INVESTIGATING MESH BASED APPROXIMATION METHODS FOR THE NORMALIZATION CONSTANT IN THE LOG GAUSSIAN COX PROCESS LIKELIHOOD (SUPPORTING INFORMATION)

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In this supporting information we derive an analytical formula for the integral $\int_\Omega \exp(Z^*(\mathbf{s}))\,d\mathbf{s}$ under the finite element method based mesh assumption

$$Z^*(\mathbf{s}) = \sum_{j=1}^{q} z_j \phi_j(\mathbf{s}), \tag{S.1}$$

as described in Section 2.2 of the main paper. The formula is found by reformulating and rewriting parts of existing quantities, essentially splitting the integrand into simpler functions and the observation domain Ω into triangles, and solving all of those separately. Recall from Section 2.2 of the main paper, that we may write

$$\Omega = \bigcup_{i}^{K} \bigcup_{k}^{L_i} T_{ik}, \tag{S.2}$$

where for each $i = 1, \ldots, K$, the $T_{i1}, \ldots, T_{iL_i}$ are the L_i disjoint triangles whose union is equal to the part of the observation domain which falls in mesh triangle $T_i^{(M)}$. As a consequence of this construction if we have $\mathbf{s} \in T_{ik}$ for some k, then we also have $\mathbf{s} \in T_i^{(M)}$.

Moving over to the basis functions $\phi_j(\mathbf{s})$, let M_j be the union of all mesh triangles $T_i^{(M)}$ where mesh node j is a corner. Writing $\mathbf{1}_{\{E\}}$ for the indicator function of an event E, we can then write the linearly independent finite element type of basis functions $\phi_1(\mathbf{s}), \ldots, \phi_q(\mathbf{s})$ on the form

$$\phi_j(\mathbf{s}) = \sum_{i=1}^{q} \mathbf{1}_{\{T_i^{(M)} \subset M_j\}} \mathbf{1}_{\{\mathbf{s} \subset T_i^{(M)}\}} f_{ji}(\mathbf{s}), \quad j = 1, \ldots, q, \tag{S.3}$$

where the $f_{ji}(\mathbf{s})$ are linear functions defined for all combinations of j and i where $T_i \in M_j$. Let us then write

$$f_{ji}(\mathbf{s}) = f_{ji}(x, y) = (1, x, y)\boldsymbol{\lambda}^{(ji)} = \alpha_{ji} + \beta_{ji}x + \gamma_{ji}y,$$

where the coefficient vector $\boldsymbol{\lambda}^{(ji)} = (\alpha_{ji}, \beta_{ji}, \gamma_{ji})^\top$ depends on the locations of the corners of triangle $T_i^{(M)}$. Let us further write x_j, y_j for the x- and y-coordinates of mesh node j, and x_{ji0}, y_{ji0} and x_{ji00}, y_{ji00} for the coordinates of the *other* two triangle points of triangle $T_i^{(M)}$. Then, as $\phi_j(\mathbf{s})$ takes the value 1 in (x_j, y_j) and 0 in both (x_{ji0}, y_{ji0}) and (x_{ji00}, y_{ji00}), the precise forms of the coefficients in $\boldsymbol{\lambda}^{(ji)}$ can be found by solving the linear system $B^{(ji)}\boldsymbol{\lambda}^{(ji)} = \mathbf{b}$, for $\boldsymbol{\lambda}^{(ji)}$ where

$$B^{(ji)} = \begin{pmatrix} 1 & x_j & y_j \\ 1 & x_{ji0} & y_{ji0} \\ 1 & x_{ji00} & y_{ji00} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Doing that gives

$$\alpha_{ji} = (x_{ji0}y_{ji00} - x_{ji00}y_{ji00})/\det(B^{(ji)}), \quad \beta_{ji} = (y_{ji0} - y_{ji00})/\det(B^{(ji)}), \quad \gamma_{ji} = (x_{ji00} - x_{ji0})/\det(B^{(ji)}),$$

where

$$\det(B^{(ji)}) = (x_{ji0}y_{ji00} - x_{ji00}y_{ji0}) - (x_j y_{ji00} - x_{ji00}y_j) + (x_j y_{ji0} - x_{ji0}y_j).$$

Since each of the basis functions only takes non-zero values within triangles where the mesh node in question is a corner point, there are always exactly three basis functions that take non-zero values within each mesh triangle $T_i^{(M)}$ (and therefore also within all subtriangles $T_{i1}, \ldots, T_{iL_i}$.) These basis functions are the ones which take the value 1 at the corner points of mesh triangle $T_i^{(M)}$. Let us denote this unordered set of mesh node indices by

$$\{i_1, i_2, i_3\} = \{j : \phi_j(\mathbf{s}) > 0 \text{ for any } s \in T_i^{(M)}\}, \quad \text{for } i = 1, \ldots, q. \tag{S.4}$$

