Computable Poincaré–Friedrichs constants for the L^p de Rham complex over convex domains and domains with shellable triangulations

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Abstract

We construct potentials for the gradient, the curl, and the divergence operators over domains with shellable triangulations. Notably, the class of shellable triangulations includes local patches (stars) in two or three dimensions. The operator norms of our potentials satisfy explicitly computable bounds that depend only on the geometry. We thus obtain computable upper bounds for constants in Poincaré–Friedrichs inequalities and computable lower bounds for the eigenvalues of vector Laplacians. As an additional result with independent standing, we establish Poincaré–Friedrichs inequalities with computable constants for the L^p de Rham complex over bounded convex domains. This is achieved via regularized Poincaré and Bogovskiĭ potential operators whose operator norms we bound. We express all our main results in the calculus of differential forms and treat the gradient, curl, and divergence operators as its particular instances. Computational examples illustrate the theoretical findings.

1 Introduction

Potentials for the differential operators of vector calculus and exterior calculus are of fundamental importance. The operator norms of these potentials are upper bounds for the Poincaré–Friedrichs constants. These quantify the fundamental stability properties of numerous partial differential equations and enter the stability and convergence theory of numerical methods. Upper bounds of the Poincaré–Friedrichs constants also provide lower bounds of the eigenvalues of the associated Laplacians. However, while potentials and Poincaré–Friedrichs constants for the gradient have been subject to extensive study, quantifiable results regarding the curl and divergence operators, or more generally the exterior derivative, are largely unavailable.

This manuscript contributes to the theory of computable estimates for Poincaré–Friedrichs inequalities for the differential operators of vector calculus. How to construct potentials for the gradient over triangulated domains is well-documented in the literature. Here, we extend this construction to the curl and divergence operators, as well as the exterior derivative. However, to proceed in the general exterior derivative case, we restrict our efforts to so-called shellable triangulations. Only contractible domains can ever admit a shellable triangulation, but having computable upper bounds for such domains is an important stepping stone towards more general situations. The class of shellable triangulations includes practically relevant triangulations: for example, local triangulations around a simplex within a larger triangulation (the so-called local patches or stars) are shellable in dimensions two and three; see also Figure 1. Additionally, we include a study of regularized Poincaré and Bogovskiĭ operators that leads to new Poincaré–Friedrichs inequalities with computable constants for the whole L^p de Rham complex over bounded convex domains.

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1.1 Conceptual overview

We give a conceptual overview of the topic before we outline the known results in the literature and our contributions in more detail. Our conceptual point of reference is the Poincaré–Friedrichs inequality for the gradient of scalar functions, which has been subject to extensive research.

1.1.1 Potentials and Poincaré–Friedrichs inequalities for the gradient

For this illustration, we let $\Omega \subseteq \mathbb{R}^3$ be a bounded connected open set. We let $L^p(\Omega)$ denote the Lebesgue space over Ω with integrability exponent $1 \leq p \leq \infty$, and we write $W^{1,p}(\Omega)$ for the first-order Sobolev space over Ω with integrability exponent p.

We are interested in a constant $C_{\mathrm{grad},\Omega,p} > 0$ such that the following holds: for every gradient vector field $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$ there exists a scalar potential $u \in W^{1,p}(\Omega)$ such that $\nabla u = \boldsymbol{f}$ and

$$\|u\|_{L^p(\Omega)} \leq C_{\mathrm{grad},\Omega,p} \|\boldsymbol{f}\|_{L^p(\Omega)}. \tag{1}$$

This inequality is called *Poincaré–Friedrichs inequality* and the constant $C_{\mathrm{grad},\Omega,p}$ is called the *Poincaré–Friedrichs constant* with exponent p. The question is therefore whether we can always find a gradient potential of sufficiently small norm so this inequality holds. One possible choice is the norm-minimizing potential

$$\Phi_{\mathrm{grad}}(\boldsymbol{f}) := \operatorname*{argmin}_{\substack{u \in W^{1,p}(\Omega) \\ \nabla u = \boldsymbol{f}}} \|u\|_{L^p(\Omega)}. \tag{2}$$

In the present setting, where Ω is connected and the potential of the different gradient potentials can thus only differ by a constant, computing $\Phi_{\mathrm{grad}}(\boldsymbol{f})$ is a one-dimensional convex minimization problem. This inequality is therefore equivalent to

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^p(\Omega)} \leq C_{\mathrm{grad},\Omega,p} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{PF}$$

Such an inequality holds if we can bound the *operator norm* of this gradient potential operator, and then the best constant is

$$C_{\mathrm{grad},\Omega,p} := \max_{u \in W^{1,p}(\Omega) \setminus \mathbb{R}} \frac{\|\Phi_{\mathrm{grad}}(\nabla u)\|_{L^p(\Omega)}}{\|\nabla u\|_{L^p(\Omega)}}. \tag{3}$$

Finding the norm-minimizing potential and the optimal Poincaré–Friedrichs constant is generally difficult. Attention has been given to *linear* potentials instead of the norm-minimizing nonlinear potential (2). The operator norm of any linear potential construction serves as an upper bound for the Poincaré–Friedrichs constant. One very simple example for such a bounded linear potential operator is the average-free potential,

$$\Phi_{\varnothing}(\boldsymbol{f}) := \operatorname*{argmin}_{\substack{v \in W^{1,p}(\Omega) \\ \int_\Omega v = 0}} \|\nabla v - \boldsymbol{f}\|_{L^p(\Omega)}, \quad \forall \boldsymbol{f} \in \nabla W^{1,p}(\Omega), \tag{4}$$

where the mean value of u is fixed to zero. Its operator norm is the Poincaré constant $C_{\varnothing,\Omega,p} > 0$ that satisfies

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C_{\varnothing,\Omega,p} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \tag{P}$$

where u_Ω denotes the average of u. We emphasize that the average-free potential is generally not the norm-minimizing potential unless $p = 2$. Hence, the optimal Poincaré constant and the optimal Poincaré–Friedrichs differ in general. Considerable research efforts have gone into determining constants in the Poincaré–Friedrichs inequalities (PF) or Poincaré inequalities (P). Upper estimates for these constants also correspond to lower bounds for the spectra of Neumann–Laplacians over those domains. The relationship between the Poincaré constant and the Poincaré–Friedrichs constant will be elaborated upon in later sections of this manuscript.

1.1.2 Potentials and Poincaré–Friedrichs inequalities for the curl and divergence

We study potentials and Poincaré–Friedrichs inequalities for the curl or divergence operators in vector calculus. There are substantial changes and much fewer results are available in the literature. We use the spaces of vector field

$$\boldsymbol{W}^p(\operatorname{curl},\Omega) := \left\{ \boldsymbol{u} \in L^p(\Omega)^3 : \operatorname{curl}\boldsymbol{u} \in L^p(\Omega)^3 \right\}, \tag{5}$$

$$\boldsymbol{W}^p(\operatorname{div},\Omega) := \left\{ \boldsymbol{u} \in L^p(\Omega)^3 : \operatorname{div}\boldsymbol{u} \in L^p(\Omega) \right\}. \tag{6}$$

Their members are those vector fields in Lebesgue spaces whose distributional curls and divergences, respectively, are in Lebesgue spaces. In contrast to the gradient, this only requires that certain sums of distributional partial derivatives are integrable, and hence these spaces are not classical Sobolev spaces of vector fields. Our objective is to find bounded potentials (i.e., right inverses) for the operators

$$\operatorname{curl} : \boldsymbol{W}^p(\operatorname{curl},\Omega) \to L^p(\Omega)^3, \qquad \operatorname{div} : \boldsymbol{W}^p(\operatorname{div},\Omega) \to L^p(\Omega),$$

We are interested in the natural analogues for the Poincaré–Friedrichs inequality of the gradient (PF), which for the curl and divergence are the inequalities

$$\min_{\substack{\boldsymbol{v}\in\boldsymbol{W}^p(\operatorname{curl},\Omega)\\ \operatorname{curl}\boldsymbol{v}=\boldsymbol{f}}} \|\boldsymbol{v}\|_{L^p(\Omega)} \leq C_{\operatorname{curl},\Omega,p}\|\boldsymbol{f}\|_{L^p(\Omega)}, \tag{7}$$

$$\min_{\substack{\boldsymbol{v}\in\boldsymbol{W}^p(\operatorname{div},\Omega)\\ \operatorname{div}\boldsymbol{v}=f}} \|\boldsymbol{v}\|_{L^p(\Omega)} \leq C_{\operatorname{div},\Omega,p}\|f\|_{L^p(\Omega)}. \tag{8}$$

The fundamental difference to the gradient is that the curl and divergence have infinite-dimensional kernels. The kernel of the gradient is the one-dimensional space of constant functions, and it is thus trivially complemented for all p, with a canonical choice of projection. By contrast, the kernels of the curl and divergence operators are generally infinite-dimensional. Moreover, it is not immediately evident that the kernels of the curl and divergence operators are complemented in the Banach space case when $p \neq 2$, and a canonical projection only exists here in the Hilbert setting $p = 2$. In that sense, there generally is no natural analogue to the Poincaré inequality (P) for the curl and divergence.

In the Banach space case, it is not even trivial whether a norm-minimizing potential actually exists. We are therefore interested in any potentials

$$\Phi_{\operatorname{curl}} : \operatorname{curl}\boldsymbol{W}^p(\operatorname{curl},\Omega) \to \boldsymbol{W}^p(\operatorname{curl},\Omega), \tag{9}$$

$$\Phi_{\operatorname{div}} : \operatorname{div}\boldsymbol{W}^p(\operatorname{div},\Omega) \to \boldsymbol{W}^p(\operatorname{div},\Omega) \tag{10}$$

and their operator norms. We remark that upper bounds for the Poincaré–Friedrichs inequality of the curl operator correspond to lower bounds for the so-called Maxwell eigenvalues. In sharp contrast to the extensive research on gradient potentials, not much attention seems to have been given to the study of computable constants in such Poincaré–Friedrichs inequalities for the curl and divergence operators.

1.1.3 Potentials and Poincaré–Friedrichs inequalities for the exterior derivative

Though we present our results in vector calculus, our main arguments are given in the formalism of exterior calculus. Exterior calculus [34, 44] is used ubiquitously in the mathematical literature of physics and engineering and has found widespread adoption in the theoretical and numerical analysis for vector-valued partial differential equations [38, 35, 3, 4, 5, 20, 30, 6]. This formalism is independent of the spatial dimension and highlights the underlying geometric structures common to the gradient, curl, and divergence operators in three space dimensions. For every $u \in W^p\Lambda^k(\Omega)$, there exists $w \in W^p\Lambda^k(\Omega)$ with $dw = du$ and such that (all the notation is fixed in detail in Section 5 later)

$$\|u\|_{L^p(\Omega)} \leq C_{k,\Omega,p}\|dw\|_{L^p(\Omega)}. \tag{11}$$

For the purpose of our discussion, this formalism allows us to leverage results from a larger body of literature in differential geometry and functional analysis.

1.2 Literature review

We review the literature on Poincaré–Friedrichs inequalities. We also identify the obstructions inherent to presently known results that we wish to overcome with our contributions.

1.2.1 General results

The qualitative existence of Poincaré–Friedrichs inequalities for the differential operators of vector calculus and exterior calculus is known for a large class of domains. There is a particularly large body of literature on the Poincaré inequality for the gradient and its numerous variants, which may include weighted integrals or boundary terms. These Poincaré-type inequalities are often derived via non-constructive arguments, such as the Rellich embedding theorem, and thus the resulting constants are not explicitly computable.

The divergence operator has received less attention. In the Hilbert space case $p = 2$, the Friedrichs inequality [13] over the Sobolev space with homogeneous Dirichlet boundary conditions along $\partial\Omega$ implies (6) by duality. However, that easy duality argument, which yields an explicit upper bound proportional to the domain diameter, seems inherently restricted to $p = 2$.

The core challenges can be found in the discussion of the curl operator in three dimensions, where Poincaré–Friedrichs inequalities have appeared under different names. Let us assume for a moment that Ω is a weakly Lipschitz domain with trivial topology, and consider only the Hilbert case $p = 2$. Then the constant in (7) agrees with the constant in the so-called Poincaré–Friedrichs–Weber inequality

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq C_{\mathrm{curl},\Omega,p} \|\operatorname{curl} \boldsymbol{u}\|_{L^2(\Omega)}, \tag{12}$$

valid for all $\boldsymbol{u} \in \boldsymbol{W}^2(\mathrm{curl}, \Omega) \cap \boldsymbol{W}^2(\mathrm{div}, \Omega)$ that satisfy $\operatorname{div} \boldsymbol{u} = 0$ and have vanishing normal or vanishing tangential trace along $\partial\Omega$. Equivalently, (12) is valid for all $\boldsymbol{u} \in \boldsymbol{W}^2(\mathrm{curl}, \Omega)$ that are L^2-orthogonal to the gradients of scalar fields in $W^{1,2}(\Omega)$ or that have vanishing tangential trace and are L^2-orthogonal to the gradients of those scalar fields in $W^{1,2}(\Omega)$ that satisfy Dirichlet boundary conditions. We refer to Equation (5) in [27], Equation (2) in [28], [64] as well as [31, Lemmas 3.4 and 3.6], [25, Proposition 7.4], and the references therein. The general case of L^p differential forms over Lipschitz manifolds subject to partial boundary conditions is discussed in [33]. However, while many of the above results rely on non-constructive estimates, we are interested in practically computable upper bounds.

1.2.2 Analytical constants over convex domains

Considerable research effort has gone into computing explicit upper estimates for the constants in the Poincaré–Friedrichs inequalities over convex domains. Notably, if the constants are required to depend on the convex domain only via its diameter, then the optimal gradient Poincaré–Friedrichs and Poincaré constants for the entire range of Lebesgue exponent $1 \leq p < \infty$ are known explicitly [55, 8, 1, 23, 26].

The literature on Poincaré–Friedrichs constants and potentials of curls and divergences is less extensive than for potentials of gradients, even over convex domains. Guerini and Savo [36] address the spectrum of the Hodge–Laplace operator on bounded convex domains with smooth boundary in the Hilbert space case $p = 2$. Their results thus pertain to the Poincaré–Friedrichs constant of the exterior derivative, and hence in particular to the curl and divergence operators. They prove that the Poincaré–Friedrichs constant for the gradient already estimates the corresponding constants for the curl and divergence operators. They also provide explicit (but not necessarily optimal) upper bounds for Poincaré–Friedrichs constants that depend only on the dimension and diameter of the convex domain. A duality argument also yields upper estimates of the Poincaré–Friedrichs constants for the gradient, curl, and divergence operators subject to Dirichlet, tangential, and normal boundary conditions, respectively, along the entire boundary. Their inequalities hold in convex Lipschitz domains, too [48]. However, no results as those of [36] are known over bounded convex Lipschitz domains and for general Lebesgue exponents $1 \leq p \leq \infty$.

1.2.3 Domains star-shaped with respect to a ball

When the domain is not convex but star-shaped with respect to a ball, then several estimates for Poincaré–Friedrichs constants are known.[1] Polynomial interpolation estimates already imply the gradient

[1] In fact, Poincaré–Friedrichs inequalities even hold over any star-shaped open bounded set; see [39, Theorem 3.1].

Poincaré–Friedrichs inequality [12, 21]. We pay particular attention to the regularized Poincaré and Bogovskiĭ potential operators for the exterior derivative, such as those of Costabel and McIntosh [19]. In principle, the operator norms of those potentials as mapping between Lebesgue spaces of differential forms are upper estimates for the Poincaré–Friedrichs constants of the domain. Here, estimates for the higher-order seminorms of these potentials are available in [37], but estimates in Lebesgue norms have not been made explicit in the literature yet, to the best of our knowledge. We particularly emphasize that these estimates for domains star-shaped with respect to a ball have in common that they rely on upper bounds for the *eccentricity* of the domain.

Let us briefly discuss the practical limitations of the aforementioned estimates for Poincaré–Friedrichs constants for convex domains or domains that are star-shaped with respect to a ball. Recall that the main objective of this manuscript is bounding Poincaré–Friedrichs constants over domains with shellable triangulations, with a key application being local stars within triangulated domains. While not all local stars describe convex subdomains, most local stars are star-shaped with respect to a ball. Even though that would enable, e.g., the averaged Poincaré and Bogovskiĭ operators [19], the estimates that rely on this geometric condition deteriorate when the aforementioned ball has radius much smaller than the domain diameter. This would not be as much a problem over local patches (stars) around interior subsimplices, where the size of the interior ball only depends on the shape regularity of the triangulation. But the interior ball can be arbitrarily small when the local patch is around a boundary simplex, even if the mesh has good shape regularity: this occurs most prominently when the domain has sharp reentrant corners. Some illustrative limit cases include the slit domain [61] and the crossed bricks domain [46], which contain local finite element patches that are not star-shaped with respect to any ball. In view of this, we refrain from treating local patches (stars) in triangulations as domains star-shaped with respect to a ball.

1.2.4 Triangulated domains

Geometric settings that admit finite triangulations enable different pathways to obtain Poincaré–Friedrichs inequalities. We review some of the main outcomes.

Computable estimates for Laplacian eigenvalues over triangulated domains have received much attention. The constant in (1) for $p = 2$ corresponds to a lower bound for the Laplace eigenvalues and quantifies the stability properties of the Laplacian on the domain Ω. Similarly, the constant in (7) for $p = 2$ corresponds to a lower bound for the Maxwell eigenvalues and quantifies the stability properties of the Maxwell system on Ω. Thus, computable upper bounds on the Poincaré–Friedrichs constants also give computable lower bounds for the eigenvalues of the associated Laplacians and vice versa. Prominent methods numerically compute guaranteed upper bounds on the Poincaré–Friedrichs constants upon solving a finite element system over a sufficiently fine triangulation and using some clever post-processing estimates. This has led to estimates for scalar Laplacian eigenvalues [14, 49] and vector Laplacian eigenvalues [29]. We remark that for the purposes of this manuscript, we aim for computable upper bounds that do not rely on the solution of (global) finite element systems.

There are numerous estimates for Poincaré–Friedrichs constants that only rely on locally computable geometric quantities, such as the diameter and volumes of simplices. Veeser and Verfürth [61] provide computable upper bounds in the case of the classical Sobolev space $W^{1,p}(\Omega)$ over triangulated domains, with focus on efficient estimates for vertex stars. Naturally, their estimates depend on the shape regularity of the mesh. A whole class of upper bounds for Poincaré–Friedrichs inequalities uses some form of passing through the triangulation and constructs the potentials step-by-step. The underlying idea is that we first construct a potential for the gradient over an initial simplex. Every time we have found a potential over a subdomain, we construct a potential over a neighboring simplex or patch: along the interfacing intersection, the two potentials will only differ by a constant, and that difference can easily be removed to ensure continuity across that interface. Cell by cell, the potential is constructed over increasing subdomains, matching along the interfacing intersections, until the entire domain is exhausted. The method is known in the finite element literature [21]. It was previously used in the context of finite volume methods [24], broken (weakly continuous) Sobolev spaces [62], or more recently in continuous–discrete comparison results [11, 22, 16, 63]. This sequential procedure applies to general triangulated domains, not only local stars, though the latter are our main interest here. Most importantly, these sequential estimates of Poincaré–Friedrichs constants generally circumvent the effect of low boundary regularity and only rely on the shape regularity.

While we thus know Poincaré–Friedrichs constants over local stars for scalar-valued Sobolev spaces, we are not aware of computable estimates for the case of $\boldsymbol{W}^p(\mathrm{curl}, \Omega)$ and $\boldsymbol{W}^p(\mathrm{div}, \Omega)$ over finite element stars.

1.3 Objectives and methodology

The main objective of this manuscript is the construction of potentials for the gradient, curl, and divergence, and more generally the exterior derivative. The operator norms of our potentials satisfy computable upper bounds, thus yielding computable upper estimates of the Poincaré–Friedrichs constants as well. As we will elaborate in what follows, we focus on domains with shellable triangulations, which include local patches (stars) in two and three dimensions as important special cases. To that end, we also devote an important effort specifically to convex domains.

1.3.1 Convex domains

Our main result for convex domains in the exterior calculus setting is the construction of regularized Poincaré and Bogovskiĭ potential operators with explicitly bounded operator norms. This is summarized in Theorem 6.2. The upper bounds for the Poincaré–Friedrichs constants are proportional to the domain diameter and are bounded in terms of the domain's eccentricity. The bounds are independent of the Lebesgue exponent $1 \leq p \leq \infty$, though the space dimension and the form degree enter the estimates.

The reasons for our study of Poincaré–Friedrichs constants over convex domains is twofold: firstly, they are of evident independent interest, and secondly, we will need them as a component for our main results on triangulations. Our exposition of regularized Poincaré and Bogovskiĭ-type potential operators follows the general methodology of Costabel and McIntosh [19]. By comparison, our variants of these operators are simplified: we only study them over convex domains, instead of domains star-shaped with respect to a ball, and we use simpler (constant) weight functions. While the resulting potentials feature generally lower regularity, they are conductive for our purposes. Crucially, this allows us to easily estimate their operator norms and thus bound Poincaré–Friedrichs constants.

1.3.2 Potentials subject to partial boundary conditions

We also address the construction of potentials for the gradient, curl, and divergence operators, and in general the exterior derivative, over a simplex subject to partial boundary conditions. We are not aware of explicit estimates or regularized potentials for these boundary conditions in the published literature. For example, given a divergence-free vector field over a tetrahedron with vanishing normal trace along three of the tetrahedron's faces, we want to find the unique vector field potential that not only is a preimage under the curl operator but also has vanishing tangential trace along the same three faces. We achieve this by constructing an auxiliary problem subject to full boundary conditions, so that we can build upon the regularized Bogovskiĭ operators and the Poincaré–Friedrichs inequalities subject to boundary conditions on the entire boundary. We thus obtain Poincaré–Friedrichs inequalities over simplices and subject to partial boundary conditions. Again, we address the entire range $1 \leq p \leq \infty$ of Lebesgue exponents and our Poincaré–Friedrichs constants are explicitly computable.

1.3.3 Main results

Our main objective remains to find potentials for the differential operators of vector calculus and exterior calculus, including the curl and divergence in particular, over triangulated domains. The operator norms of these potentials will serve as our computable Poincaré–Friedrichs constants. In light of the different approaches discussed above, we seek constants that be explicitly computed in terms of mesh geometry, that do not require the solution of global finite element problems, and that do not depend on the boundary regularity of the domain.

The sequential construction of potentials for gradient vector fields, as discussed earlier, is well-established in the literature and serves as our conceptual blueprint. Gradient potentials are easily computed over each individual simplex, but the constants of integration generally do not match: the scalar piecewise potential will belong to a broken Sobolev space. In particular, the local potentials will differ only by a constant along the simplex boundaries. However, we can produce a potential in Sobolev spaces over increasingly larger intermediate subdomains, which we construct sequentially. At each step, we select a simplex that shares a face with one of the previously processed simplices and adjust the constant of integration of the local potential. The global potential is built cell by cell until the entire domain is covered. Our main result in this context is Theorem 4.4.

We generalize this sequential construction of potentials to the curl, the divergence, and more generally, the exterior derivative. However, we need to overcome new challenges that arise due to the infinite-dimensional kernels of these differential operators, as we now explain in more detail. The basic inductive strategy remains the same. We start by constructing, say, a curl potential over a single simplex. Having already defined a potential operator over a subtriangulation, we select a neighboring simplex whose intersection with the preceding simplices includes at least one common face. We then construct a potential for the curl operator whose tangential traces along the intersection match those of the already existing potential. Repeating this procedure eventually exhausts the original triangulation. Here, Theorems 9.3 is our main result.

However, unlike in the potential construction of the gradient, it is not immediately evident whether the construction of the local curl potential with given tangential traces is a well-posed auxiliary problem. For the case of the gradient, it is sufficient that the sequential traversal of the triangulation satisfies that each new simplex shares at least one face with one of the previous simplices. For the differential operators of vector calculus and exterior calculus, we choose to be more restrictive: we require that the new simplex intersects with the existing subtriangulation along an $(n-1)$-dimensional boundary submanifold. This allows us to define a well-posed auxiliary problem and to extend the existing curl potential to the new simplex. Whether a triangulation admits such a particular traversal is a non-trivial condition and defines the class of *shellable* triangulations.

