.

Conversely, if even one of the UI bond equations is a differential equation, then the system is *dynamic*.

In the case where the system model is in terms of VS, however, the following result applies .[3]

Proposition 2.19 (Model in VS) A *necessary and sufficient condition for a system to be* instantaneous *is that the model in VS has zero order, i.e., the state vector does not exist.*

Example 2.20 Consider the resistive circuit seen in Example 2.11. This system is instantaneous because the UI bond (which in this case coincides with the output transformation of the model in VS) is worth

—

The order of such a system is clearly zero because there are no elements capable of storing energy.

In contrast, all other systems presented in Section 2.3 are dynamic.

2.4.2 Linear or non linear systems

One of the fundamental properties enjoyed by a broad class of systems (or more preci- sely models) is linearity. It is precisely on this class of systems that we focus our attention in this volume. The importance of linear systems derives from a number of practical considerations.

The first is that such systems are easy to study. Efficient analysis and synthesis techniques have been proposed for them, which are no longer applicable if linearity is lost. Second, a linear model turns out to be a good approximation of the behavior of many real systems as long as they are subjected to small inputs.

Finally, as will be discussed in Chapter 12 (see § 12.4), it is often possible to linearize a model around a working point, obtaining a *linear model to variations* that is valid for small signals.

The linearity property can be formally defined as follows.

Definition 2.21. *A system is said to be*

linear*: if the* principle of superposition of effects *applies to it. This means that if the system responds to the cause with the effect and to the cause with the effect , then the system's response to the cause is , whatever the values taken by the constants and .*
The following diagram summarizes this property:

cause effect
cause effect *cause* *effect*

[3]Note that in fact this result is true under the assumption that the model in VS is controllable and observable (see § 11.7.2).

Nonlinear*: otherwise.*

It is straightforward to determine whether a system is linear or not once the structure of the model is known, either this UI or in terms of VS.

Proposition 2.22 (IU Model) A *necessary and sufficient condition for a si- stem to be* linear *is that the IU bond be expressed by a* linear *differential equation*[4] *, i.e., for a SISO system:*

(2.25)

where in general the coefficients of the linear combination of the UI model are functions of time.

The above condition immediately extends to the case of MIMO systems. For in such a case the system is linear if and only if each of the functions , , expresses a linear combination between the -ma component of the output and its derivatives and the input variables with their derivatives.

Proposition 2.23 (Model in VS) *A necessary and sufficient condition for a system to be* linear *is that in the model in VS both the equation of state and the output transform- ation are* linear *equations:*

. .
. .

. .

ovvero

whe re *matrix* *matrix*
matrix *matrix*

[4]A differential equation in the form

is *linear* if and only if the function expresses a linear combination between the output and its derivatives and the input and its derivatives. In other words, such a differential equation is linear if the sum weighted according to appropriate coefficients of the output and its derivatives and the input and its derivatives is zero. Note that being a function of time , in general the coefficients of the linear combination are themselves a function of time .

In general, the matrices of coefficients , , and are functions of time.

Example 2.24 The hydraulic circuit model of Example 2.2 in Fig. 2.3 is linear. Indeed, if one considers its IU model, it is immediate to observe that the function binds the output and its derivatives to the input variables and their derivatives by means of a linear-type relationship. Moreover, if we consider its model in VS it is again immediate to verify that it is in the form given in Proposition 2.23 where

The R and RC circuits seen in Examples 2.11 and 2.12, respectively, are both examples of linear systems.

Similarly, the mass-spring system seen in Example 2.14 is linear. In particular, in the latter case with reference to the model in VS, the following applies

In contrast, the pendulum presented in Example 2.13 is nonlinear as can easily be seen by observing the structure of Eqs. (2.15) and (2.16). However in the case where the simplification is carried out.

valid for small fluctuations, we arrive at a linear model (see eq. (2.18) and (2.19)).

Finally, the thermal system presented in Example 2.15 is linear both in the case where the uniform temperature inside the furnace is considered and in the case where this assumption is not verified

Example 2.25 Consider the system described by the UI model.

Such a system is nonlinear. In fact, its UI model is an algebraic[5] *nonlinear* equation, where the nonlinearity arises from the presence of the term at the second mem- ber. Indeed, it cannot be placed in the form (2.25) in which neither at the first nor at the second member appear constant addends, independent of either the output variables and its derivatives or the input and its derivatives.

[5]Note that an algebraic equation is nothing more than a special case of a differential equation in which the orders of derivation are zero.

It is interesting to verify the above by showing through a simple numerical example that such a system violates the principle of superposition of ef- fects. To this end, consider the following two constant inputs: and . The output due to the first input is equal to while the output due to the second input is worth ....Now, suppose we apply to the system a input equal to the sum of the previous two inputs, *i.e.*
. The resulting output is equal to .

Example 2.26 Consider the system described by the UI model.

This system is linear in that it is in the form (2.25) where e

.

2.4.3 Stationary or non stationary systems

Another important property enjoyed by a wide class of systems is stationarity. In particular, in this text we will deal precisely with the analysis of linear and stationary systems.

Definition 2.27. *A system is said to be*

stazionario *(o* tempo-invariante*): se per esso vale il* principio di traslazione causa-effetto nel tempo*, cioè se il sistema risponde sempre con lo stesso effetto ad una data causa, a prescindere dall'istante di tempo in cui tale causa agisca sul sistema.*
The following diagram summarizes this property:

cause effect *cause effect*

Non-stationary *(or* time-varying*): otherwise.*

Fig. 2.11 shows the typical behavior of a linear system stressed by the same applied cause at two different time intervals, i.e., starting from and starting from : in the two cases the resulting effect is similar but simply originates from instants of time that differ from each other by just an amount equal to

.

Of course, in reality no system is stationary. Consider, for example, the wear and tear to which all physical components are subject and thus the variations that di- verses the system's characteristic parameters undergo over time. Nevertheless, there is a large class of systems for which such variations can be considered negligible over significantly large time intervals. This implies that within such time intervals these systems can with good approximation be considered stationary.

Just like the previous elementary properties, stationarity can also be verified through a simple analysis of the model structure.

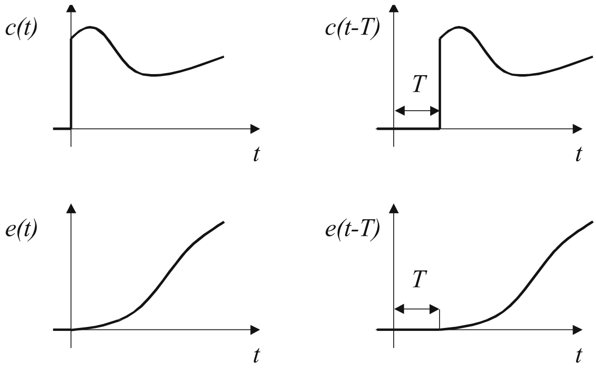


Fig. 2.11 Cause-effect translation over time**.**

Proposition 2.28 (UI model) *A necessary and sufficient condition for a si- stem to be* stationary *is that the UI bond does not depend explicitly on time, that is, for a SISO system:*

which in the case of linear systems reduces to a linear differential equation with constant coefficients:

Proposition 2.29 (Model in VS) *A necessary and sufficient condition for a system to be stationary is that in the model in VS the equation of state and the output transform- ation do not depend explicitly on time:*

which in the case of linear systems reduces to

Where , , and are matrices of constants.

Example 2.30 Consider the instantaneous and linear system

Because of the above, such a system is clearly nonstationary.

However, it is interesting to verify non-stationarity through the principle of cause-effect translation. To this end, consider the input

se
otherwise

which has the shape shown in Fig. 2.12.a. The output in response to this input has the pattern in Fig. 2.12.b.

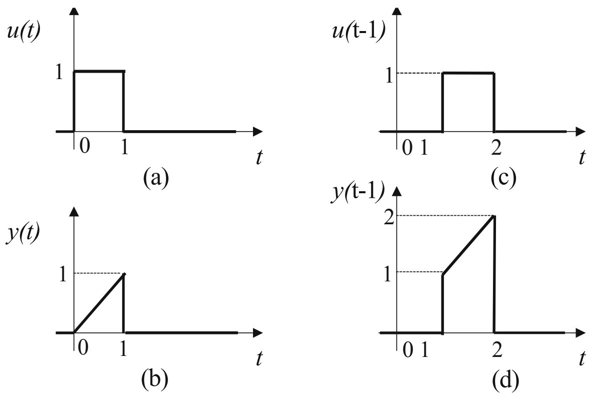


Fig. 2.12. Example 2.30

Now suppose the same input is applied to the system but with one unit of time delay: let the input signal therefore be equal to *(* see Fig. 2.12.c). It is easy to verify that the output of the system has the pattern shown in Fig. 2.12.d, which does not coincide with the previous output shifted forward by one time unit.

2.4.4 Own systems or improper

The following definition applies.

Definition 2.31. *A system is said to be*

proper*: if the* principle of causality *applies to it, that is, if the effect does not precede in time the cause that generates it;*
improper*: otherwise.*

In nature all systems are obviously proper. However, there are some patterns that correspond to improper systems.

Example 2.32 Consider the ideal capacitor of capacitance F in Fig. 2.13 where V represents the voltage at the ends of the capacitor and A the current flowing through it at time s .

Assume and .

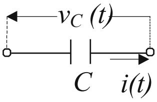
As is well known, the dynamics of a capacitor is governed by the differential equation

—

Therefore, the UI bond of such a system is

That is, by explicating the second-member derivative

This equation clearly highlights how the output at time depends *on* that is, from a value taken by the input at a later instant of time.



Input: $u(t)=v_C(t)$

Output: $y(t)=i(t)$

Fig. 2.13. Ideal capacitor

Note that in reality there is no capacitor that has only the capacitance . Every capacitor also always has its own internal resistance . If we were to account for that resistance we would have a circuit , which as is easy to verify is a system of its own .

The rules for determining whether a system is proper or improper based on the UI model or the model in VS can be stated as follows.

Proposition 2.33 (UI model, SISO system) A *necessary and sufficient condition for a SISO system to be* proper *is that in the UI bonding*

(2.26)

the order of derivation of the output is greater than or equal to that of the input, that is, it ....holdsIn particular if it holds the system is said to be strictly proper.

The extension of this result to the case of a MIMO system is immediate. For in this case for a system to be proper in none of the equations (2.2) must appear derivatives of any input variable of higher order than the derivative of the corresponding output variable. In other words, for each must result in.

Finally, for the system to be strictly its own such inequality must be strictly verified for each .

Proposition 2.34 (Model *in VS***)** *A system described by a model in VS:*

(2.27)

Is always its own.

The system is strictly its own *if the output transformation does not depend on* *:*

The model in VS of a *linear, stationary strictly proper* system therefore reduces to

Example 2.35 The ideal capacitor, which as seen in Example 2.32 is an improper system, is described by the equations

Resulting in a model in VS of the type:

which does not fall into the form defined by eq. (2.27) due to the presence of the terms .

Example 2.36 The system in Example 2.14 is strictly proper. The systems in Examples 2.11 and 2.12 are proper but not strictly proper

2.4.5 Concentrated parameter systems or distributed

The following definition applies.

Definition 2.37. *A system is said to be*

with concentrated parameters *(or* finite size*): if its state is described by a finite number of quantities (each associated with a component);*
with distributed parameters *(or* infinite dimension*): otherwise.*

Example 2.38 In an electric circuit, the state is described, e.g., by the value of the voltages in the capacitors and the currents in the inductances: in a device with a finite number of "circuit" components, the state vector also has a number of components
finished.

It should be noted, however, that representing an electrical circuit with a finite number of components is possible only following a simplification, which, however, is permissible in the case of small electrical systems: in fact, the speed of light propagates with such rapidity that in fact the quantities of interest depend only on time and not on space. For example, the current can be considered the same in all sections of a conductor representing a branch.

However, there are physical systems in which propagation is much slower so that the quantities of interest are functions of both time and space. A typical example in this regard is offered by hydraulic systems.

Example 2.39 Consider a free-flowing channel under uniform flow regime whose generic section bounded by two sluice gates, is shown in Fig. 2.14. Let e

The flow rate and liquid level in the abscissa section of the channel at time

. It can be shown that these quantities are related by the Saint-Venant equations:

i.e., from partial derivative equations where they are constants that depend on the channel geometry and conditions of motion.

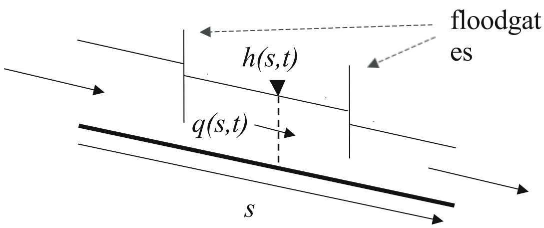


Fig. 2.14. Free-flowing channel

To describe the state of the system we need to know the level in each section of the channel so the system has infinite states

Again, it is immediate to determine whether a system is parameter concen- trined or not, from simple analysis of the structure of the mathematical model.

Proposition 2.40 (UI Model) *A* concentrated parameter *system is described by an* ordinary differential equation .[6]

[6]A differential equation is called *ordinary* when the unknowns are functions of only one real variable (e.g., time).

A distributed parameter *system is described by a* partial derivative equation .[7]

Proposition 2.41 (Model in VS) *The state vector of a* concentrated parameter *system has a* finite *number of components; in contrast, the state vector of a* distributed parameter *system has an* infinite *number of components.*

Example 2.42 The systems presented in Section 2.3 all have concen- trined parameters.

However, consider the thermal system considered in Example 2.15. In the case where the temperature inside the furnace is assumed to be uniform, the system is first-order. Assuming, on the other hand, that the temperature is non-uniform, it is possible to divide the area inside the furnace into two regions and thus obtain a more detailed second-order model. By dividing the oven area into an increasing number of regions it is possible to construct more and more accurate but higher order models. At the limit considering infinitesimal areas it is possible to define an infinite-order model in which each state variable represents the temperature at a diverse point of the furnace. The resulting model is in this case with distributed parameters.

2.4.6 Systems without delay elements or with delay elements

A delay element is formally defined as follows.

Definition 2.43. *A* finite delay element *is a system whose output at time is equal to the input at time , where it is precisely the delay introduced by the element.*

A delay element can be schematized as in Fig. 2.15.

Example 2.44 A fluid of varying temperature moves with velocity in an adiabatic pipeline of length . If at the inlet at the instant the temperature holds , at the outlet the temperature will hold equally after a time .

Recall that, as noted in Section 2.4.1, although the equation describing the UI bond of a delay element is an algebraic equation, ta- le system is not instantaneous because the output at time depends on the previous values of the input.

Proposition 2.45 *A necessary and sufficient condition for a system not to con- tain element of delay is that in the model (be it UI or in VS) all quantities have the same argument.*

---

[7] A differential equation is called *a partial derivative* equation when the unknowns are functions of several independent real variables (e.g., time and space).

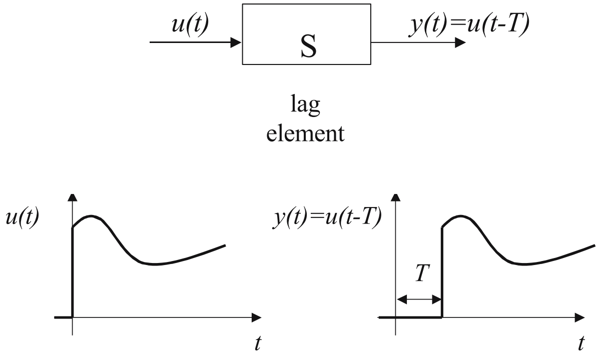


Fig. 2.15. Delay element

Example 2.46 The system described by the UI model

contains lag elements in that both quantities with argument and quantities with argument appear in the model .

Similarly contains elements of delay the model in VS

In contrast, all the systems presented in Section 2.3

do not contain delay elements.

Exercises

Exercise 2.1 The following mathematical models of dynamic systems are given.

(2.28)

(2.29)

(2.30)

(2.31)

1. Classify these models into input-output models or models in state variables, indicating the value of significant parameters (order of derivation of output, input, size of state vector, input and output).

2. Identify the structural properties that characterize them: linear or nonlinear, stationary or time-varying, dynamic or instantaneous, with concentrated or distri- buted parameters, with or without delay elements, proper (strictly or not) or improper. Motivate answers.

Exercise 2.2 Identify the general properties that characterize the structure of the following systems, assigned using the input-output model.

1.

2.

3.

4.

Exercise 2.3 The beam from a laser device, by reflection on a plane mirror, illuminates a point on a graduated rod positioned at a distance from the emitter and parallel to the emitted light beam. The position of the dot on the graduated rod can be changed by rotation of the mirror about its axis.

[rad] angle formed by the mirror with respect to the horizontal
[m] position of the illuminated point on the graduated rod
[m] distance of the rod from the laser emitter

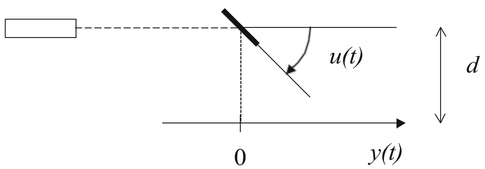


Fig. 2.16. Laser device

Determinare il modello matematico in termini di legame ingresso–uscita di tale sistema (si assuma che nella situazione in figura valga – e).

Identify the general properties that characterize the structure of such a system.

Exercise 2.4 Two cylindrical tanks of base and m are connected in the configuration shown in Figure 2.17. The height of the liquid in the two tanks is denoted and m, respectively, while the volume of liquid in them is denoted and m .

The first tank is supplied with a variable flow rate m s while a flow rate m s escapes from a valve at its base. The flow rate in uscita dal primo serbatoio alimenta il secondo serbatoio, dal quale, a sua volta, fuoriesce una portata m s .

The law of conservation of mass for an incompressible fluid dictates that the derivative of the volume of liquid contained in a reservoir is equal to the flow rate pertaining to it, i.e., said and the total sum of the inlet and outlet flow rates, is

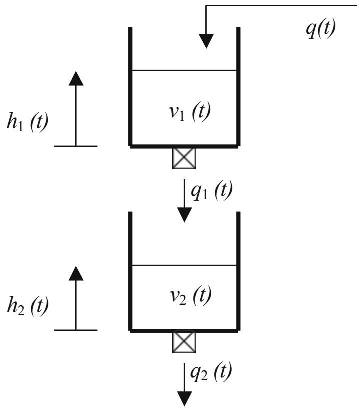


Fig. 2.17. Two tanks in cascade

1. Determine a mathematical model in terms of state variables for this system, choosing as state variables and the volume of liquid in the two tanks, as input the flow rate at the first tank, and as output *t h e* height of the second tank. Indicate the value of the matrices that constitute the representation.
2. Identify the general properties that characterize the structure of such a system.
3. Determine the mathematical model in terms of the input-output linkage of such a system.
4. Indicate how the representation in state variables is changed if it is assumed that the second tank can also be fed from outside by means of a variable flow rate (The inputs would in this case be two: e
.)

Exercise 2.5 For the study of road vehicle suspensions, it is usual to use a model called the quarter car model in which only one suspension and *t h e* suspended mass affecting it (a quarter of the total mass of the car's cor- pond) are represented. We will consider the simplest model, represented in Figure 2.18, in which the mass of the wheel is neglected.

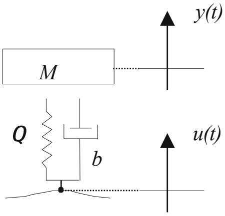


Fig. 2.18. One-degree-of-freedom model of the quarter car.

In the figure, the suspension is represented by a spring with elastic coefficient N m and by a damper with damping coefficient N s m . We con- sider as input the position of the wheel on the road surface and as output

The position of the suspended mass. The weight force is neglected by assuming that it is balanced by the spring tension in the equilibrium condition (model to variations).

1. Determine the input-output model of such a system.
2. Try to determine a mathematical model in terms of state variables for this system, choosing as state variables *and*
   Check that this choice does not allow for a model in the form standard.
3. Let the following be chosen as *state* variables *e* — e

Let it be verified that this choice allows obtaining a model in standard form. State the value of the matrices that constitute the representation.

4. Identify the general properties that characterize the structure of such a system.

Esercizio 2.6 Tre serbatoi cilindrici sono collegati nella configurazione mostrata in Figura 2.19. La superfice di base dei tre cilindri si denota m , l'altezza del liquido nei serbatoi si denota m , mentre il volume di liquido in essi contenuto si denota m dove .

A pump produces a variable flow rate m s that is distributed by −to the first tank and for to the second tank. A second pump that draws from the third reservoir and pours into the second one also allows you to generate a *variable* flow rate m s .

Infine, dalla valvola alla base di ogni serbatoio fuoriesce una portata m s . The flow rates escaping from the first and second tanks feed the third tank.

The law of conservation of mass for an incompressible fluid dictates that the derivative of the volume of liquid contained in a reservoir is equal to the flow rate pertaining to it, i.e., said and the total sum of the inlet and outlet flow rates, is

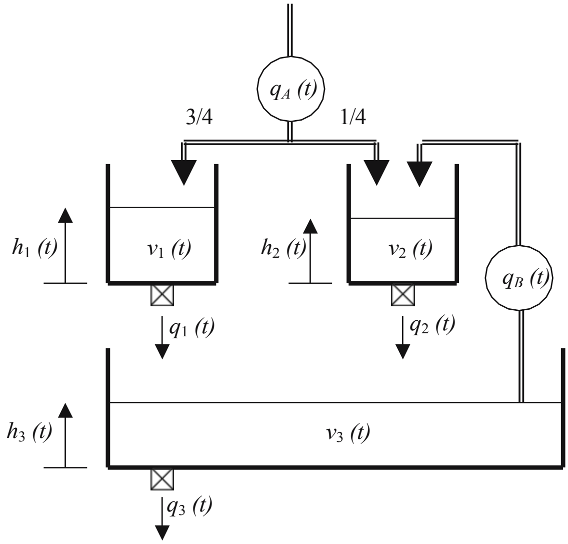


Fig. 2.19. Three tanks

1. Determine a mathematical model in terms of state variables for this system, choosing as state variables (with) the liquid vo- lume in the three reservoirs, as inputs and flow rates imposed by the pumps, and as outputs the sum of the heights of the second and third reservoirs. Indicate the value of the matrices that constitute the representation.
2. Identify the general properties that characterize the structure of such a system.

3

Time domain analysis of input-output models

In this chapter we will study linear, stationary, parameter-concentrated SISO systems described by an input-output model: this model consists of an ordinary, linear differential equa- tion with constant coefficients. The analysis techniques presented in this chapter are based on direct integration of the differential equation: we speak in this case of *time-domain analysis* or *analysis in* .

In the first section we define the *fundamental problem* of system analysis, which is to determine the output signal that satisfies a given model. Due to the linearity of the system, it will be possible to decompose this solution into the sum of two terms: *free evolution*, which depends on initial conditions alone, and *forced evolution*, which depends on the presence of a nonzero input. In the second section we preliminarily study the *homogeneous equation* associated with a given model: this allows us to define particular signals called *modes* that characterize the system's own evolution. In the third section we study free evolution, which is shown to be a linear combination of modes. In the fourth section we deal in detail with modal analysis, studying and classifying such signals. In the fifth section, a particular forced response, called *impul- sive response*, is presented: it is the forced response that delivers upon the application of a unit impulse; its importance arises from the fact that it is a canonical regime, i.e., analytical cono- scence of such a signal is equivalent to knowledge of the system model. As a result of this, an important result, the *Duhamel integral, is* presented in the sixth section: it states that the forced evolution that follows any input signal can be determined by convolution between the input itself and the impulse response. Finally, the seventh section introduces a family of canonical signals that can be obtained from the impulsive response by successive integration or derivation.

3.1 Input-output model and problem of analysis

A linear, stationary, concentrated-parameter SISO system is described by the following *input-output* (UI) *model*

(3.1)

In this expression is the independent variable, while the two signals and represent the output and input variables, respectively. The coefficients of this equation are all real, that is, for , and for .

The maximum degree of derivation of the output is called *the order of the system*. It is assumed that the system is proper and is therefore worth .

3.1.1 Fundamental problem of the analysis of systems

The fundamental systems analysis problem for the given UI model is to solve the differential equation (3.1) from an assigned initial instant. This requires determining the output trend for knowing:

the *initial conditions*

(3.2)

i.e., the value taken at the initial instant by the output and its derivatives up to order ;

the signal

per (3.3)

that is, the value taken by the *input* applied to the system from the initial instant .

The solving of a differential equation is addressed in basic courses in mathematical ana- lysis. Here we will recall some of these already known solving methods (without giving a demonstration of them) and introduce others, always emphasizing, however, their physical interpretation. The exposition of this chapter assumes that the reader is familiar with the material presented in Appendix B.

Before moving on, however, a clarification should be made regarding the link between initial conditions and initial state. The past history of the system for is summarized in the stateHowever ...., in the description of the pro- fundamental blem of systems analysis for input-output models is not assigned that state but rather the initial conditions of the output and its derivatives. The two pieces of information are mutually equivalent: in fact, the initial state of the system is uniquely[1] related to the initial conditions. In particular, the following applies.

[1]To be exact, this is true for *observable* systems. This will be better discussed later when we study the properties of controllability and observability.

If the system has zero initial state (it is then said to be initially *at rest* or *unloaded*) then the initial conditions given by (3.2) are also all zero, i.e.

If conversely, the system has nonzero initial state, then the initial conditions given by (3.2) are not all identically null, i.e.

3.1.2 Solution in terms of free evolution and forced evolution

In the previous chapter it has already been noted, in qualitative terms, that it is possible to consider the evolution of the output of a system as an *effect* determined by two different types of *causes*: causes internal to the system (i.e., the *initial state*) and causes external to the system (i.e., the *inputs*). For a linear system, the principle of so- vation of effects applies, and therefore it is also possible to say that the effect due to the simultaneous presence of both causes can be determined by summing the effect that each of them would produce if it acted alone.

It is therefore possible to write the output of the system for as the sum of two terms:

(3.4)

The term is called *free evolution* (or also *free response*, *free regime*)and represents the contribution to the response due solely to the initial state of the system at instant . This term can also be defined as the response of the system (3.1) from the initial conditions given by (3.2) if the input is identically null for .

The term is called *forced evolution* (or also *forced response*, *forced regime*) and represents the contribution to the total response due solely to the in- gress applied to the system by . This term can also be defined as the response of the system (3.1) subject to the input given by (3.3) if the initial conditions are all identically zero.

In the rest of the chapter we will study the two terms separately and show how they can be calculated.

A small simplification will almost always be made, assuming that the initial time instant considered ....isBecause the system described by (3.1) is stazio-
nary, this does not reduce the generality of the approach. In fact, if it were , one could always with a simple change of variable solve the dif- ferential equation in the variable . Indeed, the initial conditions in correspond to initial conditions in and substituting in the expression of per gives the perOnce the ana- expression is determined.
lytic of the response as a function of , substituting gives the (see Example 3.15).

3.2 Homogeneous equation and ways

This section studies a simplified form of differential equation, called *homogeneous*, in which the second member is zero. This analysis allows us to introduce the fundamental concept of a *mode*: it is a signal that characterizes the dynamic evolution of the system. The number of modes is equal to the order of the system, and the signals obtained from the linear combination of modes are the solutions of the homogeneous equation.

Definition 3.1 *Given the differential equation* (3.1)*, by setting the second member equal to zero we define the* homogeneous equation *associated with it*

(3.5)

where we recall that it is a real function and the coefficients for Are also real.

A polynomial can be associated with each homogeneous equation.

Definition 3.2 *The* characteristic polynomial *of equation* (3.5) *is the polynomial of degree of the variable*

(3.6)

Which has the same coefficients as the homogeneous equation.

According to the fundamental theorem of algebra, a polynomial of degree with real coef- ficients has real or complex conjugate roots. The roots of such a polynomial are the solutions of the *characteristic equation* . In general there will be distinct roots[2] each of multiplicity :

where *it is worth* whether and clearly .

Nel caso particolare in cui tutte le radici abbiano molteplicità unitaria, avremo

with if .

[2]The symbol used to denote the roots of the characteristic equation is because, as will be seen in the study of the transfer function, these roots are also called the *poles* of the system.

Definition 3.3. *Given a root of the characteristic polynomial of multiplicity , we define* modes *associated with that root as the functions of time*

So to a system whose characteristic polynomial has degree correspond in total ways.

Example 3.4 Consider the homogeneous differential equation

The characteristic polynomial is worth

and therefore it has roots

di molteplicità
of *multiplicity*

This polynomial corresponds to the four modes , , and .

By linearly combining the various modes with each other with appropriate coefficients, it is possible to construct a family of signals.

Definition 3.5. *A* linear combination of the modes of a system *is a signal which is obtained by summing the various modes each weighted by an appropriate coeffi- cient. In particular, each distinct root of multiplicity corresponds to a combination of terms*

(3.7)

and therefore, taking into account that there are distinct roots, a linear combination of the modes takes the form :

or

(3.8)

In the special case where all roots have unit multiplicity, one can write

(3.9)

omitting for simplicity the second subscript in the coefficients .

Example 3.6 The system studied in Example 3.4 with e has four modes , ,
.... andA linear combination of these modes
therefore takes the form

Note that although the modes are known from knowledge of the ca- racteristic polynomial, the coefficients appearing in a linear combination of them are indeterminate parameters for now: in this sense, Eq. (3.8) defines a family of signals in parameterized form. For example, in the following we will see that free evolution is a linear combination of modes. By appropriately particularizing the coeffi- cients we will be able to determine the free evolution from every possible initial condition.

We can finally give a fundamental result that explains the importance of the linear combination of modes: in fact, this family of signals constitutes the general integral of the homogeneous equation.

Theorem 3.7. *A real signal is a solution of the homogeneous equation* (3.5) *if and only if it is a linear combination of the modes associated with that equation.*

Demonstration. The fact that the general integral of a homogeneous equation such as (3.5) has the parameterization given by (3.8) is well known from mathematical analysis courses.

Without pretending to be exhaustive, we simply consider the special case in which all the roots of the characteristic polynomial have unit multiplicity and di- show the only necessary condition, namely, that a signal of the form (3.9) is a solution of (3.5). To do this, note that the derivative of the signal considered holds for :

Substituting in (3.5) gives the first member:

Now observe that for each value of the factor in parentheses cancels out; it is in fact worth

being root of the characteristic polynomial. So with the substitutions made the first member of (3.5) cancels, giving the identity sought.

3.2.1 Complex roots and conjugate

Note that in the case where the characteristic polynomial has complex roots, the corresponding modes appearing in Eq. (3.8) are also complex signals. More precisely, since it is a polynomial with real coefficients, for every complex root of multiplicity , there certainly exists a complex root to it conjugate and of ....multiplicityTherefore to such a pair of roots corresponds to a linear combination of ways:

(3.10)

Which we grouped into pairs of terms for .

For the signal to take real values for each , as desired, we require-de that even the *coefficients e* are complex and conjugate to each other for each value of : if this in fact also occurs the two terms e are complex and conjugate to each other, and their sum will give a real number.

In the case where the characteristic polynomial has complex roots, it is communi- cally possible to give a parameterization of the signal in which only real quantities appear.

Proposition 3.8 *The sum of terms given in eq.* (3.10)*, representing the contri-bute of a pair of complex conjugate roots of multiplicity to the linear combination of modes, it can also be rewritten as*

(3.11)

where instead of the complex unknown coefficients e appear the real unknown coefficients and .

Demonstration. Consider the generic term , where we have omitted the indices so as not to burden the discussion. We write the coefficients and in polar form

e

Where is the modulus of the coefficient and is its phase.

Therefore, it is worth

dove nel terzo passaggio abbiamo usato la formula di Eulero (cfr. Appendice A.3) e nel quarto abbiamo introdotto un nuovo coefficiente che vale il doppio del modulo del coefficiente . La combinazione lineare di due modi is therefore equivalent to the term , which is called *pseudoperiodic mode*.

The preceding consideration leads to the definition of a structure of the linear combination of modes equivalent to that given by (3.8), in which, however, each pair of complex, conjugate roots corresponds to a linear combination of modes in the form given by (3.11).

We number the roots of the characteristic polynomial for simplicity as follows. There are distinct real roots of multiplicity (for)

and distinct complex and conjugate root pairs[3] of multiplicity (for)

Thus, we can represent a linear combination of modes by distinguishing, thanks to (3.7) and (3.11), the modes associated with real roots and those associated with complex and conjugate root pairs

(3.12)

In the special case where all roots have unit multiplicity[4] one can write

(3.13)

omitting the second subscript in the coefficients for simplicity.

Equations (3.12) and (3.13) should therefore be seen as an alternate form of equations (3.8) and (3.9) more suited to describe the case in which the characteristic polynomial of the system has both real and complex roots.

Example 3.9 Consider a system whose homogeneous differential equation is

[3]It must, of course, apply to .

[4]In that case it applies .

The characteristic polynomial is (the known term is missing) and therefore it has roots

di molteplicità

of *multiplicity*

of *multiplicity*

Thus there are distinct real roots and distinct complex conjugate root pairs.

Thus, it can be written that a linear combination of the modes takes the form

It is also possible to pose a combination of modes associated with a complex, conjugate root pair in another standard form.

Proposition 3.10 *The sum of terms given in eq*. (3.10)*, representing the contribute of a pair of complex conjugate roots of multiplicity to the linear combination of modes, it can also be rewritten as*

(3.14)

where instead of the complex unknown coefficients e appear the real unknown coefficients and .

Demonstration. This result follows from the same considerations made for the previous proposition, bearing in mind that by placing the coefficients and

In Cartesian form, it holds:

having place and .

So with the same reasoning already seen, we can give the following expression of the linear combination of modes, distinguishing between linear combinations of modes associated with distinct real roots and linear combinations of modes associated with distinct complex and conjugate root pairs

(3.15)

In the special case where all roots have unit multiplicity, one can write

(3.16)

omitting the second subscript in the coefficients for simplicity.

Equations (3.15) and (3.16) should therefore be seen as fully equivalent to equations (3.12) and (3.13): they, too, give the parametric structure of the linear combination in the case where the characteristic polynomial of the system has both real and complex roots.

Example 3.11 The same problem as in Example 3.9 can also be solved by posing

Finally, observe that if we represent on the complex plane the two *coefficients* and as done in Fig. 3.1 it is easily demonstrated that it is worth

(3.17)

3.3 Evolution free

We now turn to characterizing free evolution, that is, the contribution to the response due to the fact that the system is not initially at rest.

Proposition 3.12 *Free evolution is a linear combination of the modes of the system.*

Demonstration. If we assume that the input applied to the system is always zero for , its derivatives of order 1, 2, etc. will also be zero. So the free evolution for of the system described by (3.1) coincides with the

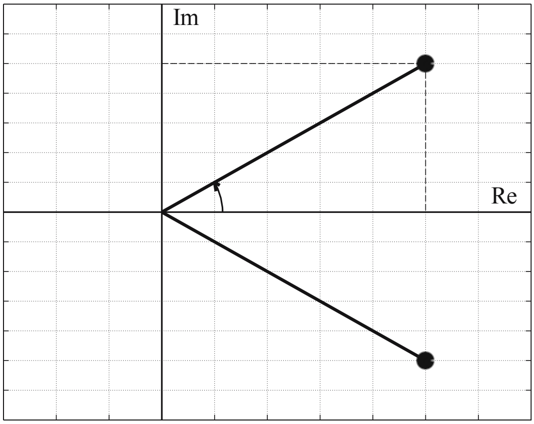


Fig. 3.1. Link between the coefficients of complex modes

solution of the associated homogeneous equation (3.5) from the initial conditions (3.2). According to Theorem 3.7, therefore, the signal is a particular linear combination of the modes.

Keep in mind that the course of the free evolution, and thus the value of the coefficients characterizing its parameterization, depends on the ini- tial conditions. So once written as a linear combination of the modes of the system, we derive the values of the unknown coefficients due to the initial conditions (3.2) by imposing

Example 3.13 You want to calculate for the free evolution of a system whose homogeneous differential equation is

from the initial conditions , and .

The characteristic polynomial is and therefore it has roots

di molteplicità
of *multiplicity*

Therefore, one can write

while deriving twice gives

e

Taking into account the initial conditions, the system is obtained :

whose solution , , allows the expression of free evolution to be written for as

(3.18)

3.3.1 Complex roots and conjugate

In case the characteristic polynomial has complex conjugate roots, it is still possible to use the same way to determine the free evolution, however, having the accuracy to express the linear combination by means of the formula given in eq. (3.12) or eq. (3.15).

Example 3.14 You want to calculate for the free evolution of a system whose homogeneous differential equation is

from the initial conditions , and .

The characteristic polynomial is (the known term is missing) and therefore it has roots

di molteplicità
of *multiplicity*
di molteplicità

Thus there are distinct real roots and distinct complex conjugate root pairs.

Thus, using the parameterization given in eq. (3.12), we can write

while deriving twice gives

e

Taking the initial conditions into account, the system is obtained:

Although the system is nonlinear in the unknowns and , it is easy to see that es- so is linear with respect to the unknowns and With these substitutions we obtain the system

which has solution , and Therefore we derive :[5]

rad and free evolution applies to :

The same problem can also be solved using the parameterization given in eq. (3.12) by posing

Deriving twice gives

[5]Keep in mind that *it is the* angle formed by the vector with the axis of abscissa. This vector is in the third quadrant having real part and imaginary part (see § A.2.3).

e

Taking into account the initial conditions, the system is obtained:

which has for solution ...., andPer can therefore be written free evolution as

As predicted by Eq. (3.17), comparing the two different forms that the solution takes, the following relations apply,

e

or vice versa: e

3.3.2 Initial instant other than 0

We end this paragraph by also giving an example showing how to calculate free evolution from an initial instant .

Example 3.15 Consider the same system examined in Example 3.13, assuming, however, that the given initial conditions hold at an *initial* instant

.

Therefore, it is desired to calculate by the free evolution of the system whose homogeneous differential equation is

from the initial conditions

With the change of variable, the problem becomes one of calculating for the free evolution of the system whose homogeneous differential equation is from the initial conditions

The solution to this problem has already been calculated in Example 3.13, and based on (3.18) it holds *for*

Finally, by substituting into the expression of the function, we obtain the sought solution, which holds for

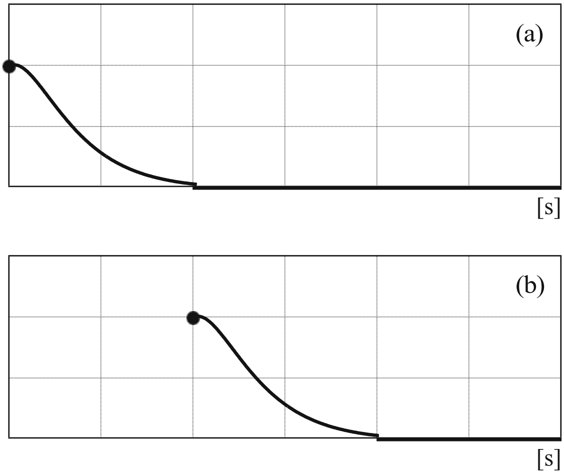


Fig. 3.2. (a) Free evolution of the system in Example 3.13 with ; **(b)** free evolution of the system in Example 3.15 *with*

The free response of the system studied in Example 3.13 from the initial instant and that of the system considered in this example, assumed , are shown in Fig. 3.2. Note that the free evolution is defined only for values of and for clarity we have indicated the initial value with a circleIt is easy to understand that because of the stationarity property, the curve
of the second evolution is obtained by translating the curve of the first one so that it starts from the new initial instant .

3.4 Classification of modes

Modes characterize the dynamics of a system, and it is important to study what form these signals take.

A first classification of modes is as follows.

Aperiodic modes: these are, by , the modes of the form

corresponding to a real root of multiplicity .

Pseudoperiodic modes: these are, by , the modes of the form

e

or equivalently of the form

corresponding to a pair of complex *conjugate* roots

Of multiplicity .

The names "aperiodic" and "pseudoperiodic" indicate that modes of the first type do not exhibit oscillatory behavior, while those of the second type exhibit oscillatory behavior (quasi-periodic, in fact).

3.4.1 Modes aperiodic

The fundamental parameter characterizing the generic aperiodic mode cor- responding to a nonzero real root is the *time constant* defined as

—

We can therefore represent the mode in the two equivalent forms

–

from which it is understood that since the exponent is a dimensionless number, it has appun- tion to the dimension of a time. In the case of a real null root , however, the time constant is not defined.

Roots of unit multiplicity

If the real root has multiplicity , only one aperiodic mode is associated with it, which takes the form of a simple

....exponentialDisti nguish three

cases:

- : in which case the mode is said to be *stable* (or *convergent*) because as *the* tends asymptotically to .
- : in which case the mode is said to be *at the stability* (or *constant*) *limit* because for each value of vale .
- : in which case the mode is called *unstable* (or *divergent*) because as *the* tends to .

In Fig. 3.3 we have shown the trend of these modes for different values of .

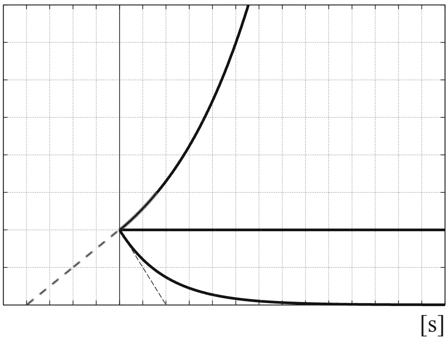


Fig. 3.3. Evolution of aperiodic modes of the type

Geometric interpretation of the time constant. The time constant previously defined has a simple geometric interpretation: it is *the sub-tangent to* the mode curve in , that is, it is the value at which the tangent line in to the mode curve intersects the x-axis. To prove this result, we determine the equation of the tangent line. The derivative of the mode in is worth:

—

therefore the tangent line to the curve in has angular coefficient . Furthermore, this line passes through the point , i.e. . We can therefore conclude that the tangent line has expression and it intersects the x-axis when , i.e. when

.

Graphically, the time constant of a mode can be derived with the construction shown in Fig. 3.3: the dashed line represents the tangent line to the mode in and is drawn until it intersects the x-axis. We observe again that the time constant takes on negative values if (unstable mode) while it takes on positive values if (stable mode). Both cases are shown in Fig. 3.3: the positive root competes with a negative time constant , while the negative root competes with a positive time constant .

Physical interpretation of the time constant. To determine the main physical meaning of the time constant, we evaluate the mode modulus for values of multiples of .

–						

Dalla tabella osserviamo che un modo stabile si può in pratica considerare estinto dopo un tempo pari a circa volte la costante di tempo : infatti il suo valore si riduce al del valore iniziale. Ad esempio, in Fig. 3.4.a si osserva che per the mode is almost extinct.

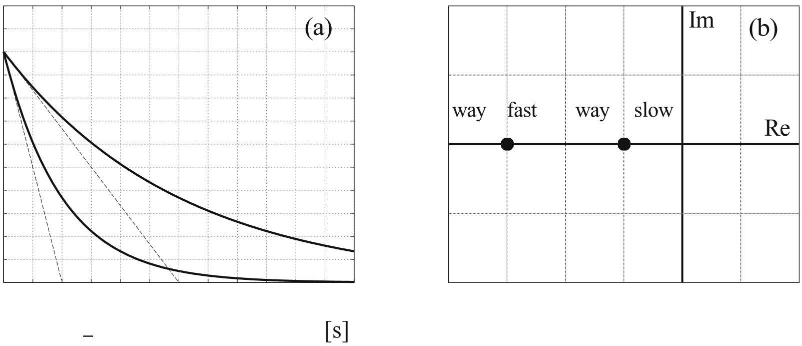


Fig. 3.4. Comparison of two stable modes with different time constants: **(a)** evolution of modes; **(b)** root representation in the plane of complex

We then define for a stable mode the *settling time at* as the time required for the value of the mode to reduce to of the initial value. This quantity is denoted Vale therefore

Note that sometimes settling time is mentioned without further specification: the 5 % settling time is then meant.

Slow modes and fast modes. A decreasing mode dies out the faster the smaller its time constant (i.e., its settling time). This allows us to compare two aperiodic decreasing modes associated with the roots respectively : the first mode is said to be *faster* than the second if or *slower* otherwise. Note again that if we represent the roots in the complex plane[6] , they lie on the negative real axis and the smaller time constant competes with the root farthest from the imaginary axis.

[6]This plan is also called *the Gauss plan* in honor of Johann Carl Friedrich Gauss (1777-1855, Germany).

For example, consider the two decreasing aperiodic modes in Fig. 3.4.a, associated respectively to the roots e . The representation in the plane complesso delle due radici è data in Fig. 3.4.b. Osserviamo che il modo associato alla radice più distante dall'asse immaginario è il più veloce: ad esso infatti corrisponde la costante di tempo più piccola – Viceversa, il modo associated with the root closest to the imaginary axis is the slowest.

Roots of multiplicity greater than one

If the real root has multiplicity , aperiodic modes are associated with it:

The first of these modes has the form already seen above while for the modes of the form with two cases can be distinguished.

If the mode is *stable* for each value of........................This result is not obvious, because studying the behavior of the mode for values of increasing we get

which for and is an indeterminate form ( ). However, *by* deriving times is achieved through the rule of l'Hospital:

and therefore we can conclude that the mode tends to for that tends to infinity.

If the mode *is unstable* for any value ofIn ....fact if the root is null ( ) the mode is worth and this function is always increasing; in particular, for you have a straight line, for *a* parabola, for a cubic parabola, etc. For , the mode diverges even more rapidly.

Examples of the evolution of such modes for and are shown in Fig. 3.5. Note that if the tangent to the mode in has unit slope; conversely, if the tangent to the mode in has zero slope.

Also for a decreasing mode of the form with we can say that the smaller the time constant , the faster the mode is to die out. However, now this quantity has an entirely different geometrical interpretation from that seen in the case : in fact *it represents the value of where the stable mode has its (only) maximum*. To demonstrate this result, we observe that the derivative of such a mode is worth

This derivative cancels for values of positive only if and . So the curve for has a maximum at the pointThose points are shown in Fig. 3.5.

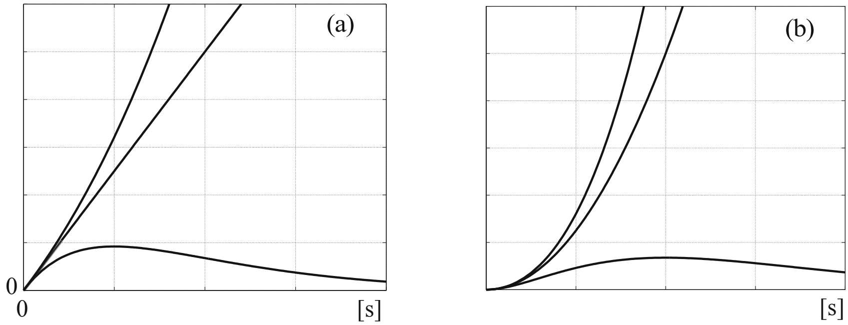


Fig. 3.5. Evolution of aperiodic modes of the type **: (a)** case ; **(b)** *case*

3.4.2 Modes pseudoperiodic

Delle diverse forme che può assumere un modo pseudoperiodico corrispondente alla coppia di radici complesse coniugate possiamo limitarci in tutta generalità a considerare la forma

The other forms are obtained from this by introducing an appropriate phase shift.

We define the following parameters.

Se , in maniera analoga a quanto fatto per i modi aperiodici si definisce la *time constant*

The *natural pulse* is defined as.

(3.19)

If we represent the pair of roots in the complex plane, assuming that Is the pole in the positive imaginary half-plane (i.e., that it is worth) as in Fig. 3.6, we observe that corresponds to the modulus of the vector joining the pole (i.e., the pole) with the origin.

The *damping coefficient* is defined as.

(3.20)

As seen in Fig. 3.6.a, in the complex plane it corresponds to the sine of the angle that the vector joining the pole with the origin forms with the semi-axis , assumed to be counterclockwise as the positive direction. Therefore such an angle is positive if , null if and negative if .

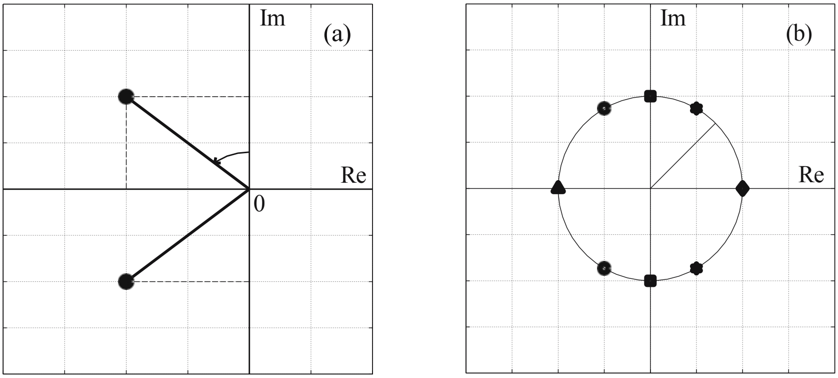


Fig. 3.6. Representation of a complex, conjugate root pair in the complex plane: **(a)** Geometric meaning of and ; **(b)** root pairs with constant as the damping coefficient varies

Finally, note that while equations (3.19) and (3.20) express the natural pulsation and damping coefficient as a function of the real and imaginary part of the roots, it is also possible to invert these relationships. We will obtain in that case

e (3.21)

Roots of unit multiplicity

If the complex conjugate root pair has multiplicity , the corresponding pseudoperiodic mode takes the form .

This mode exhibits oscillatory behavior because of the cosine factor. It is also immediate to observe that it is enveloped by the curves and ....In fact
vale

if —

if —

We distinguish three cases:

In this case, the mode is *stable* because as the envelopes tend to increase asymptotically to . Two examples of such a mode are shown in Fig. 3.7.

This mode reduces to and is also called *periodic*. In that case the mode is *at the stability limit* because as the envelopes increase, the curves are constant . Such a mode is depicted in Fig. 3.8.a.

In this case, the mode is *unstable* because as the envelopes grow, they tend to
a . This mode is depicted in Fig. 3.8.b.

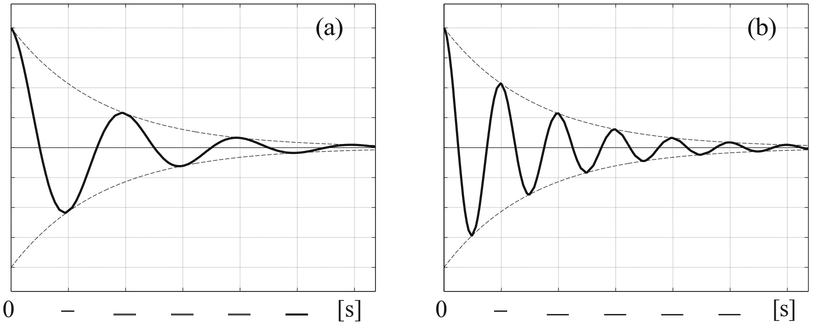


Fig. 3.7. Evolution of pseudoperiodic modes of the stable type ( ) ; the mode in figure **(a)** has the same time constant but higher damping coefficient than the mode in figure **(b)**

The time constant indicates, in a manner analogous to what has already been seen in the ca- se of an aperiodic mode, how quickly the envelope grows or decreases. It is also possible to associate a very intuitive physical meaning with the damping coefficient.

First we can observe that the damping coefficient is a real number in the range being the sine of the angle . Let us now consider several pairs of roots all characterized by the same natural pulsation but with different damping coefficient. These roots lie in the complex plane along a circumference of radius and in particular we distinguish several cases.

: se e ; in questo caso limite le due radici complesse coincidono con una radice reale negativa di molteplicità 2 a cui competono i modi aperiodici e .

: se e ; in tal caso le due radici complesse hanno parte reale negativa e ad esse corrisponde un modo pseudoperiodico stabile.

: se e ; in tal caso le due radici sono immaginarie coniugate e ad esse corrisponde un modo al limite di stabilità.

: se e ; in tal caso le due radici complesse hanno parte reale positiva e ad esse corrisponde un modo pseudoperiodico instabile.

: se e ; in questo caso limite le due radici complesse coincidono con una radice reale positiva di molteplicità 2 a cui competono i modi aperiodici e .

The various cases are summarized in Fig. 3.6.b.

Also compare two stable modes of the form e with and . These modes have the same time constant but different damping. In fact, it is worth and therefore the first mode has a higher damping coefficient. The pattern of these modes is shown in Fig. 3.7: note how the second mode, having less damping, exhibits more oscillatory behavior.

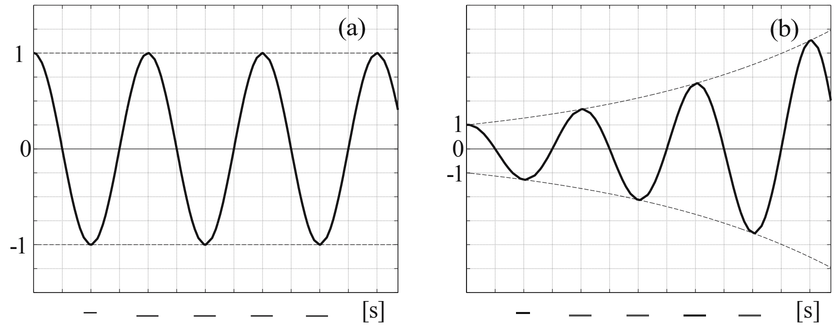


Fig. 3.8. Evoluzione dei modi pseudoperiodici del tipo : **(a)** modo al limite di stabilità ( ); **(b)** modo instabile ( )

Roots of multiplicity greater than one

If the complex conjugate root pair has multiplicity , pseudoperiodic modes are associated with it:

The first of these modes has the form already seen above. For the mode of the form with , in analogy with what was done for the aperiodic modes, we distinguish two cases.

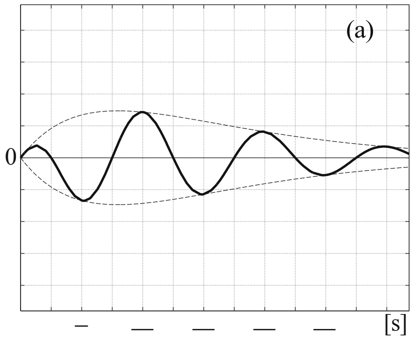
If the mode is *stable* for each value of .
If the mode is *unstable* for any value of .

Such modes are obtained by enveloping the sine function with the curves already studied. Examples of such modes are shown in Fig. 3.9.

Example 3.16 Consider a system whose input-output model has associated homogeneous equation

The characteristic polynomial of such a system is worth

And its roots are worth:



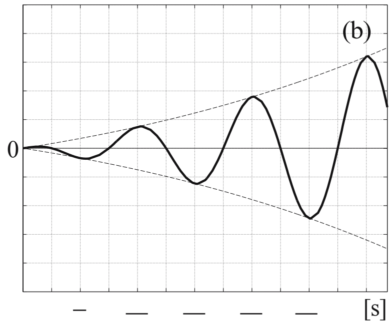


Fig. 3.9. Evoluzione dei modi pseudoperiodici del tipo : **(a)** modo stabile ( ); **(b)** unstable mode ( )

Thus, the system has an aperiodic mode corresponding to the real root and a pseudoperiodic mo- dow corresponding to the complex, conjugate root pair. These modes are valid:

: stable aperiodic mode with time constant

: modo pseudoperiodico stabile con costante di time

natural pulse

and damping coefficient

The pattern of modes is plotted in Fig. 3.10. The settling time at is worth for the first mode s. The settling time at for the second mode is worth about s: as can be seen from the figure for values of the mode value is always within the intervalThe ....first mode, having smaller

constant time and therefore less settling time, is the fastest way.

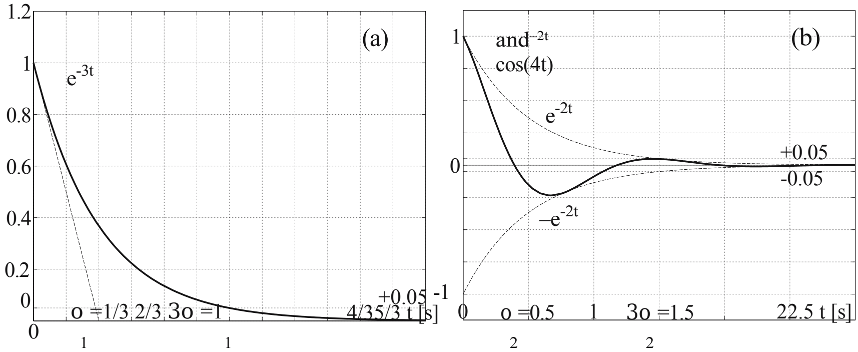


Fig. 3.10. Trend of the modes of the system in Example 3.16

3.5 The response impulsive

Before studying how the forced evolution of (3.1) can be determined with- following the application of an arbitrary input, it is useful to define a particular forced evolution. The exposition of this section assumes that the reader is fa- rily familiar with the material presented in Appendix B regarding distributions and derivatives of discontinuous signals.

Definition 3.17. *The* impulsive response *is the forced evolution that results from the application of a signal Æ , i.e., a unit impulse applied at instant .*

The importance of such a function arises from the fact that, as we shall see, it is *a canonical re- gime*: this means that analytical knowledge of this response makes it possible to determine the forced evolution of the system for any other input, and also the free evolution for any value of the initial conditions. So knowing the impulsive response of a system is equivalent to knowing its model perfectly.

3.5.1 Structure of the response impulsive

Proposition 3.18 *The impulsive response of a system described by model* (3.1) *is zero for , and for can be parameterized as a linear combination of the modes of the system and any impulsive term i.e.*

ÆÆ (3.22)

where, called the multiplicity of the generic root of the characteristic polynomial, it is worth

(3.23)

Whereas in the special case where all the roots of the characteristic polynomial have unit multiplicity, it applies:

(3.24)

In addition, the impulsive term will be present if and only if the model (3.1) *is not strictly proper and valid:*

— *if*

if

Demonstration. Let us first observe that in a causal system an effect can never precede the cause that generated it. The system described by (3.1) is cer- tainly causal (being proper) and therefore the response to an impulse applied to time must necessarily be zero for negative values of: this is imposed by the presence of the factor *Æ* in the expression of the impulsive response.

Again, we observe that an impulsive input is by definition null for . We can therefore think that the system, initially unloaded in , is found at instant in a nonzero initial state due to the- action of the impulsive input. From the instant , the input being always null, the evolution of the system is a particular free evolution with coef- ficients to be determined. This justifies the presence in the expression of of the linear combination of modes.

We will not, however, give a formal demonstration of the possible presence of the impulsive ter- mine and what has been said about the coefficient : the validity of these statements follows from the rule for determining coefficients described in the following section.

Let us now look at a simple example to motivate the presence of an impulsive term in the expression of impulsive response.

Example 3.19 Consider an instantaneous system whose model is as follows.

It is therefore a special case of model (3.1) with *(* not strictly proper system), .

Since the UI model is an algebraic equation, the characteristic polynomial has degree and therefore such a system admits no mode. We can also easily calculate the impulsive response: in fact, at the input *Æ* follows the response

−*Æ*

which is precisely in the form specified by Proposition 3.18, with..........................

Of course in the case where the characteristic polynomial of the system has distinct real roots and distinct complex and conjugate root pairs, it is always possible on the basis of what was seen in Section 3.2.1 to rewrite (3.23) in an equivalent form in which pseudoperiodic modes appear by posing

(3.25)

or

(3.26)

Finally, note that in equations (3.23), (3.25) and (3.26) we have denoted the unknown coefficients appearing in the expression of the impulsive response with the same symbols (, , ,) already used to denote the coefficients appearing in the expression of the free evolution. However, always keep in mind that the values of these coefficients will generally be different for free evolution and impulsive response. In the case of free evolution, in fact, such coefficients can assume an infinity of arbitrary values, depending on the particular initial conditions considered. In contrast, in the case of impulsive response, the coefficients depend uniquely on the characteristics of the system: their value can be determined by the procedure described in the next section.

3.5.2 Calculation of impulse response [*]

Chapter 6 will present a very simple technique for calculating the impulsive re-shift, based on the use of Laplace transforms. However, it is also possible, although less easy, to calculate the impulsive response in the time domain; for completeness, we describe a technique for doing so in this section.

The algorithm we present is based on the fact that the impulsive response has a known parameterization (Proposition 3.18) and that this response must satisfy for every value of la (3.1) when the input is an impulse. The parameterization contains unknown coefficients: the coefficients of the modes and the coefficient of the impulsive term.

Since the must satisfy eq. (3.1) for every value of , including the instant in which discontinuities or impulsive terms will typically appear, we calculate the successive derivatives of the , up to the derivative of order -mo, using the technique[7] described in Appendix B (see § B.2).

[7]Remember that if Æ is worth Æ Æ

Denoting the derivatives of the function as , , , , and remembering that Æ , for , denotes the -ma derivative of the impulse, we obtain:

The impulse response must satisfy the differential equation (3.1) being.

that is, it must satisfy the equation

$$(3.27)$$

By substituting the expression of the and its derivatives into that equation, we can impose equality between the first and second members of the coefficients multiplying the individual terms Æ Æ Æ : in fact, these signs are linearly independent of each other.

Thus, a system of equations is derived

$$(3.28)$$

where if you poseNote that in this system

le and the (for) are known coefficients that are derived from the given differential equation. Instead, the unknowns are the coefficient and the coef- ficients[8] of the which, appearing in the expression of the linear combination

, will also be present in the expression of terms .

Note that terms multiplying Æ should also appear in the first member: it can be shown, however, that these terms always cancel each other.

From the last equation of the system (3.28) it follows, as Proposition 3.18 states, that:

[8]These coefficients are gli in the case of real roots, while in the case of complex and conjugate roots the coefficients e or e will appear .

if it applies: — ;

if it counts : .

It is then possible to further simplify the calculation of the impulse response by determining the term a priori and considering that term as a known constant in the system (3.28), the last equation of which then becomes an identity.

To sum up, we can give in summary form the following procedure.

Algorithm 3.20 (Calculation of impulse response)

1. *We determine the characteristic polynomial of the homogeneous equation associated with* (3.1) *and calculate its roots.*
2. *The modes of the system are determined.*
3. *We write the in one of the parameterizations given by equations* (3.22)*,* (3.24)*,* (3.25) *or* (3.26) *as a function of the values taken by the roots of the characteristic polynomial,*

   Æ Æ

   where e is a linear combination by unknown coefficients of the modes of the system.
4. *We calculate the successive derivatives of the up to the derivative of order :*
   , , , .
5. *The following system of equations is written*

   . . .

   (3.29)

   which allows the determination of the unknown coefficients of the .

Let us now look at an application example.

Example 3.21 You want to calculate the impulse response of the system described by the UI model

—— —— —— (3.30)

The characteristic polynomial holds and has two real roots of unit multiplicity. Moreover, being in that case , we know that the will not contain the impulsive term.

So the structure of the impulsive response and its first and second derivatives holds:

Æ

ÆÆ

Æ Æ Æ

Substituting the and its derivatives into (3.30) and posing Æ gives:

Æ

Æ Æ

Æ Æ Æ

Æ Æ

Æ Æ

where it may occur that the coefficient multiplying term Æ is identically null.

We can write the system of two equations (remember that it holds being
)

which has solution and . Therefore Æ

In carrying out the previous example, it was preferred to proceed step by step by determining all the quantities that were used in the presentation of Algo- pace 3.20. It is possible to limit ourselves more simply to the steps described in the algorithm, as in the following example that deals with the case of a system having aperiodic and pseudoperiodic modes.

Example 3.22 You want to calculate the impulse response of the system described by the UI model

The characteristic polynomial is (the known term is missing) and therefore it has roots

di molteplicità
of *multiplicity*
di molteplicità

Since the system is strictly proper, the coefficient of the impulsive term in the is valid, and we can therefore pose

Æ Æ
Æ

Deriving the function twice gives:

and we can write the system of three equations

or

which with the change of variable and becomes

e has solution: , ...andThen it is also valid

| — – . rad

e infine si ricava Æ .

3.6 Forced evolution and the integral by Duhamel

This section presents a fundamental result, called the Duhamel integral[9] , which states that the forced evolution that follows the application of a generic input is determined by convolution with the impulsive response of the system. Refer to Appendix B (see § B.3) for the definition of convolution integral.

[9]Jean Marie Constant Duhamel (1797-1872, France).

3.6.1 Integral by Duhamel

In the following definition we consider a system at a remote instant : we assume that before that instant no cause could have acted on the system, which therefore is initially at rest. Also from that instant on the system acts on an input signal : knowledge of that input for an interval of time allows us to determine the output at time .

Proposition 3.23 (Duhamel's Integral) *Given a system at rest for , for each value of vale*

(3.31)

Demonstration. First define as the forced response that con- follows the application of a *finite impulse Æ* , where the formal definition of finite area im- pulse is given in Appendix B, see § B.1.3, eq. (B.4). Since according to eq. (B.5) the following applies.

Æ Æ

it is easy to see that it also applies to

as exemplified in Fig. 3.11.

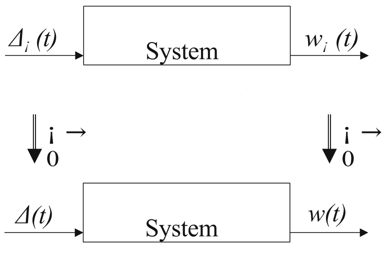


Fig. 3.11. The impulse response as the limit for of the response that follows a finite impulse Æ

Having chosen a sampling step, we then approximate the signal with a series of rectangles as shown in Fig. 3.12. The generic rectangle that makes up the input signal is a finite pulse Æ where the subscript indicates the value of the base of the rectangle, while the argument indicates that it is shifted by a quantity to the right; in addition, such a finite pulse of unit area is multipli- ed by the scaling factor that corresponds to the area of the generic rectangle

of base and heightNote that this decomposition is all the better as
the smaller it is....Said

Æ

vale .

Since the system is linear, the principle of superposition of effects applies: we can therefore approximate the total response of the system to such an input as the sum of the responses that result from the individual terms that compose it: thus

and for that tends to zero you can make the substitutions (that variable becomes real), and thus

and finally, observing that for a causal system the is null for negative values of the argument, that is, for we obtain the Duhamel integral (3.31).

Note that Duhamel's integral is a convolution integral (see § B.3) in which for convenience we have placed the upper extreme of integration equal to instead of a because, as already noted, the convolution of the two signals and is zero for values of . We can therefore also write based on the properties of the convolution integral

being for .

This equivalent form of the Duhamel integral also lends itself to a further physical interpretation. The contribution to the value taken by the output at the instant due to the value taken by the input instants of time earlier is weighted through the impulse response .

In a system whose modes are all stable the tends to zero and, for values of greater than a certain value that depends on the time constants of the system, it can in practice be considered zero. So a system whose modes are all stable loses memory of the value assumed by the input after a time greater than
From its application.

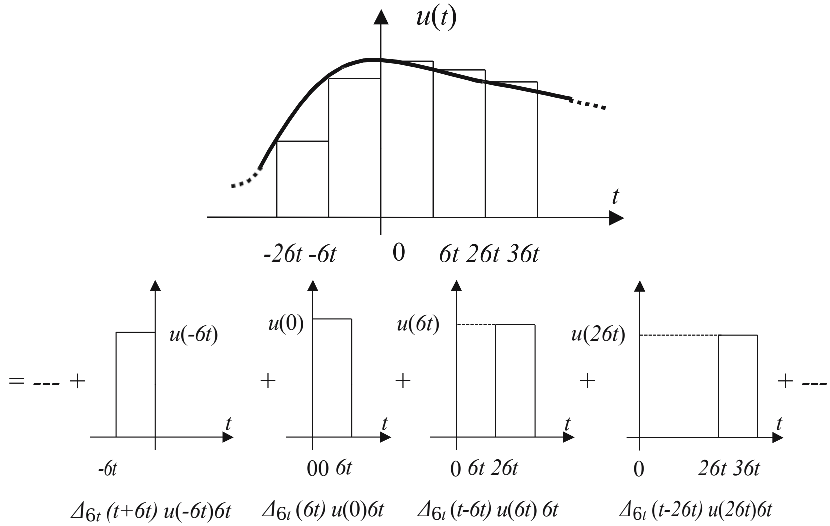


Fig. 3.12. Decomposition of a signal into a sum of finite pulses

3.6.2 Breakdown into free evolution and forced evolution

Several physical interpretations can be given to Duhamel's integral. Having chosen a ge- nerical instant of time that will be considered as the initial instant, we decompose the integral into two terms by writing for :

The first term represents the contribution to the output signal at time due to the values assumed by the input for values of time prior to the initial instant . Because of this input, the system will be at instant in a state that will generally be different from the zero state, that is, it will have a nonzero initial state (i.e., the initial conditions given by (3.2) will not all be zero). As we have defined the fundamental problem of systems analysis, the contribution of this term to the output signal at time is precisely called *free evolution*. Note that free evolution thus has a twofold interpretation: on the one hand, it can be regarded as the response of the system caused by the presence of a nonzero initial state, i.e., as the response to that input acting on the system previously at instant brought the system to that state .

The second term represents the contribution to the output signal at time due to the values assumed by the input for values of time after the initial instant : this term thus coincides with *forced evolution*. This observation allows us to thus to give a method for calculating the forced evolution of a system whose impulsive response is known.

Finally, note that Duhamel's integral makes it possible to justify the claim already made that the impulse response is a canonical regime. In fact, having assigned such a function, it is also possible to calculate the value of the output that follows the application of any other input.

3.6.3 Calculation of the forced response by convolution

Based on the considerations made in the previous paragraph, given a generic initial time instant the forced evolution can be determined by one of two equivalent formulas

(3.32)

The second formula is derived from the first with the change of variable :

If (3.32) is simplified into.

(3.33)

In the table at the end of this chapter are some notable formulas that may be useful in solving Duhamel's integral.

Finally, we conclude with two examples showing how to calculate the forced response of a system using the Duhamel integral.

Example 3.24 We wish to calculate the forced evolution of the system considered in Example 3.21 and described by the UI model (3.30), resulting from the application for

Of an input Æ The impulse response of such a system is worth

Æ .

The forced evolution will be null ....forTo determine its value at instants subsequent to the initial one, we apply the first of (3.33), keeping in mind that for ....It is therefore valid *for*

Therefore, we can write that the forced evolution is worth

Æ

Of course, the same result would also have been obtained by applying the second of (3.33). Keep in mind that this case also holds for

It therefore *applies to*

Example 3.25 For the same system as in the previous example, determine the forced response that follows the application of the input signal

se

elsewhere

also shown in Fig. 3.13.

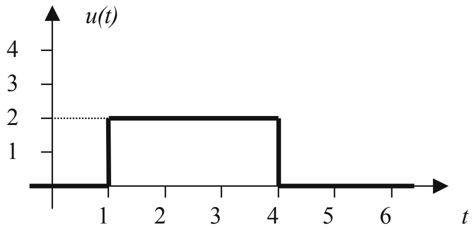


Fig. 3.13. Input signal in Example 3.25

Si è visto che la risposta impulsiva vale Æ e grazie to Duhamel's integral we can write that the forced response is worth

if

if

if

With the change of variable we see that for vale while for vale

Therefore, it is worth

se

if

if

The above example lends itself to some important considerations.

First note that although the input acts on this system only for a limited time interval , the forced response that follows from it does not cancel for values ofIn fact ...., at the instant the system will be found to have A non-zero state due to the previous action of As of that instant the evolution of the system, failing the action of the input, is reduced to a free evolution, which justifies its having the form of a linear combination of modes: .

The shape of the forced response during the time interval also has a particular structure: it is a linear combination of modes plus a constant term. This particular shape in step response will be better studied in Chapter 6 (see § 6.5) when we discuss the shape of the forced evolution for a large family of input signals.

Finally, note that the given input can also be considered as the sum of two signals: an amplitude step applied at the time instant , and an amplitude step applied at the time instantTaking ....advantage of the linearity and stationarity of the system, it would have been possible to determine the forced evolution from knowledge of the unit step response alone (see Exercise 3.7).

3.7 Other canonical regimes [*]

It has been observed that the impulsive response of a system is a canonical regime, that is, a particular evolution whose knowledge is equivalent to knowing its model perfectly. This is an immediate consequence of Duhamel's integral. However, it should be noted that there are other canonical regimes.

First we will give some definitions.

Given a signal we define its integro-differential family as follows: is the signal itself, while for vale

vdls

So it is the first derivative of the signal while is its integral, and so on for the other values of . Note that this notation is consistent with the notation used in Appendix B (see § B.1.5) to describe the family of canonical signals *Æ* , obtained by integration and pulse derivation.

In particular, given a system whose impulsive response is worth , we can definition the integro-differential family of signals , Note firstly that an important physical meaning can be associated with the generic signal.

Proposition 3.26 *Given a system described b y model* (3.1) *be its impulsive redisplacement. For , the signal is the forced response that follows the application of an input Æ .*

Demonstration. The impulsive response is the forced response that follows the application of an impulsive input *Æ* and can be written, thanks to Duhamel's integral,

Æ

According to Proposition B.8 in Appendix B, deriving or integrating this expressio- n is obtained for :

Æ

which can precisely be interpreted, again based on Duhamel's integral, as follows: upon application of the signal *Æ* the system responds with an output

.

It is finally possible to enunciate the following result

Proposition 3.27 *Given a system described b y model* (3.1) *be its impulsive redisplacement. If the system is at rest for and is the input signal applied to it, for each value of vale*

(3.34)

Demonstration. Duhamel's integral allows us to write.

while thanks to Proposition B.8 part 2, which states that the convolution between two signals does not change if one operand of the convolution is derived while the other is integrated, the following holds true

e osservando che è sempre nulla per valori negativi del suo argomento (cioè per) si può restringere l’estremo superiore della convoluzione a .

Dunque per , l’evoluzione forzata conseguente all’applicazione del segnale Æ è un regime canonico.

Exercises

Exercise 3.1. A system described by the input-output model is given.

$$— \quad — \quad — \quad — \tag{3.35}$$

Determine the modes that characterize such a system by indicating their caracteristic parameters. Evaluate the stability of the individual modes by indicating approximately the settling time. Evaluate which mode is the slowest and the fastest.

Exercise 3.2. For the system (3.35) determine the free evolution from the initial istant given the initial conditions

$$— \quad —$$

Exercise 3.3. Verify that the impulsive response of the system (3.35) holds:

$$\_ \quad \frac{\_}{—} \quad \frac{\_}{—} \quad Æ$$

Determine an equivalent expression of the impulse response according to the form given in eq. (3.25).

Exercise 3.4. Determine the forced response of the system (3.35) that results from the application of an input signal Æ .

(Apply Duhamel's integral keeping in mind that the value of the impulse response is known from the previous exercise.)

Exercise 3.5. Consider a linear, stationary system described by the following UI model:

$$\text{—— —— ——} \qquad (3.36)$$

Let it be verified that the impulse response of such a system is worth

Æ

e si calcoli la risposta forzata conseguente all'azione di un ingresso Æ .

Exercise 3.6. The following exercise is intended to show that although each stable aperiodic mode has a monotonically decreasing trend, the free evolution of a system characterized by multiple stable aperiodic modes is not necessarily monotonic.

Consider a system whose homogeneous differential equation is worth

(a) Determine the modes that characterize such a system by indicating its characteristic parameters.
(b) Let it be verified that the free evolution from the initial conditions

applies to :

(c) Plot the trend of this function and verify that it, having multiple maxima and minima, does not have a monotonic trend. How do you explain the presence of such maxima and minima even in the absence of pseudoperiodic modes?
(d) Assess the 5% settling time for the given free evolution.

Exercise 3.7. Consider the linear, stationary system described by the following UI model:

$$\text{—— —— ——} \qquad (3.37)$$

The modes of such a system were studied in Example 3.21, while in E- semple 3.25 the forced response to the application of the input signal was determined

se
elsewhere

Note that Æ Æ also applies, i.e., the given input can also be regarded as the sum of two signals: an amplitude step applied at time instant , and an amplitude step applied at time instant

.

Calculate the forced response to the signal by following an alternative procedure to that used in Example 3.25.

(a) Calculate by m e a n s of Duhamel's integral the forced response that results from the application of a unit step.
(b) Exploiting the property of linearity and stationarity, determine the forced response to *t h e Æ* signal and the *Æ* signal .
(c) Sum the two responses and verify that the total response has the same expression as determined in Example 3.25.

Tables of indefinite integrals

The following formulas are useful for calculating the forced response using Duhamel's inte- gral.

4

Time domain analysis of representations in state variables

This chapter deals with the study, in the *time* domain, of models of linear, stationary and concentrated-parameter systems described in terms of *state variables*. In the first section we recall what the *foundational problem of systems analysis* consists of for such models. To solve this problem, the *state transition matrix* must be determined: in the second section this concept is defined and a general procedure, called *Sylvester development*, for its computation is presented. In the third section, the general solution to the analysis problem is presented, which is expressed through *Lagrange's formula*. In the fourth section, a particular transformation *of* variables, called a *similarity transformation*, is studied, which makes it possible to change from a representation in state variables to a different rap- presentation in state variables always of the same system: what distinguishes the two representations is the choice of quantities assumed as state variables. A particular similarity transformation, called *diagonalization*, makes it possible in many cases to switch to a new (easier to analyze) representation in which the state matrix is in the *canonical diagonal form*: this procedure is presented in the fifth section. If it is not possible to return a given matrix to the diagonal form, it is still always possible to return it, by similarity, to a canonical block diagonal *form*, called *the Jordan form*: this procedure is described in the sixth section. In the eighth and final section, modes are also defined for systems in state variables and a physical interpretation is associated with them.

4.1 Representation in state variables and problem of analysis

A linear, stationary system of order , with inputs and outputs, has the following representation in terms of *state variables* (VS):

(4.1)

where the *state vector* and its derivative have components, the *input vector* has components, and the *output vector* has components.

The fundamental systems analysis problem for such a system is to determine the state and output trends for known:

the value of the initial state ;
the input trend for .

The solution to this problem is provided by the so-called Lagrange formula, which will be described later. Preliminarily, however, it is useful to introduce the notion of a *state transition matrix*, which is described in the following section.

4.2 The transition matrix of the state

Given a square matrix its *exponential* (see Appendix C, § C.2.6) is the matrix

The state transition matrix is a special exponential matrix whose elements are functions of time.

Definition 4.1 *Given a model in VS* (4.1) *in which the matrix has dimension , the* state transition *matrix is defined as the matrix*

$$ \tag{4.2}$$

Note that this series is always convergent and therefore the state transition matrix is well defined for every square matrix .

