Cours Analyse 3 - Aut 2016 - MT/SV ⓪

Sem

1 grad div rot

2 $\int_\Gamma f\,dl$ $\int_\Gamma F\cdot dl$ $\int_\Gamma \text{grad} f\cdot dl = f(B)-f(A)$

3 Green $\iint_\Omega \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\,dx_2 = \int_{\partial\Omega} F\cdot dl$

4 $\iint_\Sigma f\,ds$ $\iint_\Sigma F\cdot ds$

5 $\iint_\Sigma \text{rot}\, F\cdot ds = \int_{\partial\Sigma} F\cdot dl$

6 $\iiint_\Omega F\,dx_1 dx_2 dx_3$ $\iiint_\Omega \text{div}\, F\, dx_1 dx_2 dx_3 = \iint_{\partial\Omega} (F\cdot\nu)\,ds$

7 Applications : loi de conservation

8 fonctions holomorphes $f = u + iv$, Cauchy Rieman

9 Thm Cauchy $\int_\gamma f(z)\,dz = 0$

10 Form int. Cauchy $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\alpha)}{(\alpha - z)}\,d\alpha$

11 Séries de Laurent $f(z) = \sum_{n=-\infty} c_n (z-z_0)^n$ $c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\alpha)}{(\alpha - z_0)^{n+1}}\,d\alpha$

12 Thm résidus $\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z_k}(f)$

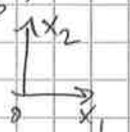
13 Calcul d'intégrales

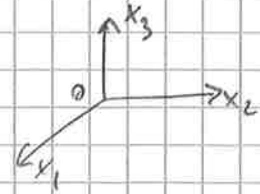
14

Sem 1

$x=(x_1, x_2, \dots, x_n)$ $\quad$ $n=2$ $\quad x=(x_1, x_2)$

$n=3$ $\quad x=(x_1, x_2, x_3)$





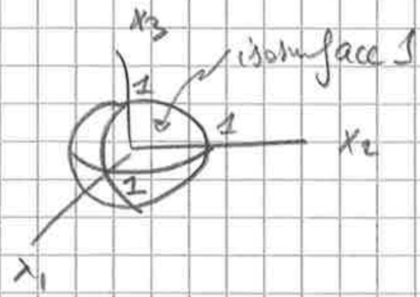
$f: \mathbb{R}^n \to \mathbb{R}$ champ scalaire

$x \to f(x)$ $\quad$ $\vec{x} \to f(\vec{x})$

$(x_1, x_2, \dots, x_n) \to f(x_1, x_2, \dots, x_n)$

Ex: $\overbrace{\ }^{n=3}$ $f(x_1 x_2 x_3) = x_1^2 + x_2^2 + x_3^2$

isosurface 1: $\{(x_1, x_2, x_3) \in \mathbb{R}^3;\ f(x_1, x_2, x_3) = 1\}$



opérateur: champ $\to$ champ

gradient: $\quad \vec{grad} : f \to \vec{grad} f$

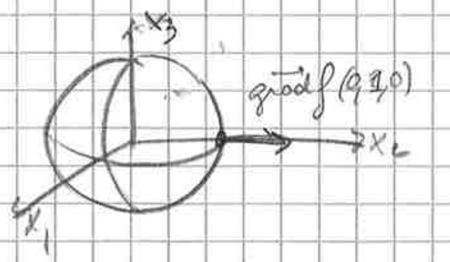
$f(\vec{x}) \to \vec{grad} f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$

$= \nabla f \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

Ex 1: $f(x_1 x_2 x_3) = x_1^2 + x_2^2 + x_3^2$

$\vec{grad} f(x) = (2x_1, 2x_2, 2x_3)$

$\vec{grad} f(0,1,0)$? $\qquad f(0,1,0) = 1 \quad \vec{grad} f(0,1,0) = (0,2,0)$



$\Delta f(x) = \frac{\partial^2 f}{\partial x_1^2}(x) + \frac{\partial^2 f}{\partial x_2^2}(x) + \frac{\partial^2 f}{\partial x_3^2}(x)$

Ex1 $\quad f(x_1 x_2 x_3) = x_1^2 + x_2^2 + x_3^2$ $\qquad$ Exercice: calc $\vec{grad} f, \Delta f$

$\Delta f = 2+2+2 = 6$ $\qquad f(x_1 x_2 x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}\ (= r)$

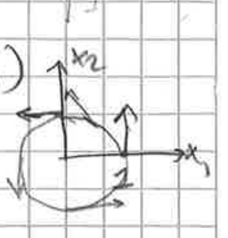
F: champ vectoriel

$F: \mathbb{R}^n \to \mathbb{R}^n$

$\vec{x} \to F(\vec{x})$

$(x_1, x_2, \dots x_n) \to F_1(x_1 x_2 \dots x_n), F_2(\quad), \dots, F_n$

Ex2: $\overbrace{\ }^{n=2}$ $F(x_1, x_2) = (-x_2, x_1)$



opérateur divergence

$\mathrm{div} : F \to \mathrm{div}\, F$

$$\vec{F}(\vec{x}) \to \mathrm{div}\, \vec{F}(\vec{x}) = \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2}{\partial x_2}(x) + \dots + \frac{\partial F_n}{\partial x_n}(x)$$

Ex 2 $\mathrm{div}\, F(x) = \mathrm{div}\,(\underbrace{-x_2}_{F_1(x_1 x_2)}\ \underbrace{x_1}_{F_2(x_1 x_2)}) = 0$

$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

$\mathrm{div}\vec{F} = \vec{\nabla} \cdot \vec{F} =$

opérateur rotationnel (curl en anglais)

$\mathrm{rot}\, F$

$n = 2$ $\quad F(x_1 x_2) = (F_1(x_1 x_2), F_2(x_1 x_2))$

$$\mathrm{rot}\, F(x_1, x_2) = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}$$

$\mathrm{rot} : \underset{\text{ch vect}}{\vec{F}} \to \underset{\text{ch scal}}{\mathrm{rot}\,\vec{F}}$

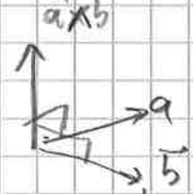
$n = 3$ $\quad \mathrm{rot} : \underset{\substack{\in \mathbb{R}^3 \\ \text{ch vect}}}{\vec{F}(x)} \to \underbrace{\mathrm{rot}\,\vec{F}(x)}_{\substack{\in \mathbb{R}^3 \\ \text{ch vect}}}$

$\vec{\mathrm{rot}}\,\vec{F} = \vec{\nabla} \times \vec{F} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \times (F_1, F_2, F_3)$

$$\vec{a} \wedge \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

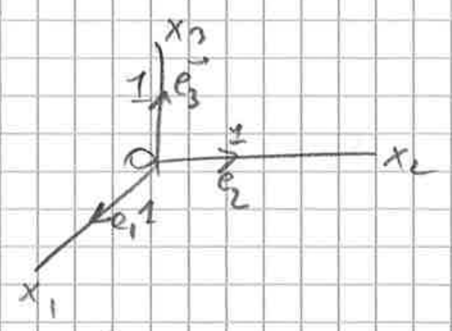
$\wedge$: $\times$ anglais

cross pro



$\vec{a} \wedge \vec{b}$ est orthog $\vec{a}$ et $\vec{b}$

$$\vec{\mathrm{rot}}\,F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



$$= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)$$

$(1,2,3) \to (2,3,1)$

Rque $\vec{F}(x_1 x_2 x_3) = (F_1(x_1 x_2), F_2(x_1 x_2), 0)$

$\vec{\mathrm{rot}}\,\vec{F} = \left(0, 0, \underbrace{\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}}_{\mathrm{rot}\,F}\right)$

Ex 2 $\quad F(x_1, x_2) = (-x_2, x_1)$

$\mathrm{rot}\, F(x_1, x_2) = 2$

$F(x_1 x_2, x_3) = (-x_2, x_1, 0)$

$\mathrm{rot}\, F(x_1, x_2, x_3) = (0, 0, 2)$

Notation ∇

$f : \mathbb{R} \to \mathbb{R}$

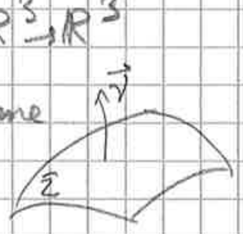
$x \mapsto f(x)$ (ex $f(x) = \sin x$)

$f'(x)$ ($\cos x$)

$\int_a^b f'(x)\,dx = f(b) - f(a)$

But : généraliser ceci dans le cas où $f : \mathbb{R}^3 \to \mathbb{R}$ $F : \mathbb{R}^3 \to \mathbb{R}^3$

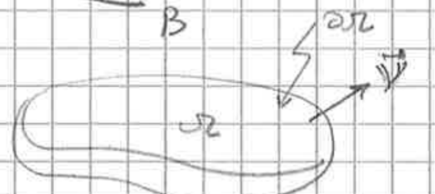
relations entre intégrales curvilignes, de surface, de volume



ex : . $\int_\Gamma \vec{\text{grad}}\, f \cdot \vec{dl} = f(B) - f(A)$

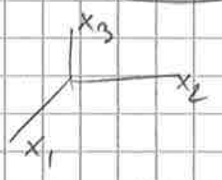
int curviligne

. $\iiint_\Omega \text{div}\, \vec{F}\, dx_1 dx_2 x_3 = \iint_{\partial\Omega} \vec{F} \cdot \vec{\nu}\, ds$



int de volume   int de surface

ou encore $\iiint_\Omega \frac{\partial f}{\partial x_1} dx_1 dx_2 dx_3 = \iint_{\partial\Omega} f \nu_1\, ds$



. $\vec{F}$ est un chp vect à div nulle :

$\iint_{\partial\Omega} \vec{F} \cdot \vec{\nu}\, ds = 0$ débit entrant = débit sortant $\forall\, \Omega$

relations ($n=3$) $\Delta = \text{div grad}$

$\text{div rot} = 0$

$\text{rot grad} = 0$

$\text{div}(fF) = f\,\text{div}F + F \cdot \text{grad}\, f$ (ex)

Navier-Stokes

$\rho(u \cdot \nabla)u - \mu \Delta u + \nabla p = f$

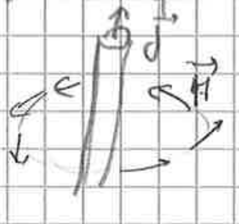
$\text{div}\, u = 0$

$\vec{u}$: vitesse

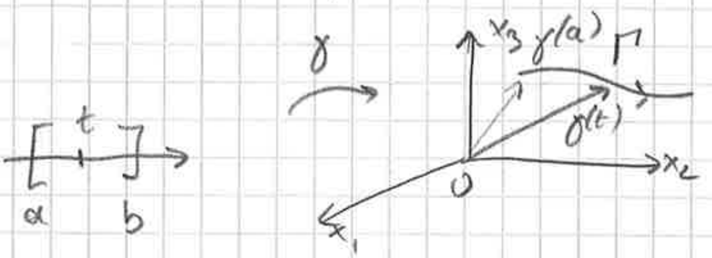
Elasticité $+\mu \Delta u + \overset{(+\mu)}{\lambda}\, \text{grad div}\, u = f$

$\vec{u}$: def par à la pos d'éq

Maxwell $\vec{\text{rot}}\, \vec{H} = \vec{J}$



Courbe simple (def 88 livre)



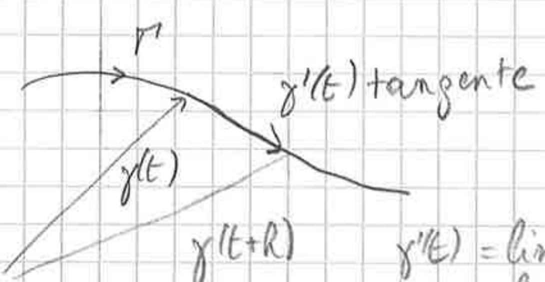
Ex : $\gamma : [0, \frac{\pi}{2}] \to \mathbb{R}^3$

$t \to \gamma(t) = (\cos t, \sin t, 0)$

1/4 cercle centre O rayon 1

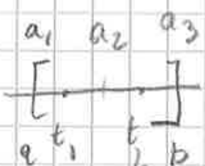
$\gamma : [0, \pi/2] \to \Gamma$ bijective, régulière

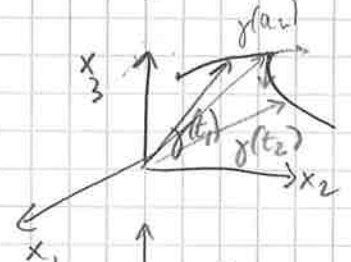
$\gamma : [0, 2\pi] \to \mathbb{R}^3$ courbe simple (bij : $[0, 2\pi[\to \Gamma$) fermée



$\gamma(t+h)$ $\qquad \gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} \neq 0$

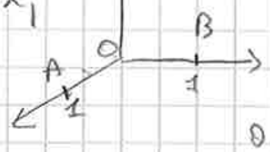
plus tard





Γ courbe régulière par morceaux

$\gamma \in C^0([a_1, a_3])$ $\gamma \in C^1([a_1, a_2]) \cap C^1([a_2, a_3])$



$\gamma : [0, 1] \to \mathbb{R}$

$t \to \gamma(t)$

$0 \leq t \leq 1/2$ $\quad \gamma(t) = \vec{OA} + 2t \vec{AO} = (1-2t)\vec{OA}$

$= (1-2t)(1, 0, 0)$

$1/2 \leq t \leq 1$ $\quad \gamma(t) = 2(t - \frac{1}{2})\vec{OB} = 2(t - \frac{1}{2})(0, 1, 0)$

$f : \mathbb{R}^n \to \mathbb{R}$

$(x_1, x_2, \dots x_n) \to f(x_1, x_2, \dots, x_n)$

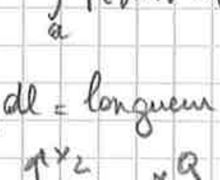
$\gamma : [a, b] \to \Gamma$

$t \to \gamma(t)$

$$\int_\Gamma f \, dl = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt = \int_a^b f(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)) \sqrt{(\gamma_1'(t))^2 + \dots + (\gamma_n'(t))^2} \, dt$$

$f = 1$ $\quad \int_\Gamma dl = \text{longueur}(\Gamma)$

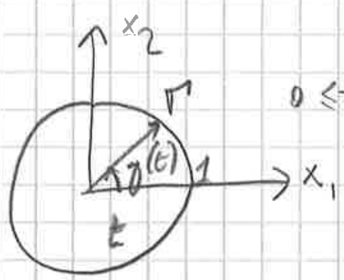
Ex $n = 2$



$0 \leq t \leq 1$

$\gamma(t) = \vec{OP} + t\vec{PQ}$ $\quad \gamma'(t) = \vec{PQ}$ $\quad \|\gamma'(t)\| = \|\vec{PQ}\|$

$$\int_\Gamma dl = \int_0^1 \|\vec{PQ}\| \, dt = \|\vec{PQ}\| \int_0^1 dt = \|\vec{PQ}\|$$



$0 \leq t \leq 2\pi \quad \gamma(t) = (\cos t, \sin t) \qquad \gamma'(t) = (-\sin t, \cos t) \quad \|\gamma'(t)\| = 1$

$$\int_\Gamma dl = \int_0^{2\pi} dt = 2\pi$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \to f(x_1, x_2) = x_1 \qquad f(\gamma(t)) = f(\gamma_1(t), \gamma_2(t)) = \cos t$$

$$\int_\Gamma f dl = \int_0^{2\pi} \cos t = 0$$

$$F : \mathbb{R}^n \to \mathbb{R}^n$$
$$(x_1 x_2 \dots x_n) \to (F_1(x_1, x_2 \dots x_n), F_2(\quad), \dots F_n(x \quad))$$

$$\int_\Gamma F \cdot dl = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \left(F_1(\gamma(t))\gamma_1'(t) + \dots + F_n(\gamma(t))\gamma_n'(t)\right) dt$$

Ex : $F : \mathbb{R}^2 \to \mathbb{R}$

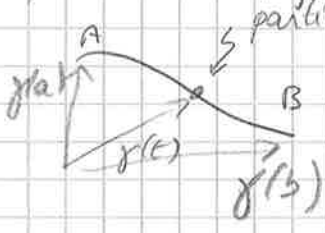
$(x_1 x_2) \to (-x_2, x_1)$

Γ cercle centre O ray 1 $\qquad F(\gamma(t)) = (-\gamma_2(t), \gamma_1(t)) = (-\sin t, \cos t)$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = 2\pi$$

Rque : il existe une infinité de param d'un courbe mais heureusement $\int_\Gamma$ fermé dep pas de la param

traj d'1 particule, masse m force F



$$m\gamma''(t) = F(\gamma(t))$$

$$\int_a^b m\gamma''(t)\gamma'(t)\,dt = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$\frac{m}{2}(\gamma'(b))^2 - \frac{m}{2}(\gamma'(a))^2 = \int_\Gamma F \cdot dl \qquad \text{thm E cinétique}$$

Ici reg/morceaux

<u>Thm 3.3bis</u> : $\Omega \subset \mathbb{R}^3$ un domaine $f : \Omega \to \mathbb{R}$ $\mathcal{C}^1$ Γ courbe simple reg par morceaux d'origine A et extr B

$$\int_\Gamma \operatorname{grad} f \cdot dl = f(B) - f(A)$$

$$\left(\int_a^b f'(x)dx = f(b) - f(a)\right)$$

<u>Dem</u> : Γ simple reg

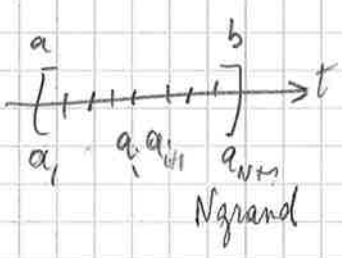
$$\int_\Gamma \operatorname{grad} f \cdot dl = \int_a^b \operatorname{grad} f(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b \left(\frac{\partial f}{\partial x_1}(\gamma(t))\gamma_1'(t) + \dots + \frac{\partial f}{\partial x_n}(\gamma(t))\gamma_n'(t)\right) dt$$

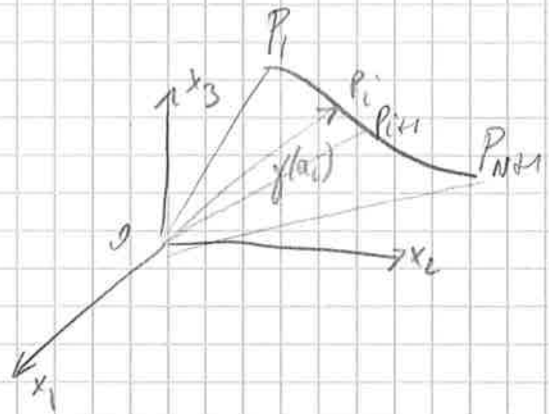
$$\frac{d}{dt} f(\gamma(t)) =$$

$$= \int_a^b \frac{d}{dt} f(\gamma(t))\,dt = f(\gamma(b)) - f(\gamma(a))$$

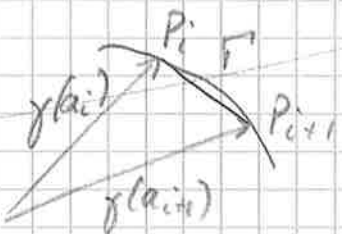
Corollaire si F dérive d'1 pot ($\exists f$ tq $F = \operatorname{grad} f$) et si Γ est fermée alors $\int_\Gamma F \cdot dl = 0$

Pourquoi $\|\gamma'(t)\|$?





definition $\int_\Gamma f\,dl = \int_a^b f(\gamma(t))\|\gamma'(t)\|\,dt = \sum_{i=1}^N \int_{a_i}^{a_{i+1}} f(\gamma(t))\|\gamma'(t)\|\,dt \simeq$