Shellable simplicial complexes, and more generally polytopal complexes, are a well-established notion in discrete geometry and combinatorics, see, e.g., Kozlov [40] and Ziegler [66], and the references therein. Any shellable simplicial complex must necessarily triangulate a contractible space. With respect to our main interest, local patches (stars) in 2D and 3D triangulations are shellable.

1.4 Notation

Whenever $x \in \mathbb{R}^n$ is a vector, we write $\|x\| = \|x\|_2$ for its Euclidean norm, and whenever $A \in \mathbb{R}^{n\times n}$, we let $\|A\|_2$ be its operator norm with respect to the Euclidean norm. Furthermore, $\mathbf{J}F$ always denotes the Jacobian of any mapping F.

1.5 Organization of this manuscript

The remainder of this manuscript is structured as follows. We review Poincaré–Friedrichs constants for the gradient over convex domains in Section 2. We also discuss there the difference with the Poincaré inequality and motivate our interest in linear potentials. We review basic notions of triangulations in Section 3. We develop computable upper bounds for the Poincaré–Friedrichs constants for the gradient over face-connected triangulated domains in Section 4. We recap Sobolev spaces in vector calculus and the calculus of differential forms in Section 5. Our regularized potentials over convex sets are then introduced in Section 6, giving rise to computable Poincaré–Friedrichs constants as their operator norms. We subsequently review shellable triangulations of manifolds in Section 7, and we construct an important geometric reflection operator in Section 8. We finally provide computable upper bounds for Poincaré–Friedrichs constants for the exterior derivative over shellable triangulations in Section 9. We present numerical examples in Section 10 and conclude by some outlook in Section 11.

2 Review of Poincaré and Poincaré–Friedrichs inequalities

This section surveys variations of the Poincaré–Friedrichs inequalities for the gradient operator, with emphasis on analytical upper bounds over bounded convex domains. We explain the difference between the Poincaré–Friedrichs inequality, which addresses the norm-minimizing potential, and the Poincaré inequality, which addresses the potential with mean value zero, and rephrase this in terms of potentials. This survey serves as a building block in constructing computable constants over triangulated domains in a combinatorial way below, but we believe it is also of independent interest.

Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set. Given any $p \in [1, \infty]$, we let $L^p(\Omega)$ denote the Lebesgue space over Ω with integrability exponent p, and we write $\boldsymbol{L}^p(\Omega) := L^p(\Omega)^n$ for the corresponding Lebesgue space of vector fields. We also write $W^{1,p}(\Omega)$ for the first-order Sobolev space over Ω with integrability exponent p.

We say that a domain $\Omega \subseteq \mathbb{R}^n$ satisfies the *Poincaré–Friedrichs inequality* with exponent $p \in [1, \infty]$ if there exists a constant $C_{\mathrm{grad},\Omega,p} \geq 0$ such that the following holds: for every vector field $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$ there exists $u \in W^{1,p}(\Omega)$ such that $\nabla u = \boldsymbol{f}$ and

$$\|u\|_{L^p(\Omega)} \leq C_{\mathrm{grad},\Omega,p} \|\boldsymbol{f}\|_{L^p(\Omega)}. \tag{13}$$

Since Ω is connected, this is equivalent to

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^p(\Omega)} \leq C_{\mathrm{grad},\Omega,p} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

We call $C_{\mathrm{grad},\Omega,p}$ the *Poincaré–Friedrichs constant* with exponent p.

2.1 Relationship with Poincaré inequalities

We wish to clarify the relationship between the Poincaré–Friedrichs inequality, in the sense introduced above, with other inequalities that are known as Poincaré inequality (or also Poincaré–Wirtinger or Friedrichs inequality) in the literature [21, Remark 3.32]. Given $p \in [1, \infty]$ and a domain $\Omega \subseteq \mathbb{R}^n$ of finite measure, we say that Ω satisfies the Poincaré inequality with exponent p if there exists $C_{\varnothing,\Omega,p} \geq 0$ such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C_{\varnothing,\Omega,p} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega),$$

where u_Ω is the *average* of u over Ω, that is,

$$u_\Omega := \mathrm{vol}(\Omega)^{-1} \int_\Omega u(x)\, dx.$$

Clearly, this Poincaré inequality implies the Poincaré–Friedrichs inequality and we have

$$C_{\mathrm{grad},\Omega,p} \leq C_{\varnothing,\Omega,p}.$$

Towards a converse inequality, let us first observe that the average of any $u \in W^{1,p}(\Omega)$ with $p < \infty$ satisfies the bound

$$\|u_\Omega\|^p_{L^p(\Omega)} = \int_\Omega \left(\mathrm{vol}(\Omega)^{-1} \int_\Omega |u(x)|\, dx \right)^p \leq \int_\Omega \mathrm{vol}(\Omega)^{-1} \int_\Omega |u(x)|^p\, dx = \|u\|^p_{L^p(\Omega)}. \tag{14}$$

Here, we have used Hölder's or Jensen's inequality. In the case $p = \infty$, any $u \in L^\infty(\Omega)$ satisfies $\|u_\Omega\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$. We conclude that taking the average is a projection within Lebesgue spaces with unit norm. The triangle inequality now shows that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq 2\|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega).$$

Thus, the Poincaré–Friedrichs inequality implies the Poincaré inequality with

$$C_{\mathrm{grad},\Omega,p} \leq 2C_{\varnothing,\Omega,p}. \tag{15}$$

In the special case $p = 2$, taking the average is an orthogonal projection, and so this improves to $\|u - u_\Omega\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ for any $u \in L^2(\Omega)$. Hence,

$$C_{\varnothing,\Omega,2} = C_{\mathrm{grad},\Omega,2}. \tag{16}$$

This improvement also follows from the projection estimate (see, e.g., [65]). Stern's generalized projection estimate [60, Theorem 4.1,Remark 5.1] implies improved estimate for all L^p spaces with $1 \leq p \leq \infty$:

since taking the average is a projection onto the constants functions with unit norm, from (14) it now follows that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq \min\left(2, 2^{|2/p-1|}\right) \|u\|_{L^p(\Omega)} = 2^{|2/p-1|} \|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega).$$

Here, we have used $1 \leq 2^{|2/p-1|} \leq 2$ for $1 \leq p \leq \infty$. We thus conclude

$$C_{\mathrm{grad},\Omega,p} \leq 2^{|2/p-1|} C_{\varnothing,\Omega,p}. \tag{17}$$

In the limit cases $p = 1$ and $p = \infty$ we reproduce (15), and in the case $p = 2$ we achieve the identity (16) once again. In summary, our notion of Poincaré–Friedrichs constant is equivalent to the common notion of Poincaré constant, up to a numerical factor that only depends on $1 \leq p \leq \infty$ and that is at most 2.

Remark 2.1. *Let us further remark why the above notion of Poincaré–Friedrichs inequality (13) suits our discussion better than the Poincaré inequality. We want to generalize the discussion to the curl and divergence operators. The kernel of the gradient is the one-dimensional space of constant functions, and is thus complemented in the Lebesgue spaces with a canonical choice of projection. By contrast, the curl and divergence operators have infinite-dimensional kernels. Hence, it is not even trivial whether these kernels are complemented subspaces and admit a projection onto them, not to mention a canonical projection.*

2.2 Relationship with potentials

There is yet another characterization of the Poincaré–Friedrichs inequality that we like to point out. We define the potential

$$\Phi(\boldsymbol{f}) := \operatorname*{argmin}_{\substack{u \in W^{1,p}(\Omega) \\ \nabla u = \boldsymbol{f}}} \|u\|_{L^p(\Omega)}, \quad \forall \boldsymbol{f} \in \nabla W^{1,p}(\Omega).$$

If there is a Poincaré–Friedrichs constant, then by definition

$$\|\Phi(\boldsymbol{f})\|_{L^p(\Omega)} \leq C_{\mathrm{grad},\Omega,p} \|\boldsymbol{f}\|_{L^p(\Omega)},$$

and this inequality is sharp by definition of Φ provided that $C_{\mathrm{grad},\Omega,p}$ is the smallest possible constant in (13). Any Poincaré–Friedrichs inequality (13) is comparable to an upper bound for the generalized (possibly nonlinear) inverse of the gradient operator $\nabla : W^{1,p}(\Omega) \to \boldsymbol{L}^p(\Omega)$.

Because $\Phi : \nabla W^{1,p}(\Omega) \to L^p(\Omega)$ is generally a nonlinear operator for $p \neq 2$, any *linear* potential operator $\overline{\Phi} : \nabla W^{1,p}(\Omega) \to L^p(\Omega)$ satisfying $\nabla \overline{\Phi}(\boldsymbol{f}) = \boldsymbol{f}$ for any $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$ must have an operator norm that obeys the lower bound

$$\max_{u \in W^{1,p}(\Omega) \setminus \mathbb{R}} \frac{\|\Phi(\nabla u)\|_{L^p(\Omega)}}{\|\nabla u\|_{L^p(\Omega)}} \leq \max_{u \in W^{1,p}(\Omega) \setminus \mathbb{R}} \frac{\|\overline{\Phi}(\nabla u)\|_{L^p(\Omega)}}{\|\nabla u\|_{L^p(\Omega)}}.$$

A natural choice is the linear operator $\Phi_\varnothing : \nabla W^{1,p}(\Omega) \to L^p(\Omega)$ that satisfies

$$\Phi_\varnothing(\nabla u) = u - u_\Omega, \quad \forall u \in W^{1,p}(\Omega).$$

Its operator norm is just the optimal Poincaré constant $C_{\varnothing,\Omega,p}$.

Remark 2.2. *Upper bounds for Poincaré–Friedrichs constants (13) are easily obtained from linear potentials. We highlight this perspective because it seems to be most promising when generalizing the discussion to the curl and divergence operators.*

2.3 Analytical constants in Poincaré–Friedrichs inequalities over bounded convex domains

We collect examples for Poincaré and Poincaré–Friedrichs inequalities for the important special case of bounded convex domains. We have the Poincaré inequalities [55, 8, 1] (or [21, Lemma 3.24])

$$\|u - u_\Omega\|_{L^1(\Omega)} \leq \frac{\delta(\Omega)}{2} \|\nabla u\|_{L^1(\Omega)}, \quad \forall u \in W^{1,1}(\Omega), \tag{18}$$

$$\|u - u_\Omega\|_{L^2(\Omega)} \leq \frac{\delta(\Omega)}{\pi} \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in W^{1,2}(\Omega), \tag{19}$$

where $\delta(\Omega)$ is the diameter of the domain Ω. These two estimates are the best possible Poincaré inequalities in the cases $p = 1$ and $p = 2$, respectively, in terms of the diameter alone. Upper bounds for the Poincaré constant over convex domains with $1 < p < \infty$ are known in the literature [18, Theorem 1.1, Theorem 1.2]:

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C_{\mathrm{CW},p}\delta(\Omega)\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \tag{20}$$

where we use an upper bound by Chua and Wheeden:

$$C_{\mathrm{CW},p} := \sup_{v \in C^\infty([0,1])\setminus\mathbb{R}} \frac{\|v - v_{[0,1]}\|_{L^p([0,1])}}{\|\nabla v\|_{L^p([0,1])}} \leq \sqrt[p]{p}2^{1-\frac{1}{p}} = 2\left(\frac{p}{2}\right)^{\frac{1}{p}}.$$

Note that (20) is generally not optimal among the upper bounds that only depend on the domain diameter and the Lebesgue exponent. As discussed above, these Poincaré inequalities imply Poincaré–Friedrichs inequalities.

We know optimal Poincaré–Friedrichs constants over convex domains ([26, Theorem 1.1], [23, Theorem 1.1]): when $1 < p < \infty$, one can show that

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^p(\Omega)} \leq C_{\mathrm{EFNT},p}\delta(\Omega)\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \tag{21}$$

where $C_{\mathrm{EFNT},p}$ is the best possible constant that only depends on p and equals

$$C_{\mathrm{EFNT},p} := \frac{p\sin(\pi/p)}{2\pi\sqrt[p]{p-1}}.$$

Note that the last inequalities from (17) imply, again when $1 < p < \infty$, the Poincaré inequalities

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq 2^{|1-\frac{2}{p}|}C_{\mathrm{EFNT},p}\delta(\Omega)\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{22}$$

When $p = 1$, then the optimal Poincaré constant also bounds the Poincaré–Friedrichs constant:

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^1(\Omega)} \leq \frac{\delta(\Omega)}{2}\|\nabla u\|_{L^1(\Omega)}, \quad \forall u \in W^{1,1}(\Omega). \tag{23}$$

When $p = \infty$, since convex domains are Lipschitz domains, Rademacher's theorem leads to

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^\infty(\Omega)} \leq \delta(\Omega)\|\nabla u\|_{L^\infty(\Omega)}, \quad \forall u \in W^{1,\infty}(\Omega). \tag{24}$$

Remark 2.3. *Any estimate for the Poincaré–Friedrichs constant implies an estimate for the Poincaré constant, via (17). Let us compare $C_{\mathrm{CW},p}$ for the Poincaré inequality with $C_{\mathrm{EFNT},p}$ for the Poincaré–Friedrichs inequality. In the case $2 \leq p$,*

$$2^{1-\frac{2}{p}}C_{\mathrm{EFNT},p} = \frac{2^{1-\frac{2}{p}}}{2}\frac{\sin(\pi/p)}{\pi/p}\frac{1}{\sqrt[p]{p-1}} \leq 4^{-\frac{1}{p}} \leq C_{\mathrm{CW},p}.$$

In the case $2 \geq p$,

$$\begin{aligned} 2^{\frac{2}{p}-1}C_{\mathrm{EFNT},p} = \frac{2^{\frac{2}{p}-1}}{2}\frac{\sin(\pi/p)}{\pi/p}\frac{1}{\sqrt[p]{p-1}} \leq \frac{2^{\frac{2}{p}-1}}{2} &= 2^{\frac{2}{p}-2} \\ &= 4^{\frac{1}{p}-1} \leq C_{\mathrm{CW},p}. \end{aligned}$$

It follows that (22) is generally a tighter estimate than (20) for $1 < p < \infty$.

Remark 2.4. *The above Poincaré and Poincaré–Friedrichs constants are optimal for the class of convex domains, but individual convex domains may allow for better constants. We refer to [50, 15, 52] for discussions; for example, triangles allow the reduction of the constant by 20%.*

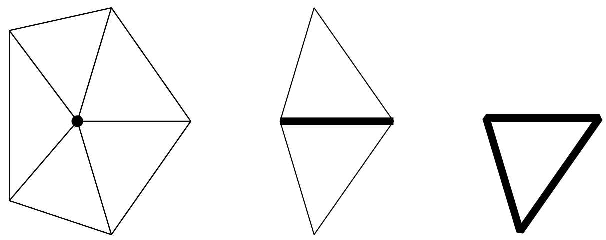

Figure 1: From left to right: local patches around a vertex, and edge, and a triangle. The local patch of any full-dimensional simplex only consists of that simplex itself (and its subsimplices).

3 Basic notions of triangulations

We gather here basic notions and definitions concerning simplicial meshes.

A k-dimensional *simplex* T is the convex hull of $k+1$ affinely independent points $v_0, v_1, \dots, v_k \in \mathbb{R}^n$. We call these points the *vertices* of the simplex T. The strictly positive convex combinations of the vertices of the simplex constitute the *interior* of the simplex, and its remaining points constitute the *boundary* of the simplex. If S is a simplex whose vertices are also vertices of another simplex T, in which case $S \subseteq T$, then we call S a *subsimplex* of T and call T a *supersimplex* of S.

A finite family of simplices $\mathcal{T}$ is a *simplicial complex* or *triangulation* if it satisfies the following conditions: (i) $\mathcal{T}$ contains all the subsimplices of its members (ii) any non-empty intersection of two members of $\mathcal{T}$ is a common subsimplex of each of them. We say that a simplicial complex $\mathcal{T}$ has *dimension* n or is n*-dimensional* if each of its simplices is a subset of an n-dimensional member of that triangulation.[2] We also write $|\mathcal{T}|$ for the *underlying set* of the simplicial complex $\mathcal{T}$, which is the union $|\mathcal{T}| = \bigcup \mathcal{T}$ of all simplices in $\mathcal{T}$. One calls any set triangulable if it is the underlying set of some triangulation.

Given any simplex T, we write $\mathcal{S}^{\downarrow}(T)$ for the simplicial complex that contains all subsimplices of T, and $\mathcal{S}^{\downarrow}_k(T) \subseteq \mathcal{S}^{\downarrow}(T)$ denotes the set of k-dimensional subsimplices of T. We write $\mathcal{V}(T) := \mathcal{S}^{\downarrow}_0(T)$ for the set of vertices of T. Whenever $\mathcal{T}$ is a simplicial complex, the set of k-dimensional simplices in $\mathcal{T}$ is denoted as $\mathcal{S}^{\downarrow}_k(\mathcal{T})$. Similarly, the notations $\mathcal{V}(\mathcal{T}) := \mathcal{S}^{\downarrow}_0(\mathcal{T})$ and $\mathcal{F}(\mathcal{T}) := \mathcal{S}^{\downarrow}_{n-1}(\mathcal{T})$ refer to the vertices and the faces (that is, members with codimension one) of this triangulation.[3] In practice, we do not always distinguish between points and singleton simplices.

When $\mathcal{T}$ is a triangulation and $T \in \mathcal{T}$, then $\operatorname{st}_{\mathcal{T}}(T)$ denotes the *local patch* or *local star* of T, which is the simplicial subcomplex of $\mathcal{T}$ that contains all supersimplices of T and their subsimplices. We write $\partial\operatorname{st}_{\mathcal{T}}(T)$ for the subset of the local patch whose members do not contain T itself. Formally,

$$\operatorname{st}_{\mathcal{T}}(T) := \bigcup_{\substack{T' \in \mathcal{S}^{\downarrow}_n(\mathcal{T}) \\ T \subseteq T'}} \mathcal{S}^{\downarrow}(T'), \qquad \partial\operatorname{st}_{\mathcal{T}}(T) := \bigcup_{\substack{T' \in \operatorname{st}_{\mathcal{T}}(T) \\ T \not\subseteq T'}} \mathcal{S}^{\downarrow}(T').$$

We also write $A_T := |\operatorname{st}_{\mathcal{T}}(T)|$ for the closed underlying set of the local patch. A crucial structural observation is the following.

Lemma 3.1. *Let $\mathcal{T}$ be an n-dimensional simplicial complex and let $S, S' \in \mathcal{T}$. Then either $\operatorname{st}_{\mathcal{T}}(S)$ and $\operatorname{st}_{\mathcal{T}}(S')$ only intersect in simplices of dimension at most $n-1$ or there exists $S'' \in \mathcal{T}$ such that*

$$\operatorname{st}_{\mathcal{T}}(S) \cap \operatorname{st}_{\mathcal{T}}(S') = \operatorname{st}_{\mathcal{T}}(S''), \qquad \mathcal{V}(S) \cup \mathcal{V}(S') = \mathcal{V}(S'').$$

Proof. Let $T \in \mathcal{T}$ be n-dimensional. We have $T \in \operatorname{st}_{\mathcal{T}}(S)$ if and only if all vertices of S are vertices of T. We have $T \in \operatorname{st}_{\mathcal{T}}(S')$ if and only if all vertices of S' are vertices of T. Consequently, $T \in \operatorname{st}_{\mathcal{T}}(S) \cap \operatorname{st}_{\mathcal{T}}(S')$ if and only if $T \in \operatorname{st}_{\mathcal{T}}(S'')$, where $S'' \in \mathcal{T}$ is the convex closure of S and S'. $\square$

[2] Simplicial complexes that we call n-dimensional are called purely n-dimensional in the literature on polytopes (cf. [66]) and simply "simplicial meshes" in the finite element literature.

[3] Our use of the term *face* as is common in classical geometry and the finite element literature [12] and is synonymous with *facet* as used in the literature on polyhedral combinatorics [57]. Notably, this terminology differs from the uses *face* and *facet* in the theory of polyhedra [66].

We introduce a specific notion of connectivity when we are given an n-dimensional simplicial complex $\mathcal{T}$. We call two n-dimensional simplices $S, S' \in \mathcal{T}$ *face-neighboring* if $S \cap S'$ is a common face of both of them. We call n-simplices $S, S' \in \mathcal{T}$ *face-connected in* $\mathcal{T}$ if there exists a sequence $S = S_0, S_1, \dots, S_m = S' \in \mathcal{T}$ such that S_i and S_{i-1} are face-neighboring for all $1 \leq i \leq m$. Such a sequence is called a *face path* from S to S' in $\mathcal{T}$. Clearly, face-connected in $\mathcal{T}$ is an equivalence relation among simplices. A *face-connected component* of $\mathcal{T}$ is an equivalence class under this equivalence relation, and we call $\mathcal{T}$ *face-connected* if it has only one face-connected component.

3.1 Shape measures and related quantities

We introduce several quantities that measure the regularity of a triangulation. These have in common that they can be computed from purely local information.

We write $\delta(T)$ and $\operatorname{vol}(T)$ for the diameter and n-dimensional volume of any n-simplex T. Moreover, $h(T)$ refers to the smallest height of any vertex of the simplex T, where the height of a vertex is defined as the distance to the affine span of its opposing face. For the purpose of the usual scaling arguments, the *n-dimensional reference simplex* $\Delta^n \subseteq \mathbb{R}^n$ is the convex closure of the origin and the n canonical unit vectors.

Whenever T is any n-dimensional simplex T, we define the *aspect shape measure* $\kappa_{\mathrm{A}}(T)$, and the *algebraic shape measure* $\kappa_{\mathrm{M}}(T)$ by

$$\kappa_{\mathrm{A}}(T) := \frac{\delta(T)}{h(T)}, \qquad \kappa_{\mathrm{M}}(T) := \sup_{\varphi : \Delta^n \to T} \|\mathbf{J}\varphi\|_2 \|\mathbf{J}\varphi^{-1}\|_2, \tag{25}$$

where the last supremum is taken over all affine transformation from the reference n-simplex onto the n-simplex T. When $\mathcal{T}$ is an n-dimensional simplicial complex, we naturally define

$$\kappa_{\mathrm{A}}(\mathcal{T}) := \sup_{T \in \mathcal{S}_n^{\downarrow}(\mathcal{T})} \kappa_{\mathrm{A}}(T), \quad \kappa_{\mathrm{M}}(\mathcal{T}) := \sup_{T \in \mathcal{S}_n^{\downarrow}(\mathcal{T})} \kappa_{\mathrm{M}}(T). \tag{26}$$

We call these the aspect and algebraic shape measure, respectively, of the triangulation.

Remark 3.2. *The ratio $\kappa_{\mathrm{A}}(T)$ measures the "shape quality" of an n-dimensional simplex T and is an instance of a so-called* shape measure. *For example, the reference triangle has aspect shape measure* 2 *and the reference tetrahedron has aspect shape measure $\sqrt{6}$. Numerous alternative shape measures have been used throughout the literature of numerical analysis and computational geometry to quantify the quality of simplices (see [10, p.61, Definition 5.1], [12, p.97, Definition (4.2.16)], [21, Definition 11.2]).*

We gather a few relationships between geometric and algebraic entities and compare the different shape measures of a single simplex.

Lemma 3.3. *Let T be an n-simplex and let $\varphi : \Delta^n \to T$ be an affine diffeomorphism from the reference n-simplex. Then*

$$\|\mathbf{J}\varphi\|_2 \leq C_{1,n} \cdot \delta(T), \qquad \|\mathbf{J}\varphi^{-1}\|_2 \leq C_{2,n} \cdot \kappa_{\mathrm{A}}(T) h(T)^{-1},$$
$$\frac{1}{\sqrt{2n}} \kappa_{\mathrm{A}}(T) \leq \kappa_{\mathrm{M}}(T) \leq n \kappa_{\mathrm{A}}(T).$$

Here, $C_{1,n} = C_{2,n} = \sqrt{n}$.

