Thus L^{-1} is also lower triangular unipotent.

Notice that a type III row operations where a row is replaced by itself plus a multiple of a higher row is performed via left multiplication by a lower triangular unipotent matrix. We will usually call these downward row operations. Here is a basic fact.

Proposition 3.14. *The class $\mathcal{L}_n$ of all lower triangular unipotent $n \times n$ matrices is a subgroup of $GL(n, \mathbb{R})$. Similarly, the class $\mathcal{U}_n$ of all upper triangular unipotent matrices is also a subgroup of $GL(n, \mathbb{R})$.*

Proof. It follows from the definition of matrix multiplication that the product of two lower triangular matrices is also lower triangular. If A and B are lower triangular unipotent, then the diagonal entries of AB are all 1. Indeed, if $AB = (c_{ij})$, then

$$c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki} = a_{ii} b_{ii} = 1,$$

since $a_{ij} = b_{ij} = 0$ if $i < j$. The identity I_n is also lower triangular unipotent, so to show $\mathcal{L}_n$ is a subgroup of $GL(n, \mathbb{R})$, it remains to show that the inverse of a lower triangular unipotent matrix A is also in $\mathcal{L}_n$. But this follows since inverting a lower triangular unipotent matrix only requires downward row operations. (Row swaps aren't needed since A is lower triangular, and dilations aren't needed either since lower triangular unipotent matrices only have 1's on the diagonal.) Thus, A^{-1} is a product of lower triangular elementary matrices of type III. But these are elements of $\mathcal{L}_n$, so A^{-1} is also in $\mathcal{L}_n$. The proof for $\mathcal{U}_n$ is similar. In fact, one can simply transpose the proof just given. $\square$

As just noted, every lower triangular unipotent matrix is the product of downward row operations. Indeed, there exist $E_1, E_2, \ldots, E_k$ of this type such that $E_k \cdots E_2 E_1 L = I_n$. Therefore, $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. But the inverse of each E_i is also a lower triangular elementary matrix of type III. Hence, $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. The analogous fact holds for upper triangular unipotent matrices.

Continuing the introduction of the cast of characters, recall from Example 3.9, that an $n \times n$ matrix which can be expressed as a product of elementary matrices of type II (i.e. row swaps) is called a *permutation matrix*. We've already seen that the set $P(n)$ of $n \times n$ permutation matrices is a matrix group, and, moreover, the inverse of a permutation matrix P is P^T. The $n \times n$ permutation matrices are exactly those matrices which