It is not usually easy to determine the state transition matrix from its definition, and it is usual to resort to other procedures that will be discussed in the following paragraphs. In one particular case, however, the calculation of turns out to be straightforward: when it is a diagonal matrix.

Proposition 4.2 *If it is a diagonal matrix of dimension*

vale

Demonstration. Vale

and, since that matrix is diagonal, the result follows from Proposition C.25.

Example 4.3 Date worth

4.2.1 Properties of the state transition matrix [*]

In this section we recall some fundamental properties enjoyed by ; these properties will enable us to prove Lagrange's formula.

Proposition 4.4 (Derivative of the state transition matrix) *Vale*

Demonstration. To prove the first equality, derive eq. (4.2); we obtain

By highlighting on the right instead, the second equality is demonstrated.

Note that an important fact also follows from the previous property: *switches with (* see § C.2.4).

Proposition 4.5 (Composition of two state transition matrices) *Vale*

Demonstration. Developing the two exponentials in the corresponding series and performing the product yields

Note that the previous result is not as trivial as it appears at first glance. For while in the scalar case it holds obviously or equivalently

, in the matrix case the relationship is not always true but holds if and only if and commute, that is, if and only if (see Exercise 4.9).

Proposition 4.6 (Inverse *of* **the state transition matrix)** *The inverse of the matrix is the matrix , i.e., it is worth*

Demonstration. According to the previous proposition, the following is true

Note that under this proposition a sta- tion transition matrix is always invertible (and therefore nonsingular) even if the matrix were singular.

4.2.2 The development of Sylvester

We now face the problem of determining the analytical expression of the state trans- siation matrix without necessarily having to calculate the infinite series defining it. The procedure we present here is based on *Sylvester's development*[1] . A second procedure, based on switching to the diagonal or Jordan form will be presented in § 4.5.1 and § 4.6.3. Finally, a third procedure, based on the use of Laplace transforms, will be presented in Chapter 6 (see Proposition 6.5).

The following result applies, the proof of which is given in Appendix G (cf. Proposition G.5).

Proposition 4.7 (Sylvester's Development) *If it is a matrix of dimension , the corresponding state transition matrix can be written as:*

(4.3)

where the coefficients of the development are appropriate scalar functions of time.

The coefficients of Sylvester's development can be determined by solving a system of linear equations. We will discuss various cases separately.

[1]James Joseph Sylvester (London, England, 1814 - 1897).

Eigenvalues of unit multiplicity

If the matrix has eigenvalues all distinct , the unknown functions
are derived by solving the following system of equations (as many equations as there are eigenvalues):

(4.4)

. .

that is, by solving the system of linear equations

(4.5)

dove is the vector of unknowns, the matrix of the coefficients is worth[2]

. . . . (4.6)
. . . .

e il vettore dei termini noti vale .

The generic component of the vector is a function of time which is called the *mode of* the matrix associated with the eigenvalue . It is easily verified that each element of the matrix is linear combination of such modes. For a complete discussion of modes, see § 4.7.1 at the end of this chapter.

Example 4.8 Consider the *matrix*

Being triangular, its eigenvalues coincide with the elements along the diago- nal. Such a matrix therefore has distinct eigenvalues and ....To determine
we write the system

from which we derive

---

[2]A matrix that takes the form of Eq. (4.6) is called *the Vandermonde matrix* in honor of Alexandre-Théophile Vandermonde (Paris, France, 1735 - 1796). The attribution arose from a misunderstanding, since the French mathematician did not study such structures.

So

As expected, each element of the matrix is a combination of the two modes
e .

Eigenvalues of non-unitary multiplicity [*]

If the matrix has eigenvalues of nonunitary multiplicity, we construct a system similar to (4.4) in which each eigenvalue of multiplicity corresponds to equations of the form :[3]

(4.7)

or

(4.8)

Again, it is possible to write a linear system of the form (4.5) do- ve to each multiplicity eigenvalue are associated rows of the coefficient matrix[4] :

[3]Note that in this expression one must calculate the derivative of the first and second members with respect to the parameter (seen as a variable) and not with respect to the variable .

[4]A matrix that takes this form is called *a confluent Vandermonde matrix*.

. . . ⋱ . ⋱ .

and rows of the vector of known terms :

Example 4.9 Consider the *matrix*

which has characteristic polynomial and therefore has eigenvalue
Of multiplicity and multiplicity . To determine we write the system

from which we derive

Dunque

Complex eigenvalues [*]

Even in the case where there are complex eigenvalues, it is possible to determine the coefficients of Sylvester's development as indicated above.

To avoid, however, working with complex numbers it is convenient to modify the procedure for calculating the coefficients as follows (we will treat only the case of au- tovalues of unit multiplicity for simplicity). Suppose that among the eigenvalues of the matrix there are 2 complex and conjugate .

In that case the two equations should appear in the system of equations (4.4)

(4.9)

However, we can replace these two equations with two equivalent equations in which no complex terms appear:

(4.10)

where and denote the real and imaginary part of a complex number. In particular, therefore, e is valid .

The first of (4.10) is obtained by adding the two equations (4.9) and dividing by . La seconda delle (4.10) si ottiene sottraendo la seconda delle equazioni (4.9) dalla prima e dividendo per . Infatti se e sono complessi e coniugati, tali saranno anche e e dunque e..La presence of the sine and cosine terms at the second member comes instead from Euler's formulas (see Appendix A.3).

The following example presents the case of a matrix with complex and conjugate eigenvalues in a particular form that will also be taken up later.

Example 4.10 Consider the *matrix*

Such a matrix has characteristic polynomial and distinct eigenvectors It is called the *matrix representation*[5] of the pair

Note that the elements along the diagonal of this matrix coincide with the real part of the eigenvalues, while the elements along the anti-diagonal coincide with the imaginary part.

To determine we write the system

from which we derive

[5]Sometimes it is defined as a matrix representation of the transposition of such matrix.

So

4.3 Formula of Lagrange

We can finally prove an important result that determines the solution to the analysis problem for MIMO systems previously stated. This result is known as *Lagrange's formula* .[6]

Theorem 4.11 (Lagrange's Formula) *The solution of system* (4.1)*, with initial state and input trend (for), holds for :*

(4.11)

Demonstration. Note preliminarily that from Proposition 4.4 it follows:

(4.12)

The equation of state of (4.1), multiplying both members by , holds:

that can rewrite itself

And, according to (4.12),

Integrating between and we have:

[6]Joseph-Louis Lagrange, born Giuseppe Lodovico Lagrangia (Turin, Italy, 1736 - Paris, France, 1813).

i.e.

and therefore

By multiplying both members by according to Propositions 4.5 and 4.6, we obtain the first of Lagrange's formulas.

The second Lagrange formula is obtained by substituting the value of so determined into the output transformation of (4.1).

4.3.1 Free evolution and evolution forced

Based on the previous result, we can also write the state evolution for As the sum of two terms:

The term

(4.13)

corresponds to the *free evolution of the state* from the initial conditions . Si noti che indica appunto come avviene la transizione dallo stato allo stato in assenza di contributi dovuti all'ingresso.

The term

(4.14)

corresponds to the *forced evolution of the state* (the second equation is proved by change of variable) . Note that in such an integral the contribution of to the state is weighed using the weighting function .

The evolution of the output for can also be written as the sum of two terms:

The term

corresponds to the *free evolution of the output* from the initial conditions .

The term

corresponds to the *forced evolution of the output*.

Observe finally that in the special case where , (4.11) reduces to

Example 4.12 Given the following representation in terms of state variables:

(4.15)

you want to calculate for *t h e* evolution of the state and output consequent to the application of an input signal *Æ* from an initial state

The state transition matrix for this representation has already been calculated in Example 4.8 and is worth

We can therefore immediately calculate the free evolution of the state, which for applies:

While the free evolution of the output for vale:

We now calculate the forced evolution of the state, which for worth

–

–

Whereas, being , *t h e* forced evolution of the output for vale:

4.3.2 Impulsive response of a representation in VS

The impulsive response of a SISO system is the forced response that results from the application of a unit impulse and therefore, posed Æ , according to Lagrange's formula it is worth

ÆÆ

Recalling the fundamental property of the Dirac function that if it is a continuous function in and belongs to this interval, it is worth

ÆÆ

you finally get

Æ

This relationship links the impulsive response to *t h e* realization matrices in state variables.

Note that as expected:

- if the system is strictly proper holds , and therefore the is a linear combination of the modes of the system;
- if the system is not strictly proper holds , and thus the is a linear combination of the modes of the system and an impulsive term.

Finally, it can be observed that the expression of the forced evolution of the output given by Lagrange's formula is quite analogous to Duhamel's integral. In fact, consider Eq. (3.33), which expresses the forced response by means of the Duhamel integral, and substitute into it the expression previously derived for ; we obtain

Æ

Æ

Which is precisely Lagrange's formula.

4.4 Transformation of similarity

The form taken by a VS representation of a given system depends on the choice of quantities that are considered as state variables. This choice is not unique, and in fact an infinite number of different representations of the same system can be given, all related by a particular type of transformation called similarity transformation. In this section we define the concept of similarity transformation and characterize the elementary relations that exist between two representations linked by similarity.

One of the main advantages of this procedure is that through particular transformations it is possible to move to new representations in which the state ma- trice assumes a *canonical form that* is particularly easy to study. Examples of canonical forms are the *diagonal form* and the *Jordan form*, which will be studied in later sections of this chapter. Other canonical forms related to controllability and observability, however, will be defined in Appendix D (cf.
§ D.2).

Definition 4.13 *Given a representation of the form* (4.1) *consider the vector related to by the transformation*

(4.16)

Where is any matrix of non-singular *constants. So there always exists the inverse of and also holds Such a transformation is called a* similarity *transformation and the matrix is called a* similarity matrix.

The similarity transformation leads to a new representation.

Proposizione 4.14 *Si consideri un sistema che ha rappresentazione in variabili di stato*

(4.17)

and a generic similarity transformation

The vector satisfies the new representation:

(4.18)

where

(4.19)

Demonstration. Deriving (4.16) we obtain.

(4.20)

and substituting (4.16) and (4.20) into (4.17) we get from which, by pre-multiplying the equation of state by the matrix , the desired result is obtained.

This we obtained is still a VS representation of the same system in which input and output are not changed, but the state is described by the vector Since there are infinite possible choices of nonsingular matrices ,
there are also infinite possible representations of the same system.

It is still said that the representations (4.17) and (4.18) are *similar* or even *related by the similarity matrix* .

Example 4.15 Given the representation in terms of state variables:

and the similarity transformation

you want to determine the representation that corresponds to the state vector .

Observe that it is worth

and also applies to

It is possible to give a simple interpretation to this transformation. Since it holds, we can write

and thus the transformation leads to a new state vector that has as its first component the second component of and as its second component the difference between the first and second components of .

Vale

There are some important relationships between two similar representations.

Proposition 4.16 (Similitude and state transition matrix) Given a matrix worth

Demonstration. We observe that.

times times

and
therefore

This result allows us to formally prove that two similar representations describe the same input-output link.

Proposition 4.17 (Invariance of UI binding by similarity) *Two representations bound by similarity subject to the same input produce the same forced response.*

Demonstration. According to Lagrange's formula, the forced response to a generic input of the system described by the representation (4.18) with

applies to :

and that is, it coincides with the forced response of the system described by the representation (4.17) subject to the same input.

Finally, the following result applies.

Proposition 4.18 (Invariance of eigenvalues by similarity) *The matrix and the matrix have the same characteristic polynomial.*

Demonstration. The characteristic polynomial of the matrix is worth

dove l'ultima eguaglianza deriva dal fatto che Le due matrici therefore have same characteristic polynomial (and therefore same eigenvalues).

This result allows us to state that two similar representations have the same modes: thus modes characterize the dynamics of a given system and are independent of the particular representation chosen to describe it. See also the discussion of modes in § 4.7.1.

Example 4.19 The matrices

e

considered in Example 4.15 and linked by a similarity transformation have both eigenvalues *a n d* and therefore modes and .

Note, however, that two matrices and , although related by a similarity relationship, do not generally have the same eigenvectors.

4.5 Diagonalization

We now consider the case of a particular similarity transformation that, under appropriate assumptions, allows us to change to a matrix in diagonal form.

A representation in which the state matrix is in diagonal *form* is called *diagonal canonical form*, and it lends itself to simple physical interpretation. For example, let us con- cide a SISO system (but the same applies to MIMO systems) whose equation of state is worth

The evolution of the -ma component of the state is governed by the equation

from which we see that the derivative of the -ma component is not affected by the value of the other components.

Possiamo dunque pensare a questo sistema come ad una collezione di sotto-sistemi di ordine 1, ciascuno descritto da una componente del vettore di stato, che evolvono indipendentemente. Il sistema corrispondente alla componente -ma ha polinomio caratteristico e ad esso corrisponde il modo Talvolta si it is also usual to define a diagonal representation by the term *decoupled* to indicate precisely the independence between the different modes.

Moving from a generic representation to a representation in diagonal form requires a particular similarity matrix.

Definition 4.20 *Given a matrix of dimension be A set of linearly independent eigenvectors corresponding to the eigenvalues We define the* modal *matrix of the matrix*

Example 4.21 Consider the matrix

which has eigenvectors and associated eigenvalues and as seen in Example C.28.

The modal matrix is worth

Of course since each eigenvector is determined minus a mol- tiplicative constant, and since the ordering of eigenvalues and eigenvectors is arbitrary, they can multiple modal matrices exist. For example, the following could also have been used as modal matrices for the given matrix

| e |

Note that if a matrix has distinct eigenvalues (as in the case of the preceding example) it certainly admits of a modal matrix: in fact, in such a case, as recalled in Appendix C (see Theorem C.64) there are certainly li- nearly independent eigenvectors. Conversely, if a matrix has eigenvalues of nonunitary multiplicity, then the modal matrix exists if and only if at each eigenvalue of multiplicity it is possible to associate linearly independent eigenvectors ....However this is not always possible as discussed in the following two examples.

Example 4.22 Consider the matrix

which has eigenvalue with multiplicity 2. To calculate the eigenvectors, one must solve the system viz.

This equation is satisfied for every value of e, and it is therefore possible to choose two linearly independent eigenvectors associated with . If, in particular, the two canonical basis vectors are chosen as eigenvectors, the modal matrix is worth

|

Example 4.23 Consider the matrix

which has eigenvalue with multiplicity 2. To calculate the eigenvalues, one must solve the system viz.

Having to pose it is possible to choose only one linearly independent eigenvector associated with , e.g.

So the given matrix does not admit modal matrix.

We can finally prove that every matrix admitting modal matrix is diagonalizable.

Proposizione 4.24 *Data una matrice di dimensione e autovalori sia una sua matrice modale. La matrice ottenuta attraverso la trasformazione di similitudine*

Is diagonal.

Demonstration. Observe meanwhile that the modal matrix, since its columns are linearly independent, is nonsingular and therefore can be inverted.

Moreover, by definition of eigenvalue and eigenvector applies *to*

and then combining all these equations

and again, by means of the formulas given in Appendix C (see § C.2.4) we can rewrite the previous equation as

or

Multiplying from the left both members of this equation by gives the result sought with

Example 4.25 Given the representation in terms of state variables already considered in Example 4.15:

(4.21)

you want to obtain by similarity a diagonal representation.

The eigenvalues of are and . The corresponding eigenvectors are worth (minus a multiplicative constant)

e

and the modal matrix and its inverse are worth, respectively,

e

So

4.5.1 Calculation of the state transition matrix by diagonalization

This section describes an alternative way to Sylvester's development to calculate the state transition matrix of a representation whose matrix can be traced by similarity to the diagonal form.

Proposition 4.26 *Given a matrix of dimension with eigenvalues , assume that it admits a modal matrix .*

Vale

$$\vdots \quad \vdots \quad \ddots \quad \vdots \tag{4.22}$$

Demonstration. According to Proposition 4.16, the following holds true.Multiply-
I give both members of this equation for to the left and to the right, the result sought is obtained.

Example 4.27 For the system in Eq. (4.21) you want to calculate by applying the formula given in Proposition 4.26.

In Example 4.25 it was seen that it is worth

e

So

Note that this expression coincides with the one already determined through Sylvester's development in Example 4.8

4.5.2 Matrices with complex eigenvalues [*]

The diagonalization procedure can also be applied to matrices with complex eigenvalues. In that case the eigenvectors corresponding to these eigenvalues are complex and conjugate, and both the modal matrix[7] and the resulting diagonal matrix are also complex. It is preferred, then, to choose a similarity matrix other than the modal matrix in order to arrive at a canonical real form in which each pair of complex and conjugate eigenvalues corresponds to a real block of order 2 along the diagonal: this block is the matrix representation of the pair of complex eigenvalues (see Example 4.10). We present this result in informal terms so as not to burden the notation.

Assume that the matrix has for simplicity a pair of eigenvalues com-
plexed and conjugate while the remaining eigenvalues are all
real and distinct. The eigenvectors and corresponding to the complex eigenvalues are also complex and conjugate and can be decomposed into real and imaginary part as follows:

It is easily shown that the vectors and are linearly independent and are also linearly independent of the eigenvectors associated with the other eigenvalues.

[7]Note that a matrix with complex eigenvalues is not diagonalizable in the real field, that is, it is not diagonalizable by a real similarity matrix.

Observe that by definition of eigenvalue and eigenvector holds:

and considering the real and imaginary parts of this equation separately, we obtain:

e

Then choose the similarity matrix in which the columns associated with the real au- tovalues are the corresponding eigenvectors (as in the case of the modal matrix) but in which the pair of complex and conjugate eigenvalues correspond to the columns and equal to the real and imaginary part of the corresponding eigenvector.

We can then write, assuming without impairing generality that the columns associated with and are the last two,

and again, by means of the formulas given in Appendix C (see § C.2.4), we can rewrite the previous equation as

or

Observe, then, that with this similarity transformation the pair of au- tovalues corresponds in the quasi-diagonal matrix to the block that represents them in matrix form

In general, we can state that if a matrix has distinct real roots (for) and distinct complex conjugate root pairs (for), by means of the matrix it is possible to trace it back to a standard quasi-diagonal form

$$. \quad \ddots \quad . \quad . \quad \ddots \quad .$$

(4.23)

$$\vdots \quad \ddots \quad \vdots \quad \vdots \quad \ddots \quad \vdots$$

where each pair of complex roots is associated with the generic
Real block representing them in matrix form

Example 4.28 Consider a system whose state matrix is worth

Such a matrix has characteristic polynomial and therefore
eigenvalues e . To these eigenvalues correspond the
eigenvectors

e

Having chosen the matrix, we finally obtain

Calculating the matrix exponential for a matrix in the form (4.23) is straightforward. Since it is block diagonal, according to Proposition C.24 it is worth

$$. \quad \ddots \quad . \quad . \quad \ddots \quad .$$

$$. \quad \ddots \quad . \quad . \quad \ddots \quad .$$

In this expression to each pair of complex roots
corresponds to the canonical block representing it in matrix form

and the matrix exponential corresponding to this particular matrix was determined in Example 4.10; it is worth

So, it is also possible to easily determine the matrix state transition matrix by the formula

analogous to (4.22).

Example 4.29 The matrix in Example 4.29 can be traced through the matrix
to the quasi-diagonal shape

Therefore, it is worth

and we also derive

4.6 Form of Jordan

Consider a matrix of dimension whose eigenvalues have nonunitary multiplicity. In that case there is no guarantee, as seen in Example 4.23, that there exist
linearly independent eigenvectors with which to construct a modal matrix: thus it is not always possible to determine a similarity transformation leading to a diagonal form.

It is shown, however, that it is always possible, by extending the concept of an eigenvector, to determine a set of linearly independent *generalized eigenvectors*. Such vectors can be used to construct a *generalized modal matrix* that allows, by similarity, to move to a matrix *in Jordan form*[8] , a canonical block generalizing diagonal form.

In this introductory discussion we will simply summarize the main results necessary for the study of Jordan's canonical form. The procedure for the computation of a generalized modal matrix will be presented in the next section, the reading of which, however, is not essential to the understanding of the material presented in this section. Indeed, recall that, given a matrix , the computation of a generalized modal matrix and the corresponding canonical Jordan form can be determined by means of the MATLAB instruction `[V,J]=jordan(A)`.

Let us begin by presenting the definition of Jordan block and Jordan form.

Definition 4.30 *Given a complex number and an integer.*
we define Jordan block of order *associated with the square matrix*

. . . ⋱ . . .

Every element along the diagonal of such a matrix is worth , while every element along the supradiagonal is worth ; every other element is null. So it is an eigenvalue of block multiplicity .

We can now define Jordan's canonical form.

Definition 4.31 *A matrix is said to be in* Jordan form *if it is a block diagonal matrix*

. . ⋱ .

where each block along the diagonal is a Jordan block associated with an eigenvalue (for).

Note that in the above definition multiple Jordan blocks can be asso- ciated to the same eigenvalue. The Jordan form is a generalization of the diagonal form: in particular, a Jordan form in which all blocks have order 1 is diagonal.

We now turn to define the notion of generalized eigenvector and chain of generalized eigenvectors in qualitative terms.

[8] Marie Ennemond Camille Jordan (La Croix-Rousse, Lyon, France, 1838-Paris, 1922).

Definition 4.32 (Structure of generalized eigenvectors) *Let there be a matrix and be an eigenvalue of multiplicity to which correspond linearly independent eigenvectors (with) . Such an eigenvalue competes with a linearly independent* generalized eigenvector *structure consisting of chains:*

chain

chain

. .

chain

The number of chains is called the geometric multiplicity[9] *of the eigenvalue . The catena ha lunghezza e termina con un autovettore...........Gli altri vettori della catena sono autovettori generalizzati ma non sono autovettori. Poiché in totale gli autovettori generalizzati sono vale anche*

The length of the longest chain is called the index
Of the eigenvalue .

Note that the multiplicity of an eigenvalue is not sufficient information to determine the corresponding generalized eigenvector structure.

Example 4.33 Let there be a matrix with eigenvalue of
....multiplicity
A such au-
tovalue corresponds one (and only one) of these generalized eigenvector structures:

The eigenvalue index holds in the first case, the second and third cases, the fourth case, and the last case.

Recall that since each chain ends with an eigenvector, the number of linearly independent au- tovectors that can be associated with a given eigenvalue coincides with the number of chains that compete with it, that is, with its geometric multiplicity . For example, in the case of the structure the eigenvalue is competed by four linearly independent eigenvectors .

[9]Be careful not to confuse the geometric multiplicity of an eigenvalue with its multiplicity : according to the definition ....valeTometimes to avoid ambiguity it is
also used to call *algebraic multiplicity*.

We finally enunciate in qualitative terms the following result, which will be taken up and demonstrated in the following section.

Proposition 4.34 *A matrix can always be traced by similarity tra- sformation to a Jordan form. If an eigenvalue is competed by a generalized eigenvector structure consisting of chains each of length (per) then that eigenvalue is competed by Jordan blocks, each of order (per). The order of the largest block coincides with the index of the eigenvalue .*

Esempio 4.35 Sia l'insieme delle matrici con autovalori di molteplicità quattro e di molteplicità singola. Una qualunque matrice può sempre essere ricondotta, mediante una trasformazione di similitudine, ad una (ed ad una sola) di queste cinque matrici in forma di Jordan:

In tutte queste matrici, all'autovalore di molteplicità singola corrisponde un solo blocco di Jordan di ordine 1. All'autovalore possono invece corrispondere uno o più blocchi a seconda della struttura di autovettori generalizzati che ad esso compete; in particolare se a compete una struttura del tipo per , (cfr. Esempio 4.33) la forma corrispondente sarà quella indicata dalla matrice .

In the matrix the eigenvalue corresponds to four Jordan blocks of order 1, and the matrix is diagonal. A matrix can be traced to this form if and only if the index of the generic eigenvalue is worth 1, that is, if . In that case, in fact, it corresponds to a number of Jordan blocks equal to its multiplicity, each of order . A matrix traceable to this form is called *diagonalizable*.

In the matrix the eigenvalue corresponds to a single Jordan block of ordi- ne 4. A matrix can be traced to this form if and only if the multiplicity of the generic eigenvalue coincides with the index , that is, ...ifIn that case in fact such an eigenvalue corresponds to a single Jordan block of order . A matrix re-conductible to this form is said to be *nonderivative*.

4.6.1 Determination of a basis of generalized eigenvectors [*]

In the previous paragraph, the concepts of generalized au- tovector and chain of generalized eigenvectors were introduced discursively. This paragra- pho will give a formal definition of these concepts and present an algorithm for determining a set of linearly independent generalized eigenvectors representing a basis of the space .

Definition 4.36 *Given a matrix of dimension , the vector is a* Generalized eigenvector (AG) of order *associated with the eigenvalue if it is worth*

(4.24)

Note that according to the above definition an eigenvalue can be seen as a particular AG of order : in fact posed the conditions given in equation (4.24) become and , which are precisely satisfied by an eigenvector and the corresponding eigenvalue .

Example 4.37 Consider the matrix

which has characteristic polynomial and therefore eigenvalue
Of multiplicity 4.
You want to determine , if it exists, an AG of order .
Vale:

e dunque

If it is an AG of order 3, it must satisfy:

e

Il primo sistema è sempre soddisfatto, mentre il secondo è soddisfatto da . Dunque, scelto e , il vettore è un AG di ordine 3. Note that other choices are also possible. For example, choosing e si otterrebbe il vettore , che è anche esso un AG di ordine 3.

The following proposition introduces the concept of AG chain and demonstrates some of its properties.

Proposition 4.38 *Given a square matrix , be a generalized eigenvector of order associated with the eigenvalue and define the sequence*

where for is worth

(4.25)

The generic vector (for) of the sequence is a generalized eigenvector of order, and in particular the vector is an eigenvector. The sequence is called a chain of generalized eigenvectors of length .

Demonstration. To prove that every vector in the chain is an AG, observe that for , if it is also valid

Then if it is an AG of order according to Definition 4.36 it is worth :[10]

And therefore it is an AG of order .

Esempio 4.39 Si consideri ancora l'Esempio 4.37. Dato l'autovettore generalizzato di ordine 3 si costruisce la seguente catena di lunghezza 3:

[10]Note that equations (4.24) remain valid even if you change the sign of both members.

It is easily verified that it is an eigenvector of .

Si noti che anche a partire dall'autovettore generalizzato è possible to construct a chain of length 3:

dove è un autovettore di . Si noti che mentre e sono linearmente indipen-teeth, on the contrary the e (*e* and) differ only by a constant
pairs
multiplicative.

A classical result (see Appendix C, Theorem C.64) states that a matrix with distinct eigenvalues has linearly independent eigenvectors. This result can be generalized as follows.

Theorem 4.40 *Given a matrix of dimension , the following properties apply:*

to each of its eigenvalues of multiplicity it is possible to associate linearly independent generalized eigenvectors;
generalized eigenvectors associated with two distinct eigenvalues and are linearly independent.

So it has linearly independent generalized eigenvectors.

Before giving an algorithm for choosing these vectors, some considerations will be made to provide an understanding of how this procedure works.

Given a matrix of dimension be one of its eigenvalues of multiplicity and consider the matrix ....We define

the *null* of the matrix (see Appendix C, § C.4). This value indicates the dimension of the vector subspace

that is, it indicates how many linearly independent vectors it is possible to choose such that their product by gives the null vector.

The parameter coincides with the *geometric multiplicity* of the eigenvalue that was previously introduced and is denoted ...byEsso has two important meanings
physical. First, it indicates the number of linearly independent eigenvectors of associated withIn addition, since each AG chain ends with an eigenvector, it also indicates the number of linearly independent AG chains that can be associated with .

Consider now the matrix and calculate its nullity

This value indicates the dimension of the vector subspace

that is, it indicates how many linearly independent vectors it is possible to choose such that their product by gives the null vector. Since if then è facile capire che vale e che, inoltre, coincide anche con il numero di AG linearmente indipendenti di ordine 1 e di ordine 2 di associated with . So it indicates the number of AGs of order 2 that can be chosen such that they are linearly independent of the eigenvectors.

Continuing the reasoning, it can be shown that we arrive for a given value of at a matrix whose nullity is worth

and which satisfies the ....relationThen this means that there exist AG of linearly independent and of order less than or equal ....toIn particular an equal number of these are AGs of order .

We observe that if there are AGs of order ( ), the number of eigenvectors of order is such that : in fact from each AG of order an AG of order can be determined by the procedure seen in Proposition 4.38. The difference indicates precisely the number of new chains of order that originate from AGs of order .

Algorithm 4.41 (Calculation of a set of linearly independent AGs)

1. *Given a matrix of dimension be one of its eigenvalues of multiplicity . Calculate for as long as it's not worth .*
2. *Let us construct the table*

Where:

element indicates the null of the matrix ;

element indicates the number of linearly independent AGs of order of the matrix and is defined as: and for ;

the element indicates the number of AG chains of matrix length and is defined as: for and .

3. *If , determine linearly independent AGs of order and calculate from each of them a chain of length .*

Attraverso questa procedura si determina un numero di catene pari a , cioè pari alla molteplicità geometrica di , che complessivamente comprendono un numero di AG pari a .

Let us now give a simple example of the application of this procedure.

Example 4.42 Consider again the matrix of Example 4.37, which has eigenvalue of multiplicity . In this case it holds:

Since it is worth . Let us therefore construct the table

Poiché , si deve scegliere un AG di ordine 3, che darà luogo ad una catena di lunghezza 3: indicheremo con il simbolo ad esponente tutti i vettori che appartengono a questa catena. Scegliendo come AG di ordine 3 il vettore , as already seen, the following chain of length 3 is obtained:

Since , no new AGs of order are determined .

Finally, since , one must also choose an AG of order 1 (i.e., an eigenvector), which will give the fourth vector sought: we will denote by the symbol ad esponente l'unico vettore che appartiene a questa seconda catena di lunghezza 1. Since an eigenvector must satisfy:

must ....applyYou could choose then and or, vice versa, e.............La prima scelta darebbe il vettore già considerato. Con la seconda si finally gets

4.6.2 Modal matrix generalized

Once you have determined linearly independent AGs using the procedure described in the previous section, you can use these vectors to construct a nonsingular matrix.

Definition 4.43 *Given a dimension matrix assume that ap- plicating Algorithm 4.41 you have determined a set of linearly independent generalized eigenvectors.*

If the generic eigenvalue corresponds to chains of generalized eigenvectors of length , we order the generalized eigenvectors associated with the eigenvalue by constructing the matrix:

catena catena catena

If the matrix has distinct eigenvalues ( ) we call the generalized modal *matrix of the matrix*

| | |

Note that in the definition of the matrix the order in which the chains are numbered is not essential: in fact this choice is arbitrary. However, it is essential that columns associated with AGs that belong to the same chain be placed next to each other and be ordered in the direction from the eigenvector to the AG of highest order.

Example 4.44 The matrix in Example 4.37 has eigenvalue of multiplicity . Applying Algorithm 4.41 it was seen that this eigenvalue is competed by two chains of AGs, one of length and one of length , given in Example 4.42.

In this case there is only one distinct eigenvalue, and the modal matrix is therefore worth

Changing the order of the chains results in a different modal matrix gener- ated

Proposition 4.45 *Given a square matrix be a generalized modal matrix of it. The matrix obtained through the similarity transformation*

Is in the Jordan form. Moreover, if the generic eigenvalue corresponds to chains of generalized eigenvectors, of length , then in the Jordan form that eigenvalue is competed by Jordan blocks, of order .

Demonstration. Observe meanwhile that the generalized modal matrix, since its columns are linearly independent, is nonsingular and therefore can be inverted.

Si consideri una generica catena di lunghezza associata all'autovalore . Per definizione di autovalore e autovettore, per il primo vettore della catena vale

mentre per il generico vettore , AG di ordine , in base alla (4.25) vale

Combining all these equations, assuming that the chain gives rise to the first columns of the matrix , we obtain

and again we can rewrite the previous equation as

or

from which it is clearly seen that the chain length corresponds in the matrix to the a Jordan block of order .

Multiplying from the left both members of this equation by gives the result sought.

Example 4.46 Consider the matrix

which has *c h a r a c t e r i s t i c* polynomial and therefore Of multiplicity 4. eigenvalue

Applying Algorithm 4.41 it was seen that such an eigenvalue is competed by two AG cate- ne, one of length and one of length , given in Example 4.42. So it is expected that such a matrix can be traced by similarity transformation to a matrix in Jordan form in which the eigenvalue corresponds to two blocks, one of order 3 and one of order 1.

This is easily verified. Indeed, having chosen the generalized modal matrix given in Example 4.42, it holds:

and finally we get

4.6.3 Calculation of the state transition matrix by Jordan form

A simple formula can be given for calculating the exponential of a matrix in Jordan form.

Proposition 4.47 *Given a matrix in Jordan form.*

$$. \quad . \quad \ddots \quad .$$

its matrix exponential is worth

(4.26)

Furthermore, if is a generic block of order

its matrix exponential is worth

Demonstration. Since the block diagonal matrix is a block matrix, the relation (4.26) follows immediately from Proposition C.24.

To prove the second result instead, preliminarily determine the power -ma of the generic Jordan block of order associated with the eigenvalue . It is easy to verify that it is worth[11]

[11]In this formula we use the binomial coefficient for , while convenzionalmente si è posto per .

$$\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots$$

As shown by direct calculation.

Moreover, since

—

we observe that the generic element of the matrix that lies along the supra-diagonal starting from the element , for , is worth precisely

— ———— — ————

— — —

The previous proposition, combined with the result of Proposition 4.16, also for- nishes an alternative way to Sylvester's development to calculate the state transition matrix.

Proposition 4.48 *Given a matrix of dimension with eigenvalues.*
, be a generalized modal matrix that allows one to go to the form of Jordan's
.... *Vale*

(4.27)

Demonstration. Similar to the demonstration of Proposition 4.26.

Example 4.49 The matrix studied in Example 4.44, by means of the generalized modal matrix given in the same example, can be traced to Jordan's form

Vale

—

and therefore also applies to

To conclude this section, note that in the case where a matrix has complex and conjugate eigenvalues its Jordan form would not be a real matrix. Again, as already seen for the diagonalization procedure, one could modify the generalized modal matrix to achieve a canonical real quasi-Jordan form. However, we will not treat this case.

4.7 State transition matrix and modes

In Chapter 3, devoted to the study of input-output models, *modes* were defined, i.e., those signals that characterize the evolution of a system. In this section we will see how the concept of mode can also be defined in the case of models in state variables.

4.7.1 Minimum polynomial and ways

Given a matrix in Jordan form, consider the corresponding state transition matrix According to Proposition 4.47 in a given block of order
associated with the eigenvalue will appear the functions of time

multiplied by appropriate coefficients. If more than one block is associated with an eigenvalue, and is the index of the eigenvalue (i.e., the order of the largest block) the term of highest order associated with the eigenvalue will therefore be .

Consider now a generic matrix Since such a matrix can always be traced by similarity to a Jordan form, its state transition matrix can be calculated mediate the formula (4.27). So each of its elements is a linear combination of the functions just described. We can therefore give the following definition.

Definition 4.50 *Given a matrix with distinct eigenvalues each of index , we define its* minimum polynomial *as.*

With each root of the minimum multiplicity polynomial we can associate the functions

we call modes. *Each element of the state transition matrix is a linear combination of such modes.*

Note that minimum polynomial and characteristic polynomial of a matrix coincide only in the case where the matrix is nonderivative (and thus, as a special case, if all eigenvalues have single multiplicity).

Example 4.51 The state matrix of the representation in eq. (4.15) has two eigenvalues and of single multiplicity and therefore, logically, of unit index. The minimum polynomial of in that case coincides with the characteristic polynomial:

The corresponding modes are therefore .... eEach element of the matrix

Is a linear combination of these modes.

Esempio 4.52 La matrice studiata nell'Esempio 4.44 può essere ricondotta alla forma di Jordan

The one eigenvalue of multiplicity has index . The characteristic polynomial and minimum polynomial are worth respectively:

e

The corresponding modes are therefore , and Each element of the matrix (cfr. Esempio 4.49) è una combinazione lineare di questi modi. Si noti, in particolare, che pur avendo l'autovalore molteplicità non compare un modo della forma .

4.7.2 Physical interpretation of eigenvectors

Given a representation in state variables (4.1) it is possible to give a very important physical meaning to the *real* eigenvectors of the state matrix .

We begin with a general result that applies to all eigenvectors, real or complex.

Proposition 4.53 *If it is an eigenvector of the matrix associated with the eigenvalue , then it applies*

Demonstration. If it is an eigenvector of the matrix associated with the eigenvalue, it is worth

Pre-multiplying both members of this expression by gives and repeating this operation we observe that *it is worth*

For each value of .

Finally we get

— —

Consider now a dynamical system described by model (4.1) whose free state evolution is to be studied from different initial conditions. Starting from a time instant and an initial state, the vector defines in the state space a curve parameterized by the value of time called the *state trajectory* or also simply *the trajectory* of the system.

Suppose that the initial condition coincides with an eigenvector of the ma- trice associated with the ....eigenvalueIn that case the free evolution of the state based on Lagrange's formula and Proposition 4.53 is worth

Dunque *il vettore di stato al variare del tempo mantiene sempre la direzione data dal vettore iniziale , mentre il suo modulo varia nel tempo secondo il modo associato all'autovalore.*

Suppose that the state matrix , of order , has a set of au- Linearly independent tovectors corresponding to the eigenvalues In such a case, if the initial condition does not coincide with an eigenvector is always possible to pose:

expressing such a vector as a linear combination, by appropriate coef- ficients , of the basis of eigenvectors. Therefore, it is also valid:

from which we see that the evolution is also a linear combination, with the same coefficients , of the individual evolutions along the eigenvectors.

Example 4.54 Consider the representation in terms of state variables already considered in Example 4.15 and Example 4.25 whose state matrix holds:

The eigenvalues of are and and the corresponding eigenvectors are

e

In Fig. 4.1 we have plotted in the plane *t h e* free evolution for different cases. Each trajectory corresponds to a different initial condition and is parameterized in the time variable : the direction of travel indicated by the arrow corresponds to increasing values.

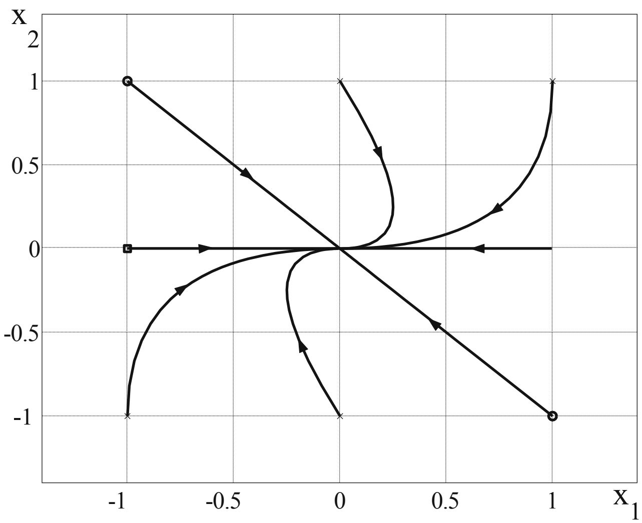


Fig. 4.1. Free evolution of the system in Example 4.54 from different initial conditions

The two initial conditions indicated by a square lie along the eigenvector . Starting from them, as time passes, the vector always maintains the same direction but its modulus decreases because the *corresponding* mode is stable: the trajectory described by the state vector is new along the segment joining the initial point with the origin.

A similar reasoning can be made for the two initial conditions indicated with a circle; they lie along the eigenvector and the trajectories originating from them are traveled according to the mode .

Le altre condizioni iniziali, indicate da un asterisco, corrispondono a combinazioni lineari di autovettori. Si osservi che le traiettorie che da essi si originano non sono rette perché le due componenti nelle direzioni degli autovettori evolvono seguendo modi diversi. In effetti si vede che al crescere del tempo tutte queste traiettorie tendono all'origine con un asintoto lungo la direzione del vettore . Ciò si spiega facilmente: poiché è il modo più veloce, la componente lungo si estingue più rapidamente e dopo un certo tempo diventa trascurabile rispetto alla componente lungo il vettore .

In the case of complex, conjugate eigenvalues, such a physical interpretation of the au- tovectors loses its meaning: the eigenvectors corresponding to them are com- plete and therefore cannot be represented in the state space of the system. We can observe, however, that in general a pair of complex eigenvalues deter- mines pseudo-periodic trajectories in the state space. The following example is for a second-order system.

Esempio 4.55 Si consideri la rappresentazione di un sistema la cui matrice di stato vale:

Tale matrice è un caso particolare di quella studiata nell'Esempio 4.10: essa ha autovalori ed è detta rappresentazione matriciale di tale coppia di numeri complessi e coniugati. In base a quanto visto nell'Esempio 4.10 vale

Si consideri un'evoluzione libera a partire dalla condizione iniziale .
In that case it applies:

This equation determines in the plane a vector that rotates clockwise with angular velocity and whose modulus reduces according to the mode La corresponding trajectory is thus the spiral curve shown in Fig. 4.2 that originates at the point indicated with a square.

All trajectories of this system, whatever the initial state, have a qualitatively similar trend. For example, the trajectory of free evolution from the initial state , indicated with a circle, is also shown in Fig. 4.2. Such a trajectory also has a spiral shape .

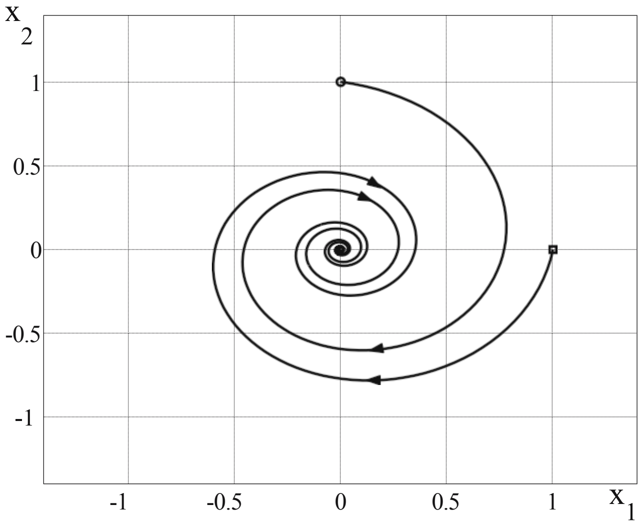


Fig. 4.2. Free evolution of the system in Example 4.55 from two different initial conditions

Exercises

Exercise 4.1. Given the matrices

e

calculate by Sylvester development the corresponding state transition matrices. Compare Example 9.13 and Example 9.28 for the solution of this exercise.

Exercise 4.2. Is given the matrix

(a) Let its ways be determined.
(b) Calculate the state transition matrix by Sylvester development, verifying that each of its elements is a linear combination of the modes.

Compare Example 9.29 for the solution of this exercise.

Exercise 4.3. A representation in state variables of a linear, stationary system is given.

(a) Given an initial instant , determine the free evolution of state and output from initial conditions , .
(b) Determine the forced evolution of state and output that follows the application of an input

Æ

Exercise 4.4. Verify that for each value of exists a similarity transform- ation that allows us to go from the representation in eq. (4.1) to the representation

Determine the corresponding similarity matrix . What physical meaning can be given to the states of the second representation?

Exercise 4.5. For the system in Exercise 4.3, determine a similarity transformation leading to a representation in which the state matrix is diagonal, determining all the matrices in the new representation.

Exercise 4.6. For the matrix in Exercise 4.2, calculate the state transition matrix by diagonalization.

Exercise 4.7. Would it be possible to calculate by diagonalization the state transition matrix for the matrix given in Exercise 4.1? And for the matrix from the same exercise?

Exercise 4.8. **[*]** Determine the representation in Jordan form of the following matrices:

Exercise 4.9. **[*]** Are given the matrices

(a) Verify that the matrices and commute.
(b) Let us calculate, by means of Sylvester's development, , and .
(c) Let it be verified that it holds .

5

The Laplace transform

In this chapter, a mathematical tool, called the *Lapla- ce transform*[1] , is presented that allows linear differential equations with constant coef- ficients to be easily solved and thus finds application in a wide variety of engineering fields. In the first section, the concept of transform is defined and the transforms of some elementary signals are computed by direct way. In the second section, some fundamental results are pre sented that allow one to become familiar with the use of transforms and also make it easy to determine the transform of a wide class of signals: in particular, the family of exponential ramps will be studied, because it contains the signals of greatest interest in the analysis of si- stems. In the third section, a technique is presented that makes it easy to antitransform a rational function : the importance of this class of functions of arises from the fact that if a function can be written as a linear combination of exponential ramps, then its transform is a rational function. In- fine in the fourth section some examples of the use of Laplace transforms for solving differential equations are presented. A table summarizing the transforms of the main signals is given at the end of the chapter.

5.1 Definition of transform and antitransform of Laplace transform

One technique often used in solving mathematical problems is the use of transforms. Suppose that a given problem can be described me- dially by time signals: for example, this is the case for a linear differential equation of the type

(5.1)

which binds the signals and and their derivatives. If the direct search for a solution to the given problem is not easy, one can consider transforming, by means of an operator

[1]Pierre-Simon Laplace (Beaumont-en-Auge, France, 1749-Paris, 1827).

, each of the signals by transforming the given *problem* into an *image problem* whose *image solution* is easier to determine. The image solution can then be antitransformed into the solution sought, by means of the inverse operator In general this procedure works if there is a two-way link between each signal and its transform.

A broad class of transforms can be described in formal terms as follows. Consider a function that has as its argument the real variable , and be given a function that has as its arguments the complex variable and the variable We call the *transform of with nucleus*

the function whose argument is the *complex* variable
so defined

Where and are appropriate extremes of integration.

5.1.1 Transform of Laplace

The Laplace transform is a special case of the operator just described, for which the following assumptions apply:

it is assumed that the function to be transformed is defined for and is locally sum-sumable, meaning that its integral exists in every finite interval of ;
we choose as extremes of integration and ;
you use the core .

Definition 5.1 (Laplace transform) *The* Laplace transform *of the real variable function is the function of the complex variable*

(5.2)

The Laplace transform of a function , is usually denoted
, or, more simply, , using the cor- responding capital letter.

In general, the integral (5.2) can be calculated only for values of belonging to an open half-plane for which it holds , as shown in Fig. 5.1. Such a half-plane is called the *region of convergence*, and the value is called the *abscissa of convergence*.

Example 5.2 (Unit step transform) Consider the unit step

Æ se
i*f*

The transform of this function is worth

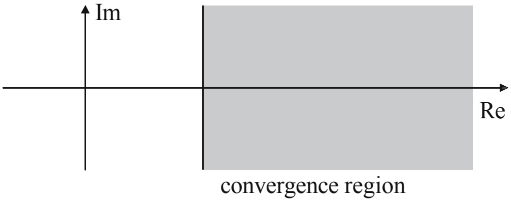


Fig. 5.1. Example of convergence region in the complex plane for the Laplace transform

The region of convergence for this integral is given by the values of con
; in fact in that case the following applies

and therefore we get the result

It has been said that the transformed function of can generally be computed only under the assumption that it belongs to the region of convergence. It is usual, however, to consider the *analytic extension of* the on all points in the complex plane where it is defined, and that is, even for values of not belonging to the convergence region.

Example 5.3 The step transform was determined under the hypothesis that it belongs to the positive real half-plane. However, the function will be considered as a function defined on almost the entire complex plane, except of course in the origin where it is not defined .

5.1.2 Antitransform of Laplace

From the function it is also possible, by inverting the operator , to redetermine the .

Definition 5.4 (Laplace's anti-transform) *If , the value of the function for each can be determined as:*

(5.3)

Where is any real value that satisfies .

Note that eq. (5.3) is in practice never used to antitransform: the interest of this formula is purely theoretical because it highlights a biunivocal relationship between a function *(* considered only for) and its transformed .

It is important to note that the trend of the function for values of is not taken into account in the calculation of the transform and, reciprocally, is not determined by the antitransform. This implies that two different functions and noncoincident for and coincident for have the same Laplace transform. Thus, there is no biunivocity relationship between function in and function in *f o r* each as would be desired. To overcome this problem, we assume that the transform describes a function that takes null values for and therefore the antitransform operator determines a function which takes null values ....forIn this way, the relationship between function and transformed becomes unique for each value of .

Example 5.5 Consider the constant function ....forThis funzione coincides with the unit step function Æ for Therefore, its transformed worth

–

However antitransforming will be defined as.

– Æ

5.1.3 Transform of signals impulsive

Many signals of interest in the study of systems analysis are distributions, or functions that may have impulsive terms (see Appendix B). To take into account the possible presence of impulsive terms in the origin, the definition of Laplace transform should be modified as follows:

(5.4)

so that the area of these terms is not neglected in the calculation of the integral.

Keep in mind that the definition in eq. (5.4) generalizes the definition given in eq. (5.2): for a function that contains no impulsive terms in the origin, the two definitions are equivalent.

Example 5.6 (Impulse transform) Consider the Dirac function Æ At based on Proposition B.9 the transform of this function is worth

Æ Æ

In the rest of this chapter we will almost always use the expression in Eq. (5.2) except in a few cases (derivative theorem and initial value theorem) where it is essential to highlight the behavior of the function in the origin.

The use of Eq. (5.2) and Eq. (5.3) to compute transforms and antitransforms is not smooth. In practice, to transform the signals of interest we simply consi- der the Laplace transforms of some canonical signals that exhaust the cases of greatest interest, while to antitransform we decompose a function into a sum of elementary functions whose antitransform can be immediately determined.

5.1.4 Calculation of the transform of the function exponential

We finally end this introductory section with the calculation of the La- place transform of a particular signal, the *exponential function*, which is defined as a function of the parameter as

Æ if
se

Proposition 5.7 *The Laplace transform of the exponential function holds.*

Æ (5.5)

Demonstration. The calculation of the transform holds

Æ

having placed

under the assumption that it is worth i.e.

. The abscissa of convergence for such a function is worth .

As a special case, if the exponential function coincides with the unit step. In such a case, it is verified that placed in eq. (5.5) we obtain precisely the unit step transform .

5.2 Fundamental properties of the Laplace transform

Some fundamental results characterizing Laplace transforms are now presented. In particular, they also make it easy to determine the Laplace transforms of all signals of interest without having to solve the integral defining that transform. The table at the end of the chapter summarizes the transforms of some remarkable functions.

5.2.1 Property of linearity

Proposition 5.8 *If its Laplace transform holds:*

Demonstration. It follows immediately from Definition (5.2) since the integral is a linear operator.

Thanks to this property, it is possible to use the transform of the exponen- tial function to calculate, for example, the transform of sinusoidal functions.

Example 5.9 (Sine and cosine transform) The transform of the cosine function is worth.

being a special case of the exponential function for . Similar reasoning applies to the sine function, the transform of which holds for .

5.2.2 Derivative theorem in

Theorem 5.10 *Given the function with Laplace transform , the transform of the function holds:*

—

Dimostrazione. Si osservi, per prima cosa, ~~c~~he vale
So the transform of the function holds by definition

—

— —

having exchanged the derivative and integral operator between them.

Based on this result, *multiplying by* in the time domain corresponds to *derive* with *respect to (* changing sign) in the domain of the Laplace variable.

The previous result makes it easy to determine the Laplace transform of an important family of functions: *exponential ramps* (or *cisoids*), which are defined by means of two parameters and as

if

— Æ

— if

Proposition 5.11 (Exponential *ramp transform) The La- place transform of an exponential ramp holds true.*

— Æ ————— (5.6)

Demonstration. It is easily demonstrated by induction.

(Initial step) Note that for the exponential ramp it coincides with the previously defined exponential function. So in that case eq. (5.6) becomes.

Æ ———

Which is true according to Eq.
(5.5).

(Inductive step) Suppose that eq. (5.6) is true for ....Let us prove that it is also true forIn ....fact, by exploiting the derivative theorem in we obtain with simple steps

— Æ — Æ — Æ

— — Æ

—

The exponential ramp is representative of a large class of functions.

Example 5.12 Consider the family of exponential ramps for which the following *applies*
eWe obtain the family of *ramp functions*, consisting of the unit step Æ and its successive integrals: the linear ramp

Æ Æ

the quadratic ramp

Æ —Æ

ecc., come mostrato in Fig. 5.2.

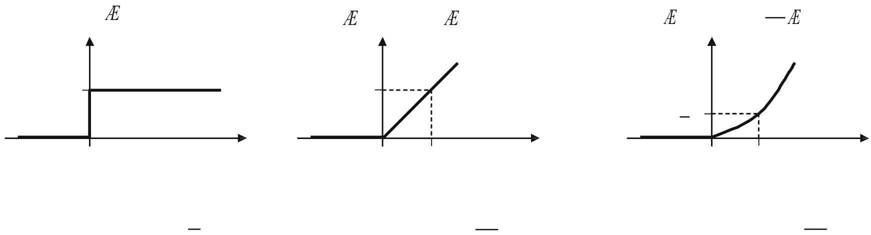


Fig. 5.2. The ramp functions *for*

More generally, the linearity property makes it easy to determine the transform of functions that can be written as a linear combination of ramps.

Example 5.13 Consider the function whose graph is plotted in Fig. 5.3. This function can be viewed as the sum of a step of amplitude and a linear ramp of slope , i.e., it can be posed

Æ Æ Æ

The transform of that function is therefore worth

Æ Æ — —

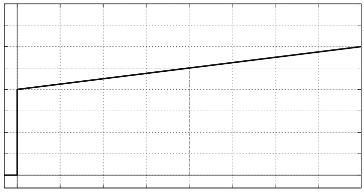


Fig. 5.3. Graph of the function Æ

5.2.3 Derivative theorem in time

Theorem 5.14. *Given a function with Laplace transform , it holds:*

—

In the case where the function is discontinuous in the origin, it should be understood to mean.
as .

Demonstration. The transform of the function holds by definition[2]

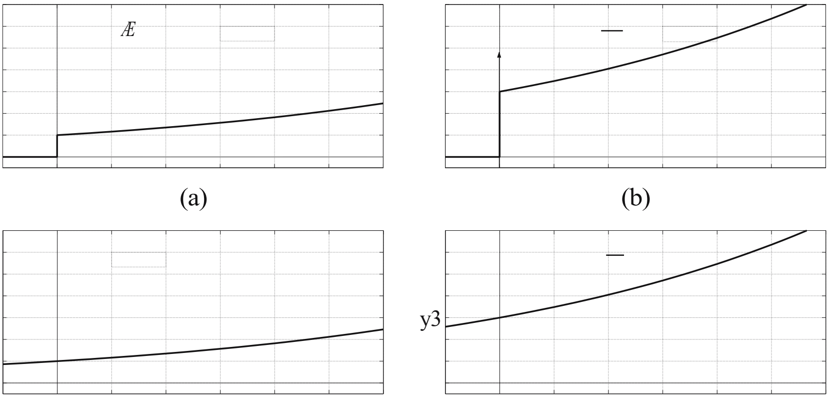
and, integrating by parts and assuming that , we get

from which the result sought is immediately obtained. Clearly if it is continuous in the origin it holds .

According to this result, *deriving with respect to* in the time domain corresponds *to multiplying by* in the domain of the Laplace variable.

Note that although for calculating the transform of a function only the values taken for , the value taken by the function in is im- portant for determining its derivative and corresponding transform. The following example clarifies this concept.

[2]Note that if the function were discontinuous in the origin, its derivative would con- tain an impulsive term; to account for this eventuality, the definition of transform given in eq. (5.4) is used.



(a) (b) (c) (d)

Fig. 5.4. (a) La funzione Æ ; **(b)** la sua derivata — Æ Æ ; **(c)** La funzione ; **(d)** la sua derivata —

Esempio 5.15 Si consideri la funzione Æ mostrata in Fig. 5.4.a la cui trasformata di Laplace vale Tale funzione non è continua in the origin since while ....It is therefore worth

This can occur immediately. The derivative of the is in fact worth [3]

ÆÆ

This function is shown in Fig. 5.4.b, where the pulse in the origin is indicated by an arrow. Transforming it gives as expected

ÆÆ

Consider now the function shown in Fig. 5.4.c, which coincides, for with and thus has identical transform . This function is continuous in the origin since .

Therefore, it is worth

[3]Recall the rule given in Appendix B (see § B.2) for calculating the derivative of a function with discontinuity.

This can occur immediately. The derivative of the is in fact worth

—

and that function is shown in Fig. 5.4.d. By transforming, we obtain as expected

The derivative theorem can also be generalized to the calculation of derivatives of higher than first order.

Proposition 5.16 *Given the function with Laplace transform , let be Its derivative -ma with respect to time for ....Vale:*

In the case where the function is discontinuous in the origin, it should be understood to mean.
as , for .

Demonstration. It is proved by repeated application of the derivative theorem since

and in steps you get the result you are looking for.

Example 5.17 Consider the function Æ shown in Fig. 5.5 whose first derivative is worth Æ and whose second derivative *is worth*
Æ The function is thus a linear ramp of slope , its first derivative is a step of amplitude and its second derivative is a pulse of area .

Based on the formulas already previously determined (see the table at the end of the chapter), it is immediate to verify that the transforms of these functions are worth respectively

— –

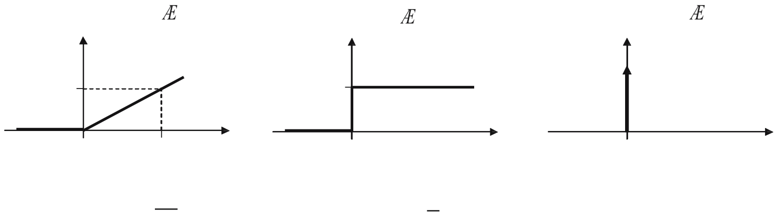


Fig. 5.5. The function Æ , its first derivative and its second derivative

Verify the values of and by applying the derivative theorem. Since is the first derivative of vale

— –

Since it is the second derivative of vale

—

where since the function is discontinuous in it is necessary to specify that the initial value is the one taken at .

5.2.4 Theorem of the integral over time

Theorem 5.18. *Given a function with Laplace transform , it holds:*

——

Dimostrazione. Se , chiaramente — e.................Detta , based on the derivative theorem holds from which the result sought is immediately obtained.

According to this result, *integrating with respect to in the time domain corresponds to dividing by in the domain of the Laplace variable*. Note the duality of this result with respect to the derivative theorem.

Example 5.19 Consider the function Æ whose Laplace transform is worthValue ....then

—— ——

This can occur immediately. The integral of the is in fact valid for :

—

and transforming yields as expected

— —

5.2.5 Translation theorem in time

Theorem 5.20. *Let be a function with Laplace transform and let , with , be a function obtained from the by "forward" translation in time. It is worth*

(5.7)

Demonstration. The transform of the function holds by definition

being null ....forWith a simple change of variable, placed , we get

According to this result, *translating forward by a quantity* in the time domain corresponds *to multiplying by* in the domain of the Laplace variable. The factor that appears in Eq. (5.7) represents a *lag element* (see Chapter 6, § 6.3.9).

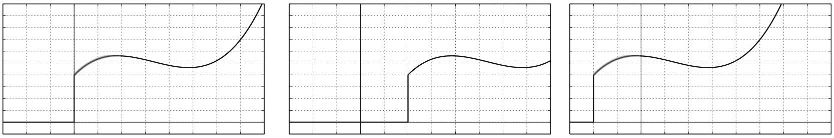
We observe that if it is nonzero between and , the theorem cannot be applied by "backward" translation in time, that is, to compute the transform of the function with : in that case, in fact, the would be identically nonzero forSee Fig. 5.6 for clarity.

In light of this theorem, it is easy to calculate the Laplace transform of functions that can be written as a linear combination of elementary signals that are also time-translated.

Example 5.21 The function in Fig. 5.7 can be thought of as the sum of three elementary functions.

Æ : an amplitude step applied in , because the function starts with value ;

Æ : a linear ramp of slope applied in , because the function decreases between and with slope ;



(a) (b) (c)

Fig. 5.6. (a) A function **; (b)** the function translated forward by ; (**c)** the function shifted backward *b y*

Æ : a linear slope ramp applied in , which counterbalances for the contribution of the previous ramp so that the function remains constant.

Therefore, it is worth

Æ Æ Æ

and transforming to term to term we get

– — —

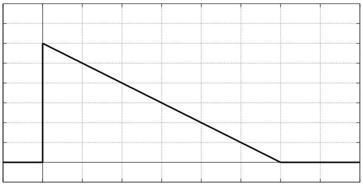


Fig. 5.7. A linear combination function of several translated elementary functions.

Finally, we conclude this section by showing how it is possible to calculate the transform of a signal that for is periodic in period , that is, such that it is worth

.

Proposition 5.22 (Transform of a Periodic Function*) Let there be a periodic function for with period . We will call the* basis function *of the function*

if

altrove

which coincides with in the first period and holds elsewhere. Let e be the transforms of and of, respectively. .... *Vale*

Demonstration. It is easy to see that it is worth

and therefore

Example 5.23 The function in Fig. 5.8 is periodic of period for ....In addition,
the basis function of is the function in Fig. 5.7, whose transform was calculated in the previous exercise. Therefore, it is worth

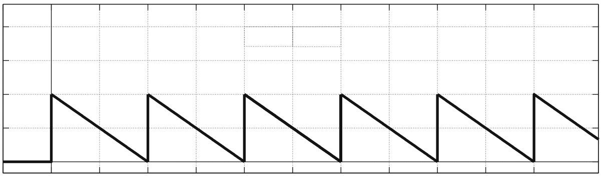


Fig. 5.8. A periodic function *for*

5.2.6 Translation theorem in

Theorem 5.24. *Let be a function with Laplace transform and let be a complex number. Vale*

(5.8)

Demonstration. The transform of the function holds by definition

With a simple change of variable, placed , we get

Note that the function is a function obtained from the by translation[4] by an amount equal to in the domain of the complex variable . According to this result, *multiplying by* in the time domain corresponds to *translating by a quantity equal to* in the domain of the Laplace variable.

Example 5.25 You want to calculate the Laplace transform of the *function* Æ che, come visto nel Capitolo 4, corrisponde ad un modo pseudo-periodico. Poiché la trasformata della funzione Æ vale (cfr. tavola alla fine del capitolo)

and, substituting with according to the previous theorem, we immediately obtain

Æ

Similarly, since the transform of the function Æ is worth

it is derived that it is worth

Æ

5.2.7 Theorem of convolution

Theorem 5.26. *Let e be two functions such that for . The Laplace transform of their convolution.*

(5.9)

vale

(5.10)

[4]The next section will introduce the concept of pole and zero of a function . It is verified that each pole of the function corresponds to a *pole* of the ....functionA similar argument applies to the
zeros of the .

Demonstration. Only the first of the two expressions of the for simplicity's sake is considered, but what will be said applies to both expressions.

First note that it is possible to give a completely equiva- lent expression to the first expression in eq. (5.9):

being for .

Therefore, according to the definition of transformed, the following applies

where in the third step we swapped the order of integration, in the fourth we moltipified by the factor, and in the sixth we used the translation-in-time theorem, which can be applied since being the is the forward translated function of .

This result is of fundamental importance in systems analysis. It i s infact seen how, thanks to Duhamel's integral, the forced evolution of the output of a system can be written as the convolution of the input with the impulsive response. Thanks to this theorem, the complicated calculation *of* a *convolution integral between two functions* is reduced, thanks to the Laplace transform, into the simple calculation of a *product between two functions*.

5.2.8 Value theorem final

The following theorem makes it possible, under certain conditions, to determine the final value of a function whose transform is known without having to antitransform.

Theorem 5.27. *Let there be a function with ....Laplace transformIf there exists finished the then*

Dimostrazione. In base al teorema della derivata vale —
So it also applies:

— —

—

from which, comparing first and last members, the result sought is obtained.

Example 5.28 Consider the function

– — —

che è la trasformata della funzione Æ mostrata in Fig. 5.9. Si verifica facilmente che vale

—

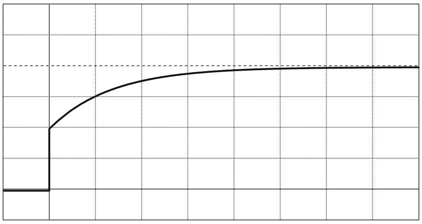


Fig. 5.9. La funzione Æ

Note that in order to be able to apply the previous theorem, it is necessary to be sure that the final value exists finite otherwise you will get incorrect results.

Esempio 5.29 Si consideri la funzione Æ la cui trasformata vale

– — —

The final value theorem does not apply, since . In that case, the following applies

—

But this value does not coincide with the final value.

Example 5.30 Consider the function *Æ* in Fig. 5.10. The final value theorem does not apply, since it does not exist. In that case, it is valid

———

But this value does not coincide with the final value.

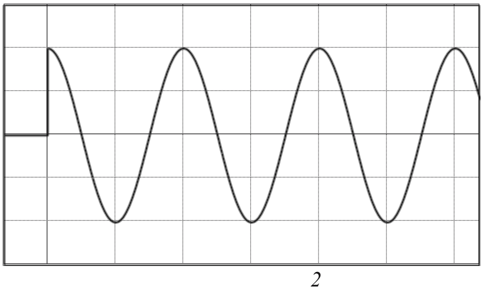


Fig. 5.10. The *Æ* function.

It is possible to state exactly the conditions under which the final value theorem is applicable, but this requires some definitions that will be presented only in the next section. We therefore defer that discussion to § 5.3.5.

5.2.9 Value theorem initial

Theorem 5.31. *Let there be a function with* ....*Laplace transformIf there exists*
finished the then

Demonstration. Observe that the following applies.[5]

— —

—

Running the limit for of the previous expression gives

—

[5]As noted in footnote 2 on page 139, if the function were discontinuous in the origin, its derivative would contain an impulsive term; the definition of transform given in Eq. (5.4) is used to account for this eventuality.

poiché il fattore dell'integrando tende a zero. Infine ricordando il teorema della derivata — vale anche:

—

from which, comparing first and last members, the result sought is obtained.

Example 5.32 Consider the function *Æ* in Fig. 5.10 whose transformed worth

It is easily verified that by applying de l'Hôpital's rule[6] , it is worth

and also . Note that this function is discontinuous in the origin, since it is worth .

The previous example highlights how in the initial value theorem it is essential to specify that the initial value is to be calculated at so that it can also be applied in the case of a discontinuous function in the origin for which .

5.3 Antitransformation of functions rational

We have seen that, given the Laplace transform of a function , it is possi- ble in principle to calculate the via the integral (5.3). In practice, this route is not smooth and we prefer to use other methods to antitransform the func- In particular, here we present a technique that allows us to determine the antitransform of any *rational proper function* in .

A *rational function* takes the form of a ratio of polynomials with real coefficients

It is called *proper* if it is worth , that is, if the degree of the polynomial at the denominator is greater than or equal to the degree of the polynomial at the numerator. As a special case, the function is called *strictly proper* if it is worth .

Rational functions are of particular importance in the field of systems analysis. Indeed, if a function can be written as a li- near combination of exponential ramps and their derivatives, then its Laplace transform is precisely a rational function.

[6]Guillaume François Antoine de l'Hôpital (Paris, France, 1661 - 1704).

The polynomial in the denominator will have real or complex conjugate roots which are called *poles*. The polynomial at the numerator will have real or complex conjugate roots that are called *zeros*. It is then possible to factor the two polynomials in the form

e

By placing the function in the form called *zeros-poles*:

(5.11)

dove .

It is still assumed that la is *in minimal form*, that is, that it has no pole coincident with a zero. For if it were worth the factor at the numerator, it could cancel with the factor at the denominator: by means of this *zero-pole cancellation*, we return the to the minimal form.

Different cases will be considered separately.

1. The function is strictly its own and all its poles have unitary multiplicity.
2. The function is strictly proper and one or more poles have multiplicity greater than one.
3. The function is its own but not strictly.
4. The function is the sum of rational functions each multiplied by a factor corresponding to a lag element.

5.3.1 Strictly eigenfunctions with poles of unit multiplicity

Suppose that the degree of the polynomial in the denominator is greater than the degree of the polynomial in the numerator, that is , and that the poles of the function are all distinct, that is, if .

Under these assumptions, the following result applies.

Proposition 5.33 *Let there be a rational function in the form* (5.11). *If it is strictly proper and its poles have unit multiplicity, it admits the following* Heaviside development*:*

(5.12)

where the real coefficient associated with the term is called the pole residue *. It is also said that in this form the is written in terms of* residue-poles.

Demonstration. The proof is constructive, but in order not to burden the notation we restrict ourselves to applying this construction to a function with only two poles. Consider.

a generic rational strictly proper function with two distinct poles that can by simple steps be brought back to the form:

(5.13)

where he placed e It is easy to verify that this function himself
admits Heaviside development. In fact,

(5.14)

and the two expressions (5.13) and (5.14) are equivalent as long as the residues and are chosen to satisfy the linear system

which always admits one and only one solution being the matrix of coefficients

nonsingular for the hypothesis ....thatThe same construction applies to a functio-
ne rational strictly proper with an arbitrary number of poles of unit multiplicity.

Heaviside development allows a rational function to be placed in a form whose antitransform is readily computed. In fact, for the generic residue-pole term, the following applies.

Æ

and therefore also applies to

Æ

The unknown residuals (for) can be calculated by the same construction used to prove the previous proposition. However, there is a simpler procedure, as the following result indicates.

Proposition 5.34 *The generic residual of the Heaviside development in eq.* (5.12)*, is worth*

(5.15)

Demonstration. Multiplying the two members of equation (5.12) by is worth

[redacted]

and, performing the limit for tending to of both members, the terms of the summation cancel, thus giving the desired result.

A simple example will help clarify the process.

Example 5.35 The rational function

has and . The poles are worth ....eSo the function can be placed in the form

and it's worth

So

e antitrasformando si ottiene Æ

Note that although in the Heaviside development the zeros of the date from (5.11) do not appear explicitly, the value of the zeros depends on the value of the residuals calculated by (5.15).

The case of a pair of complex, conjugate poles

Note that if the function has a complex pole, the residual cor- respondent will also be complex. However, at each complex pole
corresponds a complex conjugate pole whose residue is the conjugate com- plex of and the total contribution of the two poles to the will thus be given by a real term. It is possible to calculate this contribution in a relatively simple way.

Proposition 5.36 *Given a pair of complex, conjugate poles , let be the corresponding residuals. Placed*

(5.16)

vale

Æ (5.17)

Demonstration. If the modulus of the residue and its phase, the two residues have polar representation

e

and
therefore

Æ

Æ

Æ

Æ

Based on the previous proposition, it is sufficient to calculate only the pole residue , and then determine and by means of eq. (5.16) and finally calculate the antitransform using eq. (5.17).

Example 5.37 Consider the rational function

with and . The poles are worth ; ;
So the function can be placed in the form

and it's worth

So

rad

and anti-transforming gives

There is also an alternative technique, given by the following proposition.

Proposition 5.38 *Given a pair of complex, conjugate poles , let them be the corresponding residues. Placed*

(5.18)

vale

(5.19)

Demonstration. Vale:

Example 5.39 Consider the same function

studied in Example 5.37. It has already been determined that the pole residual is worth , while the pole residual is worth

I post then

the antitransform of vale

Æ

Æ

Si noti che è immediato passare dalla rappresentazione in eq. (5.17) alla rappresentazione in eq. (5.19) e viceversa ponendo

and

or vice versa:

e

5.3.2 Strictly eigenfunctions with poles of multiplicity greater than one

Now suppose that the function is, as in the previous case, strictly proper but that its poles have multiplicity that is not necessarily unitary. Under these assumptions, the following result holds.

Proposition 5.40 *Let there be a rational function in the form* (5.11). *If it is strictly proper and has distinct poles ( ) each with multiplicity it admits a development in which each pole corresponds to a sequence of residue-pole terms of the form*

(5.20)

and thus the Heaviside development of the function holds:

(5.21)

Demonstration. The proof, similar to that in Proposition 5.33, is construc- tive and is left to the reader.

Placed the in this form, it is immediate to calculate the antitransform. In fact, by antitransforming the generic term

— Æ

an exponential ramp is obtained, and therefore the following is also true

Æ

The following proposition indicates a simple procedure for calculating unknown residuals .

Proposition 5.41 *Given a pole of multiplicity , the residuals of the development in eq*. (5.21) *are worth*

and in general for vale

(5.22)

Demonstration. We preliminarily define the function

(5.23)

where it is defined in (5.20). Since it is multiplicity root of the equation , then it also applies to

(5.24)

From (5.23) while also taking into account (5.20) we get

(5.25)

a n d performing the limit for tending to of the previous expression, taking into account the first of (5.24) we obtain for the desired result.

Now calculate the successive derivatives (up to order) of Eq. (5.25). It is obtained:

and performing the limit for which tends to from the previous equations we obtain for , , , , the desired results.

A simple example will help clarify the process.

Example 5.42 The rational function

with and has poles: of multiplicity and of multiplicityTherefore it can be placed in the form

and it's worth

So

e antitrasformando si ottiene Æ

The case of a pair of complex, conjugate poles

The procedure described in the previous section to calculate the antitransform of terms associated with complex and conjugate poles can easily be extended to the case of poles of multiplicity greater than one. Indeed, the following proposition applies, the proof of which is analogous to that of Propositions 5.36 and 5.38 and for brevity is omitted.

Proposition 5.43 *Given a pair of complex, conjugate poles , let be*

where and be the residues two complex and conjugate numbers expressible in polar and Cartesian form, respectively, as follows:

Place , vale

— Æ

Place and , vale

— — Æ

Example 5.44 You want to anti-transform the function

which has poles of multiplicity 2. The Heaviside development of such a function is therefore worth

Vale

Place

e

the antitransform is worth

Æ

– Æ

An entirely equivalent form is obtained by posing

e

In that case the antitransform has expression

Æ

Æ

5.3.3 Functions not strictly own

Se il polinomio a numeratore della e il polinomio al denominatore hanno lo stesso grado , vale certamente

dove lo scalare è il quoziente dei due polinomi e il resto è un polinomio di grado .

It can therefore pose itself:

and because of what has been said, the function is strictly proper. Antitra- sforming the previous expression yields

Æ

where to anti-transform the term we used a result already seen that af- stops that the transform of the Dirac impulse function holds . The calculation of the anti- transform, on the other hand, always falls into one of the two previous cases, being strictly proper.

Thus, note an important result: *the antitransform of a rational proper but not strictly proper function contains an impulsive term*.

Example 5.45 Consider the rational function

...........................withTo perform the division of by we construct the table

from which it follows that with and .

So the function can be placed in the form

Using the procedure already seen in the previous sections, it can be easily demonstrated that the function has Heaviside development

and therefore it is worth

from which antitransforming we get

5.3.4 Antitransformation of functions with elements of delay

Rational functions, although important, do not describe all signals of interest in systems analysis. In particular, consider a function that can be written as a linear combination of *time-translated* exponential ramps. In that case, its Laplace transform contains one or more terms of the type (with) that correspond to delay elements (see Chapter 6, § 6.3.9). To antitransform these latter functions, the following result applies.

Proposition 5.46 *Let there be a function that can be written as*

Where for , the functions are rational proper functions and . Said , for , applies:

Demonstration. The result follows immediately on the basis of the linearity property and the time translation theorem.

A simple example will clarify how this result should be applied.

Example 5.47 Consider the function

– — —

Defining

− Æ — Æ

also applies to

Æ Æ Æ

This function can also be described as follows:

se
se
if
se

Its graph is shown in Fig. 5.11.

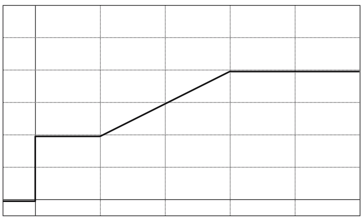


Fig. 5.11. The function *o f* Example 5.47

5.3.5 Existence of the final value of an antitransformed

Let there be a rational proper function in minimal form. You want to evaluate under what conditions you can apply the final value theorem without having to necessarily antitransform this function to assess whether there exists finite limit for Of the function .

If all the poles of the function have negative real part ( ) its antitransform for what we have seen in this section can be written as a linear combination of terms or ; these modes are all descending and therefore the limit for of the function exists and holds .

If then the function has a null real pole *o f* unit multiplicity and residual , a constant term appears in the and the limit for of the function is worth precisely .

In all other cases the does not admit finite final value. Indeed:

the presence of a null real pole of multiplicity greater than one gives rise to a term of the type , which by diverges;

the presence of pairs of imaginary poles (whether single or me- no multiplicity) gives rise to a term of the type that for admits no limit;

the presence of poles with positive real part ( ) gives rise to terms Or that by divergence.

Similar results apply if the function is a rational non proper function. We can therefore state the following result.

Proposition 5.48 *Let be a rational function in minimal form. There exists finite limit for della and therefore the final value theorem can be applied if and only if all poles of della have negative real part except at most one pole of unit multiplicity.*

Example **5.49** In Example 5.28 we saw that the final value theorem is applicable to the function

such a function has in fact a negative real pole and a null real pole .

Conversely, the theorem cannot apply to the functions given in Example 5.29 and Example 5.30. For in the former case the function to be antitransformed is worth

and it has a positive real pole and a ....null real poleIn the second
case the function to be antitransformed is worth

and it has a pair of imaginary conjugate poles .

5.4 Solving differential equations using the Laplace transform

This section presents some examples to show how Laplace transforms can be used to solve linear differential equations (or more generally integro-differential equations) with constant coefficients.

Example 5.50 Consider the circuit in Fig. 5.12.a whose evolution is described by the equation

(5.26)

This equation, taking into account the capacitor equation , becomes

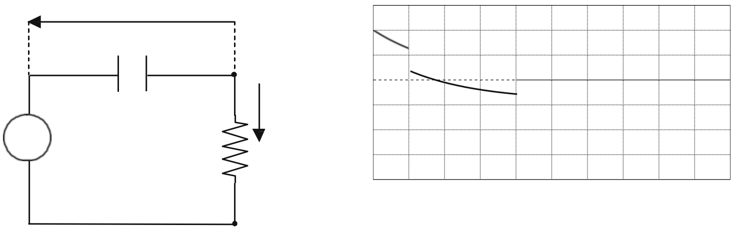
It is assumed that the applied signal is an amplitude step applied to the i- stant , i.e., it is worth Æ ; the transform of this function *is worth*

. Inoltre si suppone che nell'istante immediatamente precedente all'applicazione del segnale la tensione ai capi del condensatore valga .

Transforming the differential equation gives:

che, particolarizzando per la condizione iniziale e per il segnale assegnati diventa

and solving for finally obtains



(a) (b)

Fig. 5.12. (a) The RC circuit of Example 5.50; **(b)** trend of the voltage at the ends of the capacitor for a step applied signal Æ

The Heaviside development of such a function is worth

being

Antitransforming gives

Therefore, we can say that the trend of tension will be:

(5.27)

and the qualitative trend of this function (assumed) is as shown in Fig. 5.12.b.

