Lecture 7 – Optimality conditions for constrained optimization

Solve problems on the form

$$\min f(x)$$

subject to the constraints

$$g_i(x) \geqslant 0, \quad i = 1, .., m$$

$$\Omega = \{x \mid x \in \mathbb{R}^n,\ g_i(x) \geqslant 0 \ \forall i = 1, .., m\}$$

if we have equality constraints $g_i(x) = 0$, we can express them as

$$\begin{cases} g_i(x) \geqslant 0 \\ g_i(x) \leqslant 0 \end{cases} \Rightarrow \begin{cases} g_i(x) \geqslant 0 \\ -g_i(x) \geqslant 0 \end{cases}$$

Theorems of alternatives

Only one system can have solutions, and one of the systems has to have a solution

Gordons Theorem

m vectors $a_1, .., a_m \in \mathbb{R}^n$, the two systems

I) $a_i^T x > 0, \ i = 1, ..., m$ II) $\sum_{i=1}^{m} a_i y_i = 0, \ y_i \geqslant 0 \ \forall i$
with some $y_i > 0$

are the alternatives.

Geometric interpretation:

I) There exists a vector x, that forms an acute angle with all the vectors a_i

II) We can reach the origin with a linear combination of all the vectors a_i, where each a_i are scaled by some non-negative scalar, and at least one of the scalars has to be strictly positive.

Proof:

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$$

I) $Ax > 0$ II) $A^T y = 0$, $y \geq 0$, $y \neq 0$

Assume that both are feasible, there exists $\bar{x}, \bar{y}$ such that

$$0 < \underbrace{\bar{y}^T}_{\substack{\geq 0 \\ \neq 0}} \underbrace{A\bar{x}}_{>0} = \underbrace{(A^T\bar{y})^T}_{=0} \bar{x} = 0, \quad 0 < 0, \text{ Contradiction}$$

Forkhas Theorem

$m+1$ vectors in $\mathbb{R}^n$ $a_1, \ldots, a_m, g$

$$\text{I} \left\langle \begin{array}{ll} g^T d < 0 & \\ a_i^T d \geq 0 & \forall i = 1, \ldots, m \end{array} \right\rangle \quad \text{II} \left\langle \begin{array}{l} g = \sum_{i=1}^{m} \lambda_i a_i \\ \lambda_i \geq 0 \ , \ i = 1, \ldots, m \end{array} \right\rangle$$

Geometric interpretation:

I) Find a vector d that makes an obtuse angle with g and that makes an acute or normal angle with all the vectors a_i

II) The vector g can be written as a linear combination of all the vectors a_i with a non-negative scalar λ_i.

Proof

Assume both are feasible

We have $\bar{d}$ and $\bar{\lambda}_i$, $i = 1, \ldots, m$ as solutions

$$0 > g^T \bar{d} = \left(\sum_{i=1}^{m} \bar{\lambda}_i a_i \right)^T \bar{d} = \sum_{i=1}^{m} \underbrace{\bar{\lambda}_i}_{\geq 0} \underbrace{(a_i^T \bar{d})}_{\geq} \geq 0$$

$0 > 0$, Contradiction

KKT and linearization

Linear equality constraints