Parametric Equations

A Gentle Introduction

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Motivation

Functions are very useful. They provide a very simple and practical way for us to relate two sets of numbers. For instance, consider

$$f(x) = x^2.$$

It has a usable mapping that relates one number (e.g., 2) to another (e.g., 4). Functions may be added, subtracted, multiplied, and divided. We can even do Calculus on them (FUN!).

$$(f + g)(x) = f(x) + g(x)$$
$$\frac{df}{dx} = 2x$$

Some relationships, however, cannot be expressed simply as a function. . .

Motivation

Imagine that you're a point on the xy-plane (it's not *that* far-fetched). You start at $(1,0)$ and you can walk in any direction, take any turns or anything. Obviously, your y coordinate won't necessarily be related to your x-coordinate, but it will be related to your time. In other words, your location at time t may be expressed as two functions, one for your x coordinate and the other for your y.

$$y = f(x), x = g(x)$$

Another Example

Any regular ole' function $y = f(x)$ can be represented as a parametric curve

$$\begin{aligned} y &= f(t) \\ x &= t \end{aligned}$$

For instance, $f(x) = x^2$ can be represented as

$$\begin{aligned} y &= t^2 \\ x &= t \end{aligned}$$

The Folium of Descartes

Now for some famous curves. We start with this cool curve called the folium of Descartes. It's defined by a couple of equations something like

$$x = \frac{3 \cdot t}{t^3 + 1}$$
$$y = \frac{3 \cdot t^2}{t^3 + 1}$$

The Folium of Descartes

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t	x	y
0	0	0

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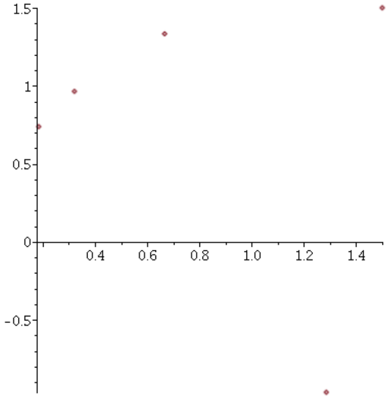
-1	undef.	undef.
-2	$6/7$	$-12/7$
$\vdots$	$\vdots$	$\vdots$

This gives us the points $(0,0), (3/2, 3/2), (2/3, 4/3), (9/28, 27/28), (12/65, 48/65), (6/7, -12/7)$ all as points on our curve (or "folium" if you will).

The Folium of Descartes

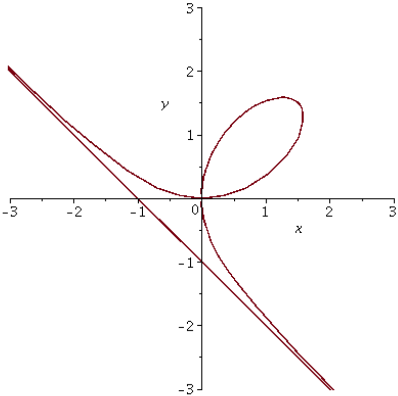
That was a lot of work to get four measly points.



And besides, we can't see the graph that well. A computer is needed for really graphing things.

The Folium of Descartes

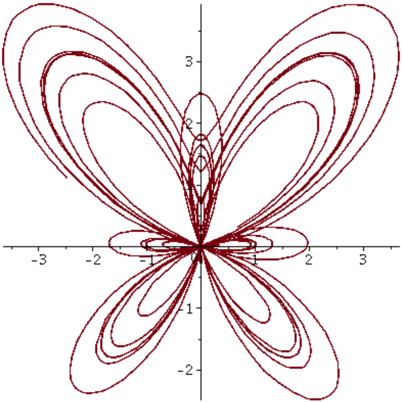
A computer nicely graphed this for me:



Now that's a graph. We're going to do this on your graphing calculators.

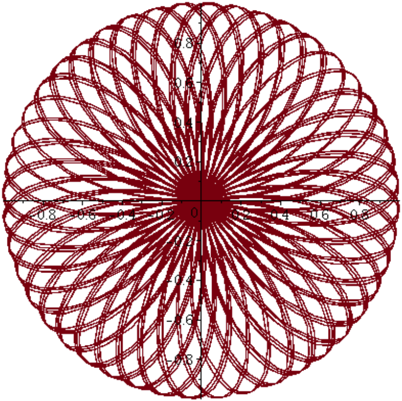
The Butterfly Curve

$$x = \sin t \left(e^{\cos t} - 2\cos 4t - \sin^5 \left(\frac{t}{12} \right) \right)$$
$$y = \cos t \left(e^{\cos t} - 2\cos 4t - \sin^5 \left(\frac{t}{12} \right) \right)$$



A Flower

$$x = \cos t \cdot \sin \pi t, 0 \leq t \leq 40 \cdot \pi$$
$$y = \sin t \cdot \sin \pi t$$



The Connection Between Parametric Equations and Implicit Functions

Earlier, we studied functions which are defined implicitly. These functions are related to parametric equations. In fact, often, they are just different representations of the same thing. For instance, take the folium of Descartes, it can be represented as an implicit function $x^3 + y^3 = 6xy$.

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Given a parametric equation, we can convert it into an implicit or explicit declaration by eliminating the variable à la Algebra I.

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y &= x^{3/2} + x^{1/2} - 3
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Calculus with Parametric Equations

Recall the implicit function declaration of the unit circle:

$$x^2 + y^2 = 1.$$

How might we go about parametrizing this?

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$$x = \cos(t)$$
$$y = \sin(t)$$

Calculus with Parametric Equations

To prove, this, we square both and add them together. What if we want to find the slope of the tangent line to the circle (i.e., the derivative). Implicit differentiation, obviously!

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$$x^2 + y^2 = 1$$
$$2x + \frac{dy^2}{dx} = 0$$

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Notice, however, that since $x = \cos(t)$ and $y = \sin(t)$, the derivative comes out to be $-\cot(t)$.

Calculus with Parametric Equations

There is a simpler way to do this than implicit differentiation, however. Remember the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

Also recall our parametric equations, $x = \cos(t), y = \sin(t)$. We take the derivative of both of these with respect to t.

$$\frac{dx}{dt} = -\sin(t)$$
$$\frac{dy}{dt} = \cos(t)$$

Calculus with Parametric Equations

Also notice that

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{-\sin(t)}.$$

Using, the chain rule, we get.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\
&= \cos(t) \cdot (-\frac{1}{\sin(t)}) \\
&= -\frac{x}{y} \\
&= -\cot(t)
\end{aligned}$$