Chapter 20

The Navier-Stokes equations

20.1 Conservation laws

A general conservation law

Consider the flow of a quantity with density $\phi(x,t)$ at $x \in \Omega$, and $\Omega \subset \mathbb{R}^n$ a bounded open domain for $n = 2$ or 3. For a time $t > 0$, the total flow of the quantity through the boundary $\partial\Omega$ is given by

$$\int_{\partial\Omega} \phi u \cdot n \, ds,$$

where n is the outward unit normal of $\partial\Omega$, and $u = u(x,t)$ is the velocity of the flow.

For an arbitrary subdomain $\omega \subset \Omega$, the change of the total quantity ϕ in ω is equal to the volume source or sink $s = s(x,t)$ minus the total flow of the quantity through the boundary $\partial\omega$,

$$\frac{d}{dt}\int_{\omega} \phi(x,t)\, dx = -\int_{\partial\omega} \phi u \cdot n \, ds + \int_{\omega} s(x,t)\, dx,$$

which by Gauss' theorem leads to

$$\int_{\omega} \left(\frac{\partial}{\partial t}\phi(x,t) + \nabla \cdot (\phi u) - s \right) dx = 0,$$

for any $\omega \subset \Omega$, and assuming the integrand is continuous we get the general continuity equation

$$\dot{\phi} + \nabla \cdot (\phi u) - s = 0, \tag{20.1}$$

for any $x \in \Omega$, and $t > 0$.

Mass conservation

Now consider the flow of mass of a continuum, with $\rho = \rho(x,t)$ the mass density of the continuum. The general continuity equation (20.1) with $\phi = \rho$, and zero sink $s = 0$, gives the equation for conservation of mass

$$\dot{\rho} + \nabla \cdot (\rho u) = 0.$$

We say that a flow is *incompressible* if

$$\nabla \cdot u = 0,$$

or equivalently if the *material derivative* is zero,

$$\frac{D\rho}{Dt} = \dot{\rho} + u \cdot \nabla \rho = 0,$$

since

$$0 = \dot{\rho} + \nabla \cdot (\rho u) = \frac{D\rho}{Dt} + \rho \nabla \cdot u.$$

Conservation of momentum

Newton's 2nd Law states that the change of *momentum* ρu, is equal to the sum of all forces, including *volume forces*,

$$\int_\omega \rho f \, dx,$$

for a force density $f = f(x,t) = (f_1(x,t), ..., f_n(x,t))$, and *surface forces*,

$$\int_{\partial\omega} n \cdot \sigma \, ds,$$

with the *Cauchy stress tensor* $\sigma = (\sigma_{ij})$, where $\sigma_{ij} = \sigma_{ij}(x,t)$, and we define $n \cdot \sigma = n^T \sigma = (\sigma_{ji} n_j)$. Gauss' theorem gives the total force as

$$\int_\omega \rho f \, dx + \int_{\partial\omega} n \cdot \sigma \, ds = \int_\omega (\rho f + \nabla \cdot \sigma) \, dx.$$

The general continuity equation with $\phi = \rho u$, and the sink given by the sum of all forces, gives the equation for conservation of momentum

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = \rho f + \nabla \cdot \sigma, \tag{20.2}$$

with $u \otimes u = uu^T$, the tensor product of the velocity vector field u. With the help of conservation of mass, we can rewrite the left hand side as

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = u(\dot{\rho} + \nabla \cdot (\rho u)) + \rho(\dot{u} + (u \cdot \nabla)u) = \rho(\dot{u} + (u \cdot \nabla)u),$$

so that we get

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f + \nabla \cdot \sigma. \tag{20.3}$$

We say that (20.2) is on *conservation form*, whereas (20.3) is on *non-conservation form.*

The Cauchy stress tensor consists of normal stresses on the diagonal, and shear stresses on the off-diagonal. We can decompose σ into a *dynamic pressure*

$$p_d = -\frac{1}{3}\operatorname{tr}(\sigma),$$

and a *deviatoric stress tensor* $\tau = \sigma + p_d I$, with I the identity matrix,

$$\sigma = -p_d I + \tau,$$

so that

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f - \nabla p_d + \nabla \cdot \tau.$$

20.2 The Navier-Stokes equations

We now consider incompressible flow, so that the velocity is divergence free, and we assume the density to be constant. To determine the deviatoric stress we need a constitutive model of the fluid.