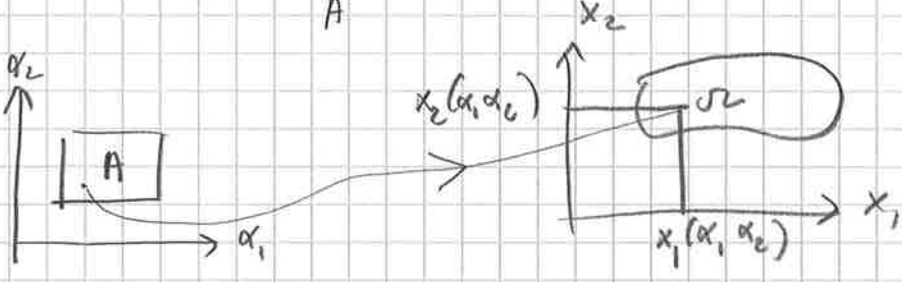
intuition : $\int_\Gamma f\,dl \simeq \sum_{i=1}^N \int_{P_iP_{i+1}} f\,dl \simeq \sum_{i=1}^N f(P_i)\|\overrightarrow{P_iP_{i+1}}\| = \sum_{i=1}^N f(\gamma(a_i))\,\|\gamma(a_{i+1})-\gamma(a_i)\|$ (1)

$\left(\int_{a_i}^{a_{i+1}} f(\gamma(t))\|\gamma'(t)\| \simeq \cancel{|a_{i+1}-a_i|}\, f(\gamma(a_i))\, \left\| \frac{\gamma(a_{i+1})-\gamma(a_i)}{\cancel{a_{i+1}-a_i}} \right\|\right.$ (2)

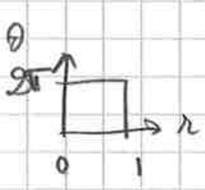
les approx (1) et (2) coïncident

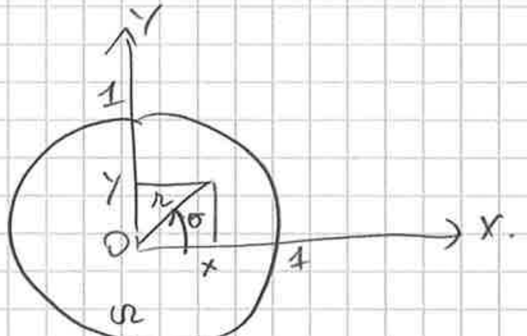
Rappel : Calcul d'intégrales dans $\mathbb{R}^2$ et changement de variable

$$= \iint_\Omega f(x_1,x_2)\,dx_1\,dx_2 = \iint_A f(x_1(\alpha_1,\alpha_2),x_2(\alpha_1,\alpha_2)) \left|\det\begin{pmatrix}\frac{\partial x_1}{\partial \alpha_1} & \frac{\partial x_1}{\partial \alpha_2} \\ \frac{\partial x_2}{\partial \alpha_1} & \frac{\partial x_2}{\partial \alpha_2}\end{pmatrix}\right| d\alpha_1\,d\alpha_2$$



Ex





$x = r\cos\theta \qquad 0 \le r \le 1$

$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$

$f(x,y) = x^2+y^2$

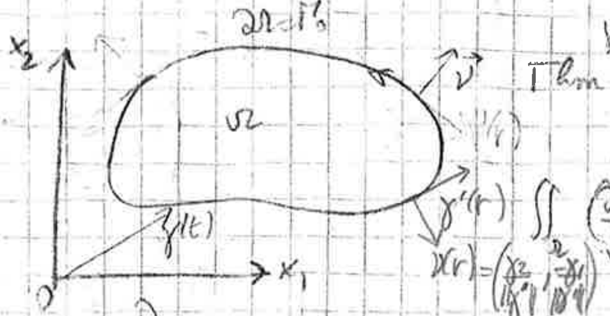
$f(r\cos\theta, r\sin\theta) = r^2$

$$\det\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix} = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r$$

$$\iint_\Omega f(x,y)\,dx\,dy = \int_0^1 dr \int_0^{2\pi} d\theta \; r^2 r$$

$$= \left[\frac{r^4}{4}\right]_0^1 2\pi = \frac{\pi}{2}$$

Sem 3 : Thm de Green



Thm (de Green) Ω domaine régulier, $\partial\Omega$ sens trigo $F : \mathbb{R}^2 \to \mathbb{R}^2$ $\mathcal{C}^1$, $(x_1,x_2) \mapsto F(x_1,x_2) = (F_1(x_1,x_2), F_2(x_1,x_2))$

$$\iint_\Omega \underbrace{\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)}_{\text{rot}\,F\ (\text{scalaire})}(x_1,x_2)\,dx_1\,dx_2 = \int_{\partial\Omega} F \cdot dl$$

Corollaire (thm de la div) ν normale unité ext à Ω

$$\iint_\Omega \underbrace{\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)}_{\text{div}\,F}(x_1,x_2)\,dx_1\,dx_2 = \int_{\partial\Omega} \vec{F} \cdot \vec{\nu}\,dl$$

Dem : $G = (-F_2, F_1)$

$$\iint_\Omega \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)(x_1,x_2)\,dx_1\,dx_2 = \int_{\partial\Omega} G \cdot dl = \int_a^b \left(-F_2(\gamma_1(t),\gamma_2(t)), F_1(\gamma_1(t),\gamma_2(t))\right) \cdot (\gamma_1'(t), \gamma_2'(t))\,dt$$

$$= \int_a^b (-F_2\gamma_1' + F_1\gamma_2')\,dt = \int_\Gamma (F\cdot\nu)\,dl = \int_a^b (F_1\nu_1 + F_2\nu_2)\|\gamma'(t)\|\,dt$$

(γ_1, γ_2)

$(\gamma_2, -\gamma_1)$

$$\int_\Gamma F\cdot\nu\,dl = \int_a^b F_1(\gamma_1(t),\gamma_2(t))\gamma_2'(t) - F_2(\gamma_1(t),\gamma_2(t))\gamma_1'(t)$$

Γ_0 $\partial\Omega = \Gamma_0 \cup \Gamma_1$

Rque : thm vrai si $\partial\Omega$ courbe fermée reg par morceaux et si trous

Ex (Thm) : $F(x_1,x_2) = (-x_2^2, x_1)$ Γ cercle centre O rayon 1

$$\text{rot}\,F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} = 1 - 2x_2$$

$$\iint_\Omega \text{rot}\,F(x_1,x_2)\,dx_1\,dx_2 = \iint_\Omega (1-2x_2)\,dx_1\,dx_2$$

chgt variable $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ $0<r<1$ $0<\theta<2\pi$

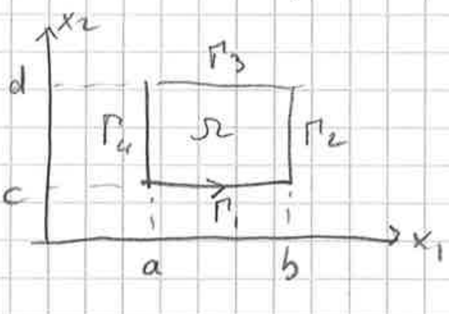
$$= \int_0^1 r\,dr \int_0^{2\pi} d\theta\,(1 - 2r\sin\theta) = \int_0^1 r\,dr \int_0^{2\pi} d\theta = \frac{1}{2}2\pi = \pi$$

$\int_{\partial\Omega} F\cdot dl$ $\gamma(t) = (\cos t, \sin t)$ $0 \leq t \leq 2\pi$

$$= \int_0^{2\pi} F(\gamma(t))\cdot\gamma'(t)\,dt = \int_0^{2\pi} (-\sin^2 t, \cos t)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \frac{1}{2}\int_0^{2\pi} (1+\cos 2t)\,dt = \frac{2\pi}{2} = \pi$$

Dem Thm Green : cas part



$$\iint_\Omega \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\, dx_2 = \int_a^b dx_1 \int_c^d dx_2 \left(\frac{\partial F_2}{\partial x_1}(x_1,x_2) - \frac{\partial F_1}{\partial x_2}(x_1,x_2)\right) dx_1\, dx_2$$

$$= \int_c^d \left(\int_a^b \frac{\partial F_2}{\partial x_1}(x_1,x_2)\,dx_1\right) dx_2 - \int_a^b \left(\int_c^d \frac{\partial F_1}{\partial x_2}(x_1 x_2)\,dx_2\right) dx_1$$

$$= \int_c^d \left(F_2(b,x_2) - F_2(a,x_2)\right) dx_2 - \int_a^b \left(F_1(x_1,d) - F_1(x_1,c)\right) dx_1$$

$$\int_{\partial\Omega} F.dl = \int_{\Gamma_1} F.dl + \int_{\Gamma_2} F.dl + \int_{\Gamma_3} F.dl + \int_{\Gamma_4} F.dl$$

Γ_1 : param	$\gamma_1(t) = (t,c)$		$a \le t \le b$	$\gamma_1'(t) = (1,0)$
Γ_2	γ_2	$= (b,t)$	$c \le t \le d$	$\gamma_2'(r) = (0,1)$
Γ_3	γ_3	$= (t,d)$	$b \le t \le a$	$(1,0)$
Γ_4	γ_4	$= (a,t)$	$d \le t \le c$	$(0,1)$

$$\int_{\Gamma_1} F.dl = \int_a^b F_1(t,c)\,dt$$

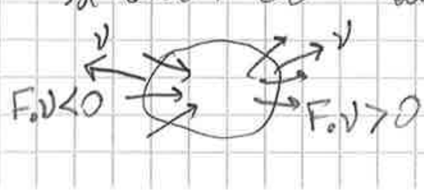
$$\int_{\Gamma_2} F.dl = \int_c^d F_2(b,t)\,dt$$

$$\int_{\Gamma_3} F.dl = \int_b^a F_1(t,d)\,dt = -\int_a^b F_1(t,d)\,dt$$

$$\int_{\Gamma_4} F.dl = \int_d^c F_2(a,t)\,dt = -\int_c^d F_2(a,t)\,dt$$

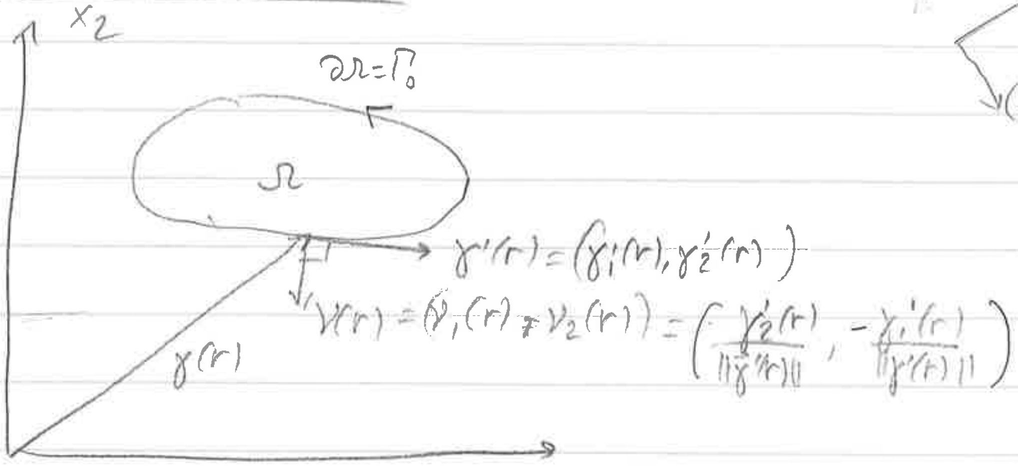
Application $F : \mathbb{R}^2 \to \mathbb{R}^2$
$(x_1 x_2) \to F(x_1,x_2) = (F_1(x_1,x_2), F_2(x_1,x_2))$

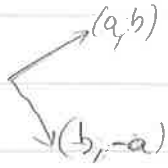
si $\operatorname{div} F = 0$ alors $\int_{\partial\Omega} F.\nu\, dl = 0$ $\forall$ domaine Ω regulier



7bis

Sem 3 : Théorème de Green





Lemme : Ω domaine régulier, $\partial\Omega$ frontière (ou bord) orientée dans le sens trigo

$f : \mathbb{R}^2 \to \mathbb{R}$ C^1
$(x_1 x_2) \to f(x_1 x_2)$

$$\iint \frac{\partial f}{\partial x_1}(x_1, x_2)\, dx_1 dx_2 = \int f \nu_1\, dl$$

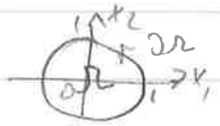
$$\iint_\Omega \frac{\partial f}{\partial x_2}(x_1 x_2)\, dx_1 dx_2 = \int_{\partial\Omega} f \nu_2\, dl$$

Corollaire : (Thm 4.2 et Corollaire 4.3 livre) $F : \mathbb{R}^2 \to \mathbb{R}^2$ C^1
$(x_1 x_2) \to (F_1(x_1 x_2), F_2(x_1 x_2))$

$$\iint_\Omega \operatorname{rot} F\, dx_1 dx_2 = \iint_\Omega \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1 dx_2 = \int_{\partial\Omega} F \cdot dl$$

$$\iint_\Omega \operatorname{div} F\, dx_1 dx_2 = \iint_\Omega \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) dx_1 dx_2 = \int_{\partial\Omega} (F \cdot \nu)\, dl$$

Ex : $F(x_1 x_2) = (-x_2^2, x_1)$ Ω = boule centre 0 rayon 1



$$\iint_\Omega \operatorname{rot} F(x_1 x_2)\, dx_1 dx_2 = \iint_\Omega (1 + 2x_2)\, dx_1 dx_2$$ coord. cylindriques

$x_1 = r\cos\theta$ $x_2 = r\sin\theta$ $dx_1 dx_2 \to r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^1 r\left[\theta + 2\sin\theta\right]_{\theta=0}^{\theta=2\pi} dr = 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=1} = \pi$$

$$\int_{\partial\Omega} F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$ $\gamma(t) = (\cos t, \sin t)$ $\gamma'(t) = (-\sin t, \cos t)$

$F(\gamma(t)) = (-\sin^2 t, \cos t)$ $\cos^2 t$ $2\cos^2 = 1+\cos 2t$

$$= \int_0^{2\pi} (+\sin^3 t + \cos^2 t)\, dt = \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = \frac{1}{2}\int_0^{2\pi} dt = \pi$$

=

Dem corollaire :

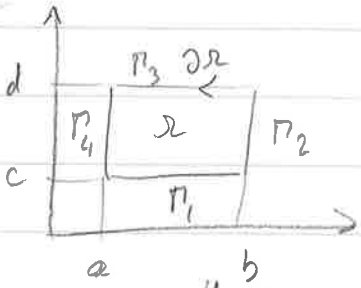
- $\iint_\Omega \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) dx_1\, dx_2 = \int_{\partial\Omega} (F_1\nu_1 + F_2\nu_2)\, dl = \int_{\partial\Omega} (F\cdot\nu)\, dl$
- $\iint_\Omega \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\, dx_2 = \int_{\partial\Omega}(F_2\nu_1 - F_1\nu_2)\, dl$

$$\int_{\partial\Omega} F\cdot dl = \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt = \int_a^b \left(F_1(\gamma(t))\gamma_1'(t) + F_2(\gamma(t))\gamma_2'(t)\right) dt$$

$$= \int_a^b \left(-F_1(\gamma(t))\nu_2(t) + F_2(\gamma(t))\nu_1(t)\right)\|\gamma'(t)\|\,dt = \int_{\partial\Omega}(F_2\nu_1 - F_1\nu_2)\, dl$$

Dem lemme :

- Ω rectangle



$$\iint_\Omega \frac{\partial f}{\partial x_1}(x_1,x_2)\,dx_1\, dx_2 = \int_c^d \left(\int_a^b \frac{\partial f}{\partial x_1}(x_1,x_2)\,dx_2\right) dx_2$$

$$= \int_c^d \left[f(x_1,x_2)\right]_{x_1=a}^{x_1=b} dx_2 = \int_c^d \left(f(b,x_2) - f(a,x_2)\right) dx_2$$

$$\int_{\partial\Omega} f\nu_1\, dl = \int_{\Gamma_1} f\nu_1\, dl + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}$$ sur $\Gamma_1 \cup \Gamma_3$ $\nu_1 = 0$

$$= \int_{\Gamma_2} f\nu_1\, dl + \int_{\Gamma_4} f\nu_1\, dl$$

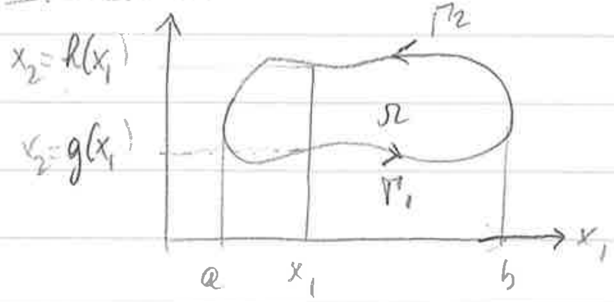
Γ_2 : $\gamma(t) = (b,t)$ $\quad c \le t \le d \quad$ $\gamma'(t) = (0,1)$ $\quad$ $\nu(t) = (1,0)$

$-\Gamma_4$: $\gamma(t) = (a,t)$ $\quad c \le t \le d \quad$ $\gamma'(t) = (0,1)$ $\quad$ $\nu(t) = (1,0)$

$$\int_{\partial\Omega} f\nu_1\, dl = \int_c^d f(b,t)\,dt - \int_c^d f(a,t)\,dt$$

idem avec $\iint_\Omega \frac{\partial f}{\partial x_2}(x_1,x_2)\,dx_1\, dx_2$

- Ω particulier



(9 bis)

$$\iint_\Omega \frac{\partial f}{\partial x_2} dx_1 dx_2 = \int_a^b \left(\int_{g(x_1)}^{h(x_1)} \frac{\partial f}{\partial x_2}(x_1 x_2) dx_1 \right) dx_2 = \int_a^b \Big(f(x_1, h(x_1)) - f(x_1, g(x_1)) \Big) dx_1$$

$$\int_{\partial\Omega} f\nu_2 \, dl = \int_{\Gamma_1} f\nu_2 \, dl + \int_{\Gamma_2} f \nu_2 \, dl$$

Γ_1 : $\gamma(t) = (t, g(t))$ $a \leq t \leq b$ $\gamma'(t) = (1, g'(t))$ $\nu(t) = \left(\frac{g'(t)}{\sqrt{1+}}, \frac{-1}{\sqrt{1+}} \right)$

$\|\gamma'(t)\| = \sqrt{1+}$

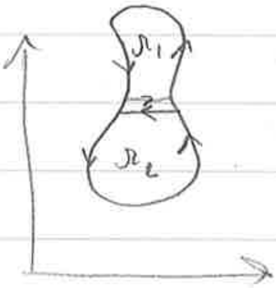
$$\int_{\Gamma_1} f\nu_2 \, dl = \int_a^b f(t, g(t)) \nu_2(t) \|\gamma'(t)\| dt = -\int_a^b f(t, g(t)) dt$$

$-\Gamma_2$: $\gamma(t) = (t, h(t))$ $a \leq t \leq b$ $\gamma'(t) = (1, h'(t))$.

