1 Fitting basics

The Time Tagger acquires clock time-tags y_i which are simply enumerated by an index x_i. The time-tags shall be fitted by a function

$$f(x) = ax + b \tag{1.1}$$

by least squares. For a set of N time-tags, we need to minimize

$$g(a,b) = \sum_{i=0}^{N-1} (y_i - f(x_i))^2 = \sum_{i=0}^{N-1} (y_i - ax_i - b)^2 \tag{1.2}$$

so the coefficients a and b need to fulfill the conditions

$$\partial_a g = -2 \sum_{i=0}^{N-1} (y_i - ax_i - b)\, x_i = 0 \tag{1.3}$$

$$\partial_b g = -2 \sum_{i=0}^{N-1} (y_i - ax_i - b) = 0. \tag{1.4}$$

Equation 1.4 directly results in

$$b = \frac{1}{N} \sum_{i=0}^{N-1} (y_i - ax_i) = \frac{S_y - aS_x}{N} \tag{1.5}$$

where we used the symbols

$$\begin{aligned} S_x &= \sum_{i=0}^{N-1} x_i \\ S_y &= \sum_{i=0}^{N-1} y_i \\ S_{xx} &= \sum_{i=0}^{N-1} x_i^2 \\ S_{xy} &= \sum_{i=0}^{N-1} x_i y_i. \end{aligned}$$

If we put 1.5 into 1.3, we obtain

$$S_{xy} - aS_{xx} - bS_x = S_{xy} - aS_{xx} - \frac{S_y - aS_x}{N} S_x = 0 \tag{1.6}$$

which results together with 1.5 in

$$a = \frac{S_x S_y - N S_{xy}}{S_x^2 - N S_{xx}} \tag{1.7}$$

$$b = \frac{S_{xy} S_x - S_y S_{xx}}{S_x^2 - N S_{xx}}. \tag{1.8}$$

2 Moving fit

The goal is now to use the offset b to correct the position of the current clock tag. For this purpose, the fit is performed over the last N time-tags. The current time-tag will always get the index $i = 0$ and will by definition be set to $x_0 \equiv 0$ and $y_0 \equiv 0$. The preceeding tag is set to $x_1 = -1$ and so on, so the fit will always be performed over the x-range $[0, -1, \ldots, -(N-1)]$ which results in

$$\begin{aligned} S_x &= \sum_{i=0}^{N-1} x_i = \sum_{i=0}^{N-1} -i = -\frac{N(N-1)}{2} && (2.1) \\ S_{xx} &= \sum_{i=0}^{N-1} x_i^2 = \sum_{i=0}^{N-1} (-i)^2 = \frac{(N-1)N(2N-1)}{6}. && (2.2) \end{aligned}$$

This simplifies expression 1.8

$$b = \frac{(2N-1)S_y - 3S_{xy}}{\sum_{i=1}^{N} i}. \qquad (2.3)$$

Because it is not a good idea to perform the calculation of S_y and S_{xy} for every clock tag from the set of N tags, we will calculate it incrementally. Here we need to distinguish two cases: At the beginning of the fit, when less than N tags already arrived, the set will grow with every new tag. If there are N tags in the set, the oldest one needs to be dropped when a new tag arrives.

To keep numbers small, only relative values are used and those are additionally reduced by the expected period T.

2.1 Growing set of tags

We start with a set of size N. When the tag $i = n + 1$ arrives, it will grow to a size of $N + 1$ and the value S_y will change (Δ_+ indicates the growing case) by:

$$\begin{aligned} \Delta_+ S_y^{(n+1)} &= \left(\sum_{i=0}^{N} y_{n+1-i} - y_{n+1} + iT\right) - \left(\sum_{i=0}^{N-1} y_{n-i} - y_n + iT\right) \\ &= -N\left(y_{n+1} - y_n - T\right) \\ &= -N d_{n+1} && (2.4) \end{aligned}$$

with the reduced step size $d_{n+1} = y_{n+1} - y_n - T$. Similarly we can calculate the increment of S_{xy}:

$$\Delta_+ S_{xy}^{(n+1)} = \left(\sum_{i=0}^{N} -i\left(y_{n+1-i} - y_{n+1} + iT\right)\right) - \left(\sum_{i=0}^{N-1} -i\left(y_{n-i} - y_n + iT\right)\right)$$

$$
\begin{aligned}
&= \left(\sum_{i=0}^{N-1} -y_{n-i}\right) + y_{n+1}\left(\sum_{i=0}^{N} i\right) - y_n\left(\sum_{i=0}^{N-1} i\right) - N^2 T \\
&= -\left(\sum_{i=0}^{N-1} y_{n-i} - y_n + iT\right) + y_{n+1}\left(\sum_{i=0}^{N} i\right) - y_n\left(\sum_{i=0}^{N} i\right) - \left(N^2 - \sum_{i=0}^{N-1} i\right) T \\
&= -S_y^{(n)} + (y_{n+1} - y_n)\left(\sum_{i=0}^{N} i\right) - \left(N^2 - \frac{N(N-1)}{2}\right) T \\
&= -S_y^{(n)} + (y_{n+1} - y_n)\left(\sum_{i=0}^{N} i\right) - \left(\frac{N(N+1)}{2}\right) T \\
&= -S_y^{(n)} + (y_{n+1} - y_n - T)\left(\sum_{i=0}^{N} i\right) \\
&= -S_y^{(n)} + d_{n+1}\left(\sum_{i=0}^{N} i\right) \qquad (2.5)
\end{aligned}
$$

2.2 Constant set size

If it is full

$$
\begin{aligned}
\Delta S_y^{(n+1)} &= \left(\sum_{i=0}^{N-1} y_{n+1-i} - y_{n+1} + iT\right) - \left(\sum_{i=0}^{N-1} y_{n-i} - y_n + iT\right) \\
&= \Delta_+ S_y^{(n+1)} - (y_{n+1-N} - y_{n+1} + NT) \\
&= -N d_{n+1} + D_{n+1} \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
\Delta S_{xy}^{(n+1)} &= \left(\sum_{i=0}^{N-1} -i\left(y_{n+1-i} - y_{n+1} + iT\right)\right) - \left(\sum_{i=0}^{N-1} -i\left(y_{n-i} - y_n + iT\right)\right) \\
&= \Delta_+ S_{xy}^{(n+1)} - \left(-N\left(y_{n+1-N} - y_{n+1} + NT\right)\right) \\
&= \Delta_+ S_{xy}^{(n+1)} - N D_{n+1} \qquad (2.7)
\end{aligned}
$$