Proof. Let $\varphi : \Delta^n \to T$ be an affine transformation. We abbreviate $M := \mathbf{J}\phi$ for its Jacobian. We begin with observing that the largest ℓ^2-norm of any column of M, which here denote by $c_{\max}(M)$, equals the maximum of the quotient $\|Mx\|_{\ell^2} / \|x\|_{\ell^1}$ over all non-zero $x \in \mathbb{R}^n$. The diameter of T is the length of its longest edge. Our first pair of inequalities follows via standard comparisons of Euclidean norms:

$$\frac{\delta(T)}{\sqrt{2}} \leq \|M\|_2 \leq \sqrt{n} \cdot c_{\max}(M) \leq \sqrt{n} \cdot \delta(T).$$

The columns of the matrix M^{-1} are the gradients of the barycentric coordinates of the vertices of T, except for the vertex $\varphi(0) \in T$. It immediately follows that

$$\|M^{-1}\|_2 \leq \sqrt{n} c_{\max}(M^{-1}) \leq \sqrt{n} \cdot h(T)^{-1}.$$

The smallest height in the reference simplex Δ^n is $h_\Delta = 1/\sqrt{n}$, whence $h(T) \geq \|M^{-1}\|_2^{-1}/\sqrt{n}$. This yields our second pair of inequalities. Notice that $h(T)^{-1} \leq \kappa_{\mathrm{A}}(T)\delta(T)^{-1}$. All relevant results follow. $\square$

We will need the maximal ratio of volumes between face-neighboring n-simplices, written $C_\rho(\mathcal{T})$, and the ratio of the diameters of any intersecting simplices, written $C_\theta(\mathcal{T})$. Formally,

$$C_\rho(\mathcal{T}) := \sup_{\substack{T,T' \in \mathcal{S}_n^{\downarrow}(\mathcal{T}) \\ T \cap T' \in \mathcal{S}_{n-1}^{\downarrow}(\mathcal{T})}} \frac{\mathrm{vol}(T)}{\mathrm{vol}(T')}, \tag{27}$$

$$C_\theta(\mathcal{T}) := \sup_{\substack{T,T' \in \mathcal{S}_n^{\downarrow}(\mathcal{T}) \\ T \cap T' \neq \emptyset}} \frac{\delta(T)}{\delta(T')}. \tag{28}$$

Finally, whenever T, T' are two n-simplices that share a common face F of codimension 1, we let $\Xi_{T,T'} : T \to T'$ denote the affine diffeomorphism that preserves F. We then define

$$C_\xi(\mathcal{T}) := \sup_{\substack{T,T' \in \mathcal{S}_n^{\downarrow}(\mathcal{T}) \\ T \cap T' \in \mathcal{S}_{n-1}^{\downarrow}(\mathcal{T})}} \|\mathbf{J}\Xi_{T,T'}\|_2 \tag{29}$$

to be the maximum of the operator norm of the Jacobian of any such diffeomorphism. This indicator quantifies how much reflection across the shared face distorts the geometry.

Lemma 3.4. *Let T_1 and T_2 be two n-simplices that share a common face F. Then*

$$\delta(T_1) \leq \kappa_{\mathrm{A}}(T_1)\delta(F) \leq \kappa_{\mathrm{A}}(T_1)\delta(T_2), \qquad \frac{\mathrm{vol}(T_1)}{\mathrm{vol}(T_2)} \leq \frac{\delta(T_1)}{\delta(T_2)}\kappa_{\mathrm{A}}(T_2) \leq \kappa_{\mathrm{A}}(T_1)\kappa_{\mathrm{A}}(T_2).$$

If $\Xi : T_1 \to T_2$ is the affine diffeomorphism that is the identity over F, then at least $n-2$ of its Jacobian's singular values equal 1, and we have

$$\begin{aligned}
\|\mathbf{J}\Xi\|_2 &\leq \frac{1}{2}\sqrt{\left(\frac{\delta(T_2)}{\delta(T_1)}\kappa_{\mathrm{A}}(T_1)+1\right)^2 + \kappa_{\mathrm{A}}(T_1)^2} + \frac{1}{2}\sqrt{\left(\frac{\delta(T_2)}{\delta(T_1)}\kappa_{\mathrm{A}}(T_1)-1\right)^2 + \kappa_{\mathrm{A}}(T_1)^2},\\
\|\mathbf{J}\Xi^{-1}\|_2 &\leq \frac{1}{2}\sqrt{\left(\frac{\delta(T_1)}{\delta(T_2)}\kappa_{\mathrm{A}}(T_2)+1\right)^2 + \kappa_{\mathrm{A}}(T_2)^2} + \frac{1}{2}\sqrt{\left(\frac{\delta(T_1)}{\delta(T_2)}\kappa_{\mathrm{A}}(T_2)-1\right)^2 + \kappa_{\mathrm{A}}(T_2)^2},\\
\det(\mathbf{J}\Xi) &= \frac{\mathrm{vol}(T_2)}{\mathrm{vol}(T_1)} \leq \frac{\delta(T_2)}{\delta(T_1)}\kappa_{\mathrm{A}}(T_1).
\end{aligned}$$

Proof. The diameter of F is at least as large as the height h_S of some other vertex of F in T_1. Now,

$$\delta(T_1)\kappa_{\mathrm{A}}(T_1)^{-1} \leq h(T_1) \leq h_S \leq \delta(F).$$

The first estimate follows. As for the second estimate, let h_1 and h_2 be the heights of F in the simplices T_1 and T_2, respectively. By the volume formula for simplices, $\mathrm{vol}(T_1) = h_1 \,\mathrm{vol}(F)/n$ and $\mathrm{vol}(T_2) = h_2 \,\mathrm{vol}(F)/n$, and thus $\mathrm{vol}(T_1)/\mathrm{vol}(T_2) = h_1/h_2$. Thus follows the second estimate:

$$\frac{\mathrm{vol}(T_1)}{\mathrm{vol}(T_2)} = \frac{h_1}{h_2} \leq \frac{\delta(T_1)}{h_2} \leq \kappa_{\mathrm{A}}(T_1)\frac{\delta(F)}{h_2} \leq \kappa_{\mathrm{A}}(T_1)\kappa_{\mathrm{A}}(T_2), \qquad \frac{\delta(T_1)}{h_2} \leq \frac{\delta(T_1)}{\delta(T_2)}\kappa_{\mathrm{A}}(T_2).$$

Lastly, let $\Xi : T_1 \to T_2$ be as stated. To estimate the Lipschitz constant of Ξ, we study the singular values of its Jacobian. Without loss of generality, F lies in the span of the first $n-1$ coordinates and contains the origin. Write $z_1 \in T_1$ and $z_2 \in T_2$ for the two vertices not contained in F. There exists a unit height vector $\hat{h}_0$ that goes in the direction of the vertex z_1. Without loss of generality, $\hat{h}_0$ is in the n-th coordinate direction. Suppose that $b_1, b_2 \in F$ such that $z_1 = b_1 + h_1\hat{h}_0$ and $x_2 = b_2 + h_2\hat{h}_0$. The mapping Ξ is linear, being the identity over F and mapping z_1 to z_2. Hence,

$$\Xi(\hat{h}_0) = h_1^{-1}(b_2 - b_1) + h_2 h_1^{-1}\hat{h}_0.$$

We see that Ξ equals the identity over the orthogonal complement of the span of $\hat{h}_0$ and $b_2 - b_1$. The only singular values of its Jacobian are the two singular values $\sigma_- \leq 1 \leq \sigma_+$ of the matrix

$$\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}, \qquad a = h_2/h_1, \qquad c = \|b_2 - b_1\|/h_1.$$

These are

$$\begin{aligned}
\sigma_\pm &= \frac{1}{\sqrt{2}} \sqrt{1 + a^2 + c^2 \pm \sqrt{\left(1 + a^2 + c^2\right)^2 - 4a^2}} \\
&= \frac{1}{\sqrt{2}} \sqrt{1 + a^2 + c^2 \pm \sqrt{\left((1+a)^2 + c^2\right)\left((1-a)^2 + c^2\right)}} \\
&= \frac{1}{2} \left(\sqrt{(1+a)^2 + c^2} \pm \sqrt{(1-a)^2 + c^2} \right).
\end{aligned}$$

We remark that this provides the upper bound

$$\sigma_+ \leq \frac{1}{2} \sqrt{\left(\frac{\delta(T_2)}{\delta(T_1)} \kappa_{\mathrm{A}}(T_1) + 1 \right)^2 + \kappa_{\mathrm{A}}(T_1)^2} + \frac{1}{2} \sqrt{\left(\frac{\delta(T_2)}{\delta(T_1)} \kappa_{\mathrm{A}}(T_1) - 1 \right)^2 + \kappa_{\mathrm{A}}(T_1)^2}.$$

The desired estimates are shown. $\square$

Remark 3.5. *While we will utilize $C_\theta(\mathcal{T})$ at numerous places throughout the manuscript, $C_\rho(\mathcal{T})$ and $C_\xi(\mathcal{T})$ will only be used throughout Section 4, the following section. Lemma 3.4 obviously shows that $C_\rho(\mathcal{T})$ of (27) is controlled by the shape measure. In a face-connected triangulation where we have an upper bound for the number of simplices sharing a vertex, this Lemma also allows, at least in principle, control of $C_\theta(\mathcal{T})$ of (28).*

4 Poincaré–Friedrichs inequalities over triangulated domains

In this section, we develop stepwise computable estimates for Poincaré–Friedrichs constants of triangulated domains. The following very classical procedure serves us as an inspiration: given a gradient vector field, we can reconstruct the scalar potential up to a constant by fixing a starting point and integrating the gradient vector field along lines emanating from that starting point. We perform a discrete analogue of this procedure over triangulated domains: having fixed a starting triangle, we traverse the triangulation along face-neighboring simplices. We always construct a gradient potential over the new simplex and fix the constant of integration using the value already known on the connecting face, thereby constructing a scalar potential over larger and larger subdomains. This basic idea has appeared in various forms before, for instance recently in [11, 22, 16, 63].

We begin with an auxiliary result with independent relevance, where we estimate the Poincaré–Friedrichs inequality when homogeneous boundary conditions are imposed along a single face of the boundary.

Lemma 4.1. *Let T be an n-simplex with a face F and $p \in [1, \infty]$. If $u \in W^{1,p}(T)$ with $\operatorname{tr}_F u = 0$, then*

$$\|u\|_{L^p(T)} \leq C_{\mathrm{PF},T,F,p} \|\nabla u\|_{L^p(T)},$$

where $C_{\mathrm{PF},T,F,p} = p^{-\frac{1}{p}} \delta(T)$ for $p < \infty$ and $C_{\mathrm{PF},T,F,\infty} = \delta(T)$.

Proof. Since the inequality follows from Rademacher's theorem in the limit case $p = \infty$, we assume $1 \leq p < \infty$. Let $u \in C^\infty(T)$ have support disjoint from F. We tacitly extend this by zero to a function $u \in L^\infty(\mathbb{R}^n)$. Without loss of generality, the segment from the midpoint of F to the opposing vertex lies on the first coordinate axis, and the minimal first coordinate among all the points of F equals 0. We write $\boldsymbol{g}$ for the trivial extension of ∇u over the entire $\mathbb{R}^n$. Using the fundamental theorem of calculus and Hölder's inequality,

$$
\begin{aligned}
\int_T |u(x)|^p \,\mathrm{d}x \, dx &\leq \int_{\mathbb{R}^{n-1}} \int_0^{\delta(T)} |u(x_1,\overline{x})|^p \, dx_1 \, d\overline{x} \\
&\leq \int_{\mathbb{R}^{n-1}} \int_0^{\delta(T)} \left| \int_0^{x_1} |\boldsymbol{g}(y,\overline{x})| \, dy \right|^p \, dx_1 \, d\overline{x} \\
&\leq \int_0^{\delta(T)} x_1^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^{x_1} |\boldsymbol{g}(y,\overline{x})|^p \, dy \, d\overline{x} \, dx_1 \\
&\leq \int_0^{\delta(T)} x_1^{p-1} \, dx_1 \cdot \int_T |\boldsymbol{g}(y,\overline{x})|^p \, dy \, d\overline{x} \leq \frac{\delta(T)^p}{p} \int_T |\nabla u(x)|^p \, dx.
\end{aligned}
$$

If $u \in W^{1,p}(T)$ has vanishing trace along F but is not necessarily smooth, then we conclude $\|u\|_{L^p(T)} \leq \delta(T) p^{-\frac{1}{p}} \|\nabla u\|_{L^p(T)}$ from approximation via members of $C^\infty(T)$ whose support is disjoint from F. We very briefly verify that density argument: There exists an affine diffeomorphism $\varphi : \Delta^n \to T$ from the reference simplex onto T that maps the convex closure of the n unit vectors onto the face F. We let $\hat{u} := u \circ \varphi$. Let $\hat{U}$ be the unit ball of the ℓ^1 metric, which contains Δ^n. We let $\tilde{u}$ be the extension of $\hat{u}$ onto $\hat{U}$ by reflection across the coordinate axes. Then $\tilde{u} \in W_0^{1,p}(\hat{U})$, and $\tilde{u}$ is the limit of a sequence $u_m \in C_c^\infty(\hat{U})$. Now $u_m \circ \varphi^{-1} \in C^\infty(\hat{U})$ approximates u within the Banach space $W^{1,p}(T)$ and has the desired support property. □

Remark 4.2. *We can improve Lemma 4.1 in the special case $p = 2$. The variational formulation of the Poincaré constant over a convex domain reveals that $C_{\mathrm{PF},T,F,p}$ lies between the the Poincaré constant without boundary conditions and with full boundary conditions. In particular,*

$$
C_{\mathrm{PF},T,F,2} \leq \frac{\delta(T)}{\pi}
$$

is an improved Poincaré inequality.

The next auxiliary result establishes Poincaré–Friedrichs constants over face patches within simplicial triangulations. We emphasize that face patches are not necessarily convex, but we can still extend the results on convex domains from Section 2.3. The following result is reasonably sharp when the two simplices have similar volumes and diameters.

Lemma 4.3. *Let $\mathcal{T}$ be a triangulation. Let $T_1, T_2 \in \mathcal{T}$ be two n-simplices whose intersection is a common face $F := T_1 \cap T_2$. Write $U := T_1 \cup T_2$. If $p \in [1,\infty]$ and $u \in W^{1,p}(\Omega)$, then*

$$
\min_{c \in \mathbb{R}} \|u - c\|_{L^p(U)} \leq C_{\mathrm{PF},T_1 \cup T_2,p} \|\nabla u\|_{L^p(U)}.
$$

Here, $C_{\mathrm{PF},T_1\cup T_2,p} = 2 C_{1,n} C_{\mathrm{EFNT},p} C_\rho(\mathcal{T})^{\frac{1}{p}} \max\left(\delta(T_1), \delta(T_2)\right)$.

Proof. Without loss of generality, F has the vertices $v_0, \dots, v_{n-1}$, and $z_1 \in T_1$ and $z_2 \in T_2$ are the remaining vertices of the two triangles. Let $\Delta_1 = \Delta^n$ be the reference n-simplex and let Δ_2 be obtained from it by flipping the n-th coordinate. We let $\varphi_1 : \Delta_1 \to T_1$ and $\varphi_2 : \Delta_2 \to T_2$ be affine transformations that map the origin to v_0, that map each unit vector e_i to v_i for $i = 1, \dots, n-1$, and that satisfy $\varphi_1(e_n) = z_1$ and $\varphi_2(-e_n) = z_2$. Write $\hat{U} := \Delta_1 \cup \Delta_2$. We have a bi-Lipschitz mapping $\varphi : \hat{U} \to U$.

Suppose that $u \in W^{1,p}(U)$. Then $\hat{u} := u \circ \varphi \in W^{1,p}(\hat{U})$. We observe

$$
\|\nabla \hat{u}\|_{L^p(\hat{U})} \leq \max\left(|\det(\mathbf{J}\varphi_1)|^{-\frac{1}{p}} \|\mathbf{J}\varphi_1\|_2, |\det(\mathbf{J}\varphi_2)|^{-\frac{1}{p}} \|\mathbf{J}\varphi_2\|_2 \right) \|\nabla u\|_{L^p(U)}.
$$

Notice that $\hat{U}$ has diameter 2 and is (crucially) convex. Thus, due to Poincaré–Friedrichs inequality (21), there exists $\hat{w} \in W^{1,p}(\hat{U})$ such that $\nabla \hat{w} = \nabla \hat{u}$ and

$$
\|\hat{w}\|_{L^p(\hat{U})} \leq 2 C_{\mathrm{EFNT},p} \|\nabla \hat{u}\|_{L^p(\hat{U})}.
$$

Next, setting $w := \hat{w} \circ \varphi^{-1}$, we find $\nabla w = \nabla u$ and

$$
\|w\|_{L^p(U)} \leq \max\left(|\det(\mathbf{J}\varphi_1)|, |\det(\mathbf{J}\varphi_2)|\right)^{\frac{1}{p}} \|\hat{w}\|_{L^p(\hat{U})}.
$$

For both $i = 1, 2$, we now recall the well-known equation $|\det(\mathbf{J}\varphi_i)| = n! \operatorname{vol}(T_i)$, and the estimate $\|\mathbf{J}\varphi_i\|_2 \leq C_{1,n} \delta(T_i)$, which is given in Lemma 3.3. We obtain the desired result. □

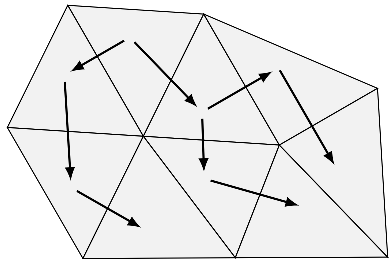

Figure 2: Face-connected triangulation of a domain. The arrows depict a spanning tree in the face-connection graph.

The main result of this section constructs a potential and gives an upper bound for the Poincaré–Friedrichs constant. It follows the same underlying principle as the "discrete mean Poincaré inequality" of [24, Lemma 3.7]. This procedure serves as the blueprint for constructing potentials of the curl and divergence operators in later sections.

Two different variations of the underlying idea are analyzed, yielding slightly different estimates. On the one hand, we can extend the scalar gradient potentials over each intermediate domain to another simplex by solving a local auxiliary problem on that new simplex, subject to partial boundary conditions. On the other hand, we can instead cover the domain with overlapping simplicial patches (such as face patches), over which local scalar potentials are easily found. Since these can only differ by constants at their overlaps, we assemble a global scalar potential piece by piece as we adjust the local constants of integration. Both estimates of Poincaré–Friedrichs constants capture the correct asymptotic behavior as p grows to infinity.

Theorem 4.4. *Let $\mathcal{T}$ be a face-connected n-dimensional finite triangulation. Suppose $1 \leq p, q \leq \infty$ with $1 = 1/p + 1/q$, and that the domain Ω is the interior of the underlying set of $\mathcal{T}$. Then for any $u \in W^{1,p}(\Omega)$ there exists $w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$ and satisfying the following estimates: there exists an n-simplex $T_0 \in \mathcal{T}$ with*

$$\|w\|_{L^p(T_0)} \leq C_{\mathrm{PF},T_0,p} \|\nabla u\|_{L^p(T_0)}.$$

For any n-simplex $T_M \in \mathcal{T}$ there exists a face path $T_0, T_1, \dots, T_M$ such that for all $1 \leq m \leq$ we have one of the following recursive estimates:

$$\|w\|_{L^p(T_m)} \leq A_m \|w\|_{L^p(T_{m-1})} + B'_m \|\nabla u\|_{L^p(T_m)} + B''_m \|\nabla u\|_{L^p(T_{m-1})},$$
$$\|w\|_{L^p(T_m)} \leq A_m \|w\|_{L^p(T_{m-1})} + B^\star_m \|\nabla u\|_{L^p(A_{F_m})},$$

where

$$A_m \leq C_\rho(\mathcal{T})^{\frac{1}{p}}, \quad B'_m \leq C_{\mathrm{PF},T_m,F_m,p}, \quad B''_m \leq C_{\mathrm{PF},T_m,F_m,p} C_\rho(\mathcal{T})^{\frac{1}{p}} C_\xi(\mathcal{T}),$$
$$B^\star_m \leq \left(1 + C_\rho(\mathcal{T})^{\frac{q}{p}}\right)^{\frac{1}{q}} C_{\mathrm{PF},A_{F_\ell},p}.$$

Here, for any $1 \leq m \leq M$, let $F_m = T_m \cap T_{m-1}$.

Proof. Let $u \in W^{1,p}(\Omega)$. We start with the Poincaré–Friedrichs inequality on the first simplex T_0. There exists $w_0 \in W^{1,p}(T_0)$ satisfying $\nabla w_0 = \nabla u$ over T_0 together with

$$\|w_0\|_{L^p(T_0)} \leq C_{\mathrm{PF},T_0,p} \|\nabla u\|_{L^p(T_0)}.$$

In particular, $c_0 := w_0 - u$ is a constant function. We then define

$$w := u + c_0.$$

Clearly, $w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$. By construction $w_{|T_0} = w_0$. We verify that w can be chosen such that it satisfies the desired recursive estimates. Suppose that $T_0, T_1, \dots, T_M$ is a face path in $\mathcal{T}$ and that

$1 \leq m \leq M$. Recall that we write $F_m := T_m \cap T_{m-1}$, which is a face of dimension $n-1$ shared by the n-simplices T_m and T_{m-1}, and that we write $A_{F_m} := T_m \cup T_{m-1}$.

We study two constructions, beginning as follows. We define $w'_m := w_{|T_{m-1}} \circ \Xi \in W^{1,p}(T_m)$, where $\Xi : T_m \to T_{m-1}$ is the unique affine diffeomorphism that leaves F_m invariant. By construction, $w'_m \in W^{1,p}(T_m)$ with

$$\operatorname{tr}_{F_m} w'_m = \operatorname{tr}_{F_m} w_{|T_{m-1}}.$$

We now define $w''_m \in W^{1,p}(T_m)$ via

$$w''_m := w_{|T_m} - w'_m = u_{|T_m} - u_{|T_{m-1}} \circ \Xi. \tag{30}$$

We crucially note that w''_m is trace-free along F_m since

$$\operatorname{tr}_{F_m} w''_m = \operatorname{tr}_{F_m} \left(w_{|T_m} - w'_m \right) = \operatorname{tr}_{F_m} w_{|T_m} - \operatorname{tr}_{F_m} w_{|T_{m-1}} = \operatorname{tr}_{F_m} u_{|T_m} - \operatorname{tr}_{F_m} u_{|T_{m-1}} = 0.$$

An application of Lemma 4.1 to the first expression in (30) gives

$$\begin{aligned} \|w''_m\|_{L^p(T_m)} &\leq C_{\mathrm{PF},T_m,F_m,p} \left(\|\nabla w\|_{L^p(T_m)} + \|\nabla w'_m\|_{L^p(T_m)} \right) \\ &\leq C_{\mathrm{PF},T_m,F_m,p} \left(\|\nabla u\|_{L^p(T_m)} + \|\nabla w'_m\|_{L^p(T_m)} \right). \end{aligned}$$

Using Lemma 3.4 as well as Definitions (27) and (29), we find

$$\begin{aligned} \|\nabla w'_m\|_{L^p(T_m)} &\leq |\det(\mathbf{J}\Xi)|^{-\frac{1}{p}} \|\mathbf{J}\Xi\|_2 \|\nabla w\|_{L^p(T_{m-1})} \\ &\leq \left(\frac{\operatorname{vol}(T_m)}{\operatorname{vol}(T_{m-1})} \right)^{\frac{1}{p}} C_\xi(\mathcal{T}) \|\nabla w\|_{L^p(T_{m-1})} = C_\rho(\mathcal{T})^{\frac{1}{p}} C_\xi(\mathcal{T}) \|\nabla u\|_{L^p(T_{m-1})}. \end{aligned}$$

Since $w_{|T_m} = w''_m + w'_m$, we finally find

$$\begin{aligned} \|w\|_{L^p(T_m)} &\leq \|w'_m\|_{L^p(T_m)} + \|w''_m\|_{L^p(T_m)} \\ &\leq C_\rho(\mathcal{T})^{\frac{1}{p}} \|w\|_{L^p(T_{m-1})} + C_{\mathrm{PF},T_m,F_m,p} \left(\|\nabla u\|_{L^p(T_m)} + C_\rho(\mathcal{T})^{\frac{1}{p}} C_\xi(\mathcal{T}) \|\nabla u\|_{L^p(T_{m-1})} \right). \end{aligned}$$

Therefrom, the first recursive estimate follows.

