Matrix exponential e^{At}

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

Properties:

1) $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$

2) $e^{(A+B)t} = e^{At} e^{Bt}$, only when $AB = BA$, comutation

3) $[e^{At}]^{-1} = e^{-At}$

$$e^{At} e^{-At} = e^{(A-A)t} = I$$

4) $A e^{At} = e^{At} A$

5) $\frac{d}{dt}(e^{At}) = A e^{At}$

$$\frac{d}{dt}\left[I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^k t^k}{k!} \cdots \right]$$

$$= A + \frac{2A^2 t}{2!} + \cdots + \frac{kA^k t^{k-1}}{k!} \cdots$$

$$= A\left(I + At + \cdots + \frac{A^{k-1} t^{k-1}}{(k-1)!} \cdots \right)$$

$$= A e^{At}$$

6) (λ, v) eigenvalue, eigenvector pair of A, we have

$$Av = \lambda v$$

$$e^{At} v = e^{\lambda t} v$$

$$\left[I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^k t^k}{k!} + \cdots\right] v$$

$$\left[A^2 v = A\underbrace{Av}_{\lambda v} = A\lambda v = \lambda \underbrace{Av}_{\lambda v} = \lambda\lambda v = \lambda^2 v, \quad A^k v = \lambda^k v \right]$$

$$= v + \lambda v t + \frac{\lambda^2 v t}{2!} + \cdots + \frac{\lambda^k v t^k}{k!} + \cdots$$

$$= v\left[1 + \lambda t + \frac{\lambda^2 t}{2!} + \cdots + \frac{\lambda^k t^k}{k!} + \cdots\right] = v e^{\lambda t} = e^{\lambda t} v$$

Realization Problems

1) Given transfer function, find state-space realization that realizes it

2) Given sequence, find state-space realization that realizes it

- Controllable Canonical Form

Given

$$H(s) = \frac{N(s)}{D(s)} = \frac{\beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0} + d_0$$

the controllable canonical form is given by

$$A_{cont} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{n-1} \end{bmatrix}, \quad B_{cont} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_{cont} = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}, \quad D_{cont} = d_0$$

- Observable Canonical Form

Dual of controllable canonical form

$$A_{obs} = A_{cont}^T$$
$$B_{obs} = C_{cont}^T$$
$$C_{obs} = B_{cont}^T$$
$$D_{obs} = D_{cont}^T$$

For the same transfer function $H(s)$, the observable canonical form is given by

$$A_{obs} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2 \\ 0 & 0 & 1 & \cdots & 0 & -\alpha_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}, \quad B_{obs} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$$

$C_{obs} = [0 \ \ 0 \cdots 0 \ \ 1]$, $D_{obs} = d_0$

PBH test

Controllability:

a system is controllable if $\forall \lambda$ eig(A):

$$\text{rank}[\lambda I - A \quad B] = n$$

Observability:

a system is observable if $\forall \lambda$ eig(A):

$$\text{rank}[\lambda I - A^T \quad C^T] = n$$

Observers

$$\begin{cases} \dot{x} = Ax + Bu, & x_0 \\ y = Cx \end{cases}$$

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) & , \hat{x}_0 \\ \hat{y} = C\hat{x} \end{cases}$$

$$e = x - \hat{x}$$

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - [A\hat{x} + Bu - L(y - \hat{y})] \\ &= Ax + Bu - [A\hat{x} + Bu - L(Cx - C\hat{x})] \\ &= Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu} + LCx - LC\hat{x} \end{aligned}$$

$$= Ax - A\hat{x} + LCx - LC\hat{x}$$

$$= A(x - \hat{x}) + LC(x - \hat{x})$$

$$= (A + LC)(x - \hat{x})$$

$$= (A + LC)\,e$$

Lagrange Formula

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$e^{At} = e^{(T\Delta T^{-1})t} = Te^{\Delta t}T^{-1}$$

$$e^{(T\Delta T^{-1})t} = \sum_{k=0}^{\infty} \frac{(T\Delta T^{-1})^k t^k}{k!}$$

$$\left[(T\Delta T^{-1})^k = (T\Delta \cancel{T^{-1}})(\cancel{T}\Delta \cancel{T^{-1}}) \cdots (\cancel{T}\Delta T^{-1}) = T\Delta^k T^{-1}\right]$$

$$= \sum_{k=0}^{\infty} T\Delta^k T^{-1}\frac{t^k}{k!} = T\left(\sum_{k=0}^{\infty} \frac{\Delta^k t^k}{k!}\right)T^{-1} = Te^{\Delta t}T^{-1}$$

