Representations of Dynamical Systems

System Representations

- Input/Output representation

  Consider relationship between input and output

  Black box view

$$y(t) = g(u(t), t) \qquad C.T.$$
$$y_k = g(u_k, k) \qquad D.T.$$

- Input/State/Output representation

  Including the systems internal states

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), t) \\ y(t) = h(x(t), u(t), t) \end{cases} \qquad C.T.$$

$$\begin{cases} x_{k+1} = f(x_k, u_k, k) \\ y_k = h(x_k, u_k, k) \end{cases} \qquad D.T.$$

Global: describes behavior accross full stat-space operating range. Used for linear systems

Local: valid around a certain operating point, approximating nonlinear systems with linearization

Properties

- Causality

  The output at time t depends only on inputs from time t and earlier.

- Time-invariance

  Behavior doesn't depend on time

  Shift in input $u(t) \rightarrow u(t+t_0)$ results in a shifted output $y(t) \rightarrow y(t+t_0)$

- Linearity

  Has to satisfy two principles

  1) Superposition

  $$f(u_1 + u_2) = f(u_1) + f(u_2)$$

  2) Homogeneity/Scaling

  $$f(\alpha u) = \alpha f(u) \quad \text{for any } \alpha$$

Shift in Basis and Similarity

- Shift in Basis

  Consider system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Choos a new basis defined by a non-singular matrix T, we have

$$z(t) = T^{-1} x(t)$$

The system in the new coordinate become

$$\begin{cases} \dot{z}(t) = T^{-1}AT\, z(t) + T^{-1}B\, u(t), & z(0) = T^{-1}x(0) \\ y(t) = CT z(t) + Du(t) \end{cases}$$

- Matrix Similarity

  A and A' is said similar if there exists a non-singular matrix T s.t.:

  $$A' = T^{-1}AT$$

  - Similar matrices represent the same linear transformations under different bases
  - Same eigenvalues, characteristic polynomial, determinant and trace

- Algebraic equivalence

  Two systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are algebraically equivalent if there exists an invertible matrix T s.t.:

  $$A_2 = TA_1T^{-1}$$
  $$B_2 = TB_1$$
  $$C_2 = C_1T^{-1}$$

$$D_2 = D_1$$

This means they represent the same input-output behavior but in different state coordinates

The matrix exponential e^{At}

Defined by power series

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

- Properties of e^{At}

1) $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$ (Linearity)
2) $e^{(A+B)t} = e^{At} e^{Bt}$ only when $AB = BA$
3) $[e^{At}]^{-1} = e^{-At}$
4) $\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$
5) (λ, v) pair of eigenvalue/eigenvector of A, then
$Av = \lambda v$
$e^{At} v = e^{\lambda t} v$

Lagrange formula

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$y(t) = Ce^{A(t-t_0)} x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t)$$

Jordan form

The Jordan form simplifies the matrix exponential for when A is not diagonalizable. A square can be decomposited into:

$$A = PJP^{-1}$$

where:

P is the matrix of generalized eigenvectors

J is the Jordan block matrix consisting of blocks J_i on the form:

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ & \lambda_i & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}$$

we can then express the matrix exponential as:

$$e^{At} = Pe^{Jt}P^{-1}$$

Matrix exponential computation

A diagonalizable ($\lambda_i \in \mathbb{R}$ and distinct)

$\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$

$v_1, v_2, \dots, v_n \in \mathbb{R}^n$

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ &\vdots \\ Av_n &= \lambda_n v_n \end{aligned}\right\} \Rightarrow A\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\Delta}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix}$$

$$AT = T\Delta$$

$$A = T\Delta T^{-1}, \quad \Delta = T^{-1}AT$$

$$e^{At} = e^{(T\Delta T^{-1})t} = Te^{\Delta t}T^{-1}$$

ARMA Models (Auto-Regressive Moving Average)

Describes an LTI system in terms of its previous outputs (auto-regression) and a moving average of the current and past inputs

AR:

$$y_k = \alpha_1 y_{k-1} + \alpha_2 y_{k-2} + \cdots + \alpha_n y_{k-n} = \sum_{i=1}^{n} \alpha_i y_{k-i}$$

MA:

$$y_k = \beta_0 u_k + \beta_1 u_{k-1} + \cdots + \beta_m u_{k-m} = \sum_{j=0}^{m} \beta_j u_{k-j}$$

ARMA:

$$y_k = \alpha_1 y_{k-1} + \cdots + \alpha_n y_{k-n} + \beta_0 u_k + \cdots + \beta_m u_{k-m}$$

$$= \sum_{i=1}^{n} \alpha_i y_{k-i} + \sum_{j=0}^{m} \beta_j u_{k-j}$$

Transfer functions

A frequency-domain representation of an LTI system.