Now we discuss the second recursive estimate. Suppose again that $1 \leq m \leq M$. We use the Poincaré–Friedrichs inequality over A_{F_m}, as given in Lemma 4.3, to find $w_{F_m} \in W^{1,p}(A_{F_m})$ such that $\nabla w_{F_m} = \nabla u$ over A_{F_m} and

$$\|w_{F_m}\|_{L^p(A_{F_m})} \leq C_{\mathrm{PF},A_{F_m},p} \|\nabla w_{F_m}\|_{L^p(A_{F_m})}. \tag{31}$$

We can define the constant

$$c_m := w_{|A_{F_m}} - w_{F_m}.$$

Now we observe that

$$\begin{aligned} \|w\|_{L^p(T_m)} &\leq \|w_{F_m}\|_{L^p(T_m)} + \|c_m\|_{L^p(T_m)}, \\ \|c_m\|_{L^p(T_m)} &= \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|c_m\|_{L^p(T_{m-1})}, \\ \|c_m\|_{L^p(T_{m-1})} &\leq \|w\|_{L^p(T_{m-1})} + \|w_{F_m}\|_{L^p(T_{m-1})}. \end{aligned}$$

In combination,

$$\begin{aligned} \|w\|_{L^p(T_m)} \leq& \|w_{F_m}\|_{L^p(T_m)} \\ &+ \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w_{F_m}\|_{L^p(T_{m-1})} + \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w\|_{L^p(T_{m-1})}. \end{aligned}$$

We sum the two integrals of w_{F_m}. When $1 < p < \infty$, recalling the complementary exponent $q = p/(p-1) \in (1,\infty)$, we use Hölder's inequality to verify

$$\begin{aligned}
\|w\|_{L^p(T_m)} &\leq \left(1 + \frac{\operatorname{vol}(T_m)^{\frac{q}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{q}{p}}}\right)^{\frac{1}{q}} \left(\|w_{F_m}\|^p_{L^p(T_m)} + \|w_{F_m}\|^p_{L^p(T_{m-1})}\right)^{\frac{1}{p}} + \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w\|_{L^p(T_{m-1})} \\
&= \left(1 + \frac{\operatorname{vol}(T_m)^{\frac{q}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{q}{p}}}\right)^{\frac{1}{q}} \|w_{F_m}\|_{L^p(A_{F_m})} + \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w\|_{L^p(T_{m-1})}.
\end{aligned}$$

Note that in the limit cases $p = 1$ and $p = \infty$ we get, respectively,

$$\begin{aligned}
\|w\|_{L^1(T_m)} &\leq \max\left(1, \frac{\operatorname{vol}(T_m)}{\operatorname{vol}(T_{m-1})}\right) \|w_{F_m}\|_{L^1(A_{F_m})} + \frac{\operatorname{vol}(T_m)}{\operatorname{vol}(T_{m-1})} \|w\|_{L^1(T_{m-1})}, \\
\|w\|_{L^\infty(T_m)} &\leq 2\|w_{F_m}\|_{L^\infty(A_{F_m})} + \|w\|_{L^\infty(T_{m-1})}.
\end{aligned}$$

The local inequality (31) now provides the second recursive estimate. The proof is complete. $\square$

The recursive construction of a gradient potential in the previous theorem, marching from simplex to simplex, can be associated with a concept of graph theory. Indeed, the face-neighbor relationship between adjacent n-simplices gives rise to an undirected graph that we call *face-connection graph*. If the potential is constructed starting from an initial n-simplex T_0, then the sequence of simplices corresponds to a path in that graph. In practice, we will pick a spanning tree for the undirected graph to describe the construction of the potentials.

We use that formalism to describe an estimate for the Poincaré–Friedrichs constant of the gradient potential, unwrapping the recursion. While the notation is a bit technical, the underlying idea is this: Unrolling the recursion gives estimates for the potential over a simplex in terms of the potential over previous simplices. The final constant can be estimated by a norm of the vector of the coefficients that appear in the recursion.

Theorem 4.5. *Let $\mathcal{T}$ be a face-connected n-dimensional finite triangulation and that the domain Ω is the interior of the underlying set of $\mathcal{T}$. Let $1 \leq p, q \leq \infty$ with $1 = 1/p + 1/q$, and suppose that $u, w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$.*

Suppose that n-simplices are enumerated as $T_0, T_1, \dots, T_M$, and that with each n-simplex $T_m \in \mathcal{T}$ we have a sequence of indices

$$0 = \imath(m,0),\ \imath(m,1),\ \dots,\ \imath(m,L_m) = m$$

such that for each $0 \leq m \leq M$ we have estimates of the following form:

$$\|w\|_{L^p(T_0)} \leq A_{m,0} \|\nabla u\|_{L^p(T_0)},$$

$$\begin{aligned}
\|w\|_{L^p(T_{\imath(m,\ell)})} \leq{}& A_{m,\imath(m,\ell)} \|w\|_{L^p(T_{\imath(m,\ell-1)})} \\
&+ B'_{m,\imath(m,\ell)} \|\nabla u\|_{L^p(T_{\imath(m,\ell)})} + B''_{m,\imath(m,\ell)} \|\nabla u\|_{L^p(T_{\imath(m,\ell-1)})}, \quad 1 \leq \ell \leq L_m.
\end{aligned}$$

Then

$$\|w\|_{L^p(\Omega)} \leq \left(\sum_{m=0}^{M} \left(\sum_{\ell=0}^{L_m} D^q_{m,\ell}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)},$$

where

$$\begin{aligned}
D_{m,L_m} &= A_{m,m}, \\
D_{m,\ell} &= \left(B_{m,\imath(m,L_m)} \cdots B_{m,\imath(m,\ell+2)}\right) \left(B_{m,\imath(m,\ell+1)} A_{m,\imath(m,\ell)} + A'_{m,\imath(m,\ell+1)}\right), \qquad 1 \leq \ell \leq L_m - 1.
\end{aligned}$$

Proof. Unwrapping the recursion for the norm over the m-th simplex, we get a total expression of the form

$$\begin{aligned}
\|w\|_{L^p(T_m)} \leq& \sum_{\ell=0}^{L_m} \left(B_{m,\imath(m,L_m)} \cdots B_{m,\imath(m,\ell+1)}\right) A_{m,\imath(m,\ell)} \|\nabla u\|_{L^p(T_{\imath(m,\ell)})} \\
&+ \sum_{\ell=0}^{L_m-1} \left(B_{m,\imath(m,L_m)} \cdots B_{m,\imath(m,\ell+2)}\right) A'_{m,\imath(m,\ell+1)} \|\nabla u\|_{L^p(T_{\imath(m,\ell)})} \\
=& A_{m,m} \|\nabla u\|_{L^p(T_{\imath(m,L_m)})} \\
&+ \sum_{\ell=0}^{L_m-1} \left(B_{m,\imath(m,L_m)} \cdots B_{m,\imath(m,\ell+2)}\right) \left(B_{m,\imath(m,\ell+1)} A_{m,\imath(m,\ell)} + A'_{m,\imath(m,\ell+1)}\right) \|\nabla u\|_{L^p(T_{\imath(m,\ell)})} \\
=:& \sum_{\ell=0}^{M} C_{m,\ell} \|\nabla u\|_{L^p(T_\ell)}.
\end{aligned}$$

Here, $C_{m,\ell}$ is the coefficient of $\|\nabla u\|_{L^p(T_\ell)}$, possibly zero, as it appears in the unwrapped recursive estimate of $\|w\|_{L^p(T_m)}$. The global Poincaré–Friedrichs inequality follows via Hölder's inequality:

$$\begin{aligned}
\|w\|^p_{L^p(\Omega)} &\leq \sum_{m=0}^{M} \|w\|^p_{L^p(T_m)} \\
&\leq \sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C_{m,\ell} \|\nabla u\|_{L^p(T_\ell)} \right)^p \\
&\leq \sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C^q_{m,\ell} \right)^{\frac{p}{q}} \sum_{\ell'=0}^{M} \|\nabla u\|^p_{L^p(T_{\ell'})} \leq \left(\sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C^q_{m,\ell} \right)^{\frac{p}{q}} \right) \|\nabla u\|^p_{L^p(\Omega)},
\end{aligned}$$

where $q \in [1,\infty]$ satisfies $1 = 1/p + 1/q$ and with obvious modifications if $p = 1$ or $p = \infty$. □

Remark 4.6. *The computable Poincaré–Friedrichs constants obtained in Theorem 4.4 depend on only a few parameters of the given triangulation: the length of any traversal from the root simplex, the ratios of the volumes of any pair of adjacent simplices, and the Poincaré–Friedrichs constants on each face patch or each simplex. Poincaré–Friedrichs constants on face patches are estimated in terms of shape regularity parameters of the triangulation; if the face patches of the triangulation are convex, then better estimates are possible.*

These computable Poincaré–Friedrichs constants increasingly overestimate the best one as the number of n-simplices in the triangulation $\mathcal{T}$ increases. Hence, we conceive their target application to be local patches (stars), in particular non-convex boundary stars. The latter occur inevitably at reentrant corners. Clearly, the same building principle applies whenever we have any non-overlapping partition of $\overline{\Omega}$ into convex local patches $\{A_m\}$ of n-simplices with an appropriate notion of connectivity. The proof proceeds verbatim, where we merely replace the simplices $\{T_m\}$ by the convex local patches $\{A_m\}$. This may allow for partitions of $\overline{\Omega}$ with significantly fewer elements, which enables a largely improved estimate of the best Poincaré–Friedrichs constant.

5 Review of vector calculus and exterior calculus

We review in this section the Sobolev spaces of vector and exterior calculus with particular emphasis on their transformation behavior. We refer the reader to Ern and Guermond [21] and Hiptmair [38] for background material on Sobolev vector analysis and to Greub [34] and Lee [44] for exterior algebra and exterior products.

5.1 Vector calculus

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set. We recall $L^p(\Omega)$, the space of scalar-valued p-integrable functions defined on Ω and that $\boldsymbol{L}^p(\Omega) := L^p(\Omega)^n$ for vector-valued functions with each component in $L^p(\Omega)$. In the three-dimensional setting, we are particularly interested in the Sobolev vector analysis. The space of scalar-valued $L^p(\Omega)$ functions with weak gradients in $\boldsymbol{L}^p(\Omega)$ is

$$W^p(\mathrm{grad},\Omega) := W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \mathrm{grad}\, v \in \boldsymbol{L}^p(\Omega)\}.$$

The space $\boldsymbol{W}^p(\mathrm{curl},\Omega)$ of vector-valued $\boldsymbol{L}^p(\Omega)$ functions with weak curls in $\boldsymbol{L}^p(\Omega)$ and the space of vector-valued $\boldsymbol{L}^p(\Omega)$ functions with weak divergences in $L^p(\Omega)$ are written

$$\begin{aligned}\boldsymbol{W}^p(\mathrm{curl},\Omega) &= \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \mid \mathrm{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)\},\\ \boldsymbol{W}^p(\mathrm{div},\Omega) &= \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \mid \mathrm{div}\, \boldsymbol{u} \in L^p(\Omega)\}.\end{aligned}$$

We are interested in transformations of these Sobolev tensor fields from one domain onto another. Suppose that $\Omega, \Omega' \subset \mathbb{R}^3$ are open sets and suppose that $\phi : \Omega \to \Omega'$ is a bi-Lipschitz mapping. We introduce the gradient-, curl-, and divergence-conforming Piola transformations, respectively, as the mappings $\phi^{\mathrm{grad}} : L^p(\Omega') \to L^p(\Omega)$, $\phi^{\mathrm{curl}} : \boldsymbol{L}^p(\Omega') \to \boldsymbol{L}^p(\Omega)$, and $\phi^{\mathrm{div}} : \boldsymbol{L}^p(\Omega') \to \boldsymbol{L}^p(\Omega)$. We also introduce $\phi^{\mathrm{b}} : L^p(\Omega') \to L^p(\Omega)$. These are defined for any $v \in L^p(\Omega')$ and $\boldsymbol{w} \in \boldsymbol{L}^p(\Omega')$ by setting

$$\begin{aligned}\phi^{\mathrm{grad}}(v) &= v \circ \phi,\\ \phi^{\mathrm{curl}}(\boldsymbol{w}) &= \mathbf{J}\phi^T(\boldsymbol{w} \circ \phi),\\ \phi^{\mathrm{div}}(\boldsymbol{w}) &= \mathrm{adj}(\mathbf{J}\phi)\,(\boldsymbol{w} \circ \phi)\,,\\ \phi^{\mathrm{b}}(v) &= \det(\mathbf{J}\phi)\,(v \circ \phi)\,,\end{aligned}$$

Here, $\mathbf{J}\phi$ is the Jacobian matrix of ϕ (see also [21, Definition 9.8]), and $\mathrm{adj}\,\mathbf{J}\phi$ denotes taking its adjugate matrix. These transformations are invertible. Bounds on the Lebesgue norms will follow from a more general result below. We use the commutativity relations

$$\mathrm{grad}\, \phi^{\mathrm{grad}}(v) = \phi^{\mathrm{curl}}(\mathrm{grad}\, v), \tag{33a}$$

$$\mathrm{curl}\, \phi^{\mathrm{curl}}(\boldsymbol{v}) = \phi^{\mathrm{div}}(\mathrm{curl}\, \boldsymbol{v}), \tag{33b}$$

$$\mathrm{div}\, \phi^{\mathrm{div}}(\boldsymbol{w}) = \phi^{\mathrm{b}}(\mathrm{div}\, \boldsymbol{w}), \tag{33c}$$

where $v \in W^p(\mathrm{grad},\Omega)$, $\boldsymbol{v} \in \boldsymbol{W}^p(\mathrm{curl},\Omega)$, and $\boldsymbol{w} \in \boldsymbol{W}^p(\mathrm{div},\Omega)$. We summarize this as a commuting diagram:

$$\begin{array}{ccccccc} W^p(\mathrm{grad},\Omega') & \xrightarrow{\mathrm{grad}} & \boldsymbol{W}^p(\mathrm{curl},\Omega') & \xrightarrow{\mathrm{curl}} & \boldsymbol{W}^p(\mathrm{div},\Omega') & \xrightarrow{\mathrm{div}} & L^p(\Omega') \\ \downarrow{\scriptstyle \phi^{\mathrm{grad}}} & & \downarrow{\scriptstyle \phi^{\mathrm{curl}}} & & \downarrow{\scriptstyle \phi^{\mathrm{div}}} & & \downarrow{\scriptstyle \phi^{\mathrm{b}}} \\ W^p(\mathrm{grad},\Omega) & \xrightarrow{\mathrm{grad}} & \boldsymbol{W}^p(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & \boldsymbol{W}^p(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^p(\Omega). \end{array}$$

Remark 5.1. *The Piola transform goes into the opposite direction of the mapping $\phi : \Omega \to \Omega'$: scalar and vector fields over Ω' are transformed into scalar and vector fields over Ω. This definition is in accordance with the notion of pullback, which we will review shortly. One advantage of that definition is that it also makes sense whenever the transformation is not bijective. However, the literature also knows the Piola transform in the direction of the original mapping.*

5.2 Exterior calculus

We now move the discussion to exterior calculus, beginning with exterior algebra. Let V be a real vector space. Given an integer $k \geq 0$, we let $\Lambda^k(V)$ denote the space of scalar-valued antisymmetric k-linear forms over V. Recall that any k-linear scalar-valued form u over V is called antisymmetric if

$$u(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)}) = \mathrm{sign}(\pi) u(v_1, v_2, \dots, v_k)$$

for any $v_1, v_2, \dots, v_k \in V$ and any permutation π of the indices $\{1, 2, \dots, k\}$. By definition, $\Lambda^1(V)$ is just the dual space of V, and $\Lambda^0(V)$ is the space of real numbers. Formally, we define $\Lambda^k(V)$ to be the zero vector space when $k < 0$.

The wedge product (or exterior product) of alternating multilinear forms is a fundamental operation in exterior algebra (see Chapter 14 in [44]). Given two alternating multilinear forms $u_1 \in \Lambda^k(V)$ and $u_2 \in \Lambda^l(V)$, their wedge product $u_1 \wedge u_2$ is a member of $\Lambda^{k+l}(V)$ defined by the formula

$$(u_1 \wedge u_2)(v_1, v_2, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\pi} \operatorname{sgn}(\pi) u_1(v_{\pi(1)}, \dots, v_{\pi(k)}) u_2(v_{\pi(k+1)}, \dots, v_{\pi(k+l)}),$$

for any $v_1, v_2, \dots, v_{k+l} \in V$. Here, the sum runs over all permutations π of the index set $\{1, 2, \dots, k+l\}$. The exterior product is bilinear and associative, and satisfies

$$u_1 \wedge u_2 = (-1)^{kl} u_2 \wedge u_1, \quad \forall u_1 \in \Lambda^k(V), \quad \forall u_2 \in \Lambda^l(V).$$

The interior product is in some sense dual to the exterior product. Given $v \in V$ and $u \in \Lambda^k(V)$, we define the interior product $v \lrcorner u \in \Lambda^{k-1}(V)$ via

$$(v \lrcorner u)(v_1, v_2, \dots, v_{k-1}) = u(v, v_1, v_2, \dots, v_{k-1}), \quad \forall v_1, v_2, \dots, v_{k-1} \in V.$$

We employ the exterior algebra only in the special case $V = \mathbb{R}^n$ of alternating forms over the n-dimensional Euclidean space. Here, it is customary to identify $\Lambda^k(V)$ with the space of antisymmetric tensors in k indices. Moreover, this particular setting comes with a canonical basis. We let $\{dx^1, dx^2, \dots, dx^n\}$ be the basis dual to the canonical unit vectors. This is a canonical basis of $\Lambda^1(\mathbb{R}^n)$. To define a canonical basis of $\Lambda^k(\mathbb{R}^n)$, we first introduce $\Sigma(k,n)$, the set of strictly ascending mappings $\sigma : \{1, \dots, k\} \to \{1, \dots, n\}$, where $k, n \in \mathbb{Z}$, and introduce the basic k-alternators

$$dx^\sigma := dx^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(k)}, \quad \forall \sigma \in \Sigma(k,n).$$

These define a basis of $\Lambda^k(\mathbb{R}^n)$. Note that $\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k}$. In particular, $\Lambda^k(\mathbb{R}^n)$ is the zero vector space whenever $k > n$.

We notice that the canonical scalar product on $\mathbb{R}^n$ gives rise to a scalar product on $\Lambda^1(\mathbb{R}^n)$, which induces a scalar product on $\Lambda^k(\mathbb{R}^n)$. The basic k-alternators are an orthonormal basis of $\Lambda^k(\mathbb{R}^n)$ with respect to that inner product.

5.3 Smooth differential forms

We let $\Omega \subseteq \mathbb{R}^n$ be any bounded open set. We write $C^\infty\Lambda^k(\Omega)$ for the space of smooth differential k-forms over $\Omega \subseteq \mathbb{R}^n$, which is the vector space of smooth mappings from Ω into $\Lambda^k(\mathbb{R}^n)$. The exterior derivative d is an operator that takes a k-form $\omega \in C^\infty\Lambda^k(\Omega)$ to a $(k+1)$-form $d\omega \in C^\infty\Lambda^{k+1}(\Omega)$. Every k-form $\omega \in C^\infty\Lambda^k(\Omega)$ can be written

$$\omega = \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma \, dx^\sigma = \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma \, dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \dots \wedge dx^{\sigma(k)}, \tag{34}$$

where $\omega_\sigma : \Omega \to \mathbb{R}$ are smooth functions. The exterior derivative $d\omega$ is defined by

$$d\omega = \sum_{\sigma \in \Sigma(k,n)} \sum_{j=1}^{n} \frac{\partial \omega_\sigma}{\partial x^j} \, dx^j \wedge dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \dots \wedge dx^{\sigma(k)}.$$

The exterior derivative is linear and nilpotent, which means $d(d\omega) = 0$ for any $\omega \in C^\infty\Lambda^k(\Omega)$. Moreover, it satisfies the Leibniz rule:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta, \qquad \forall \omega \in C^\infty\Lambda^k(\Omega), \qquad \forall \eta \in C^\infty\Lambda^l(\Omega).$$

The integral of a differential n-form is uniquely defined via

$$\int_\Omega \omega dx^1 \wedge \dots \wedge dx^n = \int_\Omega \omega(x) \, dx.$$

Remark 5.2. *In three dimensions, the calculus of differential forms is in correspondence with classical vector calculus. This is expressed formally as the commuting diagram*

$$\begin{array}{ccccccc} C^\infty\Lambda^0(\Omega) & \xrightarrow{d} & C^\infty\Lambda^1(\Omega) & \xrightarrow{d} & C^\infty\Lambda^2(\Omega) & \xrightarrow{d} & C^\infty\Lambda^3(\Omega) \\ \Big\downarrow{\scriptstyle\varpi^0} & & \Big\downarrow{\scriptstyle\varpi^1} & & \Big\downarrow{\scriptstyle\varpi^2} & & \Big\downarrow{\scriptstyle\varpi^3} \\ C^\infty(\Omega) & \xrightarrow{\text{grad}} & C^\infty(\Omega)^3 & \xrightarrow{\text{curl}} & C^\infty(\Omega)^3 & \xrightarrow{\text{div}} & C^\infty(\Omega) \end{array},$$

where ϖ^0 and ϖ^3 are the identity mappings and where

$$\varpi^1\left(u_1 dx^1 + u_2 dx^2 + u_3 dx^3\right) = (u_1, u_2, u_3),$$
$$\varpi^2\left(u_{12} dx^1 \wedge dx^2 + u_{13} dx^1 \wedge dx^3 + u_{23} dx^2 \wedge dx^3\right) = (u_{23}, -u_{13}, u_{12}).$$

In two dimensions, the calculus of differential forms can be translated into 2D vector calculus in two different ways. To the authors' best knowledge, neither convention is dominant over the other in the literature. We summarize the situation in the following commuting diagram:

$$\begin{array}{ccccc} C^\infty(\Omega) & \xrightarrow{\text{curl}} & C^\infty(\Omega)^2 & \xrightarrow{\text{div}} & C^\infty(\Omega) \\ \Big\uparrow{\scriptstyle\varkappa^0} & & \Big\uparrow{\scriptstyle\varkappa^1} & & \Big\uparrow{\scriptstyle\varkappa^2} \\ C^\infty\Lambda^0(\Omega) & \xrightarrow{d} & C^\infty\Lambda^1(\Omega) & \xrightarrow{d} & C^\infty\Lambda^2(\Omega) \\ \Big\downarrow{\scriptstyle\varpi^0} & & \Big\downarrow{\scriptstyle\varpi^1} & & \Big\downarrow{\scriptstyle\varpi^2} \\ C^\infty(\Omega) & \xrightarrow{\text{grad}} & C^\infty(\Omega)^2 & \xrightarrow{\text{rot}} & C^\infty(\Omega) \end{array}.$$

Here, $\varpi^1\left(u_1 dx^1 + u_2 dx^2\right) = (u_1, u_2)$ is the lower middle isomorphism. We introduce the rotation operator $J(x, y) = (y, -x)$ and define $\varkappa = J\varpi$ and $\operatorname{rot} = \operatorname{div} J$. The other vertical arrows are the identity. The utility of exterior calculus is that the operators of vector calculus can be translated into a common framework that does not depend on the dimension.