ARMA models

AR - auto-regressive: express the output as a function of previous outputs

$$y_k = \alpha_1 y_{k-1} + \alpha_2 y_{k-2} + \cdots + \alpha_n y_{k-n}$$

MA - moving average: express the output as a function of current and previous inputs

$$y_k = \beta_0 u_k + \beta_1 u_{k-1} + \cdots + \beta_m u_{k-m}$$

ARMA: combine these two

$$y_k = \alpha_1 y_{k-1} + \cdots + \alpha_n y_{k-n} + \beta_0 u_k + \cdots + \beta_m u_{k-m}$$

Transfer Functions

$$W_{xu}(s) = (sI - A)^{-1} B$$

$$W_{yu}(s) = C(sI - A)^{-1} B + D$$

$$x(t) = W_{xu}(s)\, u(t)$$

$$y(t) = W_{yu}(s)\, u(t)$$

$$s = \frac{d}{dt}$$

$$sI\,x(t) = s \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} s & & \\ & \ddots & \\ & & s \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

$$= Ax(t) + Bu(t)$$

$$sIx(t) = Ax(t) + Bu(t)$$

$$sIx(t) - Ax(t) = Bu(t)$$

$$(sI - A)x(t) = Bu(t)$$

$$x(t) = (sI - A)^{-1}Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

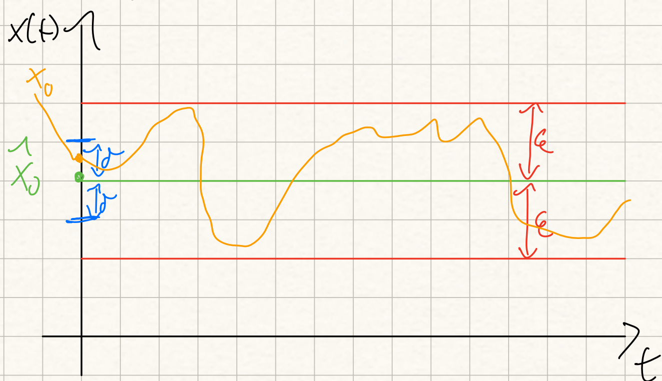
$$= C\left[(sI - A)^{-1}Bu(t)\right] + Du(t)$$

$$= C(sI - A)^{-1}Bu(t) + Du(t)$$

Local Stability

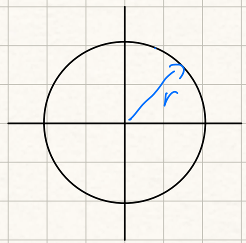
$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t.:

$$\|\hat{x}_0 - x_0\| < \delta \implies \|\hat{x}(t) - x(t)\| < \varepsilon, \ \forall t \geq 0$$



Lyapunov Criteria

Remember: $B_r = \{x \mid \|x\| < r\}$



$V(x) > 0 \quad \forall x \in \mathcal{B}_r / \{0_x\} \Rightarrow V(0_x) = 0$

For local stability:

Eq. point 0_x is locally stable if there exists a Lyapunov function $V(x) \in C^1$ s.t.:

1) $V(x) > 0 \quad \forall x \in \mathcal{B}_r / \{0_x\}$, $V(0_x) = 0$
2) $\dot{V}(x) \leq 0 \quad \forall x \in \mathcal{B}_r / \{0_x\}$

if 2) is replaced by:

2*) $\dot{V}(x) < 0 \quad \forall x \in \mathcal{B}_r / \{0_x\}$

then eq. point 0_x is locally asymptotically stable

Krasowskis Criterium for local stability

Add another condition to the Lyapunov criteria:

3) The set $\mathcal{V} = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ contains no complete trajectories other than 0_x

Lyapunov global:

Add

3) $V(x)$ is radially unbounded: $\lim_{\|x\| \to \infty} V(x) = \infty$

Reduced Lyapunov Criteria:

Need only look at eigs of A

1) if all eigs asymptotically stable:

   C.T.: $Re(\lambda) < 0$

   D.T.: $|\lambda| < 1$

   0_x is locally asymptotically stable

2) if at least one eig is unstable:

   C.T.: $Re(\lambda) > 0$

   D.T.: $|\lambda| > 1$

   0_x is unstable

3) in other cases cannot conclude anything