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Thus, utilizing the assumptions of (S.1), in addition to the simplifications in (S.2), (S.3) and (S.4), we have that for all $k = 1, \dots, L_i$ with $i = 1, \dots, K$, the integral over T_{ik} simplifies as

$$\int_{T_{ik}} \exp\left(\sum_{j=1}^{q} z_j \phi_j(\mathbf{s})\right) d\mathbf{s} = \int_{T_{ik}} \exp\left(\sum_{j=1}^{3} z_{i_j} f_{i_j i}(\mathbf{s})\right).$$

Further, since we have seen that $\int_A \exp(Z^*(\mathbf{s}))\, d\mathbf{s}$ is just a sum of such integrals, we have that

$$\begin{aligned}\int_A \exp(Z^*(\mathbf{s}))\, d\mathbf{s} &= \sum_{i=1}^{K}\sum_{k=1}^{L_i}\int_{T_{ik}} \exp(Z^*(\mathbf{s}))\, d\mathbf{s} = \sum_{i=1}^{K}\sum_{k=1}^{L_i}\int_{T_{ik}} \exp\left(\sum_{j=1}^{q} z_j \phi_j(\mathbf{s})\right) d\mathbf{s} \\ &= \sum_{i=1}^{K}\sum_{k=1}^{L_i}\int_{T_{ik}} \exp\left(\sum_{j=1}^{3} z_{i_j} f_{i_j i}(\mathbf{s})\right) d\mathbf{s}.\end{aligned} \tag{S.5}$$

Thus, we can split the full integral into a sum of integrals over the triangles in the observation domain, each of which have an integrand which is the exponential of a linear combination of three linear functions of known form. We can therefore handle one of the integrals at a time, analytically. For the reminder of this section we therefore consider solving the integral

$$\int_{T_{ik}} \exp\left(\sum_{j=1}^{3} z_{i_j} f_{i_j i}(\mathbf{s})\right) d\mathbf{s}. \tag{S.6}$$

To solve the integral in (S.6) analytically, we first apply a change of integration variable to simplify the observation domain. Denote the coordinates of the corner points of the triangle T_{ik} by $x_{ik}^{(A)}, y_{ik}^{(A)}$, $x_{ik}^{(B)}, y_{ik}^{(B)}$ and $x_{ik}^{(C)}, y_{ik}^{(C)}$. The following function will transform the triangle with corner points $(0,0), (0,1), (1,0)$ to the triangle T_{ik}: $g_{ik}(u,v) = (g_{x,ik}(u,v), g_{y,ik}(u,v))$, where

$$\begin{aligned}g_{x,ik}(u,v) &= x_{ik}^{(A)} + u(x_{ik}^{(B)} - x_{ik}^{(A)}) + v(x_{ik}^{(C)} - x_{ik}^{(A)}),\\ g_{y,ik}(u,v) &= y_{ik}^{(A)} + u(y_{ik}^{(B)} - y_{ik}^{(A)}) + v(y_{ik}^{(C)} - y_{ik}^{(A)}).\end{aligned}$$

The Jacobian determinant of $g_{ik}(u,v)$ is given by

$$J_{g,ik}(u,v) = \begin{vmatrix} x_{ik}^{(B)} - x_{ik}^{(A)} & x_{ik}^{(C)} - x_{ik}^{(A)} \\ y_{ik}^{(B)} - y_{ik}^{(A)} & y_{ik}^{(C)} - y_{ik}^{(A)} \end{vmatrix} = (x_{ik}^{(B)} - x_{ik}^{(A)})(y_{ik}^{(C)} - y_{ik}^{(A)}) - (x_{ik}^{(C)} - x_{ik}^{(A)})(y_{ik}^{(B)} - y_{ik}^{(A)}).$$

Since $J_{g,ik}(u,v)$ is constant we write it simply as $J_{g,ik}$. Applying integration by substitution using $g_{ik}(u,v)$, we have that

$$\int_{T_{ik}} \exp\left(\sum_{j=1}^{3} z_{i_j} f_{i_j i}(\mathbf{s})\right) d\mathbf{s} = |J_{g,ik}| \int_0^1 \int_0^{1-v} \exp\left(\sum_{j=1}^{3} z_{i_j} f_{i_j i}(g_{x,ik}(u,v), g_{y,ik}(u,v))\right) du\, dv \tag{S.7}$$

To give the precise formula for this integral we shall introduce some simplifying notation. Let us write the exponent as

$$\sum_{j=1}^{3} z_{i_j} f_{i_j i}(g_{x,ik}(u,v), g_{y,ik}(u,v)) = \alpha_{ik}^* + \beta_{ik}^* u + \gamma_{ik}^* v, \tag{S.8}$$

where

$$\alpha_{ik}^* = \sum_{j=1}^{3} z_{i_j}\alpha_{ijk}^*, \quad \beta_{ik}^* = \sum_{j=1}^{3} z_{i_j}\beta_{ijk}^*, \quad \gamma_{ik}^* = \sum_{j=1}^{3} z_{i_j}\gamma_{ijk}^*,$$

$$\begin{aligned}\alpha_{ijk}^* &= \alpha_{i_j i} + \beta_{i_j i} x_{ik}^{(A)} + \gamma_{i_j i} y_{ik}^{(A)}\\ \beta_{ijk}^* &= \beta_{i_j i}(x_{ik}^{(B)} - x_{ik}^{(A)}) + \gamma_{i_j i}(y_{ik}^{(B)} - y_{ik}^{(A)})\\ \gamma_{ijk}^* &= \beta_{i_j i}(x_{ik}^{(C)} - x_{ik}^{(A)}) + \gamma_{i_j i}(y_{ik}^{(C)} - y_{ik}^{(A)}).\end{aligned}$$