In base alla (5.27) vale e tale valore coincide con la condizione iniziale assegnata......................Possiamo dunque affermare che la tensione non subisce Discontinuities following the application of a step signal.

Example 5.51 Consider again the circuit in Fig. 5.12.a. We now want to determinate, given the same initial conditions and signal as in Example 5.50, the value of current .

Taking into account the capacitor equation , which we rewrite as , eq. (5.26) becomes.

By transforming the integral equation (the constant is equivalent to a step) we get:

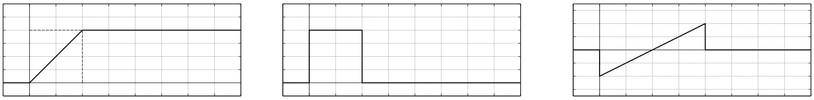
which particularized for assigned initial conditions and signal becomes

and solving for finally obtains

Since the is written as the transform of an exponential function, it is immediate to antitransform, obtaining

Therefore, we can say that the current trend will be:

(5.28)



(a) (b) (c)

Fig. 5.13. Functions to be transformed in Exercise 5.2.

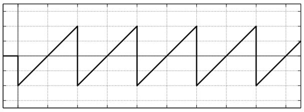


Fig. 5.14. Function to be transformed in Exercise 5.6.

Exercises

Exercise 5.1 Calculate the Laplace transform of the following time functions:

(a) Æ
(b) Æ
(c) Æ
(d) Æ
(e) Æ
(f) Æ

Exercise 5.2 Transform according to Laplace the functions assigned graphically in Fig. 5.13.

Exercise 5.3 Apply the derivative theorem to the function Æ to calculate the pulse transform. This value must coincide with that determined in Example 5.6.

Exercise 5.4 Given the function Æ , verify the final value theorem.

Exercise 5.5 Given the function Æ , verify the initial value theorem.

Exercise 5.6 Transform according to Laplace the function in Fig. 5.14. Keep in mind that this function is periodic for and its basis function is the function in Fig. 5.13.c.

Exercise 5.7 In a workshop, a punching machine repeatedly, every second, drills a hole in a metal plate. The stress to which it is subjected can thus well be represented by a train of pulses

Æ

Prove, based on Proposition 5.22, that the transform of this signal is worth

Exercise 5.8 In Appendix B we defined, for each value of , the order derivatives of the pulse

Æ Æ

Prove, by applying the derivative theorem, that Æ for every .

Exercise 5.9 Antitransform the following functions of :

(a)

(b)

(c)

(d)

(e)

Of some functions, one of the poles is shown so that the roots of the polynomial in the denominator can be easily calculated.

Exercise 5.10 Anti-transform the following function:

and graph the function If the result is correct, it is recognized in the
Graph a letter of the alphabet: which one?

Exercise 5.11 Prove that any rational function that is not proper, where is a polynomial of degree and is a polynomial of degree , can always be placed in the form

where it is a strictly proper rational function with the same poles as . This result, together with what was seen in Exercise 5.8, makes it possible to transform any rational function, not necessarily proper.

In particular, calculate the antitransform of the function

Exercise 5.12 Solve for the following differential equation

from the initial conditions and and given an applied signal
Æ .

Exercise 5.13 Solve for the following differential equation

from the initial conditions and given an applied signal
Æ .

Exercise 5.14 Consider the circuit in Fig. 5.15 and prove that the bond between the voltage at the ends of the capacitor and the voltage applied by the generator is worth

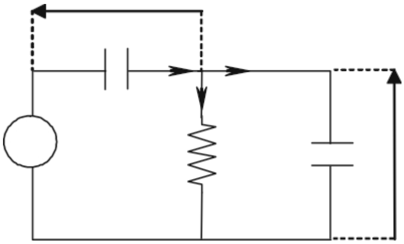


Fig. 5.15. The RC circuit of Exercise 5.9

Solve the differential equation from the *initial* conditions (initially unloaded circuit) given an input Æ .

Plot the graph of the function and discuss whether the application by the generator of a discontinuous signal at the origin causes a discontinuity in the signal .

Table of remarkable transformations

Function of time		Laplace transform
Unit pulse	Æ	
Unit step	Æ	–
Linear ramp	Æ	—
Polynomial ramp	— Æ	——
Exponential	Æ	——
Cosine	Æ	———
Breast	Æ	———
Damped cosine wave	Æ	—————
Damped sine wave	Æ	—————
Exponential ramp (or cisoid)	— Æ	————

6

Analysis in the domain of the Laplace variable

The study of the Laplace transform has been motivated by the fact that it is a useful mathematical tool for solving the differential equations that describe an important class of dynamical systems, that of linear and stationary systems. In this chapter this technique will be applied both to the analysis of input-output (UI) models and to the analysis of representations in terms of state variables (VS): this analysis is called "in the domain of the Laplace variable" or more simply still "in " to distinguish it from the study in the "time domain" or "in ".

Some of the results that will be presented here have already been obtained through time domain analysis: however, it will be useful to approach them again from the new perspective of the study in . Other results, conversely, are entirely original.

The first section describes how Laplace transforms can be applied to UI models, while the second section studies models in VS. A fundamental concept for analysis in is that of the transfer function to which the third section is devoted. This function can be factorized into various forms that you need to know about: these are described in the fourth section. The use of Laplace tra- sforms is particularly advantageous in the study of the forced response of a system as will be seen in the fifth section where the forced response is analyzed for a particularly interesting class of input signals: exponen- tial signals. This also allows us to introduce the concept of *permanent regime* and *transient regime*. Finally, as a special case of forced response, the step response (*index response*) is considered.

6.1 Analysis of input-output models using Laplace transforms

The link between the output *a n d* input of a linear, stationary SISO system is described by a linear differential equation with constant coefficients of order , of the type

(6.1)

....................withThe fundamental problem of analysis is to determine the anda-ments of the exit for knowing:

the initial conditions[1] , , ;

the input trend for .

First, it is useful to introduce the concept of *a transfer function*, which, as we shall see, plays a fundamental role in the analysis of linear and stationary systems. Given the differential equation (6.1) let be.

the characteristic polynomial of the associated homogeneous, and let be

the polynomial obtained with the coefficients of the second member of the equation. The *transfer function* of the system described by model (6.1) is called the rational eigenfunction of the variable defined as:

The importance and physical significance of this function will be discussed in section 6.3.

To solve the systems analysis problem, we transform the given differential equation according to Laplace. We denote by e the -transforms of and According to Proposition 5.16 the transform of the -ma derivative of the output vale

while, remembering that the input and its derivatives are zero at , it is also true

Therefore, the transform of (6.1) holds:

. . .

[1]Recall that in the case where there are discontinuities in the origin, those taken at are considered as initial values.

or by rearranging the terms

In the previous expression it was denoted by a polynomial of degree less than or equal to that depends on the initial conditions and whose exact expression holds:

At last we can write that the solution in the domain of of of the analysis problem (6.1) takes the form

(6.2)

By antitransforming, the solution sought in the time domain can be obtained.

In this expression of the we recognize two terms.

The term indicates the contribution to due to the presence of non-zero initial conditions: in fact, the polynomial is identically zero if and only if all initial conditions are zero. This term is thus the -transform of the free response .

The term indicates the contribution to the due to the presence of the input; thus, this term is the -transform of the forced response .

We will study the two terms separately, but first consider an example.

Esempio 6.1 Dato il sistema descritto dal seguente modello IU

with *initial* conditions and , the -transform of the input- bond. output provides the equation:

(6.3)

6.1.1 Answer free

Given (6.2), we observe that the polynomial at the deno- minator of the coincides with the characteristic polynomial of the homogeneous asso- ciated to the differential equation (6.1). So the poles of the transform of the free response characterize the modes of the system.

Considering the residue-pole form of the function , under the assumption that there are distinct poles of multiplicity according to Proposition 5.40 is obtained,

Where the generic term associated with the pole is worth

and therefore

where the residuals depend on the shape of the polynomial and thus on the initial conditions. Finally, by antitransforming, we obtain

— Æ

So, as already observed when we studied the analysis problem in the time domain, the free response is a linear combination of the modes of the system.

Example 6.2 For the system in Example 6.1, you want to calculate the free response a partire dalle condizioni iniziali e .

From (6.3) applies:

Such a function has two distinct real poles, ...eScomposing into factors e Switching to the residue-poly form gives

from which by antitransforming we derive the free answer

Æ

which as expected is a linear combination of the two modes of the system.

6.1.2 Response forced

The computation of the forced response in the time domain requires, as already seen, the computation of a convolution integral, which is not always easy. By means of Laplace tra- sforms, on the contrary, the calculation of the forced evolution is rather easy. It involves first determining the transform of the forced response

(6.4)

as the product between the input transform and the transfer function. By antitransforming this expression, we immediately derive the forced evolution

.

Esempio 6.3 Per il sistema dell'Esempio 6.1 si desidera calcolare la risposta forzata che consegue applicazione dell'ingresso Æ .

The Laplace transform of the given input is worth

and therefore

from which by antitransforming we get

Æ

6.2 Analysis of models in state variables using Laplace transforms

Given a representation in state variables

(6.5)

describing a linear, stationary MIMO system, the fundamental problem of the-analysis is to determine the state and output trends for knowing:

lo stato iniziale ;
The input trend for .

We denote by , and the -transforms of , and ....Since such vectors have , and components respectively, their transforms will also be vectors of the type

. . .

being the transform of the generic -ma component of- the input, the transform of the -ma component of the state and
The transform of the -ma component of the output.
It is immediate to transform (6.5) taking into account that the transform of the generic function holds, and in vector terms this relationship becomes:

Therefore, the transform of (6.5) holds:

(6.6)

and rearranging the equation of state we get

from which, multiplying both members by , we get ....Sostituendo this value in the output transformation, the solution in the domain of of of the analysis problem (6.5) takes the form

(6.7)

This expression is the equivalent of Lagrange's formula in the domain of : antitra-sforming it will yield the solution sought.

In the expression of and of we also recognize two terms in this case.

The e terms arise only in the presence of a nonzero initial state; these terms are thus the -transforms of the free evolution of the state
and output .

The e terms arise only in the presence of a nonidentical null input; these terms are thus the -transforms of the forced evolution of state and output .

6.2.1 The matrix solving

In the expression of Lagrange's formula in the domain of appears the matrix

which is called the *resolving matrix*. It is important to pause and study what form such a matrix takes and what its physical meaning is.

Existence of the resolving matrix

First, note that the resolving matrix is well defined whatever the value of , i.e., it is always possible to invert the ....matrixTo prove
this note preliminarily that the matrix is not a matrix of scalars but is a polynomial matrix whose generic elements are polynomials of degree 1 along the diagonal and degree 0 elsewhere. Its inverse, which is calculated by the well-known formula

exists if and only if the determinant is nonzero. Note, however, that since it is a matrix of polynomials, its determinant is also a polynomial, and in order for the solving matrix to be calculated, it must be different from the null polynomial[2] . However, it is known that the determinant is the characteristic polynomial *o f* the dimension matrix : by definition, this polynomial has degree and is therefore different from the null polynomial.

Note again that the polynomial matrix has as its elements the co-factors of which, being less than order , will be polynomials of degree less than or equal ....toBecause it is a polynomial of degree , it is possi-
bile to conclude then that the resolving matrix has rational functions for elements *strictly their own*.

[2]The *null* polynomial is the polynomial : that is, it consists of the only known term, which moreover is worth zero. Note that being a constant the null polynomial has degree 0.

Example 6.4 A representation in VS is given whose state matrix holds:

Since

the resolving matrix is worth

Note how zero-pole deletions in tervene in some elements of the solving matrix, reducing the order of the numerator and denominator polynomials .

Physical meaning of the resolving matrix

Proposition 6.5 *The resolving matrix is the Laplace transform of the state transition matrix, i.e., it is worth*

Dimostrazione. L'espressione della evoluzione libera dello stato in in base alla (6.7) vale . Nel dominio del tempo, d'altro canto, l'evoluzione libera dello stato in base alla formula di Lagrange (4.13) vale

Therefore, it is worth

and comparing the first and last members of this equation yields the result sought.

This property provides us with an additional method, in addition to those previously seen, for calculating as the antitransform of .

Example 6.6 The resolving matrix of the VS representation discussed in E-sample 6.4 is worth having given Heaviside's development of the ....termSo the State transition matrix for this representation is worth

Æ

6.2.2 Example of calculation of free evolution and forced evolution

Let the following representation be given in terms of state variables

(6.8)

You want to calculate the evolution of the state and output that follows *for* all'applicazione di un ingresso Æ a partire da condizioni iniziali .

We first calculate the resolving matrix, which is worth

while the Laplace transform of the input is worth

Æ

The Laplace transform of the state is then determined to be worth being the transform of free evolution

And the transform of forced evolution

Antitransforming , free evolution holds true

Æ

while performing the Heaviside development of each of the two elements of you get

and anti-transforming

–

Æ

–

We determine the Laplace transform of the output, which is worth

being the transform of free evolution

And the transform of forced evolution

Antitransforming , free evolution holds true.

Æ

while performing the Heaviside development of results in the following.

and anti-transforming

Æ

6.3 Function of transfer

6.3.1 Definition of function and matrix of transfer

In Section 6.1, the concept of a transfer function was introduced by referring to a SISO system described by a UI model. More generally, the following definition can be given.

Definition 6.7 *Given a linear, stationary system, a transformation matrix is defined as that variable matrix which, when multiplied by the Laplace transform of a generic input signal, gives the Laplace transform of the corresponding forced response, that is, it satisfies the equation*

(6.9)

If the input is a vector with components and the output is a vector with components, the transfer matrix has dimensions .

In the special case of a SISO system, the transfer matrix becomes a scalar function called the transfer function.

The rest of this chapter will mainly consider the case of SISO systems. The study of MIMO systems by means of transfer matrix will be dealt with only in § 6.3.6 and in Chapter 7 in § 7.2.

The transfer function, based on the definition just given, describes the external link between the input and output of a system. It is thus the counterpart of a UI model in the domain of the Laplace variable .

Given a UI model of a SISO system.

it was seen that according to eq. (6.2) it is worth

and therefore the transfer function is worth

(6.10)

Note the particular structure that the transfer function takes. It is a rational function of which has at the numerator the polynomial constructed from the coefficients of the second member of the differential equation and at the denominator the polynomial constructed from the coefficients of the first member. As already observed, since it is by definition the characteristic polynomial of the system, the poles of the transfer function coincide with the roots of the homogeneous equation and thus characterize the modes of the system. The differential equation describing the UI bond in the time domain and the transfer function contain the same information, and it is immediate to switch from one model to the other.

6.3.2 Transfer function and response impulsive

A close link exists between the transfer function and the impulsive response of a system.

Proposition 6.8 *The transfer function of a linear, stationary SISO system is the Laplace transform of the impulse response, i.e., it is worth*

Demonstration. The transform of the forced response according to (6.2) can be written as:

(6.11)

This relation states that if the application of input is followed by a forced response then the following holds true

By definition, the *impulsive* response is the forced response that follows al- the application of a unit impulse *Æ* at instant ....Turning to the domain of the
Laplace variable and remembering that the pulse transform holds *Æ* , we obtain

Æ

and comparing the first and last members of this equation yields the result sought.

This result provides a simple procedure to calculate the impulse response of a system characterized by a UI model of the type (6.1) by performing this step:

UI model

This technique, which operates in the Laplace variable domain, can be used as an alternative to Algorithm 3.20, which operates in the time domain.

Example 6.9 For the system in Example 6.1, we want to calculate the impul- sive response. Given the coefficients of the differential equation, we can directly write the transfer function of this system as

Decomposing into factors and switching to the residue-poly form gives

where

from which by antitransforming we get

Æ

6.3.3 Pulse response and input- output pattern

Recall again that the impulsive response has been called a *canonical regime*, knowledge of which is perfectly equivalent to knowledge of the model (6.1). However, it has not been discussed so far how it is possible, knowing the , to determine the UI model that corresponds to this impulsive response. This can be done by the following procedure:

modello IU,

since note the it is immediate to determine the UI model in the form (6.1).

Example 6.10 Given a system characterized by its impulsive response

Æ

you want to calculate the corresponding UI model.

Transforming the term to term yields:

taking into account the coefficients of the polynomials in the numerator and denominator, we immediately derive the UI model

...

6.3.4 Identification of the function of transfer

Again according to (6.11), the transfer function can be written as:

—— (6.12)

relationship that allows us to calculate the transfer function of a system whose forced response that follows the application of a given input is known.

Example 6.11 A system is given whose forced response as a result of the application of a signal

Æ vale Æ

You want to determine the transfer function.

By transforming the input and output signals, we obtain.

— e ——— —— – — ————

So the transfer function of such a system is

—— ————

This relationship thus allows the transfer function (and thus the UI model) to be determined on the basis of measurements of the input and output of a system initially assumed to be at rest. This procedure is called *identification*. In practice, the identification procedure is much more complex than it may seem: in fact, the condition of the system at rest may not be verified, the input and output measurements may be affected by noise, etc. This problem will not be addressed in this text.

6.3.5 Transfer function for models in state variable

Consider a SISO system described by the model in state variables (6.5). According to eq. (6.7), the forced response is worth

and taking into account (6.9) we obtain the following expression for the transfer function:

(6.13)

Note that this formula also expresses the link between input and output: however, while the coefficients characterizing the UI model appear in (6.10), the coefficients characterizing the VS model appear in (6.13).

It is easily verified that the expression from (6.13) also takes the form of a rational function. In fact, remembering that the solving matrix is a matrix whose elements are rational strictly eigenfunctions (see § 6.2.1), by indicating the dimensions of the various terms that appear in the expression of the transfer matrix we obtain:

Nella precedente espressione è un polinomio di grado , mentre ha grado ed è il polinomio caratteristico della matrice di stato . Si noti che è una funzione sempre propria, e strettamente propria se e solo se .

Example 6.12 You want to determine the transfer function of the system described by model (6.8).

Vale:

Note that the denominator of the transfer function is the character- stic polynomial of the matrix , which is worth precisely ....Th e

transfer function is not strictly its own because .

6.3.6 Matrix of transfer

In the case of MIMO systems, the input is a vector with components, while the output is a vector with components. Referring to a model in state variables, according to eq. (6.7) the following applies

and according to (6.9*)* the *transfer matrix* is worth

(6.14)

such a matrix has dimensions .

To understand what physical significance the generic element of this matrix has, we observe that the forced response that follows an input holds (omitting the subscript so as not to burden the notation) and thus the transform of the -ma component of the output is worth

Therefore, if a vector is applied to the input of the system that has as its -ma component a unit pulse and all other components zero, it is worth

Æ

and the transform of the -ma component of the output takes the value

which we call *the transfer function between the input and the output* . The antitransfer of is the response of the output to a pulse on the input and is denoted

.

Example 6.13 Consider the following representation in VS:

The transfer matrix is worth

Note that in this particular case, being , the matrix is square, but in general the number of rows in this matrix can be different from the number of columns. Again note that the element is worth zero: this indicates that the input does not affect the output .

The study of MIMO systems using the transfer matrix is also covered in Chapter 7 (see §7.2).

6.3.7 Transfer matrix and similarity

The transfer matrix (or transfer function in the case of SISO systems) describes the input-output behavior of a system in the domain of the Laplace variable. If we consider two different representations related by similarity, then- because they describe the same system it is intuitive that they must have the same transfer function. This fact is formally proved in the following proposition.

Proposition 6.14 (Invariance of the transfer matrix by similarity)
Consider two representations in VS linked by similarity

e

The two representations have the same transfer matrix. Demonstration. The transfer matrix of the second representation is worth

and this relationship can also be rewritten

and thus coincides with the transfer matrix of the first representation.

6.3.8 Moving from a model in VS to a model IU

Equation (6.13), or equivalently (6.14) for a MIMO system, allows us to easily solve the following problem[3] *: given a VS model of a linear, stationary system determine a UI model of the same system.* To solve such a problem, one can in fact follow this path:

[3]The inverse problem, which is to determine a model in VS of a system for which a UI model is known, is called the *realization* problem. It is considerably more complex and will be discussed in Chapter 7.

VS model *UI* model

that is, we first determine the transfer function that corresponds to the given representation and then determine the UI bond that corresponds to it.

Example 6.15 You want to determine the UI model that corresponds to the representation in VS given by (6.8).

The transfer function of such a system has already been calculated in Exember 6.12 and is worth

So the UI model of such a system is described by the differential equation

Even in the case of a MIMO system this procedure is of immediate application.

Example 6.16 We want to determine the UI model that corresponds to the VS rappre- sentation considered in Exercise 6.13, whose transfer matrix is worth

The forced response transform is related to the input transform by the relation

Bearing in mind that , the previous expression can be rewritten

Antitransforming gives the model

6.3.9 Systems with elements of delay

Among the various systems described in Chapter 2 there is one whose UI model takes a different form from that prescribed by (6.1). Such a system is the so-called *delay element* whose input-output link is described by the equation

(6.15)

indicating how the value taken by the output at time is equal to the value taken by the input at time (i.e. units of time before).

By transforming this relationship, remembering the time translation theorem and keeping in mind that the output coincides with the forced output, it is worth

and based on (6.9) we can write that the transfer function is worth

So the delay element transfer function is not a rational function (ratio of polynomials) but an exponential function in .

In the remainder of this chapter we will restrict ourselves to studying systems whose transfer function is a rational function. However, the technique described in Chapter 5 (see § 5.3.4) for antitransforming signals in which delay elements are present allows the results presented here to be extended to such systems.

6.4 Factorized forms of the function of transfer

As seen in the previous section, the transfer function of a linear, stationary SI- SO system without lag elements is a rational function of the variable , i.e., a ratio of two polynomials in . More precisely, we used to call the expression in eq. (6.10) *a polynomial representation* of the transfer function.

It is possible, however, to relate the transfer function to other standard representations that allow certain properties of interest to be better studied. The for- mations we consider here are the *residue-pole representation*, the *zero-pole representation* and the *Bode representation*.

6.4.1 Representation residuals- poles

Based on what we saw in the chapter on the study of Laplace transforms, from examining (6.10) we can state that if the transfer function is a strictly proper rational function, that is, if , it admits a Heaviside development of the form

$$— \qquad — \tag{6.16}$$

Where the poles are the roots of .

If vice versa the was proper but not strictly proper, that is, if , a constant term would also appear in its development, i.e.

$$— \qquad — \tag{6.17}$$

We often denote the Heaviside development of the transfer function given by Eq. (6.16) or Eq. (6.17) as the *residue-pole representation*.

Thanks to this decomposition we can easily confirm a result we have already studied in Chapter 3. Proposition 3.18 states that the impulsive response has a particular structure: it is the sum of a li- near combination of the modes of the system plus a possible impulsive term if the system is proper but not strictly so.

This result can also be demonstrated by studying the transfer function, the latter being the Laplace transform of the impulse response. If the is strictly its own, by antitransforming (6.16) we find that the will also have an expression of the type

— Æ

and therefore it is a linear combination of the modes of the system. If vice versa the was proper but not strictly proper, that is, if , by antitransforming (6.17) it is observed that an impulsive term would also appear in the expression of the:

—Æ — Æ

Note that in the latter case, as per Proposition 3.18, the impulsive term has area equal to .

6.4.2 Representation zeros- poles

A second form to which a transfer function given by (6.10) can be traced is the so-called *zeros-poles representation.* It is obtained by factoring the numerator polynomial of the transfer function and the polynomial in the denominator, to highlight zeros and poles.

If we denote by (*for*) the generic pole and by (*for*) the generic zero, it is obviously worth

e

and the transfer function (6.10) can be written as

—————— (6.18)

Where we define *gain at high frequencies* as the constant

—

Example 6.17 Consider the transfer function

——————

The polynomial of degree 1 at the numerator has coefficient and roots The polynomial of degree 3 in the denominator has coefficient and *roots* e By factoring the polynomials in the numerator and denominator we obtain the zeros-poles representation

– ——————

dove il guadagno alle alte frequenze vale –.

Minimum form

The zeros-poles representation is particularly useful for defining the concept of *minimal form*.

Definition 6.18 *A transfer function is said to be* in minimal form *if none of its poles coincides with a zero* .

Otherwise, it is always possible to reconduct a transfer function to the minimum form by performing a *zero-pole cancellation*: this cancellation reduces the order of the model.

Example 6.19 Consider the transfer function

——————

which has zero and poles ....andHe corresponds to a pattern of order .

By factoring the polynomials in the numerator and denominator we obtain which is clearly not in minimum form. Deleting the factor at numerator and denominator yields the minimal form

This model has reduced order .

Note that in a transfer function in non-minimum form not all the roots of the polynomial in the denominator of the transfer function correspond to modes of the system characterizing the impulsive response.

Example 6.20 Consider the system described by the transfer function

already studied in the previous example. Placed the in the residue-poly form we obtain:

where

while

Antitransforming gives

Æ

So, the impulsive response contains only the mode , while the mode (which corresponds to the pole coincident with zero) has zero residual and therefore does not appear.

6.4.3 Representation of Bode

The last representation of the transfer function we consider consists of a particular factorization in which the various parameters that characterize the modes appear. These parameters, as seen in Chapter 3, are: the time constant associated with a nonzero real pole; the natural pulsation and the damping coefficient associated with a pair of complex, conjugate poles.

Note that the time constant is not defined for a null real pole (we often speak of a *pole in the origin in* such a case); this requires that such poles be considered separately from the other nonzero real poles.

If the has poles be: the multiplicity of the pole in the origin, if any; the number of real poles (including those in the origin); the number of complex and conjugate pole pairs.

We can therefore rearrange the poles as follows:

real poles complex and conjugate pole pairs

poli in origin

Where:

For ;

For ;

For .

Since there are in total poles it obviously applies .

A similar argument naturally applies to zeros. We can therefore rearrange the zeros as follows:

real zeros pairs of complex and conjugate zeros

zeri in the origin

Where:

For ;

per ;

For .

Since there are in total zeros it obviously applies .

Therefore, (6.18) can be rewritten as.

(6.19)

dove Se positivo rappresenta l'eventuale numero di poli nell'origine[4], if negative, on the other hand, its absolute value is equal to the number of zeros in the origin.

It is now possible to introduce into this factorization, in place of the real and im- maginary parts of the poles and , other coefficients. They are: the in-place *time constant* of each real pole; the *natural pulsation* , and *the damping coefficient*

In place of each pair of complex conjugate poles.

[4]We assume that la is in minimal form and therefore that e are not both diversi da zero.

The generic binomial factor corresponding to a real pole can in fact be rewritten (omitting the subscript so as not to burden the notation) as

(6.20)

where represents the time constant relative to the aperiodic mode associated with the non-zero real pole .

The factors corresponding to a pair of complex *conjugate* poles.
Can instead be rewritten as:

(6.21)

where

denote the natural pulsation and the damping coefficient, respectively. Recall that by definition .

Similar parameters , and can also be introduced at the numerator in place of the real and imaginary part of each zero . Note, however, that these parameters, unlike those associated with the poles, have no physical meaning because they do not characterize the evolution of any mode.

Substituting the factorizations (6.20) and (6.21) into equation (6.19), we thus arrive at the desired form of the , called *the Bode representation*:

(6.22)

where the constant , called *the Bode gain of* the , or simply *gain* of the , is related to the constant and the other characteristic parameters of the transfer function by the relation

(6.23)

Note that the gain can also be easily calculated from the expression (6.10) of the ....In fact,

—

where and are the coefficients of the lowest degree terms in the numerator and denominator of the , i.e.

If then the has neither poles nor zeros in the origin holds .

Example 6.21 Consider the transfer function

To go to the Bode form, it is necessary to factor the characteristic polynomial in the denominator and thus

The gain is worth .

6.5 Study of the forced response using the Laplace transforms

In the previous chapters we dealt with the study of the forced response in the time domain. In particular, we saw that for input-output models the forced response can be determined by means of the Duhamel integral (see Chapter 3, § 3.6.1), while for models in state variables it can be determined by Lagrange's formula (see Chapter 4, § 4.3). In both cases it is therefore necessary to solve a convolution integral.

The use of Laplace transforms greatly simplifies this calculation because in the domain of the Laplace variable a convolution integral corresponds to a simple product of functions of . Moreover, as we shall see, it will also be possible to understand the general structure that the forced response assumes; in the cases of interest that we shall study it in fact consists of a linear combination of the modes of the system to which are also added modes introduced by the input signal.

Finally, note that although our analysis will be limited to the forced evolution of the output, the same results also apply to the forced evolution of the state. In fact, if a system in terms of state variables is characterized by the equation of state la generica componente del vettore di stato può anche essere vista come l'uscita del sistema fittizio

where the matrix is nothing but the transpose of the -th canonical basis vector or

— —

6.5.1 Forced response to canonical inputs

In this section we study the structure of the forced response under a particular assumption. It is assumed that the input signal has the form

Æ (6.24)

that is, it is assumed to belong to the family of *exponential functions* (see Chapter 5, § 5.1.4). This is equivalent to saying that its transform is worth

That is, it has a pole of single multiplicity.

The following result applies.

Proposition 6.22 *Consider a system whose transfer function is worth And subject to an input Æ The forced response of such a system can be decomposed into the sum of two terms :*[5]

The term is a linear combination of the modes of the system.

The term is a mode associated with the parameter introduced by the input signal. In particular, if the input parameter does not coincide with a pole of the transfer function, the particular integral is worth

Æ con (6.25)

while if it coincides with a multiplicity pole of the transfer function, the particular integral is worth

— Æ with (6.26)

[5]The subscript recalls that it is *a particular integral of* the differential equation describing the UI bond, while the subscript recalls it is an *integral of the* associated *homogeneous*.

Demonstration. A generic transfer function can be factorized as.

(6.27)

having distinct poles of multiplicity (.....................................forThe total number of poles
della funzione di trasferimento è e il grado del polinomio al numeratore vale .

The Laplace transform of the forced response is therefore worth

(6.28)

and has as its poles the set of poles of the and of .

The result follows immediately from the Heaviside development of the function given in eq. (6.28). In fact, the numerator of has degree and the denominator has degree, and therefore this function is strictly proper

In the case where it is not pole of the function can be written

where .

Conversely, suppose it coincides with a multiplicity pole of .
We can then assume in all generality that it is worth and and therefore

where the residual is worth

Antitransforming, the term determines a linear combination of the modes of the system, while the term determines the signal (6.25) or (6.26) as appropriate.

Based on the previous result, we can see that the forced response of the system is a linear combination of the modes of the system plus an exponential mode (or exponential ramp mode in the general case) that has the same parameter as the input signal. We can therefore think that the input signal excites the system, which evolves with its ways, but those ways are also joined by a new way introduced by entry.

This result is not surprising. Also in the case of the impulsive response it had been observed that this particular forced evolution can be decomposed into two ter- mines: a term composed of a linear combination of the modes of the system, and a possible impulsive term introduced precisely by the input.

Finally, note that although in order not to burden the exposition we have limited ourselves t o studying exponential input signals, the results of this section can be generalized to exponential ramp inputs

$$— \quad Æ \tag{6.29}$$

with . Interest in this particular family of canonical inputs arises from the fact that, as repeatedly noted, most signals of interest can be obtained by linear combinations of exponential ramps, possibly time-shifted.

Also for an exponential ramp input signal, it is possible to decompose the forced response into the sum of a linear combination term of the modes of the system and a term containing the modes introduced by the input. Since the Laplace transform of the signal (6.29) has poles, the parti- cular integral that corresponds to it is also a linear combination of terms (see Exercise 6.10).

Let us finish with two examples related to exponential type inputs. The first is related to the case where the parameter does not coincide with one of the poles of the system, and therefore the particular integral is a signal that has the same shape as the input signal.

Example 6.23 Consider a third-order system whose transfer function is worth

The ways of such a system are clearly , and .

Si vuole determinare la struttura dell'evoluzione forzata che consegue all'applicazione del segnale di ingresso Æ Poiché vale

determined the various residuals and anti-transforming we obtain

$$Æ$$

As expected the particular integral holds Æ with It is thus an exponential signal that, barring a constant, coincides with the input signal

The second example considers the more general case where instead the particular integral does not have exactly the same shape as the input signal.

Example 6.24 Consider again the system in Example 6.23 subject to the input Æ . In tal caso il parametro del segnale di ingresso coincide con un polo di molteplicità della funzione di trasferimento. Poiché

antitransforming gives

Æ

Note that in this case the particular integral has the form of a quadratic ramp perché e come atteso il residuo vale .

6.5.2 The permanent regime response and the transient response

A fundamental concept in systems analysis will now be introduced.

Definition 6.25 *The* steady-state response *to an assigned input is that function of time to which, regardless of the initial state, the output response tends as time increases*

A permanent regime is not always achieved. However, consider the following assumptions:

(A) the poles of the system transfer function are all negative real part;
(B) the applied input is a linear combination of exponential ramps.

In this case, because of what was seen in the previous section, the total output can be written as the sum of the following terms

where first the free evolution was separated from the forced evolution, and then, thanks to hypothesis (B), the forced response was decomposed into the sum of an integral of the homogeneous and a particular integral.

In this decomposition we recognize two terms.

The term asymptotically tends to zero as increasing of and is called the *transient response*. In fact, both the free evolution and the component of forced evolution are linear combinations of the modes of the system. Due to hypothesis (A), all modes are stable and holds:

e

In practice, the transient term can be considered extinct after a certain time : this value is equal to the time it takes for the *slowest* mode to become extinct. The remaining term is called the *permanent steady* state *response* because it does not depend on the initial state of the system and for is worth

i.e., the total response coincides with the particular integral and is therefore characterized by the modes introduced by the input only.

Note that this decomposition of the total response into transient and regime terms is alternative to the decomposition into free and forced evolution and has a different physical meaning. In free and forced evolution we recognize the contribution due to the initial state and input, seen as two separate causes of evolution. In the transient and regime term, on the other hand, we recognize the modes proper to the system and the input. Even the forced response alone, if assumptions A and B hold, can be decomposed into a transient term (which coincides with the integral of the homogeneous) and a regime term (which coincides with the particular integral).

Example 6.26 Consider the second-order system characterized by the transfer function

Which has two poles and a negative real part.

It is easily verified that the free evolution of such a system from the initial conditions , is worth

Æ

La risposta forzata conseguente all'applicazione dell'ingresso Æ applies instead

Æ Æ

So the transitional term and the regime term apply respectively:

Æ

Æ

In Fig. 6.1, the trend of these two signals and the com- plensive response has been plottedNote that after a time the transi- term thorium practically can be considered extinct, and the overall response coincides with the regime term. The slowest mode of the system is the one corresponding to the pole

of unit multiplicity and its time constant holds ; therefore in this case we have

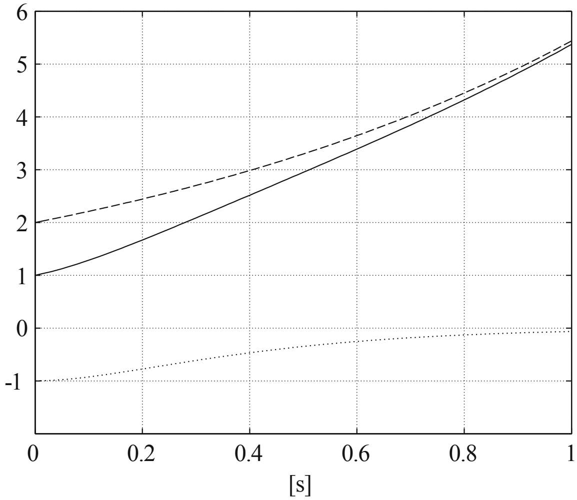


Fig. 6.1. Decomposition of the total response in transient and regime terms in Example 6.26.

6.5.3 Response indexal

In this chapter we study the particular canonical regime that results from the application of the simplest of all exponential signals: the *unit step*. The importance of such a regime, arises from the fact that in reality it often happens that a system is controlled by means of a constant (or constant step) input signal.

Definition 6.27 *The* index response *is the forced evolution that results from the application of a signal Æ, i.e., a unit step applied at instant .*

The index response , according to Proposition 3.26, is related to the impulsive response by the relationship:

— or

This result is intuitive: if a linear system at signal application Æ responde con un'uscita , all'applicazione del segnale Æ —Æ risponde con un'uscita — , essendo la derivata un operatore lineare.

First we want to characterize the behavior of the index response in . Since the unit step is a discontinuous signal in , one might wonder whether the response to that signal also exhibits a discontinuity. The following result generally applies.

Proposition 6.28 *Consider a system whose transfer function is worth*

---

e sia il guadagno alle alte frequenze.

If the system is strictly proper ( ) the index response is a continuous function in where it is worth .

Se il sistema è proprio ma non strettamente ( ) la risposta indiciale è una funzione discontinua in dove vale e .

Demonstration. The value of the index response for is certainly zero (causal system). On the other hand, the value of the index response at can be easily calculated by means of the initial value theorem. The transform of the index response
vale infatti: – e in base al teorema del valore iniziale

if

— *if*

The following result characterizes the structure of the index response.

Proposition 6.29 *The index response can be decomposed as follows.*

where it is a linear combination of the modes of the system, while it is a particular integral. The latter term if it is not a pole of the transfer function is worth

Æ *with* (6.30)

Whereas, assuming that the transfer function of the system has a pole of multiplicity , is worth

— Æ *with* (6.31)

Demonstration. The result follows immediately from Proposition 6.22, bearing in mind that a unit step is a particular exponential function in the form (6.29) of parameter .

This result takes a particularly important form if the system admits a permanent regime.

Proposition 6.30 *Consider a system whose transfer function*

has all poles except real less than zero.

In this case, the index response has a permanent regime term that is worth

$$— ÆÆ \quad (6.32)$$

Where is Bode's gain.

Demonstration. If the transfer function has all poles at real part less than zero, the general integral of the homogeneous is a transitional term while the permanent regime term coincides with the particular integral. Since the transfer function cannot have poles in the origin it holds and therefore the known term of the polynomial in the denominator holds . According to (6.30) it therefore holds.

Æ with

—

This result has an important physical interpretation. *Consider a system that has all stable modes; if you excite the system by a constant signal of unit amplitude after a transient period, the output also tends to be a constant signal of amplitude .*

Let us now look at two typical cases related to simple first-order and second-order systems.

First-order system

Consider a strictly proper first-order system. The transfer function of such a system is characterized by a single real pole . According to the Bode representation it has expression

where is the Bode gain and is the time constant associated with the one pole *(* which we assume is negative).

Observe that in zero-pole form the transfer function has representation where the gain at high frequencies is related to the Bode gain by the relation So the transform of the index response is worth

where

e dunque vale

Æ

Such a signal, as expected, contains a transient term characterized by the mode and a regime term that is worth . The mode coefficient is worth and this causes the signal to be continuous in where it holds ....The trend of that function is shown in Fig. 6.2.a, where the time scale is normalized as a function of the ....time constantNote that the figure is for the case where the
pole is negative and therefore the corresponding mode is stable: this implies the existence of a permanent regime.

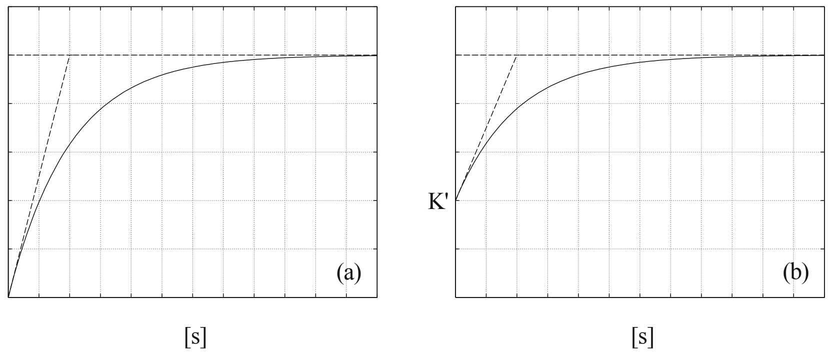


Fig. 6.2. Index response of a first-order system with stable mode: (a) strictly proper system; (b) proper but not strictly proper system.

The function tends monotonically to the steady state value, and the rate at which this value is reached obviously depends on the time constant associated with the pole . In particular, as seen in Chapter 3 (see § 3.4.1) after a *time*
il termine vale 0.95 e dunque la risposta indiciale raggiunge il 95% del regime value: this time value is called *settling time*.

More generally, we define the settling time at , which is denoted , as the instant of time from which the response reaches the del regime value. Because of what was seen in § 3.4.1, the settling time at the vale and the one at *' vale*

In the case where the system is proper but not strictly, the transfer function will also be characterized by a real zero and holds:

dove – è il parametro associato allo zero.

Con un ragionamento analogo al precedente si dimostra (cfr. Esercizio 6.11) che vale

Æ (6.33)

and thus that function has a discontinuity of amplitude in as shown in Fig. 6.2.b, also relating to a system with a pole with a negative real part.

Finally, note that for simplicity, Fig. 6.2 is for the case of systems in which In general, however, these coefficients can independently take any real value (positive or negative).

Second-order system

La funzione di trasferimento di un sistema del secondo ordine ha molti parametri e sono tanti i possibili casi da considerare. Ci si limita qui a considerare il caso del *sistema elementare del secondo ordine*, ovvero di un sistema strettamente proprio, senza zeri e caratterizzato da una coppia di poli complessi coniugati . La funzione di trasferimento di tale sistema ha la seguente rappresentazione di Bode

Where the natural pulsation holds and the damping coefficient holds (see § 6.4.3).

In tal caso, il sistema ha due modi pseudoperiodici

e

Because of what was said above, the expression of the index response will be co-composed of a constant value term plus a linear combination of the modes. It is easily demonstrated (see Exercise 6.14) that it takes the form

Æ

(6.34)

or the equivalent form

Æ (6.35)

The general form that the evolution of such a system takes is shown in Fig. 6.3 (for *)* . The time scale is normalized according to the inverse of the natural pulsation *(* which has precisely the size of the inverse of a time). The di- verse curves, on the other hand, are parameterized according to the value assumed by the damping coefficient, which is always assumed to be within the range....................Indeed:

Se il modo pseudoperiodico non sarebbe stabile e dunque non esisterebbe il regime permanente. Si noti, tuttavia, che anche in questo caso l'espressione della risposta impulsiva sarebbe data dalle eq. (6.34) e (6.35).
Se la funzione di trasferimento non sarebbe più caratterizzata da una coppia di poli complessi coniugati ma da due poli reali. In tal caso l'espressione della risposta impulsiva avrebbe una forma diversa (cfr. Esercizio 6.15 e 6.16).

Finally observe that regardless of the value of all evolutions tend to steady state value .

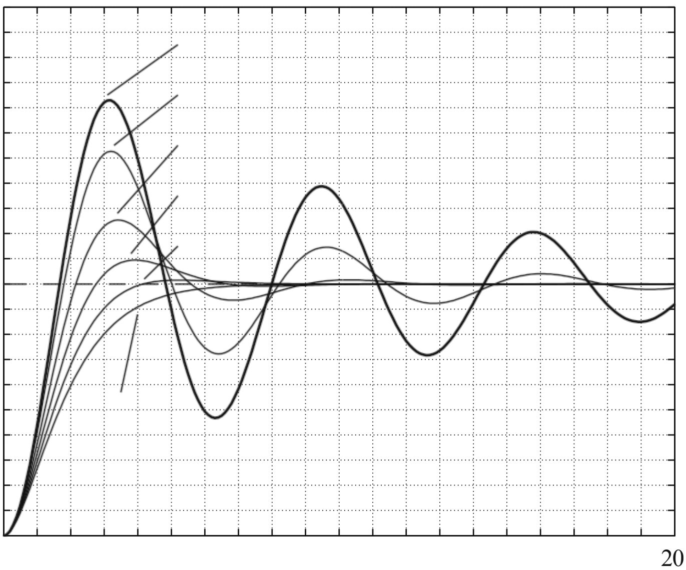


Fig. 6.3. Index response of a second-order elementary system as a function of various values of the damping coefficient.

We have seen that the transient of a first-order system always has the same form in the sense that the function tends monotonically to the steady state value and can be characterized according to a single parameter: the time constant. Conversely, the transient of a second-order system can take different forms and can be characterized according to different parameters. The most significant of these parameters are shown graphically in Fig. 6.4 (which refers to the case) and are briefly described below.

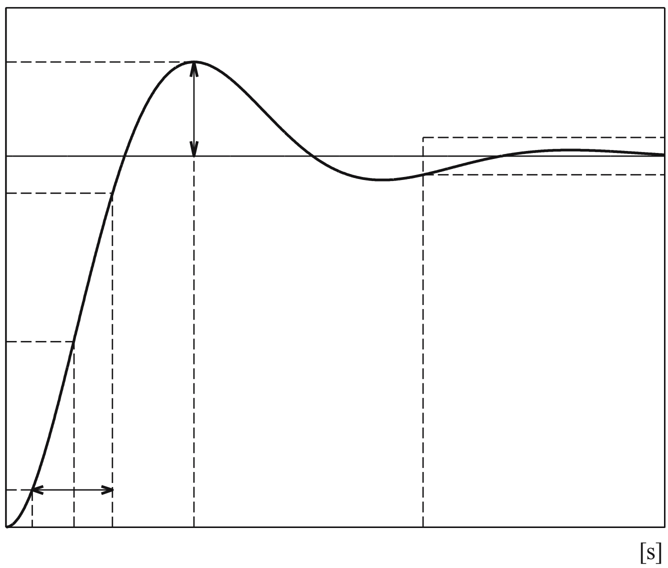


Fig. 6.4. Parameters characterizing the transient of the index response of a second-order elementary system.

Maximum value . It corresponds to the value taken by the in corri- spondence of the first maximum.

Maximum overshoot. It indicates the difference between the maximum value and the steady state value Usually, however, this parameter is expressed in percent relative to the regime value, viz.

.

Time of maximum overshoot . Indicates the instant of time at which maximum overshoot occurs.

Settling time .... It denotes the instant of time from which the response does not deviate from the steady state value ofMore ....generally, it is denoted by il tempo di assestamento al , cioè l'istante di tempo a partire dal quale la risposta non si discosta dal valore di regime di .

Tempo di ritardo . Indica il tempo necessario affinché la risposta raggiunga il 50% del valore di regime.

Tempo di salita . Indica il tempo necessario affinché la risposta passi dal 10% al 90% del valore di regime.

In the case of the second-order elementary system, it is quite easy to tie the value assumed by these parameters to the value of the damping coefficient and the natural pulsation.

Derive the expression (6.34) in order to determine the points of maximum and minimum.

Place e to simplify the notation, applies:

and this derivative cancels for

This implies and finally we derive that the maximum and minimum points of the index response are reached at the instants

at which the index response is worth

Maximum overshoot occurs for , i.e., it is worth

(6.36)

while the corresponding maximum value is worth

(6.37)

and the overshoot is worth

e (6.38)

Note, finally, that the maxima and minima of the index response lie on the due curve e , come mostrato in Fig. 6.5. Ciò consente To approximate *by* excess the exact value of the settling time to the determinando l'istante di tempo nel quale le curve dei massimi e dei minimi enter the band . This occurs when , and it applies

(6.39)

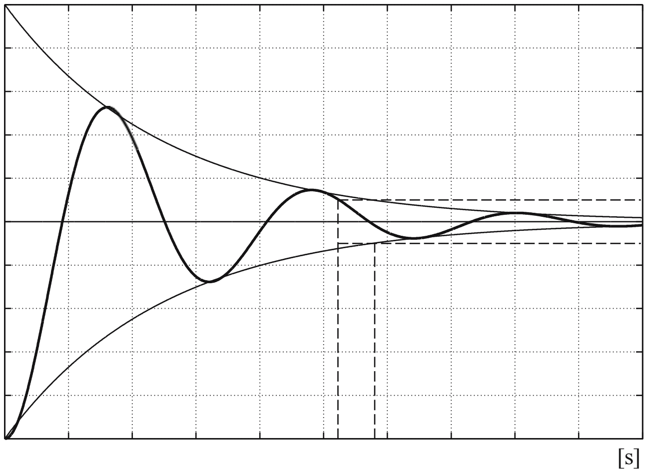


Fig. 6.5. Maxims and minima of the index response of a second-order elementary system.

Exercises

Exercise 6.1 A system described by the input-output model is given.

Determine, through the use of the Laplace transform:

(a) Free evolution from initial conditions

(b) The forced response that follows the application of the input signal

se
elsewhere

Check whether the result in (b) coincides with that obtained in Example 3.25.

Exercise 6.2 Consider the matrix

Determine the solving matrix and, by antitransforming, the matrix (see also Example 4.10).

Exercise 6.3 A system described by the model in state variables is given:

(a) Determine the transfer function.
(b) Determine, through the use of the Laplace transform, the evolution of the state and output resulting from the application of an input signal
Æ a partire da uno stato iniziale (cf. also Example 4.12).
(c) Determine an input-output model of such a system.

Exercise 6.4 For the MIMO system studied in Example 6.13, determine the evolu- output
enforced resulting from the application of an input signal Æ Æ .

Determine whether there is a permanent regime, and if so, decompose each component of the output into transitional term and regime term.

Exercise 6.5 Verify that the transfer matrices of the two similar representations studied in Example 4.15 coincide, as Proposition 6.14 implies.

Exercise 6.6 Given the system described by the UI model.

determine its transfer function and, by antitransforming, its impulse response (see Example 3.22).

Exercise 6.7 Given the transfer function

if we first determine its zero-pole representation and then its Bode representation, indicating the parameters that characterize the two representations.

Evaluate for which parameter values the given function is in non-minimum form: if so, determine the corresponding minimum form.

Exercise 6.8 Given the system described by the UI model.

Whether it determines the forced response that follows the application of an input

Æ Determine whether there is a permanent regime, and if so, ca-

so, decompose the output into transient term and steady state term, plotting its graph.

Exercise 6.9 For the system in the previous exercise, determine the total response that follows the application of input Æ from initial conditions , identifying the free and forced evolution. Determine whether a permanent regime exists, and if so, decompose the total output into transient term and regime term, plotting the graph.

Exercise 6.10 Consider a system subject to an exponential ramp-shaped input

— Æ

withShow that the forced response to such an input can be decomposed In the sum of two terms:

where is a linear combination of the modes of the system and is a linear combination of the modes associated with the parameter introduced by the input signal.

In particular, verify that if it does not coincide with any of the poles of the system transfer function, it is worth

— Æ

whereas if it coincides with a multiplicity pole of the transfer function, it is worth

— Æ

Exercise 6.11 Prove that the index response of a first-order proper (but not strictly proper) system takes the form given in eq. (6.33).

Exercise 6.12 Determine the index response of the system whose transfer function is worth

Exercise 6.13 Determine the index response of the system whose transfer function is worth

Plotting the trend of this response as a function of time, graphically evaluate the following parameters that characterize its transient: maximum value, maximum overshoot, maximum overshoot time, delay time, rise time and settling time. Check whether the value determined for the first three parameters coincides with the value that can be determined analytically using equa- tions (6.36), (6.37) and (6.38). Finally, consider whether the value determined in eq. (6.39) is a good approximation of the settling time.

Exercise 6.14 Prove that the index response of a second-order system with a pair of complex conjugate poles and no zeros takes the form given in Eq. (6.34) or the equivalent form given in Eq. (6.35).

Exercise 6.15 Prove that the index response of a second-order system with a pair of coincident real poles and no zeros characterized by the transfer function

takes the form

Exercise 6.16 Prove that the index response of a system of the second ordi- ne with a pair of distinct real poles and without zeros characterized by the transfer function

takes the form

Exercise 6.17 Prove that the generic polynomial of the second can always be written in the form as long as the roots be complex and conjugate or real but of the same sign.

In particular, prove that the roots of the polynomial have the following expression as a function of the parameters and .

If the roots are complex and conjugate of *value*

.

Se le radici sono reali e coincidenti di valore .

If the roots are real and distinct of *value*

e In this case the roots and both have opposite sign to that of .

7

Model building in state variables and analysis of interconnected systems

Two different topics are discussed in this chapter. The first section studies the problem of system *realization*, that is, of determining a model in state variables from a known input-output model. The name "realization" is a reminder that this approach was initially proposed to enable the co-instruction of a physical device, usually an electrical circuit, that allows one to si- mulate the behavior of the given system. To this end, it will be shown how the model in VS determined can directly be translated into a circuit diagram.

The topic addressed in the second section consists of the study of *interconnected systems*, i.e., consisting of several elementary components linked together. It is so- lished to represent each individual component by means of a SISO block characterized by its transfer function. The overall system will in all generality be a MIMO system of which it is possible, by means of a block algebra, to determine the transfer matrix and study the forced response. A system consisting of several interconnected components can be represented by means of a gra- fic diagram that generalizes the circuit diagram already seen in the study of the realization problem.

7.1 Implementation of systems SISO

7.1.1 Introduction

In Chapter 6 (see § 6.3.8) we discussed how it is possible to determine an input-output model of a system for which a model in state variables is known. In fact, having determined the transfer function (or the transfer matrix in the case of a MIMO system), it is immediate to derive the UI model by antitransformation.

In this section we study the inverse problem of determining a VS model of a system of which a UI model is known. The VS model obtained in this way is also called *a* system *realization*: in fact, it can be represented by means of a circuit diagram that allows direct implementation via a hardware device. In particular, this approach has been used in calculators

analog electronic analyzers (DDAs: digital differential analyzers), which in the 1950s-60s enjoyed considerable success before being finally supplanted by digital electronic calculators.

It should be noted that the problem of switching from a UI model to a VS model does not admit of a single solution: in fact, it has already been seen in Chapter 4 that several realizations in VS terms, i.e., several sets of matrices , , etc., related to each other by a similarity relation, may correspond to the same system. Here a general technique will be presented that, starting from a given UI model, makes it possible to determine, among many possible ones, a particular model *in* VS called *in canonical control form* (see Appendix D). For simplicity, only SISO systems are considered. The realization of MIMO systems is considerably more complicated, especially if the model is required to have minimum order, and is not addressed in this text.

The UI model of a linear, stationary SISO system of order can be described by a differential equation of the form:

(7.1)

Where (causal system). We will denote the first member of (7.1), while denoting its second member.

From such a model, we want to find a VS realization of the form

(7.2)

Where the *state vector* and its derivative have components,

This model can be described in a more compact form by a matrix, called the *Realization matrix*, which takes the form

Various cases are distinguished depending on the value taken by (system order) and (maximum order of derivation of the input in the UI model).

7.1.2 Case

The case (and therefore) corresponds to an instantaneous system. For such systems, (7.1) reduces to the algebraic equation

The corresponding representation in VS is degenerate: since the system is not dinamic, its state cannot be defined. So the matrices , , are not defined, while it is valid .

Example 7.1 Consider a simple electrical circuit consisting of a single res- stance and whose behavior is described by Ohm's law: ....Place and the UI model is worth

—

This equation also represents a model in VS with .

7.1.3 Case *a n d*

Consider the case of a dynamical system ( ) in which, however, is worth . In that case, the second member of (7.1) reduces to *(* no derivatives of the input appear) and thus is worth :

(7.3)

In such a case, one can choose the so-called *phase space* as the state *space* of the system, that is, one can choose the output and its first derivatives as state variables:

. .

The following *equation of state* corresponds to this choice:

where the first equations are derived from the choice of phase space and the last one is derived from (7.3). In vector form, the previous equations become

where it is worth

(7.4)

The particular form of the state matrix is called the *companion form*, and the rappre- sentation takes on a structure called *the canonical control form*[1] , as discussed in Appendix D.

The *output transformation* is derived from the definition of the first state variable and holds simply:

which in vector form is written

[1]To be precise, this representation is in canonical control form minus a constant because the non-zero term in the matrix is not necessarily unitary.

The matrix of this realization therefore holds:

(7.5)

Example 7.2 Consider a SISO system whose input-output bond is described by the differential equation

In this case it holds and and since no de- rivates of the input appear in the second member, following the procedure outlined above we can immediately give the following representation in terms of state variables:

Example 7.3 Consider a first-order SISO system whose input-output bond is described by the differential equation

In this case, e holds and and the system has the following representation in terms of state variables:

Note that in this case the state matrix is a scalar ....

(a) (b)

Fig. 7.1. Elementary components of the circuit diagram: **(a)** multiplier; **(b)** integrator

Mediated representation of circuit diagrams

The realization described by (7.5) can well be simulated by means of a *circuit diagram*, containing the two types of components shown in Fig. 7.1. A *multiplier* is characterized by a scalar and has at its output a signal that is equal to the product of the input se- gnal by . An *integrator* has at its output a signal that is equal to the integral of the input signal.

To connect these components together it will also be necessary to use branch lines and summing nodes, as shown in Fig. 7.2. A branch allows the same signal to arrive at multiple points: the branch in the figure has at its input the signal and at its output three signals , and all equal to the input signal. The summing node allows multiple signals to be algebraically summed. At the input to that node there are as many segments as there are addends, for each of which the sign with which they appear in the algebraic sum is specified; at the output, however, there is a single segment with which the result of the algebraic sum is associated. If all the ad- dends of the sum have positive signs for simplicity, the sign of the input signals is omitted, representing an inside node (see Fig. 7.3).

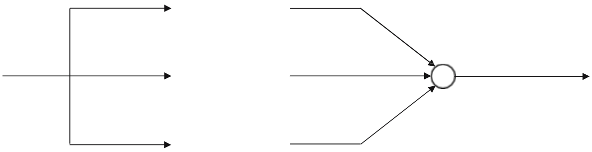


Fig. 7.2. Branch point (left) and summing node (right).

In particular, the circuit diagram that corresponds to realization (7.5) is mo- layer in Fig. 7.3. Each variable , for , corresponds to the output of an integrator that has as input , while the variable corresponds to the output of an integrator that has as input

Such a scheme can be directly used to construct a device in which individual blocks are implemented by means of operational amplifiers or even

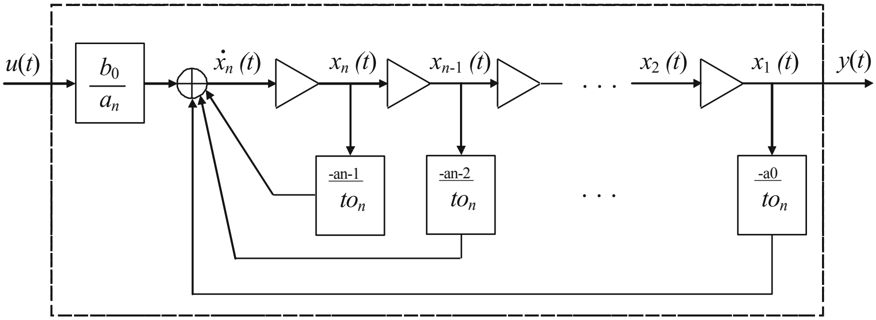


Fig. 7.3. Circuit diagram corresponding to the realization

a computational program (cf. the *Simulink* program in the MATLAB package) capace to determine the evolution of the state and output that result from a given input and given initial conditions.

Example 7.4 The circuit diagram of the realization determined in Example 7.2 is shown in Fig. 7.4

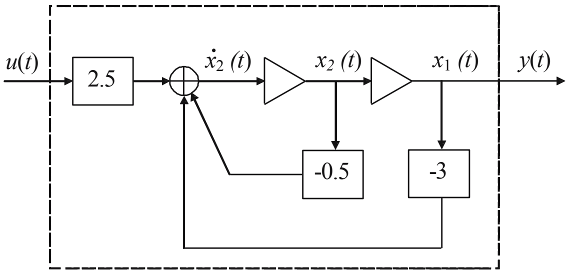


Fig. 7.4. Circuit diagram of the representation in Example 7.2

7.1.4 Case

Consider now the general case in which , that is, the case in which it contains derivatives of the input. In that case the choice as the state space of the phase space does not lead to an admissible representation, as the following example shows.

Example 7.5 Consider a SISO system whose input-output bond is described by the differential equation

In this case, e applies, and if one were to pose e, the derivatives of these variables would be worth

So the equation of state would become.

which is not in the standard form provided by eq. (7.2) because of the term..........

In order to obtain a representation in VS even in the case where it is worth , an auxiliary quantity , defined according to the following proposition, is introduced.

Proposition 7.6 *If two signals and are related by the relation* (7.1) *then there exists a signal that satisfies the two equations:*

(7.6)

e

(7.7)

Demonstration. The demonstration is unintuitive but is given for completeness. We substitute in the first member *o f* (7.1) the relation (7.7) and its derivatives, and substitute in the second member of (7.1) the relation (7.6) and its derivatives. If through these substitutions an identity is obtained, the result is proved.

With these substitutions, the first member of (7.1) becomes:

.

and grouping according to the terms becomes:

.

Substituting in the second member of (7.1) gives instead:

.

and is therefore worth .

Equations (7.6) and (7.7) can be used immediately to determine a VS representation: in particular, the first of these equations is used to determine the equation of state, while the second allows the output transformation to be determined.

Since the variable satisfies Eq. (7.6), based on what was seen in the previous section, it can be chosen as the state space :[2]

. .

This choice corresponds to the same equation of state already seen in the previous case, the only difference being that the term in (7.6) is equal to 1, i.e.

. .

This representation is also in canonical control form (barring a constant).

The output transformation is now derived from (7.7) and takes two different forms depending on whether it holds (strictly proper system) or (not strictly proper system).

Case

If , then and (7.7) gives us:

[2]This choice of state variables, unlike phase variables, has no immediate physical meaning.

The matrix of this realization therefore holds:

(7.8)

The circuit diagram corresponding to this realization is shown in Fig. 7.5. Note that in this case the output signal is a linear combination of the state variables, each multiplied by an appropriate coefficient .

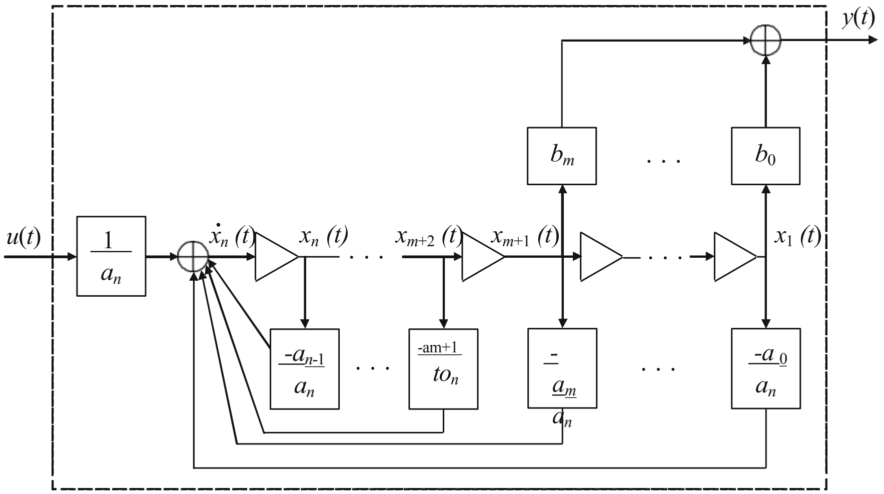


Fig. 7.5. Circuit diagram corresponding to the strictly own realization

Example 7.7 Consider a SISO system whose input-output bond is described by the differential equation

...

In this case, e holds, and following the procedure outlined above we can immediately give the following representation in terms of state variables:

The circuit diagram of this particular representation is shown in Fig. 7.6.

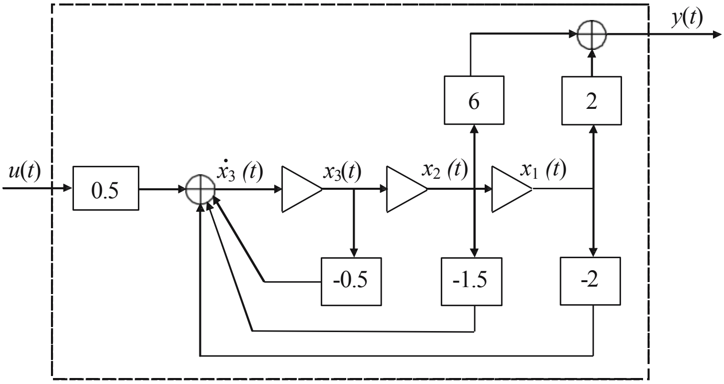


Fig. 7.6. Circuit diagram of the representation in Example 7.7

Cas_e_

If , on the other hand, (7.7) gives us:

—— —— —— ——

—— ——

—— ——

The matrix of this realization therefore holds:

(7.9)

The schematic corresponding to this realization is shown in Fig. 7.7. Observe that in that diagram the generic state component contributes to the output through two different direct paths (i.e., which do not traverse an integrator): a path that traverses the block and a path that first traverses the block and then the block . The total contribution relative to the component is therefore worth , as required. Finally, note that the input makes a direct contribution to the output through the path that first crosses the block and then the block .

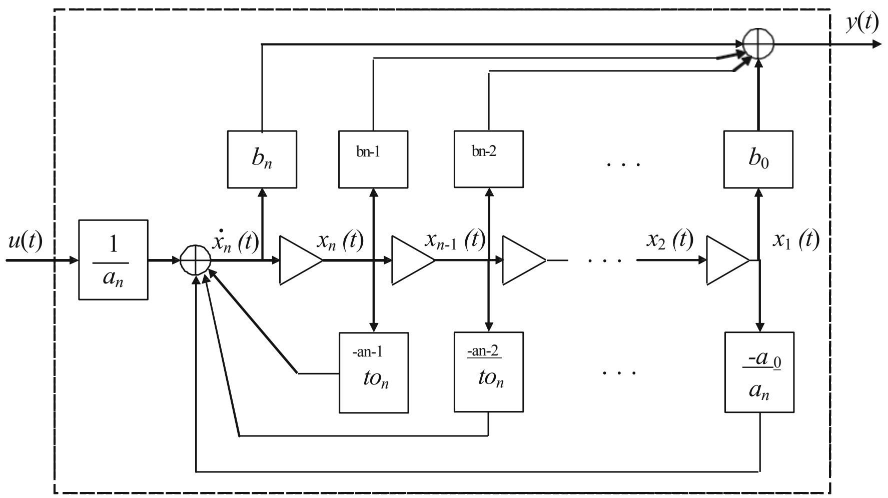


Fig. 7.7. Circuit diagram corresponding to the realization not strictly one's own

The realization can be regarded as a special case of the realization in which the coefficients are all zero. However, it was preferred to present the two cases separately for clarity.

Example 7.8 Consider a SISO system whose input-output bond is described by the differential equation

... ...

In this case it holds and following the procedure set forth above we can immediately give the following representation in terms of state variables:

The circuit diagram of this representation is shown in Fig. 7.8.

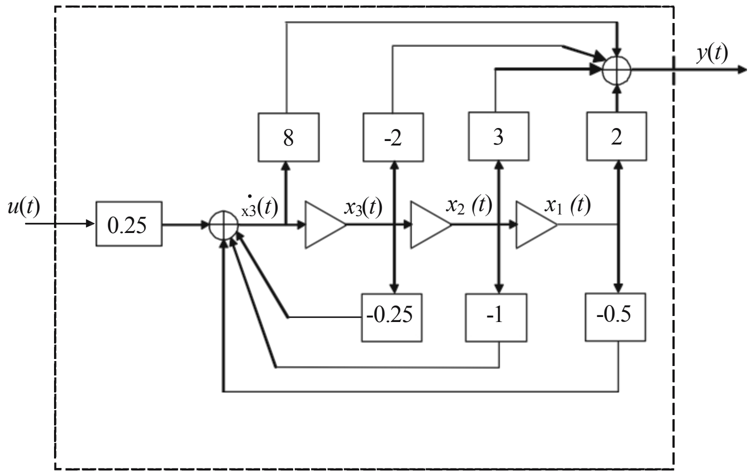


Fig. 7.8. Circuit diagram of the representation in Example 7.8

7.1.5 Transition from a set of initial conditions on the output to an initial state

It may happen that, in the transition from a UI model to a VS representation, the initial conditions of the output and its derivatives are known

—— ———— (7.10)

And it is necessary to determine the initial state that corresponds to them.

If the state space coincides with the phase space, this problem has an immediate solution because by definition the following holds true

. .

In the most general case, it is possible to determine for the VS representation an equivalent initial state according to the following proposition.

Proposition 7.9 *Consider a SISO system described by a VS representation of order*

(7.11)

where for simplicity the input was assumed to be zero. Given the initial conditions in eq. (7.10) *that can be represented by the vector:*

.

you want to determine the value of the initial state that corresponds to them.

If the matrix

.

is invertible, it is worth

Demonstration. According to (7.11), the following is true.

. . . .

Tale equazione vale per ogni e in particolare per . Il vettore incognito cercato può dunque essere determinato risolvendo il sistema lineare

as long as the matrix is invertible.

The matrix that is shown as is the well-known *observability matrix* that will be better studied in Chapter 11. Note that it is possible to apply the procedure described in the previous proposition if and only if that matrix is invertible which is equivalent to saying, for a SISO system, if and only if the representation is observable.

Esempio 7.10 Si consideri il sistema studiato nell'Esempio 7.7 e si supponga che le condizioni iniziali in dell'uscita e delle sue derivate valgano , , .

So remembering that for the representation determined in Example 7.7, the following holds true

you can write the system

Since the coefficient matrix is nonsingular, the system can be solved and has solution

7.2 Study of interconnected systems

Previous chapters have always considered stand-alone systems, neglecting the interactions they may present with other systems. In this section we

Instead, it will consider the case where a system is composed of several interconnected sub-systems, called *components*. Each component will be represented by a linear, stationary SISO input-output model: it can therefore be characterized by its transfer function (or in fully equi- valent terms, by a differential equation or by its impulsive response). Although each individual component is a SISO system, the overall system may have more than one input and one output. We will see how the transfer matrix of the overall system can be determined based on knowledge of the individual components and their interconnections.

In what follows, we will make the simplifying assumption that the connection between the different com- ponents does not affect their behavior and that each component therefore com- ponents as if it were isolated. This assumption is not always verified; compare Exercise 7.7 in this regard.

Each component will be represented according to the convention of block diagrams, that is, as a rectangular block with an input-oriented segment and an output-oriented segment, where the first segment represents the input to the system and the second the output. Finally, within the rectangular block will be specified the transfer function that binds the output to the input, as shown in Fig. 7.9.

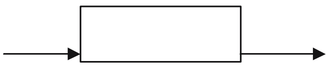


Fig. 7.9. Representation of an input-output system by block diagram

The representation by interconnected components constitutes a genera- lization of the circuit diagrams seen in the previous section. In fact, it is easy to di- show that the components used to make such diagrams, namely the multiplier and the integrator, are special cases of the blocks considered in this section.

Example 7.11 The link between the input and output of a multiplier *is worth* . In the domain of the Laplace variable it holds and therefore this component can be represented by a block with transfer function

The bond between the input and output of an integrator holds. In domino of the Laplace variable holds (see Chapter 5, § 5.2.4) and therefore this component can be represented by a block with transfer function

7.2.1 Links elementary

There are three main elementary connections: *series*, *parallel* and *feedback*.

Series

The *series* connection is shown in Fig. 7.10: In this case the output of one component coincides with the input of the next component, so the input of the overall system coincides with the input of the first component and its output coincides with the output of the last component.

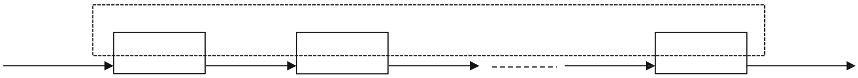


Fig. 7.10 Series connection

In such a scheme, it is usual to denote the transfer function of the single block The transfer function of the overall system holds:

In fact, with simple steps it is derived:

.

Parallel

The parallel connection is shown in Fig. 7.11: the input is the same for all components and coincides with the input of the overall system; the overall output, on the other hand, is the sum of the outputs of the individual components.

Denoting the transfer function of the single block , the transfer function of the overall system is worth

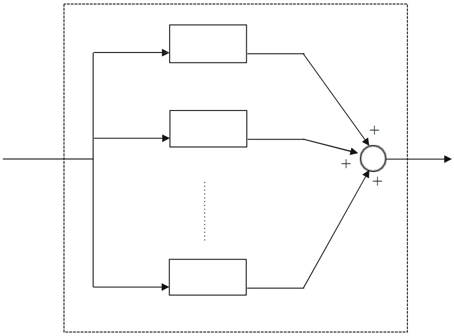


Fig. 7.11 Parallel connection

Indee
d

.

....

Counteraction

The connection in *feedback*, or more precisely *negative feedback*, which is particularly important in solving control problems, is shown in Fig. 7.12. In that diagram we recognize two blocks: the *direct chain* whose transfer function is denoted and the *feedback block* with a transfe- rection functionThe external ....output of the overall system coincides with the output of the direct chain, and in turn constitutes the input of the feedback block. The input of the direct chain consists of the difference between the external input
And the output of the feedback block.
In this case, the transfer function of the overall system is worth

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In fact, in the domain of the Laplace variable it holds:

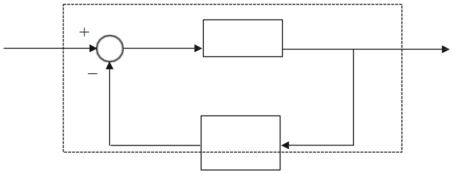


Fig. 7.12 Backreaction connection

The function is called the *closed-loop* transfer function while the

is called *the open-loop* transfer function. The latter name is justified by noting that if the block diagram is opened at the signal , it is just equal to the transfer function between the input and the output .

It is also important to point out that if in place of the negative sign in the summing block we have an additional positive sign, the connection would be said to be in *positive feedback*. By repeating similar reasoning to that just seen, it is easy to show that in this case the following would be true

Often we simply speak of feedback, without specifying whether this is negati- ving or positive. By convention in this case it is implied that the feedback is negative. Sometimes, finally, the English term *feedback* is also used to denote counterreaction. The study of feedback systems is a fundamental topic in automatic: Chapter 10 is devoted to the analysis of such systems.

Application of the rules of elementary connections

The following example shows how the results seen for elementary connections can be applied to the study of more complex systems.

Example 7.12 You want to calculate the transfer function of the system in Fig. 7.13.a characterized by the transfer functions of the individual blocks, where

Applying the parallel connection rule yields the equivalent model in Fig. 7.13.b where

and finally applying the series connection rule yields the equiva- lent model in Fig. 7.13.c, where

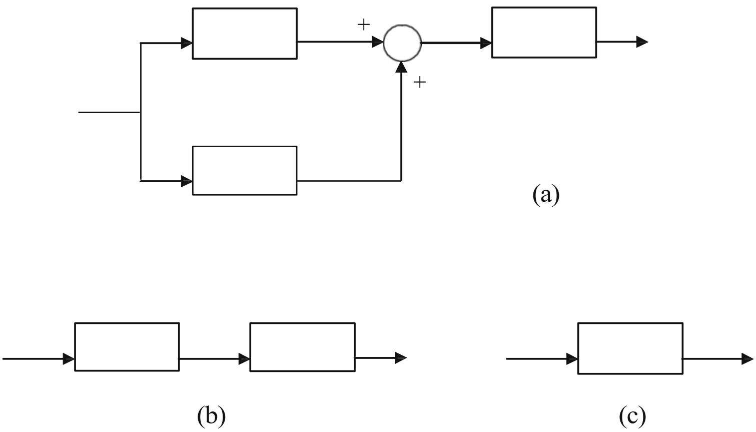


Fig. 7.13. System in Example 7.12

7.2.2 Algebra of patterns at blocks

The components of a block diagram can combine according to much more intricate linkages than those described above. In this case, it is possible to calculate the transfer function of the overall system by applying the simple rules *of block diagram algebra*. These rules are based on determining *block diagrams equivalent* to the starting one, that is, block diagrams that have the same input and output quantities, but different intermediate quantities.

These rules are summarized schematically in Figs. 7.14 and 7.15. In particular, Fig. 7.14 shows the various possible cases that originate from the displacement of a multiplier block with respect to another multiplier (case a), with respect to a summator (cases b-c) and with respect to a branch point (d-e). The most representative

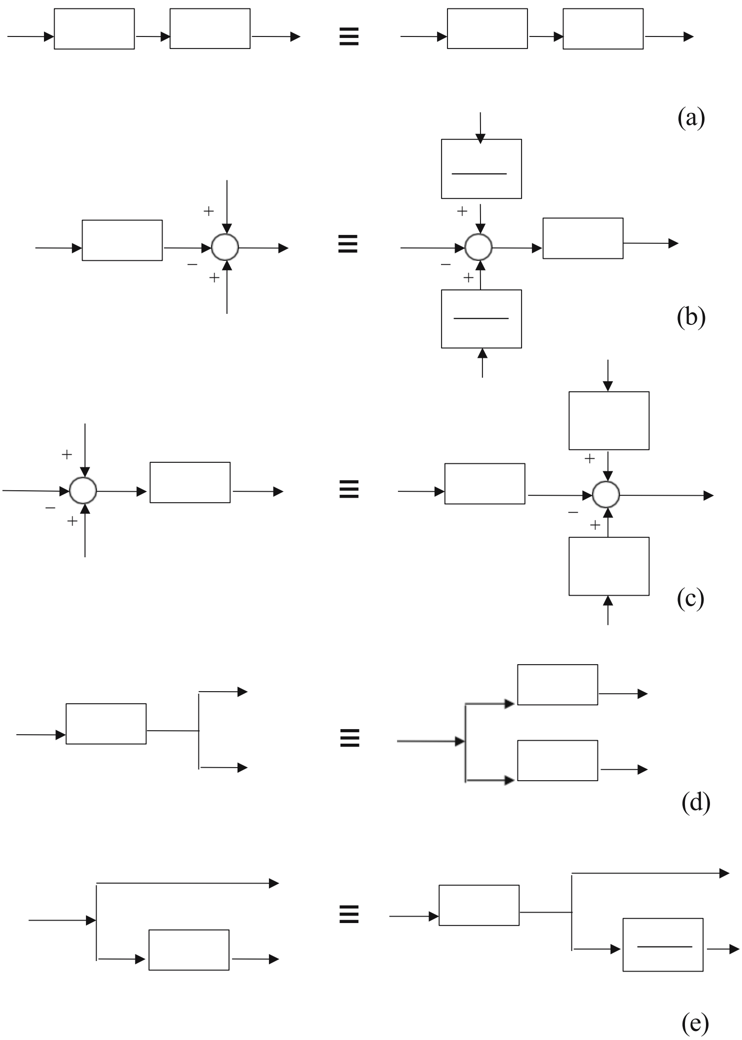


Fig. 7.14. Equivalent block diagrams originating from the displacement of a multiplier block

Possible cases that originate instead from the displacement of a summing node are shown in Fig. 7.15.

