Lecture 5 - Algorithms for unconstrained problems

Descent Algorithms

Iterative methods where the objective function decreases at every iteration. General form:

$$x_{k+1} = x_k + \alpha_k d_k$$

where:

- d_k is the descent direction
- α_k is the step size

Descent direction

d_k is a descent direction if:

$$\nabla f(x_k)^T d_k < 0$$

Convergence rate

An algorithm works well if $\|x_{k+1} - x^*\|$ is much smaller than $\|x_k - x^*\|$. The distance between the next point and the optimal point is much smaller than the distance between the current point and the optimal point.

- Linear

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq C \quad , \quad C \in (0,1)$$

- Sublinear

$C = 1$ for the inequality above

- Superlinear

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0$$

- Quadratic

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} \leq C \quad , \quad C > 0$$

Descent direction

Have to choose a direction such that

$$\nabla f(x_k)^T d_k < 0$$

Usually two options:

1) $d_k = -\nabla f(x_k) \Rightarrow \nabla f(x_k)^T d_k = -\|\nabla f(x_k)\|^2 < 0$
2) $d_k = -A_k \nabla f(x_k) \Rightarrow \nabla f(x_k)^T d_k = -\nabla f(x_k)^T A_k \nabla f(x_k) < 0$

quadratic form, A symmetric and positive definite

Step size

Line search. Consider halfline $\{x \mid x = x_k + \alpha_k d_k,\ \alpha_k > 0\}$ and determine $\alpha_k > 0$ such that

$$f(x_k + \alpha_k d_k) < f(x_k)$$

Define a function $\phi(\alpha)$ the restriction of f along the half-line

$$\phi(\alpha) = f(x_k + \alpha d_k), \quad \phi(0) = f(x_k)$$

From derivation theorem of composite functions:

$$\phi'(\alpha) = \nabla f(x_k + \alpha d_k)^T d_k$$
$$\Rightarrow \phi'(0) = \nabla f(x_k)^T d_k < 0$$

Exact line search:

$$\alpha_k = \arg\min_{\alpha > 0} \phi(\alpha)$$