5.4 Sobolev spaces of differential forms

Let us now turn our attention to Sobolev spaces of differential forms. Since the exterior product space $\Lambda^k(\mathbb{R}^n)$ carries a norm, induced from the Euclidean norm on $\mathbb{R}^n$, there are pointwise norms of differential k-forms. We let $L^p\Lambda^k(\Omega)$ be the space of differential k-forms over Ω with locally integrable coefficients such that its pointwise norm is p-integrable. The exterior derivative is defined in the sense of distributions and we introduce

$$W^p\Lambda^k(\Omega) := \{u \in L^p\Lambda^k(\Omega) \mid du \in L^p\Lambda^{k+1}(\Omega)\}.$$

We observe that $u \in L^p\Lambda^k(\Omega)$ has weak exterior derivative $f \in L^p\Lambda^{k+1}(\Omega)$ if and only if for all $v \in C^\infty_c\Lambda^{n-k-1}(\Omega)$ we have the integration-by-parts formula

$$\int_\Omega dv \wedge u = (-1)^{k(n-k)+1} \int_\Omega v \wedge f.$$

Lastly, we are also interested in differential forms whose trace vanishes along a part of the boundary. Suppose that $\Gamma \subseteq \partial\Omega$ is a relatively open subset of the boundary. We say that $u \in W^p\Lambda^k(\Omega)$ has vanishing trace along Γ if for all $x \in \Gamma$ there exists $r > 0$ such that for all $v \in C^\infty_c\Lambda^{n-k-1}(\mathbb{R}^n)$ whose support lies in the open ball $B_r(x)$ we have the integration-by-parts formula

$$\int_{B_r(x)} dv \wedge \tilde{u} = (-1)^{k(n-k)+1} \int_{B_r(x)} v \wedge \widetilde{du}.$$

If that condition is satisfied, we also write

$$\operatorname{tr}_\Gamma u = 0.$$

Accordingly, we write $\operatorname{tr}_\Gamma u = \operatorname{tr}_\Gamma u'$ for $\operatorname{tr}_\Gamma(u - u') = 0$ whenever $u, u' \in W^p\Lambda^k(\Omega)$. Lastly, we introduce the closed subspaces

$$W_0^p\Lambda^k(\Omega) := \{u \in W^p\Lambda^k(\Omega) \mid \operatorname{tr}_{\partial\Omega} u = 0\}.$$

This is a closed subspace. We know that $u \in W_0^p\Lambda^k(\Omega)$ if and only if its extension by zero $\tilde{u} : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$ is a member of $\tilde{u} \in W^p\Lambda^k(\mathbb{R}^n)$. Moreover, $dW_0^p\Lambda^k(\Omega) \subseteq W_0^p\Lambda^{k+1}(\Omega)$. We also observe that $W^p\Lambda^n(\Omega) = W_0^p\Lambda^n(\Omega) = L^p(\Omega)$. We use the abbreviation $W_0^{1,p}(\Omega) := W^p\Lambda^0(\Omega)$.

5.5 Transformations by bi-Lipschitz mappings

We are interested in transformations of Sobolev tensor fields from one domain onto another. Suppose that $\Omega, \Omega' \subset \mathbb{R}^n$ are open sets and suppose that $\phi : \Omega \to \Omega'$ is a bi-Lipschitz mapping. The pullback of $u \in L^p\Lambda^k(\Omega')$ along ϕ is the (measurable) differential form

$$\phi^* u_{|x}(v_1, v_2, \dots, v_k) := u_{|\phi(x)}(\mathbf{J}\phi_{|x} \cdot v_1, \mathbf{J}\phi_{|x} \cdot v_2, \dots, \mathbf{J}\phi_{|x} \cdot v_k) \tag{35}$$

for any $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ and any $x \in \Omega$. One can show that $\phi^* u \in L^p\Lambda^k(\Omega)$ and the following estimates.

Proposition 5.3. *Let $\phi : \Omega \to \Omega'$ be a bi-Lipschitz mapping between open sets $\Omega, \Omega' \subseteq \mathbb{R}^n$. Let $p \in [1, \infty]$ and $u \in L^p\Lambda^k(\Omega')$. Then $\phi^* u \in L^p\Lambda^k(\Omega)$ and*

$$\|\phi^* u\|_{L^p\Lambda^k(\Omega)} \leq \|\mathbf{J}\phi\|_{L^\infty(\Omega)}^k \|\det \mathbf{J}\phi^{-1}\|_{L^\infty(\Omega')}^{\frac{1}{p}} \|u\|_{L^p\Lambda^k(\Omega')}. \tag{36}$$

If ϕ is affine and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$ are the singular values of $\mathbf{J}\phi$, then

$$\|\phi^* u\|_{L^p\Lambda^k(\Omega)} \leq \sigma_1\sigma_2 \cdots \sigma_k \cdot \|\det \mathbf{J}\phi^{-1}\|_{L^\infty(\Omega')}^{\frac{1}{p}} \|u\|_{L^p\Lambda^k(\Omega')}. \tag{37}$$

Moreover, if $u \in W^p\Lambda^k(\Omega')$, then $\phi^* u \in W^p\Lambda^k(\Omega)$ and $d\phi^* u = \phi^* du$.

Proof. See [46] and Corollary 6 in [59]. □

5.6 Some approximation properties

We review a few approximation properties. Let $\mathfrak{m}$ be a non-negative scalar function whose integral equals one and whose support lies in the unit ball around the origin. Define $\mathfrak{m}_\epsilon(x) := \epsilon^{-n}\mathfrak{m}(x/\epsilon)$. We collect several approximation results that involve convolution with a mollifier. In what follows, convolutions of functions defined over domains tacitly assume that these functions have been extended by zero to all of Euclidean space.

Lemma 5.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $1 \leq p < \infty$. If $u \in L^p(\Omega)$, then the convolution $\mathfrak{m}_\epsilon \star u \to u$ in $L^p(\Omega)$ as $\epsilon \to 0$.*

Lemma 5.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $1 \leq p < \infty$. Smooth forms are dense in $W^p\Lambda^k(\Omega)$. If Ω is convex, then $C_c^\infty\Lambda^k(\Omega)$ is dense in $W_0^p\Lambda^k(\Omega)$.*

Proof. We notice $\mathfrak{m}_\epsilon \star u \in C^\infty\Lambda^k(\mathbb{R}^n)$. By the dominated convergence theorem and because u has a weak derivative, for any $v \in C_c^\infty\Lambda^{n-k-1}(\Omega)$

$$\begin{aligned}
\int_{\mathbb{R}^n} (\mathfrak{m}_\epsilon \star u) \wedge dv &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathfrak{m}(x-y) \wedge u(y) \wedge d_x v(x)\, dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} d_x\mathfrak{m}(x-y) \wedge u(y) \wedge v(x)\, dx\, dy \\
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} d_y\mathfrak{m}(x-y) \wedge u(y) \wedge v(x)\, dx\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathfrak{m}(x-y) \wedge d_y u(y) \wedge v(x)\, dx\, dy = \int_{\mathbb{R}^n} (\mathfrak{m}_\epsilon \star du) \wedge v.
\end{aligned}$$

Hence, $C^\infty\Lambda^k(\Omega)$ is dense in $W^p\Lambda^k(\Omega)$.

Next, suppose that Ω is convex. Without loss of generality, $0 \in \Omega$. Let $u \in W^p_0\Lambda^k(\Omega)$ and extend u trivially onto $\mathbb{R}^n$. Define $\varphi_t(x) = tx$ for $t > 1$. Then $\varphi_t^* u \in W^p_0\Lambda^k(t^{-1}\Omega) \subseteq W^p_0\Lambda^k(\Omega)$ and $\varphi_t^* u$ converges to u as t decreases towards 1. Given any $t > 1$, by taking the convolution with $\mathfrak{m}_\epsilon$ for $\epsilon > 0$ small enough, we approximate $\varphi_t^* u$ through members of $C_c^\infty\Lambda^k(\Omega)$. The desired result follows. $\square$

6 Regularized potentials over convex sets

We now develop bounds for Poincaré–Friedrichs constants for the exterior derivative over convex domains. Here, we consider two special cases: either the L^p de Rham complex without boundary conditions, or the L^p de Rham with full boundary conditions. The corresponding linear potentials are known as the regularized Poincaré and regularized Bogovskiĭ potentials in the literature. We build upon the discussion spearheaded by Costabel and McIntosh [19], who analyze them as pseudo-differential operators over domains star-shaped with respect to a ball. In comparison to their extensive work, our discussion is more modest: we study potentials merely over convex sets, and we are only interested in their operator norms between Lebesgue spaces. However, our goal is explicit bounds for the operator norms, giving the Poincaré–Friedrichs constants.

In the remainder of this section, $\Omega \subseteq \mathbb{R}^n$ is a bounded convex open set with diameter $\delta(\Omega) > 0$.

6.1 Regularized Poincaré and Bogovskiĭ operators

We begin by introducing the Costabel-McIntosh kernel. For any $k \in \{0, \dots, n\}$, we define the kernel $\mathcal{G}_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{G}_k(x,y) = \int_1^\infty (t-1)^{n-k} t^{k-1} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(y + t(x-y)\right) dt, \tag{38}$$

where $\chi_\Omega : \Omega \to \{0,1\}$ denotes the characteristic function of the domain Ω. Given a differential form $u \in C_c^\infty(\mathbb{R}^n, \Lambda^k)$, where $1 \leq k \leq n$, we then define the integral operators

$$\begin{aligned}
\mathcal{P}_k u(x) &= \int_\Omega \mathcal{G}_{n-k+1}(y,x)\,(x-y) \lrcorner u(y)\, dy, \\
\mathcal{B}_k u(x) &= \int_\Omega \mathcal{G}_k(x,y)\,(x-y) \lrcorner u(y)\, dy.
\end{aligned}$$

We call $\mathcal{P}_k$ the *Poincaré operator* and $\mathcal{B}_k$ the *Bogovskiĭ operator*.

We show that the integrals in the definition of $\mathcal{P}_k$ and $\mathcal{B}_k$ actually converge. In order to analyze the properties of the potentials, we first rewrite the Costabel-McIntosh kernel $\mathcal{G}_k$. Letting $x, y \in \mathbb{R}^n$ with $x \neq y$, we find

$$\begin{aligned}
\mathcal{G}_k(x,y) &= \int_0^\infty t^{n-k}(t+1)^{k-1} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(x + t(x-y)\right) dt \\
&= \int_0^\infty \sum_{\ell=0}^{k-1} \tbinom{k-1}{\ell} t^{n-k+\ell} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(x + t(x-y)\right) dt \\
&= \int_0^\infty \sum_{\ell=0}^{k-1} \tbinom{k-1}{k-1-\ell} t^{n-k+k-1-\ell} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(x + t(x-y)\right) dt \\
&= \int_0^\infty \sum_{\ell=0}^{k-1} \tbinom{k-1}{\ell} t^{n-\ell-1} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(x + t(x-y)\right) dt \\
&= \sum_{\ell=0}^{k-1} \tbinom{k-1}{\ell} \int_0^\infty t^{n-\ell-1} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(x + t(x-y)\right) dt \\
&= \operatorname{vol}(\Omega)^{-1} \sum_{\ell=0}^{k-1} \tbinom{k-1}{\ell} |x-y|^{\ell-n} \int_0^\infty r^{n-\ell-1} \chi_\Omega\left(x + r\frac{x-y}{|x-y|}\right) dr.
\end{aligned}$$

If $x \in \Omega$ and $x \neq y$, then we can restrict the inner integrals to the range $0 \leq r \leq \delta(\Omega)$, which gives

$$\begin{aligned}\mathcal{G}_k(x,y) &= \operatorname{vol}(\Omega)^{-1} \sum_{\ell=0}^{k-1} \tbinom{k-1}{\ell} |x-y|^{\ell-n} \int_0^{\delta(\Omega)} r^{n-\ell-1}\,dr \\ &= \operatorname{vol}(\Omega)^{-1} \sum_{\ell=0}^{k-1} \tbinom{k-1}{\ell} |x-y|^{\ell-n} \frac{\delta(\Omega)^{n-\ell}}{n-\ell}.\end{aligned}$$

We are now in a position to show that the potentials are bounded with respect to Lebesgue norms.

6.2 An operator norm bound with respect to the Lebesgue norm: the Poincaré case

We begin with the Poincaré operator. Let $B_{\delta(\Omega)}(0)$ be the n-dimensional ball centered at the origin. Suppose that $u \in L^\infty\Lambda^k(\Omega)$ with $1 \leq k \leq n$. We estimate $\mathcal{P}_k u(x)$ pointwise for any $x \in \Omega$ by the result of a convolution of a locally integrable function with u:

$$\begin{aligned}|\mathcal{P}_k u(x)| &= \left| \int_\Omega \mathcal{G}_{n-k+1}(x,y)\,(x-y) \lrcorner u(y)\,dy \right| \\ &\leq \int_\Omega \operatorname{vol}(\Omega)^{-1} \sum_{\ell=0}^{n-k} \tbinom{n-k}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} |x-y|^{\ell+1-n} \chi_{B_{\delta(\Omega)}(0)}(x-y) |u(y)|\,dy.\end{aligned}$$

We recall the radial integrals

$$\begin{aligned}\int_{B_{\delta(\Omega)}(0)} |z|^{\ell+1-n}\,dz &= \operatorname{vol}_{n-1}(S_1) \int_0^{\delta(\Omega)} r^{\ell+1-n} r^{n-1}\,dr \\ &= \operatorname{vol}_{n-1}(S_1) \int_0^{\delta(\Omega)} r^\ell\,dr = \operatorname{vol}_{n-1}(S_1) \frac{\delta(\Omega)^{\ell+1}}{\ell+1},\end{aligned}$$

where $S_1 \subseteq \mathbb{R}^n$ stands for the unit sphere of dimension $n-1$. One computes

$$\begin{aligned}&\int_{\mathbb{R}^n} \sum_{\ell=0}^{n-k} \tbinom{n-k}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \chi_{B_{\delta(\Omega)}(0)}(z) |z|^{\ell+1-n}\,dz \\ &= \sum_{\ell=0}^{n-k} \tbinom{n-k}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \int_{B_{\delta(\Omega)}(0)} |z|^{\ell+1-n}\,dz \\ &= \operatorname{vol}_{n-1}(S_1) \sum_{\ell=0}^{n-k} \tbinom{n-k}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \frac{\delta(\Omega)^{\ell+1}}{\ell+1} = \operatorname{vol}_{n-1}(S_1) \delta(\Omega)^{n+1} \underbrace{\sum_{\ell=0}^{n-k} \frac{\binom{n-k}{\ell}}{(n-\ell)(\ell+1)}}_{=:C_{\mathcal{P}}(n,k) \leq 2^{n-k}}.\end{aligned}$$

Here we introduce the numerical constant

$$C_{\mathcal{P}}(n,k) := \sum_{\ell=0}^{n-k} \frac{\binom{n-k}{\ell}}{(n-\ell)(\ell+1)}, \tag{39}$$

which depends only on n and k and which is bounded by 2^{n-k}.

In particular, the integral $\mathcal{P}_k u(x)$ is absolutely convergent for any choice of $x \in \Omega$ and its magnitude is pointwise dominated by the convolution of $|u|$ against an integrable function. Young's convolution inequality now implies:

$$\begin{aligned}\|\mathcal{P}_k u\|_{L^p(\Omega)} &\leq \operatorname{vol}_{n-1}(S_1) C_{\mathcal{P}}(n,k) \frac{\delta(\Omega)^n}{\operatorname{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)} \\ &\leq n C_{\mathcal{P}}(n,k) \frac{\operatorname{vol}(B_{\delta(\Omega)}(0))}{\operatorname{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)}.\end{aligned}$$

We have assumed so far that $u \in L^\infty\Lambda^k(\Omega)$. Since that space is dense in the Lebesgue spaces, a density argument establishes the following: for any $1 \leq p \leq \infty$ we have a bounded linear operator

$$\mathcal{P}_k : L^p\Lambda^k(\Omega) \to L^p\Lambda^{k-1}(\mathbb{R}^n).$$

Remark 6.1. *As already explained in the original paper [19], the operators $\mathcal{P}_k$ preserve polynomial differential forms.*

6.3 An operator norm bound with respect to the Lebesgue norm: the Bogovskiĭ case

We analyze the Bogovskiĭ potential operator by similar means. Suppose that $u \in L^\infty\Lambda^k(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq \overline{\Omega}$ and that $x \in \mathbb{R}^n$. First, if $x \notin \Omega$, then the convexity of Ω implies that $y + t(x-y) \notin \Omega$ for all $t > 1$. Hence, $\mathcal{G}_k(x,y) = 0$ and therefore $\mathcal{B}_k u(x) = 0$ in that case. Consider now the case $x \in \overline{\Omega}$. We estimate $\mathcal{B}_k u(x)$ pointwise by

$$\begin{aligned}
|\mathcal{B}_k u(x)| &= \left| \int_\Omega \mathcal{G}_k(x,y)\,(x-y) \lrcorner u(y)\,dy \right| \\
&\leq \int_\Omega \operatorname{vol}(\Omega)^{-1} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} |x-y|^{\ell+1-n} |u(y)|\,dy \\
&\leq \int_{\mathbb{R}^n} \operatorname{vol}(\Omega)^{-1} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \chi_{B_{\delta(\Omega)}(0)}(x-y) |x-y|^{\ell+1-n} |u(y)|\,dy.
\end{aligned}$$

Using once more the radial integrals discussed above, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \chi_{B_{\delta(\Omega)}(0)}(z) |z|^{\ell+1-n}\,dz \\
&= \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \int_{B_{\delta(\Omega)}(0)} |z|^{\ell+1-n}\,dz \\
&= \operatorname{vol}_{n-1}(S_1) \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \frac{\delta(\Omega)^{n-\ell}}{n-\ell} \frac{\delta(\Omega)^{\ell+1}}{\ell+1} = \operatorname{vol}_{n-1}(S_1) \delta(\Omega)^{n+1} \underbrace{\sum_{\ell=0}^{k-1} \frac{\binom{k-1}{\ell}}{(n-\ell)(\ell+1)}}_{=:C_{\mathcal{B}}(n,k) \leq 2^{k-1}}.
\end{aligned}$$

Here we introduce the numerical constant

$$C_{\mathcal{B}}(n,k) := \sum_{\ell=0}^{k-1} \frac{\binom{k-1}{\ell}}{(n-\ell)(\ell+1)}, \tag{40}$$

which depends only on n and k and which is bounded by 2^{k-1}. Similar as above, the integral $\mathcal{B}_k u(x)$ is absolutely convergent for any choice of $x \in \mathbb{R}^n$ and its magnitude is pointwise dominated by the convolution of $|u|$ against an integrable function. Young's convolution inequality now implies:

$$\begin{aligned}
\|\mathcal{B}_k u\|_{L^p(\Omega)} &\leq \operatorname{vol}_{n-1}(S_1) C_{\mathcal{B}}(n,k) \frac{\delta(\Omega)^n}{\operatorname{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)} \\
&\leq n C_{\mathcal{B}}(n,k) \frac{\operatorname{vol}(B_{\delta(\Omega)}(0))}{\operatorname{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)}.
\end{aligned}$$

We have assumed so far that u is essentially bounded. Since that space is dense in the Lebesgue spaces, a density argument yields: for any $1 \leq p \leq \infty$ we have a bounded linear operator

$$\mathcal{B}_k : L^p\Lambda^k(\Omega) \to L^p\Lambda^{k-1}(\mathbb{R}^n).$$

Moreover, $\operatorname{supp} \mathcal{B}_k u \subseteq \overline{\Omega}$, that is, the reconstructed potential has support contained within $\overline{\Omega}$.

6.4 Rewriting the potentials operators

More properties of these operators become apparent after a change of variables. We write down the full definition of these operators and perform two substitutions. For the Poincaré operator, we substitute $a = x + t(y - x)$, and then we substitute $s = (t-1)/t$, leading to

$$\begin{aligned}\mathcal{P}_k u(x) &= \mathrm{vol}(\Omega)^{-1} \int_\Omega \int_1^\infty (t-1)^{k-1} t^{n-k} \chi_\Omega\left(x + t(y-x)\right) (x-y) \lrcorner u(y)\, dt\, dy \\ &= \mathrm{vol}(\Omega)^{-1} \int_{\mathbb{R}^n} \chi_\Omega(a)\, (x-a) \lrcorner \int_0^1 t^{k-1} u\left(a + t(x-a)\right)\, dt\, da.\end{aligned}$$

For the Bogovskiĭ operator, we first substitute $a = y + t(x - y)$, and then we substitute $s = t/(t-1)$, leading to

$$\begin{aligned}\mathcal{B}_k u(x) &= \mathrm{vol}(\Omega)^{-1} \int_\Omega \int_1^\infty (t-1)^{n-k} t^{k-1} \chi_\Omega\left(y + t(x-y)\right) (x-y) \lrcorner u(y)\, dt\, dy \\ &= -\,\mathrm{vol}(\Omega)^{-1} \int_{\mathbb{R}^n} \chi_\Omega(a)\, (x-a) \lrcorner \int_1^\infty t^{k-1} u\left(a + t(x-a)\right)\, dt\, da.\end{aligned}$$

Given $a \in \Omega$, we introduce the potentials

$$\begin{aligned}\mathcal{P}_{k,a} u(x) &:= (x-a) \lrcorner \int_0^1 t^{k-1} u\left(a + t(x-a)\right)\, dt, \\ \mathcal{B}_{k,a} u(x) &:= -(x-a) \lrcorner \int_1^\infty t^{k-1} u\left(a + t(x-a)\right)\, dt.\end{aligned}$$

By definition,

$$\mathcal{P}_k u(x) = \mathrm{vol}(\Omega)^{-1} \int_\Omega \mathcal{P}_{k,a} u(x)\, da, \quad \mathcal{B}_k u(x) = \mathrm{vol}(\Omega)^{-1} \int_\Omega \mathcal{B}_{k,a} u(x)\, da.$$

6.5 Interaction of potentials with the exterior derivative

We study the interaction of the Poincaré and Bogovskiĭ operators with the exterior derivative in more detail. The main arguments are well-known and establish the exactness of several de Rham complexes. We recapitulate these arguments since our variants of the regularized potential operators are not yet included in the published literature.

We make use of the following notation [47]: for any mapping $\sigma \in \Sigma(k, n)$, we let $[\sigma] := \{\sigma(1), \dots, \sigma(k)\}$ be its image. When $p \in [\sigma]$, then the member of $\Sigma(k-1, n)$ with image $[\sigma] \setminus \{p\}$ is written $\sigma - p$.

