Clearly the rank of an $m \times n$ matrix A is at most the minimum of m and n. However, we don't yet know whether the number of corners in a matrix is well defined, since two different sequences of row operations might produce two different reduced matrices. Hence we'll need to assume for now that the rank is a well defined concept. In fact, we'll prove in Proposition 3.18 that the reduced row echelon form of an arbitrary matrix is unique.

Proposition 2.6. *Let A be an $m \times n$ matrix. Then:*

(*i*) $\mathcal{N}(A) = \{\mathbf{0}\}$ *if and only if the rank of A is n;*

(*ii*) *if $A\mathbf{x} = \mathbf{b}$ is consistent and the rank of A is n, then the solution is unique;*

(*iii*) *the linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the ranks of A and $(A \mid \mathbf{b})$ are the same; and*

(*iv*) *if A is $n \times n$ and has rank n, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.*

The converse of statement (*iv*) *is also true.*

Proof. The first statement was already proved at the end of Section 2.4.2 using (2.6). The only way to have $\mathcal{N}(A) = \{\mathbf{0}\}$ is if every variable in the system $A\mathbf{x} = \mathbf{0}$ is a corner variable, which is the same as saying A has rank n. For the second statement, let $\mathbf{u}$ and $\mathbf{v}$ be two solutions of $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{u} - \mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Thus $\mathbf{u} - \mathbf{v} \in \mathcal{N}(A)$, so $\mathbf{u} - \mathbf{v} = \mathbf{0}$ by (i). The third statement follows as in the previous example, because if the rank of $(A \mid \mathbf{b})$ is greater than the rank of A, then the last equation is equivalent to the inconsistent equation $0 = 1$. For (iv), let A have rank n. Then $(A \mid \mathbf{b})$ also has rank n, since A is $n \times n$ and hence the rank of $(A \mid \mathbf{b})$ can't exceed n. Thus $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$ by (ii) and (iii). It remains to show the converse of (iv) that if A and $(A \mid \mathbf{b})$ have the same rank for all $\mathbf{b}$, then A has rank n. But if the rank of A is less than n, one can (exactly as in Example 2.8) produce a $\mathbf{b}$ for which $(A \mid \mathbf{b})$ has rank greater than the rank of A. We will leave filling in all the details as an exercise. $\square$

Systems where $m = n$ are an important special case as they are neither under determined (fewer equations than unknowns) nor over determined (more equations than unknowns). When A is $n \times n$ of rank n, the system $A\mathbf{x} = \mathbf{b}$ is said to be *nonsingular*. Thus the nonsingular systems are the square systems which are always consistent and always have unique solutions. We will also say that an $n \times n$ matrix A is *nonsingular* if it has maximal rank n. If the rank of A is less than n, we will call A *singular*.