Example Models in Stan

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What is Stan?

- Stan is a "probabilistic programming language". Most importantly, it helps us do Bayesian inference.
- We specify our model:

$$p(\theta, y) = p(\theta) \times p(y|\theta) \tag{1}$$

- where $p(\theta)$ is our prior on the model parameters θ and $p(y|\theta)$ is the likelihood of the data y given those parameters.
- Stan will give us samples of

$$p(\theta|y) = \frac{p(\theta) \times p(y|\theta)}{p(y)} \tag{2}$$

A very simple example: linear regression

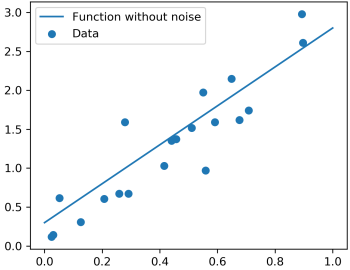
Model: Line with slope a, intercept b, and noise σ.

Prior $p(\theta)$:

$$p(a, b, \sigma) = p(a)p(b)p(\sigma)$$
$$a, b \sim \mathrm{Uniform}(-\infty, +\infty)$$
$$\sigma \sim \mathrm{Uniform}(0, +\infty)$$

Likelihood $p(y|\theta)$:

$$y|a, b, \sigma \sim \mathrm{Normal}(ax + b, \sigma^2)$$



Linear regression in Stan

```
data {
    int N; // number of data points
    vector[N] x; // Input
    vector[N] y; // Output
}
parameters {
    real a; // Slope
    real b; // Intercept
    real<lower=0> stdev; // Standard deviation of noise
}
model {
    y ~ normal(a * x + b, stdev);
}
```

Demo

Things to note:

- The $\hat{R}$ value gives us an indication of whether the model converged
- We can compute 95% credible intervals easily from the posterior samples
- We get samples from the true posterior, not an approximation
- We don't really need Stan for this – this is one of the few examples where we can compute $p(\theta|y)$ analytically
- Stan automatically puts flat priors (that is, uniform mass on the domain) on variables if none are specified.

Hierarchical models – 8 Schools

The “Hello World” of Stan!

	y	σ
School 1	28	15
School 2	8	10
School 3	-3	16
School 4	7	11
School 5	-1	9
School 6	1	11
School 7	18	10
School 8	12	18

Figure 1: Treatment effects from 8 schools. The schools have all received a coaching program designed to improve their SAT scores. We want to know how effective the coaching program is.

How do we do it?

- "Complete Pooling": *Single* treatment effect for all schools:

$$y_i|\theta \sim N(\theta, \sigma_i^2)$$

- "No pooling": *Separate* treatment effect for each school. Assume: treatment effects have nothing in common.
- Hierarchical models give us a third option: "partial pooling". Assume: treatment effects different, but drawn from the same distribution:

$$\begin{aligned} \mu &\sim \mathrm{Uniform}(-\infty, +\infty) \\ \tau &\sim \mathrm{Uniform}(0, +\infty) \\ \theta_i|\mu, \tau &\sim N(\mu, \tau^2) \\ y_i|\theta_i &\sim N(\theta_i, \sigma_i^2) \end{aligned}$$

Illustration of partial pooling

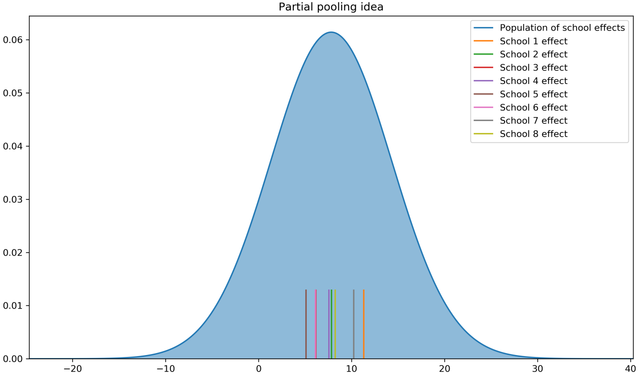


Figure 2: The idea behind partial pooling. We allow the school effects to be different, but we assume that they are all drawn from some population distribution, shown in blue.

Eight Schools demo in Stan

Things to point out:

- The mean treatment effect ends up looking similar, but we now allow for variation across schools.
- Each school gets a revised treatment estimate by “borrowing strength” from the other estimates.

AFL example

- What if we wanted to find out AFL home advantage, accounting for team strength?
- Like with eight schools, we have three options:
  - Complete pooling: one home advantage across all teams. But some teams do much better at home...!
  - No pooling: A separate home advantage for all teams. But we expect that home advantages should be roughly similar across teams.
  - Partial pooling: We draw the home advantages from a common distribution.