$W_{xu}(s)$: input-to-state transfer matrix

$W_{yu}(s)$: input-to-output transfer matrix

$$W_{xu}(s) = (sI-A)^{-1}B$$

$$W_{yu}(s) = C(sI-A)^{-1}B + D$$

ARMA model of system becomes:

$$x(t) = W_{xu}(s)\,u(t) = (sI-A)^{-1}B\,u(t)$$

$$y(t) = W_{yu}(s)\,u(t) = \left[C(sI-A)^{-1}B + D\right]u(t)$$

More on ARMA models (where s comes from)

$s = \frac{d}{dt}$, $s^k = \frac{d^k}{dt^k}$, $k=-1$: integrate

$$(\alpha_1 + \alpha_2 s + \alpha_3 s^2)\,v(t) = \alpha_1 v(t) + \alpha_2 \dot{v}(t) + \alpha_3 \ddot{v}(t)$$

$$sI\,x(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \dot{x}(t) = Ax(t) + Bu(t)$$

$$\Rightarrow \quad (sI-A)\,x(t) = Bu(t) \quad \Rightarrow \quad x(t) = (sI-A)^{-1}Bu(t)$$

Laplace Transform

Tool used for analyzing C.T. LTI systems in the frequency domain. Transforms differential equations into algebraic equations, simplifying the analysis.

For a C.T. signal $f(t)$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

where $s = \sigma + j\omega$

Transfer functions and Laplace transform

Can derive the transfer function $H(s)$ using the Laplace transform

For the differential equation:

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = b_0 u(t)$$

Taking the Laplace transform:

$$s^2 Y(s) + a_1 s Y(s) + a_2 Y(s) = b_0 U(s)$$

Solving for $H(s) = \dfrac{Y(s)}{U(s)}$:

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_2}$$

Impulse response

Characterizes how a system reacts to a single, instantaneous input - a unit impulse.

For C.T., the impulse response $h(t)$ is the output when the input is the Dirac delta function:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\delta(t) \\ y(t) = Cx(t) \end{cases}$$

For D.T., same but input is called Kronecker delta:

$$\begin{cases} x_{k+1} = Ax_k + B\delta_k \\ y_k = Cx_k \end{cases}$$

Impulse response fully describes a system's behavior. If you know the impulse response, you can find the output for any input using convolution.

Convolution

C.T.:

$$y(t) = \int_0^\infty h(\tau)\, u(t-\tau)\, d\tau$$

D.T.:

$$y_k = \sum_{n=0}^\infty h_n u_{k-n}$$

Z-transform

D.T. equivalent of Laplace transform.

For a sequence $\{x_k\}$

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k} \quad , \ z \text{ complex variable}$$

$$z u_k = u_{k+1} \ , \quad z^2 u_k = u_{k+2} \ , \quad z^{\tau} u_k = u_{k+\tau}$$

$$u_k z^{-1} = u_{k-1} \ , \quad u_k z^{-\tau} = u_{k-\tau}$$

$$(1 + 5z + 6z^2) u_k = u_k + 5u_{k+1} + 6u_{k+2}$$

$$zI x_k = Ax_k + Bu_k$$

$$\Rightarrow (zI - A) x_k = Bu_k$$

$$\Rightarrow x_k = (zI - A)^{-1} Bu_k$$

$$\begin{cases} x_k = W_{xu}(z) u_k \ , & W_{xu}(z) = (zI - A)^{-1} B \\ y_k = W_{yu}(z) u_k \ , & W_{yu}(z) = C(zI - A)^{-1} B + D \end{cases}$$

Causality condition for $W_{yu}(z)$

$W_{yu}(z)$ causal $\Leftrightarrow$ proper

Degree of numerator not higher than degree of denominator

$$\mathcal{Z}(x_{k+1}) = \mathcal{Z}(Ax_k + Bu_k)$$

$$\mathcal{Z}(y_k) = \mathcal{Z}(Cx_k + Du_k)$$

Realization problems

Problem 1:

Given a transfer function $W_{yu}(\lambda)$, $\lambda \in \{s, z\}$, find a state-space realization (A, B, C, D) that realizes it

$$W_{yu}(\lambda) = C(\lambda I - A)^{-1} B + D$$

Problem 2:

Given a sequence $\{w_0, w_1, \dots, w_k\}_{k=0}^{\infty}$, $w_i \in \mathbb{R}\ \forall i=k$ find a state-space realization (A, B, C, D) that realizes it

$$w_k = \begin{cases} D, & k=0 \\ CA^{k-1}B, & k \geq 1 \end{cases}$$

- Canonical realizations

  Specific ways to structure the matrices A, B, C, D to emphasize different structural properties of the system.