For a Newtonian fluid, the deviatoric stress depends linearly on the *strain rate tensor*

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{20.4}$$

with $\tau = 2\mu\epsilon$, where μ is the *dynamic viscosity.*

The *incompressible Navier-Stokes equations* takes the form,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f, \tag{20.5}$$
$$\nabla \cdot u = 0, \tag{20.6}$$

with the *kinematic viscosity* $\nu = \mu/\rho$, and the kinematic pressure $p = p_d/\rho$.

Non-dimensionalization

Solutions to the Navier-Stokes equations may take quite different forms, depending on the balance of the inertial and dissipative terms of the equations. To exhibit this balance, we express the Navier-Stokes equations in terms of the non-dimensional variables u_*, p_*, f_*, x_*, t_*,

$$u = Uu_*, \quad p = Pp_*, \quad x = Lx_*, \quad f = Ff_*, \quad t = Tt_*, \tag{20.7}$$

where U, P, L, T are characteristic scales of the velocity, pressure, force, length and time, respectively. The resulting non-dimensionalized differential operators are scaled as,

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t_*}, \quad \nabla = \frac{1}{L}\nabla_*, \quad \Delta = \frac{1}{L^2}\Delta_*, \tag{20.8}$$

which gives

$$\frac{U}{T}\frac{\partial}{\partial t_*}u_* + \frac{U^2}{L}(u_* \cdot \nabla_*)u_* + \frac{P}{L}\nabla_* p_* - \frac{\nu U}{L^2}\Delta_* u_* = Ff_*, \tag{20.9}$$

$$\frac{U}{L}\nabla \cdot u_* = 0, \tag{20.10}$$

or,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - Re^{-1}\Delta u = f, \tag{20.11}$$

$$\nabla \cdot u = 0. \tag{20.12}$$

Here we have dropped the non-dimensional notation for simplicity, with

$$T = L/U, \quad P = U^2, \quad F = \frac{U^2}{L}, \quad Re = \frac{UL}{\nu}, \tag{20.13}$$

where the *Reynolds number* Re determines the balance between inertial and viscous characteristics in the flow. For low Re linear viscous effects dominate, whereas for high Re we have a flow dominated by nonlinear inertial effect, and turbulence for sufficiently high Reynolds number.

Formally, in the limit $Re \to \infty$, the viscous term vanishes and we are left with the inviscid *Euler equations*,

$$\dot{u} + (u \cdot \nabla)u + \nabla p = f, \tag{20.14}$$

$$\nabla \cdot u = 0, \tag{20.15}$$

traditionally seen as a model for flow at high Reynolds numbers. Although, this simple analysis is too naive and relies on strong assumptions on the regularity of solutions to the equations, an open problem posed as one of the Clay \$1 million Prize problems.

In the limit $Re \to 0$, we obtain the *Stokes equations* as a model of viscous flow,

$$
\begin{aligned}
-\Delta u + \nabla p &= f, && (20.16)\\
\nabla \cdot u &= 0, && (20.17)
\end{aligned}
$$

with now a different scaling of the pressure and the force,

$$P = \frac{\nu U}{L}, \quad F = \frac{\nu U}{L^2}. \tag{20.18}$$

20.3 Stokes flow

The Stokes equations

The Stokes equations for a domain $\Omega \subset \mathbb{R}^n$ with boundary $\nabla\Omega = \Gamma_D \cup \Gamma_N$, and associated normal n, takes the form,

$$
\begin{aligned}
-\Delta u + \nabla p = f, &\qquad x \in \Omega, && (20.19)\\
\nabla \cdot u = 0, &\qquad x \in \Omega, && (20.20)\\
u = g_D, &\qquad x \in \Gamma_D, && (20.21)\\
-\nabla u \cdot n + pn = g_N, &\qquad x \in \Gamma_N. && (20.22)
\end{aligned}
$$

