Closest Point Method: coupled surface-bulk problems

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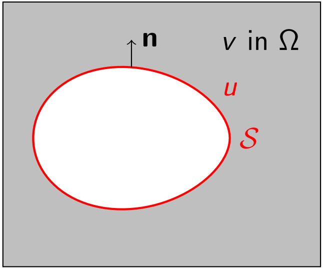
Surface bulk coupling: diffusion problem (1)



$$u_t = D\Delta_{\mathcal{S}} u - \mu u + \nu v|_{\mathcal{S}}, \text{ on } \mathcal{S} \tag{1a}$$

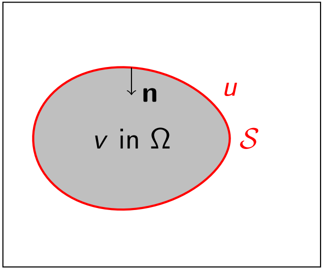
$$v_t = d\Delta v, \quad \text{in } \Omega \tag{1b}$$

Boundary condition(B.C.) for v on $\mathcal{S}$:

$$d\frac{\partial v}{\partial \mathbf{n}}\bigg|_{\mathcal{S}} = \mu u - \nu v|_{\mathcal{S}}. \tag{1c}$$

Prescribed B.C.s on four edges of the box (depending on physics and what we care).

Surface bulk coupling: diffusion problem (2)



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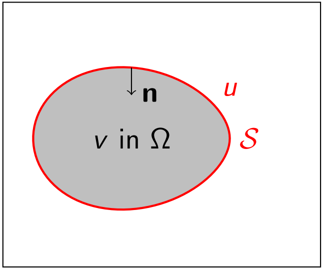

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- For convenience of illustration, we will focus on problem (2).

Big picture: forward Euler alternating time-stepping

Starting from u^0 and v^0 satisfying $d\frac{\partial v^0}{\partial \mathbf{n}}\big|_{\mathcal{S}} = \mu u^0 - \nu v^0|_{\mathcal{S}}$,
for $n = 0, 1, 2, \cdots$, alternating between the following two steps:

(1) Taking the bulk function v as known, evolve the surface function u:

$$u^{n+1} = u^n + \tau(D\Delta_{\mathcal{S}} u^n - \mu u^n + \nu v^n|_{\mathcal{S}}).$$

(2) Taking the surface function u as known, evolve the bulk function v:

$$v^{n+1} = v^n + \tau d \Delta v^n \ \text{ with B.C. } d\frac{\partial v^{n+1}}{\partial \mathbf{n}}\bigg|_{\mathcal{S}} = \mu u^{n+1} - \nu v^{n+1}|_{\mathcal{S}}.$$

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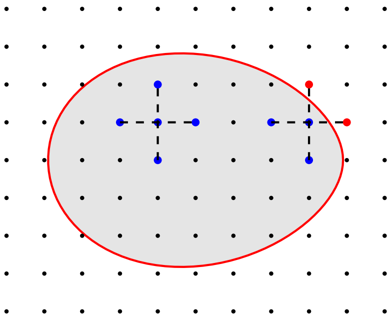
We already know how to do (1) with the Closest Point Method, now we focus on step (2).



- The Laplacian stencil of some grid points might have points outside the computational domain Ω. How do we deal with these grid points?
- How to impose the boundary condition $d\frac{\partial v}{\partial \mathbf{n}}\big|_{\mathcal{S}} = \mu u - \nu v|_{\mathcal{S}}$?

Approximation of boundary condition

According to the boundary condition $d\frac{\partial v}{\partial \mathbf{n}}\big|_{\mathcal{S}} = \mu u - \nu v|_{\mathcal{S}}$, for grid point $\mathbf{x}_i$ outside domain Ω, we have the following approximation:

$$-d\frac{v(\mathbf{x}_i) - v(\bar{\mathrm{cp}}(\mathbf{x}_i))}{\|\mathbf{x}_i - \bar{\mathrm{cp}}(\mathbf{x}_i)\|} = \mu u(\mathrm{cp}(\mathbf{x}_i)) - \nu v(\mathrm{cp}(\mathbf{x}_i)). \quad (3)$$

Here $\bar{\mathrm{cp}}(\mathbf{x}_i) = 2\mathrm{cp}(\mathbf{x_i}) - \mathbf{x}_i$ is the mirror image of $\mathbf{x}_i$ with respect to $\mathrm{cp}(\mathbf{x}_i)$.

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Since $\mathrm{cp}(\mathbf{x}_i)$ and $\bar{\mathrm{cp}}(\mathbf{x}_i)$ are not grid points, we approximate their values by Lagrange interpolation. Denote the interpolation matrix for $v(\mathrm{cp}(\mathbf{x}_i))$ by $\mathbf{E}$, denote the interpolation matrix for $v(\bar{\mathrm{cp}}(\mathbf{x}_i))$ by $\bar{\mathbf{E}}$, and denote the distance vector consisting $\|\mathbf{x}_i - \bar{\mathrm{cp}}(\mathbf{x}_i)\|$ by **dist**. Assuming **u** is known on the surface, equation (3) can be approximated by:

$$-d(\mathbf{v} - \bar{\mathbf{E}}\mathbf{v})./\mathbf{dist} = \mu\mathbf{u} - \nu\mathbf{E}\mathbf{v}. \tag{4}$$

How to evolve $\mathbf{v}$?