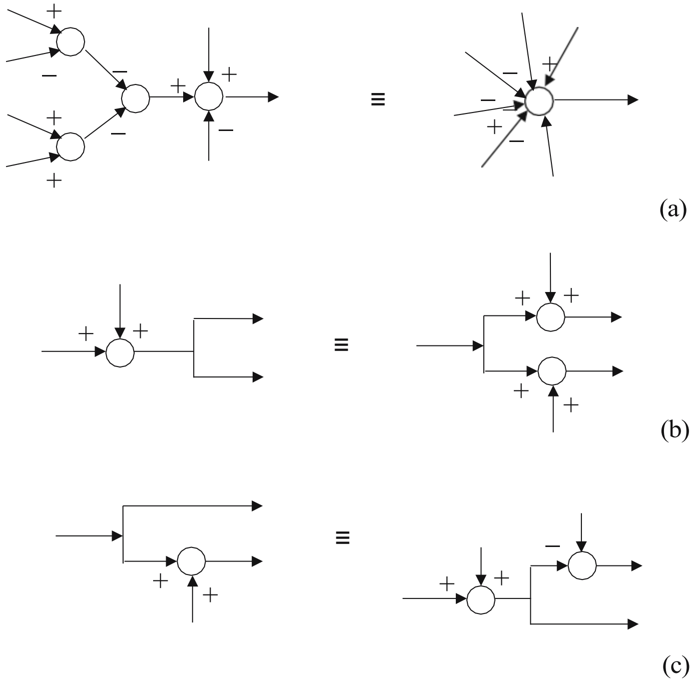


Fig. 7.15. Equivalent block diagrams originating from the displacement of a summing node

By subsequently constructing an appropriate set of equivalent block diagrams that are respectful of the rules schematically summarized in Figs. 7.14 and 7.15, and using the elementary link simplification rules, it is easy to deter- minate the transfer function of the overall system from those of the components.

Example 7.13 Consider the SISO system shown in Fig. 7.16.a. Applying the equivalence rules above, it is easy to construct the equiva- lent block diagrams shown in the same Fig. 7.16 and then determine the transfer function between the input and output shown in Fig. 7.16.d. With simple algebraic steps, this function can also be put in the simplest form

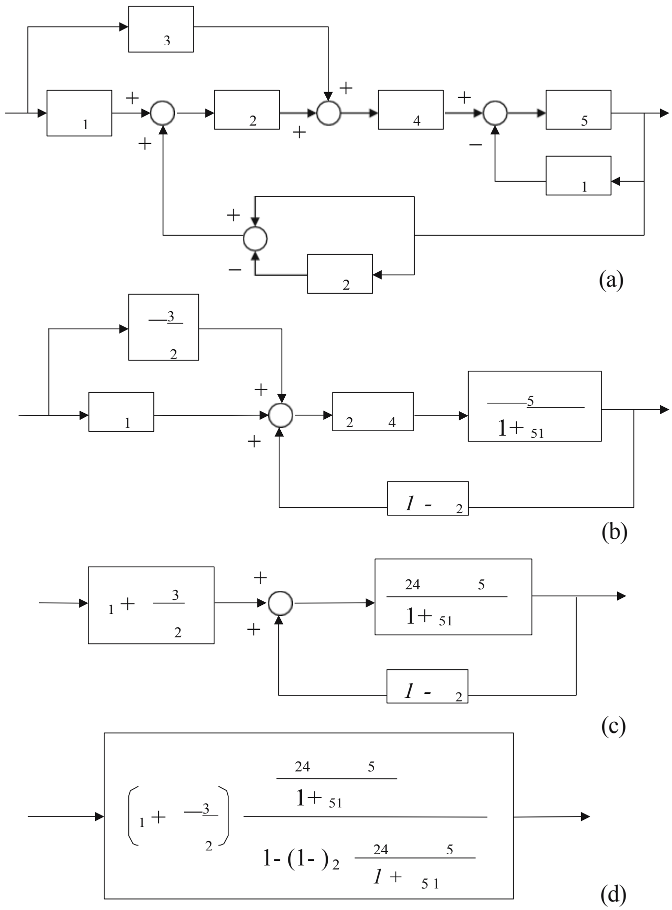


Fig. 7.16. Schemi a blocchi equivalenti nell'Esempio 7.13

7.2.3 Determination of the transfer matrix for MIMO systems

In all the cases studied in the previous sections, the overall system is an SI- SO system. Through the same techniques it is possible to determine the transfer matrix of arbitrary MIMO systems.

Recall (see § 6.3.6) that the transfer matrix of a MIMO system with inputs and outputs has dimensions and satisfies the equation:

The individual transfer function between the input a n d output can be determined by assuming that all inputs for are zero: in this case we fall back to the SISO case. Once the individual components of the transfer matrix have been determined, the value of the outputs can be determined taking into account the contribution of all inputs.

A following example will clarify how this can be done.

Example 7.14 Consider the MIMO system with two inputs and one output in Fig. 7.17. You want to determine its transfer matrix , knowing that

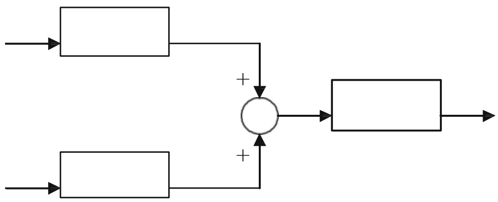


Fig. 7.17. System in Example 7.14

In this case, e applies ; ; therefore, the transfer matrix has dimension and can be written

where is the transfer function between the first input and the output, while is the transfer function between the second input and the output.

Denoting by and with the Laplace transform of the output and input vector, it is worth

Posed , the scheme of the system reduces to the SISO scheme in Fig. 7.18.a and thus

Conversely, posed , the system scheme reduces to the SISO scheme in Fig. 7.18.b and thus

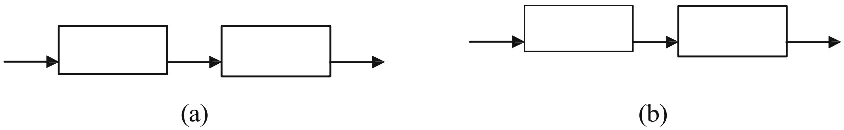


Fig. 7.18. Diagrams for the calculation of individual transfer matrices in Example 7.14

Once the transfer matrix has been determined, it is also possible to determine the forced response that follows an assigned input using Laplace transforms.

Esempio 7.15 Per il sistema studiato nell'Esempio 7.14 si desidera determinare l'uscita forzata che consegue all'applicazione dell'ingresso
whose individual components are shown Fig. 7.19.

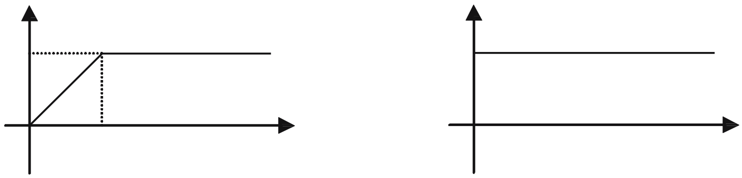


Fig. 7.19. System in Example 7.15

The input is the sum of two linear ramps, one of slope applied in and one of slope applied in (the second ramp is translated to the right by) . Therefore, it is valid:

Æ Æ

The input is the unit step and is worth

Æ

The transfer matrix was determined in Example 7.14 and is worth

So the transform of the forced response is worth and anti-transforming

Æ Æ Æ Æ

Æ Æ

Exercises

Exercise 7.1 The mass-spring system studied in Chapter 2 (see Example 2.14) is described by the input-output model:

Determine a VS representation of such a system and draw its corresponding circuit diagram. Check whether this representation coincides with the one given in Example 2.14.

Exercise 7.2 Determine a VS representation of the system described in E- semple 7.5 and draw its corresponding circuit diagram.

Exercise 7.3 Determine a VS representation and draw the corresponding circuit diagram for the following first-order systems.

(a) Supplement:
(b) Strictly own system:
(c) Not strictly your own system:

Exercise 7.4 Consider a similarity transformation in which the generic component of the new state vector is related to the component of the original state vector by the relation .

Determine the similarity matrix such that Prove that through such a transformation, the representation (7.5) can be taken back to the canonical control form given in Eq. (D.5) of Appendix D in which the nonzero coeffi- cient of the vector is worth 1. In particular, determine the matrices of the realization of the new representation as a function of the coefficients and of the input-output model.

Exercise 7.5 Consider a similarity transformation in which the generic component of the new state vector is related to the component of the original state vector by the relation .

Determine the similarity matrix such that Prove that through such a transformation, the representations (7.8) and (7.9) can be reconverted to the canonical control form given in Eq. (D.5) of Appendix D in which the nonzero coefficient of the vector is worth 1. In particular, determine the realization matrix of the two new representations as a function of the coefficients and

Of the input-output model.

Exercise 7.6 Determine the similarity matrix that allows you to go from the representation (7.8) to the representation determined by the MATLAB function `tf2ss.m`. Draw the circuit diagram of the representation determined by MATLAB and compare it with the diagram in Fig. 7.5.

Exercise 7.7 This exercise shows that *the load effect* in an electrical circuit violates a fundamental assumption that has been made for the study of interconnected systems: *the behavior of each interconnected component coincides with the behavior of the isolated component.*

The circuit in Fig. 7.20.a is a voltage divider consisting of two resistors .

(a) Prove that the transfer function between input voltage and output voltage of the divider is worth .
(b) Consider the circuit in Fig. 7.20.b given by the series of two partitions, and verify that its transfer function is worth

– –

(c) Explain this phenomenon by verifying that the input-output bond of the first partition in the series in Fig. 7.20.b is different from that of the stand-alone circuit in Fig. 7.20.a.

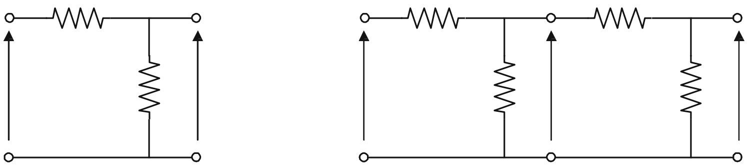


Fig. 7.20. Circuit in Exercise 7.7: (**a**) voltage divider; **(b)** series of two dividers.

Exercise 7.8 The system in Fig. 7.21 is characterized by the impulsive responses of individual blocks.

(a) Calculate the transfer functions of individual blocks.
(b) Calculate the transfer function of such a system.

Exercise 7.9 Determine the transfer matrix for the system in Fig. 7.22, characterized by the transfer functions of the individual blocks.

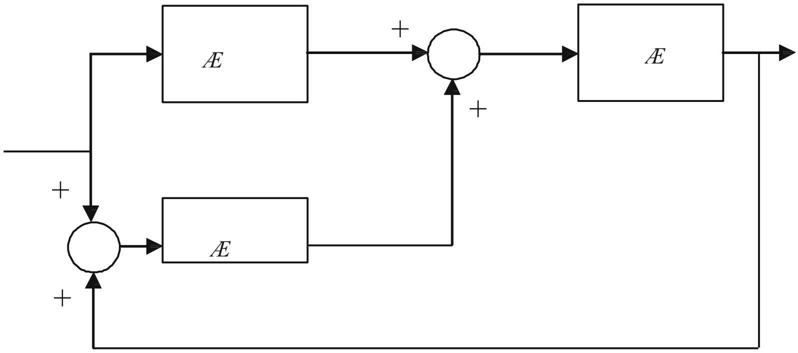


Fig. 7.21. System in Operation 7.8

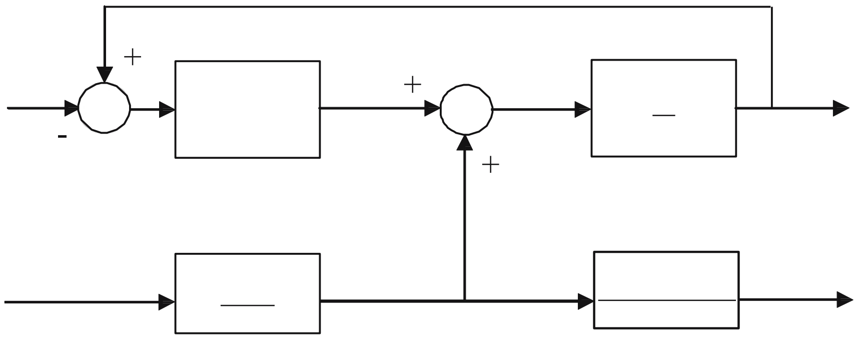


Fig. 7.22. System in Operation 7.9

8

Frequency domain analysis

This chapter will present the analysis in the frequency domain, which for linear and stationary systems is one of the most important and effective tools for studying certain properties, such as filtering properties. Analysis in that domain is based on the particular shape that the output of a linear, stationary stable system takes in response to a sinusoidal input signal. However, by virtue of the principle of superposition of effects, this type of analysis can be extended to much broader classes of input signals, that is, to all those signals that can be represented as a finite or infinite linear combination of sinusoidal components.

A formal definition of *harmonic* or *frequency response* will also be given, having in the case of systems with poles all with negative real part, a very precise physical meaning.

Within this chapter the *Bode diagram* will also be presented, which is certainly the most common way of graphically representing the harmonic response. Indeed, from such a diagram it is easy to read a number of characteristic parameters of the system under consideration. In particular, the units of measurement chosen for representing the modulus *(modulus Bode diagram*) and *phase* (*phase Bode diagram*) of the harmonic response , together with the semilogarithmic charts on which these diagrams are plotted, will first be introduced. Simple rules of thumb will also be introduced which, by exploiting the superposition principle, make it easy to plot Bode diagrams, both *asymptotic* and *exact*.

The chapter concludes with the introduction of some characteristic parameters of the harmonic response that can be read from Bode diagrams and a brief discussion of the different filtering actions that a linear, stationary system with poles all with negative real part can perform.

8.1 Response harmonic

This section will first examine the particular structure assumed by the output of a linear, stationary stable SISO system subject to a sinusoidal input signal. Based on this, the *frequency response* or *harmonic response* will also be defined.

8.1.1 Steady-state response to an input sinusoidal

Proposition 8.1 *Consider a linear, stationary SISO system having a transfer function with poles all with negative real part*[1] *. Assume that such a system is excited from the time instant by a sinusoidal-type signal having pulsation and modulus :*

Æ

Let and be, respectively, the modulus and phase of the transfer function evaluated in .

Under steady state conditions, the response of such a system is also a sinusoidal-type signal having the same pulsation as the input signal, modulus equal to the product, and whose phase shift with respect to the input signal is equal to , i.e.

Demonstration. Let us first observe that such a system admits regime. In- fact, as seen in § 6.1.1, the free evolution of a system is given by a linear combina- tion of its modes. In this case, all poles being by hypothesis negative real part, all modes of the system are extinguished by , so free evolution is extinguished by .

In addition, both

The Laplace transform of the ....input signalThe Laplace transform
Of the forced evolution is equal to

Therefore, the poles of the function are given by the poles of the transfection function plus those corresponding to the input signal, i.e., e

. In the antitransform of the the poles of the correspond to the transient term having negative real part; in contrast, the poles and corri- spond to the steady state term having zero real part. More precisely, the steady-state response holds

[1]Note that, as will be discussed in detail in Chapter 9, this condition implies a particular property of the system known as BIBO *stability*.

Where ed are the residuals corresponding to the poles and respectively, viz.

Therefore

Now, placed

also applies

to which

as it was meant to be.

Example 8.2 Consider a linear, stationary system whose transfer function is

And let such a system be subjected to the sinusoidal input

Æ

Such a system has two real and negative poles, and , therefore its free evolution tends to zero. Furthermore, because of the above, we can say that such a system responds to steady state with an output that is also sinusoidal whose pulsation is equal to that of the input sinusoid, i.e. .

In addition, as it results *and*
rad.

This implies that the amplitude of the steady-state output sine wave is equal to times the amplitude of the input sine wave, while the steady-state output sine wave is offset in advance of the rad input sine wave.

In conclusion, we can say that the value of the steady-state response is

Æ

Note that since the two characteristic time constants of the system *are* equal to and , the regime can be considered achieved as early as after a time equal to seconds.

8.1.2 Definition of response harmonic

Let us now look at the formal definition of *harmonic response* which, it is important to emphasize from the outset, is also valid for systems with positive and/or zero real-part poles, although it has no physical meaning in this case and is not measurable experimentally.

Definition 8.3. *Consider a linear, stationary SISO system having a transfer* functionHarmonic response *or* frequency response *is defined as the fun-zione of the nonnegative real variable obtained by placing in the expression of the transfer function.*

By virtue of what was demonstrated in the previous paragraph, if the has poles all to negative real part, the harmonic response enjoys a definite physical meaning. For in this case if the system having a transfer function is excited by a sinusoidal-type signal, the modulus of the harmonic response is equal to the ratio of the modulus of the input signal to the modulus of the output signal, while the phase of the harmonic response is equal to the phase shift between the input signal and the output signal. This is for each value of the characteristic pulse of the input signal.

8.1.3 Experimental determination of the response harmonic

The physical significance attributed to the in the case of systems with negative real-part poles also suggests a method for its experimental determination. For in this case it is sufficient to apply a sinusoidal signal to the input of the system, wait for the output to go to steady state, and then determine the ratio of the amplitude of the output signal to that of the input signal as well as the phase shift between the two. By repeating this operation with different input sinusoidal signals, characterized by different values of the pulse, the trend of the modulus and phase of the harmonic response in the pulse range of interest can be traced.

Clearly, this procedure is usually quite laborious because one must wait until the transient is completely exhausted in order to get a reliable estimate of the modulus and phase of the output signal. This time interval also becomes particularly long when the time constants of interest are large.

There are other alternative procedures to this for the experimental identification of the harmonic response, based on the excitation of the system by in- gress signals richer in harmonics. For example, remembering that the transfer function coincides with the Laplace transform of the impulsive response, one possibility is to experimentally determine the impulsive response and then transform it according to Laplace. However, this procedure is not feasible because of the practical difficulties involved in generating the impulsive signal. This can be remedied by rile- ving the index response instead of the impulsive response. The index response can then be derived or transformed and then multiplied by the term The latter procedure has the advantage of being much faster but, on the other hand, provides less accurate results than those obtained by directly detecting the harmonic response as the frequency changes.

8.2 Response to signals equipped with series or Fourier transform

In Appendix F we saw how there are very important classes of signals that can be decomposed into the sum of an infinite number of harmonics, i.e., sinusoidal components characterized by different values of the pulsation. By virtue of this, bearing in mind the principle of superposition of effects applicable to linear systems, it is clear that the results seen in § 8.1.1 can be extended to this type of signal. In particular, the following will examine both periodic signals do- tioned by Fourier series development and signals that are not periodic but have a Fourier transform.

Developable signals in Fourier series

Consider a linear, stationary SISO system having transfer function
With poles all negative real part. Let

His harmonic response.

Assume that such a system is excited by an *input* signal
periodic in period and developable in Fourier series :[2]

(8.1)

Where e is placed in the trigonometric form (F.5).

The output of such a system under steady state conditions is equal to

where , .

Note that this result follows immediately from the application of Propo- siation 8.1 by virtue of which the -ma harmonic present in the input (8.1) undergoes an amplification equal to and a phase shift equal to , as well as from the principle of superposition of effects.

We can therefore conclude that the output signal resulting from the application of a periodic period signal can at most contain harmonics of pulsation equal to , for
.

---

[2]As seen in § F.1.1 the assumptions that guarantee that a periodic signal is developable in a Fourier series are very mild. In particular this is true if it is defined for every value of and is continuous at intervals.

Signals equipped with the Fourier transform

What has been said for periodic signals can of course be extended to the case of nonperiodic inputs as long as they have a Fourier transform, that is, to signals that are absolutely summable. As seen in Appendix F, a signal belonging to such a class can in fact be placed in the form (F.10)

$$- \quad (8.2)$$

Where is the Fourier transform of the signal .

The signal is therefore decomposable into an uncountable infinity of armonies, with pulsations covering the entire real axis.

The presence of the integral instead of the summation does not clearly change the analysis carried out in the previous case since the principle of superposition of effects is also applicable in this case. Therefore, if a li- near and stationary SISO system with poles all with negative real part, having harmonic response is excited by means of a nonperiodic but asso- lently summing input placed in the form (8.2), it responds in steady-state output with a signal of the type

$$- \quad (8.3)$$

8.3 Diagram of Bode

The Bode diagram is undoubtedly the most widely used way to represent the harmonic response associated with a given transfer function.

Such a diagram starts from the representation of the in terms of *polar coordinates*, viz.

and involves the construction of a pair of diagrams, the *modulus Bode plot* and the *phase Bode plot*.

In the abscissa to the Bode plots is placed the pulsation expressed in decimal logarithmic scale. This choice is again dictated by the need to have a compact representation of the for large frequency (and thus pulsation) excursions. Indeed, the use of a logarithmic scale makes it easier to represent quantities that are susceptible to very wide variations, since it leads to a contraction of high values and an expansion of lower values.

The *Bode plot of the* modulus presents the modulus of *the* expressed in *decibels,* $_{\mathrm{db}}$, defined as.

$$_{\mathrm{db}}$$

Where log denotes the logarithm in base .

The *Bode plot of phase* presents on the ordinate the phase, expressed in degrees or radians, of the .

The use of the modulus representation in decibels is essentially related to the following consideration. Indeed, this choice makes it possible to apply also to the modulus *the principle of superposition* that already applies to the phase. Indeed, by placing the in factorized form, it is possible, given the representation of the individual factors, to obtain the representation of the overall function as the sum of the terms corresponding to its factors. The phase of a product is in fact equal to the sum of the phases of the individual factors, and the logarithm of a product is also equal to the sum of the logarithms of the individual factors. Therefore, since by definition$_{db}$, *if*

, it appears that

db

dbdb

where with obvious notation $_{db}$ and $_{db}$ denote the decibel modulus of and respectively.

Bode diagrams are plotted in the s e m i l o g a r i t h m i c *charts* so called because the x-axis alone is on a logarithmic scale[3] . An example of a semilogarithmic chart is shown in Fig. 8.1.

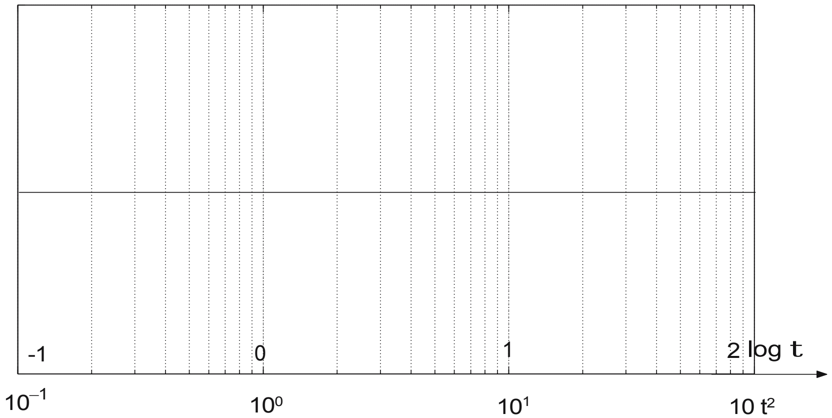


Fig. 8.1. Semilogarithmic chart

In the logarithmic scale the origin corresponds to being . To the right of the origin are the points corresponding to pulsations since for such pulsations ; to the left, however, are the points corresponding to values of

[3]Indeed, note that even though the modulus is in expressed in decibels, the y-axis of the modulus diagram is on a linear scale.

since for such pulsations it ....resultsCclearly the calibration of the axis
of the abscissae depends on the particular transfer function and must be done in such a way as to show its trend in the most significant range of .

In Fig. 8.1 we have for clarity indicated both the values of and the values of . In what follows, however, we will only indicate the values of .

Finally, observe that the null pulsation naturally never appears at the finite being

.

In the x-axis, the unit interval is named the *decade*.

Definition 8.4. *Two pulsations are one* decade *apart on the logarithmic scale when their ratio is equal to .*

Let it be in fact , then

Another significant quantity is the *octave*.

Definition 8.5. *Two pulsations are one* octave *apart in the logarithmic scale when their ratio is equal to .*

Let it be in fact , then

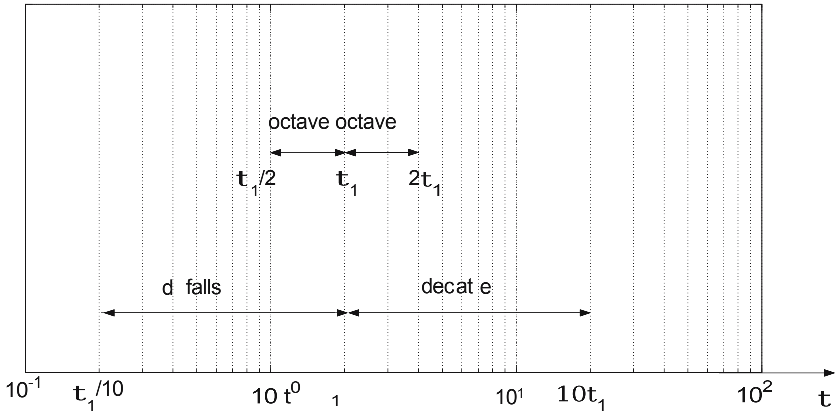


Fig. 8.2. Semilogarithmic chart: dedade and octave

The meaning of decade and octave is clearly shown in Fig. 8.2.

8.3.1 Rules for plotting the Bode diagram

There are well-established rules that make it possible to draw the Bode plot of modulus and the Bode plot of phase in a smooth and si- static way. These rules are applicable from a particular representation of the transfer function, namely the *Bode representation*, which sees the transfer function expressed as the product of a number of factors dependent on various physical parameters such as gain , time constants , natural pulsations , etc. According to this representation, introduced in Chapter 6 (see § 6.4.3), a transfer function can be written as:

(8.4)

where we refer to that chapter for the physical meaning of the individual terms. Placing in the expression above we obtain

(8.5)

and it is with reference to that expression that we will present the rules for plotting the Bode diagram.

The factorized form (8.5), together with the choice of expressing the modulus in deci- bel, allows both the modulus and phase diagrams to be constructed by resorting to the principle of superposition, that is, by summing the moduli in decibels and the phases (in degrees or radians) of each factor in (8.5).

Bode diagrams of individual factors

We will now look at the individual factors that appear in expression (8.5) and see what form the Bode diagram related to them takes. In particular we will show how for some of them it is possible to give, in addition to an exact representation, a simplified but meaningful representation that is called the *asymptotic* representation.

Gai*n*

Gain is a constant that can be either positive or negative. In particular, if we have that whereas *if*

Regardless of the sign of the modulus Bode plot is a horizontal line of ordinate$_{db}$ log .

The Bode plot of the phase is still a horizontal line but its ordinate depends on the sign of : if (as is the case in almost all applications of practical interest) the phase is worth ; if, on the other hand, the phase is .....worthThese results
Are summarized in Fig. 8.3.

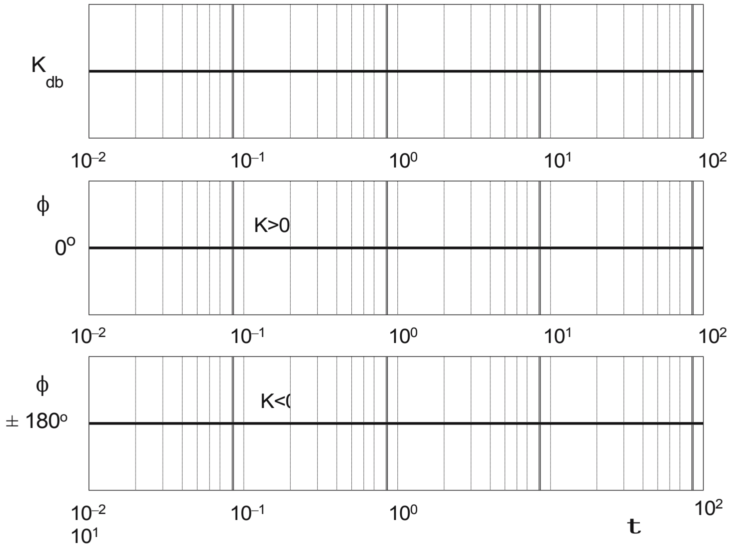


Fig. 8.3. Bode diagram of the gain K

Monomial **factor**

Now consider a monomial term in the numerator of the relative to a zero in the origin: . In expression (8.5) this corresponds to the case where .

By emphasizing the modulus and phase of we can write the monomial factor as

Modulus: The modulus expressed in decibels is then$_{db}$...................................... Such ter- mine is clearly linear with respect to This means that the diagram of the modulus corresponding to the monomial term under consideration is a line passing through the ori- gine whose slope is db per decade. That is, the line related to the diagram of the modulus passes through$_{db}$ at the pulse being

db .

Phase: Phase, on the other hand, is constant as the pulse varies and holds , so the phase diagram is a horizontal line of ordinate equal to .

The complete Bode diagram is shown in Fig. 8.4.

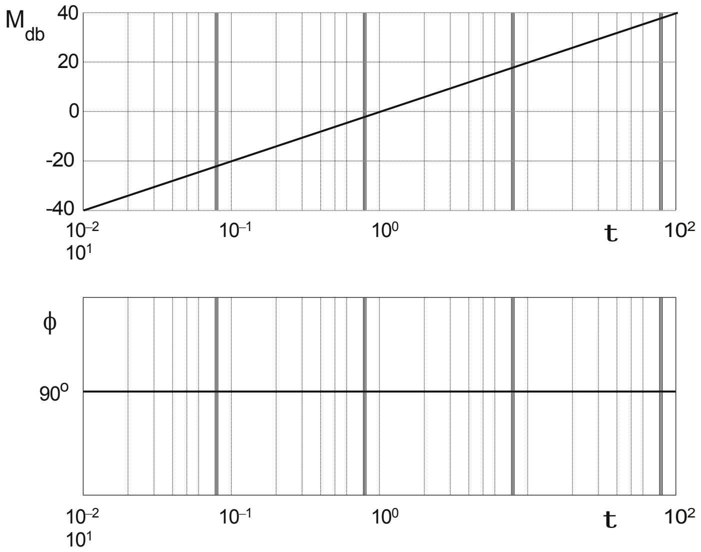


Fig. 8.4. Bode diagram of the *monomial* term

Monomial **factor**

The Bode diagram relating to a pole in the origin can be immediately derived by referring to the considerations seen about the monomial term at the numerator . In fact, the modulus and phase of the denominator factor are equal to the modulus and phase of the numerator factor, respectively, viz.

db — db

db

e

The complete Bode diagram is shown in Fig. 8.5. Note that chia-ractically, the diagrams in Fig. 8.5 are the symmetries with respect to the x-axis of the diagrams in Fig. 8.4.

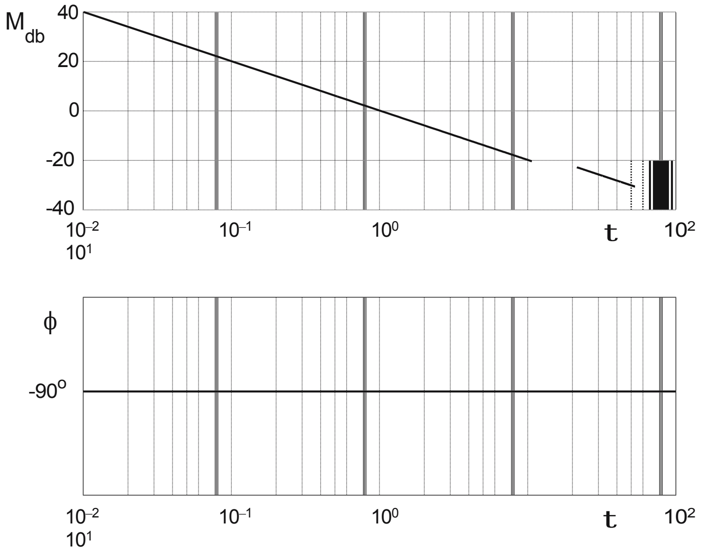


Fig. 8.5. Bode diagram of the *monomial* term

Binomial factor

Of the binomial factor at numerator , as well as of the terms presented below, it is possible to give in addition to the exact representation, a representation that is particularly convenient and effective: the *asymptotic representation*.

Module: By definition

whereby

db

When the term is negligible with respect to unity, and this occurs *when* ,

db

Conversely, when it is the unit that is negligible with respect to , that is, *when* ,

db

Therefore, the Bode diagram of the modulus for the binomial term under consideration has two asymptotes: one for ( ) and one for ( ). In particular, the x-axis is an asymptote for , while the line è un asintoto per . Quest'ultima è una retta che interseca l'asse delle ascisse nel punto e che ha pendenza pari a db per decade. I due asintoti costituiscono una spezzata che prende il nome di *diagramma asintotico del modulo* e il punto di intersezione tra le due curve, , viene detto *punto di rottura*. Tale diagramma è riportato in Fig. 8.6 con una linea spessa.

The exact diagram (thin line) is also shown in Fig. 8.6 where it can be seen that the two curves, while not exactly coincident, are nevertheless very close to each other. The maximum deviation occurs at the point of break where the exact diagram holds db while the asymptotic one vale . Moving then one octave to the right or one octave t o the left with respect to the breaking point, it is easy to calculate that the deviation of the exact diagram from the asymptotic one is db. When moving one decade away from the breaking point, the two diagrams can already be considered coincident. For the sake of completeness, the deviations between the exact modulus diagram and the asymptotic diagram as the pulsation varies are shown in Fig. 8.7, where with obvious notation$_{db}$ indicates precisely the difference between the exact and asymptotic diagrams as they vary by .

Finally, it should be noted that when the Bode plot is drawn manually, the exact modulus plot is often derived from the asymptotic plot by making corrections at only a few key points as summarized in the table below.

		——	—	—	
db					

I punti così ottenuti vengono poi raccordati facendo in modo che per pulsazioni che distano una decade o più da , il diagramma coincida esattamente con quello asintotico.

Phase: A similar argument can be repeated for the phase diagram, although it is now necessary to distinguish the case where from the case where being

Osserviamo dapprima che, a prescindere dal segno di , per valori molto piccoli della pulsazione , ossia per , per cui il semiasse negativo delle ascisse è in ogni caso un asintoto per .

Vi è poi un secondo asintoto, relativo ad , che è invece funzione del segno di . Più precisamente, se , tale asintoto coincide con la retta orizzontale di ordinata pari a essendo per

Se l'asintoto è ancora una retta orizzontale ma di ordinata pari a essendo in questo caso

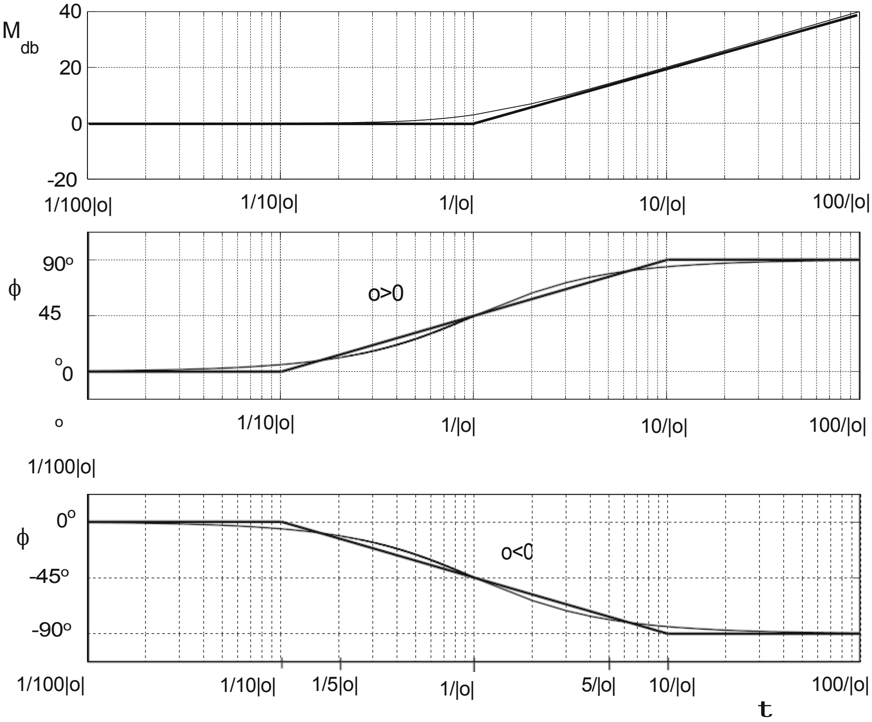


Fig. 8.6. Asymptotic (thick line) and exact (thin line) Bode diagram of the *binomial* term

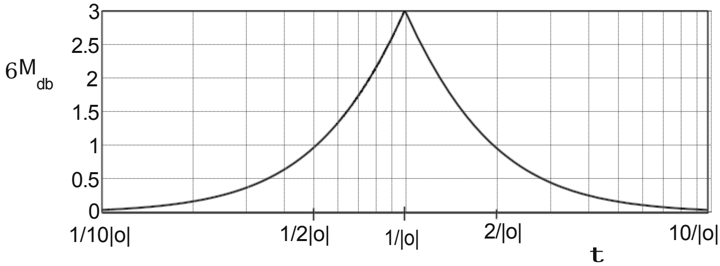


Fig. 8.7. Deviations between exact plot and asymptotic plot of the modulus relative to the *term*

The asymptotes for and for have no points of intersection. Therefore, in order to define *an asymptotic phase diagram*, it is necessary to make a connection between the two asymptotic half lines. There are several conventions for this, all of which lead to a three-sided break. However, since *for*

i*f*

i*f*

whatever convention is adopted, the third side of the broken always passes through the point of abscissa and ordinate equal to depending on the sign of .

The convention adopted in this text is to choose the third side of the spline as the segment passing through the inflection point but intersecting the two horizontal asin- tects one decade before and one decade after with respect to the breaking point, that is, at pulsations equal to and times The asymptotic diagram resultant is highlighted in Fig. 8.6 with thick stroke.

The exact phase diagram is also shown in Fig. 8.6 with thinner stroke.

Note that in plotting the phase diagram, unlike the modulus diagram, it is not convenient to use the asymptotic diagram for plotting the exact diagram. The asymptotic diagram is therefore used only when an exact evaluation of the phase for each value of When plotted manually, the exact diagram of the stage is constructed from some basic points summarized in the table below.

		——	——	—	—	—	
fo*r*	Æ	Æ	Æ	Æ	Æ	Æ	Æ
fo*r*	Æ	Æ	Æ	Æ	Æ	Æ	Æ

Binomial **factor**

The Bode diagram of the binomial factor can be easily derived by referring to the considerations just seen regarding the *term*
Indeed,

db db

db

e

so Bode diagrams of the modulus and phase of are obtained simply by flipping with respect to the x-axis the modulus and phase diagrams for the term For completeness these diagrams are given in Fig. 8.8. Again, deviations between the exact and asymptotic modulus diagrams can be evaluated by referring to Fig. 8.7 as long as these deviations have changed sign.

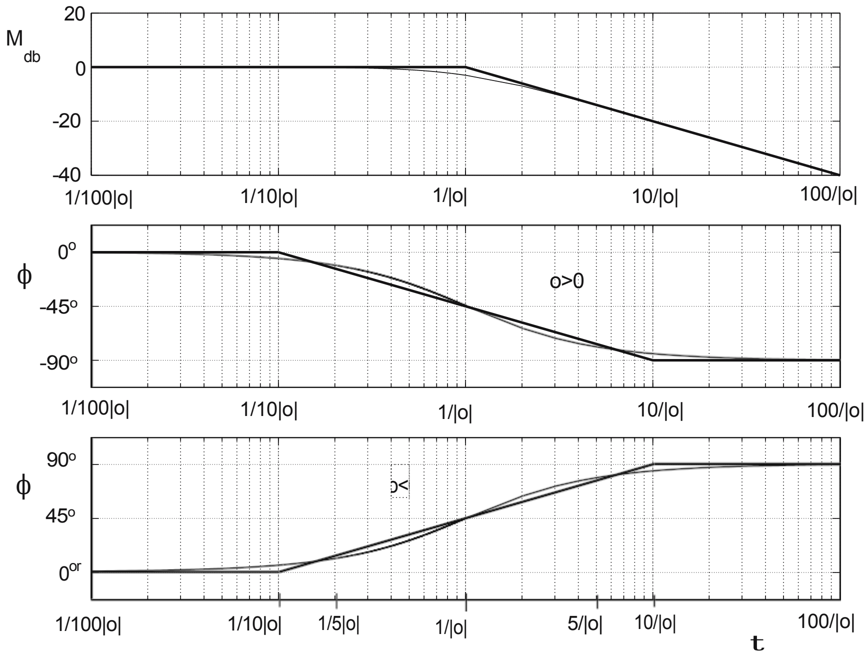


Fig. 8.8. Asymptotic (thick line) and exact (thin line) Bode diagram of the *binomial* term

Trinomial **factor**

As seen the term trinomial

corresponds to a pair of complex conjugate zeros.

Again we can give an asymptotic representation and an exact one for both modulus and phase.

Modulus: By definition, the modulus in decibels is worth

db log

The structure of the asymptotic diagram of the modulus is similar to that seen for the binomial term.

When , that is, when the terms in are negligible with respect to unity for which

db

This means that the modulus diagram has a first asymptote for, and that asymptote coincides with the x-axis.

On the other hand, when the ratio , i.e., for , the dominant terms in the expression of$_{db}$ are those related to the powers of the ratio of higher degree, so

db

We can therefore conclude that the line

is an asymptote ....forIn the chosen scale this line has slope equal to db per decade and meets the x-axis at the point relative ....toIn analogy to the case of binomial terms, the abscissa point is called *the breaking point*. The asymptotic diagram of the modulus is shown in Fig. 8.9 with the thickest line.

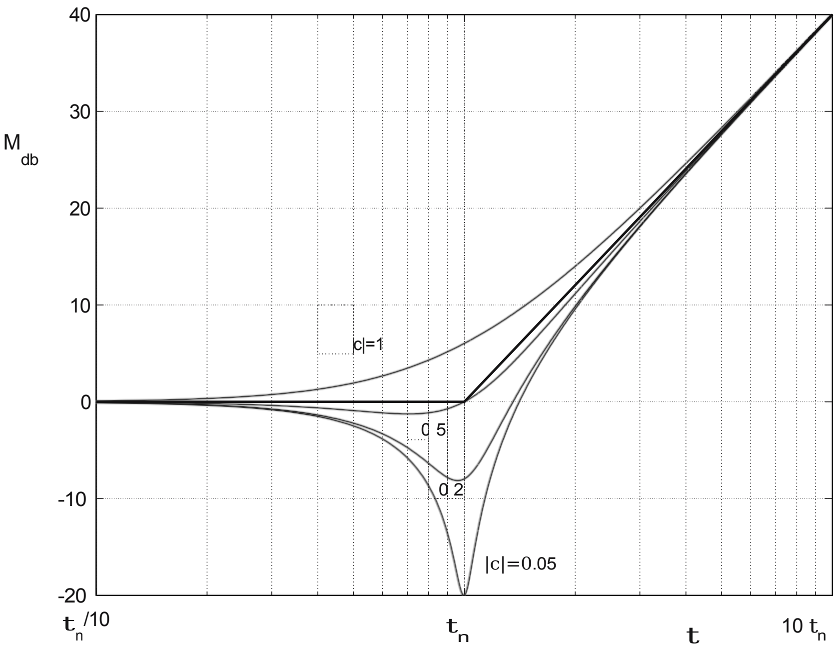


Fig. 8.9. Diagram of asymptotic (thicker line) and exact (thinner lines) modulus. relativi al termine trinomio al variare di

As for the exact diagram of the modulus, the matter is slightly more complex than that made for the binomial terms. Now in fact, as pointed out in

Fig. 8.9, it is possible to identify a family of parameterized curves as a function of the modulus of the damping coefficient . This clearly implies that the deviations between asymptotic diagram and exact diagram also constitute a family of parameterized curves as a function of These curves are shown in Fig. 8.10 at the changing pulsation As can be observed, deviations can be
either by excess or defect, depending on the value of Al decrease in the modulus of the exact modulus diagram tends to deviate more and more from the asymptotic one. The boundary condition occurs for : for such a value of the damping coefficient in fact the modulus of the trinomial factor at the breaking point is zero and thus$_{db}$.

If the trinomial term corresponds to the product of two equal binomial factors being in this case

so that for each value of the pulse the modulus in decibels is exactly equal to twice the modulus in decibels of the single binomial factor for which the considerations just seen apply, where we place in place ....ofThe maximum devi-
ment between the exact diagram and the asymptotic diagram therefore occurs in this case at the breaking point and is db. Moving instead one octave to the right or left of the breaking point, the deviation is db.

Finally, at and of the exact and asymptotic modulus diagrams are coincident with each other.

Phase: Regarding the phase diagram, we first observe that the phase of the trinomial term at the numerator is equal to

atan

and is therefore also a function of the parameter . In particular, as shown in Fig. 8.11 (see continuous curves) we will have two different families of curves depending on the sign of and thus also two different families of asymptotic diagrams (dashed lines).

The phase diagrams all have a common asymptote related to . This asymptote coincides with the x-axis being for , , whatever the sign of .

On the other hand, the sign of the damping coefficient influences the limiting behavior of the phase forIn fact , for values of

which results in an increasing function of se , and a decreasing function of se
Also

if

if

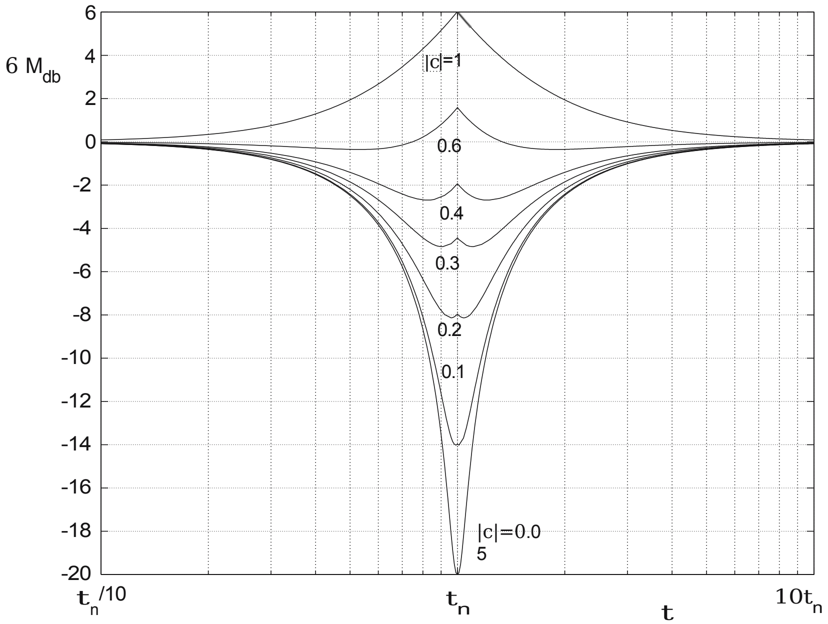


Fig. 8.10. Deviations between exact diagram and asymptotic diagram of the modulus related to the
termine al variare di e

so the second asymptote is still a horizontal line whose ordinate holds depending on whether it is greater or less than zero.

Just as already discussed in the case of the binomial term, since there are no points of intersection between the two asymptotes, in order to define an asymptotic diagram it is necessary to ef- fect a connection between the two horizontal asymptotic half lines. Again, several solutions are possible. The solution we will adopt in this text provides a good approximation between asymptotic diagram and exact diagram and is analogous to that adopted in the case of the binomial term. In this case, however, for different values of the intersection points of the connecting segment with the horizontal asymptotes are different. We denote by and the pulsations related to these intersection points, i.e., and denote the pulsations of the intersection points between the connecting segment and the horizontal asymptotes. Specifically, denotes the pulsation to the right of the breakpoint and the pulsation to the left of the breakpoint. Given the symmetry of the curves, it is evident that for any value of we have that . In what follows we shall denote by this ratio, where ....clearlyI n parti-
colar, being for the trinomial term equal to the product of two binomial terms having breaking point , it naturally applies .

It is easy to verify that a good approximation can be obtained by means of the relation

This implies that if for example , then , and
. If , then , and .

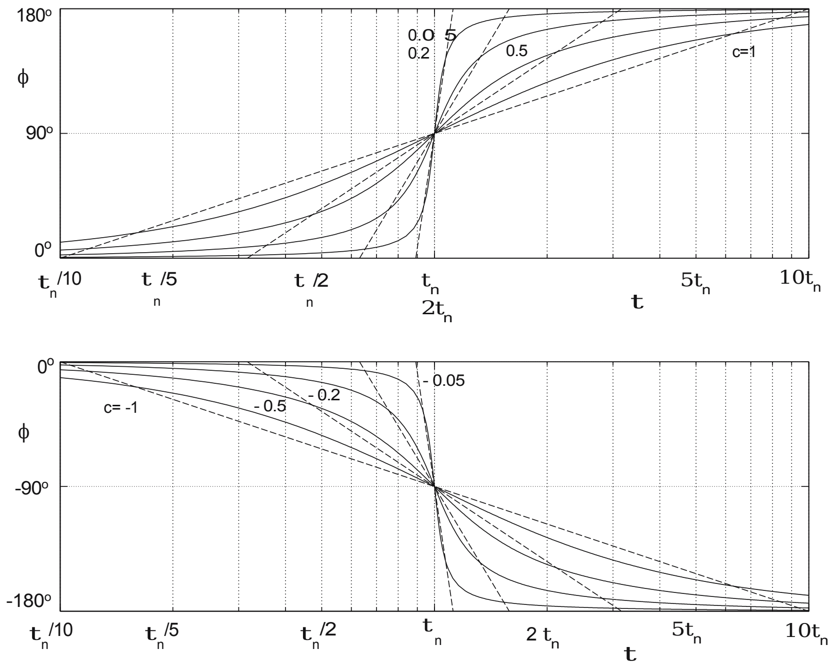


Fig. 8.11. Asymptotic (dashed lines) and exact (solid lines) phase diagram of the ter*m* al variare di

As shown in Fig. 8.11 we therefore have two different families of asymptotes depending on the sign of .

In practice, to determine on the logarithmic scale the two pulses and simply measure the length of a decade and multiply that length by The lun-
ness thus obtained is equal to the distance of and from in the chosen scale. Once the breaking point is fixed, it then becomes immediate to determine the other two pulsations and thus also the asymptotic phase diagram.

Trinomial **factor**

If the trinomial term is in the denominator, the Bode diagrams are obtained from the previous ones simply by flipping those diagrams with respect to the x-axis, as shown in Figs. 8.12 and 8.13. The deviations between the exact and asymptotic modulus diagrams as they vary from are also the opposite of the deviations from the ti that occur in the case where the trinomial term is at the numerator and are thus immediately deducible from Fig. 8.10.

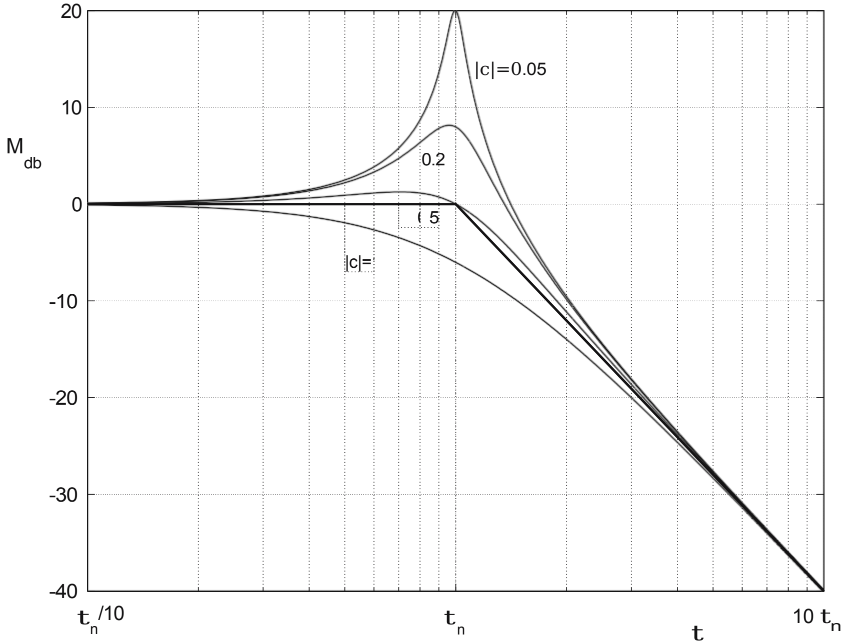


Fig. 8.12. Diagram of asymptotic (thicker line) and exact modulus (thinner lines) relativi al termine trinomio al variare di

Rules of composition

By virtue of the previous considerations regarding the superposition principle, given one in the form (8.5), the Bode diagram relating to it can be easily determined by summing the Bode diagrams relating to the individual factors. However, the introduction of asymptotic diagrams makes it possible, in the case of the modulus diagram, to follow a much quicker and easier procedure.

For the construction of the module diagram this procedure can be schematized as follows.

Determination of the modulus diagram.

1. We fix the origin in the x-axis corresponding to..........................You fix the points of breaking terms related to binomials and trinomials in the numerator and denominator.
2. We plot the modulus diagram related to gain and monomial factors; we plot the asymptotic diagram related to binomial and trinomial factors.

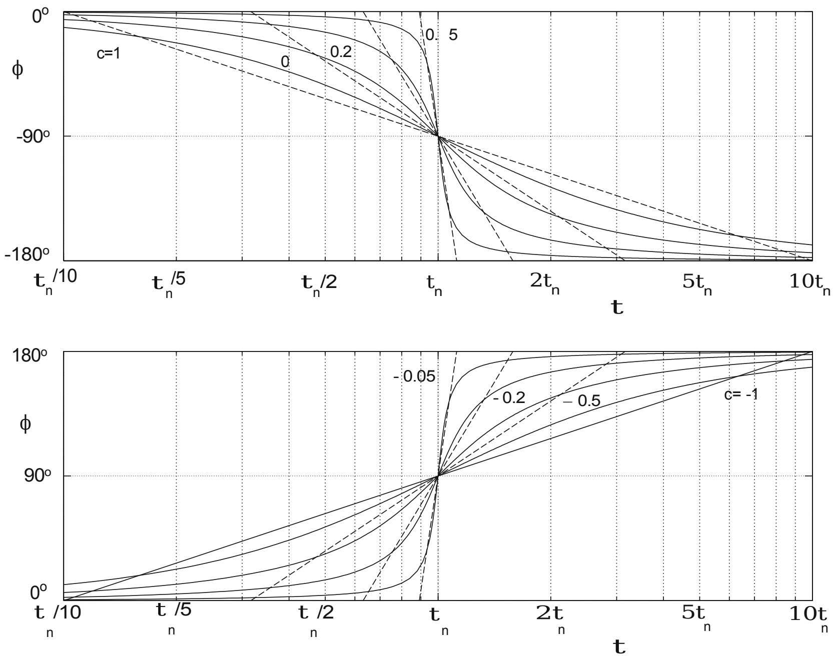


Fig. 8.13. Asymptotic (dashed lines) and exact (solid lines) phase diagram of the termine al variare di

3. The sum of these diagrams is made, taking into account the multiplicity of ciascity of each factor. This yields the asymptotic diagram of the modulus in which the last section of the spicule has a slope equal to db per decade.
4. To obtain the exact diagram, the introduction of deviations is necessary. In general, it is sufficient to take into account the deviations related to the trinomial terms, while neglecting those due to the binomial terms. To do this, reference should be made to the diagram shown in Fig. 8.10. Note that if the breakpoints are sufficiently far apart, corrections can be introduced by taking into account only one factor at a time; when, on the other hand, there are breakpoints that are less than a decade apart, it is necessary to add up the corrections related to different factors.

For the construction of the phase diagram, the approach is similar.

Determination of the asymptotic phase diagram.

1. We fix the origin in the x-axis corresponding ....toFor each termine binomial we identify the pulsation corresponding to the breaking point and the pulsations that are one decade away from it. For each trinomial term you

Identifies the pulsation corresponding to the breaking point and the two pulsations

e .

2. Asymptotic diagrams of the individual factors are plotted.
3. The sum of these diagrams is made, taking into account the multiplicity of each factor. The asymptotic diagram of the phase is thus obtained.

If, on the other hand, one wishes to construct the exact phase diagram, one starts directly from the exact diagram of the individual terms without going through the asin- totic diagram. More precisely, the exact diagrams of the binomial and trinomial terms as well as the gain and monomial terms are drawn. These are then summed taking into account their multiplicity.

8.3.2 Examples numerical

In this section, some examples of Bode diagram plotting will be presented in detail.

Example 8.6 Consider the transfer function

–

The Bode plot of the module is shown in Fig. 8.14 where the asymptotic diagrams of each term (thin solid lines), the overall asymptotic diagram (thick solid line) and the exact diagram (thick dashed line) were also plotted.

Let us first examine the individual factors.

The gain in decibels is worth$_{db}$ log db so this term gives a contribution of db for each value of .

The binomial term at numerator has breakpoint from which the modulus grows with slope db per decade.

The term diagram associated with the pole in the origin has always decreasing modulus with slope equal to db per decade and is worth db for .

The denominator binomial term, associated with the pole , has breakpoint from which the modulus decreases with slope db per decade. Note that for the asymptotic diagram of the modulus relative to the pole in the origin and that relative to the binomial term are coincident.

Finally, the denominator binomial term, associated with the pole , has breakpoint from which the modulus decreases with slope db per decade.

Following the rules of composition explained before, it is easy to derive the total asymptotic diagram- ma. From up to there is the only contribution due to the pole in the origin and the gain. For pulsations from up to *t h e* slope of the break is zero because the contribution of the denominator binomial term is added to the previous term. In the range of pulsations the slope is

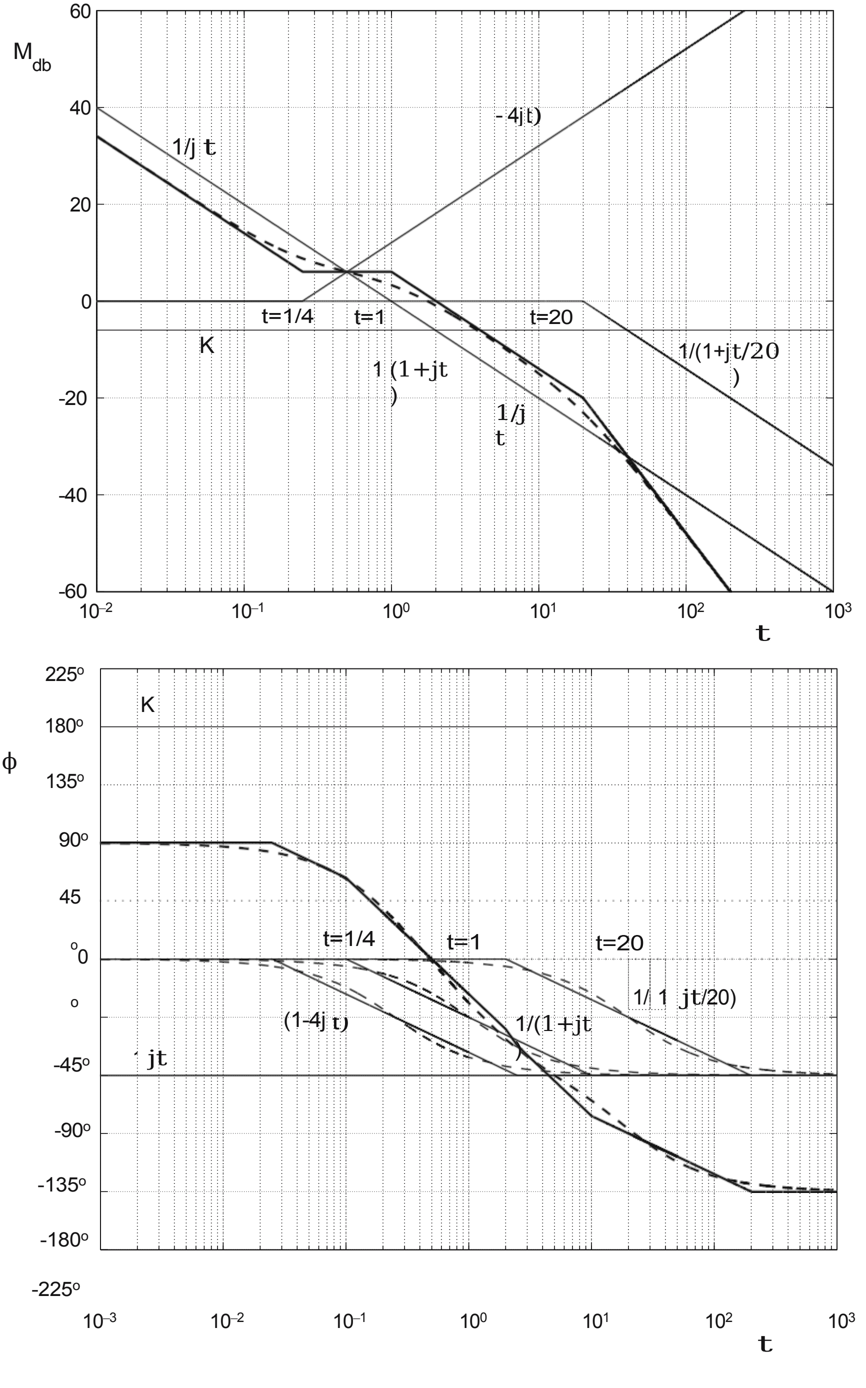


Fig. 8.14. Asymptotic and exact Bode diagram related to the transfer function o f Example 8.6.

db per decade because the contribution of a binomial term to deno- minator is added. Finally, starting from the slope is increased to db per decade because of the presence of the second binomial factor in the denominator.

For completeness, the exact modulus diagram is also shown in Fig. 8.14 with thick dashed line. As can be seen, since the different breaking points are sufficiently distant from each other, the difference between the exact and the asymptotic representation at these points is worth db.

The same Fig. 8.14 also shows the exact phase diagram (with thick dashed line), as well as the contributions related to the individual binomial factors, guadagno and pole in the origin. Finally, with continuous stroke the asymptotic diagrams (global and relative to individual factors) are plotted, while with dashed lines the exact diagrams are plotted. It can be seen that the approximation obtained re- running to the asymptotic trends is fully satisfactory

Example 8.7 Consider the transfer function

Which placed in Bode's form is equal to:

From this expression it is easy to deduce that the gain is worth , i.e. db db. The characteristic parameters of the trinomial term are and .

The Bode plot of the exact modulus (thick dashed line) is shown in Fig. 8.15 where the asymptotic diagrams of the individual terms (thin solid lines) and the total asymptotic diagram (thick solid line) are plotted.

It is easy to see that the rules of composition explained above were followed here as well.

The phase diagrams, asymptotic (thick continuous line) and exact (thick dashed line), are also shown in Fig. 8.15 where the contributions of the individual terms (solid lines) have been highlighted. With regard to the trine term, we observe that being , we have for which the abscissae of the intersection points of the slanted break side with the horizontal asymptotes are

e The resulting diagram, both in the case of the exact representation, as well as in the case of the asymptotic one, were again obtained by summing the relative contributions to the binomial term, the trinomial term and the pole in the origin. The gain in this case, being positive, makes no contribution to the phase diagram

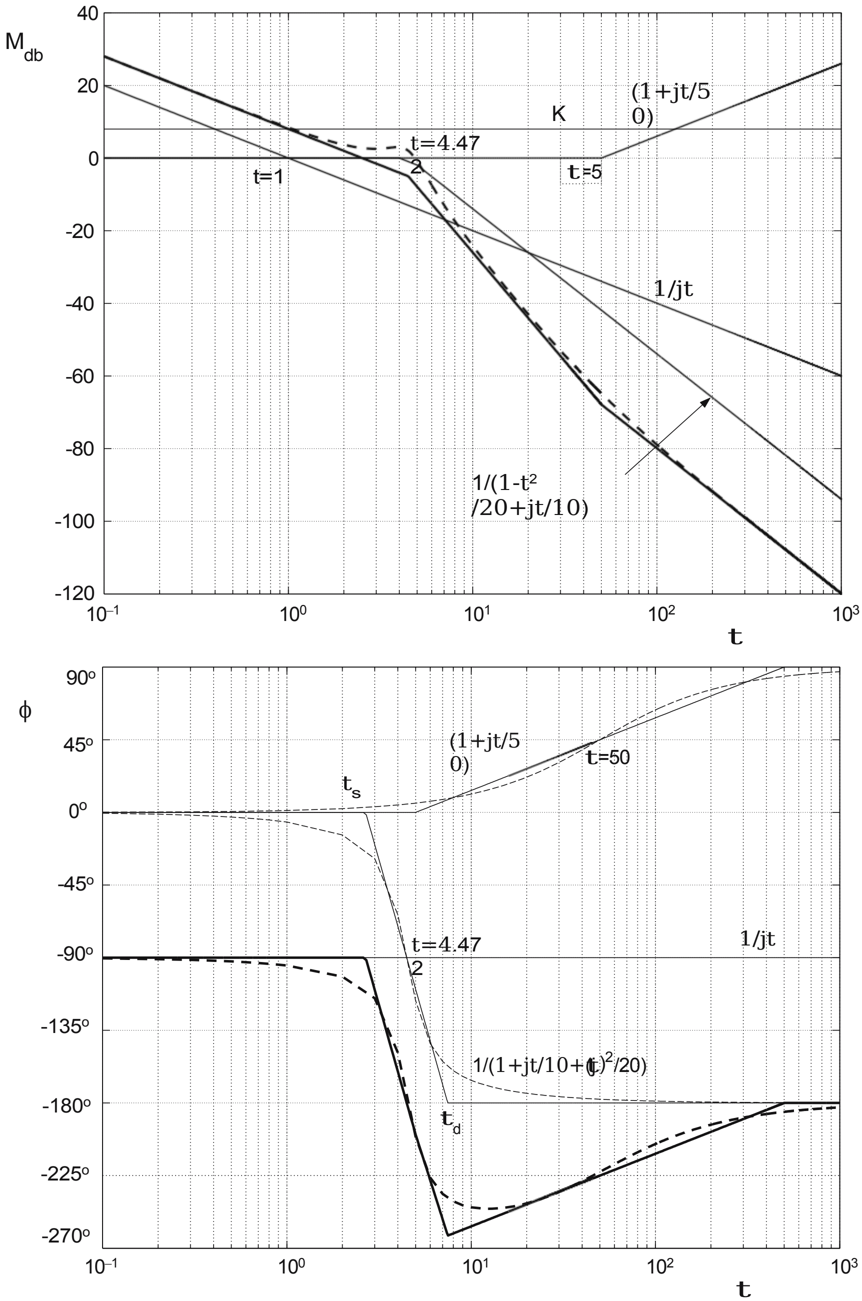


Fig. 8.15. Asymptotic and exact Bode diagram of the transfer function o f Example 8.7

8.4 Harmonic response characteristic parameters and filtering actions

The harmonic response was formally defined as the tra- fer function calculated at pure imaginary values of the Laplace variable, that is, by posing .

At the beginning of the chapter, however, it was shown how this function has a well-defined physical meaning when related to a system with poles all with negative real part. In particular, it was seen that a linear, stationary SISO system with poles all with negative real part, subjected to a sinusoidal input Æ , am- puts at steady state an output that is also sinusoidal of pulsation , amplitude, and offset from the input by an angle equal to , where and are the modulus and phase of the .

This physical interpretation of the makes Bode diagram plotting particularly important in many areas, such as Electrical Engineering and Tele-communications, where the signals involved are almost always sinusoidal, or periodic, signals, and the systems under study have negative real-part poles. In fact, Bode diagrams provide an immediate indication of what, at steady state, will be the phase shifts in advance or delay and the attenuations or amplifications that the input signal will undergo depending on its frequency. In addition, in the case of perio- dici signals, it makes it easy to assess how the spectrum of the input signal will be modified at the output.

However, the Bode diagram provides an extremely useful graphical representation even in the case of systems with positive and/or zero real-part poles, that is, for which the harmonic response does not have the physical meaning discussed earlier. Such types of systems are frequently dealt with in the field of Automatic Controls, the aim of which is precisely to determine a stabilizing control law that guarantees the satisfaction of certain specifications. In this sphere, the Bode dia- gram is precisely used as an auxiliary tool for controller synthesis since it allows the evaluation of certain parameters that are important for control purposes, such as phase margin, gain margin, etc. In what follows, however, this topic will not be addressed since it is the subject of study in specific courses geared toward controller synthesis.

8.4.1 Parameters characteristic

Let us now introduce some parameters characterizing the harmonic response of a system and directly accessible from the Bode diagram. These are particularly important because their values influence the dynamic behavior of the system and its filtering properties.

Commonly considered parameters are as follows.

Modulus at resonance *or resonance peak*: is the point of maximum (if it exists at the finite) of the modulus diagram.

Resonance pulse : is the pulse at which the modulus is equal to .

Bandwidth (or *db bandwidth*): this is expressed in Hz and indicates the frequency at which the modulus diagram has a db attenuation with respect to the modulus value in ; clearly this definition makes sense as long as the modulus of the impulse response evaluated in is finite.

Bandwidth at db and at db, and : are also expressed in Hz and indi- cate, respectively, the frequencies at which the modulus diagram exhibits db and db attenuation relative to the value of the modulus ....inAlso in this case the definition is given under the assumption that the modulus of is finite.

Phase shift at the passband: indicates the value of the phase at the pulse .

Initial slope of phase diagram .

Clearly, it is not always possible to define the parameters above. In particular, and are meaningful only for harmonic responses that exhibit a maximum at finite. This is, for example, true in the case of second-order systems characterized by a pair of complex conjugate poles with damping sufficiently smaller than The parameters , and have meaning, on the other hand, in the case of systems that do not
have poles in the origin, that is, in the case of systems whose is defined in .

Note that the values , and db are derived from the conversion of decibels to natural scale. In fact, given a value of the modulus in natural scale, an attenuation of this by an amount equal to db , leads to a new value of the modulus which expressed in decibels is worth

db log db db
log db log
log log db
log —

where

db

i.e.

—

e in particolare

–
for db
for db
for db

The following example clearly illustrates how the parameters defined above can be easily read from the Bode diagram.

Example 8.8 Consider the linear, stationary second-order system whose transfer function is with and .

Such a system has a pair of complex conjugate poles with negative real part Il diagramma di Bode di tale funzione di trasferimento è shown in Fig. 8.16.

As highlighted in the figure, from the Bode diagram we can easily rile- vate all the characteristic parameters introduced before. Let us first observe that this transfer function has a maximum at the rad/sec pulse and that this maximum is worth db. We can therefore conclude that the modulus at resonance is db, and thus , while the resonance pul- sation is rad/sec. The phase shift at resonance is equal to
Æ .

We also observe that$_{db}$ and ....thusThe bandwidth at db is then given by the value of the frequency at which the modulus is worth db. From the figure we see that this occurs for rad/sec whence Hz Hz.

Instead, the bandwidth at db is given by the value of the frequency in cor- respondence to which the modulus is worth db and is therefore equal to
Hz Hz.

The bandwidth at db is Hz Hz.

Finally, we observe that the initial slope of the phase diagram i
s , es- sing for the phase diagram tangent to the x-axis .

It is interesting to recall that the term *resonance* originates from vibration theory and was initially used with reference to oscillating systems without dissipative elements. For such systems there is in fact a particular value of the pulsation, or equivalently of the frequency, for which the modulus of the harmonic response at that frequency tends to an infinitely large value. This frequency is precisely the resonant frequency. This diction has since been extended to the case where the modu- lus of the harmonic response takes on only finite values but has a maximum at a definite value of the frequency.

8.4.2 Actions filtering

The term *bandwidth* originates from the filter theory developed in the field of electrical communications. Indeed, one of the fundamental problems in signal transmission is to ensure that only signals characterized by frequencies within a certain range (which may be finite or infinite), called *bandwidth*, are actually transmitted, while noise or other spurious signals must be filtered out. This leads to the definition of an *ideal filter* whose harmonic response is constant within the frequency range of interest and is zero elsewhere. Such a filter, however, is not physically realizable, and an attempt is then made to find an effective approximation of it.

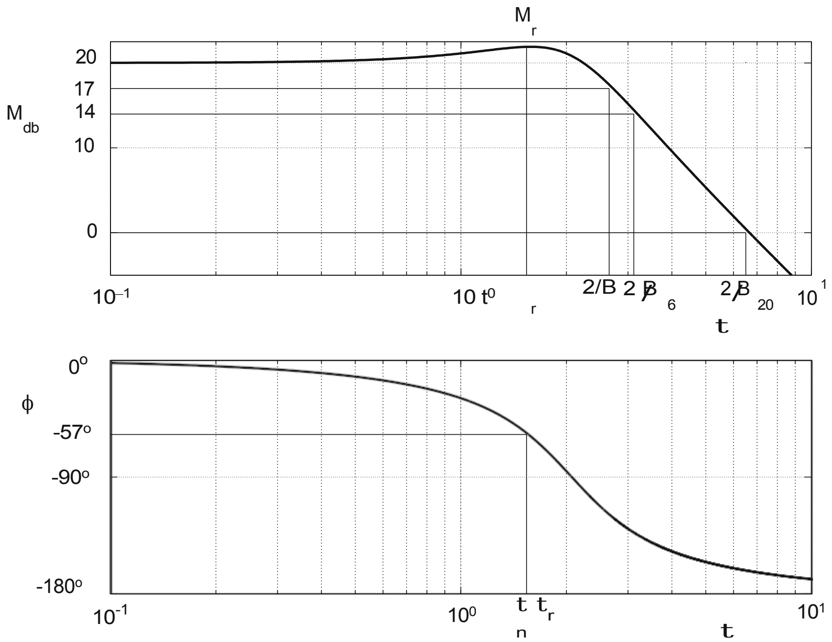


Fig. 8.16. Bode diagram of the transfer function of Example 8.8 and characteristic parameters of its harmonic response

In the following, the main types of filters are discussed and some examples of their actual approximations are given.

Low-pass filter

An ideal *low-pass* filter is one that lets through unchanged only those signals characterized by frequencies below a definite value, while signals characterized by higher frequencies are eliminated.

A filter characterized by a harmonic response whose structure is of the type shown in Fig. 8.16 is an example of an approximation of an ideal low-pass filter. Clearly, the values of the passbands , and make it possible to assess how closely the *actual* filter approximates the ideal one. In fact, when these values are very close together, it means that there is a clear separation between those frequencies at which the signal is allowed to pass and those frequencies at which there is significant attenuation.

High-pass filter

Similarly to low-pass *filters, high-pass filters* can be defined. An *ideal* high-pass fil- tro is a system that allows unaltered, or amplified by a constant quantity, all and only signals characterized by frequencies above a precise value, while signals characterized by lower frequencies are eliminated. Through the definition of appropriate parameters, it is also possible in this case to defi- nition how well a *real* filter can approximate an ideal filter. In particular, in a manner entirely analogous to the previous case, it is possible to define the passbands related to it at ,a or db, as the frequencies at which there is an attenuation of , or db with respect to the pulse .

Example 8.9 A system with a transfer function

Is a real high-pass filter. The Bode plot of the modulus of is shown in Fig. 8.17 with continuous line. In the same figure, the Bode plot of the modulus of an ideal filter approximated by the

.

In this case db and the value of the pulse at which there is an attenuation of db with respect to that value, that is, the value of the pulse at which the modulus is worth db is rad/sec. Therefore the bandwidth at db is worth Hz

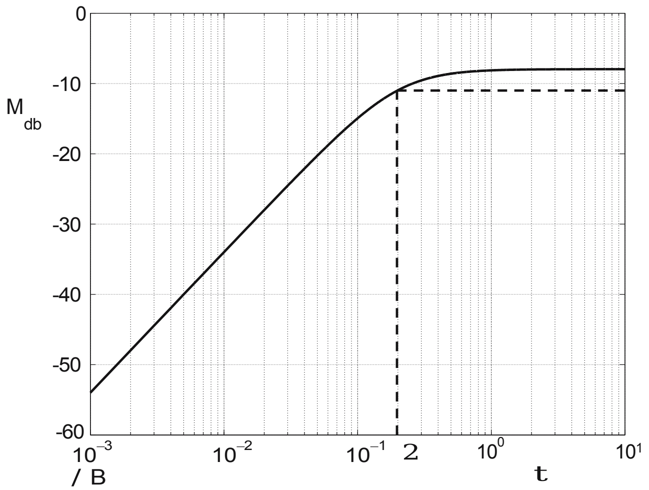


Fig. 8.17. Bode diagram of a typical high-pass filter with a transfer function.

Band-pass filter

Also extremely useful in practice is the realization of so-called *band-pass* filters, i.e., those systems capable of eliminating all signals whose pulsation is outside a certain range , while signals within that range are amplified or reduced equally. Again such behavior is purely ideal, and in practice only approximations can be made, the goodness of which can be described by parameters similar to those seen above. Note, however, that in this case two different pass-band values (a , and db) are necessary, to indicate with what goodness signals close to the pulsation are filtered and with what goodness se- gnals of pulsation close to aThe pass-band values are also defined
than the maximum value taken by the transfer function within the band. An example of a real bandpass filter and calculation of db passbands is given in the following example.

Example 8.10 A system with a transfer function

Is a real bandpass filter. The Bode plot of the modulus of the is shown in Fig. 8.18 with continuous line. In the same figure with dashed line is shown the Bode diagram of an ideal filter approximated by the .

In this case, the maximum value of the modulus, i.e., the modulus at resonance is worth

db e si ha in corrispondenza della pulsazione rad/sec.

From the Bode diagram we can easily read the values of the pulse at which there is attenuation, for example, of db, relative to , that is, the values of the pulse at which the modulus is worth db, which are rad/sec and rad/sec, respectively. We can therefore conclude that the two values of the bandwidth at db are equal to, respectively

Hz and Hz.

Exercises

Exercise 8.1 Given a system whose transfer function is worth

(one of the poles applies) .

(a) Report the transfer function of such a system in Bode form by calculating the characteristic parameters. Plot the Bode diagram of the .
(b) What is the bandwidth worth in db for this system? What is its physical significance?