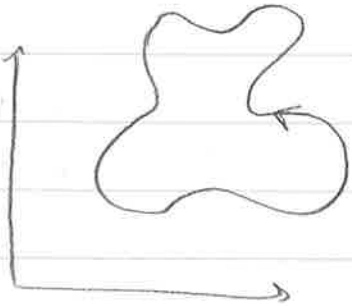
$$\int_{\Gamma_2} f\nu_2 \, dl = \int_a^b f(t, h(t)) dt$$

$$\int_{\partial\Omega} f\nu_2 \, dl = \int_a^b \Big(f(t, h(t)) - f(t, g(t)) \Big) dt$$

idem si



ou si

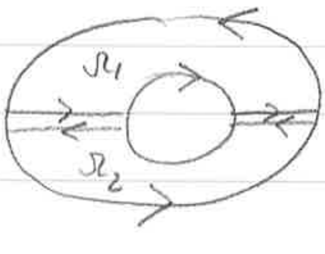


Remarque : le lemme est vrai si aussi

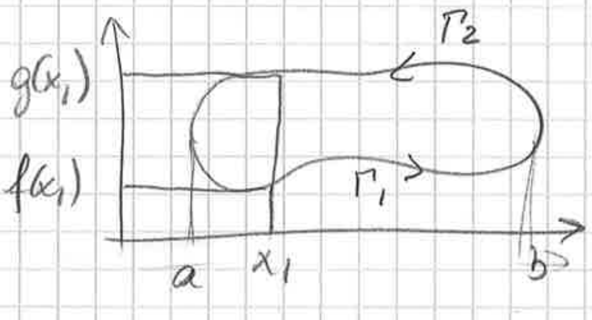
irrégulier par morceaux

$\partial\Omega = \Gamma_0 \cup \Gamma_1$

dem :



$\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$



Γ_1: $\gamma_1(t) = (t, f(t))$ $a \le t \le b$ $\gamma_1'(t) = (1, f'(t))$

Γ_2 $\gamma_2(t) = (t, g(t))$ $b \le t \le a$

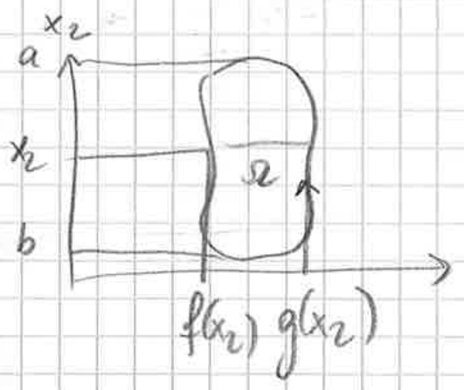
$F = (F_1, 0)$ $\displaystyle\iint_\Omega \frac{\partial F_1}{\partial x_2} = \int_a^b dx_1 \int_{f(x_1)}^{g(x_1)} \frac{\partial F_1}{\partial x_2}\, dx_2 = \int_a^b F_1(x_1, f(x_1)) - F_1(x_1, g(x_1))$

$$\int_{\partial\Omega} F \cdot dl = \int_{\Gamma_1} F \cdot dl + \int_{\Gamma_2} F \cdot dl$$

$$= \int_a^b F_1(t, f(t))\, dt + \int_b^a F_1(t, g(t))\, dt$$

$$- \int_a^b F_1(t, g(t))\, dt$$

$F = (0, F_2)$



$$\iint_\Omega \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx\, dx = -\iint \frac{\partial F_1}{\partial x_2}\, dx_1 dx_2$$

$$= \int_{\partial\Omega} F \cdot dl$$

etc...

Exercice : Soit $\Omega \subset \mathbb{R}^2$ un domaine régulier
Soit $u, v : \Omega \subset \ldots \to \mathbb{R}$ $\mathcal{C}^2$
Soit ν la normale extérieure unité à $\partial\Omega$
Montrer que

$$\iint_\Omega v \Delta u \, dx_1 dx_2 + \iint_\Omega \vec{\mathrm{grad}}\, u \cdot \vec{\mathrm{grad}}\, v \, dx dy = \int_{\partial\Omega} \vec{\mathrm{grad}}\, u \cdot \vec{\nu} \, v \, dl$$

Soit $f : \Omega \to \mathbb{R}$ $\mathcal{C}^0$ donnée, on cherche $u : \Omega \to \mathbb{R}$ $\mathcal{C}^2$ telle que

$$\begin{cases} -\Delta u = f & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

Montrer que la solution u de ce problème, si elle existe, satisfait

$$\iint_\Omega |\vec{\mathrm{grad}}\, u|^2 dx dy = \iint_\Omega f u \, dx dy$$

Montrer que la solution u de ce problème, si elle existe, est unique. Indication : soit u_1, u_2 deux solutions, montrer que $\iint_\Omega |\vec{\mathrm{grad}}\,(u_1 - u_2)|^2 dx dy$ et par conséquent que $u_1(x_1, x_2) = u_2(x_1, x_2)$ pour tout $(x_1, x_2) \in \Omega$.

Correction cf Ex 4.3.7 livre

Exercice : On considère un fluide compressible de densité volumique $\rho(x_1, x_2, t)$ et vitesse $\vec{v}(x_1, x_2, t)$ remplissant $\mathbb{R}^2$ et tel que $\frac{\partial \rho}{\partial t}(x_1, x_2, t) + \mathrm{div}\,(\rho(x_1, x_2, t)\, \vec{v}(x_1, x_2, t)) = 0$ pour tout $(x_1, x_2) \in \mathbb{R}^2$, pour tout $t > 0$. Soit Ω un domaine régulier de $\mathbb{R}^2$ de frontière $\partial\Omega$ orientée positivement. Montrer que

$$\frac{d}{dt} \iint_\Omega \rho \, dx_1 dx_2 + \int_{\partial\Omega} \rho \vec{v} \cdot \vec{\nu} \, dl = 0$$

Interpréter ce résultat.

Exercice : on considère un matériau de conductivité th $k(x_1, x_2, t)$ et de chaleur spécifique massique $c_p > 0$, et de densité volumique $\rho > 0$ et température $T(x_1, x_2, t)$ remplissant $\mathbb{R}^2$ et tq

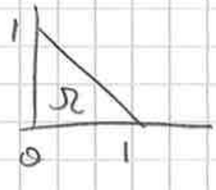
$$\rho c_p \frac{\partial T}{\partial t}(x_1,x_2,t) - \operatorname{div}\left(k(x_1,x_2,t)\,\operatorname{grad} T(x_1,x_2,t)\right) = 0$$

pour tout $(x_1,x_2) \in \mathbb{R}^2$, pour tout $t \geq 0$. Soit Ω... Montrer que

$$\frac{d}{dt}\iint_\Omega \rho c_p T(x_1,x_2,t)\,dx_1\,dx_2 + \int_{\partial\Omega} k \operatorname{grad} T \cdot \vec{\nu}\, dl = 0$$

Interpréter ce résultat.

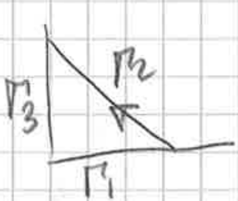
Exercice $F(x_1,x_2) = \left(0, \frac{x_1^2}{2}\right)$



$$\iint_\Omega \frac{\partial F_2}{\partial x_1} = \int_{\partial\Omega} F \cdot dl$$

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 (x_1) = \int_0^1 x_1(1-x_1)\,dx_1 = \int_0^1 (x_1 - x_1^2)\,dx$$

$$= \left[\frac{x_1^2}{2} - \frac{x_1^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



$\Gamma_1 : \gamma(t) = (t,0) \quad \gamma'(t) = (1,0) \quad 0 \leq t \leq 1$

$\Gamma_2 : \gamma(t) = (1-t, t) \quad \gamma'(t) = (-1, 1) \quad 0 \leq t \leq 1$

$\Gamma_3 : \gamma(t) = (0, 1-t) \quad \gamma'(t) = (0,-1)$ —

$$F(\gamma(t)) = \left(0, F_2(\gamma_1(t), \gamma_2(t))\right) = \left(0, \frac{(\gamma_1(t))^2}{2}\right)$$

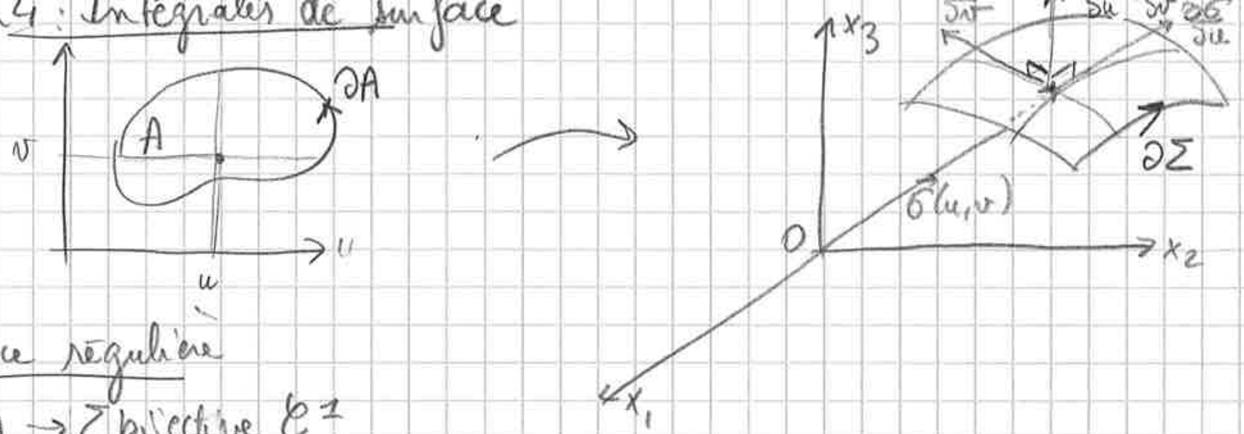
$$\int_{\partial\Omega} F\cdot dl = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} F\cdot dl$$

$$= \int_0^1 (0,t)\cdot(1,0)\,dt + \int_0^1 \left(0, \frac{(1-t)^2}{2}\right)\cdot(-1,1)\,dt + \int_0^1 (0,0)\cdot(0,-1)\,dt$$

$$= \int_0^1 \frac{(1-t)^2}{2} = \left[-\frac{(1-t)^3}{6}\right]_0^1 = \frac{1}{6}$$

Énoncé : vérifier le thm de Green pour la fonction F définie par $F(x_1,x_2) = \left(0, \frac{x_1^2}{2}\right)$ et pour $\Omega = \{(x_1,x_2) \in \mathbb{R}^2,\ 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1-x_1\}$

Sem 4 : Intégrales de surface



surface régulière

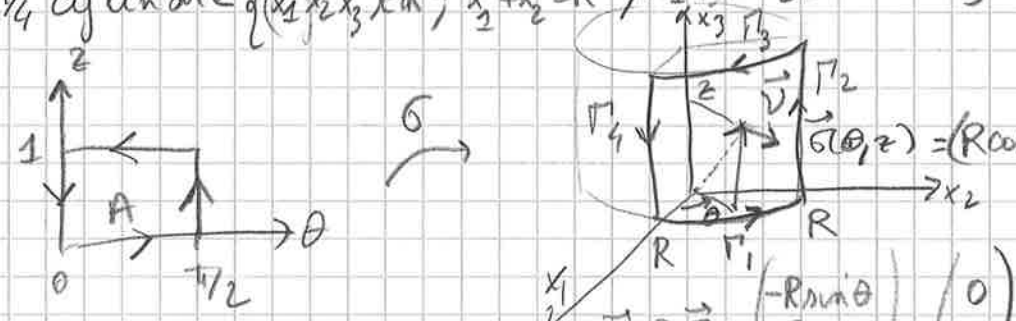
$\sigma : A \to \Sigma$ bijective $\mathcal{C}^1$

$\frac{\partial \sigma}{\partial u}(u,v) \wedge \frac{\partial \sigma}{\partial v}(u,v) \neq 0$

$\vec{\nu} = \dfrac{\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v}}{\| \quad \|}$ normale unité

bord $\partial\Sigma = \sigma(\partial A)$ orientation

Ex : 1/4 cylindre $\{(x_1, x_2, x_3) \in \mathbb{R}^3 ; x_1^2 + x_2^2 = R^2, x_1 \geq 0 \; x_2 \geq 0 \quad 0 \leq x_3 \leq 1\}$



$\vec{\sigma}(\theta, z) = (R\cos\theta, R\sin\theta, z)$

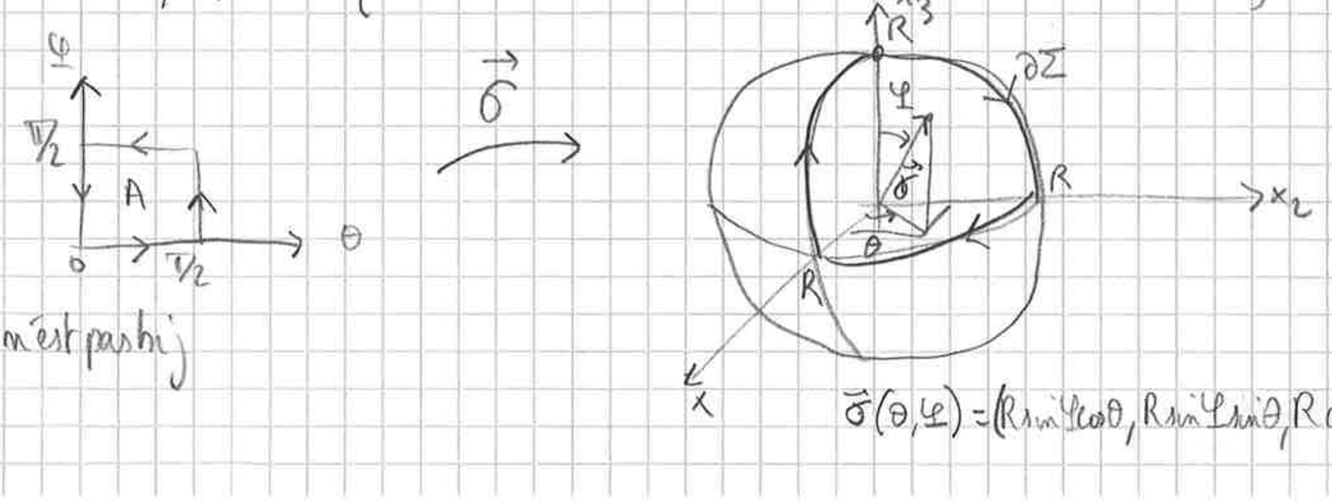
$$\frac{\partial \vec{\sigma}}{\partial \theta} \wedge \frac{\partial \vec{\sigma}}{\partial z} = \begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \end{pmatrix}$$

$\vec{\nu} = (\cos\theta, \sin\theta, 0)$

$\partial\Sigma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

orientation : règle d'Ampère, du tire bouchon du pouce

1/4 sphère $\{(x_1, x_2, x_3) \in \mathbb{R}^3 ; x_1^2 + x_2^2 + x_3^2 = R^2 \quad x_1 \geq 0 \; x_2 \geq 0 \; x_3 \geq 0\}$



n'est pas bij

$\vec{\sigma}(\theta, \varphi) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$

normale $\frac{\partial\sigma}{\partial\theta} \wedge \frac{\partial\sigma}{\partial\varphi} = \begin{pmatrix} -R\sin\varphi\sin\theta \\ R\sin\varphi\cos\theta \\ 0 \end{pmatrix} \wedge \begin{pmatrix} R\cos\varphi\cos\theta \\ R\cos\varphi\sin\theta \\ -R\sin\varphi \end{pmatrix} =$ (11)

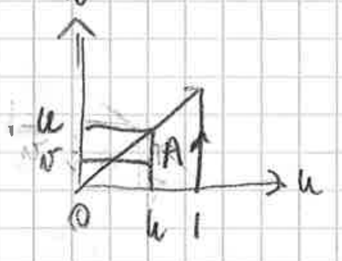
$$= \begin{pmatrix} -R^2\sin^2\varphi\cos\theta \\ -R^2\sin^2\varphi\sin\theta \\ -R^2\sin\varphi\cos\varphi \end{pmatrix} = -R\sin\varphi \ \vec{\sigma}(\theta,\varphi)$$

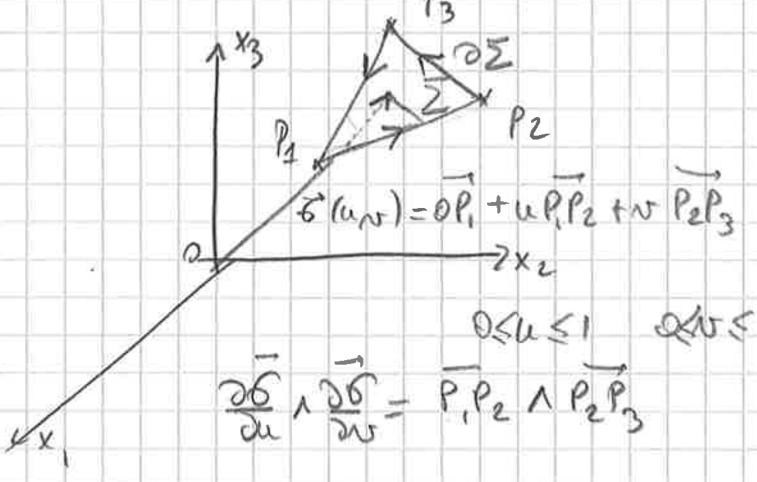
normale vers l'intérieur

Si on choisit $\vec{\sigma}(\varphi,\theta)$ = idem $\frac{\partial\sigma}{\partial\varphi} \wedge \frac{\partial\sigma}{\partial\theta} = R\sin\varphi\ \vec{\sigma}(\varphi,\theta)$

normale vers l'extérieur

triangle



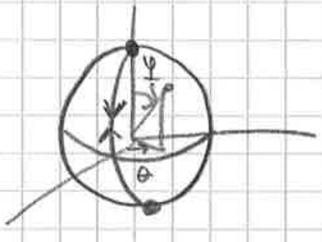


$\vec{\sigma}(u,v) = \vec{OP_1} + u\vec{P_1P_2} + v\vec{P_2P_3}$

$0 \le u \le 1 \quad 0 \le v \le u$

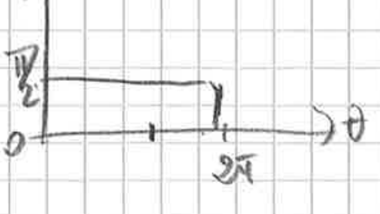
$\frac{\partial\vec{\sigma}}{\partial u} \wedge \frac{\partial\vec{\sigma}}{\partial v} = \vec{P_1P_2} \wedge \vec{P_2P_3}$

Rque sphère entière $= \{(x_1,x_2,x_3) \in \mathbb{R}^3 ;\ x_1^2+x_2^2+x_3^2 = R^2\}$ est une surface fermée (le bord $\partial\Sigma$ est ϕ)



le méridien est parcouru une fois dans un sens, une fois dans l'autre, on le supprime du bord

1/2 sphère $\{(x_1,x_2,x_3) \in \mathbb{R}^3 ;\ x_1^2+x_2^2+x_3^2 = R^2 \quad x_3 \geq 0\}$



$\partial\Sigma = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \quad x_3 = 0 \quad x_1^2+x_2^2 = R^2\}$

Cercle centre O rayon R plan Ox_1x_2

Def : intégrale de surface (12)

$\Sigma \subset \mathbb{R}^3$ surface régulière, $\vec{\sigma} : A \to \Sigma$ param,

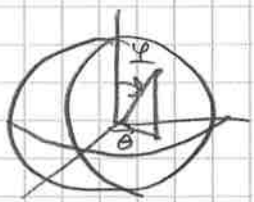
$f : \Sigma \to \mathbb{R}$ cont $\quad \iint_\Sigma f\,d\sigma = \iint_A f(\sigma(u,v)) \| \frac{\partial \vec{\sigma}}{\partial u}(u,v) \wedge \frac{\partial \vec{\sigma}}{\partial v}(u,v) \| \,du\,dv \qquad f=1 \iint_\Sigma d\sigma = \text{aire}(\Sigma)$

$F : \Sigma \to \mathbb{R}^3$ cont $\quad \iint_\Sigma \vec{F} . d\vec{\sigma} = \iint_A \vec{F}(\sigma(u,v)) . \left(\frac{\partial \vec{\sigma}}{\partial u}(u,v) \wedge \frac{\partial \vec{\sigma}}{\partial v}(u,v) \right) du\,dv$