Our goal is:

$$v^{n+1} = v^n + \tau d \Delta v^n \ \text{ with B.C. } d \frac{\partial v^{n+1}}{\partial \mathbf{n}}\bigg|_{\mathcal{S}} = \mu u^{n+1} - \nu v^{n+1}|_{\mathcal{S}}.$$

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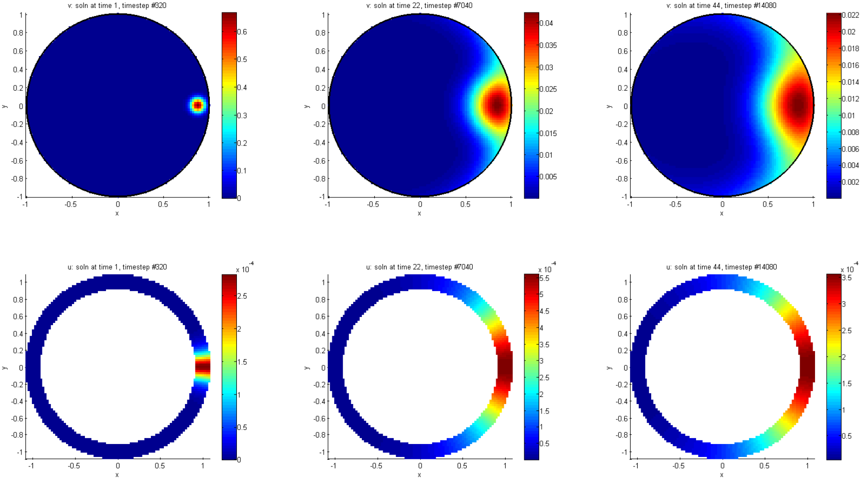
- Step 1: evolve by the normal Laplacian matrix for all grid points.

$$\mathbf{v}^{n+1} = \mathbf{v}^n + \tau d \mathbf{L} \mathbf{v}^n$$

- Step 2: correct for grid points outside Ω to match the B.C..
  Let *Band* be the set of computational grid points covering Ω, *innerBand* be the subset of *Band* containing grid points inside Ω or on $\mathcal{S}$, and *outerBand* be the subset of *Band* containing grid points outside Ω. The solution vector $\mathbf{v}$ can be written as $\mathbf{v} = [\mathbf{v}(innerBand), \mathbf{v}(outerBand)]^T$. From equation (4), taking $\mathbf{v}(innerBand)$ as known, solve for $\mathbf{v}(outerBand)$.

Surface-bulk coupling: numerical results

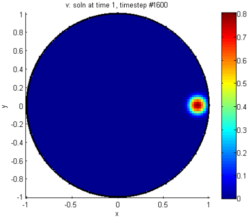


Surface bulk coupling: Fisher-KPP equation in the bulk

(Dawson 2014)

$$
\begin{aligned}
u_t &= D\Delta_{\mathcal{S}} u - \mu u + \nu v|_{\mathcal{S}}, \text{ on } \mathcal{S} \\
v_t &= d\Delta v + v(1-v), \text{ in } \Omega \\
d\frac{\partial v}{\partial \mathbf{n}} &= \mu u - \nu v,
\end{aligned}
$$

$\mathbf{n}$: unit normal pointing inwards Ω.



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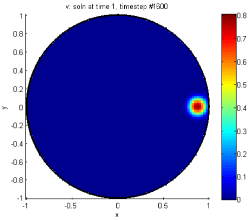

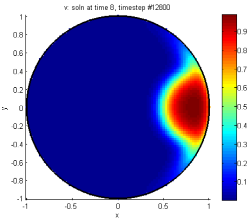

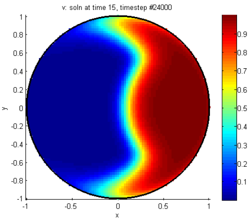

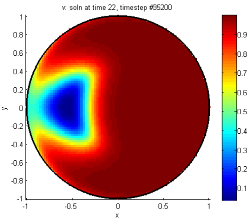

Surface bulkcoupling: reaction-diffusion process

(Macdonald, Merriman, Ruuth, 2013)

$$
\begin{aligned}
u_t &= f(u,v) + D_u \Delta_{\mathcal{S}} u - \alpha_1 u + \beta_1 U, \text{ on } \mathcal{S} \\
v_t &= g(u,v) + D_v \Delta_{\mathcal{S}} v - \alpha_2 v + \beta_2 V, \text{ on } \mathcal{S} \\
U_t &= f(U,V) + D_U \Delta U, \text{ in } \Omega \\
V_t &= g(U,V) + D_V \Delta V, \text{ in } \Omega
\end{aligned}
$$

with coupling boundary conditions:

$$
D_U \frac{\partial U}{\partial n} = \alpha_1 u - \beta_1 U, \; D_v \frac{\partial V}{\partial n} = \alpha_2 v - \beta_2 V
$$