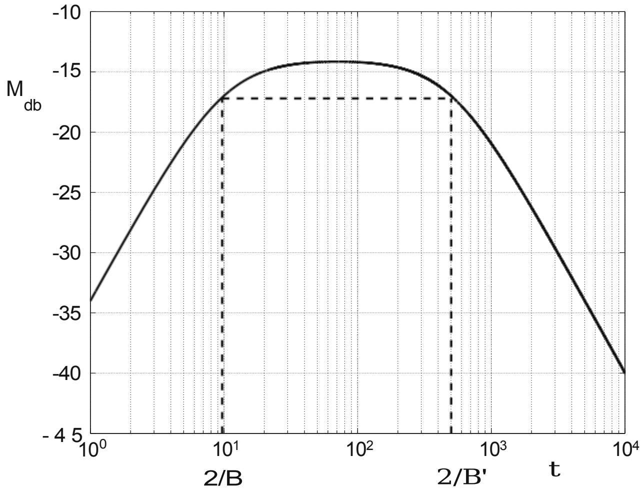


Fig. 8.18. Bode diagram of a typical bandpass filter with a transfer function.

Exercise 8.2 Repeat Exercise 8.1 with reference to the transfer function

(one of the poles applies) .

Exercise 8.3 A system described by the input-output model is given.

where it can be verified that one of the roots of the characteristic polynomial of the system holds .

(a) Calculate the transfer function of such a system and put it in Bode form, indicating all significant parameters.
(b) Draw the Bode diagram of such a function.
(c) Discuss whether it makes sense to talk about passband and resonance for such a function. If so, determine the corresponding parameters (passband a db, pulsation, modulus and phase shift at resonance).

Exercise 8.4 A system described by the input-output model is given.

(a) Determine the transfer function of such a system and put it into Bode form by calculating its significant parameters.
(b) Plot the Bode diagram of the .
(c) Based on the Bode diagram, answer the following questions. What is the frequency that undergoes the greatest attenuation between input and output? If a sinusoidal input with such a unit pulse and amplitude is assumed to be applied to the system , what is the amplitude of the steady-state output worth ?

Exercise 8.5 Consider the electrical circuit in Fig. 8.19 where [V] represents the input voltage and [A] the current in the mesh. Assume that the input to the system is.

Determine the current trend under steady state conditions.

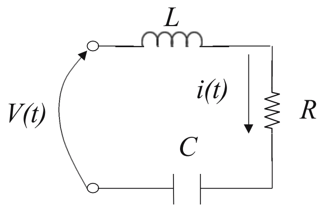


Fig. 8.19. RLC circuit

Exercise 8.6 Consider again the electrical circuit in Fig. 8.19 where [V] rappresents the input voltage and [A] the current in the mesh. Assume that the input is a square-wave signal of period [s], symmetrical with respect to the y-axis and such that [V] .

(a) Determine the current trend under steady state conditions.
(b) Repeat the same exercise assuming that the input is a triangular waveform signal symmetrical with respect to the y-axis, having the same period , but such that [V] .

9

Stability

This chapter will introduce a fundamental property in the study of dynamical systems, stability. The importance of this property derives from the fact that stability is a specification imposed on almost every controlled physical system because it implies the pos- sibility of working around certain nominal conditions without deviating too much from them.

In what follows, two different definitions of stability will be introduced: the first relating to the input-output bond (BIBO stability), the second relating to a representation in terms of state variables (Lyapunov-style stability). In the first case we will limit our analysis to linear systems only, but in the second case the definitions given are also valid in the more general case of nonlinear systems.

The third section of this chapter will then address the problem of studying stability according to Lyapunov for linear and stationary systems. In particular, an important analysis criterion based on the calculation of the eigenvalues of the state matrix (eigenvalue criterion) will be given. The section ends with a comparison between BIBO stability and Lyapunov stability.

Finally, in the fourth section, an important analysis criterion, known as Routh's criterion, will be presented, which makes it possible to evaluate the sign of the real part of the roots of a given polynomial, without resorting to calculating the roots themselves. This criterion naturally proves very useful both in the study of BIBO stability and in the study of Lyapunov stability.

9.1 Stability BIBO

Consider a SISO system and assume that such a system is at rest at the initial instant . Suppose further that such a system is perturbed by the appli- cation for of an external input of limited amplitude. The limiting assumption implies that there exists a constant such that

What is important to know is whether, over time, the output of such a system tends to diverge, or whether it also keeps itself limited. In other words, the question is whether even for the output , there is a constant such

In the case where this condition is met whatever external input is applied, as long as it is of limited amplitude, the system is called a stable BIBO[1] . More pre- cisely, the following formal definition applies while the physical meaning of the definitions given above is illustrated in Fig. 9.1.

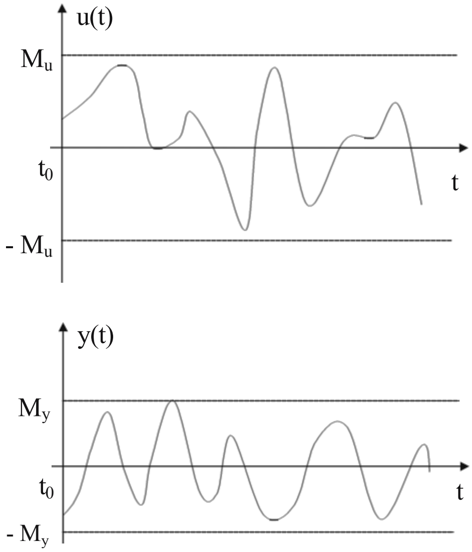


Fig. 9.1. Input and output functions of a stable BIBO system.

Definition 9.1. *A SISO system is said to be* BIBO *(*bounded-input bounded-output*)* stable *if and only if from a resting condition, each bounded input is responded to with an output that is also bounded.*

It is important to note that if a system is not BIBO stable, it does not necessarily respond with unlimited output to every limited input. However, it is true that if it is not BIBO stable, it is always possible to determine inputs whose corresponding output is unlimited, as will become evident in the following section.

[1]In the Italian literature, BIBO stability is also often referred to as ILUL (input-limited-output-limited) stability.

For *linear* and *stationary* SISO systems, the analysis of BIBO stability is greatly simplified, and there are some fundamental results in this regard. In fact, as vi- dered in previous chapters, a linear and stationary SISO system can be described in the time domain by its impulsive response. It is therefore not surprising that the property of BIBO stability is strictly dependent on the structure of the impulsive response, as shown by the following theorem.

Theorem 9.2. *Consider a SISO system that is linear and stationary. L e t its impulsive response be.* A necessary and sufficient condition *for such a system to be* BI- BO stable *is that its* impulsive response *be* absolutely summable*, i.e., exist such that*

(9.1)

Demonstration. Since BIBO stability by definition characterizes the behavior of the system from null initial conditions, we assume that the system is at rest so that its overall evolution coincides with forced evolution only. For simplicity, we assume that the instant at which the input is applied is so that, resorting to Duhamel's integral, we can write the output of the system as

(Sufficient condition.) We assume that the impulsive response is absolutely summable, that is, that (9.1) holds. The input limitation assumption allows us to state that there is a positive constant such that ,

If we consider the absolute value of the output we have that

That is, the output is also limited.

(*Necessary condition.*) To show that the absolute summability of the impulse response is also a necessary condition for BIBO stability, it is sufficient to show that, if this assumption is violated, there is at least one limited input to which the system responds with an unbounded output.

We therefore assume that the impulse response is not absolutely summable, so that

Having fixed a value of , let us further assume that the signal with

se

i*f*

the trend of which is shown in Fig. 9.2. In this case, the output at the generic instant of time holds:

where

by assumption. This implies that if the absolute summability condition of the is violated then the output resulting from the application of the limited input under consideration is not limited, so the system is not BIBO stable.

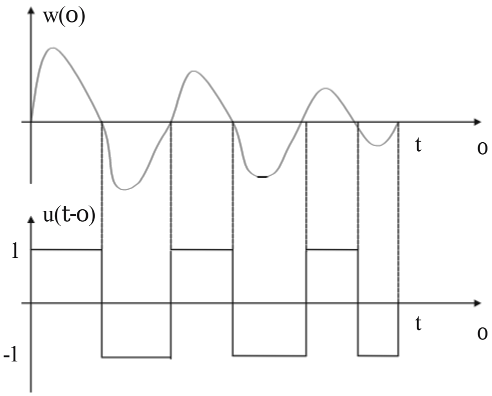


Fig. 9.2. Input function considered in the proof of Theorem 9.2

As seen in detail in Chapter 3 (see § 3.18), the impulsive response of a linear, stationary system has the form:

Æ Æ (9.2)

where they are the roots of the characteristic polynomial, that is, it is a linear combination of the modes of the system.

Note that in the following, for simplicity of discussion, we will assume that the input-output model is in *minimal form* (see § 6.4.2), that is, the transfer function has no pole coincident with a zero. Under such conditions the impulse response contains all the modes of the system i.e. , ,

.

Under such assumptions, a necessary and sufficient condition for its absolute summability is immediately derived from the examination of the structure of the immediately, which is a useful investigative tool in the study of BIBO stability in the time domain.

Theorem 9.3. A necessary and sufficient condition *for a SISO, linear, stationary, concentrated-parameter system in minimal form to be* BIBO stable *is that all the roots of the characteristic polynomial be negative real part.*

Demonstration. To prove the result, based on Theorem 9.2, it suffices to demonstrate that the impulsive response is absolutely summable if and only if the roots of the characteristic polynomial are all negative real part.

In particular, since the impulsive response is nothing but a linear combination of the modes of the system, it is easily verified that it is absolutely sum- mutable if and only if all the terms of the linear combination are absolutely sum- mutable.

It will then suffice to show that the generic mode is absolutely sum- bile if and only if it corresponds to a negative real part pole. Consider
In fact, the generic term of the impulse response , where *e*
can also be null if it is real. It is worth

if

if

BIBO stability can also be equivalently studied in terms of the transfer function if we consider that all the poles of the transfer function are also roots of the characteristic polynomial. Therefore, we can immediately state the following fundamental result.

Theorem 9.4. A necessary and sufficient condition *for a SISO, linear, stationary, concentrated-parameter system in minimal form to be* BIBO stable *is that all the poles of the transfer function are strictly negative real part.*

According to the previous theorems, a system is stable if and only if the roots of the characteristic polynomial lie in the left half-plane of the Gauss plane , i.e., they do not lie in the dashed region in Fig. 9.3.a. Note that in Fig. 9.3.a the BIBO stability region does not include the imaginary axis.

Example 9.5 Consider the system described by the input-output model

which corresponds to a simple integrator. The transfer function is worth

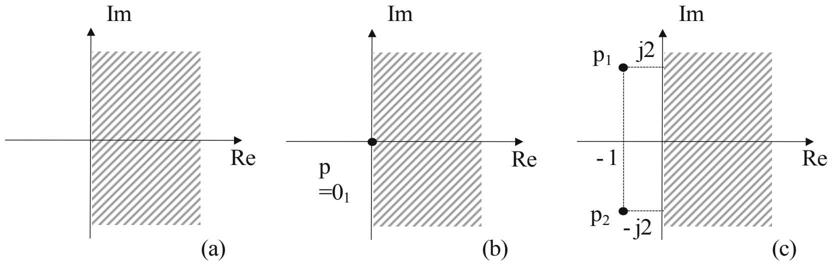


Fig. 9.3. (a) Gauss plane: the dashed region indicates the instability zone; **(b)** the root of the characteristic polynomial of Example 9.5 in the Gauss plane; **(c)** the roots of the characteristic polynomial of Example 9.6 in the Gauss plane

The only root of the characteristic polynomial is that as can be seen from Fig. 9.3.b it lies in the imaginary axis of the Gauss plane. By virtue of Theorem 9.4 we can therefore conclude that the system is not BIBO stable.

In this case it is also very easy to find a limited input to which an unlimited output corresponds. The impulse response of the system is in fact worth

Æ

so if we consider as input signal a step of amplitude , Æ , the corresponding output is worth

Æ

which diverges by .

Note, however, that there are limited inputs to which a limited output corresponds. Consider, for example, the inputIn ....that case

e .

Example 9.6 Consider the linear, stationary, concen- trated-parameter SISO system described by the input-output model

The transfer function is worth

The roots of the characteristic polynomial are and that as can be seen from Fig. 9.3.c lie in the left half-plane of the Gauss plane. The necessary and sufficient condition for BIBO stability is therefore verified so we can conclude that the system is BIBO stable

It is important to note that Theorems 9.3 and 9.4 would only give suffi- cient but not necessary conditions for stability if the input-output model is not in *minimal form*, that is, if the transfer function has one or more poles coincident with one or more zeros. See in this regard Exercise 9.2.

9.2 Stability according to Lyapunov of representations in terms of state variables

This section will introduce a different notion of stability that refers to autonomous systems described in terms of state variables and can also apply to nonlinear systems. This definition was first proposed by Russian scholar A. Lyapunov[2] and this is the reason why it is commonly called Lyapunov-style stability.

9.2.1 States of equilibrium

A dynamical system, which in this discussion we do not assume to be necessarily linear, is described in terms of state variables by a system of differential equations in the form:

(9.3)

where is a component vector function, is the state vector and
Is the vector of inputs.

The system (9.3) is said to be *autonomous* if

the input is identically zero,
the system is stationary, that is, the function does not depend explicitly on time.

In that case, equation (9.3) can be written as.

(9.4)

or

[2]Alexey Andreevich Lyapunov (Moscow, Russia, 1911 - 1973).

(9.5)

. .

For simplicity, the definition of stability that we will give below refers to autonomous systems.

The solution of eq. (9.5) starting from a time instant and an initial state defines in the state space a curve parameterized by the value of time , which we denote Such a curve is called *a state trajectory* or also Simply *trajectory* of the system.

Example 9.7 Consider the nonlinear autonomous system

(9.6)

Assume that at the instant the system is in the generic initial state and ....bothUnder such an assumption on the , it is easy to integrate the system of differential equations (9.6). In particular, posed atanh , where atanh denotes the inverse of the hyperbolic tangent, we have

(9.7)

The trajectory of the system (9.6) obtained by posing is shown in Fig. 9.4 where is indicated with an asterisk. This curve is parameterized by the value of time, and for explanatory purposes the values of time along some points have been given. As can be clearly seen from Fig. 9.4 for the trajectory converges to the point , which is also immediately verifiable from the analytical solution of the system (9.7) .

Lyapunov's stability theory is based on a fundamental notion, which is that of an equilibrium state.

Definition 9.8. *A state is an* equilibrium state*, or equivalently an* equilibrium point *for the system (9.3), if the following condition holds:*

that is, if every trajectory starting from at a generic instant of time remains in In each successive instant.

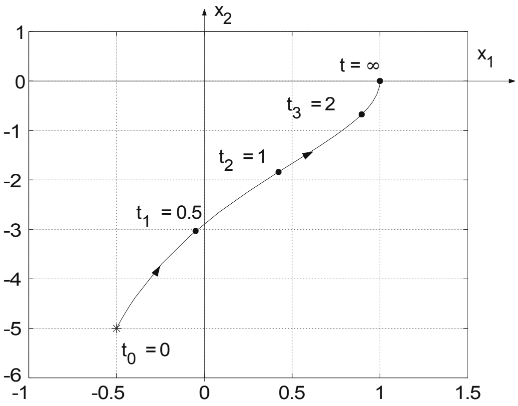


Fig. 9.4. The trajectory of the nonlinear system (9.6) from the initial condition

From a purely mathematical point of view, this implies that the constant vector is solution of the system

since in fact, if then and then the state does not vary in subsequent instants being its derivative zero.

Example 9.9 Consider again the autonomous system in eq. (9.6). It is easy to verify that such a system has 2 equilibrium states, and since these vectors are the only solutions of the systemThis ....implies that each trajectory of the system that intercepts any such point, remains at that point at each subsequent instant of time.

To better clarify the definition of equilibrium state, some trajectories of the system (9.6) obtained from different initial conditions are shown in Fig. 9.5. Note that in each trajectory a direction of travel has been given, which ap- plies how the system evolves along the trajectory as time increases. As can be seen, all trajectories that originate from a point that lies to the right of the line reach the point and since this is an equilibrium state, they remain there at each successive instant of time. Trajectories originating at any point along the line, on the other hand, terminate in the state of equili- brium Finally, trajectories originating at any point to the left of the line do not terminate in any of the equilibrium states and move away from them indefinitely as time passes

9.2.2 Definitions of stability according to Lyapunov

One of the major contributions of Lyapunov's theory is the definition of stability of an equilibrium state.

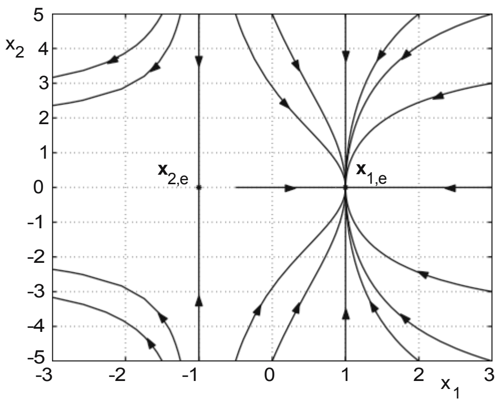


Fig. 9.5. Some trajectories of the nonlinear system (9.6) from different initial conditions

Definition 9.10. *An* equilibrium state *is said to be* stable *if for every there exists a Æ such that if Æ , then for every*
. Otherwise *is an* unstable *state of equilibrium.*

Stability in Lyapunov's sense therefore implies that if an equilibrium point is stable, the trajectory remains arbitrarily close to that point as long as the initial conditions of the system are sufficiently close to it, as illu- strato in Fig. 9.6.a. That is, taken a circle of center and radius , if the equilibrium point is stable, it is always possible to determine a new circle of radius Æ and center such that, if the initial state of the system is brought to a point in the circle of radius Æ , the system evolves with a trajectory that will never go outside the circle of radius .

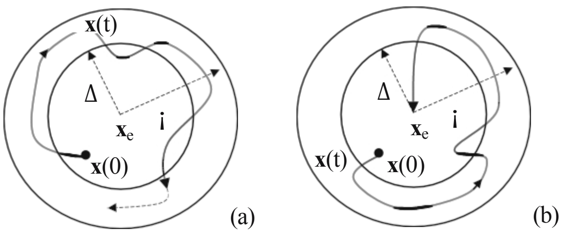


Fig. 9.6. (a) stable equilibrium state and a representative trajectory; **(b)** asymptotically stable equilibrium state and a representative trajectory

Sometimes the notion of stability is also introduced by assuming that the equilibrium state under study coincides with the origin.

Definition 9.11. *The* origin *is a state of* stable *equilibrium if for every there exists a Æ such that if Æ , then for ....everyI n case contrary the origin is a state of* unstable *equilibrium.*

Note that if a system is nonlinear, the stability of one equilibrium state does not imply the stability of the other equilibrium states.

Example 9.12 Consider the system (9.6) which, as discussed above, has two different equilibrium states and In particular, it can be shown that of these equilibrium states only is stable. In general, such a demonstration requires the ap- plication of appropriate stability criteria, which will be presented only in the following. However, in the present case this conclusion can easily be drawn by applying precisely the definition of stability of an equilibrium state.

Let us first fix our attention on the point Consider any value Of as long as it is and is assumedÆ It is immediate to show that all trajectories having origin in the circle of center and radius do not leave that circle, or equivalently, whatever is such that Æ , then for every , which is precisely the stability condition. In fact, given system (9.6), we can state the following. If the state of the system is at a point where , being at such a point , the present value of decreases until it meets the x-axis. Conversely, if the state is located in a point at which , being at that point , the value of grows until it meets the x-axis. Finally, if the state is at a point on the x-axis, it will not be able to move away from that axis by being thereAnalogue.., if the state is at a point whose abscissa is not more than , and that point is to the right (left) of , being at that point ( ), the state will evolve in the direction of .

Note that if we set a value of we could no longer assume Æ but would have to place Æ where it is any constant as long as it is , and this regardless of the particular value of chosen.

Consider now the equilibrium state In that case, it is no longer true that the traiet- tories having their origin at a point "sufficiently" close to are maintained in a neighborhood of that point. In fact, any trajectory having its origin at a point to the left of , even if arbitrarily close to it, recedes indefinitely from that equilibrium state as time passes, being at that point ....

Example 9.13 Consider the autonomous linear system

(9.8)

L’origine è chiaramente l’unico punto di equilibrio essendo l’unica soluzione del sistema

In particular, it can be shown that such an equilibrium state is stable. As clarified above, determining whether an equilibrium point is stable or not generally requires the application of appropriate criteria. However, even in this case, given its simplicity, this conclusion can be drawn on the basis of the definition of stability itself. In the case now under consideration, the matrix holds

and its eigenvalues are ....worthAs seen in Chapter 5, the solution of the
linear system with initial state holds *for*

where in the present case the state transition matrix is

The evolution of the system having its origin at a generic point is therefore governed by the equation

or

Eliminating the time dependence, we obtain the following trajectory

which coincides with an ellipse intersecting the axes at the points

Now, taken any one assumes *Æ* It is evident that - see in this regard Fig. 9.7-any trajectory having origin in a circle of radius , is always maintained within the circle of center and radius , which is precisely the stability condition of the origin .

It is interesting to note that a system can also have an *infinite* number of equilibrium states. Moreover, if the system is nonlinear such equilibrium states may also be *isolated*, and some of them may be stable and others unstable.

Example 9.14 Consider the nonlinear autonomous system

(9.9)

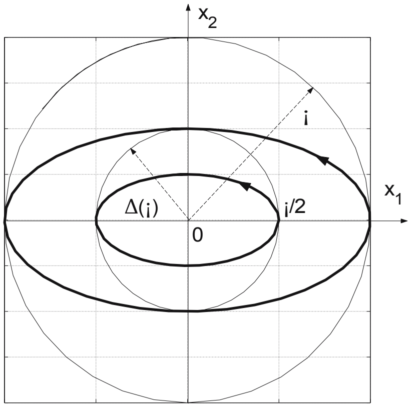


Fig. 9.7. Ellipsoidal trajectories of the system (9.8) and contours of the origin of radius and Æ

Such a system has an infinite number of equilibrium states i.e., all points

—

The infinite equilibrium points of the system (9.9) therefore all lie along the line of equation .

Through appropriate stability criteria, which we will see in the following, it is easy to dimo- strate that all equilibrium states related to odd values of are stable, while those related to even values of , including , are unstable equilibrium points (see Exercise 12.6).

A number of trajectories of the system (9.9) obtained from different initial con- dictions are shown in Fig. 9.8. The different initial conditions considered are denoted by an asterisk. Depending on these initial conditions (unless these coincide with an unstable equilibrium point) the trajectories obtained converge to one of the stable equilibrium points. The points of unstable equilibrium have been denoted with a circle instead. If the system were initially in one of these states, since these are equilibrium states, the state of the system would hold indefinitely at these points. However, if the initial conditions were to be perturbed with respect to such points, even if by an infinitesimal amount, the state of the system would move to one of the stable equilibrium points .

A stronger concept than stability is that of asymptotic stability, which also requires the satisfaction of a boundary condition. More precisely, asymptotic stability not only requires that the trajectory of the perturbed system be maintained in a neighborhood of the equilibrium point, but also requires that for that trajectory to be brought precisely to coincide with the equilibrium point. This concept is intuitively illustrated in Fig. 9.6.b and formally expressed as follows.

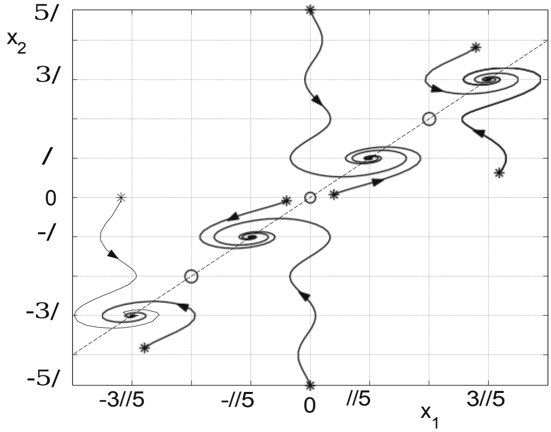


Fig. 9.8 Some trajectories of the nonlinear system (9.9) from different initial conditions

Definition 9.15. *An* equilibrium state *is said to be* asymptotically stable *if both of the following conditions hold:*

(i) For every there exists an Æ such that if Æ then
For each ;
(ii)

Condition (i) of Definition 9.15 is nothing but the stability condition so we can simply say that an equilibrium state is asymptotically stable if it is stable and if the .

Definition 9.16. *The* origin *is an* asymptotically stable *equilibrium state if both of the following conditions hold:*

(i) For every there exists an Æ such that if Æ then
For each ;
(ii) .

Example 9.17 The stable equilibrium states of systems (9.6) and (9.9) are all also asymptotically stable.

In contrast, the origin is a stable equilibrium state for the system (9.8) but not asymptotically stable. In fact, as can be seen from Fig. 9.7, the trajectories of such a system are ellipses traveled counterclockwise whose points of intersection with the axes depend on the particular initial conditions .

It is important to note that the boundary condition alone does not give any information about the stability of an equilibrium state. That is, it may happen that all trajectories originating in a neighborhood of the equilibrium state are brought to coin- cide with the equilibrium state itself even though the stability condition is not verified.

Example 9.18 Consider the nonlinear autonomous system

(9.10)

The origin is clearly an equilibrium point for the system (9.10). It can also be shown that all trajectories of such a system converge for at the origin itself. Nevertheless, the origin is not a stable equilibrium point. In fact, from the os- seration of the trajectories shown in Fig. 9.9, it is easy to be convinced that all trajectories starting from a point in the fourth quadrant (,) remain always above the dashed curve, then moving to the first quadrant where they finally converge to the origin.

So it is not true that for every there exists a Æ such that if Æ then for everyBe sure to choose any , where indicates the maximum distance of the dashed curve from the origin for this condition not to be verified .

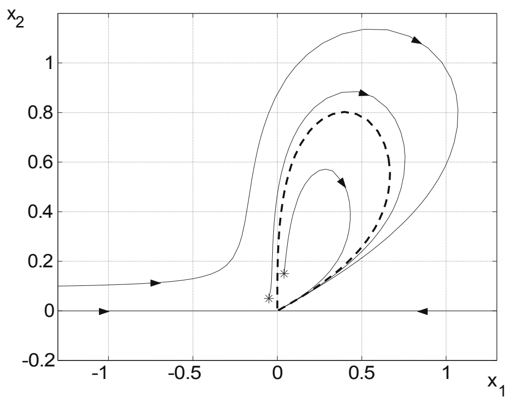


Fig. 9.9 Some trajectories of the nonlinear system (9.10) from different initial conditions

Note that an equilibrium state that verifies the boundary condition of asin- totic stability but is unstable is called a *point of attraction* for the system. Therefore, the origin is a point of attraction for the system (9.10).

All the definitions given above, referring to a neighborhood of the equilibrium point, allow us to characterize exclusively the *local* behavior of the system, that is, they make it possible to characterize its response in the case where the system is sot- toposed to small perturbations near the equilibrium state. The set of possible initial conditions from which there is asymptotic stability constitutes the *domain of attraction*. In case the domain of attraction coincides with the entire state space, one speaks of *global asymptotic stability*.

Definition 9.19. *If an equilibrium state is asymptotically stable whatever the initial state from which the system's trajectory originates, then that equilibrium state is said to be* globally asymptotically stable.

Note that if a system has a globally asymptotically stable equilibrium state, then this is also the system's *only* equilibrium state.

In real engineering problems global asymptotic stability is a desirable property although very often difficult to achieve. The problem then is relaxed and we settle for the local property alone. However, it becomes important in this case to identify the largest region of asymptotic stability, i.e., the *domain of attraction*. The solution of such a problem is in general very complex.

Example 9.20 Consider the system (9.6). As seen above, the equilibrium point is asymptotically stable. However, this point is not globally asymptotically stable because its domain of attraction does not coincide with the entire state space, but only with the half-plane to the right of the line of equation

, of which such tuition is not a part.

In contrast, the system (9.9) has an infinite number of asymptotically stable equilibrium states, none of which can therefore be globally stable. In this case, however, it is not as straightforward to determine their domains of attraction.

Example 9.21 Consider the linear autonomous system

(9.11)

Chiaramente tale sistema ha un solo punto di equilibrio che coincide con l'origine. Infatti la matrice

has eigenvalues and the state transition matrix is worth

The evolution of the system having origin at a generic point is therefore worth

or

It can be easily observed that the origin is a globally asin- totically stable equilibrium point because, regardless of the initial condition, the trajectory converges to the origin, and for every the following holds true

As an example, a trajectory of such a system obtained from the initial condition , indicated in the figure by an asterisk, is shown in Fig. 9.10. Such a trajectory shows the convergence to the origin which, by virtue of the global asymptotic stability of the origin, occurs whatever the initial condition chosen

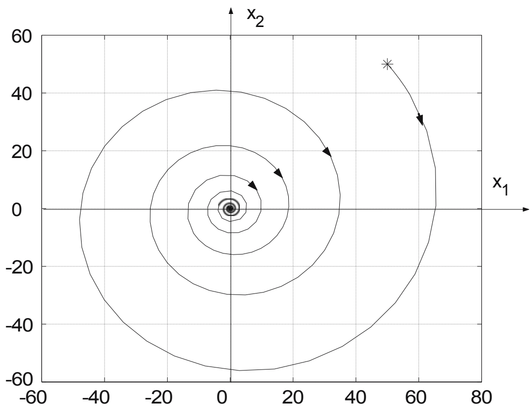


Fig. 9.10. The trajectory of the linear system (9.11) obtained from the initial condition

9.3 Stability according to Lyapunov of linear systems and stationary

In this section we will fix our attention on linear, and consistent with what we have done so far, autonomous systems.

9.3.1 States of equilibrium

For an autonomous linear system, eq. (9.5) reduces to

(9.12)

or

. .

The equilibrium states of an autonomous linear system can be characterized as follows.

Proposition 9.22 *Given an autonomous linear system , the state is an equilibrium point if and only if it is a solution of the homogeneous linear system*

The following results immediately follow from this.

If the matrix is nonsingular, the only equilibrium state of the system is That is, the origin.

Conversely, if it is singular then the system has an infinite number of equilibrium states describing a linear space: these are all the points contained in the null space of . Therefore, an autonomous linear system cannot have isolated equilibrium states (whether they are finite or infinite in number), as is possible in the case of nonlinear systems.

Demonstration. The validity of this proposition follows immediately from what we saw in Appendix C about systems of linear equations (see Theorem C.42).

Example 9.23 Consider the linear system (9.8) whose state matrix is

Such a matrix is nonsingular being . From this it follows that the origin is the only equilibrium state of the system. Note that this result is in agreement with what was said in Example 9.13 in which the possible equilibrium states of the system were computed by solving the homogeneous linear system .

Similarly, the origin is the only equilibrium state for the system (9.11) considerated in Example 9.21 being in that case the state matrix

not singular: in fact, it applies .

Example 9.24 Consider the autonomous linear system

where

In that case the system has an infinite number of equilibrium states being the *m a t r i x* singular.

In particular, equilibrium states are all points in the state space that lie on the line of equation . Such points are in fact all solutions of the homogeneous linear system

9.3.2 Stability of points of balance

The study of stability in the case of linear systems is greatly simplified. In fact, the following fundamental result, which is given without demonstration, is valid.

Proposition 9.25 *Given an autonomous linear system:*

if one equilibrium state is stable (unstable), this implies that all other possible equilibrium states are also stable (unstable);

If an equilibrium state is asymptotically stable, then the following three results hold:

1. *, that is, this state coincides with the origin;*
2. *Is the only equilibrium state of the system;*
3. *is globally asymptotically stable, that is, its domain of attraction coincides with the entire state space.*

The preceding proposition explains why in the case of linear systems it is permissible to speak of *stable system*, i.e., *asymptotically stable system*, or *unstable system*, rather than referring these properties to the generic equilibrium state.

In the area of linear and stationary systems, there are numerous stability results. The stability analysis criterion that will be presented in the following is the most commonly used and is based on the calculation of the eigenvalues of the matrix .

Theorem 9.26 (Criterion of eigenvalues).

Consider the linear and stationary system

Such a system is asymptotically stable if and only *if all the eigenvalues of the matrix have negative real part.*

Such a system is stable if and only if *the matrix has no positive real-part eigenvalues and any zero real-part eigenvalues have unit index .*[3]

[3]Recall that the index associated with an eigenvalue was introduced in Definition 4.32: it represents the length of the longest chain of generalized eigenvalues associated with an eigenvalue. Moreover, by virtue of Proposition 4.34, having taken the matrix back to its Jordan form, the index of an eigenvalue coincides with the order of the largest Jordan block among those related to the eigenvalue itself.

Such a system is unstable if and only if *at least one eigenvalue of has positive real part, or null real part and index* .

Demonstration. (Asymptotic stability.) Suppose for simplicity . As seen in detail in Chapter 4 the free evolution of the state for is worth

Where each term of the state transition matrix

Is a particular linear combination of the modes of the system.

Each real eigenvalue of index corresponds to modes of the type

Clearly

if and only if .

Moreover, at each pair of complex conjugate eigenvalues of index correspond modes of the type

whose envelope curves are ....properTherefore, such modes are also limited and

are extinguished if and only if , thus proving the validity of the first point, i.e., the necessary and sufficient condition for asymptotic stability.

(Stability.) (*If*) Suppose there are no positive real-part eigenvalues and each null real-part eigenvalue has unit index. In such a case, the elements of the state transition matrix are linear combinations of modes of two types. The modes due to any negative real-part eigenvalues, as already seen, are extinguished forEach null real-part eigenvalue corresponds to a unique

mode (its index being unit) of the type (eigenvalue) , or

(pair of eigenvalues) ; modes of this type are kept limited as time increases.

(*Only if*) If vice versa there was a null real part eigenvalue with index

The state transition matrix would in this case have elements with-holding linear combinations of terms of the type (eigenvalue), or (pair of eigenvalues) with that clearly diverge for and .

Finally, an eigenvalue with positive real part corresponds to divergent modes implying non-stability.

(Instability.) It follows immediately from what was seen in the previous points.

Example 9.27 Consider the linear system

dove

Such a system clearly has infinite equilibrium states coinciding with all points in space .

The matrix has a single eigenvalue of ....multiplicityHe is also
diagonal for which the eigenvalue index ....holdsThe system under consideration is therefore
Stable but not asymptotically stable.

It is easy to verify that the state transition matrix is worth

and thus the evolution of the system having origin at a generic *point*
vale

Therefore, whatever initial condition is chosen, the system will remain in that condition at every instant of time thereafter .

Example 9.28 Consider the linear system

dove

Such a system has infinite equilibrium states, that is, all points along the line of equation
.

Again, the matrix has a single eigenvalue of multiplicity
. Since the matrix is already in Jordan form, we can immediately affirm that it is nonderogatory and the eigenvalue index holds . Therefore, the system under consideration is unstable.

In particular, the state transition matrix is worth

and thus the evolution of the system having origin at a generic *point*
is worth , i.e.

Therefore, whatever initial condition is chosen, unless it coin- cides with any point on the line , the component tends to . In particular, we can observe that all trajectories are parallel to the line , being const. Furthermore, if the initial condition is characterized by a positive (negative) value of , the state of the system moves toward increasing (decreasing) values of .

For the sake of clarity, some trajectories of the system under consideration obtained from different initial conditions have been shown in Fig. 9.11. As usual, the initial condition of each trajectory has been denoted by an asterisk .

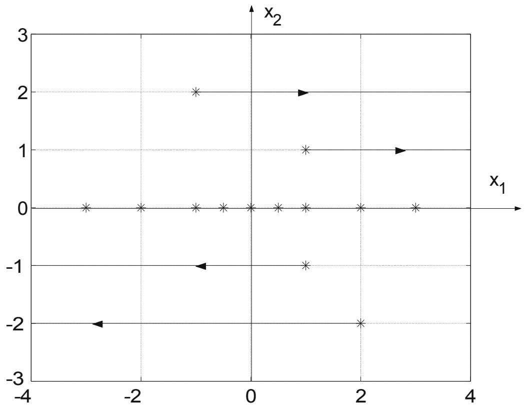


Fig. 9.11. Some trajectories of the linear system of Example 9.28 obtained from different initial conditions

9.3.3 Examples of analysis of stability

In this section we will study some examples of linear systems in detail. Note that although stability can be studied by analysis of the self-values of the state matrix alone , in some cases for greater clarity we have also integrated the equation of state and calculated the trajectories of the system.

Example 9.29 Consider the linear system of Example 9.24. The eigenvalues of the matrix are and so being singular we can immediately conclude that the system has an infinite number of equilibrium states. Moreover, since the eigenvalue is nonzero at negative real part, these equilibrium states are all stable.

This result is in agreement with what we saw in Example 9.24. Specifically, in that example we had calculated that the infinite equilibrium states all lie on the line of equation .

The solution of the linear system with initial state holds for ,
Where the state transition matrix is worth

L'evoluzione del sistema avente origine in un generico punto vale dunque ovvero

and for , it is easy to see that

By eliminating the time dependence, it is also possible to verify that the generic trajectory moves along the line of equation

This result is also highlighted in Fig. 9.12 where some trajectories of the system obtained from different initial conditions are shown. Clearly, all trajectories terminate at a point on the line of equation .

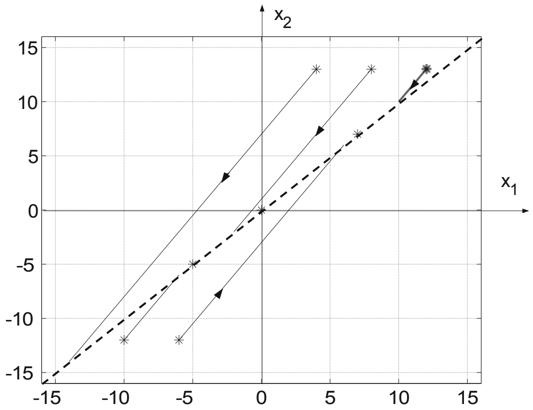


Fig. 9.12. Trajectories of the linear system of Example 9.29 from different initial conditions

Example 9.30 Consider a linear autonomous system whose matrix is worth

The eigenvalues of the matrix are so they are both positive real part. Therefore, we can say that the origin is the only equilibrium state of the system (being nonsingular) and it is unstable (being the eigenvalues of a positive real part).

It is left as an exercise to the reader to determine the evolution over time and the state trajectory similarly to what was done in Example 9.21. Note that the dynamic matrices in the two examples are the opposite of each other. By way of example, Fig. 9.13 shows the trajectory of the system obtained from the initial conditions As can be seen, the state of the system moves away from indefinitely from origin to the passage of time.

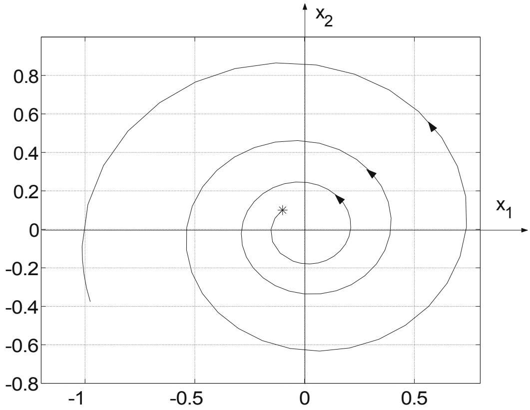


Fig. 9 .13. The trajectory of the linear system in Example 9.30 obtained from the *initial* condition

Example 9.31 Consider a linear autonomous system whose matrix is worth

Gli autovalori della matrice sono e.............................Dati i due autovalori a positive real part, again we can immediately conclude that the origin is the only equilibrium state of the system and it is unstable .

Example 9.32 Consider a linear autonomous system whose matrix is worth

Gli autovalori della matrice sono e........................La matrice è singolare so the system has an infinite number of equilibrium states. Since the two eigenvalues are positive real part, all infinite equilibrium states are unstable.

9.3.4 Comparison of BIBO stability and stability at Lyapunov

It is important at this point to relate the two different concepts of stability seen, namely BIBO stability and Lyapunov-style stability.

Consider a SISO system described by the model in terms of state variables si

Both the order of such a representation, i.e., the number of components of the vetto- ...reThe stability according to Lyapunov of such a representation can be studied
from the analysis of the ....state matrixIt is, however, possible to study for such a system
BIBO stability as well. In that case we will have to refer to its transfer function, which is worth

Proposition **9.33** *Consider a linear, stationary SISO system described me- dially by a model in terms of order state variables and be its transfer func- tion in* minimum form. *If the denominator of the is of degree then the system is* BIBO stable *if and only if it is* asymptotically stable.

Demonstration. The denominator of the coincides with or with the characteristic polynomial of the matrix . If the in minimal form has at the de- nominator a polynomial of degree then there are no zero-pole cancellations and therefore the poles of the coincide with the eigenvalues of the matrix . The rappre- sentation is asymptotically stable if and only if all the eigenvalues of are in , that is, if and only if all the poles of are in , that is, if and only if the system is BIBO stable.

Example 9.34 Consider the following linear and stationary SISO system.

where

The transfer function of such a system is worth

per cui non vi sono cancellazioni polo-zero. Essendo inoltre i poli della ,
, both with negative real part, such a system is BIBO stable. By virtue of Proposition 9.33 it follows that such a system is also asymptotically stable, or more precisely, the origin is a globally asymptotically stable equilibrium point.

Note that if we had not imposed that the transfer function be in minimal form, the result above would not have been valid. That is, it could have acca- dered that a number of modes of the representation in terms of state variables were not present in the input-output representation. If such modes were precisely the only unstable modes of the system, it would have turned out to be BIBO stable while not being stable according to Lyapunov.

Example 9.35 Consider the following linear and stationary SISO system.

where

It is easy to verify that such a system is unstable being the eigenvalues of equal to e .

However, such a system is BIBO stable being

and therefore the only root of the characteristic equation holds and has negative real part.

Finally, it is important to note that if a system is stable according to Lyapunov but not asymptotically stable, then it is not BIBO stable.

Example 9.36 Consider the system

whose state matrix, which in this case coincides with a scalar, is worth ....The system is therefore stable according to Lyapunov but not asymptotically stable.

Such a system is also non-BIBO stable. Its IU bond is *in fact* worth (see Example 9.5).

9.4 Criterion of Routh

In the previous paragraphs we have seen that if the transfer function of a system is placed in its minimum form, the BIBO stability of such a system is uniquely determined by the sign of the real part of its poles that coincide with the roots of the polynomial at the denominator of theAnalogously we have seen that the stabi-
lity at Lyapunov depends on the sign of the real part of the eigenvalues of the dynamic matrix , i.e., the sign of the roots of the ....polynomialI n both
cases then the study of stability reduces to determining the sign of the real part of the roots of a polynomial

(9.13)

The calculation of the roots of such a polynomial can become very complex when the order of such a polynomial is high. Several cry- ters have been proposed in the literature to determine the sign of the real part of the roots of such a polinomial without calculating its exact value. The most frequently used criterion, which will be presented in detail below, is called *Routh's criterion* and was proposed by Routh[4] more than a century ago, when the need to avoid root computation was even more stringent, given the lack at that time of automatic computation systems. It should be noted that the same problem can also be solved by another criterion, called *the Hurwitz criterion*, which was developed quite indipendently by the German scholar Hurwitz[5] . However, this second criterion will not be presented here.

In the remainder of the discussion we will assume that the following applies.

(9.14)

In fact, this assumption does not harm the generality of cases since if *it were* il polinomio potrebbe fattorizzarsi nella forma dove il polinomio di grado soddisfa la condizione (9.14). Potremmo dunque riferire quanto segue al solo polinomio ricordando poi che il polinomio avrà le stesse radici di con l'aggiunta, a causa del fattore messo in evidenza, di una radice di molteplicità .

Now, before presenting Routh's criterion, let us state the following important results.

The first result, also known as *Descartes' rule*[6] , relates to a second-degree polynomial and provides necessary and sufficient conditions for the roots of the polynomial to belong to the negative complex half-plane.

Theorem 9.37. Necessary and sufficient condition *so that the roots of the polynomial of the second degree*

all have negative real part is that the coefficients , , are all of the same sign.

Demonstration. Without harming the generality of the cases, suppose that it is . If it were not, it would be sufficient to divide by each coefficient of the polynomial[7] without any change in its roots.

Be , the two Vale roots obviously

or and .

[4]Edward John Routh (Quebec, Canada, 1831 - Cambridge, Britain, 1907).

[5]Adolf Hurwitz (Hanover, Germany, 1859 - Zurich, Switzerland, 1919).

[6]René Descartes (Tours, France, 1596 - Stockholm, Sweden, 1650) also known by his Latin name Descartes.

[7]Since by hypothesis the polynomial is of degree 2, it must be worth .

Suppose that the two roots are real. The coefficients and will both be positive (and therefore of concordant sign with) if and only if and , that is, if and only if the roots are both negative.

Supponiamo ora che le radici siano complesse coniugate e indichiamo tali radici come ....In tal caso vale

e

The polynomial can therefore be written as.

with

Il termine costante è certamente positivo mentre il coefficiente del termine in è positivo se e solo se , ovvero se le due radici sono entrambe a parte reale negativa.

The second result, on the other hand, relates to a generic degree polynomial and provides conditions only necessary for all roots of the polynomial to belong to the negative complex half-plane. Note that this theorem can be seen as a simplified test if it is not important to determine the number of unstable modes, if any.

Theorem 9.38. Necessary condition *for the roots of the polynomial of degree ,*

all have negative real part is that all the coefficients , for , are of the same sign.

Demonstration. Without harming the generality of the cases, suppose that it is . If not, it would be sufficient to divide by each coefficient of the polynomial [8] without resulting in any change in its roots.

Suppose the polynomial has real roots and complex root pairs. Therefore, it is worth Let us now denote by , for , the generic real root and with , for , the generic pair Of conjugate complex roots. The polynomial can be factorized as.

with (9.15)

It is evident, therefore, that if all the roots have negative real parts, that is, if for , then developing the prodactor yields only positive terms. So no coefficient of the polynomial can be negative or zero.

[8]Essendo per ipotesi il polinomio di grado , deve valere .

By virtue of Theorem 9.38, we can immediately conclude that if any coefficient of the polynomial (9.13) is negative or zero, the system under consideration is not BIBO stable or not asymptotically stable, depending on the problem studied. Conversely, since this condition is only necessary but not sufficient (except for) , in the case where all the coefficients of the polynomial are positive, the application of Routh's criterion is still necessary to complete the stability analysis.

The Routh criterion involves the construction of a table, called *a Routh table* obtained from the coefficients of the polynomial (9.13).

The first two rows of the table are formed by the coefficients of the polynomial (9.13), arranged starting with the one corresponding to the highest power. The elements in the third row are evaluated from the first and second rows as follows

until all null elements are obtained. Similarly, the coefficients of the fourth ri- ga are obtained from those of the previous two rows according to the following scheme

The process is to be iterated until ------------------------the index row is completed .

Note that in compiling the table an entire row may be divided or multiplied by a positive constant in order to simplify the calculations that follow.

Clearly, the table can be completed if and only if no null elements appear in the first column.

In case the table can be completed, Routh's criterion, formally expressed through the following theorem, allows us to evaluate the sign of the real part of the roots of the polynomial (9.13).

Theorem 9.39 (Routh's Criterion). *Given a generic polynomial, consider the Routh ta- bella that corresponds to it. The number of positive real-part roots of such a polynomial is equal to the number of sign changes of the coefficients in the first column of the table, considered consecutively. On the other hand, the number of roots with negative real part is equal to the number of permanences.*

The proof of this theorem is quite complex and will therefore not be reported.

Example 9.40 Consider the polynomial

Routh's table can be completed in this case and is as follows:

1	3	5
4	8	
1	5	
-12		
5		

In the first column we count 2 permanence and 2 sign changes. In fact, it is worth

For. per var. var.

1 4 1 -12 5

This leads to the conclusion that the polynomial under study has two negative real part roots and two positive real part roots. The roots of the polynomial are *in fact*

, , .

Example 9.41 Consider the polynomial

Routh's table in this case can be completed and is as follows:

8	7	1	
3	2		
5	3		(the line was multiplied by 3)
1			(the line was multiplied by 5)
3			

All the coefficients in the first column are positive so we do not count any sign change. We can therefore conclude that all the roots of the polynomial under consideration are negative real part. The roots of the polynomial under consideration are in fact

, .

As mentioned earlier, Routh's table may not always come com-
pletate. In particular, two *singular cases* may arise during its construction that do not allow it to be completed:

1. the first term of a row is null but the corresponding row is not identically null,
2. an entire line is null. Let us

discuss the two cases in detail.

Case 1

If the first element of a row is null but the cor- responding row is not identically null, we can immediately state that there are positive real part roots[9] . To estimate the number of such roots, simply complete the table by substituting a very small arbitrary constant of positive sign for the null term. It can be shown that the number of positive real part roots is equal to the number of sign changes that are counted in the first column of the table thus completed.

Example 9.42 Consider the polynomial

Proceeding to construct Routh's table, we see that it cannot be completed because a null element appears in the first column of the third row:

1	5	2
1	5	
0	2	

We can immediately conclude that the polynomial under study has positive rea- le roots. To know how many there are we substitute a constant and arbitrarily small for zero. We then continue with the construction of the table:

1	5	2
1	5	
	2	
2		

[9]Note that in such a case the system of which is characteristic polynomial is unstable (both in the BIBO sense and in the Lyapunov sense) since there is at least one positive real part root.

Being

we can conclude that there are 2 sign changes in the first column of the table thus obtained. Therefore, the polynomial under consideration has 2 positive real part roots.

It can also be observed that we could have equivalently imposed *that*

Had a negative sign. In this case we would have had

but we would still have counted two sign changes in the first column. By convention we always assume that it is .

It can be shown, by numerically determining the roots of the polynomial under consideration, that these are: , .

Case 2

Let us now examine the case in which a row in Routh's table is identically null. In this case the polynomial under consideration surely has either null real-part roots or even positive real-part roots. It may be useful, however, to verify that there are only null real-part roots that correspond to periodic modes: if this condition is verified, it is also usual to say, in somewhat imprecise terms, that the system whose characteristic polynomial *is at the stability limit*.

First we observe (the demonstration of this is for brevity omitted) that in the construction of the table only a row of odd index can cancel[10] . Let the index relative to that row be. It can be shown that the polynomial under consideration can be factorized as follows:

Where:

is a polynomial of ....degreeThe sign of its roots can be evaluated

examining the first column of the table relative to the first few rows (the rows above the null row). For such rows, the usual rule applies: each change in sign corresponds to a positive real part root, and each permanence corresponds to a negative real part root.

is a polynomial of degree that contains no terms of odd degree and is called *an auxiliary polynomial*. It is the polynomial in the variable constructed by the coefficients of the row preceding the null row, i.e., the index row

.

For example, if we denote by , , , , the coefficients of the index row, the auxiliary polynomial results to be

[10]Note that this is true only under the assumption that it is , that is, under the assumption that there are no roots in .

To deduce information about the sign of the real part of the roots *of* we proceed as follows. We indicate in the following as.

- The number of positive real part roots,
- The number of negative real part roots,
- the number of roots with real null part

Of the auxiliary polynomial .

We can immediately observe that the roots of the auxiliary polynomial, lacking terms of odd degree in it, are certainly arranged symmetrically with respect to the origin, as shown in Fig. 9.14. In fact, if we were to place , the auxiliary polynomial in would recur to the polynomial of degree *in*

(9.16)

The polynomial (9.16) has in general real roots (negative and positive) and immaginary roots (complex conjugate). Recalling that , we can draw the following conclusions.

Ad ogni radice reale negativa di (9.16) corrispondono due radici immaginarie pure del polinomio caratteristico disposte nel piano di Gauss come mostrato in Fig. 9.14.a.

Ad ogni radice reale positiva di (9.16) corrispondono due radici reali of the characteristic polynomial arranged in the Gauss plane as in Fig. 9.14.b.

Ad ogni coppia di radici complesse coniugate, e , di (9.16) corrispondono due coppie di radici complesse coniugate simmetriche rispetto all'origine disposte nel piano di Gauss come mostrato in Fig. 9.14.c [11].

We can then conclude that the auxiliary polynomial has as many positive real-part roots as it has negative real-part roots, that is, , plus any number of null real-part roots.

Per la determinazione del segno delle radici del polinomio ausiliario e dell'eventuale numero di radici a parte reale nulla, si procede come segue. Si deriva il polinomio ausiliario rispetto alla variabile e si sostituiscono i coefficienti di in luogo degli zeri nella riga identicamente nulla. A questo punto si completa la tabella di Routh seguendo lo schema visto in precedenza e per valutare il segno delle radici del polinomio ausiliario si considerano le righe a partire da quella di indice

[11] It can be shown that if in polar form it holds. , then the 4 roots are worth:

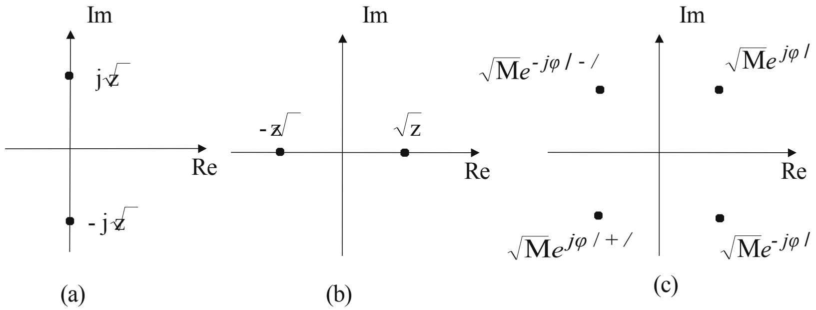


Fig. 9.14. Possibili disposizioni delle radici della equazione ausiliaria: **(a)** radice reale negativa, **(b)** radice reale positiva, **(c)** radici complesse coniugate

, that is, the one preceding the row containing the coefficients of .
In general from that line we will count a number *o f* sign variations. Based on the considerations made earlier we have seen that each variation corresponds to a positive real part root, but for reasons of symmetry each variation also corresponds to a negative real part root, and therefore

Also, having to be or

equivalently,

we can state that the possible number of null real part roots is equal to

From this it follows that if , that is, we do not count any sign changes in the- the table thus completed, then all the roots of the auxiliary polynomial are pure imaginary.

Example 9.43 Consider the polynomial

Proceeding with the construction of Routh's table, we see that it cannot be completed since the third row is found to be identically null:

	1	4 7
	1	4 7
	0	0

Since the first column has no sign changes at the first two rows, we can immediately state that at least one root is negative real part. We can also state that the other 4 roots are symmetrical with respect to the origin and that there will be at least one pair of pure imaginary roots or one root with positive real part.

To evaluate exactly the distribution of such roots we then define the auxiliary polynomial

whose derivative with respect to is worth

We substitute the coefficients of into the null row (which we rename) and continue to construct Routh's table:

	1	4 7
	1	4 7
	4 8	
	2 7	
	-6	
	7	

To derive information about the remaining 4 roots we need to examine the table thus obtained only from row 4 i.e., the row preceding the one in which the coefficients ....ofBecause we count two sign changes ( ) e due permanenze, possiamo concludere che il polinomio ausiliario ha due radici a parte reale positiva e due radici a parte reale negativa. Il polinomio di partenza ha pertanto due radici a parte reale positiva e tre radici a parte reale negativa.

Si verifica facilmente che in questo caso il polinomio in esame può venir fattorizzato come . Il polinomio ha una radice reale negativa . Il polinomio ausiliario ha radici , : such roots are symmetrical with respect to at the origin and arranged as in the case in Fig. 9.14.c.

Example 9.44 Consider the polynomial

Again, we cannot complete Routh's table because the only element in the index row 1 is null:

	1	4	3
	2	4	2
	2	2	
	2	2	
	0		

However, we can immediately state, on the basis of the first 4 rows, that the polynomial has 3 negative real part roots, since we count 3 sign permanences in those rows. We can also conclude that the other 2 roots are symmetrical with respect to the origin.

To complete the table we construct the characteristic polynomial

and substitute the coefficient of

In place of zero in index line 1:

1	4	3
2	4	2
2	2	
2	2	
4		
2		

From examination of the last 3 rows of the table thus obtained we see that in the first column there are no sign changes. By what was said earlier this means that the other 2 roots of the polynomial under study (coincident with the roots of the auxiliary polynomial) are pure imaginary. This is moreover immediate to verify being a polynomial of the second degree. In this case, therefore, the system having which characteristic polynomial is in the stability limit.

Finally, it is easily verified that it is ....worthThe *polynomial* has *roots* *e* The auxiliary polynomial has two conjugate imaginary roots .

It is important to note that, although the existence of numerical procedures that con- sent the determination of the roots of a polynomial of the type (9.13) has greatly reduced the practical importance of Routh's criterion, it continues to provide an important tool for analysis in the case where not all the parameters of the model are exactly known. See the following example for this purpose.

Example 9.45 Consider the polynomial

(9.17)

whereWe ......want to determine how the sign of the real part of the roots of the polynomial under consideration as the .

We first observe that in the case where , the polynomial under consideration can be rewritten as

Thus it is evident that this polynomial has a root coincident with the origin and, by virtue of Theorem 9.37, two negative real part roots.

Moreover, in the case where , by virtue of Theorem 9.38, we can immediately affirm that such a polynomial has at least one null real part root or one positive real part root.

To understand how the sign of the real part of the roots of that polynomial varies as the , with , we construct Routh's table:

1	6		
5			
			(la riga è stata moltiplicata per 5)

The following cases may occur.

La prima colonna della tabella di Routh non presenta variazioni di segno, ossia il polinomio in esame ha 3 radici a parte reale negativa. Ciò si verifica quando e , ossia per .

L'elemento in corrispondenza della riga di indice 1 è positivo, ma risulta negativo il termine nella riga di indice 0. Questo caso si presenta quando e fa sì che nella prima colonna si contino 2 permanenze e una variazione di segno. Il polinomio in esame ha pertanto per valori negativi di , 2 radici a parte reale negativa e una a parte reale positiva.

L'elemento in corrispondenza della riga di indice 1 è negativo, mentre il termine nella riga di indice 0 è positivo. Questo caso si presenta quanto e fa sì che nella prima colonna si contino 2 variazioni di segno e una permanenza. Il polinomio in esame ha pertanto due radici a parte reale positiva e una a parte reale negativa.

La riga di indice 1 si annulla. Ciò è vero quando................In questo caso, poiché in the first column, at the rows of index 3 and 2 we count a sign permanence, we can immediately conclude that the polynomial has a negative real-part root. The other two roots, however, will be either positive real part or zero real part, and certainly symmetric with respect to the origin. To complete our analysis, we construct the auxiliary polynomial

Essendo un polinomio di secondo grado, è immediato calcolare le sue radici, Possiamo pertanto concludere che per il polinomio under consideration has one negative real part root and 2 null real part roots, which coincide precisely with the roots of .

The results of this analysis can be briefly summarized in Table 9.1, where by , and we have denoted respectively the number of positive, negative and null real-part roots of the polynomial (9.17) as it varies from **.**

Exercises

Exercise 9.1 Consider the linear, stationary system described by the model

	1	2	0
	0	2	1
	0	3	0
	0	1	2
	2	1	0

Table 9.1. Results of Example 9.45

Assess the BIBO stability of such a system. The impulse response is also calculated and whether it is summable or not.

Exercise 9.2 Consider the linear, stationary, concen- trated-parameter SISO system described by the input-output model

$$(9.18)$$

Verify that although the roots of the characteristic polynomial are not both negative real part the system is BIBO stable.

Exercise 9.3 Consider the linear, stationary system

and assume Determine the equilibrium states, if any, of the Resulting controlled system.

Exercise 9.4 Consider the linear, stationary, autonomous system

Determine the trajectory of such a system assuming that it evolves from the initial condition .

Exercise 9.5 Evaluate whether the network in Fig. 9.15 is BIBO stable. If it is not, determine a limited input that can generate an unlimited output.

Exercise 9.6 Verify BIBO stability for the systems whose characteristic polynomials are given in the following. Also evaluate for each of these the number of any pairs of roots that are symmetric with respect to the origin.

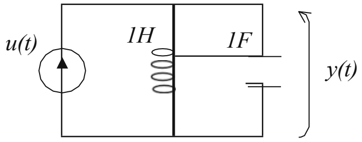


Fig. 9.15. Network related to Exercise 9.5

(a)
(b)
(c)
(d)
(e)

Exercise 9.7 Consider the linear, stationary and autonomous system described by the model

where

Assess the asymptotic stability of such a system and identify any equilibrium states.

Exercise 9.8 Let the following representation be given in terms of state variables of a linear, stationary system with concentrated parameters

Assess the stability of the system according to Lyapunov and in the BIBO sense.

Exercise 9.9 Prove by means of Routh's criterion Descartes' rule.

Exercise 9.10 Verify by means of Routh's criterion the stability of the system described by the transfer function

Is such a function in minimal form?

Exercise 9.11 Verify by Routh's criterion the stability of the system described by the following transfer function:

as the parameter changes .

Exercise 9.12 Consider the characteristic polynomial

Determine any values of for which you have BIBO stability.

Exercise 9.13 Consider the polynomial

(9.19)

where .

Check that as the parameter varies, the number of positive real-part roots , negative real-part roots , and zero real-part roots , vary as summarized in the following table.

	2	2	0
	0	2	2
	0	4	0
	0	2	2
	2	2	0
	2	2	0
	2	2	0

10

Analysis of feedback systems

In this chapter we will fix our attention on a particular scheme for linking elementary subsystems known as a *feedback scheme*. The importance of such a scheme derives from the fact that it proves particularly useful in solving many control problems.

The study of feedback systems is actually very complex and multifaceted, and in particular the determination of an appropriate transfer function that inserted in the direct chain, upstream of the process, allows the desired closed-loop specifications to be satisfied, is beyond the scope of this text. In fact, that topic is the subject of courses in *Automatic Controls* and not *Systems Analysis*.

Therefore, in this chapter we will limit ourselves to presenting some important criteria for the *analysis* of feedback systems, which are then the basis of the various *synthesis* procedures. Through these criteria it is indeed possible to obtain in a direct way some information about the global properties of the closed-loop system (in particular stability) on the basis of knowledge of the transfer functions of the component parts alone.

In this regard, both the *locus of roots* and the *Ny- quist criterion* will be presented. Finally, it will be discussed how a graphical representation of the closed-loop transfer function can be derived in the case where only a graphical representation is known of the direct chain transfer function.

10.1 Control in feedback

In Chapter 7 (see § 7.2.1), a particular connection *scheme* was introduced, which is called the *feedback* (or rather, *negative feedback*) *scheme*. It was also mentioned that this scheme is particularly useful in solving control problems. More precisely, it is particularly useful in solving those control problems whose objective is to make the controlled variable coincide with a certain *reference signal* or *set point*. The set point can be constant or time-varying: in the former case we speak of *control* problems, in the latter case we speak of *slaving* problems.

In practical reality, in effect, perfect coincidence between the controlled variable (the output) and the set point cannot be achieved, so an output signal that is a "good" approximation of the set point is considered satisfactory. The "goodness" of this approximation is measured through a set of specifications, or requirements, that the error signal, equal to the difference between the set point and the output, must meet under the operating conditions of interest.

The control scheme that best meets the required specifications in such a problem is the feedback scheme shown in Fig. 10.1.a where the following notation was used:

represents the *set point*;
Is the *input* to the process;
The *output*, that is, the controlled variable;
is the transfer function of the *process* to be controlled. It should be noted that in reality the process is subject to a number of uncertainties and variations during its operation so that in practice one never has a transfer function that describes with absolute accuracy the dynamics of the process throughout its evolution;
is the transfer function of the *measuring transducer*, if any, that per-
makes it possible to evaluate instant by instant the difference existing between the output and the set point, i.e., the signal in Fig. 10.1.a;
is the transfer function of the *controller*, or *controller*. Solving a control problem involves precisely determining an appropriate controller that, based on the difference that exists between the output and the set point, provides a signal at the process input that ensures that the desired specifications are met.

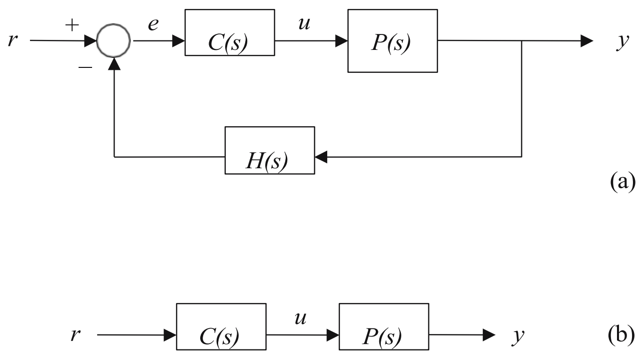


Fig. 10.1. (a) Wiring diagram of a feedback control system; **(b)** wiring diagram of an open-loop control system

The transfer function between the set point and the output holds (see § 7.2.1)

and is called *a closed-loop* transfer function. The transfer function

is instead called *an open-loop* transfer function while

Is the transfer function of the *direct chain*.

An alternative to the feedback (or closed-loop) scheme shown in Fig. 10.1.a is the *open-loop* control scheme, shown in Fig. 10.1.b, which, moreover, constitutes a special case of the feedback scheme.

However, it can be shown that the closed-loop scheme has a number of vanvantages over the open-loop scheme, which can essentially be summarized as follows:

the closed-loop scheme provides greater accuracy at steady state,
has a lower sensitivity to uncertainties and parametric variations in the process,
Has greater insensitivity to any external disturbances acting on the system.

Formal proof of these claims, as well as the practical and empirical rules for determining a transfer function to meet the desired specification, are beyond the scope of this discussion. For a det- mined discussion on this subject, we refer to specific texts oriented toward control, rather than analysis.

However, the following simple physical example shows intuitively what are the advantages of feedback control over closed-loop control.

Example 10.1 Consider the cylindrical tank schematically shown in Fig. 10.2. Let and be the inlet and outlet flow rates, respectively, and the liquid level in the tank.

Assume that initially the liquid level is m and that the Inlet and outlet pumps are not operational, viz. m /s. Yes Suppose finally that the cross section of the tank is m .

You want to bring the liquid level to the desired value m by appropriately varying the flow rates and . These flow rates then represent the input to the process, the level represents the output, and is the set point.

Una semplice soluzione a questo problema consiste nell'azionare la pompa in ingresso ottenendo una portata litro/s = m /s. In questo modo il livello sale con velocità

m/s

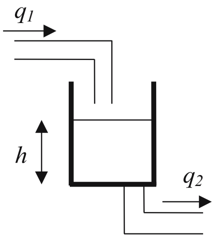


Fig. 10.2. Schematic representation of the tank considered in Example 10.1

and being m, the desired value of is reached by leaving the inlet pump open for a time s. To go from to is in fact worth
the report

—

However, such a control logic, which clearly defines an open-loop control, presents problems.

In fact, what happens if an input *disturbance* acts *on* the system (for example, operating the inlet pump does not get a flow rate of 1 liter/s but a different flow rate)?
What happens if the system *model is not exact* (for example, the cross section is not m but m)?

Clearly, in none of these cases would the desired value of the output be achieved.
Instead, a control law that allows the set point to be reached even in the presence of the above problems is as follows:

- If you open the inlet pump ( ),
- If you close the pumps ( ),
- If you open the outlet pump ( ).

Such logic realizes feedback control in that the input to the process (the flow rates and) is established instant by instant based on the difference between the set point and the output ( ).

It is then said that feedback control is *robust* in that it works well even in the presence of disturbances or errors on the model

In the following we will present some techniques for analyzing closed-loop systems, that is, we will see how it is possible to obtain in a direct way some information about the global properties of the closed-loop system (especially its stability) based on the knowledge of the transfer functions of the component parts.

10.2 Place of the roots

Root locus tracing provides a valuable tool for analysis and synthesis of linear feedback systems in the domain of .

For the definition of the locus of roots, refer to the generic feedback scheme in Fig. 10.3 whose closed-loop transfer function is worth

where it is the open-loop transfer function that is always assumed to be in minimum form.

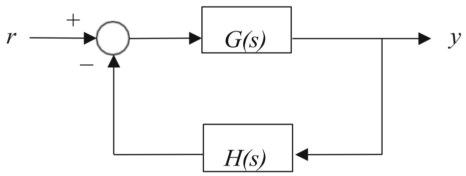


Fig. 10.3. Generic feedback scheme

Both

(10.1)

The locus of roots allows us to understand how the position of the poles of the closed-loop si- stem varies as the parameter characteristic of the direct chain transfer function varies.

The characteristic polynomial of the closed-loop system coincident with the numerator of , is equal to

(10.2)

while the characteristic equation of the closed-loop system holds

(10.3)

We can give the following definition.

Definition 10.2. *The* positive locus of the roots *is the set of lines in the Gauss plane described by the poles of the closed-loop system as the parameter varies from 0 to , where these lines are oriented in the direction of the rising .*

Example 10.3 Be it

e

In that case and the closed-loop transfer function is worth

The roots of the characteristic polynomial

are .

Il luogo positivo delle radici è il luogo dei punti nel piano di Gauss individuati dai poli al variare di da a .

For , it is valid and .

For , whereby and take real values internal to the segment . Specifically as increases from 0 to 1, it moves along the negative real semi-axis from the origin to the -1 point, while moves along the negative real semi-axis from the -2 point to the -1 point.

For the two roots coincide and it is worth .

Per , per cui le due radici sono complesse coniugate. Inoltre, la loro parte reale è pari a -1 per qualunque valore di , mentre la loro parte immaginaria tende a crescere indefinitamente in modulo al crescere di .

The positive locus of the roots therefore takes the form shown in Fig. 10.4 where the open-loop poles have been denoted by the symbol **.**

Si noti che solitamente il luogo definito come sopra viene semplicemente denominato *luogo delle radici*, senza specificare che questo è il luogo positivo. A rigore però quando si parla di luogo delle radici ci si riferisce all'insieme delle linee ottenute facendo variare da a , ossia all'insieme del luogo positivo e del luogo negativo delle radici, dove quest'ultimo è ottenuto al variare di da a 0[1]. Nel seguito della trattazione fisseremo la nostra attenzione sul solo luogo positivo delle radici che per semplicità verrà semplicemente chiamato *luogo delle radici*.

[1]The negative root locus is extremely useful if one studies negative feedback systems whose characteristic polynomial is equal to the numerator of (see § 7.2.1), or if one wishes to study the properties of the from the analytic expression of . That locus, however, will not be considered in this text.

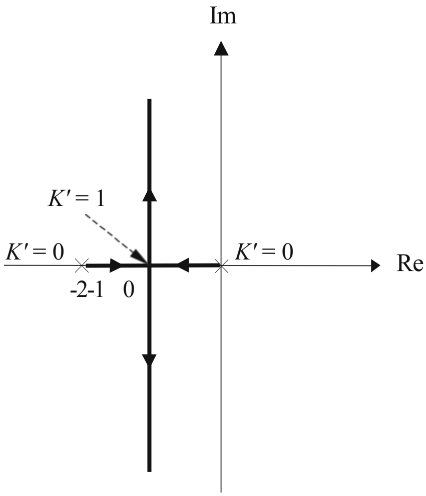


Fig. 10.4. Location of the roots of the

Equ. (10.3) is called the *vector equation of place*: since it is in fact an equa- tion in the complex variable , it can be split into two scalar equations, relating to the moduli and phases, respectively. In particular, we can give this equation an intuitive geometric interpretation. Let, in fact, be the gen- eral point in the Gauss plane. The factors ( ) can be seen as vectors joining ( ) with the coordinate point . In particular, we denote by and the moduli of the vectors and , respectively, and by and the angles that these vectors form with the positive real semi-axis. See in this regard Fig. 10.5 where the poles have been denoted by the symbol and the zeros by a small circle.

Eq. (10.3) can be decomposed into the two scalar equations :[2]

[2]Equation (b) becomes evident if we rewrite (10.3) in the form

and it is observed that and therefore , while it is even at any odd multiple of .

(10.4)

Eq. (10.4.a) is called *the modulus condition* and the (10.4.b) *the phase condition*. As is readily apparent, the phase condition does not include the parameter and this can therefore be interpreted as the equation of the locus: in fact, all and only the points of the locus satisfy this condition. This will then be used for plotting the locus. In the reverse modulus condition, appears, and in particular, for any value of this equation is satisfied by points in the locus. It then allows us to calibrate the root locus at , that is, to associate each locus point with a particular value of .

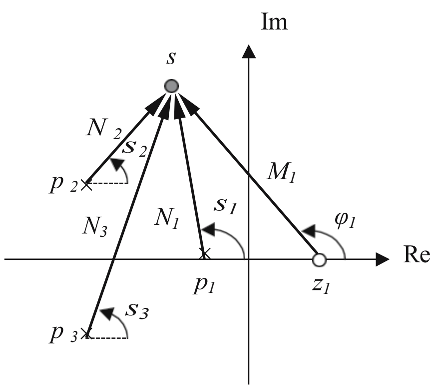


Fig. 10.5. Definition of angles , and moduli

10.2.1 Rules for tracking the place

The root locus enjoys certain properties that allow the formulation of simple rules of thumb that enable the exact tracing of some parts of it and the qualitative tracing of some other parts of it.

Note that the place obtained on the basis of these rules, although qualitative in some of its parts, provides a valuable tool for analysis and synthesis of systems in feedback. Indeed, it allows us to infer the main information about the feedback system and also allows us to understand how any change in gain and/or the addition of appropriate dynamics (i.e., appropriate poles and zeros) in the direct chain can affect the dynamics of the closed-loop system.

Note **10.4** *Note that in the following we will always assume that the system verifies* the principle of causality *so we will always refer to open-loop transfer functions such that , where and denote respectively the degree of the numerator and denominator of the .*

The rules for place tracking, some of which we will also give a formal demonstration of, can be stated as follows.

Rule 10.5 *The place of roots is branches.*

Demonstration. Being by hypothesis , Eq. (10.3) is of order and has therefore roots that depend continuously on the parameter .

Rule 10.6 *The place of roots is symmetrical with respect to the real axis.*

Demonstration. Eq. (10.3) has real coefficients: its roots are therefore either real or complex conjugate.

Rule 10.7 *Branches originate for from the poles of ....theIn particular, if a pole of the is simple from it only one branch of the locus originates; on the other hand, if has multiplicity from it branches of the locus originate.*

Demonstration. The validity of the statement follows immediately from the fact that for
Eq. (10.3) reduces to

(10.5)

whose roots are precisely the poles of the , each counted with its own multiplicity.

Rule 10.8 *For , of the branches of the place end in the zeros of the and the others tend to infinity. In particular, if one zero of the has simple multiplicity in it terminates only one branch of the locus; on the other hand, if has multiplicity in it terminates branches of the locus.*

Demonstration. It follows from the fact that if Eq. (10.3) can be rewritten as.

$$— \qquad (10.6)$$

Now, by (10.6) it becomes.

(10.7)

which has only roots that coincide precisely with the zeros of the , each taken with its multiplicity. For , these are also the only roots at the finite of eq. (10.3).

Rule 10.9 *The locus has* asymptotes *to which branches tend that end at infinity. These asymptotes intersect at a point on the real axis of abscissa equal to*

(10.8)

and form with the real axis angles equal to

(10.9)

Dimostrazione. Per semplicità la dimostrazione completa di questo risultato non viene data. Osserviamo solo, con riferimento alla Fig. 10.5, che se il generico punto tende all'infinito tutti i vettori ed assumono ampiezza infinita e una direzione comune, ossia tutti gli angoli e i divengono uguali. Ora, sia il valore comune di tali angoli. La condizione di fase diviene
, from which eq. (10.9) follows.

Note that the angles that any asymptotes form with the real axis depend only on the poly-zero excess and not on the position of the poles and zeros in the Gauss plane. In particular, the asymptotes form a star of straight lines centered in
. Such a star is regular in the sense that the angles between each pair of adjacent straight lines are equal. Since, moreover, the locus is symmetrical with respect to the real axis, such straight lines are arranged as the number changes as shown in Fig. 10.6.

Rule 10.10 *Belonging to the locus are all points on the real axis that leave to their right an odd number of poles and zeros, each counted with its own multiplicity.*

Demonstration. Consider a generic point in the complex plane belonging to the real axis as shown in Fig. 10.7. The validity of the statement follows immediately from the following observations:

- pairs of complex conjugate poles and zeros make an overall zero contribution to the phase (the sum of their respective angles is);
- poles and zeros on the real axis to the left of the point give a zero phase contribution;

the poles and zeros on the real axis to the right of the point each give a phase contribution equal ...toOnly if their number is odd does the phase condition result
therefore verified.

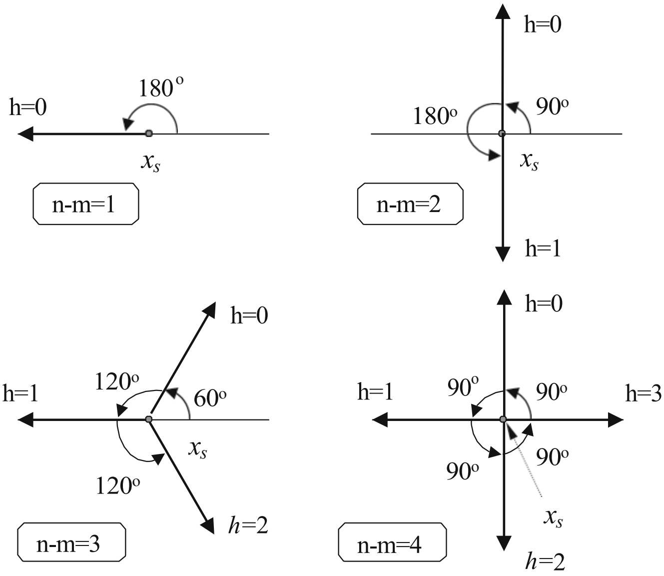


Fig. 10.6 Star of asymptotes at the variance $_{o\,f}$

Rule 10.11 *The branches of the locus may have common points corresponding to multiple roots of Eq. (10.3). In the case of double roots, the corresponding* double points *can be calculated by the equation:*

$$\text{———} \qquad \text{———} \tag{10.10}$$

In addition, at a double point the tangent of the branch going to the double point forms an angle of *with the tangent of the branch going from it.*

Demonstration. To prove the validity of this statement, recall that the roots of (10.3) coincide with the roots of

$$\tag{10.11}$$

Moreover, the double roots of Eq. (10.11) satisfy not only (10.11) but also the equation that is obtained by equalizing the derivative of the first member to zero, i.e., they are solutions of the system

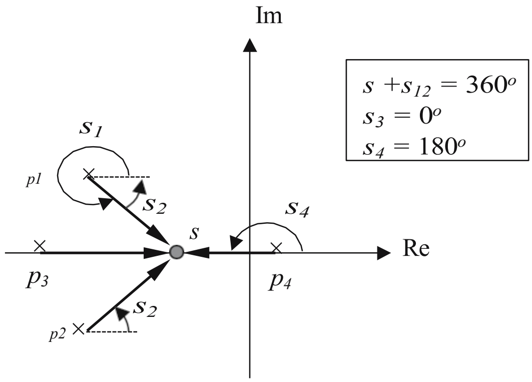


Fig. 10.7 Definition of angles in the case belonging to the real axis

(10.12)

To calculate the first derivative of the it is convenient to write the as

(10.13)

from which follows

(10.14)

Where denotes the natural logarithm. Deriving both members of (10.14) with respect to we get

(10.15)

which proves the validity of Eq. (10.10).