  - RCR 1 (Controllable Canonical Form from Transfer function)

    Built directly from the coefficients of the transfer function's denominator.

    State matrix A is structured with coefficients from the characteristic polynomial

- RCO1 (Observable Canonical form from Transfer function)
  Built from the coefficients of the transfer function but emphasizes observability instead.
  C matrix captures the output structure clearly
- RCR2 (Controllable Canonical Form from Impulse response)
  Built using the Markov parameters (impulse response sequence)
  Emphasizes controllability structure, focusing on how inputs can drive states,
- RCO2 (Observable Canonical Form from Impulse response)
  Also derived from Markov parameters but emphasizes how states affect outputs.

$$\underset{*SS}{RCR1} \underset{T}{\approx} \underset{*SS}{RCR2}$$

$$RCO1 \underset{T}{\approx} RCO2$$

$$(A, B, C, D) \underset{T}{\approx} (T^{-1}AT,\ T^{-1}B,\ CT,\ D)$$

$$(A, B, C, D) \underset{*}{\approx} (A^T, C^T, B^T, D^T)$$

- Cayley-Hamilton Theorem
  A square matrix satisfies its own characteristic polynomial.

If A is an $n \times n$ matrix and its characteristic polynomial is:

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then:

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

Helps in simplifying the computation of matrix exponentials in state-space solutions.

Free Response C.T. LTI Systems

Describes the natural behavior when there is no external input.

$$\dot{x}(t) = Ax(t) \quad , \quad x(t) = e^{At}x_0$$
$$y(t) = Cx(t)$$

- Eigenvalues and modes:

  e^{At} can be computed using the eigenvalues and eigenvectors of A. The modes of the system correspond to the eigenmodes, and their behavior is determined by the eigenvalues.
  - Eigenvalues determine growth or decay of modes
  - Eigenvectors define the shape of the modes.

1) All eigenvalues real and distinct

   $\lambda_1, \dots, \lambda_n$ real and distinct

   $v_1, \dots, v_n$ linearly independent

A can be written as

$$A = T \Delta T^{-1}$$

where

T: matrix of eigenvectors

Δ: $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Solution:

$$x(t) = T e^{\Delta t} T^{-1} x_0$$

Since Δ is diagonal:

$$e^{\Delta t} = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$$

Expanded form:

$$x(t) = \sum_{i=1}^{n} c_i v_i e^{\lambda_i t}$$

where

v_i are the eigenvectors

c_i are constants determined by the initial condition

2) Distinct complex-conjugate eigenvalues

Appear in pairs $\lambda_{1,2} = \sigma \pm j\omega$

A is not diagonalizable with real numbers, but can be handeled using real-valued decomposition.

General form of solutions

$$x(t) = c_1 e^{\sigma t} \cos(\omega t) v_1 + c_2 e^{\sigma t} \sin(\omega t) v_2$$

where

σ determine the growth/decay rate

ω determine the frequency of oscillation

v_1, v_2 real-valued vectors derived from the complex eigenvectors

3) A not diagonalizable

There exists fewer eigenvectors than eigenvalues

A Jordan canonical form is used instead

A can be written as

$$A = PJP^{-1}$$

Solution:

Matrix exponential involves both exponentials and polynomials

$$x(t) = e^{\lambda t}(c_1 v + c_2 t v)$$

— Modes

	Diagonalizable	Defective
Real λ	$e^{\lambda t}$	$t^k e^{\lambda t}$ $k = 0, 1, \ldots, n_i - 1$ $n_i = am(\lambda_i)$
Complex conj. $\lambda = \sigma + j\omega$	$e^{\sigma t}\cos(\omega t)$ $e^{\sigma t}\sin(\omega t)$	$t^k e^{\sigma t}\cos(\omega t)$ $t^k e^{\sigma t}\sin(\omega t)$ $k = 0, 1, \ldots, n_i - t$

Convergence of modes

1) Convergent mode $m(t)$: if $\lim_{t \to \infty} m(t) = 0$

2) Bounded mode $m(t)$: $\exists M > 0$ s.t.

$$|m(t)| < M, \ \forall t$$

3) Divergent mode $m(t)$: not convergent or bounded.