$= \iint_\Sigma (\vec{F}.\vec{\nu})\,ds \quad$ car $\vec{\nu} = \frac{\partial \vec{\sigma}}{\partial u} \wedge \frac{\partial \vec{\sigma}}{\partial v} / \| \frac{\partial \vec{\sigma}}{\partial u} \wedge \frac{\partial \vec{\sigma}}{\partial v} \|$

Ex : Σ sphère centre O rayon R $\Sigma = \{(x_1,x_2,x_3) \in \mathbb{R}^3 ; x_1^2 + x_2^2 + x_3^2 = R^2\}$

$\vec{\sigma}(\varphi,\theta) = (R \sin\varphi \cos\theta, R\sin\varphi \sin\theta, R\cos\varphi)$



$\partial_\varphi \vec{\sigma} \wedge \partial_\theta \vec{\sigma} = \begin{pmatrix} +R\cos\varphi\cos\theta \\ +R\cos\varphi\sin\theta \\ -R\sin\varphi \end{pmatrix} \wedge \begin{pmatrix} -R\sin\varphi\sin\theta \\ +R\sin\varphi\cos\theta \\ 0 \end{pmatrix}$

$= \begin{pmatrix} R^2\sin^2\varphi\cos\theta \\ R^2\sin^2\varphi\sin\theta \\ R^2\cos\varphi\sin\varphi \end{pmatrix} \qquad \| \partial_\varphi \vec{\sigma} \wedge \partial_\theta \vec{\sigma} \| = R^2 \sin\varphi$

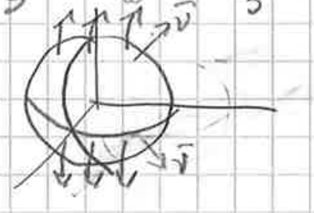
$\iint_\Sigma ds = \int_0^\pi d\varphi \int_0^{2\pi} d\theta \left(R^2 \sin\varphi \right) = 2\pi R^2 \int_0^\pi \sin\varphi\, d\varphi$

$= 2\pi R^2 \left[-\cos\varphi \right]_0^\pi = 4\pi R^2$

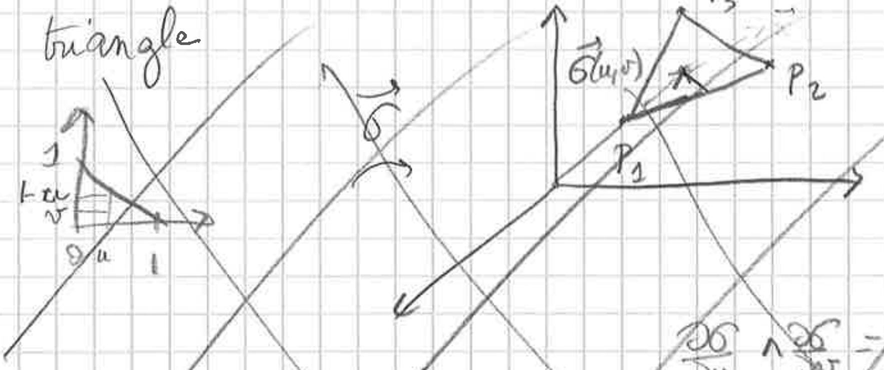
$f(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2 \quad \iint_\Sigma f\,ds = \int_0^\pi d\varphi \int_0^{2\pi} d\theta \left(R^2 \sin\varphi\, R^2 \right) = 4\pi R^4$

$F(x_1,x_2,x_3) = (0,0,x_3) \quad \iint_\Sigma F . d\sigma = \int_0^\pi d\varphi \int_0^{2\pi} d\theta \left(R\cos\varphi . R^2 \cos\varphi \sin\varphi \right)$

$= R^3 2\pi \int_0^\pi \cos^2\varphi \sin\varphi\, d\varphi = 2\pi R^3 \left[-\frac{1}{3}\cos^3\varphi \right]_0^\pi = \frac{4\pi R^3}{3}$



Rque : $\iint_\Sigma f\,ds$ ne dépend pas de la param triangle



$$\vec{\sigma}(u,v) = \overrightarrow{OP_1} + u\,\overrightarrow{P_1P_2} + v\,\overrightarrow{P_2P_3}$$

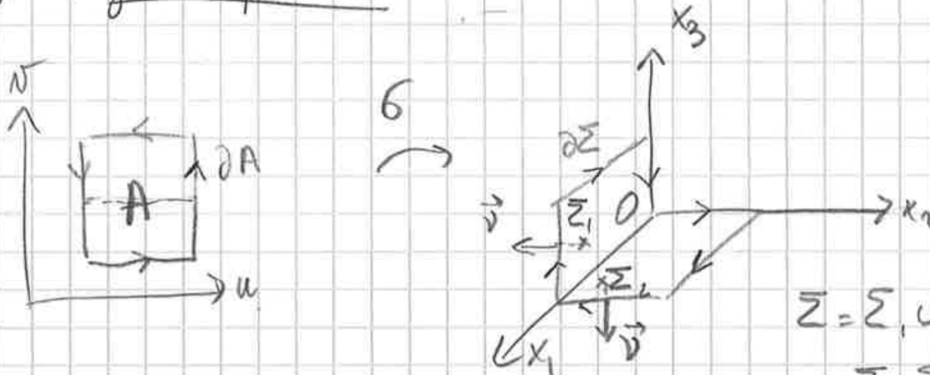
$0 \leq u \leq 1$

$0 \leq v \leq u$

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \overrightarrow{P_1P_2} \wedge \overrightarrow{P_2P_3}$$

13

surface régulière par morceaux :



$\sigma : A \to \Sigma$

C^1 par morceaux

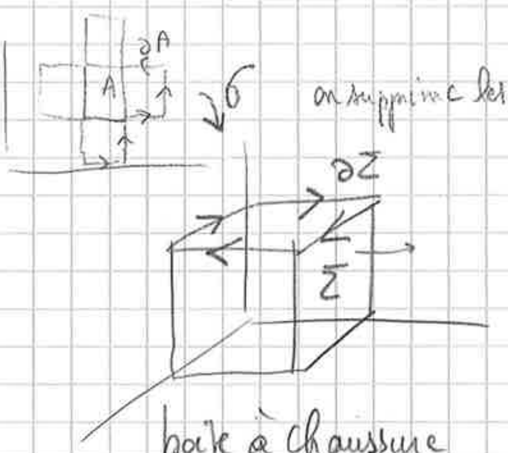
($\vec{\nu}$ cont /morceaux)

$\partial \Sigma = \sigma(\partial A)$

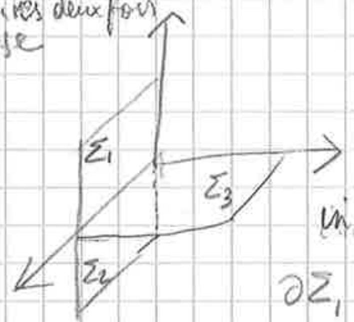
$\Sigma = \Sigma_1 \cup \Sigma_2$

Σ_1, Σ_2 surfaces régulières

on supprime les arêtes parcourues deux fois en sens inverse

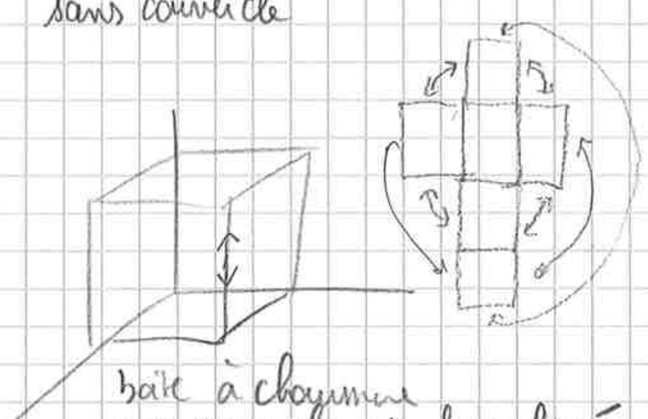


boîte à chaussure sans couvercle

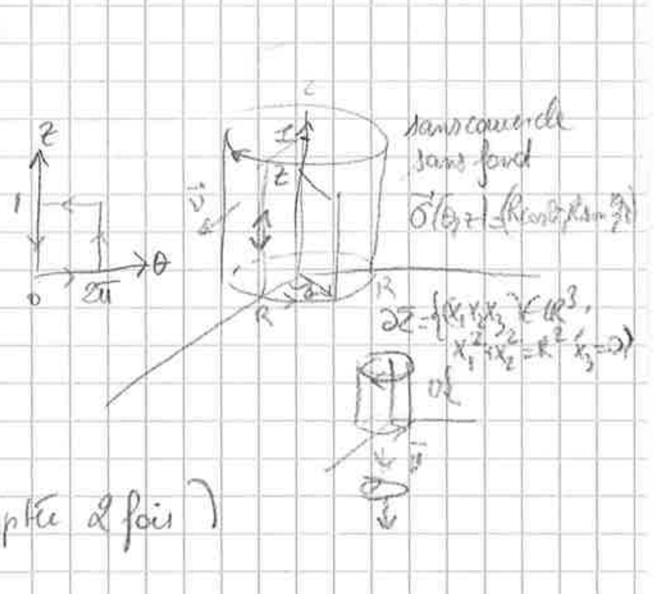


interdit (dans le cadre de ce cours)

$\partial\Sigma_1 \cap \partial\Sigma_2 \cap \partial\Sigma_3$ est 1 courbe



boîte à chaussure avec couvercle : surface fermée



sans couvercle sans fond

$\vec{\sigma}(\theta, z) = (R\cos\theta, R\sin\theta, z)$

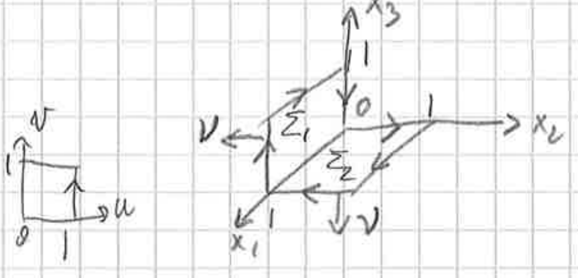
$\partial\Sigma = \{(x_1,x_2,x_3) \in \mathbb{R}^3,\ x_1^2 + x_2^2 = R^2,\ x_3 = 0\}$

$\partial \Sigma = \emptyset$ (chaque arête est comptée 2 fois)

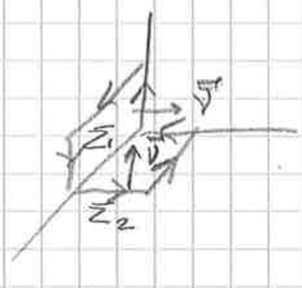
$$\iint_{\Sigma} f\,ds = \sum_{i=1}^{m} \iint_{\Sigma_i} f\,ds$$

$$\iint_{\Sigma} \vec{F}\cdot\vec{ds} = \sum_{i=1}^{m} \iint_{\Sigma_i} \vec{F}_i\cdot\vec{ds}$$

Pour faire les calculs on paramétrise chaque surface indépendamment en faisant attention à l'orientation.



ou



Σ_1 : $\sigma(u,v) = (u,0,v)$ $\qquad \dfrac{\partial\sigma}{\partial u} \wedge \dfrac{\partial\sigma}{\partial v} = \begin{pmatrix}1\\0\\0\end{pmatrix} \wedge \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$ (1)

$\sigma(u,v) = (v,0,u)$ $\qquad$ —— $\begin{pmatrix}0\\0\\1\end{pmatrix} \wedge \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ (2)

Σ_2 : $\sigma(u,v) = (u,v,0)$ $\qquad \dfrac{\partial\sigma}{\partial u} \wedge \dfrac{\partial\sigma}{\partial v} = \begin{matrix}1\\0\\0\end{matrix} \wedge \begin{matrix}0\\1\\0\end{matrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ (3)

$\sigma(u,v) = (v,u,0)$ $\qquad$ —— $\begin{pmatrix}0\\0\\-1\end{pmatrix}$ (4)

$F(x_1,x_2,x_3) = (1,1,1)$

$$\iint_{\Sigma} \vec{F}\cdot d\vec{S} = \iint_{\Sigma_1} \vec{F}\cdot d\vec{S} + \iint_{\Sigma_2} \vec{F}\cdot d\vec{S}$$

(1) et (4) $\quad = \int_0^1 du \int_0^1 dv\, (-1-1) = -2$

(2) et (3) $\quad = \int_0^1 du \int_0^1 dv\, (1+1) = 2$

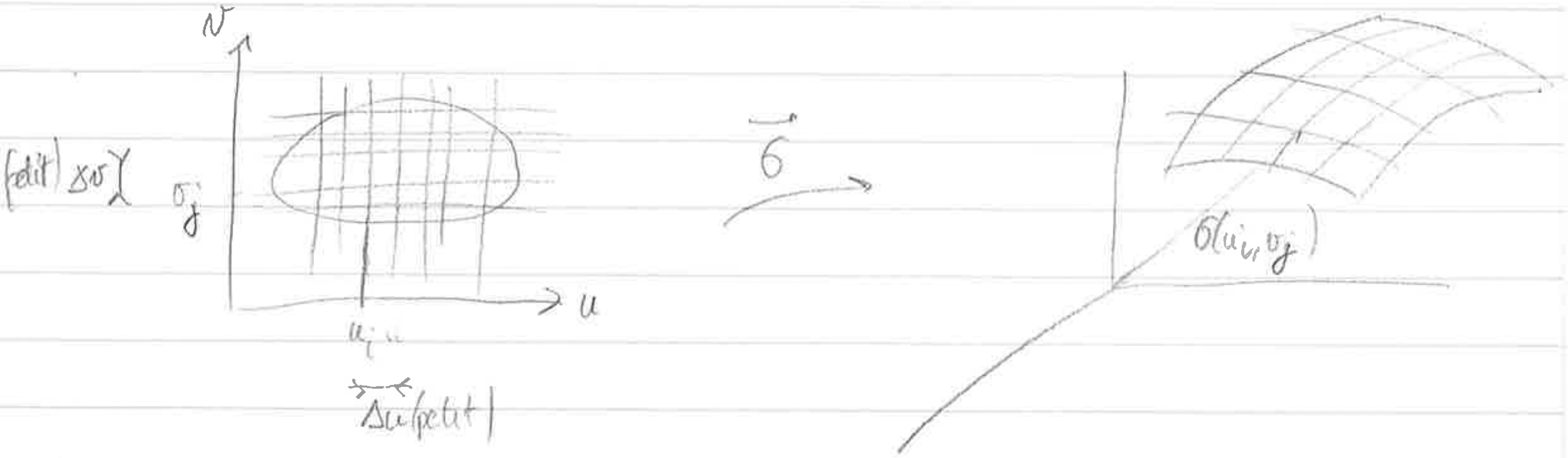
Pourquoi $\|\gamma'(t)\|$ et $\|\frac{\partial\sigma}{\partial u}\times\frac{\partial\sigma}{\partial v}\|$?

$$\int_\Gamma f\,dl \simeq \sum_i \int_{P_iP_{i+1}} f\,dl \simeq \sum_i \|\overrightarrow{P_iP_{i+1}}\|\, f(P_i)$$

$$\int_a^b f(\vec\gamma(t))\|\vec\gamma'(t)\|\,dt = \sum_i \int_{t_i}^{t_{i+1}} f(\vec\gamma(t))\|\vec\gamma'(t)\|\,dt \simeq \sum_i \Delta t\, f(\vec\gamma(t_i))\,\|\vec\gamma'(t_i)\|$$

$$\vec\gamma'(t_i) = \lim_{\varepsilon\to 0} \frac{\vec\gamma(t_i+\varepsilon)-\vec\gamma(t_i)}{\varepsilon} \simeq \frac{\vec\gamma(t_{i+1})-\vec\gamma(t_i)}{\Delta t}$$

$$\simeq \sum_i \Delta t\, f(\gamma(t_i))\frac{\vec\gamma(t_{i+1})-\vec\gamma(t_i)}{\Delta t} = \sum_i f(P_i)\,\|\overrightarrow{P_iP_{i+1}}\|$$



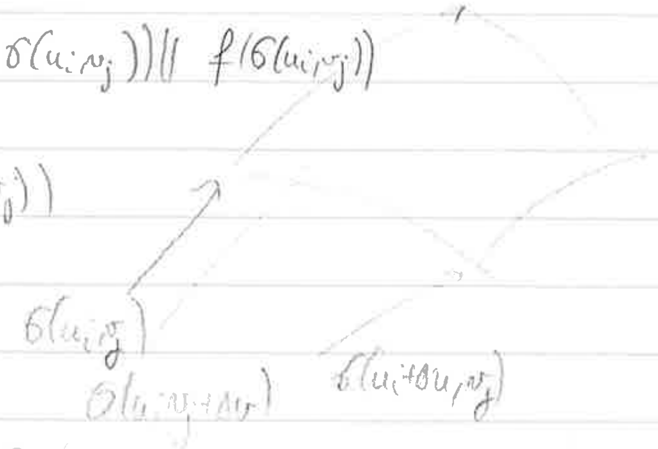
$$\iint_\Sigma f\,ds \simeq \sum_{ij} \|(\sigma(u_i+\Delta u, v_j)-\sigma(u_i,v_j)) \wedge (\sigma(u_i, v_j+\Delta v)-\sigma(u_i,v_j))\|\; f(\sigma(u_i,v_j))$$

$$\iint_A f(\sigma(u,v))\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\|\,du\,dv \simeq \sum_{i,j} \Delta u\Delta v\; f(\sigma(u_i,v_j)) \times \|(\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v})(u_i,v_j)\|$$

$$\frac{\partial\sigma}{\partial u}(u_i,v_j) = \lim_{\varepsilon\to 0} \frac{\sigma(u_i+\varepsilon, v_j)-\sigma(u_i,v_j)}{\varepsilon}$$

$$\simeq \frac{\sigma(u_i+\Delta u, v_j)-\sigma(u_i,v_j)}{\Delta t}$$

idem $\frac{\partial\sigma}{\partial v}$

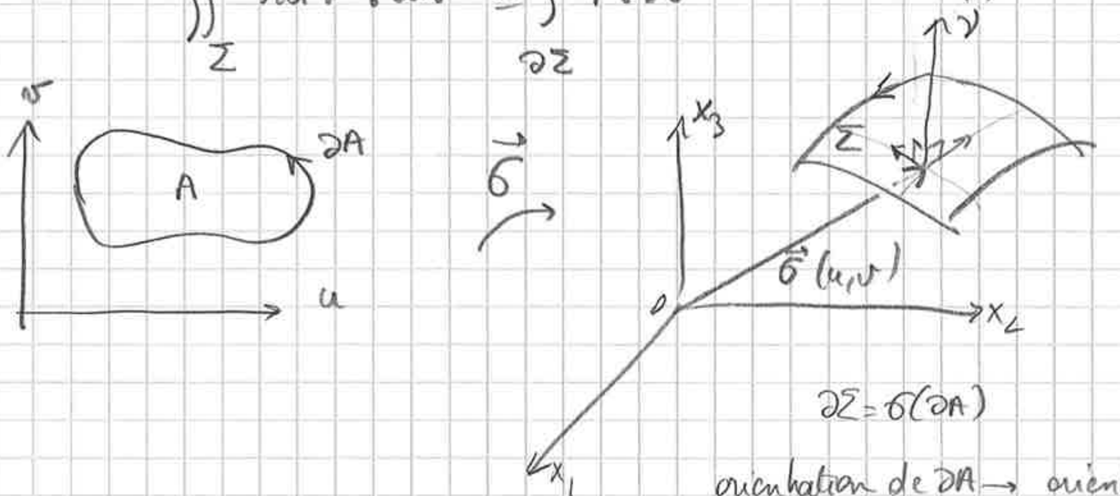


Sem 5 : Théorème de Stokes

thm : Σ surface régulière par morceaux

$\vec{F} : \Sigma \to \mathbb{R}^3 \quad \mathcal{C}^1$

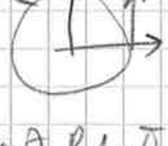
$$\iint_{\Sigma} \vec{\mathrm{rot}}\,\vec{F} \cdot d\vec{s} = \int_{\partial\Sigma} \vec{F} \cdot d\vec{\ell}$$

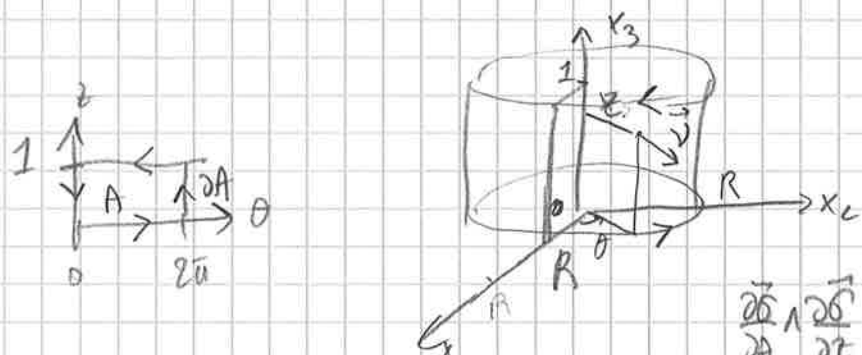


$\partial\Sigma = \sigma(\partial A)$

orientation de $\partial A \to$ orientation de $\partial\Sigma$ et de $\vec{\nu}$ selon la règle d'Ampère, de l'des. du tire-bouchon...

