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## Lecture 6: PDEs in higher dimensions

### Advection

1D:  $u_t + au_x = 0$ .

In 2D:

$$u_t + a(x, y)u_x + b(x, y)u_y = 0.$$

Or more generally, we can write this as:

$$u_t + \nabla \cdot (\vec{w}u) = 0,$$

with a vector field  $\vec{w}(x, y)$ .

[m25\_2d\_adv.m]

Advection and the wave equation are quite different from diffusion: they are hyperbolic and “information” about the solution travels along characteristics. These are the lines traced out by the vector field  $w(x, y)$ .

The numerics are a bit different too: this code uses “upwinding” finite differences which are appropriate for advection-dominated problems, but we haven’t talked about them in this course.

But we could look more carefully about constructing the matrices in this code. . .

### Heat equation

$$u_t = \nabla^2 u = u_{xx} + u_{yy}$$

on a square or rectangle. We can apply centered 2nd-order approximation to each derivative.

In the method of lines approach, we write

$$u_{xx} + u_{yy} \approx \frac{v_{i-1,j}^n - 2v_{ij}^n + v_{i+1,j}^n}{h^2} + \frac{v_{i,j-1}^n - 2v_{ij}^n + v_{i,j+1}^n}{h^2}.$$

This gives a stencil in *space* (then still need to deal with time).

$$\begin{array}{ccccc} & & (1) & & \\ & & | & & \\ & & (i, j+1) & & \\ & & | & & \\ & & | & & (i+1, j) \\ (1) & \text{-----} & (-4) & \text{-----} & (1) \\ & & | & & \\ & & (i, j) & & \\ & & | & & \\ & & (1) & & \end{array}$$

Using forward or backward Euler, accuracy is  $O(k + h^2)$ .

And a stability restriction for FE of  $k < h^2/4$ .

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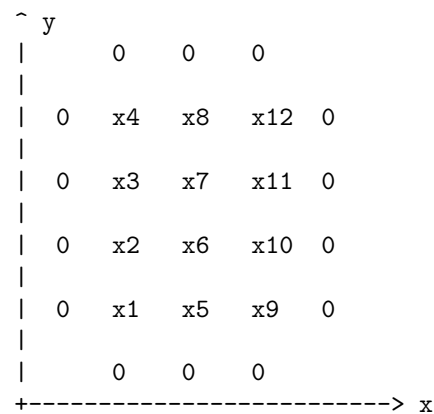
In principle, our “finite difference Laplacian” maps a matrix of 2D grid data to another such, and is thus a “4D tensor”. However, in practice we stretch out 2D to 1D, so that the tensor becomes a matrix:

$$\frac{v}{dt} = Lv.$$

How does this “stretch” work? It defines an ordering of the grid points. In Matlab: `meshgrid()` and `(:)`, see later.

### Matrix structure

Let’s look at the structure of  $L$ . We choose an ordering for the grid points (why this one? see below) and assume zero boundary conditions:



with corresponding unknowns  $v_1, \dots, v_{12}$ . The discrete Laplacian now looks like this:

$$Lv = \frac{1}{h^2} \begin{pmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$