Suppose that $u \in C^\infty\Lambda^k(\mathbb{R}^n)$. We rewrite the Poincaré potential,

$$\begin{aligned}\mathcal{P}_{k,a} u(x) &= (x-a) \lrcorner \int_0^1 t^{k-1} u\left(a + t(x-a)\right)\, dt \\ &= (x-a) \lrcorner \sum_{\sigma \in \Sigma(k,n)} \int_0^1 t^{k-1} u_\sigma\left(a + t(x-a)\right) dx^\sigma\, dt \\ &= \sum_{\sigma \in \Sigma(k,n)} \sum_{i=1}^k \int_0^1 t^{k-1} u_\sigma\left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)}\, dx^{\sigma - \sigma(i)}\, dt,\end{aligned}$$

and compute its exterior derivative:

$$\begin{aligned}d\mathcal{P}_{k,a} u(x) = &\sum_{\sigma \in \Sigma(k,n)} \int_0^1 k t^{k-1} u_\sigma\left(a + t(x-a)\right)\, dx^\sigma\, dt \\ &+ \sum_{\substack{\sigma \in \Sigma(k,n),\, 1 \leq i \leq k \\ 1 \leq j \leq n,\, j \notin [\sigma - \sigma(i)]}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)}\, dx^j \wedge dx^{\sigma - \sigma(i)}\, dt.\end{aligned}$$

We write the exterior derivative of u as

$$du(x) = \sum_{\sigma \in \Sigma(k,n)} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x)\, dx^j \wedge dx^\sigma = \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma]}} \frac{\partial u}{\partial x_j}(x)\, dx^j \wedge dx^\sigma, \tag{41}$$

and apply the Poincaré potential operator to this result, which gives

$$\begin{aligned}
\mathcal{P}_{k+1,a} du(x) &= (x-a) \lrcorner \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma]}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt \, dx^j \wedge dx^\sigma \\
&= \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma]}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt (x-a)_j \, dx^\sigma \\
&\quad - \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma] \\ 1 \le i \le k}} (-1)^{i-1} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt \, (x-a)_{\sigma(i)} \, dx^j \wedge dx^{\sigma - \sigma(i)}.
\end{aligned}$$

Next, we add the exterior derivative of the potential and the potential of the exterior derivative. Taking into account cancellations, this gives the identity

$$\begin{aligned}
& d\mathcal{P}_{k,a} u(x) + \mathcal{P}_{k+1,a} du(x) \\
&= \sum_{\sigma \in \Sigma(k,n)} \int_0^1 k t^{k-1} u_\sigma \left(a + t(x-a)\right) \, dx^{\sigma - \sigma(i)} \, dt \\
&\quad + \sum_{\substack{\sigma \in \Sigma(k,n),\, 1 \le i \le k \\ 1 \le j \le n,\, j \notin [\sigma - \sigma(i)]}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)} \, dx^j \wedge dx^{\sigma - \sigma(i)} \, dt \\
&\quad + \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma]}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt \, (x-a)_j \, dx^\sigma \\
&\quad - \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma] \\ 1 \le i \le k}} (-1)^{i-1} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt \, (x-a)_{\sigma(i)} \, dx^j \wedge dx^{\sigma - \sigma(i)} \\
&= \sum_{\sigma \in \Sigma(k,n)} \int_0^1 k t^{k-1} u_\sigma \left(a + t(x-a)\right) \, dx^{\sigma - \sigma(i)} \, dt \\
&\quad + \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \le j \le n,\, j \notin [\sigma]}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, (x-a)_j \, dt \, dx^\sigma \\
&\quad + \sum_{\substack{\sigma \in \Sigma(k,n), \\ 1 \le i \le k}} \int_0^1 t^k \frac{\partial u_\sigma}{\partial x_{\sigma(i)}} \left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)} \, dx^{\sigma(i)} \wedge dx^{\sigma - \sigma(i)} \, dt \\
&= \sum_{\sigma \in \Sigma(k,n)} \int_0^1 \frac{\partial}{\partial t} \left(t^k u_\sigma \left(a + t(x-a)\right) \right) \, dt \, dx^\sigma = \sum_{\sigma \in \Sigma(k,n)} \left(u_\sigma(x) - 0^k u_\sigma(a) \right) \, dx^\sigma.
\end{aligned}$$

We conclude that,

$$\begin{aligned}
u(x) &= \mathcal{P}_{1,a} du(x) - u(a), \qquad k = 0, \\
u(x) &= d\mathcal{P}_{k,a} u(x) + \mathcal{P}_{k+1,a} du(x), \qquad 1 \le k \le n.
\end{aligned}$$

In summary, after taking the average over $a \in \Omega$:

$$u(x) = \mathcal{P}_1 du(x) - \operatorname{vol}(\Omega)^{-1} \int_\Omega u(a)\, da, \qquad k = 0, \tag{42}$$

$$u(x) = d\mathcal{P}_k u(x) + \mathcal{P}_{k+1} du(x), \qquad 1 \leq k \leq n. \tag{43}$$

Even though the discussion for the Bogovskiĭ operator is large analogous, some modifications are needed. Suppose that $u \in C^\infty\Lambda^k(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq \overline{\Omega}$. We rewrite the Bogovskiĭ potential,

$$\begin{aligned}
-\mathcal{B}_{k,a}u(x) &= (x-a) \lrcorner \int_1^\infty t^{k-1} u\left(a + t(x-a)\right)\, dt \\
&= (x-a) \lrcorner \sum_{\sigma \in \Sigma(k,n)} \int_1^\infty t^{k-1} u_\sigma \left(a + t(x-a)\right)\, dx^\sigma \, dt \\
&= \sum_{\sigma \in \Sigma(k,n)} \sum_{i=1}^{k} \int_1^\infty t^{k-1} u_\sigma \left(a + t(x-a)\right) (-1)^{i-1} \left(x-a\right)_{\sigma(i)} dx^{\sigma - \sigma(i)}\, dt.
\end{aligned}$$

We want to take its exterior derivative, but we can generally only do that in the distributional sense. Away from the pivot point a, the form $\mathcal{B}_{k,a}u(x)$ is differentiable in x, and so we compute its exterior derivative over $\Omega \setminus \{a\}$:

$$\begin{aligned}
-d\mathcal{B}_{k,a}u(x) = &\sum_{\sigma \in \Sigma(k,n)} \int_1^\infty k t^{k-1} u_\sigma \left(a + t(x-a)\right)\, dx^\sigma \, dt \\
&+ \sum_{\substack{\sigma \in \Sigma(k,n),\ 1 \leq i \leq k \\ 1 \leq j \leq n,\ j \notin [\sigma - \sigma(i)]}} \int_1^\infty t^k \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)}\, dx^j \wedge dx^{\sigma - \sigma(i)}\, dt.
\end{aligned}$$

The derivative of $\mathcal{B}_{k,a}u$ over the whole domain, which is what we need, can only be taken in the sense of distributions. Let $\phi \in C^\infty\Lambda^{n-k}(\Omega)$ be smooth and compactly supported over Ω. We let $\epsilon > 0$ and calculate

$$(-1)^{n(k-1)} \int_{\Omega \setminus B_\epsilon(a)} \mathcal{B}_{k,a}u(x) \wedge d\phi = \int_{S_\epsilon(a)} \operatorname{tr}_{S_\epsilon(a)} \mathcal{B}_{k,a}u(x) \wedge \phi - \int_{\Omega \setminus B_\epsilon(a)} d\mathcal{B}_{k,a}u(x) \wedge \phi,$$

where $\operatorname{tr}_{S_\epsilon(a)}$ denotes the trace onto the sphere $S_\epsilon(a)$. In the limit as ϵ goes to zero, the two integrals over $\Omega \setminus B_\epsilon(a)$ in the above equation converge to the integrals of the respective integrands over $\Omega \setminus \{a\}$. To understand the derivative of $\mathcal{B}_{k,a}u$ over the domain Ω, we study the remaining surface integral. We apply several substitutions:

$$\begin{aligned}
&\int_{S_\epsilon(a)} \operatorname{tr}_{S_\epsilon(a)} \mathcal{B}_{k,a}u(x) \wedge \phi(x) \\
&= \epsilon^{n-1} \int_{S_1(a)} \operatorname{tr}_{S_1(a)} \mathcal{B}_{k,a}u(\epsilon x + a - \epsilon a) \wedge \phi(\epsilon x + a - \epsilon a) \\
&= \epsilon^{n-1} \int_{S_1(a)} \int_1^\infty \operatorname{tr}_{S_1(a)} \epsilon t^{k-1} (x-a) \lrcorner u\left(a + \epsilon t(x-a)\right) \wedge \phi(\epsilon(x-a)+a)\, dt \\
&= \epsilon^{n-1} \int_{S_1(a)} \int_\epsilon^\infty \operatorname{tr}_{S_1(a)} \epsilon^{-k+1} s^{k-1} (x-a) \lrcorner u\left(a + s(x-a)\right) \wedge \phi(\epsilon(x-a)+a)\, ds \\
&= \epsilon^{n-k} \int_{S_1(a)} \int_\epsilon^\infty \operatorname{tr}_{S_1(a)} s^{k-1} (x-a) \lrcorner u\left(a + s(x-a)\right) \wedge \phi(\epsilon(x-a)+a)\, ds.
\end{aligned}$$

We make a case distinction. When $k < n$, then the double integral itself is bounded uniformly in $\epsilon > 0$ and so the last expression vanishes as ϵ goes to zero. When instead $k = n$, then the last expression equals

$$\begin{aligned}
&\int_{S_1(a)} \int_\epsilon^\infty \operatorname{tr}_{S_1(a)} s^{n-1} (x-a) \lrcorner u\left(a + s(x-a)\right) \wedge \phi(\epsilon(x-a)+a)\, ds \\
&\quad = \int_{\mathbb{R}^n \setminus B_\epsilon(0)} u\left(y\right) \wedge \phi\left(\epsilon \frac{y-a}{\|y-a\|} + a\right)\, dy.
\end{aligned}$$

The limit of this is $\phi(a) \int_\Omega u(x)$ as ϵ goes to zero. To complete the discussion, we write the exterior derivative of u as in (41), and apply the Bogovskiĭ potential operator to this result, which gives

$$\begin{aligned}
-\mathcal{B}_{k+1,a} du(x) &= (x-a) \lrcorner \sum_{\sigma \in \Sigma(k,n)} \sum_{j=1}^{n} \int_1^\infty t^k \frac{\partial u_\sigma}{\partial x_j} (a + t(x-a)) \, dt \, dx^j \wedge dx^\sigma \\
&= \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \leq j \leq n,\, j \notin [\sigma]}} \int_1^\infty t^k \frac{\partial u_\sigma}{\partial x_j} (a + t(x-a)) \, dt \, (x-a)_j \, dx^\sigma \\
&\quad - \sum_{\substack{\sigma \in \Sigma(k,n) \\ 1 \leq j \leq n,\, j \notin [\sigma] \\ 1 \leq i \leq k}} (-1)^{i-1} \int_1^\infty t^k \frac{\partial u_\sigma}{\partial x_j} (a + t(x-a)) \, dt (x-a)_{\sigma(i)} \, dx^j \wedge dx^{\sigma - \sigma(i)}.
\end{aligned}$$

We add the (distributional) exterior derivative of the potential and the potential of the exterior derivative. In a manner that fully analogous to the discussion of the averaged Poincaré operator save for the modification when $k = n$, we come to the conclusion that

$$\begin{aligned}
u(x) &= d\mathcal{B}_{k,a} u(x) + \mathcal{B}_{k+1,a} du(x), \qquad 0 \leq k \leq n-1, \\
u(x) &= d\mathcal{B}_{n,a} u(x) - \left(\int_\Omega u(a) \right) \delta_a, \qquad k = n.
\end{aligned}$$

Here, δ_a denotes the Dirac delta at a. In summary, after taking the average over $a \in \Omega$:

$$u(x) = d\mathcal{B}_k u(x) + \mathcal{B}_{k+1} du(x), \qquad 0 \leq k \leq n-1, \tag{44}$$

$$u(x) = d\mathcal{B}_n u(x) - \left(\operatorname{vol}(\Omega)^{-1} \int_\Omega u(a) \right) \chi_\Omega, \qquad k = n. \tag{45}$$

6.6 Operator norms as bounds for the Poincaré–Friedrichs constants

We are now ready to state the main results of this section. Recall that $\delta(\Omega) > 0$ is the diameter of Ω and that $B_{\delta(\Omega)}(0)$ is the n-dimensional ball centered at the origin. The following upper bounds for the Poincaré–Friedrichs constants are proportional to the domain diameter and are independent of the Lebesgue exponent $1 \leq p \leq \infty$. However, the space dimension n and the form degree k enter the estimates, namely through definitions (39) and (40) of respectively $C_\mathcal{P}(n,k)$ and $C_\mathcal{B}(n,k)$.

Theorem 6.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex open set and let $1 \leq p \leq \infty$. We have bounded operators*

$$\mathcal{P}_k : L^p\Lambda^k(\Omega) \to \boldsymbol{W}^p\Lambda^{k-1}(\Omega), \qquad \mathcal{B}_k : L^p\Lambda^k(\Omega) \to \boldsymbol{W}^p_0\Lambda^{k-1}(\Omega).$$

They satisfy the operator norm bounds

$$\begin{aligned}
\|\mathcal{P}_k u\|_{L^p(\Omega)} &\leq C_{\mathrm{PF},\mathcal{P},k,\Omega,p} \|u\|_{L^p(\Omega)}, \\
\|\mathcal{B}_k u\|_{L^p(\Omega)} &\leq C_{\mathrm{PF},\mathcal{B},k,\Omega,p} \|u\|_{L^p(\Omega)},
\end{aligned}$$

where

$$C_{\mathrm{PF},\mathcal{P},k,\Omega,p} := C_\mathcal{P}(n,k) \operatorname{vol}_{n-1}(S_1(0)) \frac{\delta(\Omega)^n}{\operatorname{vol}(\Omega)} \delta(\Omega), \tag{46}$$

$$C_{\mathrm{PF},\mathcal{B},k,\Omega,p} := C_\mathcal{B}(n,k) \operatorname{vol}_{n-1}(S_1(0)) \frac{\delta(\Omega)^n}{\operatorname{vol}(\Omega)} \delta(\Omega). \tag{47}$$

For any $u \in \boldsymbol{W}^p\Lambda^k(\Omega)$ it holds for a.e. $x \in \Omega$ that

$$u(x) = \mathcal{P}_1 du(x) - \operatorname{vol}(\Omega)^{-1} \int_\Omega u(a) \, da \tag{48}$$

$$u(x) = d\mathcal{P}_k u(x) + \mathcal{P}_{k+1} du(x), \qquad 1 \leq k \leq n. \tag{49}$$

In particular, if $u \in W^p\Lambda^k(\Omega)$ with $0 \leq k \leq n-1$, then $w = \mathcal{P}_k du$ satisfies $dw = du$.

For any $u \in \boldsymbol{W}_0^p\Lambda^k(\Omega)$ it holds for a.e. $x \in \Omega$ that

$$u(x) = d\mathcal{B}_k u(x) + \mathcal{B}_{k+1} du(x), \qquad 0 \leq k \leq n-1, \tag{50}$$

$$u(x) = d\mathcal{B}_n u(x) - \left(\operatorname{vol}(\Omega)^{-1} \int_\Omega u(a) \right) \chi_\Omega. \tag{51}$$

In particular, if $u \in W_0^p\Lambda^k(\Omega)$ with $0 \leq k \leq n-1$, then $w = \mathcal{B}_k du$ satisfies $dw = du$.

Proof. Consider the case $1 \leq p < \infty$. Because the domain is bounded, the subspaces $C_c^\infty\Lambda^k(\Omega)$ are dense in $L^p\Lambda^k(\Omega)$, and so the stated operator norm bounds follow by an approximation argument. Taking the limit on both sides of equation then implies the inequality with $p = \infty$ because the L^∞ norm is the limit of the Lebesgue norms as p goes to infinity.

Consider now $u \in W^p\Lambda^k(\Omega)$ with $1 \leq k \leq n$. We write $\zeta := \mathcal{P}_k u$. There exists a sequence $u_i \in C^\infty\Lambda^k(\overline{\Omega})$ that converges to u in $W^p\Lambda^k(\Omega)$. For any test form $v \in C_c^\infty\Lambda^{n-k-1}(\Omega)$, we verify

$$\begin{aligned}\int_\Omega v \wedge u_i &= \int_\Omega v \wedge \mathcal{P}_{k+1} du_i + \int_\Omega v \wedge d\mathcal{P}_k u_i \\ &= \int_\Omega v \wedge \mathcal{P}_{k+1} du_i + (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathcal{P}_k u_i.\end{aligned}$$

By the continuity of bounded linear functionals, we find

$$\int_\Omega v \wedge u - \int_\Omega v \wedge \mathcal{P}_{k+1} du = (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathcal{P}_k u.$$

Hence, by definition, $\zeta \in W^p\Lambda^{k-1}(\Omega)$ with $d\zeta = u - \mathcal{P}_{k+1} du$. This shows (49), and (48) follows by an approximation argument.

Analogously, suppose that $u \in W_0^p\Lambda^k(\Omega)$ with $0 \leq k \leq n-1$. We write $\zeta := \mathcal{B}_k u$. There exists a sequence $u_i \in C_c^\infty\Lambda^k(\overline{\Omega})$ that converges to u in $W^p\Lambda^k(\Omega)$. For any test form $v \in C^\infty\Lambda^{n-k-1}(\Omega)$, which is the restriction of some member of $C^\infty\Lambda^{n-k-1}(\mathbb{R}^n)$, it holds that

$$\begin{aligned}\int_\Omega v \wedge u_i &= \int_\Omega v \wedge \mathcal{B}_{k+1} du_i + \int_\Omega v \wedge d\mathcal{B}_k u_i \\ &= \int_\Omega v \wedge \mathcal{B}_{k+1} du_i + (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathcal{B}_k u_i.\end{aligned}$$

By the continuity of bounded linear functionals, we find

$$\int_\Omega v \wedge u - \int_\Omega v \wedge \mathcal{B}_{k+1} du = (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathcal{B}_k u.$$

By definition, $\zeta \in W_0^p\Lambda^{k-1}(\Omega)$ with $d\zeta = u - \mathcal{B}_{k+1} du$. This shows (50), and (51) follows by an approximation argument.

Everything else is now apparent, and the proof is complete. □

These constants are generally not optimal. For example, when $p = 2$ and when only the divergence is considered, we have the following improved estimate.

Lemma 6.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. For each $\boldsymbol{u} \in \boldsymbol{W}^2(\operatorname{div}, \Omega)$ there exists $\boldsymbol{w} \in \boldsymbol{W}^2(\operatorname{div}, \Omega)$ with $\operatorname{div} \boldsymbol{w} = \operatorname{div} \boldsymbol{u}$ and*

$$\|\boldsymbol{w}\|_{L^2(\Omega)} \leq \delta(\Omega) \|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega)}.$$

Proof. This is a reduction to the Friedrichs inequality. The space $W_0^{1,2}(\Omega)$ is the closure of the smooth functions with support in Ω in the Hilbert space $W^{1,2}(\Omega)$. Then $\nabla : W_0^{1,2}(\Omega) \subseteq L^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is a closed densely-defined linear operator whose smallest singular value is bounded from below by $\delta(\Omega)^{-1}$, according to the Friedrichs inequality. The adjoint is the closed densely-defined linear operator $-\operatorname{div} : \boldsymbol{W}^2(\operatorname{div}, \Omega) \subseteq \boldsymbol{L}^2(\Omega) \to L^2(\Omega)$, which has the same smallest singular value. □

Remark 6.4. *The classical Poincaré operator is known for its role in proving the exactness of the smooth de Rham complex over star-shaped domains [44]. The Bogovskiĭ-type operators were first studied for the divergence operator and are a staple in the mathematics of hydrodynamics [9]. Costabel and McIntosh [19] regularize the potentials by averaging over pivot points within an interior ball using a smooth compactly supported weight function, which is why they can study domains star-shaped with respect to a ball. Their operators are pseudo-differential operators of negative order, because their averaging uses a smooth weight; this proves that the operators are bounded between a variety of function spaces. Explicit bounds for the higher-order seminorms of these pseudo-differential operators have been recently contributed by Guzman and Salgado [37]. Instead, we average over the entire domain, which requires a convex geometry, and we are interested in boundedness in the Lebesgue p-norms. We establish computable bounds on the Poincaré–Friedrichs constants, which had not been established yet, to the best of our knowledge.*

7 Shellable triangulations of manifolds

We return to the theory of triangulations, as our main objective requires some further concepts. We are interested in simplicial complexes that triangulate domains and which are *shellable*. Such simplicial complexes are constructed by successively adding simplices in a well-structured manner. Local patches (stars) within triangulations of dimension two or three are examples of such shellable complexes. The monographs by Kozlov [40] and Ziegler [66] are our main references for this section. We also refer to Lee's monograph [43] for any further background on manifolds.

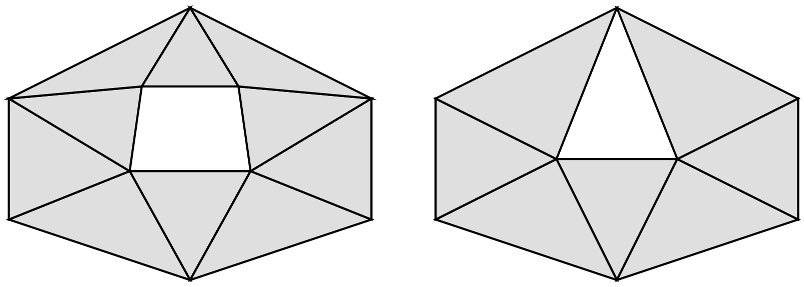

Figure 3: Left: manifold triangulation of an annulus. Right: not a manifold triangulation.

7.1 Triangulations of manifolds

Our discussion requires some notions and results concerning triangulated manifolds. We define an n-dimensional simplicial complex to be a *manifold triangulation* if the underlying set $|\mathcal{T}|$ is an n-dimensional manifold with boundary. We recall that this means that for every $x \in |\mathcal{T}|$ there exists an open neighborhood $U(x) \subseteq |\mathcal{T}|$ and an embedding $\phi : U(x) \to \mathbb{R}^n$ such that $\phi(0) = 0$ and ϕ is an isomorphism either onto the open unit ball $\mathcal{B} = \{x \in \mathbb{R}^n \mid |x| < 1\}$ or onto the half-ball $\{x \in \mathcal{B} \mid x_1 \geq 0\}$. In the former case, x is called an *interior point*, and in the latter case x is called a *boundary point*. Any simplicial complex that triangulates an n-dimensional manifold must be n-dimensional. An example of a manifold triangulation and an example which is not a manifold triangulation are given in Figure 3.

The following special cases receive particular interest: an *n-ball triangulation* is any triangulation of a topological (closed) n-ball, and we sometimes call this an *n-disk* triangulation. An *n-sphere triangulation* is any triangulation of a topological n-sphere.

We know that any manifold $\mathcal{M}$ has got a topological boundary $\partial\mathcal{M}$, possibly empty. If $\mathcal{M}$ is n-dimensional, then the $\partial\mathcal{M}$ is a topological manifold without boundary of dimension $n-1$. We gather a few helpful observations on how these notions relate to triangulations. While the reader might deem them obvious, we nevertheless include proofs.

Lemma 7.1. *Let $\mathcal{T}$ be a finite n-dimensional simplicial complex whose underlying set is a manifold $\mathcal{M}$. Then the simplices contained in the boundary constitute a triangulation of the boundary. Moreover, if $F \in \mathcal{T}$ is a face (i.e, F has dimension $n-1$), then*

- *F is not contained in the boundary if and only if it is contained in exactly two n-simplices.*

- *F is contained in the boundary if and only if it is contained in exactly one n-simplex.*

Proof. We prove these statements in several steps.

1. Let $\mathring{\mathcal{M}} := \mathcal{M} \setminus \partial\mathcal{M}$ denote the interior of the manifold. We will use the following fact:[4] if $S \in \mathcal{T}$ has an inner point that lies on $\partial\mathcal{M}$, then all inner points of S are on $\partial\mathcal{M}$. Since the boundary $\partial\mathcal{M}$ is closed, every $S \in \mathcal{T}$ is either a subset of the boundary or all its inner points lie in the interior $\mathring{\mathcal{M}}$ of the manifold.

2. We recall an auxiliary result. Suppose that Y is a topological space homeomorphic to a sphere of dimension m and that $X \subseteq Y$ is homeomorphic to a sphere of dimension $m-1$, where $m \geq 1$. As a consequence of the Jordan–Brouwer separation theorem [53, Corollary IV.5.24] [51, Corollary VIII.6.4], we know that $Y \setminus X$ has got two connected components.

3. Let now $F \in \mathcal{S}^{\downarrow}_{n-1}(T)$ be a face and let $z_F \in F$ be its barycenter. Since $\mathcal{T}$ is finite, we let $\mathring{B}_F$ be an open neighborhood around z_F homeomorphic to an n-dimensional ball so small that its closure $\overline{B_F}$ only intersects those n-simplices of $\mathcal{T}$ that already contain z_F and no faces other than F. Suppose there are distinct n-simplices $T_1, T_2, \dots, T_K$ that contain z_F. The intersection of any two of them is F, but their interiors are disjoint because otherwise they would coincide.

   If z_F is an interior point of $\mathcal{M}$, then it follows by our assumptions that $\mathring{B}_F$ is homeomorphic to an open n-ball and $\partial\mathring{B}_F$ is homeomorphic to a sphere of dimension $n-1$. Consider $X = F \cap \partial\mathring{B}_F$. If $n = 1$, then X is empty and $\partial\mathring{B}_F$ has K distinct connected components. If $n > 1$, then X is homeomorphic to a sphere of dimension $n-2$ and again $\partial\mathring{B}_F \setminus X$ has K distinct connected components. But by the auxiliary result above, $K = 2$. We conclude that F is contained in two n-simplices of $\mathcal{T}$.

   Consider the case that z_F lies on the boundary of $\mathcal{M}$ and suppose that F is contained in K distinct n-simplices of $\mathcal{T}$. By adding at least one dimension, we can double[5] the manifold $\mathcal{M}$ along the boundary and obtain the doubled manifold $\mathcal{M}'$. Similarly, we can construct a doubling of the triangulation $\mathcal{T}'$ such that F is contained in exactly $2K$ distinct n-simplices of $\mathcal{T}'$. We know that $\mathcal{M}'$ is a manifold without boundary, and hence F is an interior simplex of $\mathcal{T}'$. This implies $K = 1$. So any boundary face can only be contained in one single n-simplex.

4. Clearly, the simplices of $\mathcal{T}$ contained in the boundary constitute a simplicial complex. Every $x \in \mathcal{M}$ is an inner point of some simplex $S \in \mathcal{T}$. If $x \in \partial\mathcal{M}$ is a boundary point, then S must be a boundary simplex, so the boundary simplices triangulate all of $\partial\mathcal{M}$.

All desired results are thus proven. $\square$

Suppose that $\mathcal{T}$ is an n-dimensional simplicial complex that triangulates a manifold. Those simplices of the manifold triangulation that are subsets of the boundary of the underlying manifold are called *boundary simplices*. All other simplices of the manifold triangulation are called *inner simplices*. We have seen that the boundary simplices of a manifold triangulation constitute a triangulation of the manifold's boundary. We call this simplicial complex the *boundary complex*. It has dimension $n-1$.