Homogeneous Dirichlet boundary conditions

First assume that we have $\nabla\Omega = \Gamma_D$ and $g_D = 0$, that is, homogeneous Dirichlet boundary conditions for the velocity. We then seek a weak solution to the Stokes equations in the following spaces,

$$
\begin{aligned}
V &= [H_0^1(\Omega)]^n && (20.23)\\
Q &= \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}, && (20.24)
\end{aligned}
$$

where the extra condition on Q is needed to assure uniqueness of the pressure, which otherwise is undetermined up to a constant.

We derive the variational formulation by taking the inner product of the momentum equation with a test function $v \in V$, and the inner product of the continuity equation with a test function $q \in Q$. By Green's formula and the homogeneous Dirichlet boundary condition, we obtain the variational formulation as: find $(u,p) \in V \times Q$, such that,

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v), && (20.25)\\ b(u,q) &= 0, && (20.26) \end{aligned}$$

for all $(v,q) \in V \times Q$, with

$$a(v,w) = (\nabla v, \nabla w) = \int_\Omega \nabla v : \nabla w \, dx, \tag{20.27}$$

$$b(v,q) = -(\nabla \cdot v, q) = -\int_\Omega (\nabla \cdot v) q \, dx, \tag{20.28}$$

and

$$\nabla v : \nabla w = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}. \tag{20.29}$$

The saddle-point problem

The solution (u,p) to the Stokes equations (20.25-20.26), is also the solution to the constrained minimization problem,

$$\min J(v) = \frac{1}{2} a(v,v) - (f,v) \tag{20.30}$$

under the constraint

$$b(v,q) = 0, \tag{20.31}$$

for which we can formulate the Lagrangian

$$L(v,q) = J(v) + b(v,q), \tag{20.32}$$

so that $p \in Q$ represents a Lagrange multiplier for the constraint $\nabla \cdot u = 0$.

The Stokes equations thus represent a saddle-point problem, since

$$L(u,q) \leq L(u,p) \leq L(v,p), \quad \forall (v,q) \in V \times Q. \tag{20.33}$$

Theorem 38. *The saddle-point point problem (20.25-20.26) has a unique solution, if*

(i) the bilinear form a is coercive, i.e. that exists an $\alpha > 0$, such that

$$a(v,v) \geq \alpha \|v\|_V, \tag{20.34}$$

for all $v \in Z = \{v \in V : b(v,q) = 0, \forall q \in Q\}$,

(ii) the bilinear form b satisfies the inf-sup condition, i.e. there exists a $\beta > 0$, such that

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta. \tag{20.35}$$

Mixed finite element approximation

We now formulate a finite element method for solving Stokes equations. Since we use different approximation spaces for the velocity and the pressure, we refer to the method as a mixed finite element method. We seek an approximation $(U, P) \in V_h \times Q_h$, such that,

$$a(U, v) + b(v, P) = (f, v), \tag{20.36}$$
$$b(U, q) = 0, \tag{20.37}$$

for all $(v, q) \in V_h \times Q_h$, where V_h and Q_h are finite element approximation spaces. There exists a unique solution to (20.36-20.37), under certain conditions on the approximation spaces V_h and Q_h.

Theorem 39. *The mixed finite element problem (20.36-20.37) has a unique solution $(U, P) \in V_h \times Q_h$, if*

(i) the bilinear form a is coercive, i.e. that exists an $\alpha_h > 0$, such that

$$a(v, v) \geq \alpha_h \|v\|_V, \tag{20.38}$$

for all $v \in Z_h = \{v \in V_h : b(v, q) = 0, \forall q \in Q_h\}$,

(ii) the bilinear form b satisfies the inf-sup condition, i.e. there exists a $\beta_h > 0$, such that

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta_h, \tag{20.39}$$

and this unique solution satisfies the following error estimate,

$$\|u - U\|_V + \|p - P\|_Q \leq C \left(\inf_{v \in v_h} \|u - v\| + \inf_{q \in Q_h} \|p - q\| \right), \tag{20.40}$$

for a constant $C > 0$.

