The Osher and Rudin Shock Filter

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August 29, 2014

Transport Equation

- Transport or first-order wave equations

$$u_t(x,t) + cu_x(x,t) = 0 \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = \phi(x), \qquad -\infty < x < \infty$$

- c is a wave speed or speed of information propogation
- If $c > 0$, the information will propagate or transport to the right (positive x-direction); if $c < 0$ the information will propagate to the left.

Characteristics

- Quantity $cu_x + u_t$ is the directional derivative of u with direction $\mathbf{d} = (c, 1)$.
- In other words, u is constant along $\mathbf{d}$, and all line parallel
- These are *characteristic lines*. Parametric equation: $x - ct = x_0$
- Solution is $u(x, t) = f(x - ct)$
- $f(x)$ determined by initial conditions: $u(x, 0) = f(x) = \phi(x)$, therefore $u(x, t) = \phi(x - ct)$

Variable $c(x,t)$

- Variable wave speed, with x and t

$$u_t(x,t) + c(x,t)u_x(x,t) = 0 \qquad -\infty < x < \infty, t > 0$$

$$u(x,0) = \phi(x), \qquad -\infty < x < \infty$$

- Characteristic *curves*

$$\frac{dx(t)}{dt} = c(x(t),t), \qquad \text{with some initial condition} \qquad x(0) = x_0$$

- Calculate time derivative along these curves

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t} = 0$$

- Therefore u is constant along characteristic curves and

$$u(x(t),t) = u(x_0,0) = \phi(x_0)$$

Edge Enhancing

- The heat equation blurs or smoothes out edges
- Could we enhance edges or reverse smoothing by using inverse heat equation (i.e. replace t by $-t$)?

$$u_t = -D\nabla^2 u$$

Edge Enhancing

$$u_{n+1} = u_n + D\nabla^2 u$$

Smoothing

$$u_{n+1} = u_n - D\nabla^2 u$$

Enhancing

$$u_{n+1} = u_n - D\nabla^2 u$$

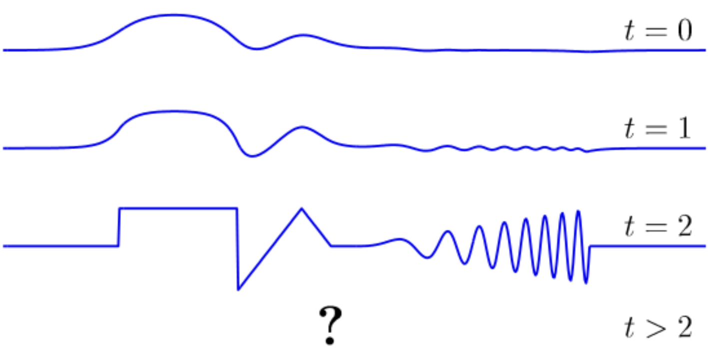
Sharpening

Edge Enhancing

- Turns out we can, but very unstable
- Small perterbations increase exponentially
- ill-posed since no continuous dependence on initial data

"The (unfinished) PDE coffee table book" by Lloyd N. Trefethen and Kristine Embree, editors, Unpublished, 2001

Fig. 1: Loss of smoothness



Shock Filtering using Transport Equation

- Want to use a Transport equation to enhance image features (edges)
- How can we chose $c(x, t)$ to do this?
- One option: shock filters
- Shock filter model proposed in 90s by Osher and Rudin: *Osher, Stanley, and Leonid I. Rudin. "Feature-oriented image enhancement using shock filters." SIAM Journal on Numerical Analysis 27.4 (1990): 919-940.*
- Good for edge enhancement. Not useful when noise is a problem (diffusion-based filters better for noisy image)

Osher and Rudin Shock Filters

- Transport equation with $c(x, t)$ using the sign of the Laplacian

$$u_t = -sign(\nabla^2 u)|\nabla u|$$

$$u(x, 0) = \phi(x)$$

- When Laplacian is negative, transport of u is towards increasing x

$$u_t = |\nabla u|$$

- When Laplacian is positive, transport of u is towards decreasing x

$$u_t = -|\nabla u|$$

- Why the absolute value of ∇u? Consider Tranport equation defined using local coordinate system based on N and T

$$u_t = c(x,t)\nabla_N u + d\nabla_T u$$

- Only want transport normal to edge, so set $d = 0$. Using $\nabla_N = N \cdot \nabla$ and $N = \nabla u / |\nabla u|$:

$$u_t = c(x,t)\frac{\nabla u \cdot \nabla u}{|\nabla u|} = c(x,t)|\nabla u|$$

Choices for $c(x, t)$

- Can be based on image values (non-linear) or initial conditions (linear)

$$c(x, t) = sign[u_{NN}(x, t)]$$
$$c(x) = sign[\phi_{NN}(x)]$$

- Can pre-smooth image with a Guassian kernel K_σ with standard deviation σ.

$$c(x, t) = sign[(K_\sigma * u)_{NN}(x, t)]$$

- Pre-smoothing with K_σ can give erroneous results for some patterns, e.g. parallel lines. Instead, can use *structure tensor*

$$J_0(\nabla u) = \nabla u \nabla u^T$$

- Eigenvectors of J_0 give local coordinate system of edge (normal and tangent). Can then smooth to get average orientations

$$J_0(\nabla u) = K_\sigma * \nabla u \nabla u^T$$

Discritisation

- Expand out Laplacian with N

$$u_{NN} = \frac{u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}}{u_x^2 + u_y^2}$$

  Can drop denominator since we are only interested in the sign
- Discretisation of u_{NN} proceeds as per anisotropic diffusion notes
- Discretisation of $|\nabla u|$ more difficult as gradient can grow unbounded due to formation of shock. Need a *flux limiter*

$$|\nabla u| \approx \sqrt{m(D_+^x u, D_-^x u)^2 + m(D_+^y u, D_-^y u)^2}$$

  where

$$m(x, y) = \begin{cases} sign[x] min(|x|, |y|) & xy > 0 \\ 0 & xy \leq 0. \end{cases}$$

  is the *minmod* flux limiter. Many others...

Shock Filter Properties

- Shocks develop at inflection points ($\nabla^2 u = 0$)
- Local extrema remain unchanged in time. No new local extrema are created.
- The steady state solution is piece-wise constant with discontinuities at inflection points
- Shock filter approximates inverse heat equation

Shock Filter Properties

- Classic version (no pre-smoothing or structure tensor) very sensitive to noise. Any noise is also enhanced by the filter

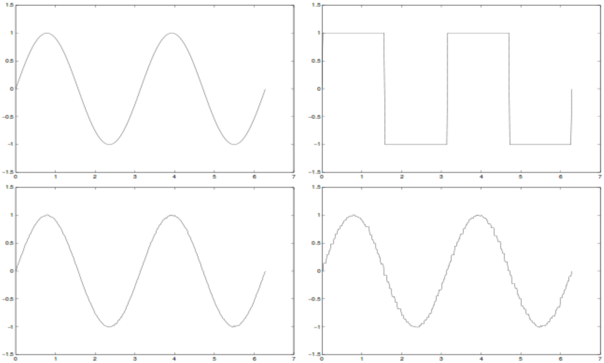


Fig. 1. Signal (sine wave) and its steady state shock filter solution without noise (top) and with very low additive white Gaussian noise, SNR=40dB (bottom).

G. Gilboa, N. Sochen, Y.Y. Zeevi, "Regularized Shock Filters and Complex Diffusion", ECCV 2002, LNCS 2350, pp. 399-313, Springer-Verlag 2002.