Ex. $\Sigma = \{(x_1 x_2 x_3) \in \mathbb{R}^3 ;\ x_1^2 + x_2^2 = R^2 \quad 0 \leq x_3 \leq 1\}$ $\quad F(x_1 x_2 x_3) = (-x_2\ x_1\ 0)$





$\vec{\sigma}(\theta, z) = (R\cos\theta, R\sin\theta, z)$

$$\frac{\partial\vec{\sigma}}{\partial\theta} \wedge \frac{\partial\vec{\sigma}}{\partial z} = \begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \end{pmatrix}$$

$$\partial\Sigma = \{(x_1 x_2 x_3) \in \mathbb{R}^3 ;\ x_1^2 + x_2^2 = R^2 \quad x_3 = 0\} \cup \{\ \ 1\} = \Gamma_1 \cup \Gamma_2$$

$$\vec{\mathrm{rot}}\,\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} \wedge \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \qquad \iint_{\Sigma} \mathrm{rot}\,F \cdot ds = \int_0^1 dz \int_0^{2\pi} \underbrace{\vec{\mathrm{rot}}\,F(\sigma(\theta,z)) \cdot \frac{\partial\sigma}{\partial\theta} \wedge \frac{\partial\sigma}{\partial z}}_{0}\, d\theta\, dz$$

Γ_1 : $\vec{\gamma}(t) = (R\cos t, R\sin t, 0) \qquad \vec{\gamma}'(t) = (-R\sin t, R\cos t, 0) \qquad 0 \leq t \leq 2\pi$

$$\int_{\Gamma_1} F \cdot d\ell = \int_0^{2\pi} dt \left(\vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) \right) = \int_0^{2\pi} dt \left(-R\sin t, R\cos t, 0\right) \cdot \left(-R\sin t\ R\cos t\ 0\right) = R^2\, 2\pi$$

Γ_2 : $\vec{\gamma}(t) = (R\cos t, -R\sin t, 1) \qquad 0 \leq t \leq 2\pi$

$$\int_{\Gamma_2} F \cdot d\ell = \int_0^{2\pi} dt\, (R\sin t, R\cos t, 0)(-R\sin t\ -R\cos t\ 0) = -2\pi R^2$$

On ajoute le fond $\tilde{\Sigma} = \Sigma \cup \underbrace{\{(x_1 x_2 x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 \leq R^2 \quad x_3 = 0\}}_{\hat{\Sigma}}$

$$\sigma(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 0) \qquad \frac{\partial\sigma}{\partial r} \wedge \frac{\partial\sigma}{\partial\theta} = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \wedge \begin{pmatrix}-r\sin\theta\\ r\cos\theta\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\r\end{pmatrix}$$

mauvais signe.

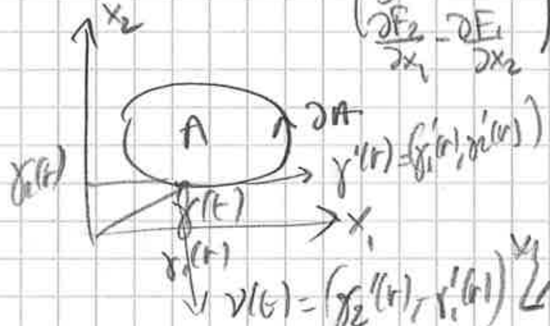
$$\iint_{\hat{\Sigma}} \overline{\mathrm{rot}}\,\vec{F}\cdot d\vec{s} = -\int_0^R dr \int_0^{2\pi} d\theta\ 2r = -2\pi R^2$$

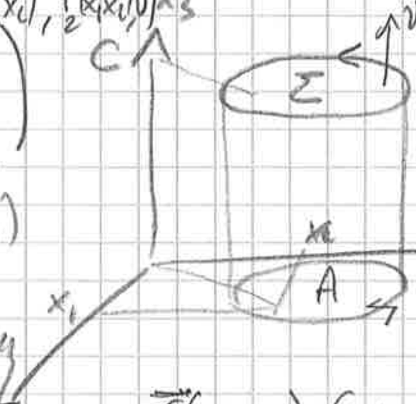
$$\iint_{\hat{\Sigma}} \mathrm{rot}\,F\, ds = -2\pi R^2 \qquad \int_{\partial\hat{\Sigma}} F\cdot dl = \int_{\Gamma_1} F\cdot dl = -2\pi R^2$$

<u>Dem thm Stokes</u>: $\vec{F}(x_1 x_2 x_3) = (F_1(x_1 x_2),\ F_2(x_1 x_2), 0)$ x_3

$$\mathrm{rot}\,F = \begin{pmatrix}0\\0\\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\end{pmatrix}$$

$\Sigma = \{x_1 x_2 x_3 \quad (x_1 x_2) \in A \quad x_3 = c\}$





$\vec{\sigma}(x_1 x_2 x_3) = (x_1\ x_2\ C) \qquad x_1 x_2 \in A \qquad \frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \begin{smallmatrix}1\\0\\0\end{smallmatrix} \wedge \begin{smallmatrix}0\\1\\0\end{smallmatrix} = \begin{smallmatrix}0\\0\\1\end{smallmatrix}$

$$\iint_\Sigma \mathrm{rot}\,\sigma\, d\sigma = \iint_A dx_1 dx_2 \left(\frac{\partial F_2}{\partial x_1}(x_1 x_2) - \frac{\partial F_1}{\partial x_2}(x_1 x_2)\right)$$

$$\partial\Sigma:\ \gamma(t) = (\gamma_1(t), \gamma_2(t), C) \qquad \gamma'(t) = (\gamma_1'(t), \gamma_2'(t), 0)$$

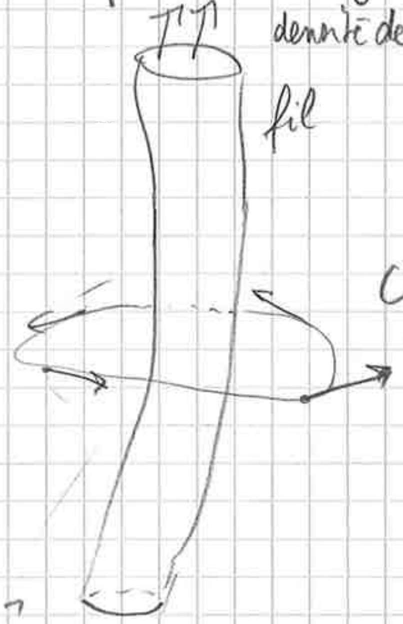
$$\int_{\partial\Sigma} F\cdot dl = \int_a^b dt\ \Big(F_1(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + F_2(\gamma_1(t)\gamma_2(t))\gamma_2'(t)\Big)$$

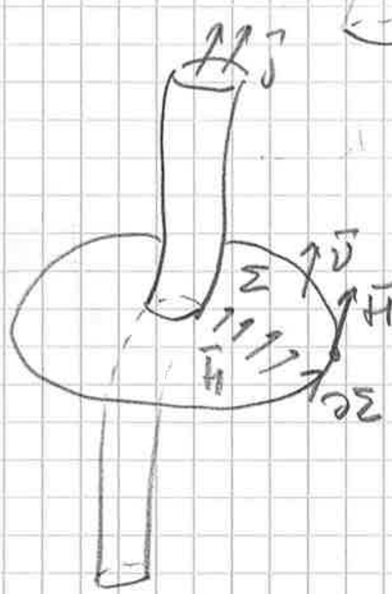
D'après le thm de Green ces 2 intégrales sont identiques. En effet

$$\iint_A \frac{\partial F_2}{\partial x_1} dx_1 dx_2 = \int_{\partial A} F_2 \nu_1\, dl = \int_a^b F_2(\gamma_1(t), \gamma_2(t))\gamma_2'(t)\, dt$$

$$-\frac{\partial F_1}{\partial x_2} \qquad F_1 \nu_2 \qquad -F_1 \qquad \gamma_1'$$

Application : Loi d'Ampère ($\mathrm{rot}\,\vec{H} = \vec{j}$)



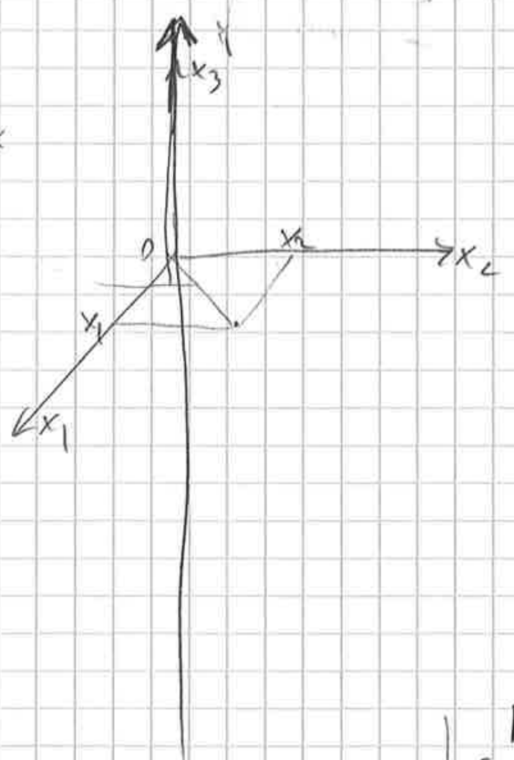


On a : $\mathrm{rot}\,\vec{H} = \vec{j}$

$$\underbrace{\iint_\Sigma \mathrm{rot}\,\vec{H}\cdot d\vec{s}}_{\int_{\partial\Sigma} \vec{H}\cdot d\vec{\ell}} = \underbrace{\iint_\Sigma \vec{j}\cdot d\vec{s}}_{\text{courant total à travers le fil}}$$

$\int_{\partial\Sigma} \vec{H}\cdot d\vec{\ell}$: circulation de $\vec{H}$ sur $\partial\Sigma$

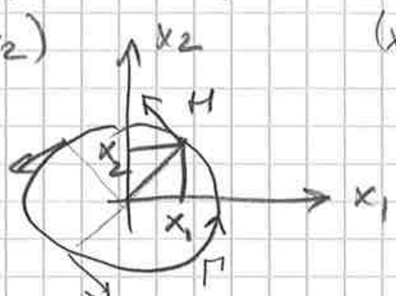
Ex



fil infiniment long, courant J

chp magn $\vec{H}$ indep x_3

$H(x_1, x_2)$ (x_1, x_2) fixés



$$H(x_1, x_2) = \alpha\,(-x_2, x_1, 0)$$

α fct de $\sqrt{x_1^2 + x_2^2}$

$\Gamma : \vec{\gamma}(t) = \left(\sqrt{x_1^2+x_2^2}\cos t, \sqrt{x_1^2+x_2^2}\sin t, 0\right)$ $0 \le t \le 2\pi$

$H(\gamma(t)) = \alpha\left(-\sqrt{}\sin t, \sqrt{}\cos t, 0\right)$

$\int_\Gamma \vec{H}\cdot d\vec{\ell} = J$

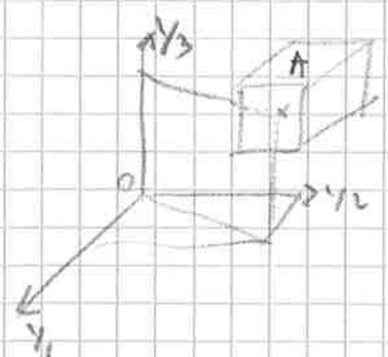
$= \int_0^{2\pi} \alpha\,(x_1^2+x_2^2)\,dt = 2\pi\alpha(x_1^2+x_2^2)$

$\alpha = \dfrac{J}{2\pi(x_1^2+x_2^2)}$ $\vec{H} = \dfrac{J}{2\pi(x_1^2+x_2^2)}(-x_2, x_1, 0)$

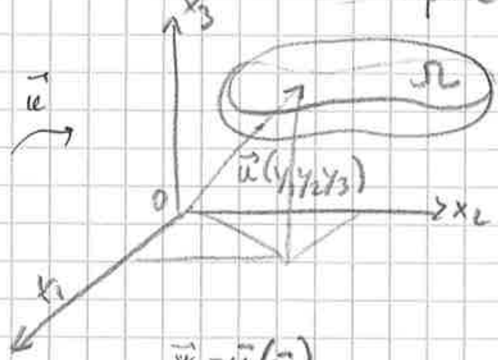
Chap 6 Thm de la divergence

Rappel : calcul d'intégrales dans $\mathbb{R}^3$ et chgt de variable (8.5 line)

espace des paramètres



espace physique



$\vec{x} = \vec{u}(\vec{y})$

$x_1 \quad u^1(y_1, y_2 y_3)$

$x_2 = \quad u^2(\text{——})$

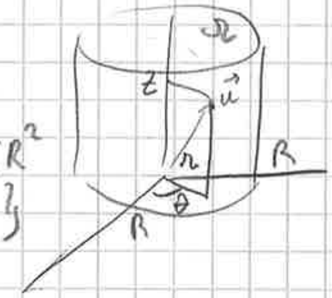
$x_3 \quad u^3(\text{——})$

$$\iiint_\Omega f(x_1 x_2 x_3)\,dx_1 dx_2 dx_3 = \iiint_A f(\vec{u}(\vec{y}))\,|\det \nabla u(\vec{y})|\,d\vec{y}$$

$$\nabla u(y) = \begin{pmatrix} \frac{\partial u^1}{\partial y_1} & \frac{\partial u^1}{\partial y_2} & \frac{\partial u^1}{\partial y_3} \\ \frac{\partial u^2}{\partial y_1} & \frac{\partial u^2}{\partial y_2} & \frac{\partial u^2}{\partial y_3} \\ \frac{\partial u^3}{\partial y_1} & \frac{\partial u^3}{\partial y_2} & \frac{\partial u^3}{\partial y_3} \end{pmatrix}$$

Ex :

$\Omega = \{(x_1 x_2 x_3) \in \mathbb{R}^3 ;\ x_1^2 + x_2^2 \leq R^2 \quad 0 \leq x_3 \leq 1\}$



$x_1 = r\cos\theta$

$x_2 = r\sin\theta$

$x_3 = z$

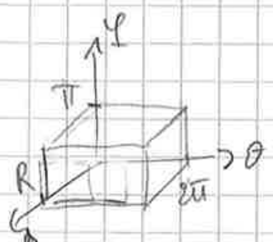
$0 \leq r \leq R$

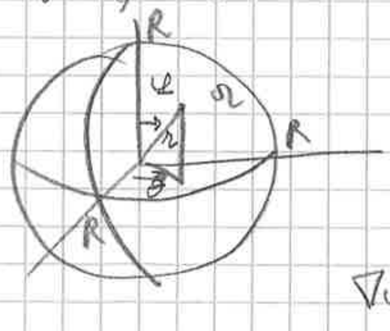
$0 \leq \theta \leq 2\pi$

$0 \leq z \leq 1$

$$\nabla u = \begin{pmatrix} \cos\theta & -r\sin\theta & 1 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Volume}(\Omega) = \iiint_\Omega dx_1 dx_2 d_3 = \int_0^R r\,dr \int_0^{2\pi} d\theta \int_0^1 dz = \frac{R^2}{2} 2\pi = \pi R^2 \qquad |\det \nabla u| = r$$





$x^1 = r\sin\varphi\cos\theta \qquad \vec{x} = \vec{u}(r, \theta, \varphi)$

$x^2 = r\sin\varphi\sin\theta$

$x^3 = r\cos\varphi$

$$\nabla u = \begin{pmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{pmatrix}$$

$$\text{Volume}(\Omega) = \iiint dx_1 dx_2 dx_3$$

$$= \int_0^R r^2\,dr \int_0^{2\pi} d\theta \int_0^{1} \sin\varphi\,d\varphi$$

$$= \frac{R^3}{3}\ 2\pi\ \ 2 = \frac{4\pi R^3}{3}$$

$$|\det \nabla u| = |-\sin\varphi\cos\theta\, r^2\sin^2\varphi\cos\theta$$

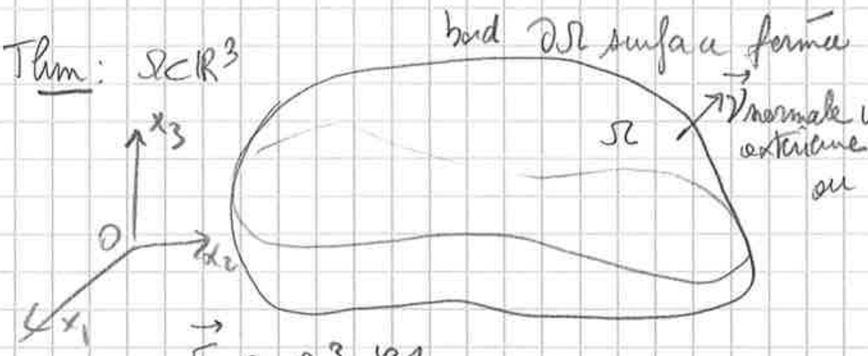
$$+ r\sin\varphi\sin\theta\,(-r\sin^2\varphi\sin\theta - r\cos^2\varphi\sin\theta)$$

$$r\cos\varphi\cos\theta(-r\sin\varphi\cos\varphi\cos\theta \quad |$$

$$= |r^2\sin\varphi\,(\cos^2\theta\sin^2\varphi + \sin^2\theta + \cos^2\varphi\cos^2\theta)\,|$$

$$= +r^2\sin\varphi$$

Thm : $\Omega \subset \mathbb{R}^3$



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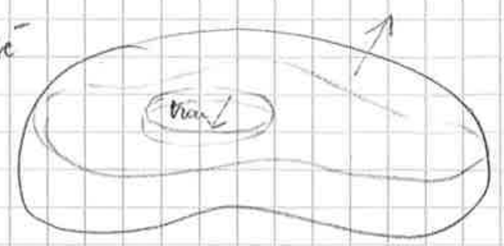


$\vec{F} : \Omega \to \mathbb{R}^3$ $\mathcal{C}^1$

$$\iiint_\Omega \operatorname{div} \vec{F} \, dx_1 dx_2 dx_3 = \iint_{\partial\Omega} \vec{F} \cdot \vec{\nu} \, ds \left(= \iint_{\partial\Omega} \vec{F} \cdot \vec{ds} \right) \quad (1)$$

montrer — montrer

Rque : $(1) \underset{\forall F}{\Leftrightarrow} \iiint_\Omega \frac{\partial f}{\partial x_i} dx_1 dx_2 dx_3 = \iint_{\partial\Omega} f \nu_i \, ds \quad i = 1,2,3 \quad \forall f$

En effet :

$\Rightarrow$: $\vec{F} = (f, 0, 0)$, $= (0, f, 0)$, $= (0, 0, f)$, $f : \Omega \to \mathbb{R}$

$\Leftarrow$: $\vec{F} = (F_1, F_2, F_3)$

$$\iiint \frac{\partial F_1}{\partial x_1} = \iint F_1 \nu_1$$
$$\frac{\partial F_2}{\partial x_2} = F_2 \nu_2$$
$$\frac{\partial F_3}{\partial x_3} = F_3 \nu_3$$
$$\overline{\iiint \operatorname{div} F = \iint F \cdot \nu}$$

II ci Dem thm.