The pair of approximation spaces must be chosen to satisfy the inf-sup condition, with the velocity space sufficiently rich compared to the pressure space. For example, continuous piecewise quadratic approximation of the velocity and continuous piecewise linear approximation of the pressure, referred to as the Taylor-Hood elements. On the other hand, continuous piecewise linear approximation of both velocity and pressure is not inf-sup stable.

Schur complement methods

We seek finite element approximations in the following spaces,

$$V_h = \{v = (v_1, v_2, v_3) : v_k(x) = \sum_{j=1}^{N} v_k^j \phi_j(x), k = 1, 2, 3\} \tag{20.41}$$

and

$$Q_h = \{q : q(x) = \sum_{j=1}^{M} q^j \psi_j(x)\}, \tag{20.42}$$

which leads to a discrete system in matrix form,

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{20.43}$$

with u and p vectors holding the coordinates of U and P in the respective bases of V_h and Q_h.

The matrix A is symmetric positive definite and thus invertible, so we can express

$$u = A^{-1}(f - Bp), \tag{20.44}$$

and since $B^T u = 0$,

$$B^T A^{-1} B p = B^T A^{-1} f, \tag{20.45}$$

which is the *Schur complement* equation. If $\mathrm{null}(B) = \{0\}$, then the matrix $S = B^T A^{-1} B$ is symmetric positive definite and can also be inverted.

Schur complement methods take the form

$$p_k = p_{k-1} - C^{-1}(B^T A^{-1} B p_{k-1} - B^T A^{-1} f), \tag{20.46}$$

where C^{-1} is a preconditioner for $S = B^T A^{-1} B$. The Usawa algorithm is based on C^{-1} as a scaled identity matrix, which gives

1. Solve $Au_k = f - Bp_{k-1}$,
2. Set $p_k = p_{k-1} + \alpha B^T u_k$.

Stabilized methods

Approximation spaces of equal order is possible, by stabilization of the standard Galerkin finite element method: find $(U, P) \in V_h \times Q_h$, such that,

$$\begin{aligned} a(U, v) + b(v, P) &= (f, v), && (20.47) \\ b(U, q) + s(P, q) &= 0, && (20.48) \end{aligned}$$

for all $(v, q) \in V_h \times Q_h$, where $s(P, q)$ is a pressure stabilization term. The resulting discrete system takes the form,

$$\begin{bmatrix} A & B \\ B^T & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{20.49}$$

where the stabilization term is chosen so that the matrix S is invertible. For example, the *Brezzi-Pitkäranta* stabilization takes the form,

$$s(P, q) = \int_\Omega h^2 \nabla P \cdot \nabla q \, dx. \tag{20.50}$$

20.4 The transient Navier-Stokes equations

Semi-discretization

We now formulate a finite element method for solving the unsteady Navier-Stokes equations (20.5)-(20.6) by semi-discretization.

For each $t > 0$, we seek approximations $(U(t), P(t)) \in V_h \times Q_h$, with $U(t) = (U_1(t), U_2(t), U_3(t))$, of the form,

$$U_k(x, t) = \sum_{j=1}^{N} U_k^j(t) \phi_j(x), \; k = 1, 2, 3, \quad P(x, t) = \sum_{j=1}^{M} P^j(t) \psi_j(x), \tag{20.51}$$

such that

$$(\dot{U}, v) + c(U; U, v) + a(U, v) + b(v, P) - b(U, q) = (f, v), \tag{20.52}$$

for all $(v, q) \in V_h \times Q_h$, where the bilinear forms are defined by (20.27)-(20.28), with the trilinear form,

$$c(u; v, w) = ((u \cdot \nabla)v, w) = \int_\Omega (u \cdot \nabla)v \cdot w \, dx. \tag{20.53}$$

