SENSAI+: README file

Simon Tavener and Rachel Popp

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1 Introduction

SENSAI+ is an enhancement of the earlier SENSAI package with additional bifurcation capabilities for systems of differential equations. (Bifurcation capabilities for maps will be added to the next release.) SENSAI+ computes paths of solutions to parametrized systems of nonlinear equations and simultaneously computes the sensitivities and elasticities to all parameters.

Let $n\,(= \mathrm{M})$ and K be respectively the number of variables and number of parameters of the basic, underlying system of nonlinear equations

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{0}, \quad \boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^{\mathrm{K}} \mapsto \mathbb{R}^n. \tag{1}$$

Let m and $\widehat{K}$ be respectively the number of variables and parameters of the "extended system" of nonlinear equations

$$\boldsymbol{h}(\boldsymbol{u}, \widehat{\boldsymbol{p}}) = \boldsymbol{0}, \quad \boldsymbol{h} : \mathbb{R}^m \times \mathbb{R}^{\widehat{\mathrm{K}}} \mapsto \mathbb{R}^m. \tag{2}$$

SENSAI+ performs arclength continuation on $\boldsymbol{h}(\boldsymbol{u}, \widehat{\boldsymbol{p}})$ for one of five different types of extended systems, specified by the variable `imap`. These are:

- Regular solutions `(imap=2)`
- Bifurcation from a trivial solution `(imap=3)`
- Limit points `(imap=4)`

- Hopf bifurcation points `(imap=5)`
- Symmetry breaking bifurcation point `(imap=6)` ... coming soon??
- Step to branch `(imap=13)`

Note: Bifurcations of maps will require different extended systems that will again be selected through different choices of `imap`.

2 Solution types

2.1 Regular solutions (imap=2)

Let

$$m = n,\ \boldsymbol{u} = \boldsymbol{x} \in \mathbb{R}^n,\ \widehat{\mathrm{K}} = \mathrm{K},$$

and

$$\boldsymbol{h}(\boldsymbol{u}, \widehat{\boldsymbol{p}}) = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{0}, \quad \boldsymbol{h} : \mathbb{R}^n \times \mathbb{R}^{\mathrm{K}} \mapsto \mathbb{R}^n. \tag{3}$$

2.2 Bifurcation from a trivial solution (imap=3)

Let

$$m = (n+1),\ \boldsymbol{u} = \begin{pmatrix} \boldsymbol{\phi} \\ \mu \end{pmatrix} \in \mathbb{R}^{n+1},\ \widehat{\mathrm{K}} = (\mathrm{K} - 1)$$

and

$$\boldsymbol{h}(\boldsymbol{u}, \widehat{\boldsymbol{p}}) = \begin{pmatrix} \boldsymbol{g_x \phi} \\ \boldsymbol{l}^\top \boldsymbol{\phi} - 1 \end{pmatrix} = \boldsymbol{0}, \quad \boldsymbol{h} : \mathbb{R}^{n+1} \times \mathbb{R}^{\widehat{\mathrm{K}}} \mapsto \mathbb{R}^{n+1}. \tag{4}$$

2.3 Limit points (imap=4)

Let

$$m = (2n+1),\ \boldsymbol{u} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\phi} \\ \mu \end{pmatrix} \in \mathbb{R}^{2n+1},\ \widehat{\mathrm{K}} = (\mathrm{K} - 1)$$

and

$$\boldsymbol{h}(\boldsymbol{u}, \widehat{\boldsymbol{p}}) = \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{g_x \phi} \\ \boldsymbol{l}^\top \boldsymbol{\phi} - 1 \end{pmatrix} = \boldsymbol{0}, \quad \boldsymbol{h} : \mathbb{R}^{2n+1} \times \mathbb{R}^{\widehat{\mathrm{K}}} \mapsto \mathbb{R}^{2n+1}. \tag{5}$$

2.4 Hopf bifurcation points (imap=5)

Let

$$m = (3n+2),\ \boldsymbol{u} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \omega \\ \mu \end{pmatrix} \in \mathbb{R}^{3n+2},\ \widehat{\mathrm{K}} = (\mathrm{K}-1)$$

where

$$\boldsymbol{\phi} = \boldsymbol{\alpha} + i\boldsymbol{\beta}$$

and

$$\boldsymbol{h}(\boldsymbol{u}, \widehat{\boldsymbol{p}}) = \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{g}_{\boldsymbol{x}}\boldsymbol{\alpha} + \omega\boldsymbol{\beta} \\ \boldsymbol{g}_{\boldsymbol{x}}\boldsymbol{\beta} - \omega\boldsymbol{\alpha} \\ \boldsymbol{l}^\top\boldsymbol{\alpha} \\ \boldsymbol{l}^\top\boldsymbol{\beta} - 1 \end{pmatrix} \quad \boldsymbol{h} : \mathbb{R}^{3n+2} \times \mathbb{R}^{\widehat{\mathrm{K}}} \mapsto \mathbb{R}^{3n+2}. \tag{6}$$

2.5 Symmetry breaking bifurcation point (imap=6)

... coming soon ...