We continue with a few more observations about the topology of local patches (stars) of manifold triangulations. This topic is surprisingly non-trivial. We only gather some results that are hard to find in the literature.

Lemma 7.2. *Let $\mathcal{T}$ be a finite n-dimensional simplicial complex whose underlying space is a manifold $\mathcal{M}$. Suppose that $1 \leq n \leq 3$. Then the following holds:*

- *If $S \in \mathcal{T}$ is an inner simplex, then $\mathrm{st}_{\mathcal{T}}(S)$ is a simplicial n-ball with S as an inner simplex and $\partial\mathrm{st}_{\mathcal{T}}(S)$ is a simplicial $(n-1)$-sphere.*

- *If $S \in \mathcal{T}$ is a boundary subsimplex, then $\mathrm{st}_{\mathcal{T}}(S)$ is a simplicial n-ball with S as a boundary simplex, and $\partial\mathrm{st}_{\mathcal{T}}(S)$ is a simplicial $(n-1)$-ball.*

[4] To see this, one easily constructs a continuous deformation of $\mathcal{M}$ into itself to move any chosen point on S to any other chosen point on S.

[5] The reader is referred to Lee's textbook [44] for more background and the technicalities.

Proof. The lemma is obvious if $n = 1$, so we assume $n \geq 2$ in what follows. We prove these statements in several steps. The reader is assumed to have some background in topology.

1. Let S be any simplex with vertices $v_0, v_1, \dots, v_k$, with barycenter z_S, and dimension k. Let $\mathcal{S} := \mathrm{st}_{\mathcal{T}}(S)$ be its star. Each l-dimensional simplex $T \in \mathcal{S}$ that contains S has vertices v_0, v_1, $\dots$, v_k, v^S_{k+1}, $\dots$, v^S_l. For any such simplex, we introduce a decomposition $T_0, \dots, T_k$, where each T_i has vertices v_0, $\dots$, v_{i-1}, z_S, v_{i+1}, $\dots$, v_k, v^S_{k+1}, $\dots$, v^S_l. The collection $\mathcal{S}^*$ of these simplices and their subsimplices constitute a simplicial complex that triangulates the same underlying set as $\mathcal{S}$. Moreover, $\mathcal{S}^* = \mathrm{st}_{\mathcal{S}^*}(z_S)$. In particular, z_S is a boundary vertex of $\mathcal{S}^*$ if and only if S is a boundary simplex of $\mathcal{S}$. So it remains to study the topology of vertex stars.

2. Suppose that $2 \leq n \leq 3$ and that $\mathcal{M}$ is a manifold without boundary. Under these assumptions, as explained in the proof of Theorem 1 in [58], the set $\partial\mathrm{st}(V)$ is a triangulation of a sphere of dimension $n-1$ for any inner vertex V. There exists a homeomorphism from the closed cone of $|\partial\mathrm{st}(V)|$ onto the local star $|\mathrm{st}_{\mathcal{T}}(V)|$. But then that closed cone and hence $|\mathrm{st}_{\mathcal{T}}(V)|$ are homeomorphic to an n-dimensional ball.

3. If $2 \leq n \leq 3$ and $\mathcal{M}$ has a non-empty boundary, then we use an approach as in the proof of Lemma 7.1: we let $\mathcal{M}'$ denote the doubling of the manifold and $\mathcal{T}'$ be the doubling of the triangulation $\mathcal{T}$. Let $V \in \mathcal{T}$ be a vertex. If V is an inner vertex of $\mathcal{T}$, then $\partial\mathrm{st}(V) \subseteq \mathcal{T} \subseteq \mathcal{T}'$ triangulates a sphere of dimension $n-1$ and $\mathrm{st}_{\mathcal{T}}(V) \subseteq \mathcal{T} \subseteq \mathcal{T}'$ triangulates a ball of dimension n, as discussed above. If V is a boundary vertex of $\mathcal{T}$, then it is an inner vertex of $\mathcal{T}'$, and so $\partial\mathrm{st}_{\mathcal{T}}(V) \subseteq \mathcal{T}'$ triangulates a sphere of dimension $n-1$ and $\mathrm{st}_{\mathcal{T}}(V) \subseteq \mathcal{T}'$ triangulates a ball of dimension n. We also know that $\partial\mathrm{st}_{\partial\mathcal{T}}(V) \subseteq \partial\mathcal{T}$ triangulates a sphere of dimension $n-2$ and $\mathrm{st}_{\partial\mathcal{T}}(V) \subseteq \partial\mathcal{T}$ triangulates a ball of dimension $n-1$. The embedding of $\partial\mathrm{st}_{\partial\mathcal{T}}(V) \subseteq \partial\mathcal{T}$ is homeomorphic to the standard embedding of the $(n-2)$-dimensional unit sphere into the $(n-1)$-dimensional unit sphere, by the topological Schoenflies theorem [54] It follows that $\partial\mathrm{st}_{\mathcal{T}}(V)$ triangulates a topological ball of dimension $n-1$. Since the closed cone of $|\partial\mathrm{st}(V)|$ is homeomorphic to the star $|\mathrm{st}_{\mathcal{T}}(V)|$, we conclude that $\mathrm{st}_{\mathcal{T}}(V)$ triangulates an n-dimensional ball.

All relevant results are proven. □

Lemma 7.3. *Let $\mathcal{T}$ be a finite n-dimensional simplicial complex whose underlying space is a manifold $\mathcal{M}$. If the underlying space of $\mathcal{T}$ is connected, then $\mathcal{T}$ is face-connected.*

Proof. We first show that each vertex star is face-connected via a short induction argument. Clearly, any simplicial 1-ball and simplicial 1-sphere are face-connected. Now, if $n \geq 1$ and $V \in \mathcal{T}$, then the n-simplices in $\mathrm{st}_{\mathcal{T}}(V)$ are in correspondence to the $(n-1)$-simplices in $\partial\mathrm{st}_{\mathcal{T}}(V)$. Hence, $\mathrm{st}_{\mathcal{T}}(V)$, a triangulation of dimension $n-1$, is face-connected if and only if $\partial\mathrm{st}_{\mathcal{T}}(V)$, a triangulation of dimension $n-1$, is face-connected. The induction argument implies that each vertex star in $\mathcal{T}$ is face-connected.

We assume that the underlying space $|\mathcal{T}|$ is connected, and hence path-connected. Given n-simplices $S, S' \in \mathcal{T}$, there exists a path $\gamma : [0,1] \to |\mathcal{T}|$ from the barycenter of S to the barycenter of S'. We can choose a sequence of n-simplices $S = \hat{S}_0, \hat{S}_1, \dots, \hat{S}_m = S' \in \mathcal{T}$ without repetitions that cover the path and whose intersections with the path are homeomorphic to $[0,1]$. For any index $1 \leq i \leq m$, the intersection $\gamma([0,1]) \cap S_i$ and $\gamma([0,1]) \cap S_{i+1}$ intersect at one point lying in a common subsimplex of S_i and S_{i+1}. We deduce that each two consecutive simplices in the sequence $\hat{S}_0, \hat{S}_1, \dots, \hat{S}_m$ will have at least one vertex in common. As each vertex star is face-connected, there exists a sequence $S = S_0, S_1, \dots, S_m = S' \in \mathcal{T}$ where $S_i \cap S_{i-1}$ is a face of both S_i and S_{i-1} for all $1 \leq i \leq m$. This just means that $\mathcal{T}$ is face-connected. □

Remark 7.4. *Triangulations with the property that all vertex stars are homeomorphic to a ball are also called* combinatorial *[7, Section 1]. All manifolds of dimension up to three admit smooth structures and smooth manifolds admit combinatorial triangulations. There are triangulations of manifolds in more than three dimensions where not every vertex star is homeomorphic to a ball.*

Not every simplicial complex is the triangulation of some (embedded) topological manifold with or without boundary. When the dimension is at least five, then there are manifolds for which no computer algorithm, given a finite simplicial complex as input, decides whether the input is the triangulation of that manifold [17]. Going further, it has been shown that deciding whether a simplicial complex triangulates *any manifold cannot be decided by any computer algorithm [56]. We therefore are not in pursuit of any easy combinatorial property that indicates whether a simplicial complex (without any further specific assumptions) triangulates a manifold.*

Conversely, not all topological manifolds, even if compact, can be described as a triangulation. Such manifolds, some even compact and simply-connected, appear in dimension four and higher [2].

7.2 Shellable simplicial complexes

The notions of shelling and shellable triangulation have been discussed widely in combinatorial topology and polytopal theory. Formally, a triangulation is shellable if its full-dimensional simplices can be enumerated such that each simplex intersects the union of the previously listed simplices in a codimension one triangulation of a manifold. This forces the intermediate triangulations to be particularly well-shaped. We build upon the notion of shelling as introduced in [66, Definition 8.1], where our definition of shelling is equivalent to the notion of the shellings of simplicial complexes, see also [66, Remark 8.3].

Suppose that $\mathcal{T}$ is an n-dimensional simplicial complex and that we have an enumeration of the n-simplices $T_0, T_1, T_2, \dots \in \mathcal{S}_n^{\downarrow}(\mathcal{T})$. For any enumeration, we call

$$\Gamma_m := \big(T_0 \cup T_1 \cup \dots \cup T_m\big) \cap T_{m+1}$$

the m-th *interface set*. We call the enumeration a *shelling* if each interface set Γ_m is a triangulated manifold of dimension $n-1$.

The reason of our interest in shellable simplicial complexes is that they can be constructed via successive adhesion of simplices. The resulting succession of simplicial complexes consists of simplicial balls or spheres.

Lemma 7.5. *Let $\mathcal{T}$ be an n-dimensional simplicial complex with a shelling $T_0, T_1, T_2, \dots, T_M$, such that each simplex of dimension $n-1$ is contained in at most two simplices. Then*

$$X_m := T_0 \cup T_1 \cup \dots \cup T_m$$

is a triangulated manifold with boundary for all $0 \leq m \leq M$. In particular, X_m is a topological n-ball when $m < M$, and X_M is either a topological n-ball or topological n-sphere.

Proof. We prove this claim by induction. Certainly, if $\mathcal{T}$ contains only one single n-simplex, then it is a shellable triangulation of a topological n-ball. Next, let $1 \leq m \leq M$ and suppose that

$$X_{m-1} := T_0 \cup T_1 \cup \dots \cup T_{m-1}$$

is already known to be a topological n-ball. Let T_m be the next n-simplex in the shelling. By definition, $\Gamma_m := X_{m-1} \cap T_m$ is a submanifold of ∂T_m, triangulated by some faces of T_m and their subsimplices.

Let F be such a face. By assumption, F must be contained in exactly one n-simplex of $T_0, T_1, \dots, T_{m-1}$, and F is in the boundary of X_{m-1}. We conclude that Γ_m triangulates a submanifold of the boundary of X_{m-1}.

On the one hand, if Γ_m is the entire boundary of T_m, then it is a topological sphere of dimension $n-1$. Since Γ_m is a submanifold of the boundary of X_{m-1}, which by induction assumption is also a topological sphere of dimension $n-1$, we conclude that Γ_m is the whole boundary of X_m. By basic geometric topology, X_m is a topological n-sphere, and T_m can only be the last simplex in the shelling, $m = M$. On the other hand, if Γ_m is a proper subset of the entire boundary of T_m, then $X_{m-1} \cup T_m$ is still a topological n-ball. $\square$

Remark 7.6. *We interpret a shelling as the construction of a triangulation by successively attaching simplices such that the intermediate triangulations are well-behaved. Conversely, the reverse enumeration describes a successive decomposition of the triangulation, hence the name "shelling".*

Remark 7.7. *Whether a simplicial complex is shellable can be checked, in principle, simply by trying out all the possible enumerations. That we cannot do much better than this is captured in the result that testing for shellability is NP-complete [32]: this complexity result is even true if we merely consider simplicial complexes of dimension two embedded in some Euclidean space.*

We now collect important examples of shellable triangulations. Essentially, in two space dimensions, interesting triangulations are shellable, but starting from three space dimensions, non-shellable situations can arise. Our main interest are local patches (stars) within triangulations: these are shellable up to three space dimensions, but not necessarily beyond.

Example 7.8. *Any simplex T (trivially) has a shelling, consisting only of T itself. The boundary complex $\partial\mathcal{T}(T)$ has a shelling: any enumeration of the boundary faces of T constitutes a shelling; see Example 8.2.(iii) in [66].*

Example 7.9. *The standard triangulation of the 3-dimensional cube by six tetrahedra, the Kuhn triangulation [41], is shellable, as are its higher-dimensional generalizations.*[6]

Example 7.10. *There exists a non-shellable triangulation of a tetrahedron and of a cube in $n = 3$, see [66, Example 8.9].*

Lemma 7.11. *Any simplicial 2-ball is shellable. Any simplicial 2-sphere is shellable.*

Proof. First, let $\mathcal{S}$ be any triangulation of a 2-sphere. By removing any triangle $S \in \mathcal{S}$, we obtain a triangulation $\mathcal{T}$ of a 2-ball. Any shelling of that triangulation $\mathcal{T}$ can be extended to a shelling of $\mathcal{S}$ by re-inserting the first triangle S. So it remains to show that any triangulation $\mathcal{T}$ of a two-dimensional ball is shellable. We will construct the shelling in reverse.

Write $M = |\mathcal{T}|$. There is nothing to show if $\mathcal{T}$ contains only one triangle. We call a triangle $T \in \mathcal{T}$ *good in* $\mathcal{T}$ if it intersects the boundary ∂M in a non-empty union of edges. Hence, a triangle is good in $\mathcal{T}$ if its intersection with ∂M is either one, two, or three edges, and a triangle is not good in $\mathcal{T}$ if that intersection is either empty, only some of its vertices, or a vertex and the opposite edge. We show by an induction argument over the number of triangles that every triangulation of a 2-ball that contains at least two triangles also contains at least two good triangles.

Clearly, this is the case if the triangulation of the 2-ball contains two triangles. Now suppose the induction claim is true when the triangulation includes at most N triangles, and assume that $\mathcal{T}$ includes $N+1$ triangles. As we travel along the boundary, we traverse along edges of at least two simplices, and therefore there are at least two triangles with an edge on the boundary. Suppose that $\mathcal{T}$ does not have at least two triangles that are good in $\mathcal{T}$. Then there exists a triangle T' that intersects ∂M in one edge and its opposite vertex. Removing T' splits the manifold into two face-connected components, each of which is a topological 2-ball. By the induction assumption, each of those components contains at least two triangles that are good in the respective component. So each component has at least one triangle that is also good in $\mathcal{T}$. Hence, $\mathcal{T}$ contains two good triangles, which completes the induction step.

We conclude that whenever $\mathcal{T}$ triangulates a 2-ball, it contains a good triangle T. If T has three edges in the boundary, then $T = M$ and we are trivially done. If T intersects with the boundary in exactly one or two edges, then $\overline{M \setminus T}$ is still a topological 2-ball. The triangulation $\mathcal{T}'$ that is obtained by removing T is a triangulation of some 2-ball that intersects T only at either two or one edges. Any shelling of $\mathcal{T}'$ can in this way be extended to a shelling of $\mathcal{T}$, and the proof is complete. □

Lemma 7.12. *Let $\mathcal{T}$ be a 3-dimensional manifold triangulation and $S \in \mathcal{T}$. Then $\mathrm{st}_{\mathcal{T}}(S)$ is shellable.*

Proof. The statement is trivially true if S is a tetrahedron. The statement is clear if S is an inner or boundary face of $\mathcal{T}$, where we only need to enumerate either one or two tetrahedra. The statement is still easily verified if S is an inner or boundary edge of $\mathcal{T}$: one chooses a starting tetrahedron (with a boundary face if S is a boundary edge) and rotates around the edge in a fixed direction to create a suitable enumeration. When S is an inner vertex, then the faces (triangles) of $\mathrm{st}_{\mathcal{T}}(S)$ that do not contain V constitute a simplicial 2-sphere. Similarly, when S is a boundary vertex, then the faces (triangles) of $\mathrm{st}_{\mathcal{T}}(S)$ that do not contain V constitute a simplicial 2-ball. Both these 2-dimensional complexes are shellable by Lemma 7.11, and any such shelling immediately yields a shelling of $\mathrm{st}_{\mathcal{T}}(S)$ since there is a one-to-one correspondence between the tetrahedra in $\mathcal{T}$ and the triangles. [7] □

Lemma 7.13. *Let $\mathcal{T}$ be an n-dimensional shellable triangulation and $V \in \mathcal{V}(\mathcal{T})$ be a vertex. Then $\mathrm{st}_{\mathcal{T}}(V)$ is shellable.*

[6] We remark that Kuhn attributes this triangulation to Lefschetz [45].

[7] For S an inner vertex, [22, Lemma B.1] also yields the claim.

Proof. This is Lemma 8.7 in [66]. $\square$

Remark 7.14. *Not all triangulable sets admit a triangulation that is shellable. Moreover, even if a set admits a shellable triangulation, not all of its triangulations are shellable. For example, if we extend the non-shellable triangulation of the tetrahedron from [66, Example 8.9] to a triangulation of a hypertetrahedron by suspending it from a new point $v_\star$, then the resulting new triangulation is non-shellable and coincides with the patch around $v_\star$. This demonstrates that patches around boundary simplices are not necessarily shellable when the space dimension n is larger than three.*

Remark 7.15. *Not all triangulable sets admit a triangulation that is also shellable. Moreover, even if a set admits a shellable triangulation, not all of its triangulations might be shellable. We refer to [66, Example 8.9] for an example of non-shellable triangulations of cubes and tetrahedra in three dimensions.*

A major structural feature of shellable simplicial complexes is that each time an n-simplex is added, stars around lower-dimensional simplices gets completed.

Lemma 7.16. *Suppose that an n-dimensional manifold triangulation $\mathcal{T}$ has a shelling $T_0, T_1, T_2, \dots, T_M$. For $0 \leq m \leq M$, write*

$$X_m := T_0 \cup T_1 \cup \dots \cup T_m.$$

For $1 \leq m \leq M$, write

$$\Gamma_m := X_{m-1} \cap T_m.$$

Then Γ_m is a union of ℓ different faces of T_m, $1 \leq \ell \leq n+1$. If $m < M$, then the intersection of those faces is an interior simplex $S_m \in \mathcal{T}$ of dimension $n - \ell$ that satisfies

$$\operatorname{st}_{X_m}(S_m) = \operatorname{st}_{\mathcal{T}}(S_m).$$

Proof. We know Γ_m is a triangulated submanifold of the boundary of T_m, and so it must be a collection of ℓ faces of T, $1 \leq \ell \leq n+1$. Note that $\ell = n+1$ can only happen for the last enumerated simplex, $m = M$, if $\mathcal{T}$ triangulates an n-sphere. Γ_m also constitutes a local patch (star) of $(n-1)$-dimensional simplices around some simplex S_m of dimension $n - \ell$ in Γ_m. By definition, S_m is a boundary simplex of X_m, and it is an interior simplex of X_{m+1}. But then S_m cannot be a subsimplex of any of the simplices $T_{m+1}, \dots, T_M$, which means that $\operatorname{st}_{X_{m+1}}(S_m) = \operatorname{st}_{\mathcal{T}}(S_m)$. $\square$

8 Reflections and Deformations on shellable stars

This section is devoted to geometric operations that are crucial for our main result in Section 9 below. Consider the situation where we have an n-dimensional local patch (star) around some simplex S and some n-dimensional simplex T within that local star. We construct a homeomorphism going from the simplex T onto its complement within the local star around S, similar to the two- and three-color maps in [22, Sections 5.3 and 6.3] and the symmetrization maps in [16, Section 7.6] This homeomorphism, which we interpret as a nonlinear reflection, is required to preserve the interface. We ensure that the homeomorphism is bi-Lipschitz, and we are particularly interested in the norms of its Jacobian. This nonlinear reflection will be used subsequently in generalizing the discussion in Section 4 to the setting of differential forms. Additionally, this endeavor produces another geometric tool: we construct a bi-Lipschitz deformation which contracts the entire star into the complement of the newly completed star. This deformation will enable additional estimates of Poincaré–Friedrichs constants.

We will use the following observation, which we state without proof, that controls the volume and some heights when a simplex is partitioned via barycentric subdivision.

Lemma 8.1. *Let T be an n-dimensional simplex with an ℓ-dimensional subsimplex S and let z_S be the barycenter of S. Let T' be one of the n-dimensional simplices obtained by splitting T in accordance with the barycentric subdivision of S at z_S.*

- $\operatorname{vol}(T') = \operatorname{vol}(T)/(\ell + 1)$.
- *The height vector of $v \in \mathcal{V}(T) \setminus \mathcal{V}(S)$ in T' is the height vector v in T.*

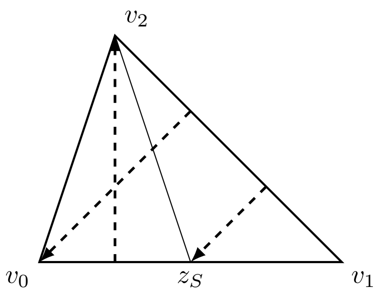

Figure 4: Illustration of Lemma 8.1. Left: the triangle $T = [v_0, v_1, v_2]$ is bisected at the edge $S = [v_0, v_1]$, leading to two new triangles. The height vector to v_2 in all three triangles remains the same. The height vector to z_S in the new triangle $[z_S, v_1, v_2]$ is one half of the height vector to v_0 in the original triangle. Right: the triangle T is trisected, leading to three new triangles. The height vector to z_T from the opposite edge in any triangle is one third of the original height vector of that edge.

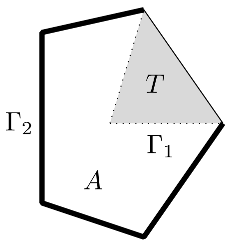

Figure 5: Sketch of the geometric situation in the proof of Proposition 8.2. The triangle T completes the star around an interior vertex. A is the complement of the triangle within the star. Γ_1 is the interface between the two sets, and left invariant by the reflection. Γ_2 is left invariant by the deformation. Ξ_1 maps T into A while preserving the interface Γ_1, and Ξ_2 maps the whole patch into A while preserving the boundary Γ_2.

- *The height vector of z_S in T' is the height vector of the single vertex $v \in \mathcal{V}(T) \setminus \mathcal{V}(T')$ in T, scaled by $(\ell+1)^{-1}$.* □

We now provide the desired bi-Lipschitz transformation: on the one hand, the nonlinear reflection across the interface between the selected simplex and the remainder of the local star, and on the other hand, the bi-Lipschitz contraction from the local star onto the complement of the selected simplex. We give detailed estimates for the singular values of their Jacobians.

Proposition 8.2. *Let $\mathcal{T}$ be a triangulation of an n-dimensional domain. Let $S \in \mathcal{T}$ be an inner simplex of dimension $\ell < n$, let $T \in \operatorname{st}_{\mathcal{T}}(S)$ be of dimension n, and let*

$$A := \overline{|\operatorname{st}_{\mathcal{T}}(S)| \setminus T}, \qquad \Gamma_1 := A \cap T, \qquad \Gamma_2 := \overline{\partial T \setminus \partial A}.$$

The following holds, where the constants on the right-hand sides are as stated in the proof.