The semi-discretization (20.52) is a system of ODEs, which we can solve by a suitable time-stepping method. For $\nabla \cdot u = 0$, we have that

$$c(u; v, w) = \bar{c}(u; v, w), \tag{20.54}$$

with

$$\bar{c}(u; v, w) = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}(v, (u \cdot \nabla)w). \tag{20.55}$$

We may alternatively use this form in (20.52), where in particular we note that $c(u; v, v) = 0$.

The θ-method

Semi-discretizaton by the θ-method takes the form: for each time interval $I_n = (t_{n-1}, t_n)$, with the time step length $k_n = t_n - t_{n-1}$, find $(U_n, P_n) = (U(t_n), P(t_n)) \in V_h \times Q_h$, such that

$$\frac{1}{k_n}((U_n, v) - (U_{n-1}, v)) + c(U_\theta; U_\theta, v) + \nu a(U_\theta, v) + b(v, P_\theta) - b(U_\theta, q) = (f, v),$$

for all $(v, q) \in V_h \times Q_h$, with

$$U_\theta = (1-\theta)U_n + \theta U_{n-1}, \quad P_\theta = (1-\theta)P_n + \theta P_{n-1}. \tag{20.56}$$

Here e.g. $\theta = 0$ corresponds to the Implicit Euler method, and $\theta = 0.5$ corresponds to the Trapezoidal method.

20.5 Stabilized finite element methods

Stabilization techniques

Previously we have found that the inf-sup condition of the Stokes equations could be circumvented by a pressure stabilization technique to allow for equal order approximation spaces for the velocity and the pressure. The idea of stabilization through a small regularizing perturbation is fundamental for finite element methods, not only in the context of saddle-point problems.

Whereas low Reynolds number flow is dominated by viscosity, for high Reynolds numbers the dominating phenomenon is nonlinear transport. Standard Galerkin finite element methods are not optimal for discretization of transport dominated equations, instead we will use stabilization techniques to formulate suitable finite element methods.

Linear transport

To understand the basic mechanisms, we first study a linear transport equation for a scalar quantity $u = u(x, t)$, convected by a divergence-free vector field $\beta = \beta(x, t)$,

$$\dot{u} + (\beta \cdot \nabla)u - \epsilon \Delta u = f, \qquad (x, t) \in \Omega \times I, \tag{20.57}$$

$$\nabla \cdot \beta = 0, \qquad (x, t) \in \Omega \times I, \tag{20.58}$$

with suitable initial and boundary conditions, and $\epsilon > 0$ a small diffusion coefficient.

Model problem

We study the simple model problem in one space,

$$-\epsilon u'' + u' = 0, \quad x \in (0,1), \tag{20.59}$$

$$u(0) = 1,\, u(1) = 0, \tag{20.60}$$

for which we formulate a standard Galerkin finite element method: find $U \in V_h$ such that,

$$\int_0^1 \epsilon u'v'\ dx + \int_0^1 u'v\ dx = 0, \tag{20.61}$$

for all test functions $v \in V_h^0$, with

$$V_h = \{v \in H^1(0,1) : v(0) = 1, v(1) = 0\}, \tag{20.62}$$

$$V_h^0 = \{v \in H^1(0,1) : v(0) = 0, v(1) = 0\}. \tag{20.63}$$

Divide the interval $(0,1)$ into M uniform subintervals $I_i = (x_{i-1}, x_i)$ of length $h = x_i - x_{i-1}$, with nodes $\{x_i\}_{i=0}^{M+1}$ and associated piecewise linear basis functions $\phi_i = \phi_i(x)$.

