Structural Properties

Reachability - DT.

Describes whether it is possible to drive a system's state from the origin (or an initial state) to a desired state using a sequence of control inputs over a finite number of steps.

$$x_{k+1} = Ax_k + Bu_k$$

The system is reachable if there exists a finite sequence of inputs $\{u_0, u_1, .., u_{T-1}\}$ that can drive the state from the zero state $x_0 = 0$ to a desired state x_T in exactly T steps.

$$x_T = A^T x_0 + \sum_{j=0}^{T-1} A^{T-j-1} B u_j$$

Starting from $x_0 = 0 \Rightarrow A^T x_0 = 0$

Reachable subspace X^r

Set of all states that can be reached from the origin using some input sequence is called the reachable subspace X^r

$$X^r = \text{span}\{B, AB, A^2B, \dots, A^{n-1}B\}$$

The subspace X_i^r satisfy the chain of inclusion

$$X_1^r \subseteq X_2^r \subseteq \cdots \subseteq X_k^r \subseteq \cdots \subseteq X_n^r = X_{n+i}^r, \quad \forall i = 1, \ldots, \infty$$

Reachability matrix

$$R = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

System is completely reachable if

$$\text{rank}(R) = n$$

where n is the dimension of the state vector

Controllability vs. Reachability

A system is controllable if it is possible to drive the system from any initial state x_0 to the origin using a finite sequence of inputs.

X^c the subspace of controllable states

Reachability in T-step

$$x_T = A^T x_0 + \sum_{j=0}^{T-1} A^{T-j-1} B u_j$$

$$R_T = [B \quad AB \quad \cdots \quad A^{T-1}B]$$

Reachability from $x_0 \neq 0$

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j$$

state x_k can be reached from a non-zero x_0 if Controllability matrix is full rank an A is invertible

called transferability:
starting from x_0 move the system to x in T steps
$T \geq n$ with minimal energy

$$x_T = A^T x_0 + R_T \begin{bmatrix} u_0 \\ \vdots \\ u_{T-1} \end{bmatrix} \qquad T \geq n \quad \operatorname{rank} R_n = n$$

Reachability - C.T.

$$\dot{x}(t) = Ax(t) + Bu(t)$$

state $x(t)$ is said reachable from the origin if there exists
a finite input sequence $u(t)$, $t \in [0, T]$ s.t.:

$$x(T) = \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau$$

in continous time

$$\text{Reachability} = \text{Controllability}$$

Controllability Gramian

a tool for determining controllability over a continuous interval

$$W_c(t) = \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)}\, d\tau$$

a system is controllable if $W_c(t)$ is positive definite for some $t > 0$.

$W_c(t)$ positive definite: all directions in the state space can be influenced by the input.

Numerical tool to check controllability without needing to compute eigenvectors directly.

Minimal Energy Transferability

finding the least effort control input to transfer the state from one point to another

minimize energy of the control input:

$$J(u) = \int_0^T ||u(t)||^2 dt$$

optimal input

$$\hat{u}(t) = B^T e^{A^T(t-\tau)} W_c^{-1}(t)\, x_f$$

where x_f is the desired final state

optimal trajectory is

$$\hat{x}(t) = e^{At} W_c^{-1}(t)\, x_f$$

Properties of Reachable LTI Systems

1) (A,B) reachable $\Rightarrow$ $(T^{-1}AT, T^{-1}B, CT, D)$ reachable

2) RCR1, RCR2 represent reachable systems

Canonical Reachability Representation

$T = [\mathcal{B}_r \mid \mathcal{B}_{\bar{r}}] \in \mathbb{R}^{n \times n}$ invertible

