

Lecture 6: PDEs in higher dimensions

Advection

1D: $u_t + au_x = 0$.

In 2D:

$$u_t + a(x, y)u_x + b(x, y)u_y = 0.$$

Or more generally, we can write this as:

$$u_t + \nabla \cdot (\vec{w}u) = 0,$$

with a vector field $\vec{w}(x, y)$.

- ▶ Advection and the wave equation are quite different from diffusion: they are hyperbolic and “information” about the solution travels along characteristics. These are the lines traced out by the vector field $w(x, y)$.
- ▶ The numerics are a bit different too: this code uses “upwinding” finite differences which are appropriate for advection-dominated problems, but we haven’t talked about them in this course.
- ▶ But we could look more carefully about constructing the matrices in this code. . .

Heat equation

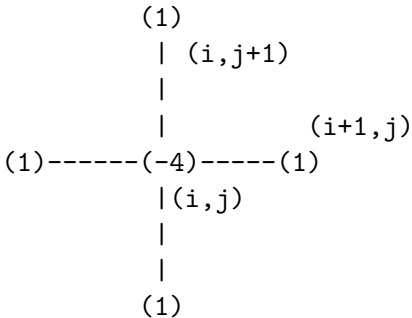
$$u_t = \nabla^2 u = u_{xx} + u_{yy}$$

on a square or rectangle. We can apply centered 2nd-order approximation to each derivative.

In the method of lines approach, we write

$$u_{xx} + u_{yy} \approx \frac{v_{i-1,j}^n - 2v_{ij}^n + v_{i+1,j}^n}{h^2} + \frac{v_{i,j-1}^n - 2v_{ij}^n + v_{i,j+1}^n}{h^2}.$$

This gives a stencil in *space* (then still need to deal with time).



- ▶ Using forward or backward Euler, accuracy is $O(k + h^2)$.
- ▶ And a stability restriction for FE of $k < h^2/4$.
- ▶ In principle, our “finite difference Laplacian” maps a matrix of 2D grid data to another such, and is thus a “4D tensor”. However, in practice we stretch out 2D to 1D, so that the tensor becomes a matrix:

$$\frac{v}{dt} = Lv.$$

- ▶ How does this “stretch” work? It defines an ordering of the grid points. In Matlab: `meshgrid()` and `(:)`, see later.

Matrix structure

Let's look at the structure of L . We choose an ordering for the grid points and assume zero boundary conditions:

\hat{y}					
		0	0	0	
	0	x4	x8	x12	0
	0	x3	x7	x11	0
	0	x2	x6	x10	0
	0	x1	x5	x9	0
		0	0		
+					\hat{x}

with corresponding unknowns v_1, \dots, v_{12} .

The discrete Laplacian now looks like this:

$$L_v = \frac{1}{h^2} \begin{pmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}$$



see [m26_head2d.m] and [m27_head2d_rectangle.m]