Then we can write the finite element approximation as

$$U(x) = \sum_{j=1}^{M} u_j\phi_j(x) + u_0\phi_0(x) + u_{M+1}\phi_{M+1}(x), \tag{20.64}$$

with $u_j = u(x_j)$ (since we have a nodal basis), and from the boundary conditions we have that

$$U(x) = \sum_{j=1}^{M} u_j\phi_j(x) + \phi_0(x). \tag{20.65}$$

The discrete system takes the form $Ax = b$, with $A = (a_{ij})$, $b = (b_i)$ and $x = (x_j)$,

$$a_{ij} = \int_0^1 \epsilon\phi_j'(x)\phi_i'(x)\ dx + \int_0^1 \phi_j'(x)\phi_i(x)\ dx, \tag{20.66}$$

$$b_i = \int_0^1 \epsilon\phi_0'(x)\phi_i'(x)\ dx + \int_0^1 \phi_0'(x)\phi_i(x)\ dx. \tag{20.67}$$

$$\tag{20.68}$$

Equation i takes the form

$$\sum_{j=1}^{M} a_{ij}x_j = x_{i-1}\left(-\frac{\epsilon}{h} - \frac{1}{2}\right) + x_i\frac{2\epsilon}{h} + x_{i+1}\left(-\frac{\epsilon}{h} + \frac{1}{2}\right) = 0. \tag{20.69}$$

We observe two different regimes,

$$-\frac{\epsilon}{h} >> \frac{1}{2} \Rightarrow -x_{i-1} + 2x_i - x_{i+1} = 0, \tag{20.70}$$

$$-\frac{\epsilon}{h} << \frac{1}{2} \Rightarrow -x_{i-1} + x_{i+1} = 0, \tag{20.71}$$

with a combination of the two when $\epsilon \approx h$. In the convection dominated case, the boundary conditions lead to two cases depending on if M is an odd or even number; either no solution exists, or the solution oscillates between 0 and 1.

To obtain a finite element approximation that is close to the exact solution in the convection dominated case, we stabilize the method by an artificial diffusion $\epsilon = h/2$. We also refer to this as an *upwind method*, since the resulting equation takes the form

$$-x_{i-1} + x_i = 0, \tag{20.72}$$

where information is propagated from the upwind direction.

Streamline diffusion stabilization

For the Navier-Stokes equations we can use artificial viscosity to stabilize the finite element method for high Reynolds numbers. But there are also more accurate, less diffusive, stabilization methods.

For example, we may use streamline diffusion stabilization, where we add artificial viscosity in the streamline direction β only, which is enough to remove the spurious oscillations. Combined with pressure stabilization to allow for equal order approximation spaces, the method takes the following form.

For each $t > 0$, find $(U(t), P(t)) \in V_h \times Q_h$, such that

$$(\dot{U}, v) + \bar{c}(U; U, v) + a(U, v) + b(v, P) - b(U, q) + s_1(U; U, v) + s_2(P, q) = (f, v),$$

for all $(v, q) \in V_h \times Q_h$, with the stabilization terms

$$s_1(U; U, v) = (\delta_1 (U \cdot \nabla) U, (U \cdot \nabla) v), \tag{20.73}$$

$$s_2(P, q) = (\delta_2 \nabla P, \nabla q), \tag{20.74}$$

with stabilization parameters $\delta_1 \sim h/U_{n-1}$ and $\delta_2 \sim h$.

By choosing $(v, q) = (U, P)$, we obtain a stability estimate of the method,

$$\frac{d}{dt}\frac{1}{2}\|U\|^2 + \|\sqrt{\nu}\nabla U\|^2 + \|\sqrt{\delta_1}(U \cdot \nabla) U\|^2 + \|\sqrt{\delta_2}\nabla P\|^2 = 0, \tag{20.75}$$

where we can observe the regularizing effect of the stabilization terms.

Least squares stabilization of the residual

The Galerkin Least Squares (GLS) method is based on a combination of Galerkin's method with a least squares minimization of the residual of the Navier-Stokes equations. GLS is a *consistent* method in the sense that all terms in the method are based on the residual of the equations, no artificial stabilization terms are added.