For simplicity, however, the property of tangents at double points is not demonstrated.

It should be noted that in general the determination of double points is by no means semi-simple. Indeed, Eq. (10.10) has solutions, not all of which, moreover, belong to to the locus of the roots[3] . Once you have then solved Eq. (10.10) you need to figure out which roots actually belong to the locus, and this is usually possible based on the information obtained by applying the previous rules.

The most frequent case is when the double points belong to the real axis and lie between two real poles. In these cases the presence of the double point is immediately deduced from the presence of two branches that run along the segment between the two poles in opposite directions. Such branches, after meeting at the double point separate continuing outside the real axis. An example of this is provided by the root locus shown in Fig. 10.4.

By means of the above 7 rules it is therefore possible to obtain a good approssimation of the root locus trend. Next, with regard to the calibration of the locus at we recall that this can be easily done by means of the modulus condition.

Rule 10.12 *Given a generic point belonging to the place, the following holds true in it*

$$\text{———} \qquad (10.16)$$

Where, in accordance with the previous notation, and .

Demonstration. It follows immediately from the decomposition of the vetto- rial equation of the place in the two conditions of modulus and phase.

Finally, it is useful to make the following observation.

Remark 10.13 *Any crossing points of the imaginary axis by place can be determined by applying Routh's criterion to the algebraic equation (10.3). More precisely, one constructs Routh's table related to that equation and calculates the value(s) of for which a row becomes iden- tically null. Any points through the imaginary axis occur at one or more of these values of .*

We now present some significant examples in order to clarify the rules for place de- termination and explain how place allows us to draw useful information about the dynamics of the closed-loop system.

Example 10.14 Let it be.

[3]Note that this statement is true in that here, as clarified at the beginning of the chapter, in speaking of the locus of roots we are in fact referring to the *positive* locus of roots. For the sake of completeness we specify that all roots in the double point equation that do not belong to the positive locus of roots belong to the negative locus of roots.

Such a transfer function has a zero and two poles: and , so it is worth and .

The site therefore has branches.

These branches originate for each from a pole of the *P e r*
one branch tends to zero and the other tends to infinity.

In particular, the branch tending to infinity forms with the real axis an angle equal to Note that since there is in this case only one asymptote coincident with the axis
negative real there is no point in calculating the value of .

By virtue of Rule 10.10 also belong to the locus all points on the real axis to the left of and those inside the ....segmentI points to the left of
in fact leave to their right the 2 poles and the zero of the ; the points inside the segment leave instead to their right the pole in the origin.

The locus of the roots in this simple example therefore lies all on the real axis.

It clearly has no double points. This fact can be deduced from the sem- pline observation that there are no branches that tend to meet, and can still be easily verified by calculating the roots of the equation of double points, which in this particular case holds true:

And verifying that its roots ( ) do not belong there.

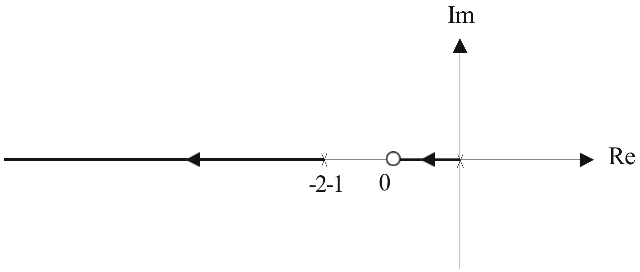


Fig. 10.8. Root location of the

Therefore, the root site has the shape and orientation shown in Fig. 10.8.

From examining the locus, we can draw the following information in terms of the dynamics of the closed-loop system having as its open-loop transfer function.

Per i poli a ciclo chiuso coincidono con i poli a ciclo aperto. L'evoluzione libera del sistema ha pertanto una forma del tipo:

where the constants and depend on the initial conditions of the system. For closed-loop poles, both are negative real part. The free evolu- tion has a form of the type:

where and . In particular as increases , it tends to values sempre più prossimi a e a valori sempre più grandi in valore assoluto. Il sistema a ciclo chiuso è pertanto stabile per ogni valore di .

Example 10.15 Be it

Tale funzione di trasferimento non ha zeri ( ) e ha tre poli reali e distinti ( ): , , .

Il luogo pertanto ha rami.

Tali rami hanno origine per ciascuno da un polo della e terminano per tutti all'infinito.

In particolare, i rami tendono all'infinito lungo diversi asintoti. Gli asintoti si intersecano in un punto sull'asse reale di ascissa pari a

and form with the real axis angles equal to

for

for

For .

In virtù della Regola 10.10 inoltre appartengono al luogo tutti i punti dell'asse reale alla sinistra di e quelli interni al segmento . I punti alla sinistra di lasciano infatti alla loro destra poli, ossia tutti i poli della ; i punti interni al segmento lasciano invece alla loro destra il solo polo nell'origine.

The equation of the double points is:

whose roots are: , i.e., and The point belongs to the place beingOn the contrary the point does not belong at the place since e Observe that the presence of a point double internally to the segment was to be expected since two branches of the locus depart from the ends of that segment, one directed in the opposite direction to the other. From the modulus condition it is also straightforward to calculate that at the double point the following applies

.

The root site thus has the shape shown in Fig. 10.9 in which the direction of travel of the branches has also been highlighted.

From that figure it is also easy to observe (see the dotted line perpendicular to the x-axis at) that at the double point, the tangent of the branch going to the double point forms an angle of with the tangent of the branch starting from it.

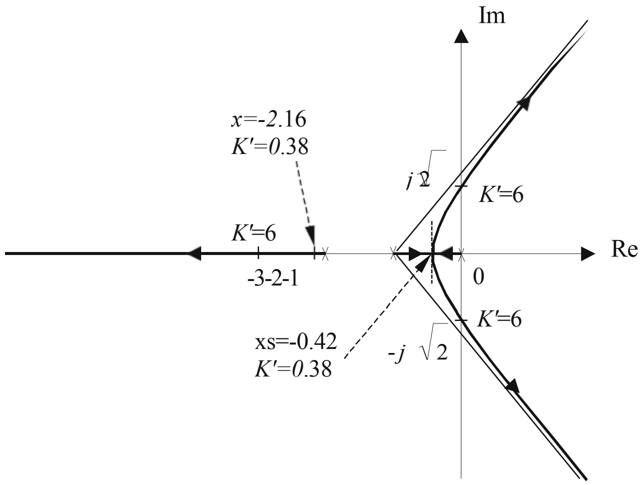


Fig. 10.9. Location of *the* roots of *the*

It is also evident that the locus crosses the imaginary axis at two points. These crossing points can be easily determined by applying Routh's criterion to the algebraic equation (10.3), which in this particular example is worth

(10.17)

From that equation we construct the table

1	2	
3		
6-		(the line was multiplied by 3)

whose index row cancels for . For that value of the auxiliary polynomial constructed from the coefficients of the previous row holds:

The zeros of such a polynomial are clearly pure imaginary numbers and are worth: − This implies that, as shown in Fig. 10.9, the location of the roots attraversa l'asse immaginario nei punti di ordinata .

Observe that the third point of the place corresponding to can easily calculate by tending to the fact that it too is a solution of the third-degree equation (10.17) where we pose ....Since we know that two of the roots of that equation sono è immediato calcolare che la terza radice vale ....Possiamo pertanto concludere che il terzo punto del luogo per si trova nell'asse reale e vale .

Si noti infine che con un ragionamento del tutto analogo è anche immediato calcolare il terzo punto del luogo per il quale vale , dove è il valore di per il quale si ha un punto doppio. Particolarizzando infatti l'eq. (10.17) con e tenendo conto che due delle radici dell'equazione così ottenuta valgono , è immediato calcolare che la terza radice vale -2.16.

From examining the locus, we can then draw the following conclusions in terms of the dynamics of the closed-loop system having an open-loop transfer function.

Per i poli a ciclo chiuso coincidono con i poli a ciclo aperto. L'evoluzione libera del sistema ha pertanto una forma del tipo:

where the constants , for , clearly depend on the initial conditions of the system.

Per i poli a ciclo chiuso sono reali, distinti e tutti a parte reale negativa. L'evoluzione libera del sistema ha pertanto una forma del tipo:

dove , e .

Per il sistema a ciclo chiuso ha un polo reale negativo con molteplicità doppia e uno reale negativo semplice. In particolare, il polo reale con molteplicità doppia coincide con il punto doppio e il polo semplice vale . L'evoluzione libera del sistema ha pertanto una forma del tipo:

Per il sistema a ciclo chiuso ha una coppia di poli complessi coniugati a parte reale negativa e un polo semplice a parte reale negativa nel ramo che parte da e tende a . La forma della evoluzione libera è:

−

where , , , and , and depend on the initial conditions.

Per il sistema a ciclo chiuso ha una coppia di poli complessi coniugati a parte reale nulla e ancora un polo reale negativo pari a . La forma della evoluzione libera è

–

Per la parte reale dei poli complessi coniugati diviene positiva per cui il sistema a ciclo chiuso diviene instabile. La forma della evoluzione libera è

–

where , and .

Example 10.16 Be it

La non ha zeri ( ) e ha poli distinti: coincidente con l'origine e due poli complessi coniugati .

Il luogo ha quindi 3 rami: ciascun ramo parte per da uno dei poli e termina per all'infinito. Vi sono 3 diversi asintoti le cui direzioni sono chiaramente , e Gli asintoti si intersecano in un punto sull'asse reale abscissa

–

Appartengono al luogo tutti i punti nell'asse reale negativo, compresa naturalmente l'origine da cui parte uno dei rami. I punti all'interno del segmento lasciano infatti alla loro destra il polo nell'origine; i punti appartenenti alla semiretta lasciano invece alla loro destra i tre poli della .

In this case, as is easily guessed since there are no branches of the locus that tend to meet, there are no double points. This is in agreement with the fact that the equation of double points

–

–

Has like roots That do not belong to the place.

Possiamo pertanto concludere che il luogo ha la forma mostrata in Fig. 10.10 [4].

[4] It can be shown that the place branches starting from the pole have tangents $_{i\ n}$ which form with the real axis angles equal to

(10.18)

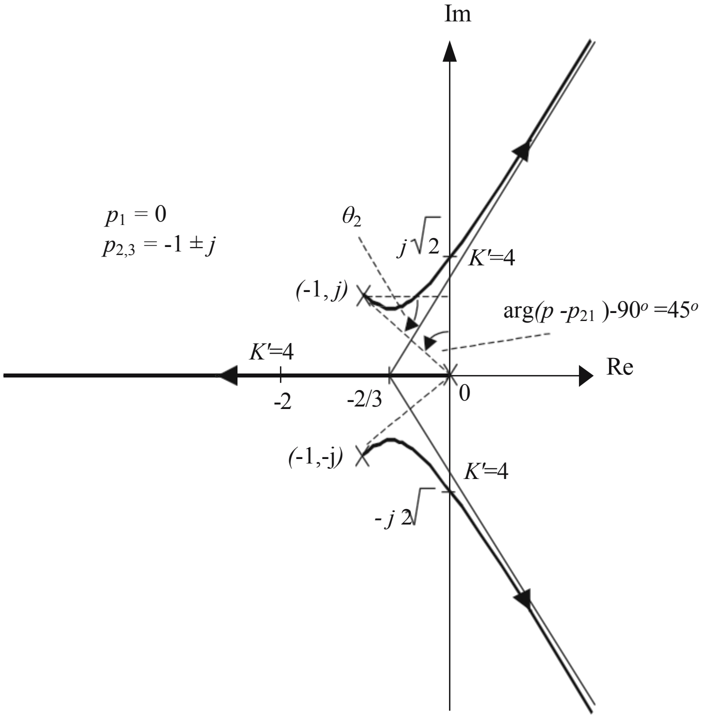


Fig. 10.10. Location of the roots of the

The locus clearly crosses the imaginary axis at two points, which again can be determined by applying Routh's criterion to the algebraic equation:

This makes it possible to give a justification for the direction of departure of the branches from the com- plex poles. Consider, for example, the pole (given the symmetry of the locus a similar argument applies to its complex conjugate). The pole has simple multiplicity so the relation (10.18) is defined only for and holds:

Essendo l'angolo alla base di un triangolo rettangolo isoscele (si veda la Fig. 10.10), tale angolo è pari a , ossia Pertanto dalle eguaglianze sopra segue che Il ramo del luogo che ha origine dal polo part then tangent to the ray that originates at the pole and passes coincident with the origin.

From that equation we construct the table

1	2	
2		
4-		(the line was multiplied by 2)

whose index line 1 cancels for . For that value of the auxiliary polynomial constructed from the coefficients of index row 2 holds:

whose roots are that coincide with the points at which the locus crosses the imaginary axis.

Ripetendo inoltre un ragionamento analogo a quello visto negli esempi precedenti, si determina immediatamente che il terzo punto del luogo per cui vale è il punto sull'asse reale di ascissa pari a .

At this point it is then immediate to understand for what values of the closed-loop system having as open-loop transfer function is stable or unstable and what is the structure of its free evolution .

Example 10.17 Let it be.

Tale funzione di trasferimento ha uno zero e 4 poli: , e .

The locus therefore has 4 branches, one of which ends in zero and the others at infinity along the directions identified by the angles: , andThe star center of the
asymptotes has as abscissa

From the pole in the origin, having it double multiplicity, two branches naturally depart.

Appartengono inoltre all'asse reale tutti i punti interni al segmento e i punti della semiretta .

It is therefore intuitable that the site has the shape shown in Fig. 10.11.

The only points of intersection with the imaginary axis are the two poles coincident with the origin.

There are then naturally two double points in the real axis: one inside the segment e uno alla sinistra del punto . In tali zone dell'asse reale infatti vi sono due rami del luogo diretti in verso opposto. Risolvendo l'equazione dei punti doppi (una equazione di quarto grado) è possibile verificare che i punti doppi valgono .... eIt is left as an exercise to the reader to determine the values of

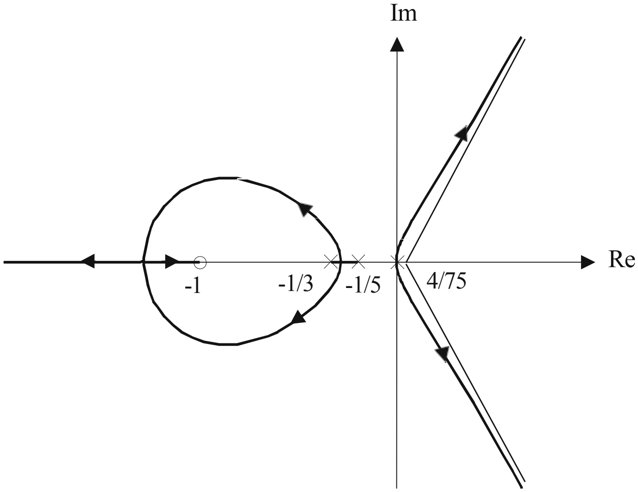


Fig. 10.11. Location of the roots of Example 10.17

at such points. Observe that at double points the tangent of the branch going to the double point forms an angle of with the tangent of the branch starting from it.

With similar considerations to those seen in the previous examples, it is easy at this point to draw conclusions about the dynamics of the closed-loop system as varies.

10.3 Criterion of Nyquist

The Nyquist criterion constitutes one of the fundamental criteria for the analysis and synthesis of linear and stationary feedback systems based on the frequency response of the open-loop transfer function. This criterion is based on the drawing of a particular diagram, called Nyquist diagram, shown in the following section.

10.3.1 Diagram of Nyquist

Given a generic transfer function that is always assumed to be in minimum form, let the function obtained by posing .

The Nyquist diagram of the is the locus of points in the com- plex plane as it varies from to . It is therefore a parameterized curve in which a direction of travel is associated as the pulsation increases .

The following property demonstrates the symmetry of the Nyquist diagram with respect to the real axis of the complex plane, which greatly simplifies its plotting.

Property 10.18 Given a transfer function , either

For each value of the pulse, it is worth

i.e., the modulus of is an even function of while the phase is an odd function of .

Demonstration. The validity of the statement follows from the following simple geometric consideration. Since la is given by the ratio of two polynomials in , it can be written as

which in terms of modulus and phase, becomes

Let e be with a generic point on the imaginary positive semiaxis of the Gauss plane. The factors e can be seen as vectors joining e with the point .

Assume for simplicity that the has no zeros and has three poles arranged as in Fig. 10.12.a where and are naturally conjugate complex poles. In this case

if
if

where

Consider now the ....pointIn this case (see Fig. 10.12.a and b)

being

Also

whether ,
whether .

But as it is

worth

whether ,

whether .

from which it follows that

as it was meant to be.

An entirely similar line of reasoning can of course be repeated in the case where the
also has real and/or complex conjugate zeros.

By virtue of this property, the Nyquist diagram can be plotted by initially taking into consideration only the pulsations Then, being the
digram relating to pulsations the symmetrical with respect to the real axis of the diagram relating to , its plotting is straightforward.

The Nyquist diagram can of course be derived for points from the Bode di-gram. There are, however, simple rules of thumb that, together with considerations about the range of phase change and the trend of the modulus as it increases by , allow its qualitative tracing with a good approximation at least at high and low frequencies. These rules can be stated as follows.

Note that in the following we will assume for simplicity that the poly-zero excess in the o- rigine is always greater than or equal to zero. Therefore, with reference to the Bode representation of (see eq. (6.22)), we will assume that it holds , this being the most frequent case in practice.

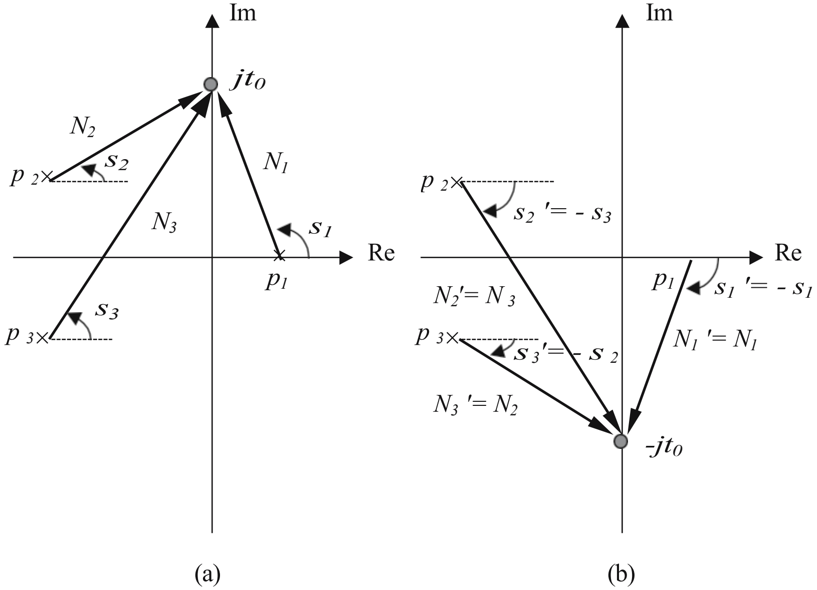


Fig. 10.12. Demonstration of Property 10.18

Rule 10.19 *If valid , the Nyquist diagram starts for from the coordinate point with a phase equal to*

se

if

Where is the gain of[5] *.*

If the diagram starts from an improper point in the complex plane with a phase equal to

se

whether .

Demonstration. As is well known, the transfer function is given by the ratio of two polynomials in , viz.

---

[5]Recall that the gain of a generic transfer function has been defined as (see § 6.4.3)

If , then e

—

which proves the first part of the statement.

If instead , then for which

———

which implies, in terms of modulus and phase, that

——

e

——

Rule 10.20 *If the transfer function is strictly proper then the Nyquist diagram ends for in the origin of the complex plane.*

If the is proper the Nyquist diagram ends for in the coordinate point .

It also applies to

if

se

Demonstration. Clearly

———

whereby

—— se

if

e

——— if

if .

Let us now illustrate the use of such rules through some simple examples that also highlight how in general it is necessary to evaluate within what range the phase varies and what is the trend of the modulus as , in order to draw, even if only qualitatively, the Nyquist diagram of a certain transfer function.

Example 10.21 Consider the transfer function

Which has only pole and no zero (e).

Suppose initially that it is and . Based on the two rules stated above, we can immediately state that:

Nyquist diagram starts for from the coordinate point in the real positive half axis with phase ;

ends for in the origin with phase *being*

.

We also know from knowledge of the Bode diagram of that transfer function that as the phase varies, it is always between and

, so we can conclude that the Nyquist diagram lies all in the fourth quadrant of the complex plane[6] . Moreover, the modulus remains practically constant until values of equal to , after which it tends to decrease until it reaches zero. We can therefore conclude that the Nyquist diagram for has the pattern shown in the fourth quadrature of the complex plane in Fig. 10.14.a.

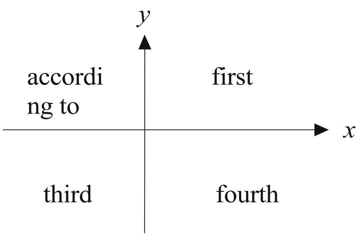


Fig. 10.13. Quadrant numbering

Finally, by virtue of Property 10.18, it is immediate to complete this dia- gram for negative values of , which results in the symmetrical with respect to the real axis of the previous diagram.

Therefore, the complete Nyquist diagram is as shown in Fig. 10.14.a.

Suppose now that it is and . Based on the two rules stated above, we can immediately state that:

Nyquist diagram starts for from the coordinate point in the real positive half axis with phase ;

[6]Remember that by convention the quadrants of a generic Cartesian plane are numbered as shown in Fig. 10.13.

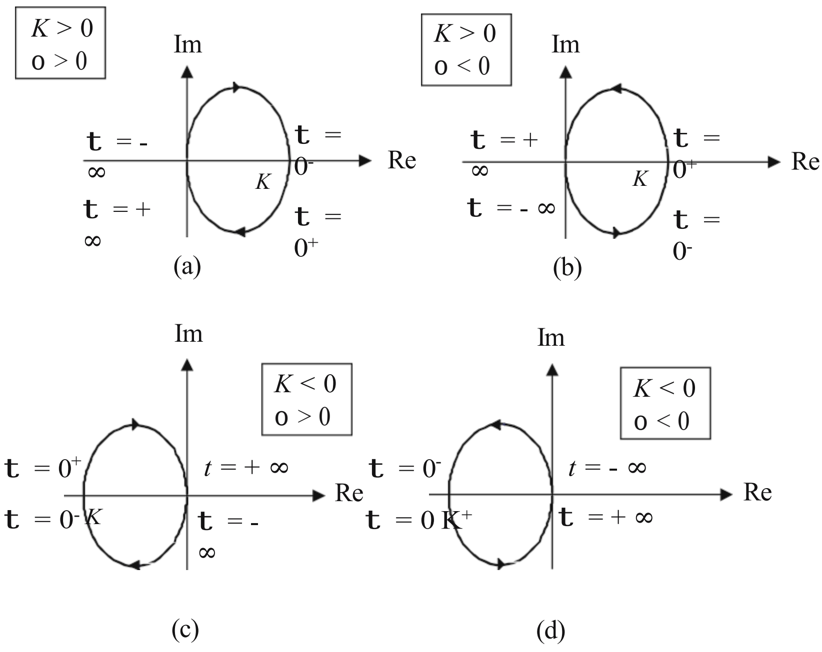


Fig. 10.14. Nyquist diagram of varying the sign of *e*

ends for in the origin with phase (or equivalently) being e

.

We also know from knowledge of the Bode diagram of that transfer function that as the phase varies, it is always between and

, so we can conclude that the Nyquist diagram lies all in the first quadrant of the complex plane. The modulus of course has a trend similar to the previous case, since it does not depend on the sign of . We can therefore conclude that the Nyquist diagram for has the trend shown in the first quadrature of the complex plane in Fig. 10.14.b.

Finally, the Nyquist diagram for negative values of is the symmetrical with respect to the real axis of the previous diagram so the complete diagram has the pattern shown in Fig. 10.14.b.

By repeating similar reasoning for the other two sign combinations of and , namely , and , , it is easy to verify the trends shown in Figs . 10.14.c-d.

Example 10.22 Consider the transfer function

———

which has two poles, one of which is in the origin, and no zero.

The characteristic parameters for the purpose of plotting the Nyquist diagram are:
, e....................Therefore,

Nyquist diagram starts for from an improper point in the complex plane with phase ; ends for in the origin with phase .

Such information is, of course, not sufficient for tracing the Nyquist diagram- ma because it does not tell us in which quadrants the diagram is actually located.

However, plotting, also very qualitatively, the Bode diagram of the allows us to state that the phase for is all between and so the diagram is all in the third quadrant of the comples- sion plane. Moreover, the modulus is strictly decreasing as the pulsation increases. Finally, taking into account the symmetry of the diagram with respect to the real axis, the qualitative trend of the complete diagram is as shown in Fig. 10.15.

Note that determining the position of the asymptote to which the diagram- ma tends for (and therefore also) does not follow from the previous considerations. For this purpose it is necessary to calculate

from which it follows that the asymptote sought intersects the real axis in .

Example 10.23 Consider the transfer function

In such a case the following applies: , , and , so

Nyquist diagram starts for from an improper point in the complex plane with phase ; ends for in the origin with phase .

From plotting, even qualitatively, the Bode diagram, it is easy to realize that the phase for positives is all between ....eThe diagram
of Nyquist relative to positives therefore has the trend shown in Fig. 10.16. Note that also in this case for the determination of the position of the asymptote to which the diagram tends per it is necessary to repeat a similar reasoning to that seen in the previous example for which we calculate

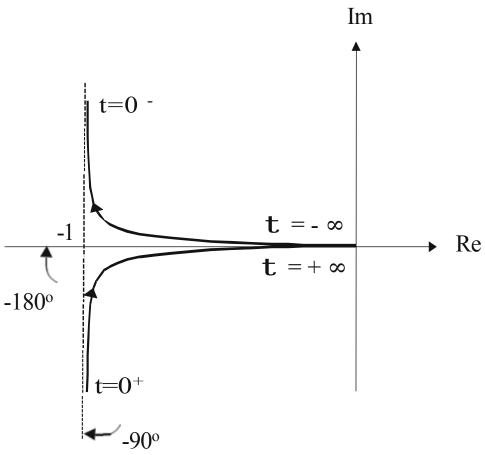


Fig. 10.15. Nyquist diagram of

We can therefore conclude that the asymptote sought coincides with the positive imma- ginary semi-axis.

Finally, by virtue of Property 10.18, the complete diagram has the pattern shown in the same Fig. 10.16

Example 10.24 Be it

Vale , , and , so.

Nyquist diagram starts for from the coordinate point of the complex plane with phase ;

ends for in the origin with phase .

Plotting even qualitatively the Bode diagram, it is easy to realize that the phase for positives is between and so the diagram affects the first three quadrants. The Nyquist diagram relating to positives therefore has the pattern shown in Fig. 10.17.a. This diagram was plotted using `Matlab` software so that in fact it is not possible to appreciate the actual trend of the curve at high frequencies. To this end, the part of the diagram relating to high values has therefore been highlighted in the same figure: it is thus evident that the phase with which the diagram ends at the origin is equal to

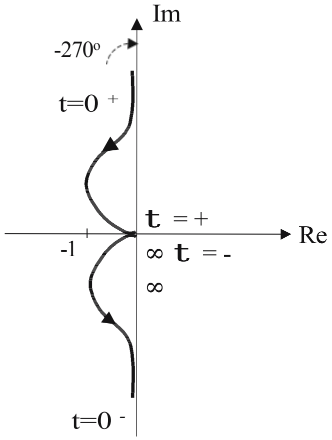


Fig. 10.16. Nyquist diagram of ———

and not equal to as it might apparently appear from the com- plete diagram. By virtue then of Pro- pect 10.18, the diagram for the has the pattern shown in Fig

10.17.b.

10.3.2 Criterion of Nyquist

The Nyquist criterion makes it possible to determine whether a given closed-loop system is stable from the Nyquist diagram of the open-loop transfer function.

Before formally stating Nyquist's criterion, it is useful to make some preliminary observations.

Consider the generic polynomial

Assume that it has no roots at real null part. Let be any point on the imaginary axis. As shown in Fig. 10.18, as the value of
from to each vector having origin in the generic root of and terminating at the point undergoes a phase change equal to:

, if it lies in the left half-plane of the complex plane,
, if it lies in the right half-plane of the complex plane.

Based on this observation and again under the assumption that it has no roots on the imaginary axis, we can state the following. Given the number of roots a positive real part of , the phase change it undergoes as it varies from to is equal to

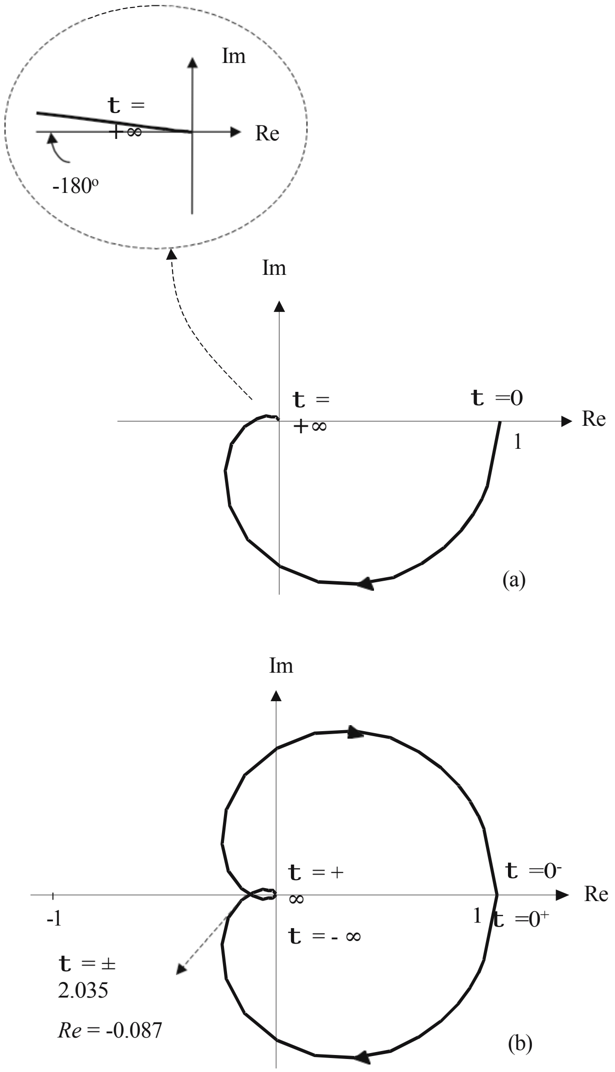


Fig. 10.17. Nyquist diagram of

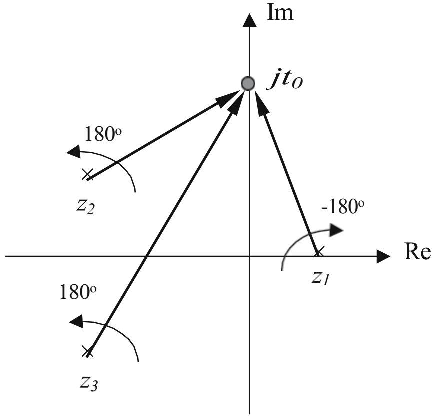


Fig. 10.18. Phase change of the vectors when varying from $_{t\ o}$

(10.19)

where it is naturally equal to the number of negative real part roots.

Example 10.25 Be it

where , and are arranged as in Fig. 10.18.

By virtue of Rule 10.20, being and , we can state that

Furthermore, it follows from the symmetry of the Nyquist diagram *that.* Therefore

which result is in agreement with (10.19) being .

Now, be the open-loop transfer function related to the feedback system whose closed-loop stability we want to study.

Assume that the is a transfer function proper for which . We define *difference* function associated with , the transfer function

Of course, since the given is the ratio of two polynomials in , *the* is given by the ratio of two polynomials in . In particular, let be where and are two polynomials of degree and , respectively. Then

that is, it is the ratio of two polynomials both of degree (being by hypothesis).

Let there also be

the number of positive real-part zeros of the polynomial , that is, the number of positive real-part zeros of the ;
the number of positive real-part zeros of the polynomial , that is, the number of positive real-part poles of the (and thus also of the).

By placing in the , we can calculate the phase change that undergoes as it varies from to , which is worth

where, with obvious notation and denote respectively the phase changes of the polynomials and as they vary from to . By virtue of Eq. (10.19),

whereby

(10.20)

This phase change can equivalently be expressed in terms of the number of turns (positive if counterclockwise) that the representative vector *of* performs around the origin as it varies from to . For this purpose it is sufficient to divide the last member of (10.20) by , whereby

(10.21)

Example 10.26 Be it

It is easy to verify (this task is left to the reader) that the Nyquist diagram of the- la has the pattern shown in Fig. 10.19. From that diagram it is also evident that the vector representing the completes around the origin a clockwise turn as da to . Therefore, according to the notation introduced before we can write that .

This result is clearly in agreement with (10.21) being and .

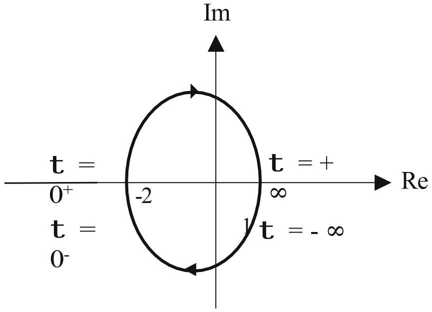


Fig. 10.19. Nyquist diagram of

Consider now the feedback system having as its open-loop transfer function where , is *t h e* transfer function of the direct chain and feedback. As is well known

so omitting the dependence on we can write, without ambiguity in the notation,

From this it follows that the poles of the depend on the function alone (equal to the product of by) but not on the individual functions and .

We can therefore state that.

is also equal to the number of positive real-part poles of .

By virtue of these considerations, we can evaluate the possible number of positive real part poles of ( ) from the Nyquist diagram of the ....Indeed ,

Given the transfer function , we define the difference function associated with it. We plot the Nyquist diagram of that function and count the number of revolutions *(*positive if counterclockwise) that the extremum of the vector representing the completes around the origin as it varies from to A this point, known the number of positive real part poles of the *(* which as seen coincide with those of the *)* , from eq. (10.21) it is immediate to calculate the number of positive real part poles of the , i.e.

If we can conclude that the closed-loop system is stable. Conversely, if
, the closed-loop system is unstable.

Before formally stating Nyquist's criterion, however, it is important to make one last observation that allows for its more immediate application that does not require calculation of the difference function.

Being , for each value of :

that is, the two real parts differ by one unit and the two imaginary parts coincide. Therefore, given the Nyquist diagram of the , to obtain the Nyquist diagram of the it is sufficient to translate the Nyquist diagram of the by one unit to the right. The diagram of the is then located with respect to the origin in the same relative position as the Nyquist diagram of the is located with respect to the point Of coordinates . The number of revolutions that the representative vector *of* makes around the origin as da a varies is therefore equal to the number of revolutions that the representative vector di makes around the point always as da a varies .

By virtue of the latter observation, Nyquist's criterion can therefore be formally enunciated as follows.

Theorem 10.27 (Nyquist Criterion). *Consider a feedback system and let be The open-loop transfer function. Let it be a transfer function proper, without poles in the imaginary axis and such that its Nyquist diagram* does not *pass through the coordinate point .*

Necessary and sufficient condition *for the corresponding loop system closed is stable is that the number of revolutions (positive if counterclockwise) that the extreme of the representative vector of the compie around the point , for which varies from to , be equal and opposite to the number of real-part poles positive of the , i.e. .*

In case the closed-loop system is unstable, the number of poles apart positive real of the closed-loop transfer function is equal to .

Example 10.28 Consider the transfer function

———

already considered in Example 10.21 and whose Nyquist diagram is shown in Fig. 10.14 at varying the sign of and . It is desired to study the stability of the closed-loop system having as open-loop transfer function.

Let us now consider the four cases separately.

Let.....................As shown in Fig. 10.14.a, the Nyquist diagram *of the* remains all to the right of the point which implies that . Th*e*

also has in that case only one negative real-part pole for ....whichIn virtue
of Nyquist's criterion we can then conclude that that is, the closed-loop system is stable.

Let and . Again (see Fig. 10.14.b) the diagram
Nyquist's is all to the right of the point for which . Now
however, holds soWe ....can therefore conclude that the system a closed cycle is unstable and has a positive real-part pole.

Let e In this case we need to distinguish three different situations:
, e . Let us leave out for now the case where *it is.*
since in such a case the Nyquist diagram of the passes through the coordinate point .
In the case that is , the Nyquist diagram of the surrounds the point
In addition, the vector representing the compie as it varies from da
to a clockwise turn around the ....pointIn that case then

. Finally, being , we can therefore conclude *that*
that is, the closed-loop system is unstable and has a positive real-part pole.
Conversely, if the system is closed-loop stable since we are in a case entirely analogous to that seen in the first point ( ).

Both and Again we have to distinguish three different situations: , andLet us ....leave for the moment the case in ....whereIn fact, in such a case the Nyquist diagram of the *pass through the*
coordinate point .
In the case that is , the Nyquist diagram of the surrounds the pointAlso , the representative vector of the compie as it varies
Of gives a counterclockwise turn around the point.....................................Then

. Being , we can conclude that that is the system at closed cycle is stable.
On the contrary, if the system is unstable closed-loop since we are in an entirely analogous case to that seen in the second point (,).

Example 10.29 Consider the transfer function

already considered in Example 10.24 and whose Nyquist diagram is shown in Fig. 10.17. It is desired to study the stability of the closed-loop system having as open-loop transfer function.

The Nyquist diagram of the intersects the x-axis at two points,
one of which is related to . In order to evaluate, it is necessary to determine the second point of intersection of the diagram with the horizontal axis to figure out whether or not the point is internal to the diagram itself. For this purpose, it is sufficient to solve the algebraic equation and then calculate at which

values of the pulse you have the intersection sought[7] . Then, at these values we calculate the value assumed ....byIn this particular example it is easy to verify that the second point of intersection of the curve with the x-axis occurs at the pulsations, *and the* abscissa of the point of intersection holds .

The Nyquist diagram of the is therefore all to the right of whereby

. Finally, since , it is , that is, the closed-loop system is stable.

Suppose now that in place of the previous transfer function we have

The Nyquist diagram of the naturally has the same shape as the Nyquist diagram of the However the abscissa and ordinate of each point must be multiplied by Specifically, the diagram will intersect the positive real semi-axis at the abscissa point 20 (20 being the new gain value) at the pulse and the negative real semi-axis at the abscissa point *a t* the *pulse*

In this case then the point remains internal to the diagram as shown in Fig. 10.20 and is worth Being , we can conclude that the closed-loop system is unstable in this case since it has two positive real-part poles ( )

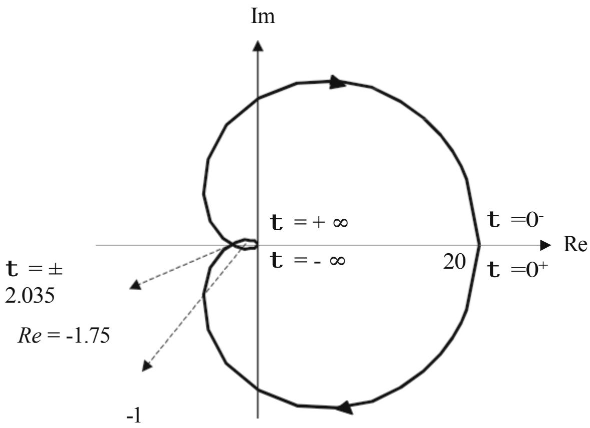


Fig. 10.20. Nyquist diagram *of* where

Let us now discuss 2 critical cases, namely the case in which the has poles in the origin, or more generally imaginary poles, and the case in which the Nyquist diagram of the passes through the coordinate point .

[7]Observe that, given the symmetry of the diagram with respect to the x-axis, *if* is solution of the equation , also is solution of that equation.

First critical case: has imaginary poles

Let us now see how to proceed in the case where the has poles on the imaginary axis.

If la has a *pole in the origin*, la is not clearly defined in . It is then assumed by convention that when the transfer function has a pole in the origin, the variable travels along the imaginary axis of the Gauss plane along a hooked path as shown in Fig. 10.21.a, where the amplitude of the deviation from the origin is infinitesimal. When the point varies along the hooked path thus defined, it is clear that the vector starting from the pole in the ori- gine and ending at the point has an infinitesimal amplitude for values of near zero. This implies that the modulus of the for values of close to ze- ro has an infinite amplitude being the factor with at denominator of the .

In addition, since by convention the pole in the origin is left to the right of the hooked path, it is assimilated to the other poles with negative real part and thus gives rise to a phase change of , or equivalently of clockwise, when it goes from negative values (small in modulus) to positive values (small in modulus), *t h a t* is, for ranging from to .

It is then said that a pole in the origin involves in the Nyquist diagram a closure to infinity of clockwise *f o r* ranging from to .

Of course then if the pole in the origin has multiplicity , the closure at infinity will always occur clockwise but with a phase change equal to

.

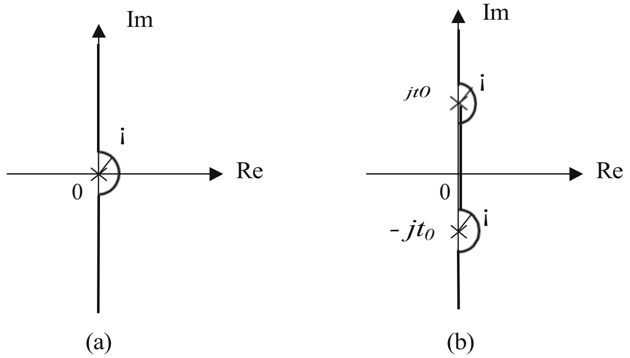


Fig. 10.21. Path followed by the variable in the Gauss plane **(a)** when it has a pole in the origin and **(b)** when it has a pair of poles in the imaginary axis

Analogamente, se la ha una *coppia di poli immaginari puri* ,
the is not clearly defined in .

We then assume a convention similar to that seen in the previous point, that is, we assume that the variable travels along the imaginary axis of the Gauss plane along a hooked path as shown in Fig. 10.21.b, where the amplitude of the deviation from the imaginary poles is infinitesimal. The pair of pure imaginary poles is then left to the right, and these poles are therefore assimilated to the other negative real-part poles.

Ripetendo un ragionamento analogo a quello appena visto per un polo nell'origine, possiamo affermare che ogni coppia di poli nell'asse immaginario comporta nel diagramma di Nyquist una chiusura all'infinito di in senso orario sia per che varia da a , sia per che varia da a .

Clearly then if such poles have multiplicity , then the closures at the in- finite will always occur clockwise but with a phase change equal to

.

Example 10.30 Be it

This transfer function has already been examined in Exemple 10.22 above, and its Nyquist diagram is shown in Fig. 10.15.

Such a transfer function has a pole in the origin and thus starts from an improper point in the complex plane.

In order to apply Nyquist's criterion and count the number of revolutions it is necessary to "close" the diagram. Having the one pole in the origin, the closure will occur with a phase change equal to clockwise at values o f ranging from to . The Nyquist diagram then takes the form shown in Fig. 10.22.

The Nyquist diagram is all to the right of the point for which .
Being , it holds that the closed-loop system having as open-loop transfer function is stable .

Example 10.31 Consider the transfer function

whose Nyquist diagram is shown in Fig. 10.16.

Having the a pole in the origin with multiplicity we must in this case introduce a closure at infinity of clockwise at which varies from to . The Nyquist diagram thus completed thus takes the form shown in Fig. 10.23. The point is therefore internal to the diagram and

is worth Being (the pole in the origin is assimilated in fact to the poles apart from the negative real), results and we can conclude that the closed-loop system has 2 positive real-part poles .

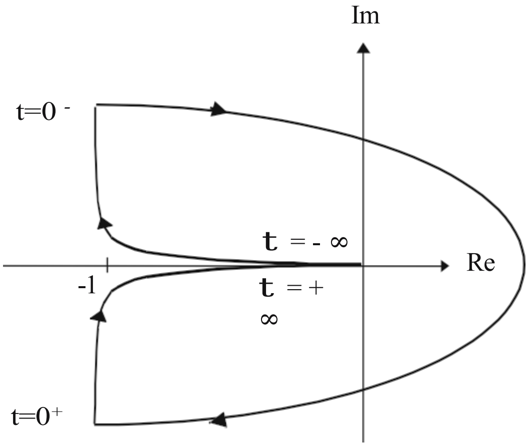


Fig. 10.22. Nyquist diagram of ——— With the introduction of the closure to the infinity of clockwise due to the pole in the origin

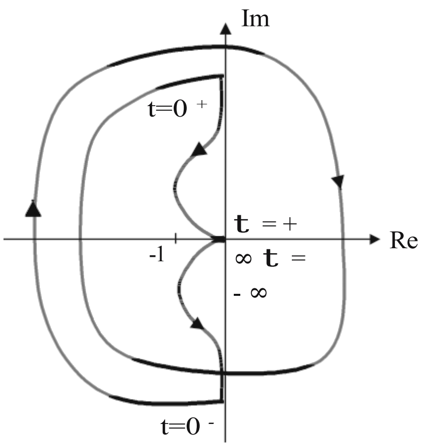


Fig. 10.23. Nyquist diagram of with the introduction of the infinity closure o f clockwise due to the pole in the origin of multiplicity 3

Second critical case: the Nyquist diagram of passes *by*

Another critical situation arises when the Nyquist diagram *of the*
passa per il punto di coordinate ....In questo caso naturalmente possiamo subito affermare che il sistema a ciclo chiuso è instabile in quanto esso ha almeno un polo a parte reale nulla. Sia infatti il valore di in corrispondenza del quale il diagramma di Nyquist passa per . Allora o equivalentemente
and thus is root of the closed-loop characteristic equation and thus pole of the
.

However, it may be important to assess whether there are any positive real-side poles.

Of course, the Nyquist criterion as seen above cannot be appli- cated: if the Nyquist diagram passes through one cannot in fact evaluate
.

A solution to this problem exists, and it is to modify the diagram in such a way that it is then possible to apply the Nyquist criterion with reference to the modified diagram.

Specifically, the diagram is warped so that the point lies to the left of the curve when it is traveled in the direction of the crescents. The Nyquist criterion is then applied on the basis of the curve thus obtained, and the resulting value of indicates the number of positive real-part poles of the .

Example 10.32 Consider a closed-loop system with unit feedback
and both

the open-loop transfer function coincident with the transfer function of the direct chain, where and . It is desired to evaluate the stability of the closed-loop system and the possible number of poles with positive real part of the
.

As already seen (Example 10.28) if the Nyquist diagram of the passes through the coordinate point at the pulsation . We can therefore say that a pole of the closed-loop system lies in the origin so that the closed-loop system is unstable.

To evaluate any positive real part poles of the we can apply the Nyquist criterion after modifying the Nyquist diagram according to the rule stated above, as shown in Fig. 10.24.

Since this diagram is all to the right of the point , it is worth . Essen-give then we can conclude that the closed-loop system has no positive real-part poles ( ).

As a verification, note that

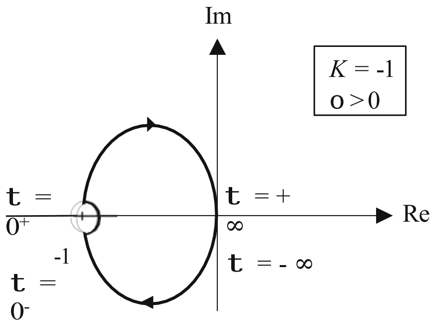


Fig. 10.24. Nyquist diagram of ——— with e

Example 10.33 Let it be.

—

In that case , ...eIn accordance with the rules seen for tracking
of the Nyquist diagram, it is immediate to observe that this diagram starts from an improper point in the complex plane with phase and ends in the origin also with phase .

It is also immediate to observe that the , for any value of nonzero, is equal to a negative real number. In fact,

—

Therefore, the Nyquist diagram of such a transfer function coincides with the negative real semi-axis of the complex plane, including the origin. In particular, for positive values of that axis it is traveled in the positive direction; for negative values of it is traveled in the negative direction.

The presence of the dual pole in the origin also results in the clockwise infinity closure of *f o r* ranging from to .

Finally, since the diagram passes through the point , this digram must be modified in accordance with the above rule.

Therefore, the resulting diagram adapted for the purpose of applying the Nyquist criterion is as shown in Fig. 10.25.

Note that just to provide greater clarity in the representation the dia- gram was plotted close to but not coincident with the negative real semi-axis.

In reality, however, it coincides with that semi-axis for both positive and negative pulsations.

Applying the Nyquist criterion with reference to such a curve, it is easy to verify that the closed-loop system has no positive real-part poles ( ).
However, it is unstable since it has zero real part roots, which follows from the Nyquist diagram passage for the point .

As a verification, observe that the denominator of the transfer function at closed cycle is equal *to* It has therefore
Two conjugate complex poles with zero real part, .

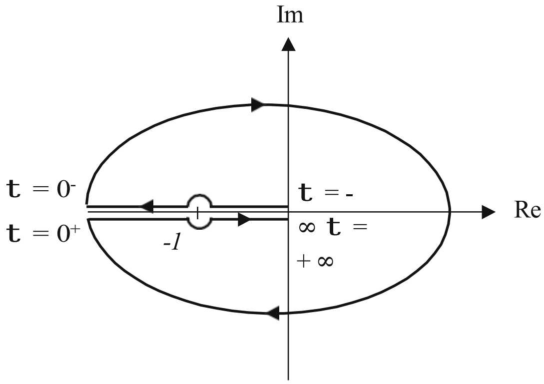


Fig. 10.25. Nyquist diagram of the modified for the purpose of applying the Nyquist criterion.

10.4 Places to calculate when graphically assigned

In many practical applications it may happen that given a certain feedback system one does not know the analytical form of the direct chain transfer function, but has only the graphical representation relating to the domain of that function. It is possible in this case, through the use of appropriate locations, to easily derive a graphical representation of the closed-loop transfer function.

Let us assume initially that the feedback is unitary, viz. ....Vedre-mo then that the proposed procedures can be applied even when this assumption is not met.

We distinguish two different cases:

you have the Bode diagram of the ,
you have the Nyquist diagram of the .

In the first case, therefore, the modulus and phase of , in the second case, on the other hand, the real and imaginary parts of the

.

10.4.1 Paper by Nichols

Consider a closed-loop system with unit feedback and transfer function

$$ \qquad (10.22) $$

Where is the transfer function of the direct chain. In particular, let be.

$$ \qquad (10.23) $$

e

$$ \qquad (10.24) $$

The polar representations of and of , respectively.

We call *Nichols diagram* (or *Nichols representation*) of the the curve obtained in the Cartesian *plane* (called *Nichols plane*) in which in abscissa is placed the phase expressed in degrees and in ordinate is placed the modulus expressed in decibels. Such a curve is therefore parameterized in and is immediately obtainable from the Bode diagram of the .

Plotting the Nichols diagram of the on an appropriate chart, called *Nichols chart*, allows the determination of the Bode diagram of the

.

The Nichols chart is in fact an abacus comprising two different families of curves in the Nichols plane: the first family is given by the set of closed-loop constant modulus (constant) curves; the second family is given by the set of closed-loop constant phase (constant) curves. By plotting the Nichols diagram of the on the Nichols chart, it is therefore immediate to read for each value of the the corresponding value of and , that is, of the modulus and closed- cycle phase corri-spondent to that value of the pulsation. For each point on Nichols' paper corresponds to a well-defined closed-loop constant modulus curve and a well-defined closed-loop constant phase curve. Plotting the thus obtained values of and for the values of of interest in semilogarithmic paper, we obtain the Bode diagram of the .

Let us now see how closed-loop constant modulus and phase curves are defined in the Nichols plane.

By virtue of (10.22) and keeping in mind (10.23) and (10.24), we can write where for simplicity of notation the dependence on . Two scalar equations follow from this vector equation, namely, the equation with constant modulus and phase.

The equation of place at constant modulus is equal to

which after simple manipulations can rewrite itself in compact form as[8]

(10.25)

Since we assume that we have the Bode diagram of the , in that equation will have to be substituted for its expression in decibels, viz.

Furthermore, since we are interested in parameterizing constant modulus curves with values of expressed in decibels, instead of in (10.25) we will have to put

Note that since , and functions of , every constant modulus curve in the Nichols plane is therefore a curve parameterized by the pulsation .

The constant-phase equation of place is easily obtained by manipulating the scalar equa- tion

and is equal to[9]

[8]The module condition

can be rewritten as

from which follows

[9]The phase condition

Can be rewritten as follows

asi$n$

The constant a anda locations in the Nichols plane constitute the Nichols map, shown in Fig. 10.26. These places obviously repeat identically for vertical bands with periodicity equal ....toAlso within each band of periodicity both families of curves have a symmetrical trend with respect to the vertical passing through the midpoint of the band itself (i.e., the abscissa point).

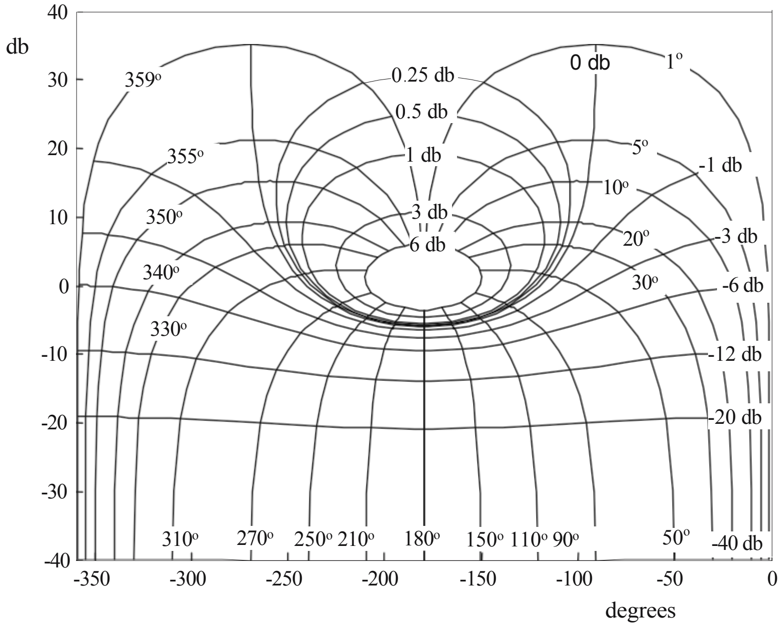


Fig. 10.26. Nichols paper.

atan

atan ata$n$

asi$n$
asi$n$

In summary, Nichols' paper can be used as follows. Suppose we have a closed-loop system with unit feedback and direct chain transfer function of which we know only the Bode diagram in a certain interval of . We then plot by points the Nichols diagram of the Every
point on that diagram intersects a constant modulus curve and a constant phase curve. For example, let be the value of the pulsation relative to a certain point on the Nichols diagram, and let e be the values assumed by the intersecting curves at that point. Let these values of and give respectively the modulus (in decibels) and phase (in degrees) of the closed-loop transfer function for È possible then to obtain by points the Bode diagram of the in the considered interval.

Example 10.34 Consider a closed-loop system in unit feedback. Let it be supposed that only the Bode diagram of della is known in a certain frequency range but not its analytical expression. In particular, let the Bode diagram of the be the one shown in Fig. 10.27.

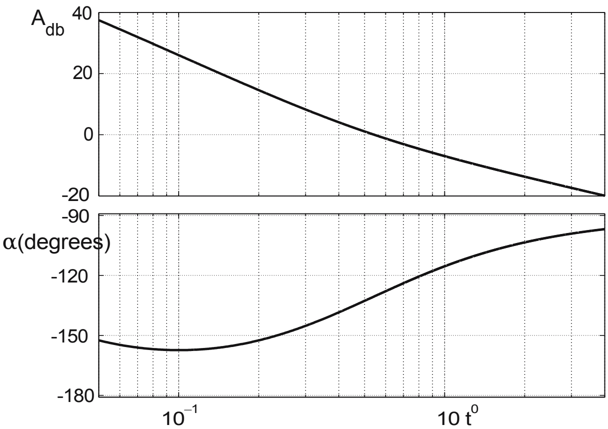


Fig. 10.27. Bode diagram of the socket under consideration in Example 10.34

From that diagram it is possible to construct the following table obtained by evaluating for some significant points of the modulus and phase of .

Plotting these points on the Nichols chart, it is easy to read for the same values of , the values of the modulus (in decibels) and phase (in degrees) of the function of closed-loop transfer as shown in Fig. 10.28. More precisely, in that figure, the data points in the table above were indicated with small circles. By interpolation, the Nichols diagram of the .

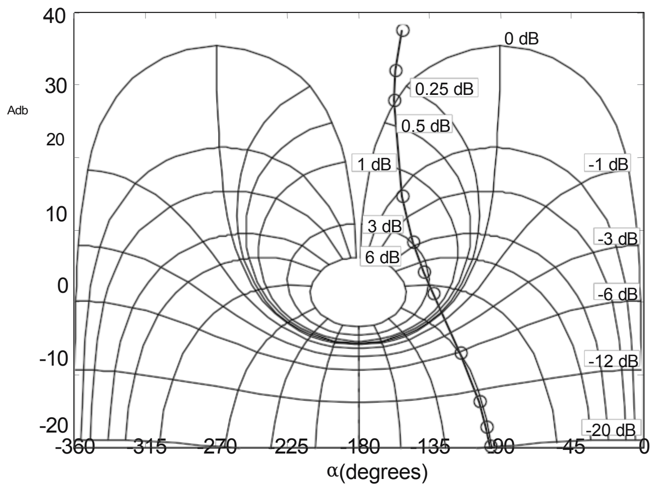


Fig. 10.28. Nichols diagram of the socket under consideration in Example 10.34.

The modulus and phase values of the relative to the sockets under consideration for plotting the Nichols diagram are summarized in the following table:

Finally, by plotting the corresponding values of and of in the semi-logarithmic chart for the different considered, the Bode diagram of the in Fig 10.29 i s obtained by points.

Note that from the Nichols diagram of the it is easy to deduce some im- portant observations related to the behavior of the closed-loop system, such as the modulus at resonance, the resonance pulsation, the bandwidth and the relative pulsation.

Example 10.35 Consider again the unit feedback closed-loop system whose direct chain transfer function is equal to the one considered in Example 10.34 above. The Nichols diagram of that function has been plotted by points and is shown in Fig. 10.28.

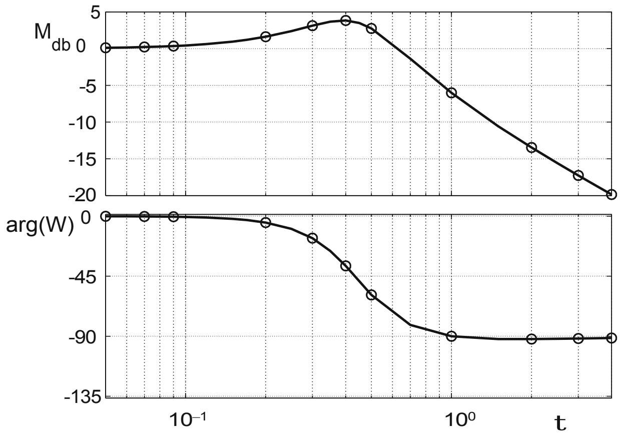


Fig. 10.29. Bode diagram of the relative to Example 10.34.

From such a diagram it is easy to realize that the maximum value that the modulus of the can take is approximately equal to db: the constant modulus curve corresponding to db is in fact the curve to the left of the Nichols diagram tangent to it[10] . In particular, such a tangency point occurs for as can easily be deduced from the last table seen in Example 10.34. We can therefore conclude that the modulus at resonance and the pulsation at resonance are worth respectively

rad/s

The validity of this result can of course be verified by looking at the Bode plot of the shown in Fig. 10.29.

From the Nichols diagram of the we finally deduce that and in
.................................particularRemembering that the bandwidth at db is equal to the value of the
frequency at which there is an attenuation of db with respect to the value of the modulus in
, we can say that the bandwidth at db is in this case equal to the value of at which db, divided naturally
....From
diagram in Fig. 10.28 we therefore conclude that.

Hz

By entirely similar reasoning, we can calculate the bandwidth at db, at db, etc .

[10]In fact, the curve parameterized by the db value is not shown in Fig. 10.28. However, this value can be deduced in good approximation by taking into account that this curve lies between that at db and that at db and in particular is closer to that at db.

10.4.2 Places on the floor of Nyquist

Both

Since by assumption the feedback is unitary, omitting for simplicity the dependence on , we can write

Re Im

Re Im

or equivalently

By posing and equal to constants, we obtain the constant-modulus and constant-phase places in the Nyquist plane, respectively. It is easy to verify that both constant modulus and constant phase curves are circumferences .[11]

In particular, constant modulus curves have radius and center coordinates given by the following expressions:

[11]Let us remember as a first thing that

is the equation of a circumference in the plane . Specifically, this circumference is centered at the point of coordinates

and has radius

Now, since the condition of modulus

vale

which is the equation of a circumference of center

and radius

Their qualitative trends are shown in Fig. 10.30. The center of such circumferences thus always lies on the x-axis and for tends to coincide with the origin. As the center shifts to the right as the value of , the center moves to the right until it reaches infinity for . Then, for values of , the center moves along the negative real axis and for As can therefore be seen from the
Fig. 10.30 the place characterized by the value of (i.e., the vertical line passing through the point of abscissa) is an axis of symmetry for such a family of circumferences. Moreover, circumferences symmetrical with respect to such a line are characterized by inverse values of.

In contrast, constant-phase curves have radius and *center* coordinates given by the following expressions :[12]

The qualitative trend of these curves is as shown in Fig. 10.31 where we can observe their symmetry with respect to the x-axis. In particular, two circumferences that are symmetrical with respect to the x-axis are characterized by values of opposites.

La carta dei luoghi a costante e a costante è infine riportata in Fig. 10.32. Tale carta è ottenuta mediante l'unione delle due famiglie di circonferenze riportate

[12]The phase condition is

but being

vale

n

ata atan

which is the equation of a circumference of center

and radius

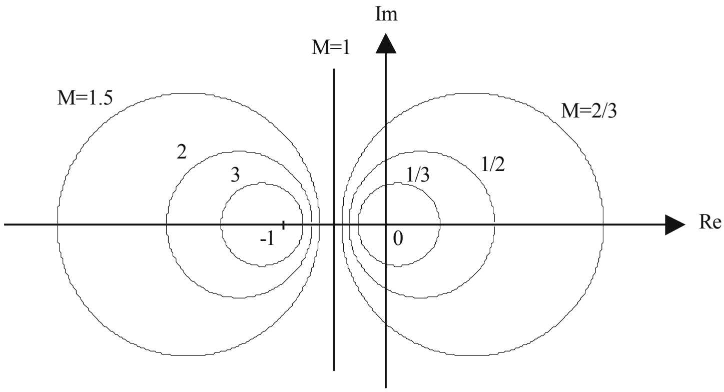


Fig. 10.30. Place ad constant in Nyquist's plane.

in Figs. 10.30 and 10.31 and can be used quite analogously to what we have seen regarding Nichols' paper. In fact, suppose we have a closed-loop system with unit feedback and an open-loop transfer function of which we know the trend relative to the frequency domain only by graphical means. We then plot by points the Nyquist diagram of the on the paper in Fig. 10.32. Each point on the diagram thus obtained intersects a constant modulus curve and a constant phase curve. For example, let be the value of the pulsation relative to a certain point on the Nyquist diagram, and let e be the values associated with the intersecting curves at that point. Such values of and provide the modulus and phase, respectively, of the closed-loop transfer function for . From these values it is then easy to construct by points the Nyquist diagrams of the closed-loop transfer function.

Finally, it is important to note that although the Nichols chart and the chart in Fig. 10.32 were obtained under the assumption that it is , these charts can also be used when the feedback is not unitary, as long as at least one graphical representation of it is known. For this purpose, it is sufficient to use the following contrivance. Omitting the dependence on , the closed-loop transfer function

can in fact be rewritten as

It is then possible to use recital cards in place of la , the transfer function being

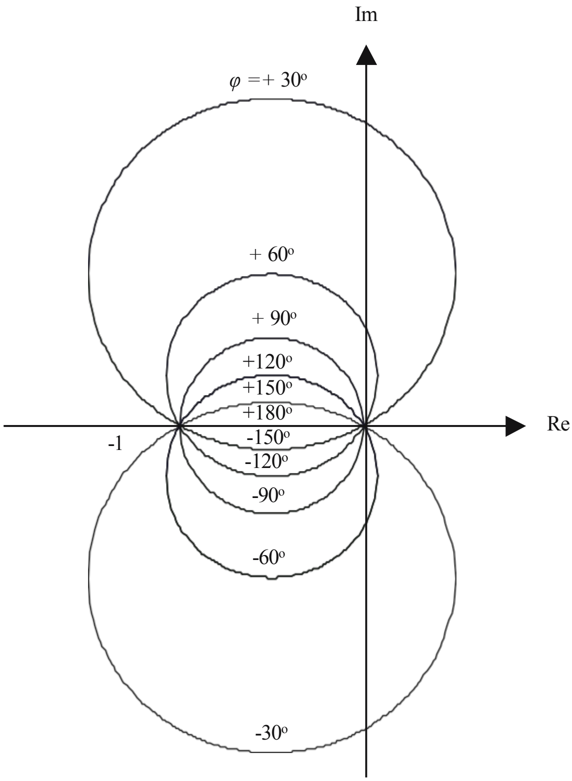


Fig. 10.31. Place ad constant in the Nyquist plane.

into the desired shape. One then constructs by points the Nichols (or Nyquist) diagram of and then multiplying the values that takes at certain values of by the value that takes at those same values of , one obtains the corresponding diagram of .

Exercises

Exercise 10.1 Verify by Routh's criterion the stability of the closed-loop system with unitary counter reaction corresponding to the following open-loop transfer function:

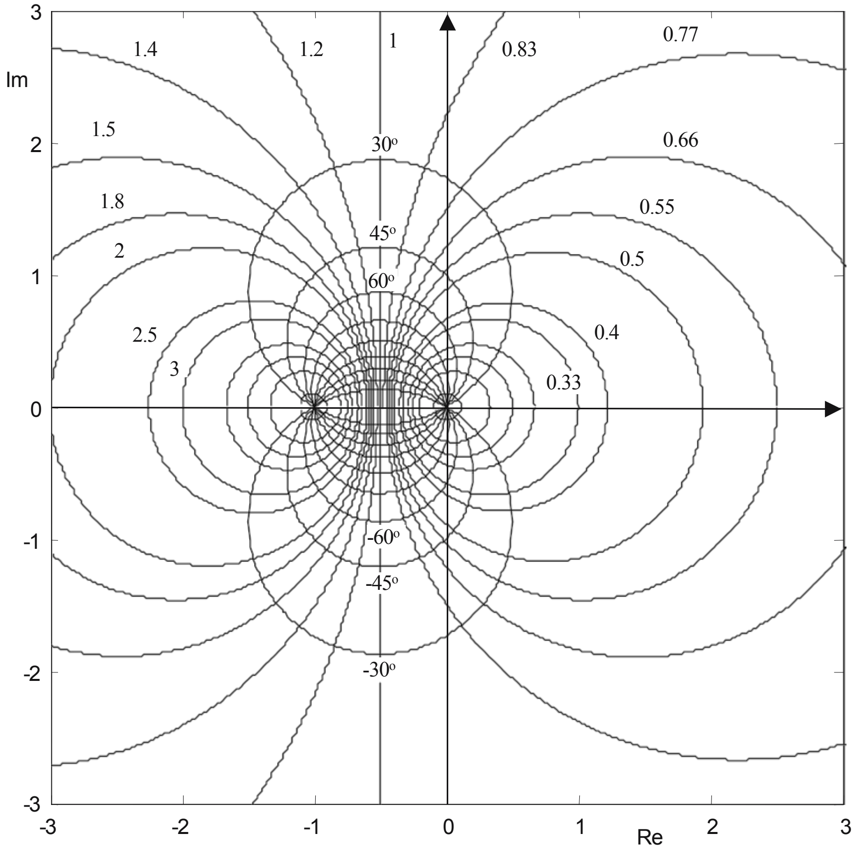


Fig. 10.32. Place ad and constants in the Nyquist plane.

Exercise 10.2 Plot the locus of roots for the following assigned systems me- dialed by their open-loop transfer functions:

(a)

(b)

(c)

(d)

(e) ______________

(f) ______________

(g) ______________

(h) ____________________

Exercise 10.3 Plot the Nyquist diagram of the transfer function from Exercise 10.1 and study the stability of the closed-loop system by applying Nyquist's criterion.

Exercise 10.4 Analyze using Routh's criterion the stability as *t h e* system parameter in Fig. 10.33 changes. Interpret the results obtained using Nyquist's criterion.

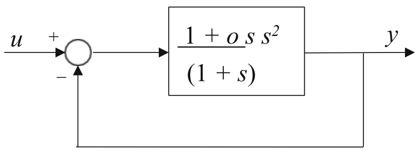


Fig. 10.33. Feedback system related to Exercise 10.4.

Exercise 10.5 Plot the locus of the roots as the parameter changes for the system from the previous exercise.

Exercise 10.6 Plot the Nyquist diagram of the following transfection functions from Exercise 10.2.

Also study the stability of closed-loop systems in unit feedback having such direct chain transfer functions.

Exercise 10.7 Let it be seen by Nyquist's criterion that there is no value of so that the feedback system with direct chain transfer function

____

results in stability.

Exercise 10.8 Consider a unitary feedback system having

As the transfer function of the direct chain. Draw the Bode diagram of the closed-loop transfer function using Nichols' paper. Also plot the same diagram from the analytical expression of the closed-loop transfer function.

11

Controllability and observability

This chapter introduces two fundamental properties of dynamical systems, namely, the properties of *controllability* and *observability*. Specifically, controllability indicates the possibility of bringing the state of the system from a generic initial condition to a desired final value, while observability indicates the possibility of reconstructing the value of the initial state of the system on the basis of observing its output. In general, controllability and observability depend on:

From the particular initial state of the system,
From the initial instant of time,
From the objective state (in the case of controllability).

However, this dependence is lost in the case of *linear and stationary* systems, which are the class of systems on which we will focus our attention. For this reason in the rest of the discussion we will always speak of controllability and observability *of the system* and refer to an initial instant of time .

Si noti inoltre che tali proprietà sono date con riferimento a sistemi in termini di variabili di stato, pertanto in questo capitolo ci riferiremo sempre a sistemi nella forma:

In particular, in the following we provide necessary and sufficient conditions for controllability and observability and show how these properties are invariant with respect to any similarity transformation.

In an asterisked section, a number of important ca- nonic forms are also introduced, such as the *controllable canonical Kalman form* and *the observable canonical Kalman form*, to which any linear, stationary system can be brought back through appropriate similarity transformations. Also in in-depth sections, the concept of state feedback is introduced and it is shown how the controllability property is equivalent to the possibility of arbitrarily assigning the eigenvalues of the closed-loop system. Also introduced is the concept of an asymptotic observer of the state, which is necessary within the feedback loop in case the state is not directly measurable. In particular, it is shown how such an observer can always be constructed in the case where the system is observable.

The link between controllability properties, observability and input-output transformations is finally discussed at the end of the chapter.

11.1 Controllability

In this section a formal definition of a controllable linear and stationary system will first be given. Criteria for analyzing this property will then be given, the first two of a general nature, the third relating to the case where the matrix is in diagonal form with eigenvalues all distinct.

Since the controllability of a linear, stationary system depends on the pair of matrices alone in the following we will not specify the output transformation.

Definition 11.1. *A linear, stationary system*

is said to be controllable *if and only if it is possible, by acting on the input, to transfer the state of the system from* any *initial state to* any *other state , called* zero state *or* objective state*, in a* finite *time* .

Let us now look at a simple physical example in order to intuitively illustrate the controllability property.

Example 11.2 Consider the network in Fig. 11.1 and assume that this network is ini- tially at rest. Assume as state variables the currents at the ends of the two inductors, i.e., let e .

It is easy to verify that it is not possible, by simply acting *on* the input voltage, to impose any value on these currents. In fact, it follows from the symmetry of the network that whatever . The system is therefore not controllable. It is interesting to note, however, that it is possible to impose the desired value on one of the two state variables, subject to the constraint that the other will also assume that value. Indeed, we observe that the following relationships hold:

—

from which it follows that

Where the second equality follows from the fact that .

Therefore, the differential equation governing the evolution of the first component of the state is equal to

Now, remembering Lagrange's formula, we have that

$$(11.1)$$

Suppose now that the input is a step of amplitude . We show that for every possible choice of parameters of the system, for every possible choice of and of the target state , there exists a value of such that the first compo- nent of the state carries in at time . Substituting into equation (11.1) the expression Æ and integrating we obtain in fact that

To impose the desired value on the state it is then sufficient to assume

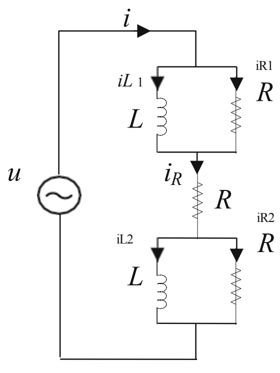


Fig. 11.1. The uncontrollable network of Example 11.2.

11.1.1 Checking controllability for arbitrary representations

In this section, two different criteria for analyzing controllabi- lity will be proposed, both based on the calculation of appropriate matrices. Specifically, the first criterion is based on the definition *of* the *controllability gramian*, while the second criterion is based on the

On the definition of the *controllability matrix*. As we shall see, the second criterion provides a more straightforward analysis procedure since the determination of the controllability matrix simply requires the unfolding of some matrix products. However, the first criterion is extremely useful because, in the case where the system is controllable, knowledge of the controllability gramian makes it possible to derive a control law capable of transferring the state of the system from its initial value to the desired final value.

Definition 11.3 *Given a linear, stationary system***.**

(11.2)

where and , we define Gramian[1] of controllability *the matrix*

(11.3)

Theorem 11.4. *The system* (11.2) *is controllable if and only if the controlla- bility gramian is nonsingular for every .*

Demonstration. (Sufficient condition*.)* The proof of the sufficient condition is co- structive, i.e., assuming that the controllability gramian is nonsingular for , an input is determined to switch between any state to any final state in a time . In particular, assume

In this expression, the left factor is a function of the variable time , while the second factor is a *constant* vector which depends on the final time instant and the initial and final states.

It is immediate to verify that such an input leads to the state ; in fact according to Lagrange's formula it is worth

[1]Another term used to denote Gramian is *Gram matrix*. The name comes from Jorgen Pedersen Gram (Nustrup, Denmark, 1850 - Copenhagen, 1916).

Where the variable change was used .

(Necessary condition.) Suppose there exists such a that the gramian is singular in . This implies, by virtue of Theorem E.5 (see Appendix E), that the rows of are linearly dependent in Then there exists a vector [2]
constant such that

We now show that chosen as an initial state and chosen an end instant there is no input to bring the state into . In fact, according to Lagrange's formula if a vector is reachable *at* time from the state , it holds:

where in the last step the change of variable .

Pre-multiplying both members of that equation by , it follows

(11.4)

Whatever the input signal , the integral at the second member is zero and the previous equation becomes : this equation cannot be satisfied being .

Thus, we have shown that if the gramian is singular in it is not possible to reach the state at any instant of time However, the
Proof of the following Theorem 11.6 shows that if the gramian is singular at a given instant of time then it is singular at *every* other instant of time
From this naturally follows the non-controllability of the system.

Example 11.5 Consider the linear, stationary system

Both It is desired to verify the controllability of such a system and deter-
undermine an appropriate control law capable of bringing the system into the state
at the time .

The state transition matrix for this system is worth

So the Gramian of controllability holds.

[2]It is easily verified that the vector belongs to the null space of the gramian.

e

So the system is controllable.

It also applies to

e

Therefore assumed *for*

the system moves to the desired state at the time.

Theorem 11.4 provides a criterion for verifying controllability that is constructive and shows how to choose an appropriate input that allows a desired state to be reached. However, if you only want to determine whether a given system is controllable, it is easier to use the following criterion.

Theorem 11.6. *Given a linear, stationary system*

Where and , we define the controllability *matrix as the matrix*

A necessary and sufficient condition *for the system to be* controllable *is that it is worth*

Demonstration. From Theorem E.5, we know that the controllability gramian is nonsingular for every if and only the rows of are linearly indi- pendent inAs a consequence, by virtue of Theorem 11.4, to prove the validity of this theorem, it is sufficient to prove the equivalence of the following two conditions.

(a) All rows of are linearly independent in .
(b) The controllability matrix has rank equal to .

To this end, we preliminarily observe that the elements of are linear combinations of terms of the type where is eigenvalue of , so they are analytic functions ....inWe can therefore apply Theorem E.7 according to which the rows of are linearly independent in if and only if

For each .

Since the matrix has full rank for every , the equation above reduces to

Based on the Cayley-Hamilton Theorem (see Appendix G, Proposition G.5) we know that the function with can be written as a linear com- bination of , , , ; therefore the columns of con are linearly independent of the columns of , , , ...Consequently

which proves the validity of the statement.

Example 11.7 Consider the linear, stationary system described by the equation of state

(11.5)

Vale ....eThe controllability matrix has *dimension*
and it's worth

Columns 1, 2 and 4 form a nonsingular minor of order 3. Therefore, it is worth

So the system is controllable.

11.1.2 Verification of controllability for diagonal representations

Let us now see how we simplify the checkability check in the case where the matrix has eigenvalues that are all distinct and is in diagonal form.

Theorem 11.8. *Consider a linear, stationary system with e described by the following equation of state*

$$\vdots \quad \vdots \quad \ddots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \ddots \quad \vdots$$

in which, that is, the matrix is in diagonal form. Let the eigenvalues of all be distinct, i.e., for each .

A necessary and sufficient condition *for the system to be* controllable *is that the matrix has no identically null rows.*

Demonstration. (*Necessary condition.*) Suppose that the -th line of is identically zero. In this case we have that

that is, the -ma component of the state evolves in free evolution and cannot be controlled by the input.