Assuming neither $\beta_{\mathrm{ik}}, \gamma_{\mathrm{ik}}$, nor $\beta_{\mathrm{ik}} - \gamma_{\mathrm{ik}}$ are zero, we can, using the simplified notation in (S.8), express (S.7) as

$$
\begin{aligned}
|\mathrm{J}_{\mathrm{g,ik}}| \int_0^1 \int_0^{1-\mathrm{v}} \exp\left(\alpha^*_{\mathrm{ik}} + \beta^*_{\mathrm{ik}}\mathrm{u} + \gamma^*_{\mathrm{ik}}\mathrm{v}\right) \mathrm{d}\mathrm{u}\,\mathrm{d}\mathrm{v} &= |\mathrm{J}_{\mathrm{g,ik}}| \exp(\alpha^*_{\mathrm{ik}}) \int_0^1 \exp(\gamma^*_{\mathrm{ik}}\mathrm{v}) \left[\frac{1}{\beta^*_{\mathrm{ik}}}\left(\exp(\beta^*_{\mathrm{ik}}(1-\mathrm{v})) - 1\right)\right] \mathrm{d}\mathrm{v} \\
&= |\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{\beta^*_{\mathrm{ik}}} \left[\frac{\exp(\beta^*_{\mathrm{ik}})}{\gamma^*_{\mathrm{ik}} - \beta^*_{\mathrm{ik}}}(\exp(\gamma^*_{\mathrm{ik}} - \beta^*_{\mathrm{ik}}) - 1) - \frac{1}{\gamma^*_{\mathrm{ik}}}(\exp(\gamma^*_{\mathrm{ik}}) - 1)\right] \\
&= |\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{\beta^*_{\mathrm{ik}}\gamma^*_{\mathrm{ik}}(\gamma^*_{\mathrm{ik}} - \beta^*_{\mathrm{ik}})} \left[\beta^*_{\mathrm{ik}}(\exp(\gamma^*_{\mathrm{ik}}) - 1) - \gamma^*_{\mathrm{ik}}(\exp(\beta^*_{\mathrm{ik}}) - 1)\right].
\end{aligned}
$$

For the special cases where either $\beta_{\mathrm{ik}}, \gamma_{\mathrm{ik}}$ or $\beta_{\mathrm{ik}} - \gamma_{\mathrm{ik}}$ are exactly zero, the integral takes even simpler forms, computed analogously. The final expression for the sub-integral in (S.6) is therefore

$$
\int_{\mathrm{T}_{\mathrm{ik}}} \exp\left(\sum_{\mathrm{j}=1}^{\mathrm{q}} \mathrm{z}_{\mathrm{j}}\phi_{\mathrm{j}}(\mathbf{s})\right) \mathrm{d}\mathbf{s} = \begin{cases}
|\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{2}, & \text{if } \beta^*_{\mathrm{ik}} = \gamma^*_{\mathrm{ik}} = 0, \\
|\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{(\gamma^*_{\mathrm{ik}})^2} \left[\exp(\gamma^*_{\mathrm{ik}}) - 1 - \gamma^*_{\mathrm{ik}}\right], & \text{if } \beta^*_{\mathrm{ik}} = 0, \gamma^*_{\mathrm{ik}} \neq 0, \\
|\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{(\beta^*_{\mathrm{ik}})^2} \left[\exp(\beta^*_{\mathrm{ik}}) - 1 - \beta^*_{\mathrm{ik}}\right], & \text{if } \beta^*_{\mathrm{ik}} \neq 0, \gamma^*_{\mathrm{ik}} = 0, \\
|\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{(\beta^*_{\mathrm{ik}})^2} \left[1 + \exp(\beta^*_{\mathrm{ik}})(\beta^*_{\mathrm{ik}} - 1)\right], & \text{if } \beta^*_{\mathrm{ik}} = \gamma^*_{\mathrm{ik}} \neq 0, \\
|\mathrm{J}_{\mathrm{g,ik}}| \frac{\exp(\alpha^*_{\mathrm{ik}})}{\beta^*_{\mathrm{ik}}\gamma^*_{\mathrm{ik}}(\gamma^*_{\mathrm{ik}} - \beta^*_{\mathrm{ik}})} \left[\beta^*_{\mathrm{ik}}(\exp(\gamma^*_{\mathrm{ik}}) - 1) - \gamma^*_{\mathrm{ik}}(\exp(\beta^*_{\mathrm{ik}}) - 1)\right], & \text{otherwise.}
\end{cases} \tag{S.9}
$$

The final formula for $\int_\Omega \lambda(\mathbf{s})\,\mathrm{d}\mathbf{s}$ is thus found inserting (S.9) into (S.5).