$\mathcal{B}_r$ basis for X^r

$\mathcal{B}_{\bar{r}}$ basis for $X^{\bar{r}}$

$x = Tz$

$$z = \left[\begin{array}{c} x_r \\ \hline x_{\bar{r}} \end{array}\right]$$

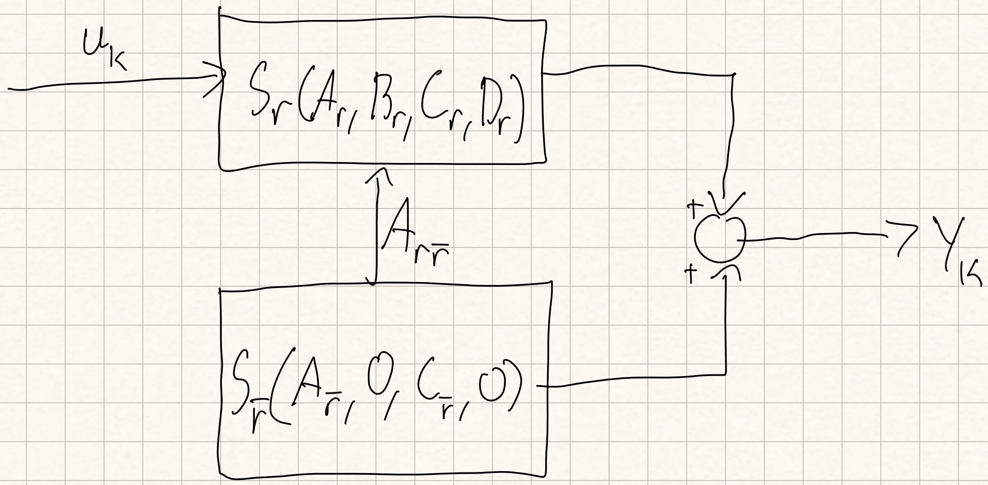
$$\begin{bmatrix} x_{r,k+1} \\ x_{\bar{r},k+1} \end{bmatrix} = \left[\begin{array}{c|c} A_r & A_{r\bar{r}} \\ \hline 0 & A_{\bar{r}} \end{array}\right] \begin{bmatrix} x_{r,k} \\ x_{\bar{r},k} \end{bmatrix} + \left[\begin{array}{c} B_r \\ \hline 0 \end{array}\right] u_k$$

$$y_k = [C_r \mid C_{\bar{r}}]\, z_k + D_r u_k$$

Two subsystems:

$$x_{r,k+1} = A_r x_{r,k} + A_{r\bar{r}} x_{\bar{r},k} + B_r u_k$$
$$x_{\bar{r},k+1} = A_{\bar{r}} x_{\bar{r},k}$$

$$y_k = C_r x_{r,k} + C_{\bar{r}} x_{\bar{r},k} + D_r u_k$$



$$W_{yu}(\lambda) = C_r(\lambda I - A_r)^{-1} B_r + D_r$$

Observability - D.T.

determines whether the internal state of a system can be fully determined from knowledge of the output measurements over a finite number of steps.

$$x_{k+1} = A x_k + B u_k$$
$$y_k = C x_k + B u_k$$

Observability in T-step:

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i$$
$$y_k = C A^k x_0 + \sum_{i=0}^{k-1} C A^{k-i-1} B u_i + D u_k$$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

system is completely observable if $\mathcal{O}$ has full rank

$$\text{rank}(\mathcal{O}) = n$$

where n is the dimension of the state vector

Subspace of non-observability

$X^{\bar{o}} = \ker \mathcal{O}$

$x \in X^{\bar{o}} \Rightarrow x$ is not observable

$X^{\bar{o}} = \{0_x\} \Rightarrow \text{rank}\,\mathcal{O} = n \Rightarrow$ full observability

Properties:

1) $X^{\bar{o}} \subset \ker C \Rightarrow \forall x \in X^{\bar{o}} \Rightarrow Cx = 0$
2) $X^{\bar{o}}$ is A-invariant

Reconstructability

asks whether the current state can be reconstructed from past and current output measurements.

Observability $\Rightarrow$ reconstructability

Reconstructability + A full rank $\Rightarrow$ observability

Minimum squares error determination of the Initial State

estimate initial state using imperfect measurements with noise or when exact measurements of the output are not available

output after N steps

$$Y = \mathcal{O}_N x_0 + H_N U + V$$

Y stacked vector of measurements

U is stacked input sequence

$\mathcal{O}_N$ observability matrix

H_N matrix accounting for inputs

Minimize square error between measured and predicted:

$$J(x_0) = \| Y - \mathcal{O}_N x_0 \|^2$$

Optimal solution

$$\hat{x}_0 = (\mathcal{O}_N^T \mathcal{O}_N)^{-1} \mathcal{O}_N^T Y$$

System must be observable for $\mathcal{O}_N$ to be full rank and the inverse to exist.