(*Sufficient* condition.) We give for simplicity the demonstration of this condition only in the case where the input is scalar (*)* . The controllability matrix can be written in full as.

$$\vdots \qquad \ddots \qquad \ddots$$

that is, as the product of two nonsingular matrices, where the first is a diagonal matrix whose elements are all nonzero being equal to the elements of , while the second matrix is equal to the Vandermonde matrix (see § 4.2.2). Therefore, the controllability matrix is also nonsingular.

Example 11.9 Consider the linear, stationary system described by the equation of state

$$(11.6)$$

Whose diagonal matrix has all distinct eigenvalues: , e . Since the third row of identically null, we can immediately conclude that the system is not controllable .

11.1.3 Controllability and similarity

In the case of linear and stationary systems, controllability is not a property of the particular realization and is therefore invariant with respect to any similarity transformation. This is the reason why, in the case of linear and stationary systems, it is permissible to speak of controllability of the system and not controllability of the realization.

Theorem 11.10. *Consider two representations of the same system of order :*

e

bound by the similarity transformation , where is a nonsingular matrix. Therefore it is worth: and .

La prima realizzazione è controllabile se e solo se *la seconda è controllabile.*

Dimostrazione. La matrice di controllabilità della seconda rappresentazione vale:

times

and being nonsingular, the controllability matrices of the two representations have the same rank.

Example 11.11 Consider the linear, stationary system described by the equation of state

Vale eThe similarity transformation , with

e

said and leads to the system

The controllability matrix of the first system and the second system are respectively valid:

| | |

Both matrices are square and have full rank:

so the two representations are both controllable.

Note that since controllability is invariant with respect to the particular realization, in fact Theorem 11.8 provides an alternative criterion for analyzing con- trollability even when the matrix is not in the diagonal form, as long as it has eigenvalues that are all distinct. In fact, in this case it is always possible to define a similarity tra- sformation in which the dynamic matrix of the new realization is in the diagonal form. At this point, Theorem 11.8 can be used for the study of pair controllability where and are the coefficient matrices of the realization in . By virtue of Theorem 11.10, the conclusions reached for the realization in are then valid for the original representation.

Example 11.12 Consider the linear, stationary system described by the equation of state

The matrix is not diagonal, but having eigenvalues all distinct, i.e. , and , it is possible to define a *similarity* transformation such that in the realization in the new dynamic matrix is diagonal. In particular, assuming

(where the columns of are eigenvectors of) , the dynamic matrix of the realization in is diagonal. More precisely, it is worth

Since we do not have identically null rows, we can conclude that the original representation is controllable.

It is left to the reader to verify this example through the use of the controllability matrix.

11.1.4 Kalman controllable canonical form [*]

We now introduce a particular canonical *form*, called *the Kalman controllable canonical form*, which emphasizes the controllability properties of a given linear and stationary system, in a manner quite analogous to how the Jor- dan canonical form emphasizes the stability properties. Of course, in this case, since controllability is a property of the pair , the canonical form concerns the structure of both matrices and not of the matrix alone .

Definition 11.13. *A linear, stationary system***.**

(11.7)

in the Kalman controllable canonical form *is characterized by the following structure of the coefficient matrices:*

where is a square matrix of order equal to the rank of the matrix of con- trollabilità e è una matrice il cui numero di righe è anch'esso pari ad . In particular, the pair

Is controllable.

The above definition implies that the state vector of a realization in the canonical Kalman form can be rewritten as

dove e . Di conseguenza il sistema (11.7) può essere decomposto in due sottosistemi secondo lo schema in Fig. 11.2 dove

the controllable part is the subsystem of order governed by the differen- tial equation:

the non-controllable part is the subsystem of order governed by the differential equation:

which cannot be influenced in any way by the input, either directly or indirectly through .

Note that any linear, stationary system can be traced to the canonical Kalman form. The following result applies in this regard.

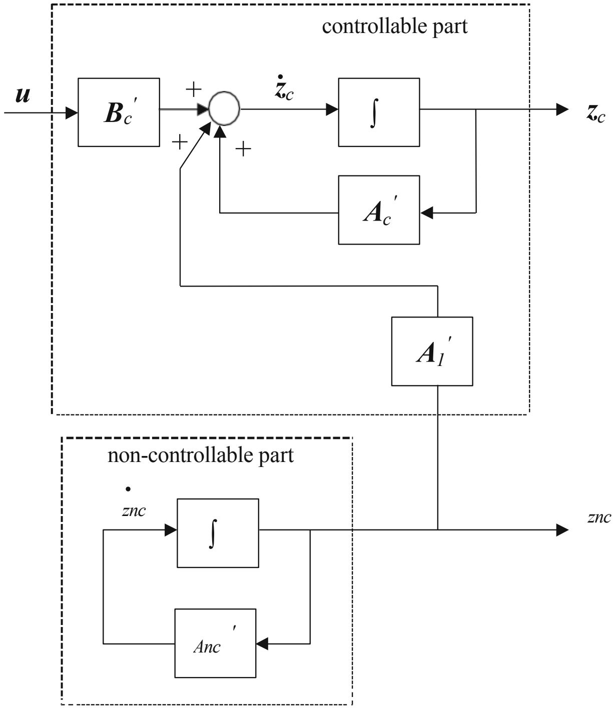


Fig. 11.2. Controllable canonical form of Kalman

Theorem 11.14 *Given a generic system*

this can be traced back to the canonical controllable Kalman form through a simple similarity transformation , where it is a nonsingular matrix whose first columns coincide with linearly in-dependent columns of the controllability matrix and whose remaining columns are equal to
columns linearly independent of each other and linearly independent of the previous columns.

It is therefore clear that the similarity transformation that allows a system to be brought into the canonical Kalman form is not unique.

Example 11.15 Consider the system in Example 11.9 whose control- lability matrix is worth

....................................ePo determine the first 2 columns of the matrix we need to select 2 linearly independent columns ....ofAd example, proceeding from
Left to right we get the two vectors:

The third column of must be a vector linearly independent of and . The simplest choice is to choose such a vector so that it is orthogonal to the two preceding vectors[3] . This implies that, said

such a vector, it must be solution of the system of equations:

From such a linear system of 2 equations in 3 unknowns, it results *that* while can be any nonzero real number. The vector can therefore be any vector in the form

If we assume for simplicity , by means of the transformation where

the system is placed in the canonical controllable Kalman form. In particular, the new realization is obtained

dove

[3]Recall that two vectors and are orthogonal to each other when their scalar product is zero i.e. .

La terza componente dello stato nella nuova realizzazione è quindi in evoluzione libera. Si noti che in effetti questo è un caso particolare poiché non influenza la parte controllabile essendo .

11.2 Status feedback [*]

In the feedback control scheme seen in Chapter 10, it was assumed that feedback occurred on the output. This in fact is not the only possibility. Indeed, in many cases it is more advantageous to perform feedback on the state of the system rather than on its output. The state is in fact the set of physical quantities that determine, known the external input, the future evolution of the system. It is therefore intuitive that in order to obtain the desired evolution of the system, or equivalently the satisfaction of the imposed specifications, it is generally more advantageous to make the input depend on the state rather than on the output.

The feedback scheme in this case takes on the structure shown in Fig. 11.3 where it indicates the *set point*, i.e., the signal to be reproduced. In the following we will assume for simplicity that it is .

The feedback control law is defined by a matrix whichis in general a function of time. Particularly in the case where the set point is zero, this law holds .

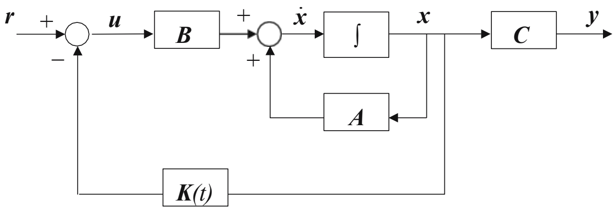


Fig. 11.3 Wiring diagram of a state feedback control system

In this case, the controlled system is governed by the differential equation

where the matrix is called *a closed-loop* dynamic matrix.

There are several procedures for determining the feedback matrix , dependent on the particular objective of the control. In the following we will set the

our attention to a particular class of problems whose objective is to im- pose the desired dynamics to the closed-loop system through an appropriate choice of its eigenvalues. In this case the feedback matrix is a constant matrix for which the control law takes a form of the type ....In this regard
The following fundamental result applies.

Theorem 11.16 *The system*

with and , is controllable if and only if *chosen any set of real numbers and/or pairs of conjugate complex numbers , there exists a feedback matrix such that the eigenvalues of the closed-loop matrix are equal to .*

In other words, the controllability property coincides with the possibility of being able to as- sign at will the eigenvalues of the closed-loop system through constant feedback on the state. In the following for clarity we will study separately the case where the input is scalar from the case where the input is a generic vector at , with .

11.2.1 Input scalar

The determination of the feedback matrix leading to the desired eigenvalues proves particularly simple in the case where the input is scalar and the matrix both in the canonical form of control (see Appendix D, eq. (D.4)).

Theorem 11.17. *Consider the system*

(11.8)

With ....eSia such a system in the canonical form of control, viz.

(11.9)

Let , , be any set of real eigenvalues and/or cop- pie of conjugate complexes. Let , , , be the coefficients of the characteristic polynomial related to these eigenvalues.

Chosen as feedback matrix

the cycle system has as eigenvalues , .

Demonstration. Recall that, called the eigenvalues of a matrix, the coefficients , , , of its characteristic polynomial are related to the eigenvalues of the matrix by the relation

The dynamic matrix of the closed-loop system is equal to

.

for which the eigenvalues of coincide precisely with the desired eigenvalues. In fact, the matrix is also in canonical control form so that the coefficients of the last row coincide with the coefficients of its characteristic polynomial.

Note that this result provides a constructive procedure for determining the feedback matrix even when the system is not in canonical control form. In fact, as seen in Appendix D, any controllable system can be placed by appropriate similarity transformation , in the canonical control form for which the determination of the feedback matrix, which we denote as , is immediate. At this point, multiplying to the right the matri- ce by the inverse of the transformation matrix ( ) , we obtain the retraction matrix for the starting realization. This procedure is illustrated through the following simple example.

Example 11.18 Consider the system

It is clearly controllable: the matrix is indeed diagonal, its eigenvalues are distinct, and the vector has no null elements.

Si desidera determinare una matrice di retroazione tale per cui il sistema a ciclo chiuso sia stabile e i suoi autovalori valgano , .

For this purpose, we first calculate a *similarity* transformation such that the new realization in is in the canonical controllable form.

Following the procedure outlined in Appendix D, it is easily obtained (verification of this is left as an exercise to the reader) that

to which the new realization corresponds

In canonical form of control.

The feedback matrix for this realization is worth

so the retraction matrix for the starting system is

It is immediate to verify that the eigenvalues of the matrix (coincident naturally with the eigenvalues of the matrix) are precisely equal to ,

.

11.2.2 Input not scaled

In the case where the input is not scalar but the system is controllable, several procedures can be followed for determining a feedback matrix to impose the desired closed-loop eigenvalues. In the following for brevity only one such procedure will be presented, which also involves the determination of a particular canonical form.

Before defining this canonical form, however, it is necessary to introduce some preliminary definitions.

Consider the generic pair where ....eSia the -ma column of the matrix and

The controllability matrix associated with the pair .

Now, select the linearly independent columns of according to the following criterion: starting from left to right, discard all the columns of that can be written as a linear combination of the columns of that lie to their left.

At this point we can define a new matrix obtained by reordering the columns of selected as follows:

(11.10)

where each integer , , denotes the number of linearly indepen- dent columns associated with . Clearly

Where the equality holds if and only if it is controllable.

The indices are called *controllability indices* of .

Example 11.19 Be it

so that and . Following the above procedure we select the following linearly independent columns:

from which it follows that ,.......................The system is therefore controllable being

.

Finally, the matrix is worth

In the case where the pair is controllable, it is possible to define a canonical part- form that is extremely useful in assigning closed-loop eigenvalues. Any controllable system can be placed in such a form through a uniquely defined similarity transformation once the matrix is known
. Indeed, the following result applies, which for simplicity we give without demonstration.

Theorem 11.20. *If the system*

$$(11.11)$$

with and , is controllable, then it can be placed, through an appropriate similarity transformation , in the form

$$(11.12)$$

with

$$(11.13)$$

where it indicates a generic element that can be nonzero while corresponding to empty positions there are zeros.

System (11.12) is said to be in the canonical form of multivariable control.

Note that it can be shown that the last rows of each block of are always *linearly independent of* each other. This property proves to be essential in the assignment of closed-loop eigenvalues.

The similarity transformation is determined as follows. Let
l’inversa della matrice definita in (11.10); si nominino le sue righe come:

L'inversa della matrice relativa alla trasformazione di similitudine cercata è definita in funzione delle ultime righe di ciascun blocco di come

Example 11.21 Consider again the system in Example 11.19, which as seen is controllable and for which the controllability indices are worth , , . By calculating the inverse of the matrix it is immediate to derive the row vectors

and then

The matrices and of the canonical multivariable control form are equal to

It is easy to verify that the last few lines of each block of are indeed li- nearly independent

If the system is in the canonical form of multivariable control, the determination of a feedback matrix to impose the desired eigenvalues on the closed-loop system is immediate. It requires solving a linear algebraic system of equations in unknowns (the elements of the feedback matrix). For simplicity, in order to avoid the introduction of notation that would be rather cumbersome, this procedure is illustrated directly through a numerical example.

Example 11.22 Consider again the pair defined in Example 11.21. Let

the eigenvalues you wish to impose on the closed-loop system.

Ora, a causa della forma di , tutte le righe di , fatta eccezione delle righe di indice pari a , e , non vengono modificate dalla retroazione. Inoltre, poiché le righe di di indice pari a , e sono tra loro linearmente indipendenti, le corrispondenti righe di possono essere assegnate ad arbitrio. In particolare possiamo scegliere in modo tale che la matrice del sistema a ciclo chiuso sia pari a

$$\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad .$$

where , , , , , ,

, , coincide with the coefficients of the polynomial having the desired closed-loop eigenvalues as roots.

Ora, sia la matrice di retroazione cercata. Indichiamo con ( ) l'elemento of place of the matrix ( ). Given the structure of the matrix , it is immediate to verify that

where

Therefore satisfies the imposed specification the matrix whose elements are solution of the linear algebraic system of equations in 27 unknowns

i.e., the matrix

—— —— ——

—— ——

In the case where the system to be controlled is not in the canonical form of multivariable control, it is sufficient to first determine a similarity transformation that will bring it into the desired canonical form. Said matri- ce defining such a transformation, the feedback matrix sought will be equal to where is the feedback matrix related to the system in the canonical form of multivariable control.

11.3 Observability

In this section we will introduce another fundamental property in the study of dynamical si- stems, namely the property of *observability*. Again we will limit our analysis to time-continuous, linear and stationary systems.

In particular, because observability depends on the pair of matrices alone in the following we will limit ourselves to considering autonomous systems, that is, systems whose external input is zero.

Definition 11.23. *A linear, stationary system*

is said to be observable *if and only if, whatever its initial state , that state va- lour can be determined on the basis of observation offree evolution for a* finite *time .*

Let us now look at a simple physical example to illustrate this concept intuitively.

Example 11.24 Consider the network in Fig. 11.4 where the voltage at the ends of the capacitor was assumed as the state variable, i.e. Given the symmetry of the network it is easy to verify that whatever the initial value of the voltage at the ends of the capacitor, the output voltage is zero. In fact for any , it holds being whatever the initial value of the voltage at the ends of the capacitor. Thus, measuring the output for a given time interval does not allow us to trace the initial state of the system. This means that the system is not observable .

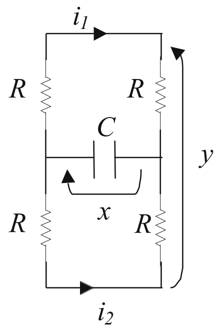


Fig. 11.4. The uncontrollable network of Example 11.24

11.3.1 Observability verification for arbitrary representations

Two different analysis criteria will also be given for observability, both of which are based on the calculation of appropriate matrices. The first criterion is based on the verification of the non-singularity of a matrix, called *the observability gramian*, while the second is based on the calculation of the rank *of* the *observability matrix*. Exactly as in the case of controllability, the second criterion is of much more immediate application. However, the first criterion is extremely important because it provides a construc- tive procedure for determining the initial state of the system, known its output variable for a finite time interval.

Definition 11.25. *Given a linear, stationary system*

(11.14)

Where and , we define the observability gramian *as the matrix*

(11.15)

Theorem 11.26. *The system (11.14) is observable if and only if the observability gramian is nonsingular for every .*

Demonstration. (Sufficient condition*.)* The proof of the sufficient condition is co-structive, i.e., assuming that the observability gramian is nonsingular for , we determine a system of equations that allows us to determine on the basis of the observed value of the output between and .

By virtue of Lagrange's formula, it is worth

dove è l'unica variabile incognita. Moltiplicando ambo i membri di tale equazione a sinistra per e integrando da a , otteniamo

from which it follows, Gramian being by hypothesis nonsingular,

This expression gives the value of the initial state as a function of the inverse of the gramiano and of the *integral* which can be immediate-mente calcolato in base all'osservazione dei valori assunti dall'uscita.

(Necessary condition.) Suppose there exists such that the observability gramian is singular in . By virtue of Theorem E.5 this implies that the columns of are linearly dependent inTherefore ...., there exists a constant vector such that , for .

Consider as the initial state of the system any vector in the direction of , i.e., either with The output of the system holds identically null for any whatever value of being This means that based on the observational-zione of the output in the time interval we are unable to distinguish among the infinite possible values of the initial state in the direction of .

It follows from the proof of the following Theorem 11.28 that if the observability gra- mian is singular at a given instant of time then it is singular at every instant of time This implies the unobservability of the system.

Example 11.27 Consider the linear, stationary system

con

It is desired to verify the observability of such a system. Also having observed *f o r* the output of the system in free evolution and having seen that it *is worth* , you want to determine the value of the initial state .

The state transition matrix for this system is worth

Therefore, the Gramian of observability holds.

e

So the system is observable.

In addition,

e

while it is worth

Therefore we derive

An alternative criterion for verifying observability is as follows.

Theorem 11.28. *Given a linear, stationary system*

(11.16)

Where and , we define the observability *matrix as the matrix*

.

.

A necessary and sufficient condition *for the system to be* observable *is that it is worth*

Demonstration. From Theorem E.5 we know that the observability gramian is nonsingular for every if and only if the columns of are linearly indi- pendent inAs a consequence, by virtue of Theorem 11.26, to prove
the validity of this theorem, it is sufficient to prove the equivalence of the following two conditions:

(a) all rows of are linearly independent in ;
(b) the observability matrix has rank equal to .

This can be done by a demonstration entirely analogous to that seen for Theorem 11.6.

Note that the validity of this theorem can alternatively be proved by relying on the duality principle (see Theorem 11.38 below).

Example 11.29 Consider the linear, stationary system whose model is

wit*h*

Vale ....eThe observability matrix has *dimension*
and it's worth

The first three rows form a nonsingular minor of order 3. Therefore it holds:

So the system is observable.

11.3.2 Observability verification for diagonal representations

We now present an important criterion for observability analysis based on sem- plice inspection of the structure of the coefficient matrix. This criterion is applicable when the matrix is in the diagonal form and with distinct eigenvalues.

Theorem 11.30. *Consider a linear, stationary system described by the model*

where ,.........................Suppose it is in diagonal form and has eigenvalues all distinct, viz.

. . ⋱ . . . ⋱ .

Where , for each .

A necessary and sufficient condition *for the system to be* observable *is that the matrix has no identically null columns.*

Demonstration. (*Necessary condition.*) Suppose that the -th column of is identically zero. In this case we have that it does not directly influence any output variable. Moreover, since the matrix is diagonal, the other state variables do not influence either. So whatever the initial value
, no component of the output is affected by it either directly or indirectly through other components of the state.

(Sufficient condition.) It is demonstrated in much the same way as seen for the demonstration of the sufficient condition in Theorem 11.8, or alternatively by relying on the duality principle (see Theorem 11.38 below).

Example 11.31 Consider the linear, stationary system whose model is

con

In this case the matrix has distinct eigenvalues and is in the diagonal form so we can apply Theorem 11.30 and conclude that the system is unobservable being the second column of identically null .

Example 11.32 Consider the linear, stationary system whose model is

with

Vale and .

Si noti che in questo caso la matrice è in forma diagonale, tuttavia il Teorema 11.30 non è applicabile in quanto i suoi autovalori non sono distinti.

For the study of observability, we then calculate the observability matrix, which has dimension and is worth

Such a square matrix is singular and holds:

so the system is unobservable.

Note that it is easy to give an intuitive explanation for the unobservability of such a system. In fact, the free evolution holds

So it is not possible to reconstruct exactly the value of e but only their weighted sum. Two different initial states and indeed produce the same outputIn contrast, such a problem does not exist for
Two modes associated with distinct eigenvalues.

11.3.3 Observability and similarity

In a manner entirely analogous to that seen for controllability, it is possible to demonstrate that observability is also not a property of the particular representation and is therefore invariant with respect to any similarity transformation.

Theorem 11.33. *Consider two representations of the same system of order*
:

e

bound by the similarity transformation , where is a nonsingular matrix. Therefore it is worth: and .

The first realization is observable if and only if *the second is observable.*

Demonstration. By similar reasoning to that seen for the con- trollability property, it is shown that the observability matrices of the two representations are related by the relation and therefore have the same rank being nonsingular.

Example 11.34 Consider the linear, stationary system whose model is

con

Vale eThe similarity transformation , with

e

leads to the system

with

La matrice di osservabilità del primo sistema e del secondo valgono rispettivamente:

e

Both matrices are square and have full rank:

so the two representations are both observable.

11.3.4 Observable canonical form of Kalman [*]

Similarly to what we have seen for controllability, we now introduce a par- ticular canonical *form*, called the *Kalman observable canonical form*, which emphasizes the observability properties of a given continuous-time, linear, stationary system. Of course, observability being a property of the pair the canonical form concerns the structure of both matrices and .

Definition 11.35. *A linear, stationary system*

(11.17)

in the canonical observable Kalman form *is characterized by the following structure of the coefficient matrices:*

where is a square matrix of order equal to the rank of the observational matrix-vabilità e è una matrice il cui numero di colonne è anch'esso pari ad . In particular, the pair

Is observable.

The following definition implies that the state vector of a realization in the canonical Kalman observable form can be rewritten as

where and Consequently, the system (11.17) can be Decomposed into two subsystems according to the diagram in Fig. 11.5 where

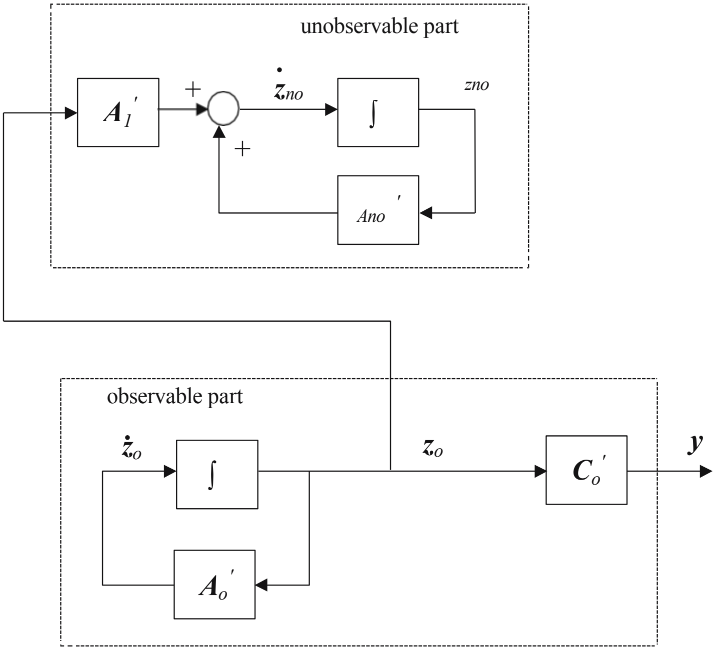


Fig. 11.5 Observable canonical form of Kalman.

the observable part is the subsystem of order governed by the differen- tial equation:

the unobservable part is the subsystem of order governed by the differential equation:

the output transformation is governed by the algebraic equation:

The output is therefore in no way affected, either directly or indirectly, by the vector .

Note that any linear, stationary system can be brought back to the canonical Kalman observable form. In particular, the following result holds.

Theorem 11.36. *Given a generic system* *this can be brought back to the canonical observable Kalman form through a simple similarity transformation , where it is a nonsingular matrix whose first columns coincide with the transpositions of li- nearly independent rows of the observability matrix and whose remaining columns are equal to linearly independent of each other and linearly independent of the previous columns.*

Therefore, just as in the case of the Kalman controllable canonical form, even- that the similarity transformation that allows a system to be placed in the Kalman observable canonical form is not unique.

Example 11.37 Consider the linear, stationary system already considered in Example 11.31 whose model is

with

(11.18)

The observability matrix of such a system is worth

e . By selecting 2 linearly independent rows f r o m top to bottom and transposing them, we obtain the two column vectors

Again, we determine the third column of so that this is orthogonal to and . That is, called that column, this must be solution of the linear system of 2 equations in 3 unknowns (the components of):

It is easy to verify that a solution of such a system is any vector of the type

We assume for ....simplicityMediating the transformation where the system is placed in the observable canonical Kalman form. In particular, the new realization is obtained

in cui

Thus, the third state component in the new realization does not affect the free evolution of the system in any way. Note that this is actually a case particolare poiché non influenza la parte non osservabile essendo

11.4 Duality between controllability and observability

Consider the linear, stationary system:

Where , , , , , and its system
dual

Where , , and , , denote the transpositions of , and .

Theorem 11.38 (Duality Principle). *The system is controllable (observable)* if and only if *the system is observable (controllable).*

Demonstration. Given and the controllability and observability matrix of the system. , for , it is easy to prove that it applies:

.

Similarly, the following applies.

Poiché per una generica matrice vale , il risultato deriva immediatamente dai Teoremi 11.6 e 11.28.

Example 11.39 Consider the linear, stationary system described by the model

the dual of which is worth

The controllability matrix of the first system and the observability matrix of the second system are valid:

These matrices are each the transpose of the other and have rank equal to the order of the system. So the first system is controllable while the second is observable.

The observability matrix of the first system and the controllability matrix of the second system are valid:

These matrices are each the transpose of the other and have rank less than the order of the .....systemHence, the first system is unobservable while the second is unobservable
controllable.

11.5 Asymptotic observer of the state [*]

In § 11.2, the control scheme based on state feedback was presented. In particular, that section discussed how an appropriate *state feedback law*

(11.19)

can be determined in the case where the system is controllable by arbitrarily assigning the eigenvalues of the closed-loop matrix .

The physical realizability of such a control law is, of course, conditional on the possibility of measuring at each instant of time the value of all components of the state. This, however, is not in general possible. Thus arises the need to realize a device that is capable of providing instant by instant a "satisfactory" estimate of the state of the system on the basis of knowledge of only the measurable quantities of the controlled system, namely, its output and input

.

If the starting system is observable, a simple solution to this pro- blem exists as long as we simply impose that coincidence between the state vector and its estimate occurs-whatever the initial unknown state of the system-for and not after a finite time interval. This is the reason why in the following we will speak of *asymptotic* state estimation. In particular, the following definition applies.

Definition 11.40. *Consider the linear, stationary system*

(11.20)

with , and .

A linear, stationary dynamical system

(11.21)

with , , ,and

(11.22)

is an asymptotic observer *(or* estimator*)* of the linear system (11.20) *if*

(11.23)

For all possible input functions and all possible initial states e .

An observer so defined is also called a *Luenberger observer*[4] and its determination consists of determining the constant matrix . Such an os- servator is therefore a linear, stationary system having the same order as the system whose state is to be estimated; its input is given by the input and output of that system, and its output transformation is analogous to that of the observed system.

The representative diagram of the structure of an asymptotic estimator is shown in Fig. 11.6, the structure of which is evident if we rewrite the equation of state of the observer in the form

(11.24)

Of course, not all systems with a structure of the type shown in Fig. 11.6 are asymptotic estimators for the system (11.20). Indeed, the boundary condition (11.23) must be verified. In this regard, it is easy to prove the following result.

Theorem 11.41. *A linear, stationary system whose dynamics is governed by equations of the type (11.21) and (11.22) is an* asymptotic estimator *of system (11.20)* if and only if *the matrix has all its eigenvalues with negative real part.*

[4]David G. Luenberger (Los Angeles, California, 1937).

Demonstration. We denote by

The *estimation error*, which measures the difference existing between the state and the estimated state

.

Dimostreremo che l'errore segue una dinamica autonoma ed è retto da una equazione differenziale del primo ordine la cui matrice dinamica è pari a .

Subtracting member by member (11.21) from (11.20) we get

That is, the dynamics of the error is regulated by the autonomous system

from which the validity of the statement follows.

The simplest criterion for choosing an appropriate matrix that defines the asymptotic observer is to impose the desired eigenvalues on the autonomous system that governs the error dynamics. This criterion is always applicable as long as the system whose state we wish to estimate is observable. Indeed, the following theorem holds.

Teorema 11.42. *Il sistema*

with and , is observable if and only if *chosen any set of real numbers and/or pairs of conjugate complex numbers , there is a Matrix such that the eigenvalues of the matrix are equal to*

.

Dimostrazione. La validità dell'enunciato segue immediatamente dal Teorema 11.16 e dal principio di dualità. Infatti, per il principio di dualità la coppia è osservabile se e solo se la coppia è controllabile. Ma per il Teorema 11.16 la coppia è controllabile se e solo se esiste una matrice costante tale che gli autovalori di possano essere fissati ad arbitrio. Inoltre gli autovalori di una matrice coincidono con gli autovalori della sua trasposta per cui poter fissare ad arbitrio gli autovalori di è equivalente a poter fissare ad arbitrio gli autovalori di , da cui segue la validità dell'enunciato.

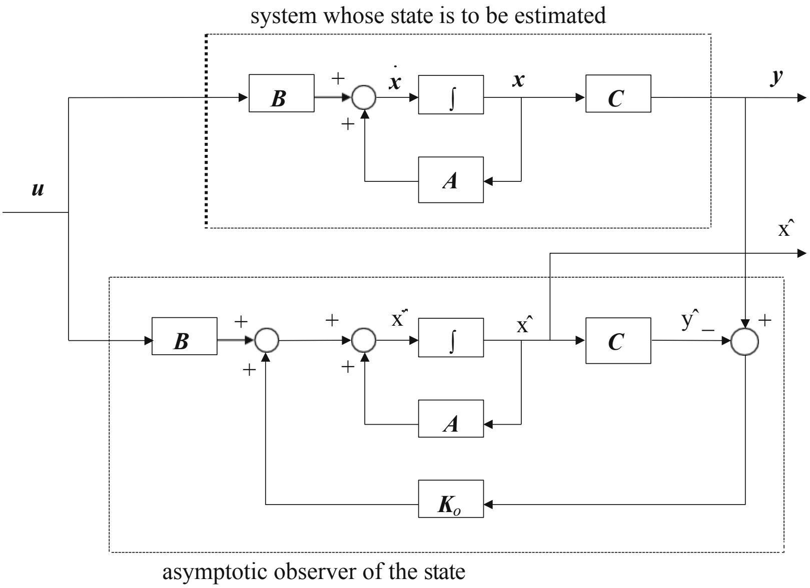


Fig. 11.6. Structure of an asymptotic observer of the state

By virtue of Theorem 11.42 we can conclude that the property of observability coincides with the possibility of being able to arbitrarily assign the eigenvalues of the autonomous system governing the dynamics of the estimation error, just as controllability coincides with the possibility of being able to arbitrarily assign the eigenvalues of the closed-loop system.

Le procedure viste in § 11.2.1 e § 11.2.2 per la determinazione di una opportuna matrice in retroazione che permetta di assegnare gli autovalori desiderati alla matrice possono pertanto essere utilizzate anche per la determinazione della matrice al fine di assegnare gli autovalori desiderati alla matrice . Assegnare gli autovalori desiderati alla matrice coincide infatti con l’assegnare gli autovalori desiderati alla matrice

What was said in § 11.2.1 and § 11.2.2 is therefore repeated identically in the case where the output is scaled or unscaled, respectively, provided that in place of its transpose and in place of the matrix .

Example 11.43 Consider the SISO system.

whose equation of state coincides with the equation of the system considered in E- simple 11.18. Such a system is clearly observable: the matrix is indeed diago- nal, its eigenvalues (, and) are distinct, and the vector has no null elements.

You wish to determine a matrix such that the eigenvalues governing the dynamics of the error, that is, the eigenvalues of the matrix are equal to , .

Following a procedure analogous to that seen in Example 11.18 with refe- rience, however, to the pair , we determine the transformation matrix

In addition,

being (see Example 11.18) and

from which , ...andFrom this it follows that

It is left to the reader to verify that the eigenvalues of the matrix are precisely equal to the desired eigenvalues .

The Luenberger observer is also called *a full-order observer* because its order coincides with the order of the observed system. It can actually be shown that given an observable system of order , with outputs (provided they are linearly independent of each other), there exists an asymptotic order estimator whose self-values are arbitrarily assignable. Such an asymptotic observer is therefore called a *reduced-order observer*.

However, the problem of determining a reduced-order observer is beyond the scope of this discussion and will therefore not be examined.

11.6 State feedback in the presence of an observer [*]

In the case where state feedback is to be realized but that state is not measurable, the need to construct an observer arises. As seen above a simple solution to the state estimation problem exists if the system is os- servable and an asymptotic observer is constructed. The objective of this section is to show that the estimate obtained by means of an asymptotic observer can be used in the calculated feedback law assuming that the *state*

Is measurable. In other words, we will now show that in the case where the state is not measurable it is possible to take as a law in feedback

(11.25)

where

is the matrix obtained by appropriately assigning the desired eigenvalues to the matrix ,

is the state estimate obtained by an asymptotic observer whose matrix is chosen so as to assign appropriate eigenvalues to the matrix

.

To this end we first present the following result.

Theorem 11.44. *Consider the linear, stationary system*

(11.26)

where , , ,

(11.27)

and is the estimate obtained through the asymptotic observer

(11.28)

The resulting closed-loop system is an order system whose eigenvalues are given by the union of the eigenvalues of the matrix and the eigenvalues of the matrix .

Demonstration. The resulting closed-loop system is clearly an eigen-system of order whose equation of state is

Consider now the similarity transformation

where

It is immediate to verify that

so the coefficient matrix of the new closed-loop realization is block triango- lare. Its eigenvalues are therefore given by the union of the eigenvalues of the individual blocks along the diagonal (see Appendix C). Finally, recalling that a similarity tra- sformation leaves the eigenvalues of the dynamic matrix unchanged, it follows from this that the statement is valid.

Finally, from Theorem 11.44 and Theorems 11.16 and 11.42, the following fundamental result follows, which, in the case where the system is both controllable and observable, allows for state feedback with observer *by separately* determining the controller and observer matrices by the above criteria.

Theorem 11.45. *The system*

with , and , is controllable and observable if and only if *chosen any two sets of real numbers and/or pairs of conjugate complexes*

e there exists a matrix and a ma- trice such that the eigenvalues of the matrix are equal to and the eigenvalues of are equal to .

The structure of the closed-loop system with observer is shown in Fig. 11.7.

It is important to note that the eigenvalues of the system are naturally chosen in such a way as to best meet the desired specification. In particular, it is common practice to choose the eigenvalues relative to the observer in such a way that the error dynamics is significantly faster than that of the closed-loop system: in general, it is done in such a way that the response of the observer is 2 to 5 times faster than that of the closed-loop system. This is usually possible because the observer is not a physical structure but rather a structure implemented to a computer, and the rapidity of its response is in fact limited only by the sensitivity of the estimator itself to possible errors in the measurement of external quantities.

11.7 Controllability, observability and relationship input-output

We conclude this chapter by examining what link exists between the controllability and observability properties and the input-output relationship of the system.

For this purpose, the preliminary definition of a particular canonical *form*, known as the *canonical Kalman form,* is essential.

11.7.1 Canonical form of Kalman

The canonical Kalman form is a generalization of the con- trollable and observable canonical Kalman forms introduced in the previous paragraphs. In particular, the following result applies.

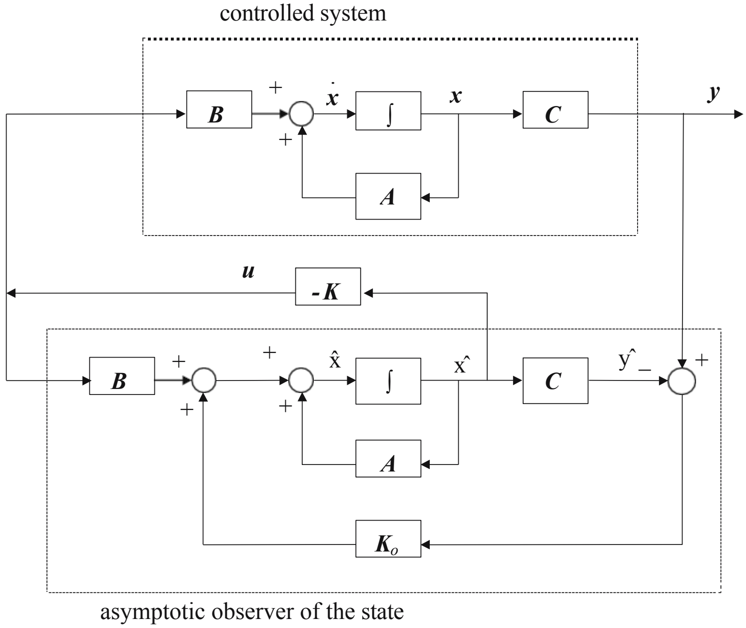


Fig. 11.7. Structure of a feedback system with asymptotic state observer

Theorem 11.46. *Given any linear, stationary system in the form*

(11.29)

it is always possible to define a similarity transformation such that the realization in has the following structure:

(11.30)

and the linear, stationary system

(11.31)

Is controllable *and* observable.

The proof of this theorem also provides a constructive procedure for determining the matrix . However, this demonstration will not be reported as it is beyond the scope of this discussion.

However, it is important to reiterate that in the case where a system is both controllable and
osservabile la dimensione della matrice è pari all'ordine del sistema. Al contrario, if the system is neither controllable, or unobservable, or neither controllable né osservabile la dimensione di è strettamente inferiore all'ordine del sistema.

11.7.2 Relationship input-output

Let us now look at the relationship between the controllability and observability properties and the UI bond in the case of a SISO system.

Theorem 11.47. *Consider a SISO system with ,*

(11.32)

The transfer function expressing the UI bond depends only on the part with- trollabile e osservabile di tale sistema. In particolare, se , e sono defined as in Theorem 11.46, it is worth

(11.33)

Demonstration. By virtue of Theorem 11.46 there is a similitudi- ne transformation that allows the system (11.32) to be placed in the equivalent canonical form of Kalman (11.30).

Because of what we saw in Chapter 6 (see § 6.3.7), the transfer functions related to two equivalent representations are identical to each other. Therefore, if we denote by , and the matrices of the coefficients of the system in the canonical form (11.30), we can write

Being triangular upper block, also and consequently , are upper triangular. In particular, the following applies.

where it indicates the presence of elements that may be nonzero, which is not important to specify, however.

Taking into account the structure of the vectors and it is immediate to verify that:

as it was meant to be.

From Theorem 11.47 the following result immediately follows.

Theorem 11.48. *Consider a SISO system with ,*

whose input-output transfer function is worth

A necessary and sufficient condition *for the system to be* controllable *and* observable *is that the denominator of the expressed in minimum form has degree equal to the order of the system.*

Example 11.49 Consider the linear, stationary system described by the model

Since the matrix is diagonal and has distinct eigenvalues, it is immediate to observe that such a system is uncontrollable and unobservable.

Furthermore, the characteristic polynomial of vale

while the transfer function is equal to

which in minimal form has a denominator of order .

Example 11.50 Given a linear, stationary system, consider two of its possi- ble representations in terms of state variables:

e

It is easy to verify that the first representation is controllable but not observable, while the second representation is observable but not controllable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the starting system has a polo-zero cancellation in the transfer function (which is of course the same in the two cases), in fact

If a cancellation occurs in the transfer function, then controllability and observability vary, depending on how the state variables are chosen. In order for any representation to be controllable and observable, the transfer function must admit no pole-zero cancellation

Finally, note that if the transfer function in minimum form has a denominator of lower order than the order of the system we can certainly con- clude that the system is either unobservable or uncontrollable. However, from the analysis of the we cannot conclude whether the system is uncontrollable, unobservable or neither controllable nor observable.

11.8 Reachability and reconstructability [*]

We conclude this chapter by giving a brief mention of two other important properties of dynamical systems, *reachability* and *reconstructability*. These properties will be only briefly introduced because in the case of linear and stationary continuous-time systems, that is, for the class of systems considered in this text, they coincide with the properties of controllability and observability, respectively.

11.8.1 Controllability and reachability

As seen in detail in this chapter, the controllability problem is related to the possibility of transferring in a finite time interval the current state of the system to a predetermined state (target state) by appropriately acting on the input. In general, the possibility of transferring the state of the system to a desired value depends not only on the desired value, but also on the initial state and the initial instant of time. Thus, assuming for simplicity that zero state is as- sumed as the target state, a generic dynamic system may be controllable to zero state from certain initial conditions, assumed at certain instants of time, while it may not be controllable from different initial conditions, or even from the same initial conditions assumed, however, at different instants of time.

Given a generic dynamical system, therefore, it makes no sense to refer controllability to the system, since it is not a property of the system; rather, assuming the target state is fixed, it is a property of the initial state and the initial time instant. In particular, the following definition applies.

Definition 11.51. *A state of a dynamical system is zero controllable (or simply, controllable) from instant if there exists a finite and an input , , capable of bringing the system from the state to the state at time .*

In the case where the system is linear is stationary, if some state is controllable at a given instant of time, then every state is controllable at any instant of time. This allows us to relate Definition 11.1 valid for a linear and stationary system to Definition 11.51 referring to a generic dynamical system and thus to understand why for a linear and stationary system controllability is a property of the system.

Reachability, on the contrary, concerns the possibility of being able to reach in a finite time interval any state from a predetermined state (e.g., from the zero state), again by appropriately acting on the input. More precisely, the following definition applies.

Definition 11.52. *A state o f a dynamical system is* reachable *from zero (or simply, reachable) at the instant if there exists an instant , , and an input that acting on the system in the time interval is capable of bringing the system from the zero state to the state .*

In general these properties are not related to each other, in the sense that the validity of one does not imply the validity of the other. However, in the case of linear and stationary systems every state controllable to the zero state is also reachable from the zero state. Moreover, for linear, stationary and continuous-time systems the reverse is also true: every controllable state is also reachable. This implies that for that class of systems the two properties are completely equivalent.

11.8.2 Observability and reconstructability

In this chapter we have seen that observability concerns the possibility of determining the initial state of the system on the basis of observing the external quantities of the system (the output alone in the case of an autonomous system) for a finite time interval.

There is also an important other property, *reconstructability*, which instead implies that the state can always be reconstructed based on knowledge of the external quantities of the system for a finite time interval .

Obviously, the state can be derived from by solving the differential equation , so observability naturally implies reconstruc- tability. The opposite implication, however, is true only for a narrow class of dynamical systems that includes linear, stationary, and time-continuous systems, that is, for the class of systems of interest in this book. This is the reason why in the present discussion we have restricted ourselves to talking about observability.

Exercises

Exercise 11.1 Given the representation in terms of state variables of a linear, stationary system

Determine whether this representation is controllable and observable.

In particular, carry out the analysis both by calculating con- trollability and observability gramians and by calculating controllability and observability matrices.

Exercise 11.2 Consider the system in Exercise 11.1 and *assume*
Determine a control law that can bring the system to the point
at the instant of time .

Exercise 11.3 Given a representation in terms of state variables of a linear, stationary system

Determine whether this representation is controllable and observable.

Exercise 11.4 Consider the system in Fig. 11.8 where ed denotes the po- positions of the barycenters of the two carriages with respect to a fixed reference. Determine the model of such a system in terms of state variables by assuming the following as state variables

and as the output quantity the position of the center of gravity of the whole system, i.e.

Let it be verified that such a system is unobservable.

Exercise 11.5 Given the representation in terms of state variables you

we study the controllability and observability of this representation as the parameters change .

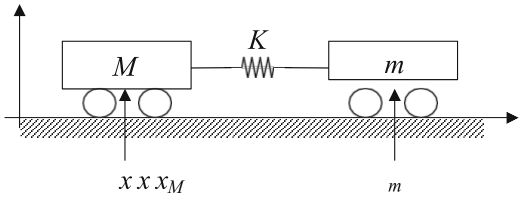


Fig. 11.8. Two-carriage system from Exercise 11.4.

Exercise 11.6 Given the representation in terms of state variables in Exercise 11.3, trace it back to the canonical controllable Kalman form.

Exercise 11.7 Given the representation in terms of state variables in Exercise 11.3, trace it back to the canonical Kalman observable form.

Exercise 11.8 Given the representation in terms of state variables in Exercise 11.1, determine whether it is possible to determine an appropriate feedback law tale per cui gli autovalori del sistema a ciclo chiuso siano assegnabili ad arbitrio. Nel caso in cui questo sia possibile, si determini la matrice tale per cui gli autovalori del sistema a ciclo chiuso siano pari a , .

Repeat the exercise with reference to the system in Exercise 11.3.

Exercise 11.9 Given the representation in terms of state variables in Exercise 11.1, determine, if possible, an asymptotic observer of the state such that the system representing the dynamics of the estimation error has eigenvalues equal to

, .

Exercise 11.10 Calculate the transfer function related to the representation in terms of state variables in Exercise 11.3. Discuss the result obtained in relation to the controllability and observability properties of this representation.

12

Analysis of nonlinear systems

In reality, all physical systems, whether mechanical, electrical, hydraulic, etc., present nonlinear links between the different physical variables involved. The main characteristic of a nonlinear system is that it does not satisfy the principle of superposition of effects. For a nonlinear system, therefore, it is not possible to calculate the response to an external input given by the sum of two signals, separately calculating the system's response to each signal and then summing the results thus obtained.

This chapter will first discuss the most common causes of nonlinearity and the typical effects they cause on the behavior of systems. The most common methods of analyzing nonlinear systems, namely the two *Lyapunov methods,* will then be presented: one based on the definition of an appropriate scalar function of the state, known precisely as *the Lyapunov function*; the other based instead on linearizing the nonlinear system in a neighborhood of the equilibrium point whose stability is to be studied.

12.1 Typical causes of non linearity

The main causes of nonlinearity include: *saturation*, *on- off nonlinearity*, *threshold* or *dead zone*, *hysteresis*, etc. In the following, the characteristics and effects of the most important and frequent nonlinearities are briefly described. We will restrict ourselves to considering instantaneous systems in which the output depends solely on the value as- sumed by the input at time . Nonlinearities may of course also be present in dynamic systems.

Saturation

A physical system is subject to saturation when it exhibits the following behavior: for small increases in the input variable it exhibits proportional increases in the output; however, when its output variable reaches a de- termined level, a further increase in the input causes no change

in the output. In other words, once a certain threshold is reached, the output variable settles down to around its maximum attainable value. A typical input-output pattern in the presence of nonlinearity due to saturation is shown in Fig. 12.1.a, where the thicker line indicates the actual behavior of the system, while the thinner line is representative of ideal saturation. A number of physical systems, including elastic springs, dampers, magnetic amplifiers, etc., exhibit such behavior.

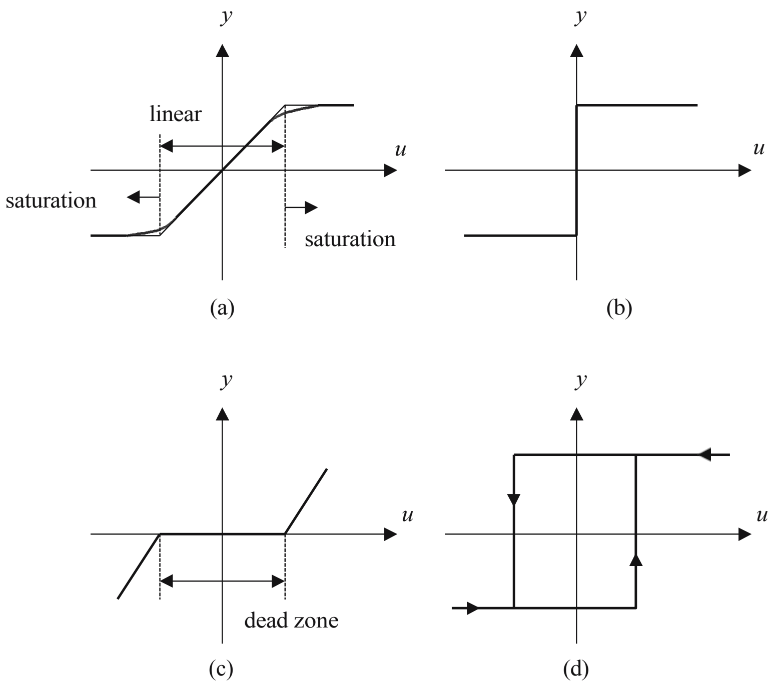


Fig. 12.1. **(a)** nonlinearity due to saturation; **(b)** on-off nonlinearity; **(c)** dead zone; **(d)** hysteresis

On-off linearity

A limiting case of saturation is on-off nonlinearity. This occurs when the linearity field is of zero amplitude and the curve in that area is vertical, as shown in Fig. 12.1.b. Such behavior is typical of electrical relays.

Dead zone

In many physical systems, the output is zero until the amplitude of the input signal exceeds a certain value. The set of values of the non-sufficient input to produce a response from the system define what is called a *so- glia* or *dead-zone*. The input-output relationship in this case is of the type shown in Fig. 12.1.c. Such behavior is typical of all DC motors: because of friction, in fact, until the voltage at the ends of the armature windings reaches a given threshold value, no rotation of the motor axis occurs.

Hysteresis

Hysteresis is a typical example of a multi-valued nonlinearity, that is, the output of the system- ing is not uniquely determined by the value of the input. The input-output relationship in this case has a pattern of the type shown in Fig. 12.1.d. Such a behavior is frequently found in magnetic-type devices. This type of nonlinearity usually involves energy storage within the system resulting in self-oscillation and thus instability.

12.2 Typical effects of non linearity

The possible consequences of nonlinearities are many. In particular, a nonlinear system may have a finite or infinite number of isolated equilibrium points, may have limit cycles, bifurcations, may exhibit chaotic effects, etc. For completeness, in the following these phenomena will be briefly discussed and illustrated through some simple numerical examples.

Isolated equilibrium points

As mentioned in Chapter 9 on stability, a nonlinear system can, unlike a linear system, have a finite or infinite number of isolated equilibrium states. In particular, some of these states may be stable and others unstable. See in this regard Examples 9.9 and 9.14.

Limit cycles

Nonlinear systems can exhibit oscillations of constant amplitude and period even in the absence of external stresses. Such self-powered oscillations are called *limit-cycles*. As is well known, oscillatory effects can also be observed in the case of linear self-contained systems if they have zero real-part poles. Note, however, that there is a fundamental difference between the limit cycles of nonlinear systems and the oscillations of linear systems: the amplitude of self-oscillations in nonlinear systems is independent of the initial conditions; in contrast, the oscillations that can occur in linear systems with poles in the imaginary axis depend strictly on the initial conditions of the system.

An example of a limit cycle is illustrated through the following classic example from the literature.

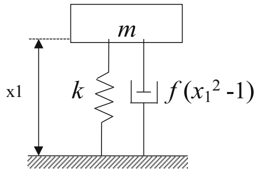


Fig. 12.2. Mass-spring-damper system of Example 12.1.

Example 12.1 (Van der Pol oscillator[1]) Consider the second-order nonlinear autonomous system

(12.1)

— —

which describes the behavior of the mass-spring-damper system depicted in Fig. 12.2 where it represents the change in the equilibrium position of the mass , or equivalently the deformation of the spring and damper.

The nonlinearity of the system is due to the damper whose damping coefficient varies as the position of the mass varies : for values *of* in modulus greater than one, the damping coefficient is positive and the damper absorbs energy from the system; for small values of *(* in modulus less than one) in- contrast, the damping takes on negative values and provides energy to the system. Such a law of variation of the damping means that the deformation of the spring, and thus also of the damper, can never grow indefinitely, nor go to zero: this deformation tends to oscillate with an amplitude and period that do not depend on the initial conditions of the system, as illustrated in Fig. 12.3 where the limit cycle to which the evolution is taken is indicated by the thick-stroke curve. More precisely, in that figure, the trend of some state trajectories obtained by assuming Ns/m, N/m and Kg h as been shown.

Bifurcations

By varying some characteristic parameter of a system (even a linear one), it may succe- d that the asymptotic behavior of the system, that is, the behavior it exhibits for very large times, may be of a different kind. It may, for example, ac- cept that a system is, for a given value of a parameter, at a stable equilibrium point; as that parameter increases, however, the equilibrium point per- dutes its stability and the system achieves periodic motion or even exhibits

[1]Balthazar Van der Pol (Utrecht, Netherlands, 1889 - Wassenaar, Netherlands, 1959).

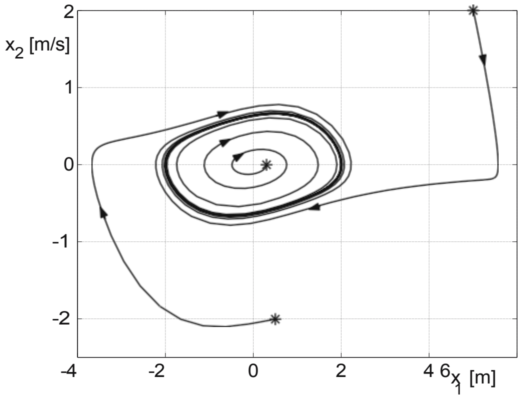


Fig. 12.3 Van der Pol oscillator trajectories from different initial conditions

chaotic behavior. The change in asymptotic behavior that is ve- rienced as a given parameter varies, which in the following will be denoted by the letter , is called a *bifurcation*.

One way to visualize the effect of bifurcation is to represent some measure of the asymptotic behavior of the system as the parame- tro changes . In the case of first-order systems, an obvious choice is to represent any equilibrium points in the ....planeFor higher-order systems on the other hand, there is no general rule: depending on the particular system under study, it is appropriate to represent one of the coordinates of the equilibrium points; in other cases, however, it may be more meaningful to show the trend of the Euclidean norm of these points as the .

By way of example we see in the following some types of bifurcation. In particular, in order to provide a more intuitive graphical representation these will be illustrated with reference to first-order systems.

Saddle node bifurcation is the simplest type of bifurcation and shows how equilibrium points can be created or destroyed as a certain parameter changes .

Example 12.2 (Bifurcation with saddle node) Consider the first-order system

(12.2)

For negative values of the system has two equilibrium points: one stable and the other unstable; for the system has a single equilibrium point that coincides with the origin; finally, for positive values of the system has no equilibrium point. All this is summarized in Fig
12.4.a.

Another example of bifurcation is *transcritical bifurcation* (*transcritical bifurcation*), which neither creates nor destroys equilibrium points as the parameter changes . Simply, there is a value of where the stability properties of different points of

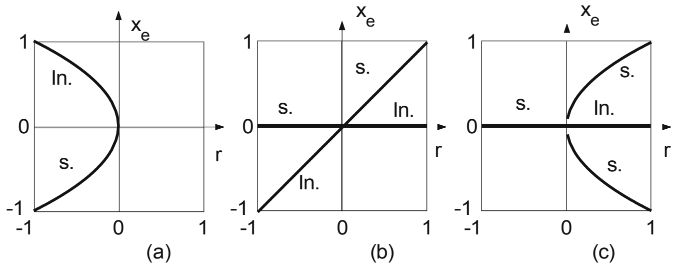


Fig. 12.4. (**a**) saddle bifurcation (s.: stable, in.: unstable); **(b)** transcritical bifurcation; **(c)** fork bifurcation

equilibrium are reversed, that is, stable equilibrium points become unstable and unstable equilibrium points become stable.

Example 12.3 (Transcritical bifurcation) Consider the first-order system

(12.3)

For every value of , except for , the system has two different equilibrium points, one stable and the other unstable. More precisely, one of the two equilibrium points coincides with the origin; the other takes positive or negative values depending on the sign of . Moreover, for *t h e* stable equilibrium point is the one coincident with the origin ; conversely, for the equilibrium point coincident with the origin becomes unstable, as summarized in Fig. 12.4.b .

Pitchfork bifurcation (*pitchfork bifurcation*) is a symmetric bifurcation, so it occurs in numerous problems that have some symmetry with respect to a given partition of the state space. Pitchfork bifurcation causes a single equilibrium point to give rise to three different equilibrium points as a given parameter changes, one coincident with the original equilibrium point and having stability properties opposite to it, and the other two having the same stability properties and being symmetric with respect to it.

Example 12.4 (Fork bifurcation) Consider the first-order system

(12.4)

For each value of the system has a single equilibrium point coincident with the origin. For *t h e* origin it becomes an unstable equilibrium point and two other stable equilibrium points also a rise, as shown in Fig 12.4.c.

Recall also the *Hopf bifurcations* (*Hopf bifurcations*) that occur in nonlinear oscillators. In such a case, an equilibrium point may turn into a limit cycle, or conversely, a limit cycle may collapse into a fixed point. Such bifurcations are much more complex, however, and their study is beyond the scope of this chapter.

Chaos

Nonlinear systems can exhibit behavior that is called chaotic-that is, it can happen that seemingly negligible differences in the input variables produce very large, and unpredictable, differences in the output variables. It is important to emphasize that such behavior is not at all stochastic. In fact, in stochastic systems the system model or the external inputs or initial con- ditions contain uncertainties, and as a consequence the output cannot be predicted exactly. In contrast, in chaotic systems, the model of the system as well as the input variables and initial conditions are deterministic.

A well-known example of chaotic behavior is given by the Chua circuit .[2]

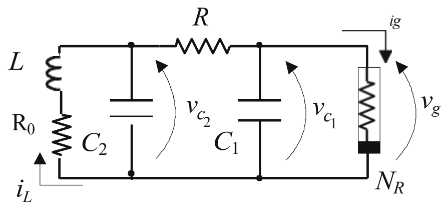


Fig. 12.5. Chua's oscillator

Example 12.5 (Chua's Circuit) Chua's circuit consists of a line- re inductor , two linear capacitors and , a linear resistor and a voltage-controlled resistor . Adding a linear resistor in series with the inductor results in the Chua oscillator shown in Fig. 12.5. The oscillator is completely described by a system of three ordinary differential equations. Through a simple change of variables, the dimensionless equations of state of the Chua oscillator become

(12.5)

where

(12.6)

[2]Leon O. Chua (Philippine Islands, 1936).

If the dimensionless parameters are set equal to: , , , and the system (12.5) exhibits chaotic behavior. This fact is clearly evidenced in Fig. 12.6, which shows two different evolutions of the system obtained from very close initial conditions, *viz.* e As can be seen, after a short time interval the two evolutions are completely different from each other.

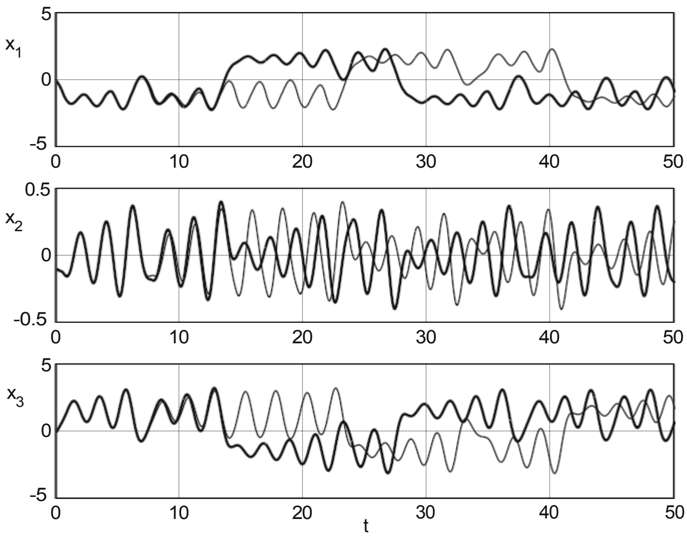


Fig. 12.6. Due diverse evoluzioni del sistema (12.5) ottenute a partire dai punti (linea spessa) e (linea sottile)

For completeness, Fig. 12.7 also shows the trajectory of the system obtained starting from Indeed, such a trajectory shows a typical trend of chaotic systems, known as *double scroll*, that is, the trajectory tends to rotate alternately around two points, called *attractors* without ever converging to any one of them and without ever crossing the same point in the state space more than once.

12.3 Study of stability by function of Lyapunov

In the following, the most well-known stability analysis criteria for nonlinear systems will be explained. In particular, an important stability analysis criterion, known as *Lyapunov's direct method*, or also as *Lyapunov's second criterion*, will first be presented. More precisely, this criterion provides sufficient conditions for the stability and asymptotic stability of an equilibrium state of an autonomous system.

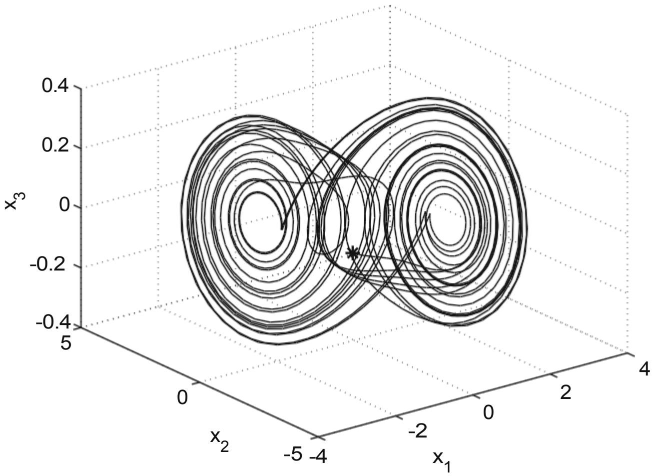


Fig. 12.7. The trajectory of the system (12.5) obtained from the point (indicato in figura con un asterisco)

Before enunciating Lyapunov's direct method, however, it is essential to recall some basic definitions.

12.3.1 Positive defined functions or negative

Definition 12.6. *A continuous scalar function is* defined as positive *in if there exists a region of the state space (constituting a circular surrounding of) such that for all states in* ....*eSe coincides with the entire state space, then it is called* globally defined positive.

It is useful to give a geometric interpretation of this concept. For this purpose we assume for simplicity that it is . In this case . Fig. 12.8.a is
An example of a typical shape of the in a three-dimensional space and in a surrounding
circolare del punto : in questo caso la ha la forma di un paraboloide ri-face upward where the point of minimum is zero and is right at .

A second geometric representation can be given in state space, that is, in the plane . For this purpose, consider Fig. 12.8.b. Contour lines define a set of closed curves around the equilibrium point. Such curves are nothing but the intersections of the paraboloid with horizontal planes, projected into the planeNote that the contour curve relative to a constant value
smaller is internal to that related to a larger constant value. Finally, note that these curves can never intersect. For otherwise the would not be a uniquely defined function because it would take on two different values at the same point .

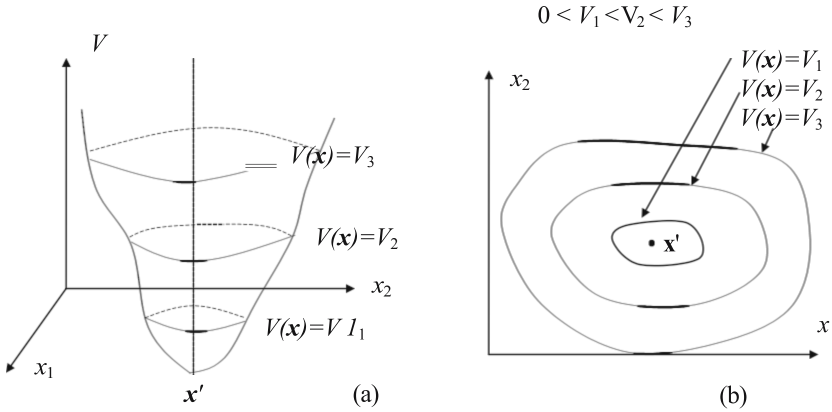


Fig. 12.8 Typical form of a positive definite function in

A simple example of a function defined positive in the origin is given *by* , with . In particular, the has the shape of a paraboloid Elliptical facing upward and having vertex right in the origin.

Definition 12.7 *A continuous scalar function is* semidefinite positive *in Whether there exists a region of the state space (constituting a circular surrounding of) such that for all states in ....eSe coincides with the entire state space, then it is called* globally semidefinite positive.

Un semplice esempio di funzione in semidefinita positiva nell'origine è dato da .

Definition 12.8 *A continuous scalar function is (globally*) defined negative *in if it is (globally) defined positive in .*

Definition 12.9. *A continuous scalar function is (*globally*)* semidefinite negative *at if it is (globally) semidefinite positive at .*

It is easy to associate such concepts with a geometric interpretation similar to that just seen for positive definite functions.

12.3.2 Direct method by Lyapunov

Before giving the formal statement of Lyapunov's direct method, let us recall that ta-le method is inspired by the fundamental principles of Mechanics. For as is well known, if the total energy of a mechanical system is dissipated continuously over time, then the system tends to settle into a well-defined equilibrium condition. Furthermore, the total energy of a system is a positive definite function, and the fact that this energy tends to decrease with the passage of time implies that its derivative time is a negative definite function. Lyapunov's criterion is based precisely on the generalization of these observations: if a system has an asymptotically stable equilibrium point and is perturbed in a neighborhood of that point, as long as it is within its domain of attraction, then the total energy stored by the system will tend to decrease until it reaches its minimum value precisely at the asymptotically stable equilibrium state.

It is clear, however, that the application of this principle is not straightforward where the definition of the "energy" function is not straightforward, as is the case in the vast majority of cases where systems are known only through a purely mathematical model. To overcome this difficulty Lyapunov introduced a "dummy" energy function, known precisely as the *Lyapunov function* and denoted by convention by the letter . In general it is a function of state and time , i.e. . When associated with an autonomous system the Lyapunov function does not depend explicitly on time , i.e. Note, however, that even in this case the depends on time, albeit indirectly, that is, through the ....In Following we will limit our attention to the stand-alone case.

The following theorem, whose proof we also give for completeness, constitutes the formal statement of Lyapunov's direct method.

Theorem 12.10 (Lyapunov's Direct Method). *Consider an autonomous system described by the vector equation*

dove il vettore di funzioni è continuo con le sue derivate parziali prime , per . Sia un punto di equilibrio per tale sistema, ossia per ogni .

If there exists a continuous scalar function together with its partial prime derivatives, defined positive at and such that

is semidefinite negative at , then is a stable *equilibrium state.*

If it is also defined as negative at , then is an equilibrium state asymptotically stable.

Demonstration. Let us again assume that the system is of the second order so that a clear geometric interpretation can be provided. Observe for this purpose Fig. 12.9 where we have highlighted the equilibrium state and some level lines of the function .

To prove that is a point of stable equilibrium, it is sufficient to show that for every there exists a *Æ* such that all trajectories beginning at a point that satisfies the condition *Æ* , i.e., all trajectories

beginning in a circle of center and radius Æ , denoted in the following as
Æ , evolve within a circle of center and amplitude , i.e., in
.

Being by assumption continuous and positively defined in , its level lines have a structure of the type shown in Fig. 12.9. Therefore there always exist closed lines entirely contained in . Fixed one such line , let Æ be the radius of the circle of center and tangent internally to that curve. Such a circle is by definition entirely contained in the level line .

Consider trajectories whose initial state is contained in Æ In such points and by assumption. Such trajectories can therefore never intersect contour lines characterized by constant values greater than and will remain in the region bounded by the curve , internal by construction to , which shows that is a state of stable equilibrium.

If we finally assume that the trajectories originating in Æ will intersect contour lines parameterized by smaller and smaller values of until they come to , which shows that is in this case an asymptotically stable equilibrium point.

The function that satisfies the conditions of Theorem 12.10 is called a *Lyapunov function*.

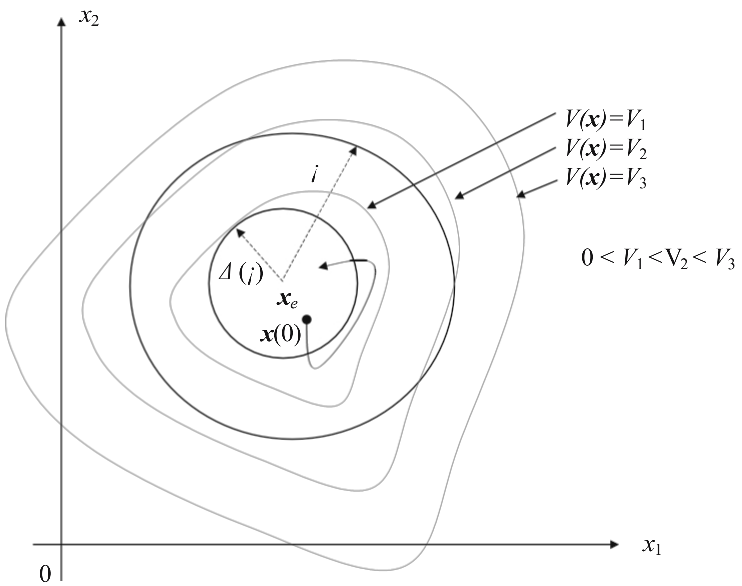


Fig. 12.9. Geometric interpretation of Lyapunov's direct method.

It is important at this point to make some clarifications. The theorem just stated provides *sufficient conditions* for the stability and asymptotic stability of an equilibrium state. However, these conditions are not necessary. This means that if a positive definite function is determined in a given equilibrium state , but whose first derivative is not semidefinite (or definite) negative at , this does not imply that is not a stable (or even asymptotically stable) equilibrium point. A example to that effect is presented in Fig. 12.10. Looking at the trajectory of the system, it is evident that is an asymptotically stable equilibrium point. However, it is also evi- dent that the chosen function, some contours of which are shown in the figure, does not allow us to draw any conclusions about the stability of this equilibrium state. In fact, its derivative is neither defined nor semidefinite negative in the vicinity of .

Determining a Lyapunov function that then allows one to draw the do- vid conclusions about the stability of an equilibrium state is in general a very complex problem, particularly when dealing with high-order systems. This constitutes the strongest limitation of Lyapunov's direct method. Note that several procedures for the systematic construction of Lyapunov functions have been proposed in the literature, but the usefulness of these procedures is in fact limited to particular classes of systems.

Example 12.11 Consider the nonlinear autonomous system

It is easy to verify that the origin is an equilibrium state being solution of the system

To study the stability of the origin, we choose as the Lyapunov function

This function is in fact continuous with its partial prime derivatives and is strictly positive throughout the state space, except in the origin where it cancels. If we derive the with respect to time, we obtain

which is defined as negative in the origin. In fact, if we assume ,
is strictly negative in , which constitutes a circular circle of the origin.
Furthermore, ...We can therefore state that the origin is a point of equilibrium asymptotically stable.

Note that there is also an extension to the previous theorem that allows conclusions to be drawn about the instability of an equilibrium state. That theorem is in the se- ction given below. Its proof, however, is for brevity omitted but can easily be deduced by considerations similar to those seen for Theorem 12.10.

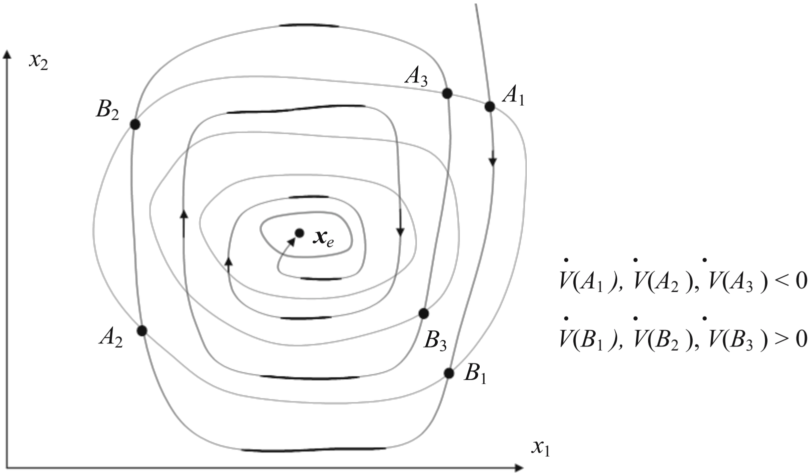


Fig. 12.10. Example of an unrepresentative Lyapunov function**.**

Theorem 12.12 (Instability criterion). *Consider an autonomous system described by the vector equation*

dove il vettore di funzioni è continuo con le sue derivate parziali prime , per . Sia un punto di equilibrio per tale sistema. Se esiste una funzione scalare continua insieme alle sue derivate prime, definita positiva in and such that it is defined as positive at , then is an equilibrium state unstable.

Example 12.13 Consider the nonlinear system

It is easy to verify that the origin is an equilibrium state. If we then choose as a Lyapunov function.

we can also conclude that the origin is an unstable equilibrium state, being

defined positive in the origin.

12.4 Linearization around an equilibrium state and stability

This section will present another important stability criterion, also first proposed by Lyapunov and often cited in the literature as *Lyapunov's first criterion*. The main advantage of this method is that, unlike the direct method, it can be applied systematically.

This approach is based on *linearizing* the nonlinear system under consideration around the equilibrium state whose stability is to be studied. Typical analysis techniques for linear systems can then be applied to the linear system thus obtained. The information derived in this way then allows conclusions to be drawn about the behavior of the original system in a neighborhood of the equilibrium state under consideration.

Consider the generic nonlinear and autonomous system

(12.7)

and be a state of its own equilibrium.

Suppose that at a generic time instant the system is near the equilibrium state . Specifically, let Æ be , where Æ is a measure of the distance of the perturbed state from the equilibrium state at time instant . Similarly, we denote by Æ the generic value assumed by the state at time instant . Since the state of the system evolves according to (12.7), Æ must be solution of (12.7) at each instant of time , i.e.

Æ Æ (12.8)

or equivalently

Æ Æ Æ Æ

Æ Æ Æ Æ

.

Æ Æ Æ Æ

(12.9)

Moreover, since is an equilibrium state, by definition

for which, denoted as

Æ — Æ

we can write

Æ Æ Æ Æ

Æ Æ Æ Æ (12.10)

.

.

Æ Æ Æ Æ

At this point if the functions , , , are developable in Taylor series in a neighborhood of , by stopping the series development at the first-order terms we obtain the following system of equations:

Æ

—— Æ —— Æ —— Æ

Æ

Æ Æ Æ

.

Æ

Æ Æ Æ

(12.11)

This system is further simplified by taking into account that since is an equilibrium point, for each...Therefore

(12.12)

Which placed in matrix form becomes:

(12.13)

or even

(12.14)

where

(12.15)

is the *Jacobian matrix* or *Jacobian* of computed in .

The system (12.14) is called a *linearized system*, and its dynamic matrix coincides with the Jacobian of computed at the equilibrium state . Instead, the state variables of the linear system indicate the differences between the state coordinates of the nonlinear system and those of the equilibrium point. Of course, this is true only in the vicinity of the equilibrium state itself, i.e., for small values of Æ within which the approximation resulting from neglecting terms of higher than first order in the Taylor series development is valid.

At this point we can state the following theorem due to Lyapunov, which states that the stability of the equilibrium state of a nonlinear system can stu- dy itself, barring critical cases, by simply analyzing the stability of the linearized system.

Theorem 12.14 (Lyapunov's First Criterion). *Let Æ Æ be the linear system obtained by linearization of around the equi- librium state . If the matrix has eigenvalues all with negative real part, then the equilibrium state is* asymptotically stable.

If the matrix has one or more positive real-part eigenvalues, then the equilibrium state is unstable.

Example 12.15 Consider the nonlinear system in Example 12.11. Through Lyapunov's direct method we have shown that the origin is an asymptotically stable equili- brium state for such a system. We can come to the same conclusion through the linearization-based method. Indeed, the Jacobian of evaluated in the origin holds.

whose eigenvalues are the roots of the algebraic equation

so they are clearly both negative real part, this equation being of the second order and all coefficients at the first member being strictly positive.

Such a system also has a second equilibrium state coincident with the point

also solution of the nonlinear system

To assess the stability of such an equilibrium state, we need to calculate the Jacobian of at , which is worth

Since the roots of the equation are

equal to and , we can conclude that it has a positive real-part eigenvalue, which allows us to say that is an unstable e quilibrium state.

The only case in which no conclusion can be drawn about the stability of an equilibrium state of a nonlinear system based on Theorem 12.14 is when the Jacobian matrix has, in addition to any number of negative real part eigenvalues, one or more zero real part eigenvalues. In such a case it is necessary to resort to other stability analysis criteria. A first possibility is, of course, to apply Lyapunov's Direct Method (see Example 12.17 in this regard). An alternative to this is to apply an important theo- rema known in the literature as the *Center Manifold Theorem*, which is based on the analysis of a nonlinear system of reduced order with respect to the starting system, and in particular of order equal to the number of eigenvalues of a real null part. However, this result will not be presented as it is beyond the scope of this discussion.

Example 12.16 Consider the first-order system

The origin is clearly an equilibrium point for such a system. In particular, it is the u- nual equilibrium point if , whereas if the system is linear and has an infinite number of equilibrium points, that is, every .

We now want to study the stability of the origin.

The Jacobian of evaluated in the origin is equal to

—

that is, it has an eigenvalue lying on the imaginary axis. Therefore, linearization does not allow us to draw any conclusions about the stability of the origin. The o- rigine may in fact be asymptotically stable, stable or even unstable, depending on the value of .

More precisely, if , the origin is an asymptotically stable equilibrium point as can be easily demonstrated by Theorem 12.10. Assuming in fact

, for .

If the system is linear with a zero real part eigenvalue and unit index, so the system (and thus all its equilibrium points) is stable but not asymptotically stable.

Finally, if the origin is an unstable equilibrium point as can be easily demonstrated by Theorem 12.10. In fact, if we still assume ,

For .

Example 12.17 Consider the simple pendulum already presented in Example 2.13 and shown in Fig. 2.8. Assume that no external mechanical torque acts on the system. Under this assumption, assumed as state variables e

, as seen in Example 2.13, the VS model of such a system is given by the differential equations:

— — (12.16)

Such a system has two isolated equilibrium points that satisfy the *equations* and , that is, and The Jacobian of holds in this case

Per valutare la stabilità nell'origine calcoliamo lo Jacobiano in :

whose eigenvalues are

For every value of these eigenvalues have real part , so the origin is an asymptotically stable equilibrium point.