2.6 Step to branch (imap=13)

[from bifurcation from a trivial solution]

Let

$$m = n,\ \boldsymbol{u} = \boldsymbol{x},\ \widehat{\mathrm{K}} = \mathrm{K}.$$

Restore trivial solution $\boldsymbol{x}_0$ at λ_0 and $\boldsymbol{\phi}$ and μ from the “bifurcation from a trivial solution” solution vector. Construct initial guess $\boldsymbol{u} = \boldsymbol{x}_0 + \epsilon\phi$ at $\lambda = \lambda_0 + \epsilon$. Change imap from 13 to 2 and perform continuation of a regular point.

3 Paths using arclength continuation

$$\boldsymbol{H}(\boldsymbol{w}(s), \widehat{\boldsymbol{p}}) = \begin{pmatrix} \boldsymbol{h}(\boldsymbol{u}(s), \lambda(s)) \\ N(\boldsymbol{u}(s), \lambda(s), s) \end{pmatrix} = \boldsymbol{0}, \quad \boldsymbol{H} : \mathbb{R}^{m+1} \times \mathbb{R}^{\widehat{\mathrm{K}}} \mapsto \mathbb{R}^{m+1} \tag{7}$$

where

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{u} \\ \lambda \end{pmatrix} \in \mathbb{R}^{m+1}. \tag{8}$$

Here

- Regular solutions (imap=2)

$$m = n, \boldsymbol{u} = \boldsymbol{x} \text{ and } \widehat{\mathrm{K}} = (\mathrm{K} - 1)$$

- Bifurcation from a trivial solution (imap=3)

$$m = (n+1), \boldsymbol{u} = \begin{pmatrix} \boldsymbol{\phi} \\ \mu \end{pmatrix} \text{ and } \widehat{\mathrm{K}} = (\mathrm{K} - 2)$$

- Limit points (imap=4)

$$m = (2n+1), \boldsymbol{u} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\phi} \\ \mu \end{pmatrix} \text{ and } \widehat{\mathrm{K}} = (\mathrm{K} - 2)$$

- Hopf bifurcation points (imap=5)

$$m = (3n+2), \boldsymbol{u} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \omega \\ \mu \end{pmatrix} \text{ and } \widehat{\mathrm{K}} = (\mathrm{K} - 2)$$

- Symmetry breaking bifurcation points (imap=6)

... coming soon ...

- Step to branch (imap=13)

$$m = n, \boldsymbol{u} = \boldsymbol{x} \text{ and } \widehat{\mathrm{K}} = (\mathrm{K} - 1)$$

Note: This may be unnecessary repetition

4 Sensitivities along solution paths

Differentiating (7) with respect to p_k

$$\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{w}} \frac{\partial \boldsymbol{w}}{\partial \widehat{p}_k} + \frac{\partial \boldsymbol{H}}{\partial \widehat{p}_k} = \mathbf{0}, \quad k = 1, 2, \ldots, \widehat{K} \tag{9}$$

or

$$\frac{\partial \boldsymbol{w}}{\partial \widehat{p}_k} = - \left[\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{w}} \right]^{-1} \frac{\partial \boldsymbol{H}}{\partial \widehat{p}_k}, \quad k = 1, 2, \ldots, \widehat{K}$$

where

$$\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{w}} = \begin{pmatrix} \dfrac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}} & \dfrac{\partial \boldsymbol{h}}{\partial \lambda} \\ \\ \dfrac{\partial N}{\partial \boldsymbol{u}} & \dfrac{\partial N}{\partial \lambda} \end{pmatrix}.$$

The matrix $\partial \boldsymbol{H}/\partial \boldsymbol{w}$ is already available since it is required to solve (7) via Newton's method. The (1,1)-block, the matrix $\partial \boldsymbol{h}/\partial \boldsymbol{u}$ is constructed in `dhvec_duvec.m`. The (1,2)-block, the vector $\partial \boldsymbol{h}/\partial \lambda$ is constructed in `dhvec_dparam.m`. In fact $\partial \boldsymbol{h}/\partial p_k$ is constructed in `dhvec_dparam.m` *for all* values $k = 1, \ldots,$ K and previously only the values for λ was selected. Sensitivities and elasticities with respect to all parameters are now computed along all paths of solutions computed using Keller arclength continuation.

Sensitivities and elasticities of functionals $Q(\boldsymbol{w}(\widehat{\boldsymbol{p}}), \widehat{\boldsymbol{p}})$ can be computed as before. (... coming immediately ...) For example, the sensitivity and elasticity of the *frequency* at the Hopf bifurcation point with respect to all parameters can be computed through an appropriate choice of the functional Q.

5 Examples

5.1 Limit points - Examples/BIFN_examples/Combustion

Examples/BIFN_examples/Combustion
Run MuFile
Create Matlab files

JOBNAME = ONE
IMAP = 2, ilambda=1, imu=2, ds = 0.1, nstep=50, nu=1
IMAP = 3, ilambda=2, imu=1, ds = 0.01, nstep=100, nu=1

5.2 Hopf bifurcation points

Examples/BIFN_examples/Oscillator
Run MuFile
Create Matlab files

5.2.1 Limit points

JOBNAME = ONE
IMAP = 2, ilambda=2, imu=1, ds = 0.01, nstep=3, nu=-1
IMAP = 3, ilambda=1, imu=2, ds = 0.01, nstep=150, nu=-1

5.2.2 Hopf points

JOBNAME = TWO
IMAP = 2, ilambda=2, imu=1, ds = 0.001, nstep=7, nu=1
IMAP = 4, ilambda=1, imu=2, ds = 0.01, nstep=350, nu=-1