Observability - C.T.

$$y(t) = Ce^{At}x$$

Same as for D.T.

$$X^{\bar{o}} = \ker \mathcal{O}, \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$X^{\bar{o}} = \{0_x\} \Rightarrow \operatorname{rank} \mathcal{O} = n \Rightarrow$ full observability

$X^{\bar{o}}$ does not depend on the time $[0, T]$

In C.T.:

Observability and Reconstructability same concept

Observability Gramian

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

System is observable over time interval $[0, T]$ if $W_o(t)$ positive definite

Properties of observable state-space realizations

$$(A, B, C, D) \underset{T}{\approx} (T^{-1}AT, T^{-1}B, CT, D)$$

(A, C) observable $\iff$ $(T^{-1}AT, CT)$ observable

Dual Systems

$$\underset{\text{primal}}{(A, B, C, D)} \overset{*}{\approx} \underset{\text{dual}}{(A^*, B^*, C^*, D^*)}$$

$$A^* = A^T, \quad B^* = C^T, \quad C^* = B^T, \quad D^* = D$$

$$\text{rank}\underbrace{[B \;\; AB \; \cdots \; A^{n-1}B]}_{R} = n \implies R^T = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix} = \begin{bmatrix} C^* \\ C^* A^* \\ \vdots \\ C^* (A^{n-1})^* \end{bmatrix} = \mathcal{O}^*$$

$$\text{rank } R^T = n \implies \text{rank } \mathcal{O}^* = n$$

Primal reachable $\iff$ Dual observable

Primal observable $\iff$ Dual reachable

Canonical Representation

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \end{bmatrix} = \left[\begin{array}{c|c} A_o & 0 \\ \hline A_{o\bar{o}} & A_{\bar{o}} \end{array}\right] \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \left[\begin{array}{c} B_o \\ \hline B_{\bar{o}} \end{array}\right] u$$

$$y = [C_o \mid 0] \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + D_o u$$

PBH test

For Controllability:

a system is controllable if $\forall \lambda$ eig(A):

$$\text{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$$

For Observability:

a system is observable if $\forall \lambda$ eig(A):

$$\text{rank}\begin{bmatrix}\lambda I - A^T & C^T\end{bmatrix} = n$$

The Kalman Canonical

$x = Tz$

$$z = \begin{bmatrix} x_{ro} \\ x_{r\bar{o}} \\ x_{\bar{r}o} \\ x_{\bar{r}\bar{o}} \end{bmatrix}, \quad T = \begin{bmatrix} \mathcal{B}_{ro} & \mathcal{B}_{r\bar{o}} & \mathcal{B}_{\bar{r}o} & \mathcal{B}_{\bar{r}\bar{o}} \end{bmatrix} \in \mathbb{R}^{n \times n} \text{ invertible}$$

$$\mathcal{B}_{r\bar{o}} = \mathcal{B}_r \cap \mathcal{B}_{\bar{o}} \qquad (X^r \cap X^{\bar{o}})$$

$$T^{-1}AT = \left[\begin{array}{cc|cc} A_{ro} & 0 & A_{13} & 0 \\ A_{21} & A_{r\bar{o}} & A_{23} & A_{24} \\ \hline 0 & 0 & A_{\bar{r}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{r}\bar{o}} \end{array}\right], \quad T^{-1}B = \begin{bmatrix} B_{ro} \\ B_{r\bar{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$C T = \begin{bmatrix} C_{ro} & 0 & C_{r\bar{o}} & 0 \end{bmatrix}, \qquad D_{ro}$$

$$W_{yu}(\lambda) = \begin{bmatrix} C_{ro} & 0 & C_{r\bar{o}} & 0 \end{bmatrix} \begin{bmatrix} \lambda I - A_{ro} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}^{-1} \begin{bmatrix} B_{ro} \\ B_{r\bar{o}} \\ 0 \\ 0 \end{bmatrix} + D_{ro}$$

$$= C_{ro}(\lambda I - A_{ro})^{-1} B_{ro} + D_{ro}$$

We say that (A, B, C, D) is a minimal realization when it is fully reachable and observable

For minimal realizations:

- $W_{yu}(\lambda)$ has dimension n
- Internal Stability $\iff$ External Stability