If, on the other hand, we cannot draw any conclusions about the stability of the origin using the linearization criterion. We must therefore proceed by other means. One possibility is to use Lyapunov's Direct Method (Theorem 12.10). Let us assume, for example.

Clearly, e is defined as positive in the *surroundings* of origin. In addition,

so that it is semidefinite negative in the considered neighborhood of the origin. We can therefore conclude that the origin is a stable equilibrium point.

Valutando infine lo Jacobiano in è facile vedere che vi è un autovalore a parte reale per ogni valore di (la verifica di ciò è lasciata per brevità al lettore). Possiamo pertanto concludere che per ogni valore del coefficiente di smorzamento il punto di equilibrio è instabile.

Exercises

Exercise 12.1 Consider the linear system

And evaluate its asymptotic stability.

It is also assessed whether the Lyapunov functions.

(a)

(b)

are significant for the purpose of analyzing the stability of this system.

Exercise 12.2 Consider the electrical circuit in Fig. 12.11. Recalling the elementary laws binding voltages and currents in an inductor ( ) and a capacitor ( ), let be the voltage at the ends of the capacitor and the current in the inductor.

Prove that the second-order nonlinear system

—

—

Is representative of the network under consideration.

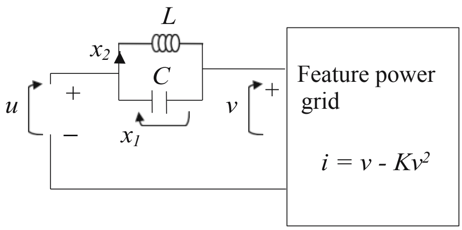


Fig. 12.11 System from Example 12.2

Determine any equilibrium states for (input short circuit) and study the stability of these states.

(It is easy to show that the origin is the only equilibrium state. For the study of the stability can then be assumed as a Lyapunov function that coincides with the sum of capacitive and inductive energy).

Exercise 12.3 Consider the first-order system

——

Study the stability of the origin as the parameters and .

Exercise 12.4 Consider the nonlinear system of Exercise 12.17. Prove by Lyapunov's first criterion the asymptotic stability of the origin and the instability of the point .

Exercise 12.5 Consider the nonlinear system

Determine the equilibrium points, if any, and study the stability of these points by Lyapunov's first criterion.

Exercise 12.6 Consider the system

Prove that the points

—

are equilibrium points for such a system. Prove further that for odd values of such points are of stable equilibrium, whereas for even values of , including , Such equilibrium points are unstable.

Appendices

A

Recalls to complex numbers

This appendix aims to summarize in a compact form the already known concepts related to number sets with special emphasis on the set of complex numbers. For a more complete discussion, please refer to the texts adopted in the Mathematical Analysis courses.

A.1 Definitions elementary

The set of *natural numbers* is .

The set of *integers* is .

The set of *rational numbers* is —.

L'insieme dei *numeri reali* si denota : a differenza degli insiemi precedenti, non può essere enumerato.

Denoting the *imaginary unit* , we also define the set of the *imaginary numbers*

as the set of numbers obtained by multiplying the imaginary unit by any real number .

A.2 The numbers complex

A.2.1 Representation Cartesian

The set of *complex numbers* is

A generic complex number

(A.1)

consists of two terms: the *real part* and the *imaginary part*
It can be represented by a vector in the plane as mo-
layer in Fig. A.1. The abscissa represents the real part, while the ordinate represents the imaginary part. The representation (A.1) is also called the *Cartesian representation*.

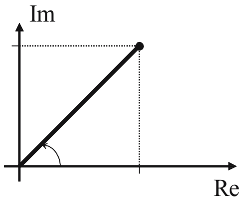


Fig. A.1. Cartesian and polar representation of a complex number

Observe again that a real number can be seen as a com- plex number with zero imaginary part ( ) , while an imaginary number can be seen as a complex number with zero real part ( ) . Therefore, the following applies.
e .
It is usual to denote by superscript , , and the restriction of that set to positive, null or negative values of the real part. For example, , etc.
Given a complex number , the number is said to be its *conjugate* if it has same real part of and opposite imaginary part, i.e., if

A.2.2 Exponential imaginary

It is possible to give another equally intuitive representation of a com- plex number. However, preliminarily it is necessary to define a particular com- plex number that is obtained by raising the real number base of natural logarithms by an imaginary exponent.

Proposition A.1 *Given an imaginary number worth*

that is, the exponential of that imaginary number is a complex number that has real part and imaginary part .

Demonstration. We know that given any scalar is worth

In the particular case where you get

In the first sum it is easy to recognize the McLaurin development of the cosine function

while in the second it is easy to recognize McLaurin's development of the sine function

Which proves the statement.

A.2.3 Representation polar

Consider again Fig. A.1. Given a complex number, we can define its *modulus* and *phase* as

(A.2)

As observed from the figure, it is the modulus of the representative vector of , while is the angle that this vector forms with the real axis, taking the counterclockwise direction as positive.

The inverse formulas that allow us to derive real part and imaginary part known modulus and phase also apply, of course:

(A.3)

It is therefore possible to write

where the result of Proposition A.1 was used in the last step. A *polar* representation of a complex number is defined as the representation in temrini of modulus and phase

(A.4)

Given a complex number , its conjugate has same modulus of and opposite phase, i.e.

Example A.2 Given complex numbers whose Cartesian representation is given in the first row of the following table, the corresponding polar representation was calculated using formulas (A.2) and given in the second row:

		–		–		–		–

Vectors in the complex plane representative of these numbers are shown in Fig. A.2

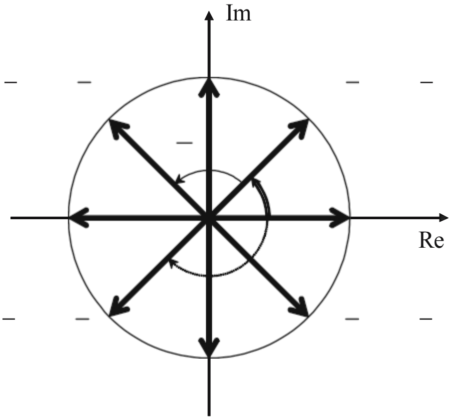


Fig. A.2. Representative vectors of complex numbers in Example A.2.

Some clarifications regarding this example.

Note that since the cosine and sine functions are periodic with period , it applies to :

and thus a complex number can give rise to several polar representations, all equivalent to each other, if at its f as and multiples are added or subtracted of . For example, the admits all polar representations following number

Nella tabella si è scelto di rappresentare la fase nell'intervallo .
Un numero reale positivo ha modulo e fase .
Un numero reale negativo ha modulo e fase .
Moltiplicare un numero complesso per equivale a ruotare il suo vettore rappresentativo di radianti in senso antiorario lasciando il suo modulo inalterato; viceversa, moltiplicare per equivale a ruotare il vettore di radianti in senso timetable. At d example, note that multiply ndo by the complex number you *get* (see Fig. A.2).

Care should be taken in using the formulas (A.2) for calculating of the fa if of a nume ro com plete. Consider, for example, the numbers e , whose representative vectors lie, respectively, in the first and third quadrants. Given and the corresponding phases are worth

as also shown in Fig. A.2.

Usually a pocket calculator does not allow the two topics to be specified and for the calculation of producing in both the two cases discussed here the identical resultIn fact, ...if the argument of the *function* è il calcolatore determina un angolo sempre compreso nell'intervallo and if the vector lay in the third quadrant you need to subtract (or add) al valore ottenuto col calcolatore. Analogamente il calcolatore non distingue fra la fase di un numero il cui vettore giace nel secondo e quarto quadrante: se l'argomento della funzione è il calcolatore determina un angolo sempre compreso nell'intervallo : se il vettore giace nel secondo quadrante occorre sottrarre (o sommare) a tale valore.

A.3 Formulas of Euler

Finally, we recall some elementary relationships that allow a periodic function to be written as a sum of exponential functions.

Proposition A.3 *The following relationships apply.*

Demonstration. They are easily proved by Proposition A.1 and by remembering that cosine is an even function, while sine is an odd function. In fact, it is worth

while

B

Signals and distributions

The purpose of this appendix is to describe some *signals*, i.e., functions
of the real variable called *time*, of particular importance in the a- nalysis of systems. Such signals often exhibit discontinuities, and in order to treat them analytically, it is necessary to introduce a new mathematical tool, the *distribution*, which precisely generalizes the function concept.

B.1 Signals canonical

B.1.1 The step unit

We begin by defining the *unit step* function, which we denote *Æ* The e-analytical spression of such a function is worth

Æ se

i*f* (B.1)

and its graph is shown in Fig. B.1.a. This function is continuous everywhere except at the origin, where it has a discontinuity of amplitude 1.

Note **B.1** *Note that given a generic function , through the unit step we can also easily define the function*

Æ *if*

se

which is obtained from it by canceling the values for (see Fig. B.2).

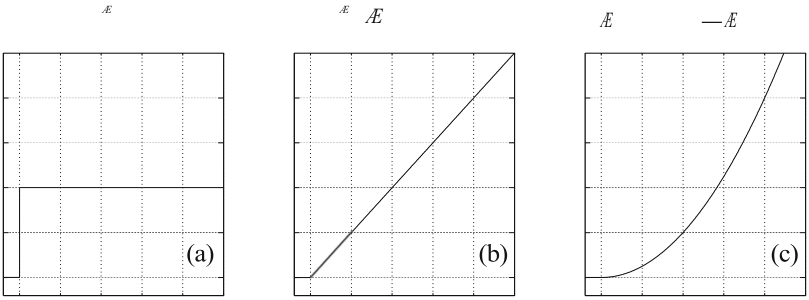
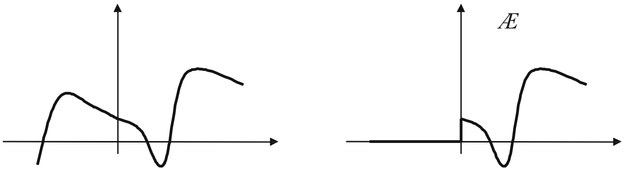


Fig. B.1. (a) Unit step; **(b)** Linear ramp; **(c)** Quadratic ramp.



(a) (b)

Fig. B.2. (a) A generic function **; (b)** the function Æ

B.1.2 Ramp functions and the ramp exponential

We can also easily define the integral of the unit step, which we call *Unit ramp* and we denote Æ Clearly applies:

Æ Æ Æ if se (B.2)

and the graph of that function is shown in Fig. B.1.b.

In general, we can recursively define for the family of *ramp* functions:

Æ Æ Æ times

or Æ if if (B.3)

For we have the *quadratic ramp* shown in Fig. B.1.c

Æ —Æ

for we have the *cubic ramp*

Æ —Æ

etc.

A generalization of the ramp function is the *exponential ramp* (or *cisoid*) defined by means of the two parameters and as

if

— Æ

— if

The exponential ramp makes it possible to represent the whole class of possible mo- diatures characterizing the dynamics of a physical system. Special cases of the exponential ramp include the following.

If and exponential ramp describes the *unit step*.
If when the exponential ramp varies, it generates the family of *ramp functions* previously defined. Note also that linear combinations of ramps can be used to describe *polynomial functions*, e.g.,

If and exponential ramp describes an *exponential function* .
If and , linear combinations of exponential ramps can be used to describe *sinusoidal functions*, e.g.,

B.1.3 The impulse

We now want to extend the family of canonical signals by considering the derivatives of the unit step (first derivative, second derivative, etc.). To do this, the results of classical analysis cannot be used, in the sense that the derivative of a discontinuous function is not defined.

We then proceed as follows. Having fixed a generic , we first define the function

se

Æ se

if

This function, shown in Fig. B.3, can be regarded as an approximation of the unit step, in the sense that Æ Æ .

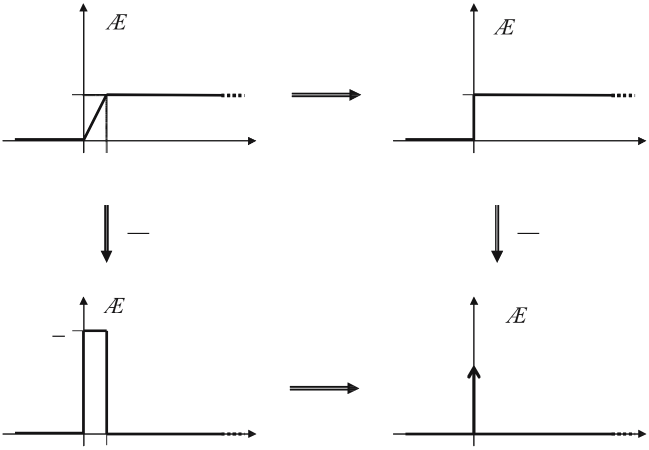


Fig. B.3. Relationship between the unit step Æ, the impulse Æ, and the functions Æ and Æ

However, since it is continuous, we can calculate its derivative, which is worth:

$$Æ \quad Æ \qquad \text{if} \qquad \text{otherwise.} \qquad \text{(B.4)}$$

The function Æ , also shown in Fig. B.3, is called the *finite base impulse* : it is a rectangle of base and height ; therefore, its area is worth 1 regardless of the value of .

We can then define the derivative of the unit step, which takes the name of *unit impulse* (or *Dirac function*), as

$$Æ \quad —Æ \quad — \quad Æ \quad —ÆÆ \qquad \text{(B.5)}$$

Note that this definition of a function through a limit is not formally correct in the sense of classical mathematical analysis but only makes sense if we admit to generalizing the concept of a function as *distribution theory* does. Pro- prly speaking, therefore, the impulse is not a function in the classical sense but a distribution.

The Æ pulse has these properties:

is zero outside the origin, i.e.

$$Æ \qquad \text{if} \qquad \text{(B.6)}$$

assumes an infinite value in the origin;
its area is worth 1, i.e.

Æ Æ (B.7)

By convention we represent Æ by an arrow centered on the origin as in Fig. B.3. In contrast, the function Æ represents an impulse centered on the point

.

To conclude we recall a property that will be useful later.

Proposition B.2 *If it is a continuous function in , its product for the impulse is worth*

ÆÆ (B.8)

and in general if it is a continuous function in vale

ÆÆ (B.9)

Demonstration. It is sufficient to observe that according to (B.6) the values assumed by the function

for are not significant since the impulse is zero at such points.

B.1.4 The derivatives of the pulse

Using the same reasoning in the limit, we define the successive derivatives of the impulse. First, we observe that it is also possible to define the impulse as

ÆÆ

Where the functionÆ shown in Fig. B.4, is worth

if

Æ se

otherwise.

Note that this function, like the Æ , has unit area but, because it is conti- nuous, can be derived at intervals. Deriving it yields the function Æ , also shown in Fig. B.4.

We can then define the derivative of the pulse, which is called the *doublet*, as

Æ —Æ — Æ —Æ Æ (B.10)

Similarly, we can define for each value of the derivatives of order of the impulse

Æ —Æ —Æ (B.11)

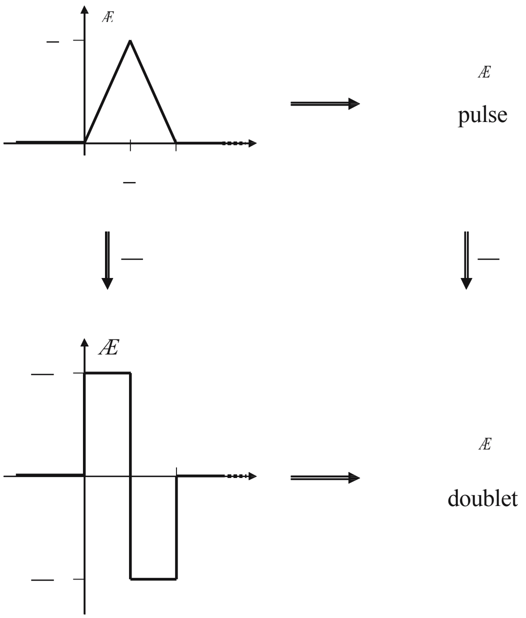


Fig. B.4. Relazione fra l'impulso unitario Æ , il doppietto Æ , e le funzioni Æ e Æ

B.1.5 Family of signals canonical

It is therefore possible to define the *family of canonical signals Æ* for , do- ve by convention Æ Æ Negative values of correspond to the integrals Of the pulse, that is, to the step and ramps. Positive values of correspond to the derivatives of the pulse.

Note that these signals are *linearly independent of* each other[1] . So given a function

Æ

if such a function is identically zero on an interval (with) necessarily holds for every .

B.2 Calculation of derivatives of a discontinuous function

The mathematical formalism just introduced is used to compute, in the sense of the theo- ry of distributions, the derivatives of discontinuous functions. In particular, in the analysis

[1]The formal definition of linear independence between functions and some criteria for testing this property are given in Appendix E.

of systems one often encounters signals that are null for and continuous for , but may have discontinuities in the origin.

Given, for example, a continuous function , consider the function Æ (see Fig. B.2): this function has a discontinuity in the origin *if* . You want to calculate the successive derivatives of that function .[2]

The first derivative is worth

— Æ — Æ Æ — Æ Æ

(B.12)

that is, it consists of the derivative of the original function multiplied by Æ plus an impulse in the origin multiplied by : the latter term is missing only if .

The second derivative is worth

— Æ —Æ Æ — Æ —

Æ Æ Æ

(B.13)

that is, it consists of the second derivative of the original function multiplied by Æ , plus an impulse in the origin multiplied by , plus a doublet multipli- cated by .

In a similar way, next-order derivatives are calculated.

—Æ Æ ÆÆ

(B.14)

Æ Æ

Where we denote Æ Æ .

Example B.3 Consider the function Æ The first derivative of that funzione is worth

— Æ —Æ ÆÆ Æ

The second derivative is worth

— Æ — Æ ÆÆ

ÆÆ

[2]In order not to burden the notation we denote the first, second, , but, derivative of the function .

Esempio B.4 Si consideri la cisoide Æ La derivata prima di tale funzione vale

— Æ Æ Æ Æ Æ

The second derivative is worth

— Æ Æ Æ Æ

Æ Æ

B.3 Integral of convolution

The convolution integral is an operator of fundamental importance in the stu- dents of signals and systems because it provides the mathematical tools to solve numerous problems.

Definition B.5 *Given two functions we define* convolution of con *a new function of the real variable*

This function is also indicated to specify that it is constructed by applying the "convolution integral" operator, denoted by , to the two functions and .

It is possible to give a graphic interpretation of the convolution between two functions.

Example B.6 Consider the two functions.

se *if*
otherwise altrimenti

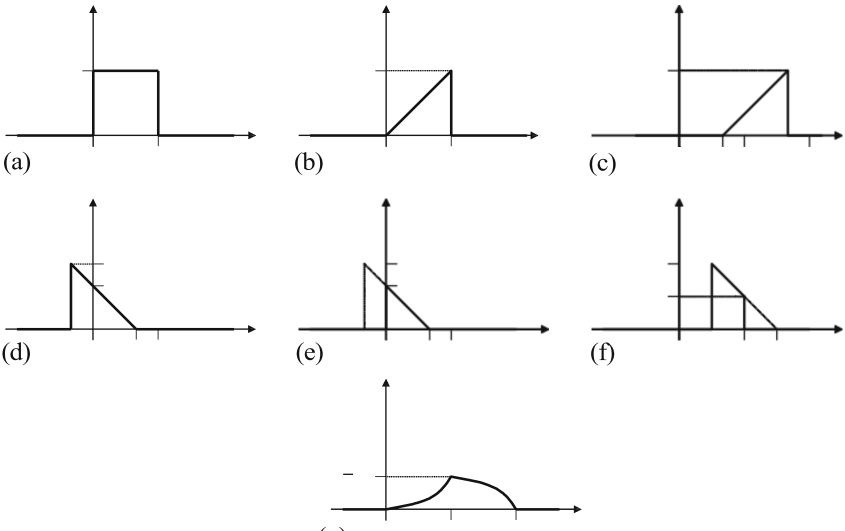
shown in Fig. B.5.a-b. To calculate

se
otherwise

you shift by the second curve, as shown in Fig. B.5.c: if you shift to the right and, vice versa, if you shift to the left. To calculate

se
otherwise

one flips the curve representative of around a vertical axis passing through , as shown in Fig. B.5.d.



(a) (b) (c) (d) (e) (f) (g)

Fig. B.5. Graphical interpretation of *convolution*

Finally, we can calculate the ....product functionFor such

signal takes the form given in Fig. B.5.e: its area is therefore worth ....For such a signal takes the form given in Fig. B.5.f: its area is *therefore* worth For all other values of the product function is identically zero.

It is therefore possible to conclude that it applies:

se

if

otherwise

and that function is shown in Fig. B.5.g.

It is easy to observe that the following property applies.

Proposition B.7 *The convolution operator is commutative, i.e.*

Demonstration. With a change of variable, one can write

This also allows it to be defined as "convolution between and " (instead of "convolution of with ") since the order of the operands is irrelevant.

The following proposition links the convolution operator with the integration and derivation operators.

Proposition B.8 *Given two functions , let them be*

— *e* —

their derivatives[3] *and be*

e

their integrals. The following relationships apply:

1. *The derivative of the convolution between two functions is obtained by performing the convolu- tion between one of the two functions and the derivative of the other, i.e.*

—

2. *The integral of the convolution between two functions is obtained by performing the convolu- tion between one of the two functions and the integral of the other, i.e.*

3. *a convolution integral does not change if one of the two operands is derived while the other is integrated, viz:*

Demonstration. To prove the first result, note that the following applies.

— — —

On the other hand, because of the commutativity property given in Proposition B.7, the following also applies

i.e.

— — —

[3]If the functions are not continuous these derivatives are understood to be determined by means of the theory of distributions, as seen in section B.2.

where the commutativity property was still used in the last step.

The second result follows from the first. In fact, to prove that the three functions of given are identical, it suffices to observe that evaluated in them cancel, men- tioned their derivatives coincide for every value of In fact by definition of integral

— , while under the first part of the proposition

vale

— —

e

— —

The third result is also derived from the first. In fact, it is obtained according to the first part of the proposition

— — —

while deriving gives

— — —

B.4 Convolution with canonical signals

We finish with some results related to the convolution of a function with a canonical signal.

Proposition B.9 (Convolution with impulse) *Given a function continued in vale*

Æ (B.15)

and more generally given any interval containing valid

Æ (B.16)

Demonstration. Observe that Æ Æ is an impulse centered in . Therefore, it is worth

Æ Æ Æ

where in the first step eq. (B.9) was used and in the last step eq. (B.7). Note that eq. (B.16) is derived immediately from eq. (B.15) since Æ if

.

The previous result states that by performing convolution of a *signal* with a pulse centered in we derive the value taken by the signal in . This result can be generalized to the -ma derivative of the pulse, from the convolution of which we derive the value of the -ma derivative of the given signal.

Proposition B.10 *Given a continuous function in together with its derivatives up to order -mo, the following holds true*

$$— \quad Æ \tag{B.17}$$

Demonstration. Observe that Æ holds, and thus by repeatedly deriving and using the result of Proposition B.8 part 1 we obtain:

$$— \quad — \quad Æ \quad —Æ \quad Æ$$

$$— \quad — \, Æ Æ$$

$$\cdot \quad \cdot$$

$$— \quad — \, Æ Æ$$

C

Elements of linear algebra

This appendix recalls the fundamental concepts of linear algebra that will serve in the study of systems analysis. It was preferred to include in this appendix only recalls to material that is assumed to be already known: such material inclu- de the definition of matrix and vector, the main matrix operators, the determinant and rank of a matrix, solving systems of linear equations, the inverse matrix, eigenvalues and eigenvectors.

Other elements of linear algebra are also necessary for the study of systems analysis, which, since they are perhaps less well known, have been preferred to be covered in the various chapters of the text, particularly in Chapter 4. This material includes, for example, the matrix exponential, the diagonalization procedure, the Jordan form, and canonical forms in general. Finally, an important result known under the name of the Cayley-Hamilton Theo- rema and some approaches that descend from it are presented in Appendix G.

C.1 Matrices and vectors

Definition C.1 *A dimension* matrix *is a table of elements arranged on rows and columns*

$$\vdots \quad \vdots \quad \ddots \quad \vdots$$

Notation is also used to indicate that the matrix has element
At the intersection of row ma and column ma.

Here we will consider real matrices, that is, matrices in which each element belongs to the set of real numbers . In addition, a capital letter will be used To denote a matrix. It is written to *d e n o t e*
is a matrix of dimension .

Example C.2 Consider the *matrix*

for which applies , , , , ,...

Some special cases of matrices are noteworthy.

Definition C.3 *A* scalar *is a matrix of dimension .*
A vector *is a matrix in which one of the two dimensions is unitary. Distinguish:*

Column vector*: is a matrix consisting of only one column.*
Row vector*: is an array consisting of a single row.*

You will use a lowercase letter , to denote a vector. To denote that it is a column vector of size you will write .

Example C.4 Consider the vectors

The dimension vector is a column vector with 3 components. The *vector* of dimension is a row vector with 4 components.

A dimension matrix can think of itself as composed of column vectors of dimension and is written

$$| \quad | \quad |$$

to indicate that is the -th column. Similarly, it can be thought of as consisting of dimension line vectors and you write

per indicare che è la -ma riga.

Example C.5 The *matrix*

has three columns

and two
lines

A matrix of dimension with is called rectangular. A particularly important case is that of square matrices.

Definition C.6 *A matrix is said to be* square *if it has dimension , that is, if it has as many rows as columns. In that case the matrix is also said to have* order .

The diagonal *of a square matrix of order is the set of elements Which have equal row and column numbers.*

Example C.7 The following is a square matrix of order whose diagonal is composed of the elements :

Definition C.8 *A square matrix is called:*

Diagonal: if all elements not belonging to the diagonal are worth zero.

Block diagonal*: if square blocks can be identified along the diagonal and all elements not belonging to those blocks are worth zero.*

Lower *(resp.,* upper*)* triangular*: if all elements above (resp., below) the diagonal are worth zero.*

Triangular lower (resp., upper) blocks*: whether square blocks can be identified along the diagonal and all elements above (resp., below) these blocks are worth zero.*

Identity matrix*: if it is diagonal and the elements along the diagonal are all worth one. In that case the matrix is denoted or, if necessary to specify that it is of order , denoted .*

Example C.9 The following square matrices are given:

The matrix is diagonal, the matrix is lower triangular, the matrix is upper triangular, and the matrix is the identity matrix of order 3.

Instead, the following matrix is block diagonal:

it consists of three blocks, one of order 2 and two of order 1.

Finally, the following matrix is block triangular:

i due blocchi diagonali e hanno entrambi ordine 2.

C.2 Operators matrices

C.2.1 Transposition

The transpose of a matrix is obtained by exchanging the rows of the matrix with each other for its columns.

Definition C.10 *Given a matrix of dimension we define.*
trasposta *di la matrice di dimensione che ha lungo the ma row the elements of the ma column of , and along the ma column the elements of the ma row of , i.e.*

$$. \quad . \quad \ddots \quad .$$

Example C.11 Given the dimension matrix below, it is worth

e

Note that the following properties apply.

If it is any diagonal matrix, it holds .

The transpose of a lower triangular matrix is an upper triangular matrix and vice versa.

The transpose of a row vector is a column vector and vice versa.

If it is also true , that is, by applying to a generica matrix twice the transposition operator we again obtain the matrix .

C.2.2 Sum and difference

The sum or difference of two matrices is obtained by summing term to term.

Definition C.12 *Given two matrices and both of dimension we define* sum *of and as the matrix also of dimension that has for element the sum of elements and , i.e.*

$$. \quad . \quad \ddots \quad .$$

Similarly, we define difference *of and the matrix also of size .*

Note that the sum or difference of two matrices is defined if and only if they have the same dimension.

Example C.13 Given the matrices and of size given below:

e

vale

e

C.2.3 Product of a matrix by a scalar

It is possible to multiply a matrix by a scalar.

Definition C.14 *Given a number and a matrix of dimension We define product of by , the matrix also of dimension that is obtained by multiplying each element by , i.e.*

$$. \quad . \quad \ddots \quad .$$

Example C.15 Given , for the matrix of size given below, the following applies.

e

C.2.4 Matrix product

The product between two matrices requires special attention because it is not obtained by multiplying term by term but by performing a product called row by column.

Definition C.16 *Given a matrix of dimension and a matrix di dimension definiamo* prodotto *di e la matrice of size*

$$. \quad \ddots \quad .$$

Note that *the* generic matrix element is calculated by performing the *scalar product* of the vector , representing the row ma of , with the vector representing the column ma of and is defined as follows:

$$.$$

that is, it is calculated by multiplying the first element of with the first element of , the second element of with the second element of , etc., and doing the sum of the individual products.

Example C.17 Given matrices of dimension , and dimension :

e

vale

It is clear that it is possible to perform the product of if and only if the two matrices conform so that the number of columns of coincides with the number of rows of . More generally, the product of multiple matrices is possible if they conform appropriately:

For each dimension matrix holds:

that is, multiplying left or right the matrix by the identity matrix of appropriate order still yields the matrix .

It is also important to note that while the product in classical algebra (between scalars) satisfies the commutative property, that is, it holds , the matric product does not necessarily satisfy this property. In fact, if i t conforms with and therefore the product is definite, $i\,t$ does not necessarily conform with and the product is definite . Consider, for example, the case of matrices and in E-sample C.17. While it conforms with by having three columns and three rows, the product is not defined by having two columns and three rows.

The following result applies.

Proposition C.18 *Two matrices and are said to* commute *if* .

A necessary *condition for two matrices to commute is that they are both square and of the same order.*

Demonstration. For both product and product to be defined , if the matrix has dimension necessarily the matrix must have dimension

. Also, in that case the first product would have dimension , while the second would have dimension . For them to be identical it must hold .

The previous condition is necessary but not sufficient as the following example shows.

Example C.19 Consider the matrices

e

Vale

Note that a *diagonal* matrix of dimension commutes with every other matrix of dimension .

Finally, the following elementary relations whose validity is verified by direct multiplication apply.

Proposition C.20 *Let there be a matrix of dimension*

whose -ma row is worth ; let it be a matrix of dimension

whose -ma column is worth ; be

e

Diagonal matrices of order and , respectively.

The following identities apply:

C.2.5 Power of a matrix

Definition C.21 *Given a square matrix of order we define* power of degree *of the matrix*

times

also square of order that is obtained by multiplying by itself times.

Note that for is worth . Also defined , that is, the power of degree of any matrix is the identity matrix.

Example C.22 Given the *matrix* of size is worth

C.2.6 The exponential of a matrix

Given a scalar , its exponential holds by definition:

and it is shown that such a series is always convergent. By analogy, we extend this concept to the case of square matrices.

Definition C.23 *Given a matrix its* exponential *is a matrix defined as*

It is shown that such a series is always convergent.

In the case of block diagonal matrices, the following result applies.

Proposition C.24 *Given a generic block diagonal matrix.*

vale

Demonstration. It is easy to verify that for every vale

and therefore

from which the result sought is derived.

Thus, as a special case of the previous proposition, the following result applies, which allows the immediate calculation of the exponential of diagonal matrices.

Proposition C.25 *Given a generic diagonal matrix*

. . . *vale* . . . .
. . .
. . . .

Example C.26 The exponential of the *diagonal* matrix.

vale

Determining the exponential of arbitrary matrices by calculating the infinite series is generally not straightforward. Appendix G (see § G.3) presents a simple procedure to solve this problem.

C.3 Determinant

It is possible to associate a square matrix with a real number, called the determinant, to which many properties of the matrix are closely related.

Let us begin by defining the concept of the minor of a matrix.

Definition C.27 *Consider a square matrix of order . Its the* lower *is the square matrix of order obtained by deleting the -th row and the -th column. Such a minor is denoted .*

Example C.28 Given the matrix is worth

etc.

Definition C.29 *Consider a square matrix of order . The* determinant *of is a real number that is denoted by.*

or

And is defined as follows.

If , place applies .

For a generic applies

(C.1)

where we denote the cofactor *of the element , that is, the determinant of the lesser multiplied by (each is a scalar).*

If the matrix is called singular*, otherwise it is called* nonsingular.

The preceding definition makes it possible to recursively calculate the determinant of an order matrix as a function of determinants of order matrices (the minors in fact), which in turn can be calculated as a function of determinants of lower order, up to determinants of order matrices

.

Example C.30 For a generic matrix of order ,

is worth e , therefore ....eWe can finally
pose[1]

Come esempio numerico, data vale

Example C.31 Given the matrix

We first calculate the cofactors of the elements along the first column. Vale:

[1]Note the difference between the notation and the notation . While indicates a matrix, denotes the determinant of that matrix, i.e., a scalar.

So adding the product of each element along the first column by its cofactor yields the well-known formula

Equation C.1 is just one of many formulas for calculating the de- termant: in that formula the calculation of the determinant *develops* along the elements of the first column. There are similar formulas that develop along the elements of any column: fixed an arbitrary column index, it is in fact worth

(C.2)

Finally, it is also possible to develop the calculation of the determinant along a row; fixed an arbitrary row index, it is in fact worth

(C.3)

Finally, some elementary properties are mentioned that are given without demonstration.

Proposition C.32 *The following relationships apply.*

(a) The determinant of a diagonal or triangular matrix, is equal to the product of the elements along the diagonal, i.e. .
(b) The determinant of a block diagonal or block triangular matrix is equal to the product of the determinants of the blocks along the diagonal.
(c) The determinant of a product of square matrices is equal to the product of the deter- minants, that is, if then .

Example C.33 Consider the matrices defined in Example C.9.

The determinant of the three matrices , e is equal to the product of the elements along the diagonal and thus always holds being the first diagonal and the others triangular.

The determinant of the identity matrix holds: this result holds, of course, whatever its order.

Anche per le due matrici diagonali e triangolari a blocchi definite nello stesso esempio è agevole il calcolo del determinante. Per la prima matrice vale For the second matrix, the following applies

.

Finally, consider the matrices and of Example C.19. It is easily verified that it is true of and and therefore it is *also* true of

C.4 Rank and nullity of a matrix

Definition C.34 *The* rank *of a dimension matrix is equal to the number of columns (or rows) of the matrix that are linearly independent. This value is denoted .*

If we define *minor*[2] of a matrix , any matrix that is obtained from by deleting an arbitrary number of rows and columns, we can give the following characterization of the rank of a matrix.

Proposition C.35 *The rank of a matrix is equal to the order of the largest nonsingular least square.*

According to the preceding proposition to determine the rank of a matrix one can proceed as follows. Given a matrix of dimension we consi- der all least squares of order : if at least one of them were nonsingular we can conclude that the matrix has ....rankIf they were all
singular, we go on to consider all minors of order , etc.

Example C.36 Consider the two matrices of dimension :

e

The determinants of the three possible minors of order of the first matrix are

e

therefore not being all null is worth: .
The determinants of the three possible minors of order of the second matrix are

e

Since they are all null, the minors of order . are evaluated. Since there are nonzero elements, it holds: .

In the particular case of a square matrix , the greatest minor consists of the matrix itself and is singular. So we start by calculating the determinant of the matrix and if it is nonsingular we conclude that its rank is equal to . Otherwise we proceed to calculate the determinants of the minors of order , etc.

Example C.37 Consider the square matrix

[2]Definition C.27 introduced *minors* , i.e., the set of minors of orderIn general, we denote by the name of minor any submatrix.

The determinant of the matrix holds and therefore this matrix is singular is certainly has rank < 2. Since not all elements of the matrix are null, there are nonsingular minors of order and therefore this matrix has rank .

A concept related to the rank of a matrix is that of null space and nullity.

Definition C.38 *Given a matrix of dimension is defined as* null space *the whole*

consisting of all the vectors in that multiplied to the left by produce the null vector.

Such a set is a vector space; its dimension is *called matrix* nullity *and is denoted*

Note that the null vector always belongs to and if it is the only vector that belongs to this set, it holds If the null space contains other vectors instead, the nullity is equal to the number of linearly independent vectors that you can choose from it.

Example C.39 Consider the matrix :

already studied in Example C.36.

Its null space includes all vectors such that

i.e., satisfying the system

(C.4)

It is immediately observed that the second equation is identical to the first equation multiplied by 2, and is therefore redundant. It is therefore worth

Since there is only one equation that binds three unknowns, there are two degrees of freedom to choose two linearly independent vectors that satisfy that equation. E.g., one can arbitrarily fix the first two components choosing and , or and , obtaining the two vectors

e

and therefore it is worth

The following theorem, which is given without proof, links together rank and nullity of a matrix.

Theorem C.40 *Given a matrix with columns worth*

(C.5)

Example C.41 Consider the matrix already studied in Examples C.36 and C.39. Such a matrix, as seen, has , and number of columns , so Eq. (C.5) is satisfied.

An intuitive explanation of the previous theorem is that the rank of the matrix is equal to the number of linearly independent equations of the system (C.4), while it represents the number of unknowns of the system. So the null of the matrix, which consists of the number of linearly independent solutions of the si- stem (C.4), is just equal to the difference between e and its rank

C.5 Systems of equations linear

It is possible to determine whether a linear system of equations in unknowns admits one and only one solution, based on analysis of the determinant of the matrix of coefficients.

The following theorem, which we do not demonstrate, applies.

Theorem **C.42** *Consider a linear system of equations in unknowns.*

where of dimension is the matrix of coefficients*, of dimension is the* vector of known terms *and dimension is the* vector of unknowns.

1. *If the matrix is nonsingular, the system admits one and only one solution;*
2. *If the matrix is singular, let it be the matrix that is obtained by adding to the matrix an additional column formed by the vector . Worth:*
   (a) If the system admits no solutions;
   (b) If the system admits infinite solutions.

Example C.43 The linear system of equations in unknowns is given.

which can also be rewritten in the matrix form with

The determinant of the matrix has already been calculated in Example C.30 and is worth : therefore this system admits one and only one solution.

It is possible to solve such a system by substitution. By deriving from the first equation and substituting in the second equation, we obtain

– –

Dunque la soluzione del sistema dato è , , ovvero in forma matriciale .

The following example presents the case of a linear system that admits no solution.

Example C.44 The system of linear equations in unknowns is given.

which can also be rewritten in the matrix form with

The matrix , as seen in Example C.37 is singular and has rank . The matrix coincides with the matrix studied in Example C.36, which has rank . Therefore, such a system admits no solution.

In fact, deriving from the first equation and substituting in the second equation gives

or

Leading to a glaring inconsistency: .

The following example presents the case of a system that admits infinite solutions.

Example C.45 The linear system of equations in unknowns is given.

which can also be rewritten in the matrix form with

The matrix in this example is identical to that in the previous example and, as already seen, is singular and has rank . The matrix coincides with the matrix studied in Example C.36 which has rank . So such a system admits infinite solutions.

Deriving from the first equation and substituting in the second equation gives

or

La seconda equazione è sempre soddisfatta (non dipende dalle incognite) mentre la prima è soddisfatta da una infinità di soluzioni della forma , dove è un numero arbitrario, e......................Dunque tale sistema ammette infinite soluzioni

Of the form .

C.6 Reverse

Definition C.46 *Given a square matrix of order , we define* inverse *of the matrix , also square and of order , which enjoys the following property:*

The inverse matrix of exists if and only if it is nonsingular; moreover, when it exists it is unique.

Before presenting a procedure for calculating the inverse, it is necessary, first to define the matrix of cofactors and the addition of a matrix.

Definition C.47 *Consider a square matrix of order .*

The matrix of the cofactors *of is the square matrix of order that has the cofactor of . as its element:*

The added matrix *of is the square matrix of order that is obtained by transposing the matrix of cofactors:*

Example C.48 Given the matrix is worth

Therefore, it is worth

e

Definition C.49 *Consider a square matrix of nonsingular order.*
If , place *applies* .

For a generic applies

Example C.50 Consider a generic second-order matrix

the addition of which is worth

If it is worth

As a numerical example, consider the matrix studied in Example C.44, whose determinant ....holdsCalculate the inverse of :

It is easy to verify that it is worth

Example C.51 Consider the matrix studied in Example C.48, whose determi- nant is worthThe ....addition of this matrix has already been calculated and the inverse
is therefore worth

——— ——

The inverse also allows us to solve a linear system whose coefficient matrix is nonsingular.

Theorem C.52 *Given a linear system of equations in unknowns , if the matrix is nonsingular the unique solution of the system is worth*

Demonstration. Multiply both members of the equation from the left byYou get:

given that and that according to the properties of the identity matrix.

Example C.53 Consider again the system in Example C.43 where

e

The inverse of the matrix was calculated in Example C.50.
Therefore, it is worth

This value coincides, as expected, with that determined in Example C.43 by solving the system by substitution

Finally, some elementary properties are recalled.

Proposition C.54 *The following relationships apply.*

(a) Given a generic nonsingular diagonal matrix, its inverse is obtained by inverting the elements along the diagonal, i.e.

$\vdots \quad \vdots \quad \ddots \quad \vdots \qquad \vdots \quad \vdots \quad \ddots \quad \vdots$

(b) Given a generic nonsingular block diagonal matrix, its inverse is obtained by inverting the blocks along the diagonal, i.e.

$$\vdots \quad \vdots \quad \ddots \quad \vdots \qquad\qquad \vdots \quad . \quad \ddots \quad \vdots$$

(c) Given two matrices and of nonsingular order, the following is valid

(d) Given a matrix of nonsingular order, it is worth

Demonstration. The first three properties are self-evident and are demonstrated by verifying that (in the first two cases) or (in the third). To prove the last property, note that since according to Proposition C.32.(c) it is also true

From which we derive the relationship sought.

Example C.55 Consider the matrices defined in Example C.9.

The inverse of the diagonal matrix is worth

Inoltre mentre si può verificare che vale .

The inverse of the block diagonal matrix is worth

C.7 Eigenvalues and eigenvectors

Another concept of fundamental importance that can be defined only for square matrices is the following.

Definition C.56 *Given a square matrix of order , both a scalar and a column vectorIf ....it is worth*

(C.6)

then it is said to be an eigenvalue *of with which the* eigenvector *is associated* .

A square matrix of order whose elements are all real numbers has autovalori[3] che possono essere numeri reali oppure presentarsi a Pairs of complex and conjugate numbers.

Proposition C.57 *Let a matrix be* diagonal *or* triangular. *Its eigenvalues are the elements (for) present along the diagonal.*

Example C.58 The following matrices are given.

Observing that each of them is triangular or diagonal, we can immediately deduce that:

the matrix has eigenvalues and ;
the matrix has eigenvalues , and ;
the matrix has eigenvalues , and .

More generally, for a generic square matrix the eigenvalues can be calculated as follows. Let us first give the following definition.

Definition C.59 *The* characteristic polynomial *of a square matrix of order is the polynomial of degree in the variable defined as* .

Example C.60 Given the matrix

we first calculate the matrix whose elements contain the variable :

The determinant of such a matrix holds*:*
and therefore the characteristic polynomial of is the polynomial of second *degree*
.

[3]It might happen that some of the eigenvalues coincide, e.g., ...If vice versa for then the matrix is said to have eigenvalues of unit multiplicity.

Proposition C.61 *The eigenvalues of a square matrix of order are the roots of its characteristic polynomial, that is, the solutions of the equation*

.

Furthermore, if it is an eigenvalue of order , the eigenvector corresponding to it is a nonzero solution of the linear system

(C.7)

where it is a column vector whose elements are all worth zero.

Demonstration. An eigenvalue with its eigenvector must satisfy Eq. (C.6) from which Eq. (C.7) is immediately derived.

Ora, in base al Teorema C.42, l'eq. (C.7) ammetterà come soluzione (oltre alla soluzione ovvia) anche una soluzione se e solo se la matrice è singolare. Ciò implica che e dunque è radice del polinomio caratteristico della matrice .

Example C.62 Consider again the matrix

taken into consideration in the previous example. Its eigenvalues are the solutions of the equation viz.

Determine the corresponding eigenvectors.

The eigenvector

corresponding to the eigenvalue must satisfy the system

or

The two equations are linearly dependent: if the first is satisfied, the second will automatically be satisfied as well. Such linear dependence between the equations of the system (C.7) always occurs.

We can limit ourselves to considering only the first of the two equations, which impo- ne the relation . So chosen an arbitrary first component, the eigenvector must have as its second component .

Chosen is worth

The eigenvector

corresponding to the eigenvalue must satisfy the system

or

Again, as expected, the two equations are linearly dependent, and we can limit ourselves to considering only the first of them, which imposes the relationship So chosen an arbitrary first component, the eigenvector must Have as a second component . Chosen is worth

As seen in the example, the system (C.7) always admits an infinity of solutions (from which a nonzero solution will always be chosen) because an eigenvector is determined minus a multiplicative constant. In fact, it is easy to see that if
is an eigenvector associated with the eigenvalue , the vector-which is obtained by multiplying by a nonzero scalar-is also an eigenvector associated with . To prove this, observe that

and the result is obvious by comparing the first and last members.

The following classical result applies, the proof of which we do not report.

Theorem C.63 *Let there be eigenvectors of a generic matrix and suppose that the corresponding eigenvalues are distinct. Then the eigenvectors Are linearly independent.*

The following result follows immediately from this theorem.

Theorem C.64 *If an order matrix has distinct eigenvalues, then there exists a set of linearly independent eigenvectors, which thus forms a basis for .*

D

Matrices in companion and canonical forms

Given a representation in state variables, it is always possible to switch, medianting a similarity transformation, to a representation in which the state matrix assumes a canonical diagonal or Jordan form. There are, however, other canonical forms in which the state matrix takes on a particular structure called the *companion form*. In the first section of this appendix, companion form matrices are defined and a first elementary result concerning the determini- nation of their characteristic polynomial is presented. In the second section we present two canonical forms that are of particular importance: these are the *control canonical form* and *the observation canonical form*. Finally, in the last section we pre- sent some results concerning the determination of the eigenvectors of a matrix in companion form that greatly simplify its study.

D.1 Matrices in the form companion

A matrix is said to be *in companion* form if it takes the form

$$\begin{matrix} \cdot & \cdot & \ddots & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{matrix} \tag{D.1}$$

where is a column vector of zeros, is the identity matrix of order , and is a row vector of arbitrary coefficients. Note that this matrix contains free parameters , while all other matrix elements are fixed and are worth o or .

More generally, we also speak of companion form for the matrix that is obtained by transposing (D.2) and which takes the form

(D.2)

Example D.1 The matrices

e

Are all fit companion.

D.1.1 Polynomial characteristic

The particular structure of matrices in companion form allows their characteristic polynomial to be determined in a direct way.

Proposition D.2 *Given a generic matrix in companion form* (D.1) *or* (D.2)*, its characteristic polynomial holds:*

(D.3)

Demonstration. The two companion forms (D.1) and (D.2) are each the transpose of the other, and it is well known that the characteristic polynomial of a matrix coincides with that of its transpose. So it is sufficient to prove the result only for the first of the two forms.

The characteristic polynomial of the matrix in (D.1) is worth

Developing according to the cofactors in the last line is worth

where is the matrix obtained from removing the -th row and the -but column. Vale dove è una matrice triangolare superiore il cui determinante vale , mentre è una matrice triangolare inferiore il cui determinante vale . Essendo diagonale a blocchi vale

and therefore

Based on the previous result, the *monic* characteristic polynomial (i.e., such that the coefficient of the term of the highest degree is worth 1) of a matrix in compa- gn form (D.1) is a polynomial whose coefficients, from degree to degree , appear neatly in the bottom row of the matrix changed sign.

Example D.3 The matrix of Example D.1 has characteristic polynomial

and thus eigenvalues , , .

D.2 Canonical forms of representations in state variables

In this appendix, reference is always made to a SISO system whose model in state variables is worth

(D.4)

D.2.1 Canonical form of control

The VS representation in eq. (D.4) is said to be *in canonical control form* if the realization matrix holds:

(D.5)

Such a representation is thus characterized by a state matrix in the companion form (D.1). The matrix is preassigned, while the matrix and the scalar can take arbitrary values.

Transfer function

Proposition D.4 *The transfer function of a representation in canonical control form* (D.5) *holds:*

(D.6)

Dimostrazione. Detto il polinomio caratteristico di si dimostra preliminarmente che vale

or in completely equivalent terms

In fact, taking into account that it is worth

(D.7)

$$. \quad . \quad . \quad \ddots \quad . \quad .$$

mentre in base alla Proposizione D.2 vale , si ha che

$$\vdots \qquad \vdots \qquad \vdots$$

Therefore, it is worth

$$\vdots \qquad \overline{}$$

and developing this expression gives (D.6).

Controllability

Proposition D.5 *A representation in canonical control form is always controllable.*

Demonstration. The controllability matrix of the representation (D.5) holds:

$$| \quad | \quad | \qquad . \quad . \quad . \quad \ddots \quad . \quad .$$

(D.8)

wher
e

e

Such a matrix always has rank having unit determinant.

The inverse of the controllability matrix of the canonical form of control has a very simple form that we give explicitly because it will be used later.

$$\vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \tag{D.9}$$

Finally, note that a representation in canonical control form is os- servable if and only if its transfer function in minimal form has order , according to Theorem 11.48.

Transition to the canonical form of control

The following proposition states that any controllable representation can be traced, through similarity, to the canonical form of control.

Teorema D.6. *Si consideri una generica rappresentazione descritta dalla* (D.4) *e sia il polinomio caratteristico della matrice . Se tale rappresentazione è controllabile, sia*

$$| \quad | \quad |$$

its controllability matrix and pose

dove la matrice definita in eq. (D.9) *dipende dai coefficienti del polinomio characteristic.*

The similarity transformation leads to a representation

In the canonical form of control.

Dimostrazione. Si osservi in primo luogo che l'ipotesi di controllabilità del sistema è essenziale per poter applicare la procedura. Infatti la matrice può essere usata come matrice di similitudine se e solo se essa è non singolare, e ciò richiede che entrambi i suoi fattori siano non singolari. Mentre la matrice è sempre non singolare poiché ha determinante unitario, la matrice è non singolare se e solo se la rappresentazione è controllabile.

Place

it is easily verified that the generic column satisfies

.

(D.10)

Si vuole dimostrare che la matrice di similitudine data porta ad una rappresentazione in cui la matrice di stato vale

. . . ⋱ . . .

To do this we show that , showing that *for*
Indeed

.

.

.

.

where in the calculation of was placed according to the Cayley-Hamilton theorem (see Theorem G.1).

Finally, it remains to be demonstrated that and this occurs immediately being

Example D.7 Consider the representation given by

(D.11)

Where the matrix has characteristic polynomial

We can pose:

while

and finally

It is immediately verified that the *representation*
vale

and thus takes the canonical form of control.

D.2.2 Canonical form of observation

The VS representation in eq. (D.4) is said to be *in canonical observation form* if the realization matrix holds:

$$\cdots \quad (D.12)$$

Note that the canonical form of observation (D.12) is the dual representation della forma canonica di controllo (D.5). Infatti vale , , e Ciò consentirà di semplificare notevolmente la prova delle proprietà Of such representation.

Transfer function

Proposition D.8 *The transfer function of a representation in canonical form of observation* (D.12) *holds:*

$$\qquad (D.13)$$

Demonstration. The transfer function of (D.12) holds.

and since that expression is a scalar it is equal to its transpose. Thus

And the result follows from Proposition D.4.

Observability

Proposition D.9 *A representation in canonical form of observation is always observable.*

Demonstration. This result follows immediately from the fact that the representation (D.5) is controllable as was shown in Proposition D.5. So the representation (D.12) is observable according to the duality principle (see Theorem 11.38).

Specifically, according to the duality principle, the observability matrix of (D.12) is related to the controllability matrix of (D.5) by the relationship

(D.14)

where the last relation comes from the fact that is symmetrical. Therefore, the matrix
coincides with the matrix given in (D.8), and its inverse is worth

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{array} \qquad \text{(D.15)}$$

Finally, note that a canonical-form representation of observation is controllable if and only if its minimum-form transfer function has order
, according to Theorem 11.48.

Transition to the canonical form of observation

The following proposition states that any observable representation can be traced by similarity to the canonical form of observation.

Teorema D.10. *Si consideri una generica rappresentazione descritta dalla* (D.4) *e sia il polinomio caratteristico della matrice . Se tale rappresentazione è osservabile, sia*

its observability matrix, and let us pose *dove la matrice definita in eq.* (D.15) *dipende dai coefficienti del polinomio characteristic.*

The similarity transformation leads to a representation

In the canonical form of observation.

Demonstration. Consider the dual representation of the given representation that holds and is controllable since the given representation is os- servable. According to Theorem D.6, the dual representation can be placed in canonical control form by the transformation

| | |

where given by (D.9) coincides with given by (D.15).

Therefore, it is worth

:

dove è una matrice nella forma compagna (D.1). Trasponendo le precedenti equations are obtained

che tendendo conto che Can be rewritten:

where it is a matrix in the companion form (D.2). This shows that the similarity matrix transforms the given representation into a representation in the canonical form of observation.

Example D.11 Consider the representation given in (D.11) and already considered in Example D.7.

We can pose:

while the matrix has already been determined in Example D.7. Finally

—

Si verifica immediatamente che la rappresentazione
vale

and therefore takes the canonical form of observation.

D.3 Eigenvectors of a matrix in the form companion

We now consider the problem of determining the eigenvectors of a matrix in companion form by first referring to the companion form (D.1). In the last section we extend these results to the transposed form (D.2).

D.3.1 Autovectors

Proposition D.12 *Consider a generic matrix in the companion form given in eq.* (D.1) *and be one of its eigenvalues. The vector*

.

Is an eigenvector associated with .

Demonstration. It is immediately verified by substitution that the *equation* is satisfied. In fact, if it is root of the characteristic polynomial (D.3), it is also worth

So taking into account the particular shape of the matrix in eq. (D.1) it is worth

$\vdots$ $\vdots$

Example D.13 The matrix from Example D.1 has, as seen in Example D.3, eigenvalues , ...,A these eigenvalues are associated with, respectively.
mind, eigenvectors:

The previous result also allows us to state that if a matrix in companion form has distinct eigenvalues a modal matrix of it is worth

. . ⋱ .

A matrix in this form is the transpose of a Vandermonde matrix (see Chapter 4, § 4.2.2).

D.3.2 Generalized eigenvectors [*]

The following proposition makes it possible to calculate the generalized eigenvalues (see Ca- pital 4, § 4.6) of a matrix in companion form with eigenvalues of multiplicity greater than one.

Proposition D.14 *Consider a generic matrix in the companion form given in eq.* (D.1) *and let it be an eigenvalue of multiplicity . That eigenvalue corresponds to the following chain of generalized eigenvectors:*

— ———
. .

Demonstration. Having proved in the previous proposition that it is a self vector associated with , It is sufficient, based on Proposition 4.38 to prove that for
vale .

First observe that if it is root of multiplicity of the polynomial

then it will also be the root of the polynomials obtained by deriving up to order .
This implies:

—

—

. . .

Consider the ....caseThanks to the first of the previous equations the following holds true

. .
. .

. .
. .

Similarly, the other reports are shown.

Example D.15 Consider the matrix

which has characteristic polynomial and eigenvalues

Of double multiplicity and single multiplicity. A generalized modal matrix for the matrix (having eigenvalues and AGs for columns) is worth

Such a matrix can therefore be placed in the form of Jordan:

Note that to each multiplicity eigenvalue of a matrix in a com- plex form, there corresponds a single AG chain of length . So such a matrix is always traceable to a nonderivative Jordan form (see Example 4.35) in which the eigenvalue corresponds to a single Jordan block of order .

D.3.3 Matrices in companion form transposed

To determine a modal (generalized) matrix for a matrix in companion form (D.2), the following result can be exploited.

Proposizione D.16 *Data una generica matrice in forma compagna* (D.2) *sia la sua trasposta. Se è una matrice modale (generalizzata) di , allora la matrix*

Is a modal (generalized) matrix of .

Dimostrazione. Si consideri dapprima il caso in cui (e dunque) sia diagonalizzable and is the diagonal matrix containing its eigenvalues. By definition of a modal matrix, the following holds.

Transposing that equation yields the equation[1]

(D.16)

and multiplying both members from the left and right by you get

Which demonstrates the intended result.

Nel caso in cui la matrice sia riconducibile alla forma di Jordan tramite the modal matrix applies instead and, transposing this equation and Multiplying both members from left and right by , we obtain

Note that the transformation characterized by the matrix leads to a transposed Jordan form, i.e., lower triangular.

Sulla base del precedente risultato, per determinare una matrice modale per una matrice in forma compagna trasposta (D.2) è possibile dapprima determinare la matrice modale della sua trasposta (che per quanto visto precedentemente assume la forma di una matrice di Vandermonde) per poi determinare l'inversa della sua trasposta.

[1]Equation (D.16) is interpreted by saying that is a left modal matrix for the matrice . Infatti la generica riga della matrice è un autovalore sinistro della matrice , that is, it satisfies the equation for an appropriate eigenvalue .

Example D.17 Consider the matrix

la cui trasposta vale

Le due matrici hanno autovalori , , (cfr. Esempio D.13) e ha matrice modale

So it has modal matrix

—

E

Linear independence of time functions

In this appendix, some definitions related to the linear independence of functions of time are first recalled. Next, two theorems useful in the analysis of controllability and observability are given.

Definition E.1. *Consider a set of scalar functions of real value.*

Such functions are said to be linearly independent *in the interval if and only if there exist real numbers , not all of which are null, such that*

Otherwise, such functions are said to be linearly independent *in* .

Example E.2 Consider the two functions and defined as.

fo*r*

fo*r*

per

These are linearly dependent in each interval with......................................In fact, If we assume , results for each .

They are also linearly dependent in each interval with................................In fact, If we assume , it results for each . On the contrary they are linearly independent in each interval with

e .

The example above clearly highlights that two or more functions can be linearly dependent in one interval but be linearly independent in a larger interval. Conversely, linear independence in a given interval implies linear independence in every other interval that contains it.

The concept of linear independence of scalar functions also extends to the case of vector functions.

Definition E.3. *Consider a set of real-valued vector functions.*

Such functions are said to be linearly independent *in the interval if and only if there exist real numbers , not all of which are null, such that*

Otherwise, they are said to be linearly independent *in .*

In other words, vector functions are linearly dependent in if and only if there exists a constant, nonzero vector

such that, defined

.

the matrix function having as -ma row the function , is worth

.

.

Example E.4 Be it

for , where...................................The two functions above are linearly dependent in if and only if there exists a nonzero vector such that

This equation is naturally verified if and only if and for each i.e. if and only if and , or equivalently if and only if .

Let us now look at two theorems that provide necessary and sufficient conditions for linear independence of vector functions. Note that only the first will be given as a proof. The second theorem, on the other hand, will be given without proof since this is beyond the scope of this discussion.

Theorem E.5. *Let , for , continuous real-valued vector functions defined ....inBoth the matrix having ---------------- comema line. We define*

(E.1)

Functions are linearly independent *in if and only if it is nonsingular.*

Demonstration. (Necessary condition.) Let us reason by absurdity. Assume that the functions are linearly independent in but that is sin- gular. This means that there exists a nonzero row vector such that
, which implies that either or even

(E.2)

Since the integrand is a nonnegative scalar function for each
, equation (E.2) implies that for each ....This with- betrays the assumption of linear independence of the so that if they are linearly independent in then .

(Sufficient condition.) Suppose it is nonsingular but the are linearly dependent ....inThen by definition there exists a vector nonzero and constant row such that for each Of consequently we have

That contradicts the assumption that it is nonsingular. Therefore, if it is n o n s i n g u l a r then the are linearly independent in ....

Let us now introduce a particular class of functions, called *analytic functions*, and then present a theorem related to them.

Definition E.6. *Let there be an open interval in the real axis and let there be a real-valued function defined in that interval.*

A function of real variable is called an element of class in if its -ma derivative exists and is continuous for every in .... is the class of Functions having derivatives of any order.

A function of real variable is said to be analytic *in if it is an element of and if for every there exists a positive real number such that, for every , is representable in Taylor series around the point , i.e.*

Theorem E.7. *Assume that , is an analytic function. in . Let the matrix having as -ma riga. Sia la -ma derivata di . Le , , sono linearmente independent in and of itself and only if*

$$| \quad | \quad | \quad | \tag{E.3}$$

For each .

F

Fourier series and integral

Harmonic analysis of a periodic signal, consists of its development into an infinite sum of elementary signals of sinusoidal form, called a *Fourier series*. The advantage of such decomposition in the study of linear systems is immediate: since the response of the system to each individual elementary signal can be determined easily, the total response is determined by summing the individual responses. Finally, this analysis can generalize to a broader class of signals, not necessarily periodic: in that case, the signal is described by means of the *Fourier integral*. All results presented in this appendix are given without demonstration.

F.1 Series of Fourier

F.1.1 Form exponential

Consider a signal defined for all values of and continuous at times[1] . Assume that this function is periodic in period , i.e., that it is worth

for *each*

and we define pulsation of that signal as the magnitude

—

Under these assumptions, it is possible to decompose the given signal into the following *Fourier series in exponential form*:

(F.1)

[1]This condition is sufficient but not strictly necessary for Fourier series decomposition. It can be replaced by more general regularity conditions.

Where the coefficients of the development are worth[2] , for ,

(F.2)

The Fourier coefficients given by (F.2) are complex scalars; the set of such coefficients is called the *spectrum* of the function. In fact, it should be pointed out that relation (F.1) is verified at all points where the signal is continuous is derivable. At the points of discontinuity the series converges to the mean value between left and right limits of the signal. The Fourier series (F.1) has the following interpretation: the functions (for) constitute a basis of infinite dimension for the period functions . Any periodic function can therefore always be represented by a linear combination of such basis functions with appropriate coefficients given by (F.2).

Example F.1 Consider the complex function of period defined *by* by

if

if

This function has pulsationIts Fourier coefficients are worth

per .

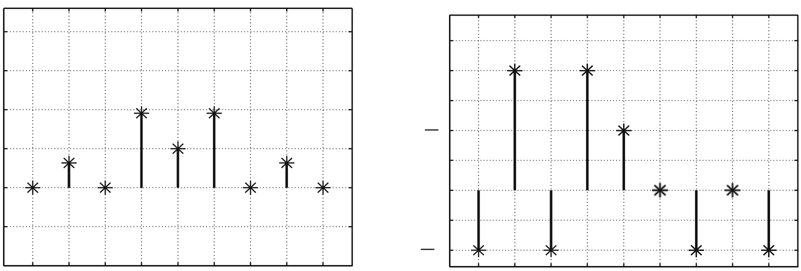


Fig. F.1. Spectrum of the amplitudes and phases of the function in Example F.1

[2]Note that although in eq. (F.2) integration and , have been chosen as extremes, it is possible to calculate Fourier coefficients by integrating at any period .

The amplitude and phase spectrum of that function, i.e., the moduli and phases of its Fourier coefficients, are shown in Fig. F.1 for values of between and...........

F.1.2 Form trigonometric

Where the signal is a real function the preceding development can be traced to a more intuitive form. Note, first, that in that case the Fourier coefficients associated with the integers e are complex and conjugate, viz.

In that case it is sufficient to calculate the coefficients only for values of and it is possible to rewrite the series (F.1) as follows:

This is equivalent to saying that a real periodic signal can be decomposed into a *Fourier series in trigonometric form*:

(F.3)

Where the coefficients of the development are worth

for (F.4)

for

These coefficients are real and are related to the coefficients of the development in exponential form by the simple relations :

e for

Note that can be thought of as the coefficient associated with the function which represents the constant function that holds over the entire rea- le axis: it represents the average value assumed by the signal. The Fourier series (F.3) has the following interpretation: the constant function and the e-functions form a basis of infinite dimension for real functions of period . The pulsation component is called the *fundamental harmonic* of the signal, while a component whose pulsation holds is called the *-ma harmonic*.

The following properties are easily demonstrated.

- If it is an even function, that is, if for every value of , then the coefficients of exponential development are real numbers; this implies that in trigonometric development the coefficients are zero.
- Conversely, if it is an odd function, that is, if for every value of , then the coefficient is zero and the remaining coefficients are imaginary numbers; this implies that in trigonometric development the coefficients for Are null.

Finally, an entirely equivalent development to that given in eq. (F.3) is as follows.

(F.5)

Where the coefficients of the development are worth, for ,

Example F.2 Consider the square wave-shaped signal, of period and parame- tro , defined by

if

if

if

shown in Fig. F.2. Recalling that according to (F.2) the following applies.

its Fourier coefficients of exponential development are worth

for

As expected, since this function is even all the coefficients of the exponen- tial development are real. In that case, the coefficients of the trigonometric development are worth

per

The spectrum of such a function is represented in Fig. F.2. Since the Fourier coefficients in this particular case are all real, there is no need to represent the spectrum of amplitudes and phases separately.

Note, finally, that since the given signal is real it is worth : the spectrum is therefore symmetrical with respect to the x-axis .

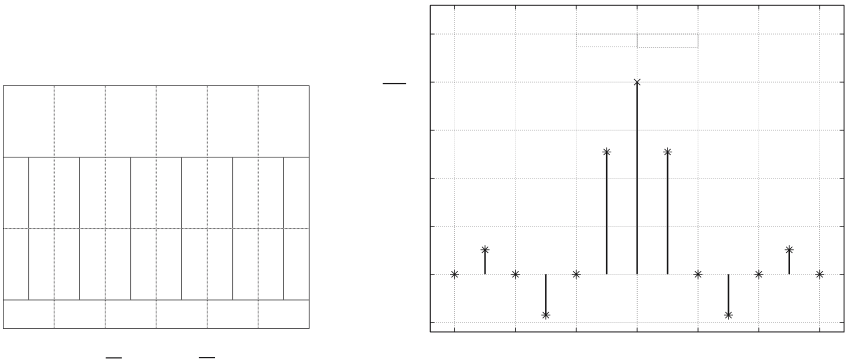


Fig. F.2. Square wave studied in Example F.2 and its spectrum

F2 Integral and transform of Fourier transform

F2.1 Form exponential

The procedure previously described for decomposing a signal into Fourier se- riors can be applied only to periodic signals. Consider, however, a signal that is not periodic but satisfies the condition of absolute summability

It can be regarded as the limiting case of a periodic signal in which the period tends to and the fundamental pulsation tends to .

In that case place , and said

the increment between two successive pulses, the integral (F.1) can be rewritten as follows

— —

Where in addition to the substitution also placed

— — —

e

–

–

Considering the limit for , the variable becomes continuous, and its increment becomes an infinitesimal ....From the above expressions we obtain
The decomposition of the signal into a *Fourier integral in exponential form*:

— (F.6)

Where the function continues,

(F.7)

is called the *Fourier transform* of the given signal.

Eq. (F.6) has the following interpretation: a nonperiodic function can be decomposed into a linear combination of an infinite and continuous set of basis functions ....forThe coefficients of such a combination are given by the function
continuous function that is also called the *spectrum* of the function in analogy with what we have seen for periodic functions.

Example F.3 Consider a finite pulse defined by

se

otherwise

shown in Fig. F.3. This signal is absolutely summable (the area under the function is finite). Its transform is worth

and *for*

— —

such a function is represented in Fig. F.2. Since in this case the spectrum is real, there is no need to represent the spectrum of amplitudes and phases separately, as would be necessary in the more general case .

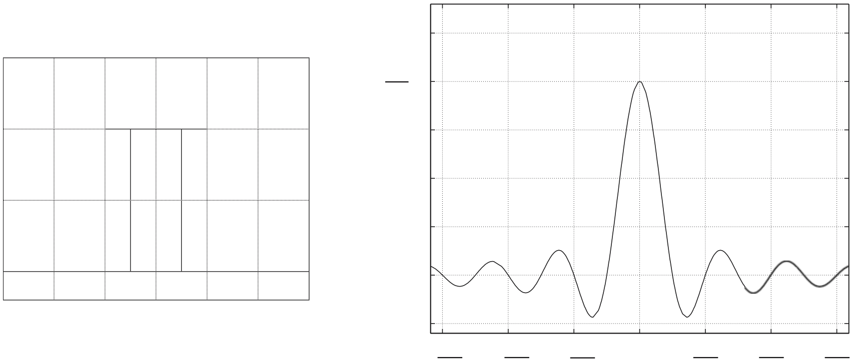


Fig. F.3. Finite pulse studied in Example F.3 and its spectrum

F.2.2 Form trigonometric

Where the signal is a real function the Fourier integral can be reduced to a more intuitive form. Note, first, that in such a case the function
enjoys the following property

that is, it takes complex conjugate values for and .
In that case, the integral (F.6) can be rewritten

which provides, in a manner analogous to that seen for the Fourier series, a *Fourier integral in trigonometric form*

(F.8)

Where the coefficients of the development are the real continuous functions

(F.9)

related to the Fourier transform by the simple relations:

e

Finally, an entirely equivalent development to that given in eq. (F.8) is as follows.

– (F.10)

wher
e

Example F.4 The Fourier integral of the function studied in Example F.3 in trigonometric form has coefficients:

— e

or

— e

The function is identically zero because the signal is an even function.

F3 Relationship between Fourier transform and Laplace transform

In the previous section, it was preferred to introduce the Fourier transform as the limiting case of the Fourier series. This is in order to make clear the interpretation of the transform as a harmonic component of the signal relative to a given frequency
.

One could have defined the transform directly by means of eq. (F.7) in a similar ma- ner to what was done in Chapter 6 where the Laplace transform was defined[3]

(F.11)

In such a case, the integral (F.6) takes on the meaning of the Fourier antitransform, that is, it allows the determination of a signal whose spectrum is known .

Comparing the two different transformation operators according to Laplace and according to Fourier defined by eq. (F.11) and eq. (F.7), respectively, the following differences are noted.

[3]In this section we denote the Laplace transform to distinguish it from the Fourier transform.

The core of the Laplace transform is worth with , while that of the Fourier transform is worth with . So it is a complex function of the complex variable , while it is a complex function of the real variable .
The Fourier transform requires that the function to be transformed be absolutely summable. This restriction is not necessary for the Laplace transform, which exists as long as the integral (F.11) converges to a subset of the com- plex plane called the convergence region (which is the case for most signals of interest).
The Laplace transform requires the function to be transformed to be null for
This restriction is not necessary for the Fourier transform.

Consider now a signal that can be transformed according to Laplace. There are two cases of interest.

Case A: The signal is absolutely summable. In this case, the Fourier transform of the signal can also be calculated and is worth , i.e., the Fourier transform coincides with the restriction of the Laplace transform to the imma- ginary axis onlyNote that if the signal is absolutely summable, the axis
imaginary belongs to the convergence region of the Laplace transform.

Example F.5 Consider a finite pulse defined by

se

otherwis*e*

Such a signal is absolutely summable, and its Fourier transform is worth

and *for*

On the other hand, such a signal can be thought of as the sum of an amplitude step and a step of amplitude shifted to the right of , i.e.

Æ Æ

and its Laplace transform is worth

Comparing the two transforms, it is verified as expected that .

Case B: The signal is not summable at all. In that case it is not tra- sformable according to Fourier, and the imaginary axis does not belong to the region of convergence of the Laplace transform. It is always possible to evaluate the function
, i.e., the analytical extension of the Laplace transform along the imagi- nary axis, excluding at most any null real-part poles of the However such function does not have the meaning of signal spectrum .

Example F.6 Consider the signal Æ that coincides with the unit step. Such a signal, not being absolutely summable, is not Fourier-transformable. In fact, applying (F.7) we obtain.

and such a limit does not exist. On the other hand, the signal has Laplace transform

– and it's worth

The function is defined for each value of but does not represent the spectrum of the signal .

G

Cayley-Hamilton theorem and calculation of matrix functions

G.1 Theorem of Cayley-Hamilton

The following important result, which is named the Cayley-Hamilton theorem[1] , defines the concept of a polynomial function of a square matrix and states that a matrix is root of its own characteristic polynomial.

Theorem G.1. *Given a square matrix of order , let be*

its characteristic polynomial. The matrix is root of its own characteristic polynomial*, that is, it satisfies the equation*

where it is a square matrix of order whose elements are all worth zero.

Demonstration. For simplicity we will prove the theorem only under the assumption that the ma- trice has distinct eigenvalues : in that case it is always possible to associate linearly independent eigenvectors with them (see Appendix C, Theorem C.64). In the case where the matrix has eigenvalues with multiplicity greater than one, an even stronger result applies, as shown in Theorem G.3.

Recall that the characteristic polynomial of has eigenvalues as roots and therefore can also be written in the form

Substituting the matrix yields the matrix polynomial

[1] Arthur Cayley (1821-1895, England), William Rowan Hamilton (1805-1865, Ireland).

Where it is important to note that the various factors commute with each other.

Let us now consider the matrix product for a generic eigenvector . Recalling relation (C.7), which must be satisfied by each eigenvalue and corresponding eigenvector and states that , it is worth

Since the product cancels for every , and the vectors cost- ures a basis of we can say that the matrix is identically null.

Esempio G.2 La matrice del secondo ordine ha polinomio caratteristic . Since it is verified that it is worth

G.2 Cayley-Hamilton theorem and polynomial minimum

In the case of matrices with eigenvalues with nonunitary multiplicity, in Chapter 4 (cf. § 4.7.1) the concept of a minimal polynomial was introduced, which is generally a factor of the characteristic polynomial. The Cayley-Hamilton theorem in its strongest version can be stated for the minimal polynomial.

Theorem G.3. *Given a square matrix of order , let be*

its minimum polynomial in which denotes the index of the eigenvalue . The matrix satisfies the equation

Demonstration. In Chapter 4 (see § 4.6) we saw that given a square matrix it is always possible to determine a basis consisting of linearly independent generalized eigenvectors. In particular, if it is an eigenvector of multiplicity to that basis will belong generalized eigenvectors (for). Such eigenvectors will construct in chains and, if the eigenvalue has index , each chain has length less than or equal to : this implies that each of the vectors is a generalized eigenvector of order less than or equal to and therefore satisfies the equation

According to Definition 4.36.

Consider now the matrix product for a generic generalized eigenvector in a manner analogous to that seen in the proof of Theorem G.1
vale

and since this product is null for each of the generalized eigenvectors that co-constitutes a basis for , we can say that the matrix is identically null .

Example G.4 The fourth-order matrix

(as verified by inspection, the matrix being in Jordan form) has two distinct autovalues: of multiplicity and index , and of multiplicity and indexHis minimum polynomial therefore holds:

Poiché

e

it is easily verified that it is worth

G.3 Analytical functions of a matrix

In the previous sections, the concept of a qua- drata matrix polynomial was defined. More generally, consider an analytic scalar function[2] in a region of the complex plane. Within such a region the function can be developed as a polynomial series and is worth

(G.1)

It is possible in such a case to extend this function to the field of square matrices by defining an equivalent matrix function through the series

(G.2)

Examples of analytical functions are polynomial functions, the exponentiation function defined in Appendix C (see § C.2.6), trigonometric functions, and , the inverse function , etc.

The following proposition presents a simple technique, based on the Cayley-Hamilton Theorem, to determine without resorting to series calculation.

Proposition G.5 *Given an analytic function , for every square matrix of order there exist scalars such that:*

(G.3)

If the matrix has distinct eigenvalues , the unknown coefficients are determined by solving the system

(G.4)

in which each eigenvalue corresponds to an equation.

If the matrix has eigenvalues of nonunitary multiplicity, the unknown coefficients are determined by solving a system of equations in which each eigenvalue of multiplicity corresponds to the following equations:

---

[2]The formal definition of analyticity for a function of complex variable i s not given in this text. However, compare the definition of real analytic function given in Appendix E (see Definition E.6).

(G.5)

Demonstration. It is first shown that the matrix can be parameterized in the form given in eq. (G.3). Next we show how to compute the unknown parameters.

Parameterization

Let the characteristic polynomial of the matrix , of degree . By the assumptions made, it is always possible to rewrite (G.1) as

(G.6)

where is the quotient polynomial between and , while

is the polynomial remainder that has degree less than or equal to .
Similarly, (G.2) also has a similar decomposition

Recalling the Cayley-Hamilton Theorem, which states that the matrix is radi- ce of its own characteristic polynomial, that is, it satisfies the equation , we can finally pose:

The problem of determining the function was thus reduced to the calculation of the coefficients , i.e., the polynomial remainder .

Determination of the polynomial remainder

The polynomial remainder can be determined by the classical long division algorithm. However, this technique is not always smooth, especially if the series (G.1) has infinite terms. There is a simpler technique, which takes advantage of the definition of a characteristic polynomial.
Be an eigenvalue of the matrix, and consider eq. (G.6) calculated for .
Vale:

since , being by definition an eigenvalue is the root of the characteristic polynomial. This makes it possible to write for each eigenvalue an equation in the unknowns . If the matrix has distinct eigenvalues, we obtain the system in eq. (G.4), in which matrix of coefficients is called the Vandermonde matrix (see Chapter 4, § 4.2.2).

If it is an eigenvalue *o f* the multiplicity matrix it is root of multiplicity of the characteristic polynomial, which is equivalent to saying that it is root of polynomials:

Denoting for simplicity the derivative -ma with respect to of the polynomial , deriving times eq. (G.6) gives:

By evaluating these expressions in the terms in square brackets cancel out and the system of equations takes the form given in eq. (G.5)

The first example is for the case of a matrix with eigenvalues of unit multiplicity.

Esempio G.6 Si vuole determinare per la matrice che ha

two distinct eigenvalues and .

The matrix has order two and is therefore valid where the unknown coefficients are determined by solving

from which we derive

So

The second example considers the case of a matrix with eigenvalues of nonunitary multiplicity.

Example G.7 You want to determine for the matrix which has

characteristic polynomial and therefore has eigenvalue *o f* multiplicity and multiplicity .

La matrice ha ordine tre e vale dunque Nel sistema di equazioni che consente di determinare i coefficienti incogniti all'autovalore compete two equations; the first is

while the second is obtained by deriving the first with respect to and is worth

Therefore, the system can be written

da cui si ricava

Therefore, the following is obtained

We conclude with three observations.

1. The same technique used to determine the constant matrix also makes it possible to determine the matrix function of the variableCompare a this regard Example 4.9 where it is calculated for the same matrix considerated in Example 4.9. In that case the scalar coefficients are functions of the real variable .

2. Note that Sylvester's development presented in Chapter 4 (see Proposition 4.7) is a special case of Proposition G.5.
3. Note that Proposition G.5 also applies to the computation of matrix powers and polynomials. For example, given a matrix of order there are coefficients such that the power with can always be rewritten in the form

as a linear combination of the matrices , , ,..............................More generally, if is any polynomial of degree there are coefficients such that

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