Application : on considère un fluide incompressible non visqueux de densité $\rho > 0$ occupant $\mathbb{R}^3$ et sol des eq d'Euler (1753) $\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3$ $p : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{cases} \rho\left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}\right) + \frac{\partial p}{\partial x_1} = 0 \\ \rho\left(\text{—} v_2 \quad \text{—} v_2 \quad \text{—} v_2\right) + \frac{\partial p}{\partial x_2} = 0 \\ \rho\left(\text{—} v_3 \quad \text{—} v_3 \quad \text{—} v_3\right) + \frac{\partial p}{\partial x_3} = 0 \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \end{cases}$$

Soit Ω un domaine de bord $\partial\Omega$ une surface fermée. On a :

$$\iiint_\Omega \left(\rho\left(v_1 \frac{\partial v_1}{\partial x_1} + \qquad \right) + \frac{\partial p}{\partial x_1}\right) dx_1 dx_2 dx_3 = 0$$

$$+ \iiint_\Omega \left(\rho \, v_1 \frac{\partial v_2}{\partial x_1} + \qquad \right) + \frac{\partial p}{\partial x_2} \Big)$$

$$+ \iiint_\Omega \rho \, v_1 \frac{\partial v_3}{\partial x_1} + \qquad \Big) + \frac{\partial p}{\partial x_3} \Big)$$

or $v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = \frac{\partial}{\partial x_1}(v_1 v_1) + \frac{\partial}{\partial x_2}(v_1 v_2) + \frac{\partial}{\partial x_3}(v_1 v_3)$

$$\underbrace{- v_1 \frac{\partial v_1}{\partial x_1} - v_1 \frac{\partial v_2}{\partial x_2} - v_1 \frac{\partial v_3}{\partial x_3}}_{0}$$

Donc $\iiint_\Omega \left(\rho \left(\frac{\partial}{\partial x_1}(v_1 v_1) + \frac{\partial}{\partial x_2}(v_1 v_2) + \frac{\partial}{\partial x_3}(v_1 v_3) \right) + \frac{\partial p}{\partial x_1} \right) dx_1 dx_2 dx_3 = 0$

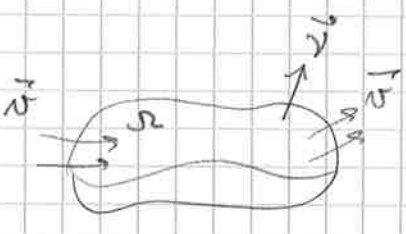
$$\iiint_\Omega \left(\rho \left(\frac{\partial}{\partial x_1}(v_1 v_2) + \frac{\partial}{\partial x_2}(v_2 v_2) + \frac{\partial}{\partial x_3}(v_2 v_3) \right) + \frac{\partial p}{\partial x_2} \right)$$

$$\iiint_\Omega \rho \left(\frac{\partial}{\partial x_1}(v_1 v_3) + \frac{\partial}{\partial x_2}(v_2 v_3) + \frac{\partial}{\partial x_3}(v_3 v_3) + \frac{\partial p}{\partial x_3} \right)$$

thm div $\iint_{\partial\Omega} \left(\rho \underbrace{(v_1 v_1 \nu_1 + v_1 v_2 \nu_2 + v_1 v_3 \nu_3)}_{v_1 (\vec v \cdot \vec \nu)} + p \nu_1 \right) ds = 0$

$$\iint_{\partial\Omega} \left(\rho \quad v_2 (\vec v \cdot \vec \nu) \quad + p \nu_2 \right) ds = 0$$

$$\iint_{\partial\Omega} \left(\rho \quad v_3 (\vec v \cdot \vec \nu) \quad + p \nu_3 \right) ds = 0$$

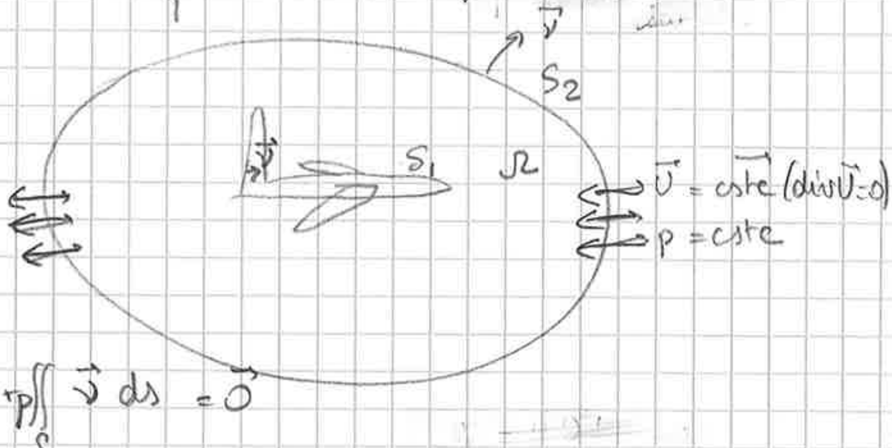


ie $\iint_{\partial\Omega} \left(\rho \vec v (\vec v \cdot \vec \nu) + p \vec \nu \right) ds = 0$

Conservation de la quantité de mvt.

Conséquence

Paradoxe d'Alembert :



$\partial\Omega = S_1 \cup S_2$

$$\iint_{\partial\Omega} \left(\rho \vec v (\vec v \cdot \vec \nu) + p \vec \nu \right) ds = 0$$

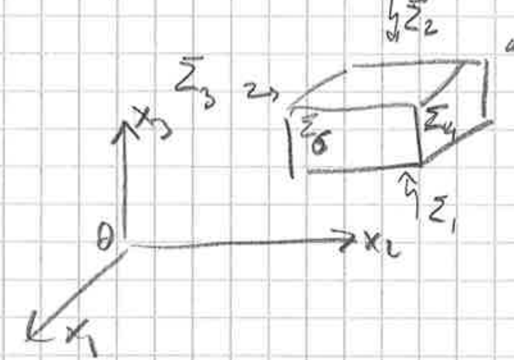
or $\iint_{S_2} \left(\rho \vec v (\vec v \cdot \vec \nu) + p \vec \nu \right) ds = \rho \vec v \iint_{S_2} \vec v \cdot \vec \nu + p \iint_{S_2} \vec \nu \, ds = \vec 0$

donc $\iint_{S_1} \left(\rho \vec v \underbrace{(\vec v \cdot \vec \nu)}_{0} + p \vec \nu \right) ds = 0$: l'avion n'a pas de traînée ou de portance !

$\rightarrow$ il faut ajouter un terme visqueux $-\mu \Delta v_i$ $i = 1, 2, 3$

Navier 1822

Dem thm



$\Omega =]a_1, a_2[\times]b_1, b_2[\times]c_1, c_2[$

parallélépipède rectangle

$$\iiint_\Omega \frac{\partial f}{\partial x_1} dx_1 dx_2 dx_3 = \int_{b_1}^{b_2} dx_2 \int_{c_1}^{c_2} dx_3 \int_{a_1}^{a_2} dx_1 \frac{\partial f}{\partial x_1}(x_1, x_2 x_3).$$

$$= \int_{b_1}^{b_2} dx_2 \int_{c_1}^{c_2} dx_3 \left(f(a_2, x_2, x_3) - f(a_1, x_2 x_3) \right)$$

$$\iint_{\partial\Omega} f \nu_1 \, ds = \iint_{\Sigma_6} f \, ds - \iint_{\Sigma_5} f \, ds = \iint$$

$b_1 \leq u \leq b_2 \quad c_1 \leq v \leq c_2$

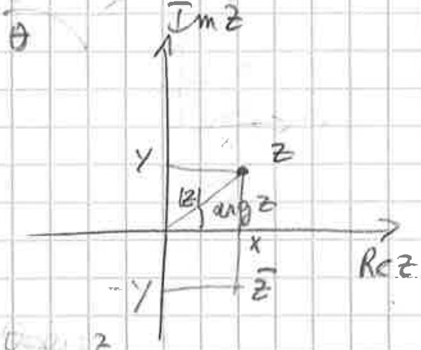
Σ_6 : param $\vec{\sigma}(u,v) = (a_2, u, v)$ $\quad \frac{\partial \vec{\sigma}}{\partial u} \wedge \frac{\partial \vec{\sigma}}{\partial v} = \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \wedge \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} = \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \quad \vec{\nu} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Σ_5 : (a_1, v, u) $\quad \frac{\partial \vec{\sigma}}{\partial u} \wedge \frac{\partial \vec{\sigma}}{\partial v} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{\nu} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$$\iint_{\Sigma_6} f ds = \int_{b_1}^{b_2} du \int_{c_1}^{c_2} dv \; f(a_2, u, v) \, du \, dv$$

$$\iint_{\partial\Omega} f \nu \, ds = \int_{b_1}^{b_2} dx_2 \int_{c_1}^{c_2} dx_3 \left(f(a_2, x_2, x_3) - f(a_1, x_2 x_3) \right) = \iiint_\Omega \frac{\partial f}{\partial x_1} dx_1 dx_2 dx_3$$

Chap 9 fonctions holomorphes - Eq Cauchy-Riemann



Rappels : calculs dans $\mathbb{C}$: i tq $i^2 = -1$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$ $\quad \theta \in \mathbb{R}$

- $z \in \mathbb{C}$ $\quad z = x + iy$ $\quad \mathrm{Re}\, z = x$ $\quad \mathrm{Im}\, z = y$
- $z \in \mathbb{C}$ $\quad \bar{z} = x - iy$ conjugué complexe
- $z \in \mathbb{C}$ $\quad |z| = \sqrt{x^2+y^2}$ module $\quad \arg z$ argument

$z = |z| e^{i\arg z}$ $\quad \arg z$ déf à $2k\pi$ près $k \in \mathbb{Z}$

valeur principale de l'argument de z : $-\pi < \arg z \leq \pi$

ex : $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2$

$= x_1 + x_2 + i(y_1 + y_2)$

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$

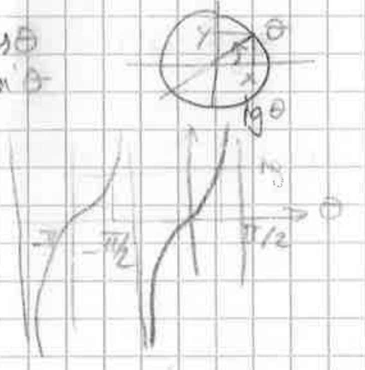
$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$

$z_1 = |z_1| e^{i\theta_1}$ $\quad z_2 = |z_2| e^{i\theta_2}$

$z_1 z_2 = |z_1||z_2| e^{i(\theta_1+\theta_2)}$

$|z_1 z_2| = |z_1||z_2|$

si on note $\theta = \arg z$ $\quad z = |z| e^{i\theta}$ s'écrit $\quad x = |z|\cos\theta$, $\; y = |z|\sin\theta$

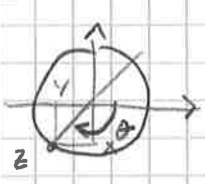


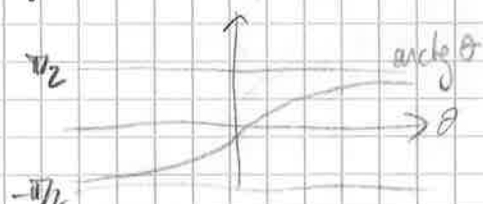
θ fct de x, y ?

pex si $0 < \theta < \pi/2$ (ie si $x > 0$ et $y > 0$) $\quad \mathrm{tg}\,\theta = \frac{y}{x}$

$\theta = \arg z = \mathrm{arctg}\, \frac{y}{x}$

si $-\pi < \theta < -\pi/2$ (ie si $x < 0$ et $y < 0$) $\theta = \arg z = -\pi + \mathrm{arctg}\, \frac{y}{x}$





Ex : $z \in \mathbb{C}$ tq $z^2 = 1$ $\quad z = |z| e^{i\theta}$ $\quad z^2 = |z|^2 e^{i2\theta} = 1$ $\quad -\pi < \theta \leq \pi$

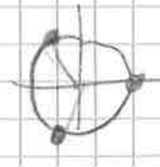
$|z|^2 = 1$ $\quad e^{i2\theta} = 1$ donc $2\theta = 2k\pi$ $k \in \mathbb{Z}$

mais puisque $-\pi < \theta < \pi$ $\quad k = 0$ ou $k = 1$ $\quad \theta = 0$ $\quad \theta = \pi$

$z \in \mathbb{C}$ tq $z^3 = 1$ $\quad z = |z| e^{i\theta}$ $\quad z^3 = |z|^3 e^{i3\theta} = 1$

$|z|^3 = 1$ $\quad e^{i3\theta} = 1$ $\quad 3\theta = 2k\pi$ $k \in \mathbb{Z}$ $\quad k = 0$ $\; k = 1$ $\; k = -1$

$\theta = 0$ $\quad \theta = \frac{2\pi}{3}$ $\quad \theta = -\frac{2\pi}{3}$



fonctions complexes

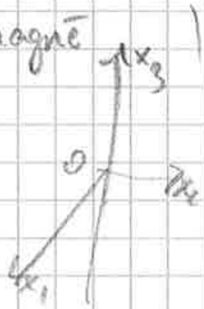
$$f : \mathbb{C} \to \mathbb{C}$$
$$z \to f(z)$$
$$x+iy \to \underbrace{u(x,y)}_{\mathrm{Re}f} + i\,\underbrace{v(x,y)}_{\mathrm{Im}f}$$

$u : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to u(x,y)$ $\quad v : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to v(x,y)$

Ex1 : $f(z) = \bar{z} = x - iy \qquad u(x,y) = x \qquad v(x,y) = -y$

Ex2. $f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2ixy}_{v(x,y)}$

Ex3 : $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$ $\left(= \frac{\bar{z}}{|z|^2}\right)$ (Champ magné



Rque :

$f : \mathbb{C} \to \mathbb{C}$, $z \to f(z)$, $(x,y) \to u(x,y) + iv(x,y)$ $\xleftrightarrow{\text{correspondance}}$ $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1,x_2) \to \vec{F}(x_1,x_2) = (F_1(x_1,x_2), F_2(x_1,x_2))$

Ex4 : $f(z) = e^z = e^{x+iy} = \underbrace{e^x \cos y}_{u(x,y)} + i\,\underbrace{e^x \sin y}_{v(x,y)}$

Puisque $e^z = e^{z+i2k\pi}$ $k \in \mathbb{Z}$ $f(z) = e^z$ n'est pas injective

Ex5 : la fonction logarithme

On veut def la fct inverse de e^z :

$$z = |z| e^{i \arg z} \qquad -\pi < \arg z \leq \pi$$

$$\log z = \log |z| + i \arg z \qquad (\log ab = \log a + \log b \quad a,b > 0;\ \log e = 1)$$

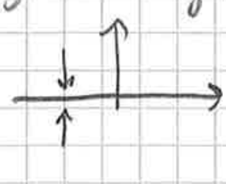
on a bien $e^{\log z} = e^{\log|z| + i\arg z} = |z| e^{i\arg z} = z$

attention : - $\log : \mathbb{C}\setminus\{0\} \to \mathbb{C}$

- $\log(z_1 z_2)$ peut être diff de $\log z_1 + \log z_2$

ex $z_1 = -1 \quad z_2 = -1 \quad \log z_1 z_2 = \log 1 = 0 \quad \log z_1 = \log z_2 = i\pi$

- log n'est pas cont sur $\mathbb{C}\setminus\{0\}$



Ex 6. $f(z) = z^{\gamma} = e^{\gamma \log z} = e^{\gamma(\log|z| + i\arg z)} = e^{\gamma \log|z|} e^{i\gamma \arg z}$

Ex 7 $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ $\quad \sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

$\cosh z = \dfrac{e^{z}+e^{-z}}{2}$ $\quad \sinh z = \dfrac{e^{z}-e^{-z}}{2}$

$$\cos z = \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2} = \frac{e^{ix}e^{-y}+e^{-ix}e^{y}}{2} = \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}$$

$$= \cos x \,\frac{e^{y}+e^{-y}}{2} - i\sin x\,\frac{e^{y}-e^{-y}}{2} = \cos x \cosh y - i \sin x \sinh y$$

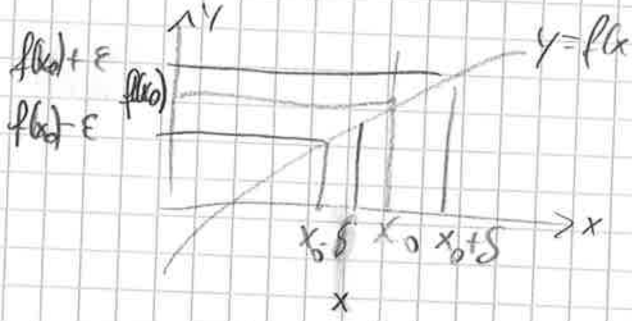
Limite

Def : f est continue en z_0 et on écrit $\lim_{z\to z_0} f(z) = f(z_0)$ si

$\forall \varepsilon > 0\ \exists \delta > 0$ si $|z-z_0| < \delta$ alors $|f(z)-f(z_0)| < \varepsilon$

(Rappel $f : \mathbb{R} \to \mathbb{R}$ cont en x_0

$\forall \varepsilon > 0\ \exists \delta > 0$ si $|x-x_0| < \delta$ alors $|f(x)-f(x_0)| < \varepsilon$



$-\varepsilon < f(x) - f(x_0) < \varepsilon$

$f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$)

Si f et g sont cont en z_0 alors $f+g$ est cont en z_0
$f \cdot g$ (exercice)

Dérivée

$\log z$ n'est pas cont en $z_0 = -1$: $t \to 0$ $\log(-1+it) = \log\sqrt{1+t^2} + i\arg(-1+it) \xrightarrow{t\to 0} 0 + i\pi$

$\log(-1-it) = \ldots + i\arg(-1-it) \to 0 - i\pi$

Def (3.1 livre) $f : \mathbb{C} \to \mathbb{C}$ dérivable en z_0 si

$\lim_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ existe et est finie (la même limite pour tout $z \to z_0$)

Si f dérivable $\forall z_0 \in \mathbb{C}$ on dit que f est holomorphe

Idem $f : O \to \mathbb{C}$ $\quad O \subset \mathbb{C}$ ouvert

Règles de dérivation : les mêmes que dans $\mathbb{R}$

$O \subset \mathbb{C}$ $\quad f, g : O \to \mathbb{C}$ holomorphes $\quad \forall z_0 \in O$ on a

