Lecture 6: PDEs in higher dimensions

Advection

1D: $u_t + au_x = 0$.

In 2D:

$$u_t + a(x,y)u_x + b(x,y)u_y = 0.$$

Or more generally, we can write this as:

$$u_t + \nabla \cdot (\vec{w}u) = 0,$$

with a vector field $\vec{w}(x,y)$.

[m25_2d_adv.m]

Advection and the wave equation are quite different from diffusion: they are hyperbolic and "information" about the solution travels along characteristics. These are the lines traced out my the vector field $w(x,y)$.

The numerics are a bit different too: this code uses "upwinding" finite differences which are appropriate for advection-dominited problems, but we haven't talked about them in this course.

But we could look more carefully about constructing the matrices in this code…

Heat equation

$$u_t = \nabla^2 u = u_{xx} + u_{yy}$$

on a square or rectangle. We can apply centered 2nd-order approximation to each derivative.

In the method of lines approach, we write

$$u_{xx} + u_{yy} \approx \frac{v_{i-1,j}^n - 2v_{ij}^n + v_{i+1,j}^n}{h^2} + \frac{v_{i,j-1}^n - 2v_{ij}^n + v_{i,j+1}^n}{h^2}.$$

This gives a stencil in *space* (then still need to deal with time).

```
         (1)
          | (i,j+1)
          |
          |        (i+1,j)
(1)------(-4)-----(1)
          |(i,j)
          |
          |
         (1)
```

Using forward or backward Euler, accuracy is $O(k + h^2)$.

And a stability restriction for FE of $k < h^2/4$.

In principle, our "finite difference Laplacian" maps a matrix of 2D grid data to another such, and is thus a "4D tensor". However, in practice we stretch out 2D to 1D, so that the tensor becomes a matrix:

$$\frac{v}{dt} = Lv.$$

How does this "stretch" work? It defines an ordering of the grid points. In Matlab: meshgrid() and (:), see later.

Matrix structure

Let's look at the structure of L. We choose an ordering for the grid points (why this one? see below) and assume zero boundary conditions:

```
^ y
|       0    0    0
|
|  0   x4   x8   x12  0
|
|  0   x3   x7   x11  0
|
|  0   x2   x6   x10  0
|
|  0   x1   x5   x9   0
|
|       0    0    0
+--------------------------> x
```

with corresponding unknowns $v_1, \ldots, v_{12}$. The discrete Laplacian now looks like this:

```
                |-4  1  0  0    1  0  0  0    0  0  0  0|   | v1  |
                | 1 -4  1  0    0  1  0  0    0  0  0  0|   | v2  |
                | 0  1 -4  1    0  0  1  0    0  0  0  0|   | v3  |
                | 0  0  1 -4    0  0  0  1    0  0  0  0|   | v4  |
                |                                       |   |     |
            1   | 1  0  0  0   -4  1  0  0    1  0  0  0|   | v5  |
Lv   =     ---  | 0  1  0  0    1 -4  1  0    0  1  0  0|   | v6  | .
           h^2  | 0  0  1  0    0  1 -4  1    0  0  1  0|   | v7  |
                | 0  0  0  1    0  0  1 -4    0  0  0  1|   | v8  |
                |                                       |   |     |
                | 0  0  0  0    1  0  0  0   -4  1  0  0|   | v9  |
                | 0  0  0  0    0  1  0  0    1 -4  1  0|   | v10 |
                | 0  0  0  0    0  0  1  0    0  1 -4  1|   | v11 |
                | 0  0  0  0    0  0  0  1    0  0  1 -4|   | v12 |
```