Data & Model

- We'll use AFL data from 2014 onwards.
- We give each team i a separate skill θ each year k (no pooling):

$$\theta_{ki} \sim N(0, 1)$$

- We give each team i a separate home advantage γ (assumed constant across seasons; flat priors on μ and σ):

$$\gamma_i | \mu_h, \sigma_h \sim N(\mu_h, \sigma_h^2)$$

- The likelihood for team i to beat team j at home in year k is:

$$\text{i beats j in year k} | \theta, \gamma \sim \text{Bernoulli}(\text{logit}^{-1}(\theta_{ki} - \theta_{kj} + \gamma_i))$$

Complications

- Sometimes the venue is home for both teams. In that case, assume no home advantage.
- The Bernoulli likelihood needs win or loss, so we discard draws.

Mean and standard deviation of home advantage in AFL since 2014

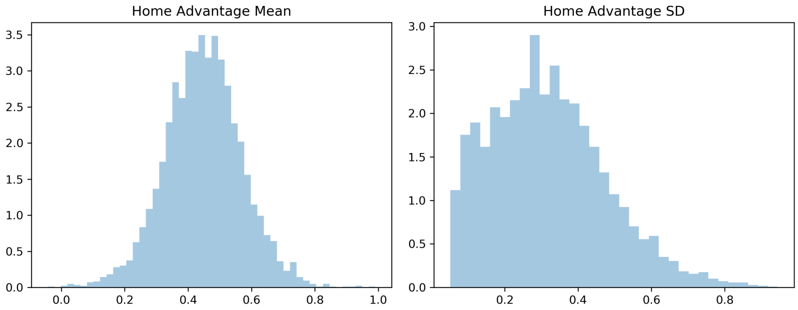


Figure 3: Mean and standard deviation of home advantage in the AFL. The means of these posteriors suggest the distribution is about 0.4 ± 0.3. For equal teams, 0.4 translates to $+10\%$ win probability.

Home advantage estimates by team

	2.5%	25%	median	75%	97.5%
Geelong	0.22	0.50	0.69	0.94	1.51
GWS Giants	0.13	0.42	0.57	0.76	1.19
West Coast	0.13	0.41	0.57	0.76	1.23
Adelaide	0.08	0.40	0.54	0.72	1.18
North Melbourne	0.07	0.38	0.53	0.72	1.16
Fremantle	0.07	0.39	0.53	0.70	1.11
Hawthorn	0.02	0.36	0.52	0.71	1.14
Port Adelaide	-0.00	0.33	0.48	0.64	1.03
Carlton	-0.09	0.31	0.46	0.63	1.03
Collingwood	-0.08	0.30	0.45	0.62	1.02
Essendon	-0.11	0.29	0.45	0.61	1.03
St Kilda	-0.17	0.24	0.41	0.57	0.92
Richmond	-0.19	0.23	0.39	0.54	0.92
Western Bulldogs	-0.21	0.22	0.39	0.55	0.89
Gold Coast	-0.17	0.23	0.39	0.54	0.89
Sydney	-0.39	0.06	0.26	0.41	0.67
Melbourne	-0.57	-0.03	0.21	0.40	0.67
Brisbane	-0.60	-0.09	0.18	0.37	0.62

Figure 4: Home advantage estimates γ for all AFL teams.

Best teams this year

	2.5%	25%	median	75%	97.5%
Richmond	0.34	1.01	1.34	1.69	2.35
Sydney	-0.12	0.49	0.83	1.19	1.84
West Coast	-0.07	0.52	0.82	1.15	1.80
Hawthorn	-0.31	0.26	0.60	0.94	1.62
Collingwood	-0.38	0.25	0.56	0.88	1.52
GWS Giants	-0.46	0.17	0.53	0.90	1.56
Melbourne	-0.49	0.14	0.48	0.80	1.41
Geelong	-0.64	-0.03	0.31	0.65	1.30
Port Adelaide	-0.74	-0.06	0.27	0.61	1.24
Essendon	-0.79	-0.15	0.19	0.51	1.14
Adelaide	-0.87	-0.20	0.14	0.46	1.04
North Melbourne	-0.98	-0.33	0.01	0.33	0.95
Western Bulldogs	-1.43	-0.81	-0.47	-0.17	0.42
Fremantle	-1.48	-0.83	-0.49	-0.15	0.46
Brisbane	-1.99	-1.26	-0.91	-0.56	0.07
St Kilda	-2.12	-1.41	-1.03	-0.67	-0.06
Gold Coast	-2.47	-1.74	-1.34	-0.98	-0.34
Carlton	-2.89	-2.16	-1.79	-1.45	-0.79

Figure 5: Team skills θ for this season.

Conclusions

- Stan is extremely flexible. If you can write down the model, Stan can (probably) sample it.
- Hierarchical models are often useful. They allow you to "borrow strength" between estimates.
- Quick plug: Stan is not the only probabilistic programming language. In particular, in R, Nick is working on a framework called greta which uses tensorflow to do sampling more quickly