For each $t > 0$, find $(U(t), P(t)) \in V_h \times Q_h$, such that

$$(\dot{U}, v) + \bar{c}(U; U, v) + a(U, v) + b(v, P) - b(U, q) + s_1(U; U, v) + s_2(U, v) = (f, v),$$

for all $(v, q) \in V_h \times Q_h$, with the stabilization terms

$$\begin{aligned} s_1(w; U, v) &= (\delta_1(\dot{U} + (w \cdot \nabla)U + \nabla P), \dot{v} + (w \cdot \nabla)v + \nabla q) \\ s_2(U, v) &= (\delta_2 \nabla \cdot U, \nabla \cdot v), \end{aligned}$$

with stabilization parameters $\delta_1 \sim h/U_{n-1}$ and $\delta_2 \sim hU_{n-1}$.

By choosing $(v, q) = (U, P)$, we obtain a stability estimate of the method,

$$\frac{d}{dt}\frac{1}{2}\|U\|^2 + \|\sqrt{\nu}\nabla U\|^2 + \|\sqrt{\delta_1}(\dot{U} + (U \cdot \nabla)U + \nabla P - \nu \Delta U)\|^2 + \|\sqrt{\delta_2}\nabla \cdot U\|^2 = 0,$$

where we can observe the regularizing effect of the stabilization terms.

20.6 A posteriori error estimation

The model problem

Consider the linear model problem: find $u \in V$ such that,

$$a(u, v) = L(v), \tag{20.76}$$

for all $v \in V$. The Galerkin finite element method takes form: find $U \in V_h$ such that,

$$a(U, v) = L(v), \tag{20.77}$$

for all $v \in V_h$, with $V_h \subset V$. Galerkin orthogonality is expressed as

$$a(e, v) = 0, \quad \forall v \in V_h, \tag{20.78}$$

for the error $e = u - U$.

The adjoint equation

We define an output functional of the form

$$M(u) = (u, \psi), \tag{20.79}$$

and to estimate the output error

$$M(u) - M(U), \tag{20.80}$$

we introduce the adjoint problem: find $\varphi \in V$, such that

$$a(v, \varphi) = M(v), \quad \forall v \in V. \tag{20.81}$$

We have that

$$M(u) - M(U) = a(u, \varphi) - a(U, \varphi) = L(\varphi) - a(U, \varphi) = r(U, \varphi), \tag{20.82}$$

with the weak residual

$$r(U, \varphi) = L(\varphi) - a(U, \varphi). \tag{20.83}$$

We can split the integral over the elements K in the mesh $\mathcal{T}^h$, so that

$$M(u) - M(U) = r(U, \varphi) = \sum_{K \in \mathcal{T}^h} r_K(U, \varphi) = \sum_{K \in \mathcal{T}^h} \mathcal{E}_K, \tag{20.84}$$

with the error indicator

$$\mathcal{E}_K = r_K(U, \varphi). \tag{20.85}$$

To approximate the error indicator we can compute an approximation to the adjoint problem, $\Phi \approx \varphi$, so that

$$\mathcal{E}_K \approx r_K(U, \Phi). \tag{20.86}$$

The Navier-Stokes equations

The weak form of the Navier-Stokes equations can be formulated in terms of the weak residual, as the problem to find $\hat{u} = (u, p) \in V \times Q$, such that,

$$r(\hat{u}, \hat{v}) = 0, \tag{20.87}$$

for all $\hat{v} = (v, q) \in V \times Q$. The Galerkin finite element method then takes the form: find $\hat{U} = (U, P) \in V_h \times Q_h$, such that,

$$r(\hat{U}, \hat{v}) = 0, \tag{20.88}$$

for all $\hat{v} = (v, q) \in V_h \times Q_h$.

The a posteriori error analysis can then be extended also to the Navier-Stokes equations, with modifications for the nonlinearity of the equations, and possible stabilization terms in the finite element method.