Free Response D.T. LTI Systems

$$x_{k+1} = A x_k$$

$$x_k = A^k x_0$$

1) All eigenvalues real and distinct

Can diagonalize A the same way as for C.T.

$$A = T \Lambda T^{-1}$$

This time the solution becomes:

$$x_k = T \Lambda^k T^{-1} x_0$$

Since Λ diagonal:

$$\Lambda^k = \mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k)$$

2) Distinct Complex-Conjugate eigenvalues

Eigenvalues appear in pairs $\lambda_{1,2} = \sigma \pm j\omega$

Solution:

The complex eigenvalues can be written as:

$$\lambda = \rho e^{j\theta} = \rho \cos^{j\theta} + j\rho \sin^{j\theta}$$

where

$\rho = |\lambda|$ is the magnitude (growth/decay rate)

θ is the angle (frequency of oscillation)

The solution becomes:

$$x_k = \rho^k (c_1 \cos(k\theta) v_1 + c_2 \sin(k\theta) v_2)$$

3) A Not Diagonalizable

Use the Jordan canonical form

Solution $$x_k = \lambda^k (c_1 v + c_2 k v)$$

- Modes

	Diagonalizable	Defective
Real λ	λ^k	$k^i \lambda^k$ $i = 0, 1, \ldots, n_i - 1$
Complex conj. $\rho e^{\pm j\omega}$	$\rho^k \cos(\omega k)$ $\rho^k \sin(\omega k)$	$k^i \rho^k \cos(\omega k)$ $k^i \rho^k \sin(\omega k)$ $i = 0, 1, \ldots, n_i - 1$

$|\lambda| = \rho$

$|\lambda| < 1 \Rightarrow m_k$ convergent

$$|\lambda| = 1 \begin{cases} \text{boundedness if diagonalizable} \\ \text{divergence (weak) if defective} \end{cases}$$

$|\lambda| > 1 \Rightarrow$ divergence (strong)

Forced Response

Describes how a system responds to an external input

$$\begin{cases} x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\ y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau + D u(t) \end{cases}$$

$$\hat{y}(s) = C(sI-A)^{-1} x_0 + [C(sI-A)^{-1}B + D] \hat{u}(s)$$

$$y(t) = \mathcal{L}^{-1}(\hat{y}(s))$$

$$u(t) = u e^{\lambda t}, \quad \lambda \in \mathbb{R}/\mathbb{C}, \quad u \in \mathbb{R}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \frac{1}{2j} e^{j\omega t} + \frac{-1}{2j} e^{-j\omega t}$$

$$u(t) = u e^{\lambda t} \Rightarrow \hat{u}(s) = u \frac{1}{s-\lambda}$$

Techical Lemma if $\lambda \notin sp(A)$

$$(sI-A)^{-1} \frac{1}{s-\lambda} = (\lambda I - A)^{-1} \frac{1}{s-\lambda} - (sI-A)^{-1}(\lambda I - A)^{-1}$$

$$y(t) = C e^{At}(x_0 - \hat{x}_0) + W_{yu}(\lambda) u e^{\lambda t}$$

$$x_f(t) = \underbrace{-e^{At}\hat{x}_0}_{\text{Transient part}} + \underbrace{\mathcal{W}_{xu}(\lambda)\, u e^{\lambda t}}_{\text{Permanent part}}$$

$$y_f(t) = \underbrace{-Ce^{At}\hat{x}_0}_{\text{Transient part}} + \underbrace{\mathcal{W}_{yu}(\lambda)\, u e^{\lambda t}}_{\text{Permanent part}}$$

Step response

$$u(t) = ue^{0t} \quad (\lambda = 0)$$

$$x(t) = e^{At}(x_0 - \hat{x}_0) + \mathcal{W}_{xu}(0)\,u$$

$$y(t) = Ce^{At}(x_0 - \hat{x}_0) + \mathcal{W}_{yu}(0)\,u$$

$$\hat{x}_0 = \mathcal{W}_{xu}(0)\,u$$

$\mathcal{W}_{xu}(0) = \mathcal{W}_{xu}(s)\big|_{s=0}$ input-to-state DC gain

$\mathcal{W}_{yu}(0) = \mathcal{W}_{yu}(s)\big|_{s=0}$ input-to-output DC gain

For a stable system:

$$u(t) \equiv u, \ \forall t \geq 0 \implies \lim_{t\to\infty} y(t) = \mathcal{W}_{yu}(0)\,u$$

$$\lim_{t\to\infty} x(t) = \mathcal{W}_{xu}(0)\,u$$

Frequency Response

Describes how a system responds to a sinusoidal input.
Provides information about how different frequencies are amplified or attenuated by the system.

input $u(t) = e^{j\omega t}$

stedy-state output can be described using the transfer function

$$W_{yu}(s) = C(sI - A)^{-1}B + D$$

frequency response is obtained by just substituting

$$W_{yu}(j\omega) = C(j\omega I - A)^{-1}B + D$$

Magnitude response $|W_{yu}(j\omega)|$

Phase response $\angle W_{yu}(j\omega)$

$$\Rightarrow W_{yu}(j\omega) = |W_{yu}(j\omega)|\, e^{j\angle W_{yu}(j\omega)}$$