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$
$$(fg)'(z_0) = f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0)$$
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{\left(g(z_0)\right)^2} \qquad \text{(exercice)}$$
$$(f \circ g)'(z_0) = f'(g(z_0))\,g'(z_0)$$

on veut montrer : $(e^z)' = e^z \qquad (\cos z)' = -\sin z$ etc...

$$\frac{e^z - e^{z_0}}{z - z_0} = \frac{e^{(x+iy)} - e^{(x_0+iy_0)}}{x - x_0 + i(y-y_0)} = \frac{e^x \cos y - e^{x_0}\cos y_0 + i e^x \sin y - e^{x_0}\sin y_0}{x - x_0 + i(y-y_0)} \;??$$

théorème 9.2 : $O \subset \mathbb{C}$ ouvert $\quad f : O \to \mathbb{C}$
$$z \to f(z)$$
$$x+iy \to u(x,y) + i v(x,y)$$

$$\left(f \text{ holomorphe}\right) \iff \left(u, v \in \mathcal{C}^1(O) \text{ et } \forall (x,y) \in O \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ et } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right.$$
$$\left. f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) \right)$$

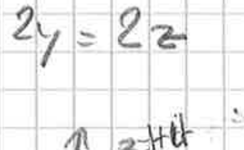
Ex 2 : $f(z) = z^2 = x^2 - y^2 + 2ixy$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

f holomorphe dans $\mathbb{C}$ $\quad f'(z) = 2x + i\,2y = 2z$

Ex 1 : $f(z) = \bar{z} = x - iy$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1 \quad f$ n'est pas holom.



$$z = 1 + it \quad t \to 0 \qquad \frac{f(z) - f(z_0)}{z - z_0} = \frac{1 - it - 1}{1 + it - 1} = -1$$

$$z = 1 + t \quad t \to 0 \qquad \frac{f(z) - f(z_0)}{z - z_0} = \frac{1 + t - 1}{1 + t - 1} = 1$$

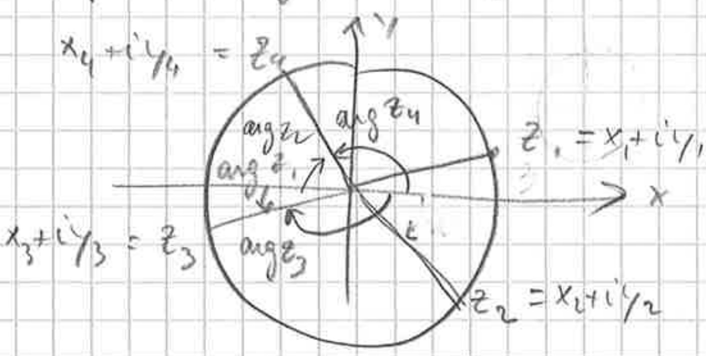
Ex 3 : $f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$
$$\frac{\partial u}{\partial x} = \frac{x^2+y^2 - 2x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2}$$
$$\frac{\partial v}{\partial x} = \frac{y\,2x}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = -\frac{x\,2y}{(x^2+y^2)^2}$$

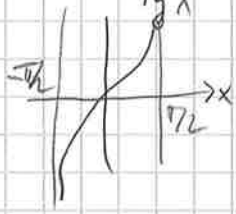
$$f'(z) = \frac{y^2 - x^2 + 2ixy}{(x^2+y^2)^2} = -\frac{1}{z^2}$$

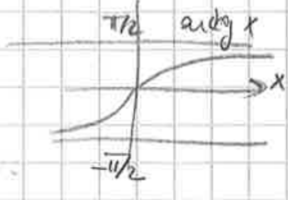
Ex 4 $f(z) = e^z = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ f holomorphe dans $\mathbb{C}$

Ex5 $f(z) = \log z = \log|z| + i \arg z$







$x>0$: $\arg z_1 = \mathrm{arctg} \frac{y_1}{x_1}$ $\quad \arg z_2 = \mathrm{arctg} \frac{y_2}{x_2}$ $\quad -\pi/2 < \arg z_1, \arg z_2 < \pi/2$

$x<0\ y<0$: $\arg z_3 = -\pi + \arg z_1 = -\pi + \mathrm{arctg} \cancel{\frac{y_1}{x_1}} = -\pi + \mathrm{arctg} \frac{y_3}{x_3}$

$x<0\ y>0$: $\arg z_4 = \pi + \arg z_2 = +\pi + \mathrm{arctg} \frac{y_2}{x_2} = \pi + \mathrm{arctg} \frac{y_4}{x_4}$

exercice

pex $x>0\ y>0$ $\quad \log z = \underbrace{\log \sqrt{x^2+y^2}}_{u(x,y)} + i \underbrace{\mathrm{arctg} \frac{y}{x}}_{v(x,y)}$

$\frac{\partial u}{\partial x}(x,y) = \frac{1}{\sqrt{x^2+y^2}} \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} = \frac{x^2}{x^2+y^2} \frac{1}{x} = \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = -\frac{y}{x^2+y^2}$

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i \frac{y}{x^2+y^2} = \frac{1}{z}$

Si $z = x \quad x<0 \qquad \log z = \log|x| + i\pi = \log(-x) + i\pi$

$\frac{\partial u}{\partial x} = \frac{1}{-x}(-1) \qquad \frac{\partial v}{\partial y} = 0$

Conclusion : f holomorphe $O \to \mathbb{C}$ $\quad O = \{z \in \mathbb{C};\ \mathrm{Re}\, z \leq 0\ \mathrm{Im}\, z = 0\}$ et $f'(z) = \frac{1}{z}$

Ex 6 : idem $f(z) = z^\gamma \quad \gamma \in \mathbb{C}$

E 7 : exercice

Dem Thm 3.2 : $f: O \to \mathbb{C}$ $\quad f(z) = u(x,y) + iv(x,y)$

On va dém que si f est holomorphe et si $u, v \in C^1$ alors $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ et $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

f holomorphe : $\forall z_0 \in O$: $\quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$z = z_0 + ih$, $z = z_0 + h$

$z_0 = x_0 + iy_0$ $\quad z = z_0 + h \quad h > 0 \quad h \to 0$
$= x_0 + h + iy_0$

$$f'(z_0) = \lim_{\substack{h>0 \\ h \to 0}} \frac{f(z_0+h) - f(z_0)}{z - z_0} = \lim_{\substack{h>0 \\ h \to 0}} \frac{u(x_0+h, y_0) + iv(x_0+h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x_0 + h + iy_0 - x_0 - iy_0}$$

$$= \lim_{\substack{h>0 \\ h \to 0}} \left(\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i \frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \right)$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \qquad (1)$$

$z = z_0 + ih \quad h > 0 \quad h \to 0$
$= x_0 + i(y_0 + h)$

$$f'(z_0) = \lim_{\substack{h>0 \\ h \to 0}} \frac{f(z_0+ih) - f(z_0)}{z - z_0} = \lim_{\substack{h>0 \\ h \to 0}} \frac{u(x_0, y_0+h) + iv(x_0, y_0+h) - u(x_0, y_0) - iv(x_0, y_0+h)}{x_0 + i(y_0+h) - x_0 - iy_0}$$

$$= \lim_{\substack{h \to 0 \\ h>0}} \left(\frac{u(x_0, y_0+h) - u(x_0, y_0)}{ih} + i \frac{v(x_0, y_0+h) - v(x_0, y_0+h)}{ih} \right)$$

$$= -i \left(\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \right) \qquad (2)$$

On égale (1) et (2).

donc $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$.

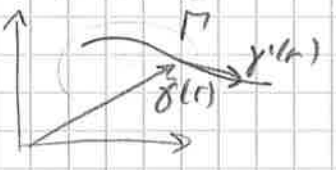
Chap 10 Intégration complexe

Thm de Cauchy - Formule intégrale de Cauchy

Rappel : $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$

$(x_1, x_2) \to \vec{F}(x_1, x_2) = \left(F_1(x_1,x_2), F_2(x_1,x_2)\right)$

Γ courbe [illegible] rég.
param $\gamma : [a, b] \to \mathbb{R}^2$
$t \to \gamma(t)$



alors $\displaystyle\int_\Gamma \vec{F} \cdot d\vec{l} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\,dt$

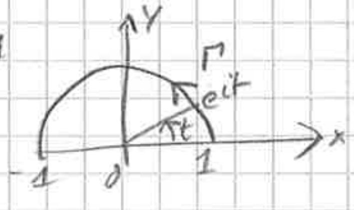
Déf 10.1 : Soit $\Gamma \subset \mathbb{C}$ courbe reg param $\gamma : [a,b] \to \Gamma$

Soit $f : \Gamma \to \mathbb{C}$ cont. On note

$$\int_\Gamma f(z)\,dz \underset{\text{abus de notation}}{=} \int_\gamma f(z)\,dz \underset{\text{def}}{=} \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

produit de deux nombres complexes

Exemple 1 $f(z) = z^2$ Γ demi cercle rayon 1 centre 0



$\gamma : [0, \pi] \to \mathbb{C}$
$t \to e^{it} = \cos t + i \sin t$

$\gamma'(t) = i e^{it}$

$$\int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz = \int_0^\pi (e^{it})^2\, i e^{it}\,dt = i\int_0^\pi e^{i3t}\,dt = \frac{i}{3i}\left[e^{3it}\right]_0^\pi$$

$$= \frac{1}{3}\left(e^{3i\pi} - 1\right) = \frac{1}{3}(-1-1) = -\frac{2}{3}$$

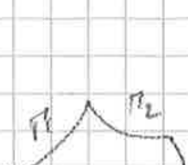
Exemple 2 $f(z) = z^2$ Γ cercle centre 0 rayon 1

$$\int_\Gamma f(z)\,dz = \frac{i}{3i}\left[e^{3it}\right]_0^{2\pi} = 0$$

Exemple 3 $f(z) = \frac{1}{z}$ Γ cercle centre 0 rayon 1

$$\int_\Gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it}\,dt = i\,2\pi$$

Rque : Si Γ est régulière par morceaux $\Gamma = \bigcup_{k=1}^{m} \Gamma_k$



$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{m} \int_{\Gamma_k} f(z)\,dz$$

Thm 10.2 (thm de Cauchy)

Soit $D \subset \mathbb{C}$ un domaine simplement connexe

Soit $f : D \to \mathbb{C}$ holomorphe $\Gamma \subset D$ une courbe simple fermée regulière (par morceaux)

On a alors
$$\int_\Gamma f(z)dz = 0$$

cf Exemple 2 $\quad f(z) = z^2 \quad \Gamma$ cercle centre 0 rayon 1 $\quad \int_\Gamma f(z)dz = 0$

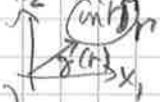
Exemple 3 : le thm ne s'applique pas. Par contre si Γ cercle centre 2 rayon 1 $\quad \gamma(t) = 2 + e^{it}$
$$\int_\Gamma \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{2+e^{it}} ie^{it}dt = \text{on doit trouver } 0$$
$$= \left[\ln(2+e^{it})\right]_{t=0}^{t=2\pi} = \ln 3 - \ln 3 = 0$$

Ex 4 : $f(z) = \frac{1}{z^2}$ Γ cercle centre 0 rayon 1 : le thm ne s'applique pas mais
$$= \int_\Gamma \frac{1}{z^2}dz = \int_0^{2\pi} \frac{ie^{it}}{e^{2it}}dt = i\int_0^{2\pi} e^{-it} = \frac{i}{-i}\left[e^{-it}\right]_0^{2\pi} = 0$$

$f : D \to \mathbb{C}$ holomorphe satisfait CR $f = u + iv \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Dem : $\gamma : [a,b] \to \mathbb{R}$

$t \to \gamma(t) = \alpha(t) + i\beta(t)$
$$\int_\Gamma f(z)dz = \int_a^b \underbrace{\left(u(\alpha(t),\beta(t)) + iv(\alpha(t),\beta(t))\right)}_{f(\gamma(t))}\underbrace{\left(\alpha'(t) + i\beta'(t)\right)}_{\gamma'(t)}dt$$
$$= \int_a^b \left(u(\alpha(t),\beta(t))\alpha'(t) - v(\ )\beta'(t) + i\,v(\ )\alpha' + u(\ )\beta'\right)dt$$



Thm Green (4.2) $F : \mathbb{R}^2 \to \mathbb{R}^2$

$(x_1, x_2) \to F(x_1, x_2) = (F_1(x_1 x_2), F_2(x_1 x_2)) \qquad \gamma(t) = (\gamma_1(t), \gamma_2(t))$
$$\int_\Gamma F \cdot d\ell = \int_a^b F_1(\gamma_1(t),\gamma_2(t))\gamma_1'(t) + F_2(\gamma_1(t),\gamma_2(t))\gamma_2'(t)$$
$$= \iint_{\text{int}\,\Gamma} \left(\frac{\partial F_2}{\partial x_1}(x_1 x_2) - \frac{\partial F_2}{\partial x_1}(x_1 x_2)\right)dx_1 dx_2$$
$$\int_\Gamma f(z)dz = \int_\Gamma F \cdot d\ell + i\int_\Gamma G \cdot d\ell \qquad F = (u, -v) \quad G = (v, u)$$
$$= \iint_{\text{int}\,\Gamma} \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{0}dxdy + i\iint_{\text{int}\,\Gamma} \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{0}dxdy \quad \text{d'après CR}$$
$$= 0$$

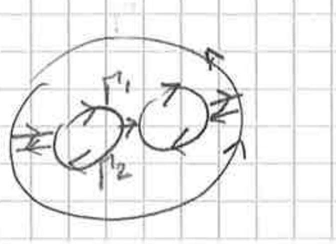
Corollaire

$D = D_0 \setminus (D_1 \cup D_2)$

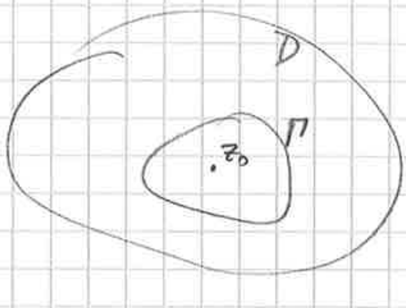
$f : D \to \mathbb{C}$ holom
$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

Dem $\int_{\Gamma_1} f(z)dz = 0 \quad \int_{\Gamma_2} f(z)dz = 0$

on somme



Thm 10.2. (Formule intégrale de Cauchy) ⑨



D simplement connexe

$f : D \to \mathbb{C}$ holomorphe

Γ courbe simple fermée régulière

$\forall z_0 \in \operatorname{int} \Gamma \quad f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz$

Exemples of exercices

$\int_\Gamma \frac{\cos 2z}{z}\, dz$ et $\int_\Gamma \frac{e^{2z}}{z}\, dz$ (calcul direct pas simple)

$f(z) = \cos 2z$, $z_0 = 0$ $\qquad f(z) = e^{2z}$, $z_0 = 0$

Dem : (idée) corollaire Thm Cauchy

C : cercle centre z_0 rayon r (qui va tendre vers 0)

$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz = \int_C \frac{f(z)}{z - z_0}\, dz$$

param de C : $\gamma(t) = z_0 + re^{it} \quad 0 \le t \le 2\pi$

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + re^{it})\, ire^{it}}{re^{it}}\, dt = i \int_0^{2\pi} f(z_0 + re^{it})\, dt \xrightarrow{r \to 0} 2\pi i\, f(z_0)$$

donc $\int_\Gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$

Thm 10.2 (Formule intégrale de Cauchy – suite et fin)

D simpl conn

$f : D \to \mathbb{C}$ hol

$\Gamma \subset D$ fermé

$f^{(n)}$ existe $\forall n \in \mathbb{N}$ et $\forall z_0 \in \text{int}\,\Gamma$ $\quad f^{(n)}(z_0) = \dfrac{n!}{2\pi i} \displaystyle\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

Dém : par induction (récurrence) sur n

$n = 1$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\substack{h \to 0\\ h\neq 0}} \frac{f(z_0+h)-f(z_0)}{h}$$

on sait déjà : $f(z_0+h) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(z)}{z-(z_0+h)}\,dz$

et $f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(z)}{z-z_0}\,dz$

donc $f(z_0+h) - f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \left(\frac{f(z)}{z-(z_0+h)} - \frac{f(z)}{z-z_0} \right) dz$

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-(z_0+h))(z-z_0)}$$

On prend la $\lim_{h \to 0}$ pour obtenir $\quad f'(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(z)}{(z-z_0)^2}\,dz$

Ex Γ cercle centre 0 rayon 1 $\quad \displaystyle\int_\Gamma \frac{1}{z^2}\,dz = \frac{2\pi i}{1!} \underbrace{f'(z_0)}_{0} = 0$

$f(z) = 1 \quad z_0 = 0$

Ex : discuter en fct de Γ $\quad \displaystyle\int_\Gamma \frac{e^{z+z}}{(z-2)^3}$

Si $2 \notin \text{int}\,\Gamma$ $\quad \dfrac{e^{z+2}}{(z-2)^3}$ est holomorphe $\quad \displaystyle\int_\Gamma \frac{e^{z+z}}{(z-2)^3} = 0$

$2 \in \text{int}\,\Gamma$ $\quad f(z) = e^{z+2}$ $\quad z_0 = 2$, $n = 2$ $\quad \displaystyle\int_\Gamma \frac{e^{z+2}}{(z-2)^3} = \frac{2\pi i}{2!} \underbrace{f''(2)}_{e^4} = \pi i e^4$

Chap 11 Séries de Laurent

Rappels dans $\mathbb{R}$: $f : \mathbb{R} \to \mathbb{R}$ (ou $f :]a,b[\to \mathbb{R}$) $x_0 \in \mathbb{R}$ (ou $a < x_0 < b$)

- **dév limité** : $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \dots$

Plus précis : $f(x) = f(x_0) + \text{———} + \underbrace{(x-x_0)^2 \varepsilon(x)}$

avec $\varepsilon(x) \xrightarrow{x \to x_0} 0$

ou encore : $\exists\ 0 < \theta < 1$

$\text{———} + \frac{f^{(3)}(\theta x + (1-\theta)x_0)}{3!}(x-x_0)^3$

- **Série de taylor de f en x_0** :

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$

Question : A-t-on $f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$?

Réponse : – oui si $f(x) = e^x$ $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad \forall x \in \mathbb{R}$ (dév lim en 0 : $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ …)

– oui si $f(x) = \frac{1}{1-x}$ $\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad |x| < 1$ $(-1 < x < 1)$

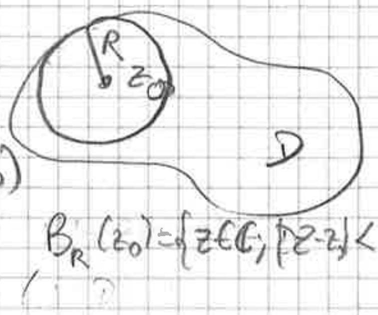
– non si $f(x) = e^{-1/x^2}$ $\quad x > 0$
$\qquad\qquad = 0 \quad x = 0$

$f(0) = 0 \quad f'(0) = \lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{e^{-1/x^2}}{x} = 0 \quad f^{(n)}(0) = 0\ \forall n$

On ne peut donc pas avoir $\underbrace{f(x)}_{} = \sum_{n=0}^{+\infty} \underbrace{\frac{f^{(n)}(0)}{n!}}_{0} x^k \quad x > 0$

car $\neq 0$

Série de Taylor dans $\mathbb{C}$:



D simplement connexe, Soit $f : D \subset \mathbb{C} \to \mathbb{C}$ holomorphe, soit $z_0 \in \mathbb{C}$
soit R tq $B_R(z_0) \subset D$ (si $D = \mathbb{C}$, alors $R = +\infty$)
Alors $\forall z \in B_R(z_0)$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$B_R(z_0) = \{z \in \mathbb{C}, |z - z_0| < R\}$

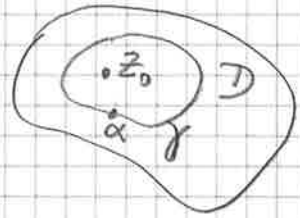
Ex : $f(z) = e^z$ holomorphe dans $\mathbb{C}$, $z_0 = 0$ $\quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \forall z \in \mathbb{C}$

$f(z) = \frac{1}{1-z}$ $z_0 = 0$ $\quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \forall z$ tq $|z| < 1$

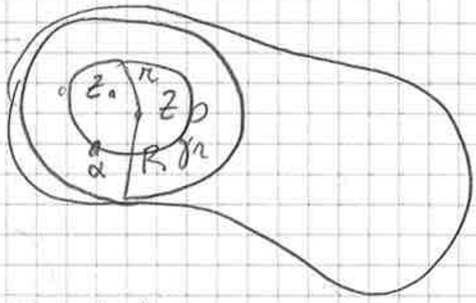
$f(z) = \frac{1}{1+z^2}$ $z_0 = 0$ $\quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad \forall z$ tq $|z| < 1$

Rque : $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$
$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

avec $c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\alpha)}{(\alpha - z_0)^{n+1}} \, d\alpha$



Dem :



$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\alpha)}{\alpha - z} \, d\alpha$$

$$\alpha - z = \alpha - z_0 + z_0 - z \qquad |z - z_0| < |\alpha - z_0| \qquad \left| \frac{z - z_0}{\alpha - z_0} \right| < 1$$

$$\frac{1}{\alpha - z} = \frac{1}{\alpha - z_0 + z_0 - z} = \frac{1}{(\alpha - z_0)\left(1 - \frac{z - z_0}{\alpha - z_0}\right)}$$

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n \qquad \forall\, |q| < 1$$

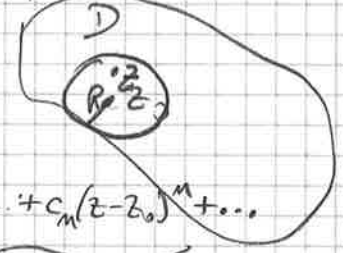
$$\frac{1}{\alpha - z} = \frac{1}{\alpha - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\alpha - z_0} \right)^n$$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\alpha)}{\alpha - z} \, d\alpha = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\alpha)}{\alpha - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\alpha - z_0} \right)^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\alpha)}{(\alpha - z_0)^{n+1}}}_{\frac{f^{(n)}(z_0)}{n!}} (z - z_0)^n$$

Série de Laurent dans $\mathbb{C}$

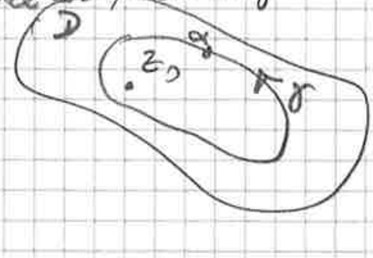
Thm II.1 $D \subset \mathbb{C}$ simplement connexe, $z_0 \in D$, $f : D \setminus \{z_0\} \to \mathbb{C}$ holomorphe, $z \in B_R(z_0)$, on a :



$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n$$

$$\underbrace{\cdots + \frac{c_{-n}}{(z-z_0)^n} + \cdots + \frac{c_{-1}}{z-z_0}}_{\text{partie singulière}} \underbrace{+ c_0 + c_1(z-z_0) + \cdots + c_n(z-z_0)^n + \cdots}_{\text{partie régulière}}$$

où $c_n = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(\alpha)}{(\alpha - z_0)^{n+1}}\, d\alpha$ avec γ courbe fermée $\subset D$, $z_0 \in \operatorname{int}\gamma$



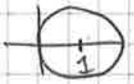
Rque : $c_{-1} = \dfrac{1}{2\pi i} \displaystyle\int_\gamma f(\alpha)\, d\alpha$

Ex 1 : Si $f : D \to \mathbb{C}$ holomorphe $f(z) = \sum_{n=0}^{+\infty} c_n (z-z_0)^n$, la partie singulière est nulle, $c_n = 0$ $n < 0$. On dit que z_0 est un point régulier

Ex 2 : $f(z) = \frac{1}{z}$ $D = \mathbb{C} \setminus \{0\}$ $z_0 = 0$ $f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$ $\forall z \neq 0$

On dit que $z_0 = 0$ est un pôle d'ordre 1 $c_n = 0$ $n \neq -1$ $c_{-1} = 1$

Ex 3 : $f(z) = \frac{1}{z}$ $z_0 = 1$ $f(z) = \dfrac{1}{1 - \underbrace{(-z+1)}_{q}} = \sum_{n=0}^{\infty} q^n = \sum_{n=0}^{\infty} (-(z-1))^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$ $\forall z, |z-1| < 1$



la partie singulière est nulle, $z_0 = 1$ pt régulier

Ex 4 : $f(z) = \frac{1}{z^2}$ $z_0 = 0$ $f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$ $c_n = 0$ $n \neq -2$ $c_{-2} = 1$ $\forall z \neq 0$

$z_0 = 0$ pôle d'ordre 2

Ex 5 : $f(z) = \frac{1}{z^2} + \frac{2}{z} = \sum_{n=-\infty}^{+\infty} c_n z^n$ $c_n = 0$ $n \neq -1$ et -2 $c_{-2} = 1$ $c_{-1} = 2$ $z_0 = 0$ $\forall z \neq 0$

$z_0 = 0$ pôle d'ordre 2

Ex 6 : $f(z) = \dfrac{1}{z(z+1)} = \dfrac{1}{z} - \dfrac{1}{z+1} = \dfrac{1}{z} - \dfrac{1}{1-(-z)} = \dfrac{1}{z} - \sum_{n=0}^{+\infty} (-z)^n$ $0 < |z| < 1$

$= \frac{1}{z} - \sum_{n=0}^{\infty} (-1)^n z^n$ $z_0 = 0$ $c_{-1} = 1$ $c_n = (-1)^{n+1}$

pôle d'ordre 1.

Ex 7 : $f(z) = e^{1/z}$ $z_0 = 0$ $f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \frac{1}{n!} = \underset{\text{partie rég}}{1} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n! z^n}}_{\text{partie sing.}}$ $\forall z \neq 0$

Ex 8 : $f(z) = \dfrac{\sin z}{z} = \dfrac{z - \frac{z^3}{3!} + \cdots}{z} = 1 - \dfrac{z^2}{3!} + \cdots$ z_0 pt régulier

$\sin z / z^2$ $= \frac{1}{z} - \frac{z}{3!} \ldots$ pôle d'ordre 1

D simpl conn.

<u>Def</u> : • $f : D\setminus\{z_0\} \to \mathbb{C}$ holomorphe

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n \quad \forall z \in B_R(z_0)$$

c_{-1} = Résidu de f en z_0 = $\mathrm{Res}_{z_0}(f)$

• z_0 pôle d'ordre m : $f(z) = \sum_{n=-m}^{+\infty} c_n (z-z_0)^n$

$$= \frac{c_m^{\neq 0}}{(z-z_0)^m} + \dots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0)$$

$$+ c_n (z-z_0)^n + \dots$$

<u>Prop 11.4</u> : z_0 pôle d'ordre m $\quad c_{-1} = \mathrm{Res}_{z_0}(f) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

<u>Dém</u> Par récurrence sur m

$m=1$ $\quad f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots + c_n(z-z_0)^n + \dots$

$$(z-z_0)f(z) = c_{-1} + c_0(z-z_0) + c_1(z-z_0)^2 + \dots + c_n(z-z_0)^{n+1} + \dots$$

$\lim_{z\to z_0}$ ―――― $= c_{-1}$

$m=2$: $\quad f(z) = \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots + c_n(z-z_0)^n + \dots$

$$(z-z_0)^2 f(z) = c_{-2} + c_{-1}(z-z_0) + c_0(z-z_0)^2 + \dots$$

$\frac{d}{dz}$ ―――― $= c_{-1} + 2c_0(z-z_0)$

$\lim_{z\to z_0}$ ―――― $= c_{-1}$

$m=3$ $\quad f(z) = \frac{c_{-3}}{(z-z_0)^3} + \frac{c_{-2}}{(z-z_0)^2} + c_{-1} + c_0 + c_1(z-z_0) + \dots$

$$(z-z_0)^3 f(z) = c_{-3} + c_{-2}(z-z_0) + c_{-1}(z-z_0)^2 + c_0(z-z_0)^3 + \dots$$

$\frac{d^2}{dz^2}$ ―――― $= 2c_{-1} + 3.2.c_0(z-z_0) + \dots$

$\lim_{z\to z_0}$ ―――― $= 2\,c_{-1}$

$(m-1)!$

Rque : on peut aussi faire $e^z = e^2 e^{z-2} = e^2(1 + z - 2 + \dots$

Ex : $f(z) = \frac{e^z}{(z-2)^2}$ $\quad z_0 = 2$ pôle d'ordre 2 $\quad \mathrm{Res}_2(f) = \lim_{z\to 2} \frac{d}{dz}\left((z-2)^2 f(z)\right) = \lim_{z\to 2} \frac{d}{dz} e^z = e^2$

Ex : $f(z) = \frac{\sin z}{1+z^2}$ = $\quad z_0 = i$ $\quad \sin i = \frac{\frac{1}{e} - e}{\dots} \neq 0$ z_0 pôle d'ordre 1 $\quad \mathrm{Res}_i(f) = \lim (z-i)f(z) = \lim \frac{\sin z}{\dots} = \frac{\sin i}{\dots}$

Chap 12 Thm des résidus et applications

Thm 12.1 $D \subset \mathbb{C}$ simplement connexe

γ courbe fermée $\subset D$

$z_1, z_2, \dots, z_m \in \operatorname{int} \gamma$

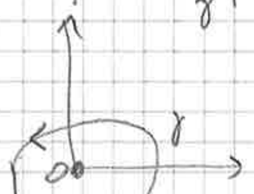
$f : D \setminus \{z_1, z_2, \dots, z_m\}$ holomorphe

Alors

$$\int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}_{z_1}(f) + \operatorname{Res}_{z_2}(f) + \dots + \operatorname{Res}_{z_m}(f) \right)$$

Rque : $f : D \to \mathbb{C}$ holomorphe $\int_\gamma f(z)\,dz = 0$

Ex :



$f(z) = \frac{1}{z}$ $z \neq 0$

$0 \in \operatorname{int} \gamma$ $\operatorname{Res}_0(f) = 1$

$$\int_\gamma f(z)\,dz = 2\pi i$$

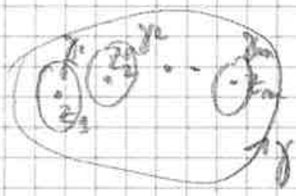
$$\int_\gamma f(z)\,dz = 0$$

$f(z) = \frac{1}{z(z+1)}$ $z_1 = 0$ $z_2 = -1$

$f(z) = \frac{1}{z} - \frac{1}{z+1}$ $\operatorname{Res}_0(f) = 1$ $\operatorname{Res}_{-1}(f) = -1$

$\int_\gamma = 0$ $\int_\gamma f(z)\,dz = 2\pi i\left(\operatorname{Res}_0(f) + \operatorname{Res}_{-1}(f)\right) = 0$ $\int_\gamma = 2\pi i$ $\int_\gamma = -2\pi i$

Dem :



Coroll thm Cauchy

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} + \int_{\gamma_2} + \dots + \int_{\gamma_m}$$

$$= 2\pi i \left(\operatorname{Res}_{z_1}(f) + \dots + \operatorname{Res}_{z_m}(f) \right)$$

Ex : $f(z) = \frac{2}{z} + \frac{3}{z-1} + \frac{1}{z^2}$

$\operatorname{Res}_0(f) = 2$ $\operatorname{Res}_1(f) = 3$

$\int_\gamma f(z)\,dz = 2\pi i\,(2+3) = 10\pi i$ $= 2\pi i \cdot 2$ $= 2\pi i \cdot 3$ $= 0$

Si γ passe par 0 ou 1, pas bien déf.

Application au calcul d'intégrales

Cas général $I=\int_0^{2\pi} f(\cos\theta,\sin\theta)\,d\theta$

γ : param $\gamma(t)=e^{i\theta}$ $0<\theta<2\pi$

On cherche $\tilde f(z)$ de sorte que

$$I=\int_\gamma \tilde f(z)\,dz$$

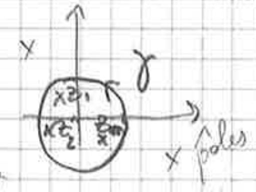
$$\int_\gamma \tilde f(z)\,dz=\int_0^{2\pi}\tilde f(e^{i\theta})\,ie^{i\theta}\,d\theta$$

On identifie : $\tilde f(e^{i\theta}) = \dfrac{f(\cos\theta,\sin\theta)}{ie^{i\theta}}$

$$= \frac{f\left(\frac{e^{i\theta}+e^{-i\theta}}{2},\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)}{ie^{i\theta}}$$

$z=e^{i\theta}$ $\quad \tilde f(z)=\dfrac{f\left(\frac{z+\frac1z}{2},\frac{z-\frac1z}{2i}\right)}{iz}$

$$I=\int_\gamma \tilde f(z)\,dz = 2\pi i\left(\mathrm{Res}_{z_1}(\tilde f)+\dots+\mathrm{Res}_{z_m}(\tilde f)\right)$$



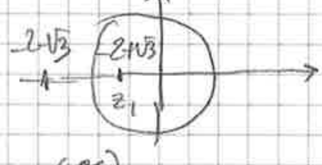
Ex : $I=\int_0^{2\pi}\dfrac{d\theta}{2+\cos\theta}$ $\qquad f(\cos\theta,\sin\theta)=\dfrac{1}{2+\cos\theta}$

$$\tilde f(z)=\frac{\frac{1}{2+\frac{z+\frac1z}{2}}}{iz}=\frac{\frac{2}{4+z+\frac1z}}{iz}$$

$$=\frac{2}{i(z^2+4z+1)}$$

$\Delta=16-4=12$

$$z_{1,2}=\frac{-4\pm\sqrt{12}}{2}=-2\pm\sqrt3$$



$$I=2\pi i\,\mathrm{Res}_{z_1}(\tilde f)$$

$\mathrm{Res}_{z_1}(\tilde f)=?$

$$\tilde f(z)=\frac{2}{i(z^2+4z+1)}=\frac{2}{i(z+2+\sqrt3)\underbrace{(z+2-\sqrt3)}_{z-z_1}}$$

$$\mathrm{Res}_{z_1}(\tilde f)=\lim_{z\to z_1}(z-z_1)\tilde f(z)$$

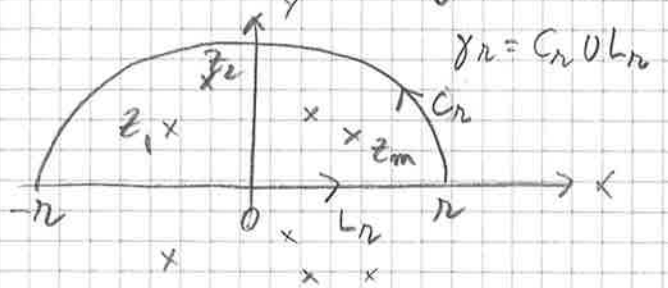
$$=\frac{2}{i(z_1+2+\sqrt3)}=\frac{1}{i\sqrt3}$$

$$I=\frac{2\pi}{\sqrt3}$$

- Cas général $\int_{-\infty}^{+\infty} R(x) e^{iax} dx \quad a \geq 0$

avec $R(x) = \frac{P(x)}{Q(x)}$ P, Q polynômes tq

$Q(x) \neq 0 \quad x \in \mathbb{R} \quad \deg Q \geq \deg P + 2$



Les pts singuliers de $f(z) = R(z) e^{iaz}$ sont situés comme sur la fig pour r suff grand. n'y en a pas sur l'axe réel ($Q(x) \neq 0$)

Donc $\int_{-r}^{r} R(x) e^{iax} dx + \int_{C_r} R(z) e^{iaz} dz$

$$= 2\pi i \left(\operatorname{Res}_{z_1}(R(z)e^{iaz}) + \dots + \operatorname{Res}_{z_m}(R(z)e^{iaz}) \right)$$

On va montrer que si $r \to \infty$ $I_r = \int_{C_r} R(z) e^{iaz} dz = 0$ de sorte que

$$\int_{-\infty}^{+\infty} R(x) e^{iax} dx = 2\pi i \left(\operatorname{Res} \right)$$

$$I_r = \int_0^{2\pi} \frac{P(re^{i\theta})}{Q(re^{i\theta})} e^{ia(r\cos\theta + ir\sin\theta)} ire^{i\theta} d\theta$$

$$\leq \int_0^{2\pi} \underbrace{\left|\frac{P(re^{i\theta})}{Q(re^{i\theta})}\right|}_{\leq \frac{C}{r^2}} \underbrace{\left| e^{-ar\sin\theta} \right|}_{\leq 1} r \, d\theta$$

car $\deg Q \geq \deg P + 2$ (C indep r et θ)

$$\leq \int_0^{2\pi} \frac{C}{r} d\theta \xrightarrow{r \to +\infty} 0$$

$$|16 + r^4 e^{i2\theta}|^2 = |16 + r^4(\cos 2\theta + i \sin 2\theta)|^2$$
$$= (16 + r^4 \cos 2\theta)^2 + r^8 \sin^2 2\theta$$
$$= 16^2 + r^8 + 2 \cdot 16 r^4 \cos 2\theta$$
$$\geq 16^2 + r^8 - 2 \cdot 16 r^4 \cos 2\theta = (16 - r^4)^2$$

$|16 + r^4 e^{i2\theta}| \geq |16 - r^4|$ $\quad \left(\sqrt{a^2} = |a|, \ \sqrt{(-2)^2} = 2 \right)$

$$\frac{1}{|16 + r^4 e^{i2\theta}|} \leq \frac{1}{|16 - r^4|}$$

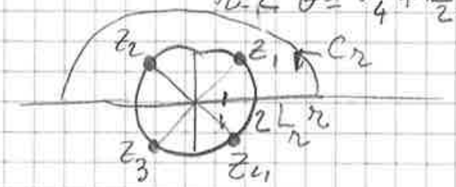
Ex $\int_{-\infty}^{+\infty} \frac{x^2}{16 + x^4} dx \quad (a = 0)$

$f(z) = \frac{z^2}{16 + z^4}$ $(a = 0)$ pôles $16 + z^4 = 0$

$z^4 = -16$

$r^4 e^{i4\theta} = 16 e^{i(\pi + 2k\pi)}$

$r = 2 \quad \theta = \frac{\pi}{4} + \frac{k\pi}{2}$



$$\operatorname{Res}_{z_1}(f) = \lim_{z \to z_1} (z - z_1) f(z)$$
$$= \lim_{z \to z_1} \frac{z^2}{(z - z_2)(z - z_3)(z - z_4)}$$
$$= \frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)}$$
$$= \frac{1 + i}{8i\sqrt{2}}$$

$$\operatorname{Res}_{z_2}(f) = \frac{1 - i}{8i\sqrt{2}}$$

$$\int_{-\infty}^{+\infty} \frac{x^2}{16 + x^4} dx = 2\pi i \frac{2}{8i\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

$$\int_{C_r} \frac{z^2}{16 + z^4} dz = \int_0^{2\pi} \frac{r^2 e^{i2\theta}}{16 + r^4 e^{i2\theta}} r e^{i\theta} d\theta$$

$$\leq \int_0^{2\pi} | \quad | d\theta$$

$$= \int_0^{2\pi} \frac{|r^2 e^{i2\theta}| \, |re^{i\theta}|}{|16 + r^4 e^{i2\theta}|} d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{|16 + r^4 e^{i2\theta}|} d\theta$$

justif

$$\leq \frac{r^3}{|16 - r^4|} 2\pi \xrightarrow{r \to \infty} 0$$