1. *There exists a bi-Lipschitz piecewise affine mapping*

$$\Xi_1 : T \to \Xi_1(T) \subseteq A$$

which is the identity along $\Gamma_1 = T \cap A$. At any point, the singular values $\sigma_1 \geq \cdots \geq \sigma_n$ of its Jacobian satisfy

$$\sigma_1 \leq C_{5,n,\ell}(\mathcal{T}), \quad \sigma_2, \ldots, \sigma_n \leq C'_{5,n,\ell}(\mathcal{T}), \quad |\det \mathbf{J}\Xi_1| \leq C^{\det}_{5,n,\ell}(\mathcal{T}),$$
$$\sigma_n^{-1} \leq C_{6,n,\ell}(\mathcal{T}), \quad \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1} \leq C'_{6,n,\ell}(\mathcal{T}), \quad |\det \mathbf{J}\Xi_1^{-1}| \leq C^{\det}_{6,n,\ell}(\mathcal{T}).$$

Moreover, for any $0 \leq k \leq n$ and $p \in [1, \infty]$,

$$\sigma_1 \cdots \sigma_k |\det \mathbf{J}\Xi_1|^{-\frac{1}{p}} \leq C_{5,n,k,\ell,p}(\mathcal{T}), \qquad \sigma_n^{-1} \cdots \sigma_{n-k+1}^{-1} |\det \mathbf{J}\Xi_1|^{\frac{1}{p}} \leq C_{6,n,k,\ell,p}(\mathcal{T}).$$

2. There exists a bi-Lipschitz piecewise affine mapping

$$\Xi_2 : |\mathrm{st}_{\mathcal{T}}(S)| \to \Xi_2(|\mathrm{st}_{\mathcal{T}}(S)|) \subseteq A$$

which is the identity along $\partial A \setminus \partial T$. At any point, the singular values $\sigma_1 \geq \cdots \geq \sigma_n$ of its Jacobian satisfy

$$\sigma_1 \leq C_{7,n,\ell}(\mathcal{T}), \quad \sigma_2, \ldots, \sigma_n \leq C'_{7,n,\ell}(\mathcal{T}), \quad |\det \mathbf{J}\Xi_2| \leq C^{\det}_{7,n,\ell}(\mathcal{T}),$$
$$\sigma_n^{-1} \leq C_{8,n,\ell}(\mathcal{T}), \quad \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1} \leq C'_{8,n,\ell}(\mathcal{T}), \quad |\det \mathbf{J}\Xi_2^{-1}| \leq C^{\det}_{8,n,\ell}(\mathcal{T}).$$

Moreover, for any $0 \leq k \leq n$ and $p \in [1, \infty]$,

$$\sigma_1 \cdots \sigma_k |\det \mathbf{J}\Xi_2|^{-\frac{1}{p}} \leq C_{7,n,k,\ell,p}(\mathcal{T}), \qquad \sigma_n^{-1} \cdots \sigma_{n-k+1}^{-1} |\det \mathbf{J}\Xi_2|^{\frac{1}{p}} \leq C_{8,n,k,\ell,p}(\mathcal{T}).$$

Proof. We derive the estimate in several steps. In what follows, we use the notation $\hat{z}$ for the normalization of any vector $z \in \mathbb{R}^n$.

- Without loss of generality, after shifting the coordinate system, the barycenter z_S of S is the origin.

  We fix the subsimplex $S' \subseteq T$ that is complementary to S within the simplex T. Note that $\Gamma_2 \subseteq \partial T$ is the union of exactly those faces of T that contain S', whereas $\Gamma_1 = A \cap T$ is the union of exactly those faces of T that contain S. We let $z_{S'}$ be the midpoint of S'.

  We apply barycentric refinement to the simplex S and then the complementary simplex S'. This produces a triangulation $\mathcal{G}$ of $|T|$ whose n-dimensional simplices contain the vertices $z_S = 0$ and $z_{S'}$.

- There exists $\varrho > 0$ such that $\mathrm{st}_{\mathcal{T}}(S)$ is star-shaped with respect to the closed ball $\overline{\mathcal{B}_\varrho(z_S)}$. We show that, if we let $\mathcal{K}$ denote the simplicial complex obtained by barycentric refinement of S, then the minimum height of z_S in any n-simplex $K \in \mathcal{K}$ is a possible choice of ϱ. Given any n-simplex $K \in \mathcal{K}$, the face $F \subseteq K$ opposite to z_S is some boundary face of $\mathrm{st}_{\mathcal{T}}(S)$. The height of F in K is the height of F in the original simplex $T \in \mathrm{st}_{\mathcal{T}}(S)$ with $K \subseteq T$ scaled by $(\ell + 1)^{-1}$.

  Assuming that $\varrho > 0$ is at most the minimum height of z_S in any n-simplex $K \in \mathcal{K}$, suppose that $z \in \mathcal{B}_\varrho(z_S)$. Let $x \in \mathrm{st}_{\mathcal{T}}(S)$ be in the interior of the local patch. Suppose that the line segment I from z to x is not wholly contained in $\mathrm{st}_{\mathcal{T}}(S)$. Then I intersects with a boundary face $F \subseteq K$ of the local patch. The straight line segment from z_S to x lies within one open half-space defined by the affine hull of F. Since I intersects with F, this open-half space cannot contain z, but it must contain $\mathcal{B}_\varrho(z_S)$. That is a contradiction. Hence, the straight line segment from z to x bust lie within the interior of the local patch. We deduce that $\mathrm{st}_{\mathcal{T}}(S)$ is star-shaped with respect to $\mathcal{B}_\varrho(z_S)$.

- Let $\rho \in (0, 1]$, yet to be determined. We define $y = -\rho z_{S'}$ as vector in the opposite direction of $z_{S'}$ and with length $\rho \|z_{S'}\|$. Henceforth, we assume $\rho \leq \varrho / \|z_{S'}\|$ so that $y \in \overline{\mathcal{B}_\varrho(z_S)}$. By construction, the closed line segment from z_S to y is contained in $\overline{\mathcal{B}_\varrho(z_S)}$, and we conclude that the convex closure of T and y must lie in $\mathrm{st}_{\mathcal{T}}(S)$.

  We define another triangulation $\mathcal{G}^c$ as follows: given any n-simplex $G \in \mathcal{G}$, we replace its vertex z'_S by the vertex y at the opposite position, thus obtaining a new n-simplex G^c. Indeed, that $\mathcal{G}^c$ is a simplicial complex follows easily from $\mathcal{D}$ being a simplicial complex.

  We let the simplicial complex $\mathcal{G}^*$ be the union of $\mathcal{G}$ and $\mathcal{G}^c$. By construction, all its n-simplices contain z_S as a subsimplex, which is an inner vertex of that triangulation. In particular, $\mathcal{D}^*$ is its own star around z_S.

- We introduce a new mapping $\Theta : |\mathcal{G}| \to |\mathcal{G}^c|$ between the underlying sets as follows. Let $G \in \mathcal{G}$ be an n-simplex and $G^c \in \mathcal{G}^c$ be constructed from G. We let $A_G : \Delta^n \to G$ and $A_{G^c} : \Delta^n \to G^c$ be affine reference transformations that agree on the vertices common to G and G^c. We then define the piecewise affine mapping

$$\Theta_{|G} := A_{G^c} \circ A_G^{-1}.$$

It follows that

$$|\det(\mathbf{J}\Theta_{|G})| = \frac{\operatorname{vol}(G^c)}{\operatorname{vol}(G)}.$$

We want to further characterize the singular values of this transform. Let $h_z \in \mathbb{R}^n$ be the height vector of $z_{S'}$ inside the simplex G, that is, the vector pointing to $z_{S'}$ and standing orthogonally on the affine hull of the face G that opposes $z_{S'}$ We verify that

$$\Theta_{|G}(x) = x - (1+\rho)\frac{\langle \hat{h}_z, x\rangle}{\langle \hat{h}_z, \hat{z}_{S'}\rangle}\hat{z}_{S'}.$$

Indeed, the right-hand side equals x whenever x lies in the plane orthogonal to h_z, and when $x = z_{S'}$, then

$$\begin{aligned}\Theta_{|G}(z_{S'}) &= z_{S'} - (1+\rho)\hat{z}_{S'}\frac{\langle \hat{h}_z, z_{S'}\rangle}{\langle \hat{h}_z, \hat{z}_{S'}\rangle} \\ &= z_{S'} - (1+\rho)\hat{z}_{S'}\|z_{S'}\| = z_{S'} - (1+\rho)z_{S'} = -\rho z_{S'} = y.\end{aligned}$$

If we orthogonally decompose $z_{S'} = h_z + b_z$ for some $b_z \in \mathbb{R}^n$, then

$$\begin{aligned}\Theta_{|G}(h_z) &= h_z - (1+\rho)\frac{\langle h_z, h_z\rangle}{\langle h_z, z_{S'}\rangle}z_{S'} \\ &= h_z - (1+\rho)\frac{\langle h_z, z_{S'}\rangle}{\langle h_z, z_{S'}\rangle}z_{S'} = h_z - (1+\rho)z_{S'} = -\rho h_z - (1+\rho)b_z.\end{aligned}$$

Evidently, the transformation $\Theta_{|G}$ equals the identity on the orthogonal complement of the span of h_z and b_z. Let β be the angle between $z_{S'}$ and h_z. Then $\|h_z\| = \cos(\beta)\|z_{S'}\|$ and $\|b_z\| = \sin(\beta)\|z_{S'}\|$. We study the singular values of the matrix

$$M_{\Theta,G} := \begin{pmatrix} -\rho & 0 \\ -(1+\rho)\|b_z\|/\|h_z\| & 1 \end{pmatrix} = \begin{pmatrix} -\rho & 0 \\ -(1+\rho)\tan(\beta) & 1 \end{pmatrix}.$$

Its two singular values are:

$$\sigma_{\max}(\Theta, G) := \frac{1}{2}\sqrt{(1+\rho)^2 + (1+\rho)^2\tan(\beta)^2} + \frac{1}{2}\sqrt{(1-\rho)^2 + (1+\rho)^2\tan(\beta)^2}, \tag{52}$$

$$\sigma_{\min}(\Theta, G) := \frac{1}{2}\sqrt{(1+\rho)^2 + (1+\rho)^2\tan(\beta)^2} - \frac{1}{2}\sqrt{(1-\rho)^2 + (1+\rho)^2\tan(\beta)^2}. \tag{53}$$

All other singular values of $\mathbf{J}\Theta_{|G}$ equal 1. The singular values $\sigma_{\max}(\Theta, G) \geq 1$ and $\sigma_{\min}(\Theta, G) \leq 1$ is monotonically increasing and decreasing, respectively, in $\tan(\beta)$. Hence, they must the extremal maximal singular values of $\mathbf{J}\Theta_{|G}$. It is evident that $\rho = \operatorname{vol}(G^c)/\operatorname{vol}(G)$.

We develop explicit bounds for these singular values. The definition of the tangent and the decomposition $z_{S'} = h_z + b_z$ imply that $\tan(\beta) = \|b_z\|/\|h_z\|$.

Recall that $G \in \mathcal{G}$ is obtained from T via barycentric subdivisions: first at z_S, leading to some intermediate n-simplex T', and then at $z_{S'}$, leading to $G \subseteq T'$. Let $F_z \subseteq G$ be the face opposite to the vertex $z_{S'}$, which is some face of T'. Now, the height of F_z in G is $(n-\ell)$-th of the height of F_z in T' since S' has dimension $n-\ell-1$. The height of the face F_z in T' is just the height of its opposing vertex, which lies in S', and which equals the height of that vertex in T. Thus,

$$\frac{\|b_z\|}{\|h_z\|} = \frac{\|b_z\|}{(n-\ell)^{-1}\|(n-\ell)h_z\|} = \frac{\|z_{S'}\|}{(n-\ell)^{-1}\|(n-\ell)h_z\|} \leq (n-\ell)\kappa_{\mathrm{A}}(T).$$

We abbreviate

$$\mu_{T,\ell} := \frac{1}{2}\sqrt{(1+\rho)^2 + (1+\rho)^2(n-\ell)^2\kappa_{\mathrm{A}}(T)^2} + \frac{1}{2}\sqrt{(1-\rho)^2 + (1+\rho)^2(n-\ell)^2\kappa_{\mathrm{A}}(T)^2}. \tag{54}$$

This establishes bounds on the singular values of the transformation. In summary, the singular values of the Jacobian of Θ at almost every x satisfy

$$\sigma_1(\Theta, x) \leq \mu_{T,\ell}, \quad \sigma_2(\Theta, x) = \cdots = \sigma_{n-1}(\Theta) = 1, \quad \sigma_n(\Theta, x)^{-1} = \frac{\sigma_1(\Theta, x)}{\left|\det \mathbf{J}\Theta_{|G}\right|} \leq \mu_{T,\ell}/\rho. \tag{55}$$

- We introduce another mapping $\Phi : |\mathcal{G}^*| \to |\mathcal{G}^c|$ as follows. Consider any n-simplex $G \in \mathcal{G}$ and let $G^c \in \mathcal{G}^c$ be its image under Θ. We construct a bi-Lipschitz mapping

$$\Phi_G : G \cup G^c \to G^c.$$

The construction will be such that the union of Φ_G for all n-simplices $G \in \mathcal{G}$ will define the desired bi-Lipschitz mapping $\Phi : |\mathcal{G}^*| \to |\mathcal{G}^c|$, which will be the identity along $\partial |\mathcal{G}^*| \setminus \partial |\mathcal{G}|$.

Once again, h_z denotes the height of $z_{S'}$ within G. Here, we let $Q \subseteq G \cap G^c$ be the subsimplex that is complementary to the line segment from the origin to $z_{S'}$ in G. Equivalently, Q is complementary to the line segment from the origin to y in G^c. From the definition of simplices, we now conclude that any $x \in G \cup G^c$ has a unique representation

$$x = \lambda z_{S'} + \mu x_Q, \quad \lambda \in [-\rho, 1], \quad \mu \in [0,1], \quad |\lambda| + \mu \leq 1, \quad x_Q \in Q,$$

Since μx_Q lies in the hyperplane spanned by the origin and Q, we have

$$\lambda = \lambda(x) := \frac{\langle h_z, x \rangle}{\langle h_z, z_{S'} \rangle}.$$

Based on that observation, we define

$$\begin{aligned}
\Phi_G(x) &:= \mu x_Q + \frac{\rho}{\rho+1} (\lambda(x) - 1) z_{S'} \\
&= \mu x_Q + \frac{\rho}{\rho+1} \lambda(x) z_{S'} - \frac{\rho}{\rho+1} z_{S'} \\
&= x - \frac{\rho+1}{\rho+1} \lambda(x) z_{S'} + \frac{\rho}{\rho+1} \lambda(x) z_{S'} - \frac{\rho}{\rho+1} z_{S'} \\
&= x - \frac{1}{\rho+1} \lambda(x) z_{S'} - \frac{\rho}{\rho+1} z_{S'} \\
&= x - \frac{1}{\rho+1} \frac{\langle h_z, x \rangle}{\langle h_z, z_{S'} \rangle} z_{S'} - \frac{\rho}{\rho+1} z_{S'}.
\end{aligned}$$

We readily verify that this transformation is a bi-Lipschitz mapping from $G \cup G^c$ onto G^c that satisfies the desired mapping properties. It remains to analyze its Jacobian and get explicit estimates for its singular values.

We once more introduce an orthogonal decomposition $h_z + b_z = z_{S'}$ for some $b_z \in \mathbb{R}^n$. With that,

$$\begin{aligned}
\mathbf{J}\Phi_G(x) &= \mathrm{Id} - \frac{1}{1+\rho} \frac{z_{S'} \otimes h_z^t}{\langle h_z, z_{S'} \rangle} \\
&= \mathrm{Id} - \frac{1}{1+\rho} \frac{z_{S'} \otimes h_z^t}{\langle h_z, h_z \rangle} \\
&= \mathrm{Id} - \frac{1}{1+\rho} \frac{h_z \otimes h_z^t}{\langle h_z, h_z \rangle} - \frac{1}{1+\rho} \frac{b_z \otimes h_z^t}{\langle h_z, h_z \rangle} \\
&= \mathrm{Id} - \frac{1}{1+\rho} \hat{h}_z \otimes \hat{h}_z^t - \frac{\|b_z\|/\|h_z\|}{1+\rho} \hat{b}_z \otimes \hat{h}_z^t.
\end{aligned}$$

The Jacobian acts as the identity over the orthogonal complement of the span of h_z and $z_{s'}$. We write β for the angle between h_z and $z_{S'}$. Hence, $\tan(\beta) = \|b_z\|/\|h_z\|$. Direct computation shows

$$\mathbf{J}\Phi_G(x) \cdot \hat{b}_z = \hat{b}_z, \qquad \mathbf{J}\Phi_G(x) \cdot \hat{h}_z = \frac{\rho}{1+\rho} \hat{h}_z - \frac{\|b_z\|/\|h_z\|}{1+\rho} \hat{b}_z. \tag{56}$$

It remains to study the singular values of the matrix

$$M_{\Phi,G} = \begin{pmatrix} \frac{\rho}{1+\rho} & 0 \\ -\frac{\tan(\beta)}{1+\rho} & 1 \end{pmatrix}.$$

By building the symmetric matrix $M_{\Phi,G}^t M_{\Phi,G}$ and computing its eigenvalues, we obtain the singular values

$$\sigma_{\max}(\Phi, G) = \frac{1}{2(1+\rho)} \sqrt{(2\rho+1)^2 + \tan(\beta)^2} + \frac{1}{2(1+\rho)} \sqrt{1+\tan(\beta)^2},$$
$$\sigma_{\min}(\Phi, G) = \frac{1}{2(1+\rho)} \sqrt{(2\rho+1)^2 + \tan(\beta)^2} - \frac{1}{2(1+\rho)} \sqrt{1+\tan(\beta)^2}.$$

These are monotonically increasing from 1 and decreasing from 0.5, respectively, in $\tan(\beta)$. Hence, these are also the maximal and minimal singular values of the Jacobian $\mathbf{J}\Phi_G$, the remaining singular values being equal to 1. Notice that

$$\det \mathbf{J}\Phi = \sigma_{\max}(\Phi, G)\sigma_{\min}(\Phi, G) = \frac{\rho(1+\rho)}{(1+\rho)^2} = \frac{\rho}{1+\rho}.$$

We now recall that the height of h_z in $G \in \mathcal{G}$ equals $(\ell+1)^{-1}$ multiplied with the height of some vertex of S within T. Similar as above, we use the upper bound

$$\tan(\beta) = \frac{\|b_z\|}{\|h_z\|} \leq (\ell+1)\kappa_{\mathrm{A}}(T).$$

We conclude that the singular values of the Jacobian of Φ at almost every x satisfy

$$\sigma_1(\Phi, x) \leq \frac{1}{2(1+\rho)} \sqrt{(2\rho+1)^2 + (\ell+1)^2\kappa_{\mathrm{A}}(T)^2} + \frac{1}{2(1+\rho)} \sqrt{1+(\ell+1)^2\kappa_{\mathrm{A}}(T)^2}, \tag{57}$$
$$\sigma_2(\Phi, x) = \cdots = \sigma_{n-1}(\Phi, x) = 1, \tag{58}$$
$$\sigma_n(\Phi, x)^{-1} = \frac{\sigma_1(\Phi, x)}{|\det \mathbf{J}\Phi|} \leq \frac{1}{2\rho} \sqrt{(2\rho+1)^2 + (\ell+1)^2\kappa_{\mathrm{A}}(T)^2} + \frac{1}{2\rho} \sqrt{1+(\ell+1)^2\kappa_{\mathrm{A}}(T)^2}. \tag{59}$$

This finishes the discussion of the transformation Φ.

Having shown all the desired estimates, and the proof is complete. □

Remark 8.3. *We notice that the mappings Ξ_1 and Ξ_2 above are not only bi-Lipschitz, but in fact also piecewise affine, where piecewise refers to an essentially non-overlapping decomposition of their respective domains into convex polytopes. While the reflection and deformation mappings in Proposition 8.2 serve our purpose, it might be possible to improve the analysis or construction and lower the Jacobian estimates.*

Remark 8.4. *The estimates in Proposition 8.2 should be considered as* proof-of-concept *but not as optimal in any sense. Some improvements are immediately possible if $\ell = n-1$. Then the reflection Ξ in Lemma 3.4 satisfies the same properties as the mapping Ξ_1, mapping the simplex T bijectively onto the A and being the identity along the common face S. However, at any point, the singular values $\sigma_1 \geq \cdots \geq \sigma_n$ of the Jacobian of Ξ satisfy*

$$|\det \mathbf{J}\Xi| \leq C_\rho(\mathcal{T}),$$
$$\sigma_1, \sigma_n^{-1} \leq \frac{1}{2}\sqrt{\left(C_\theta(\mathcal{T})\kappa_{\mathrm{A}}(\mathcal{T})+1\right)^2 + \kappa_{\mathrm{A}}(\mathcal{T})^2} + \frac{1}{2}\sqrt{\left(C_\theta(\mathcal{T})\kappa_{\mathrm{A}}(\mathcal{T})-1\right)^2 + \kappa_{\mathrm{A}}(\mathcal{T})^2},$$
$$\sigma_2, \dots, \sigma_{n-1}, \sigma_n \leq 1.$$

In particular,

$$\sigma_1 \cdots \sigma_k |\det \mathbf{J}\Xi_2|^{-\frac{1}{p}} \leq \left(\frac{1}{2}\sqrt{\left(C_\theta(\mathcal{T})\kappa_{\mathrm{A}}(\mathcal{T})+1\right)^2 + \kappa_{\mathrm{A}}(\mathcal{T})^2} + \frac{1}{2}\sqrt{\left(C_\theta(\mathcal{T})\kappa_{\mathrm{A}}(\mathcal{T})-1\right)^2 + \kappa_{\mathrm{A}}(\mathcal{T})^2} \right) C_\rho(\mathcal{T})^{\frac{1}{p}}.$$

These estimates improve over the ones in Proposition 8.2 when the reflection is over a single face. We believe that better estimates can be computed from the geometric data also in the other cases.

9 Constructive estimates of Poincaré–Friedrichs constants

We are now in the position to develop Poincaré–Friedrichs constants for the exterior derivative over domains with shellable triangulations; this includes the curl and divergence operators in three dimensions. These results use local Poincaré–Friedrichs constants over simplices, with or without boundary conditions, as subcomponents.

The analysis of the Poincaré potential operators in Section 6 has produced Poincaré–Friedrichs constants for convex domains and their analogues for the case of full boundary conditions. We also need a Poincaré–Friedrichs inequality for differential forms over a simplex but subject to homogeneous boundary conditions along a collection of faces.

Lemma 9.1. *Let T be an n-simplex and let $\Gamma = F_0 \cup \cdots \cup F_\ell$ be the union of $\ell+1$ different faces of T. Suppose that $u \in W^p\Lambda^k(T)$ such that $\operatorname{tr}_{T,\Gamma} u = 0$. Then there exists $w \in W^p\Lambda^k(T)$ such that $\operatorname{tr}_{T,\Gamma} w = 0$ and*

$$dw = du, \quad \|w\|_{L^p(\Omega)} \leq C_{\mathrm{PF},\Gamma,\ell,k,p}(T)\|du\|_{L^p(\Omega)}.$$

Here, $C_{\mathrm{PF},\Gamma,\ell,k,p}(T) > 0$ is a constant such that

$$C_{\mathrm{PF},\Gamma,\ell,k,p}(T) \leq n! \cdot 2^{\ell+1} C_{\mathcal{B}}(n,k+1)\operatorname{vol}_{n-1}(S_1(0)) \cdot \kappa_{\mathrm{M}}(T)^{k-1} C_{1,n}(T)\delta(T).$$

Proof. There exists an affine bijection $\varphi : \Delta^n \to T$ that maps the convex closure of the n unit vectors onto the face F_0. We can also assume that the face of Δ^n orthogonal to the i-th coordinate axis is mapped onto the face F_i. In what follows, we let $\widetilde{A}$ be the convex set obtained from reflecting Δ^n along the coordinate axes $\ell+1$ through n. We see that $\operatorname{vol}(\widetilde{A}) = 2^{n-\ell}\operatorname{vol}(\Delta^n) = 2^{n-\ell}/n!$

We let $\hat{u} := \varphi^* u \in W^p\Lambda^k(\Delta^n)$ and define $\hat{g} \in L^p\Lambda^{k+1}(\Delta^n)$ via $\hat{g} := d\varphi^* u$. Then $d\hat{u} = \hat{g}$. We let $\tilde{u} \in W^p\Lambda^k(\widetilde{A})$ be the extension of $\hat{u}$ onto $\widetilde{A}$ by reflection across the coordinate axes, and let $\tilde{g} \in L^p\Lambda^{k+1}(\widetilde{A})$ is the extension of $\hat{g}$ onto $\widetilde{A}$ by reflection across the coordinate axes. By construction, $\tilde{u} \in W^p_0\Lambda^k(\widetilde{A})$ with $\tilde{g} = d\tilde{u}$. We observe

$$\|\tilde{g}\|_{L^p(\widetilde{A})} \leq 2^{\frac{n-\ell}{p}}\|\hat{g}\|_{L^p(\Delta^n)}.$$

By the analysis for the Bogovskiĭ potential operators, Theorem 6.2, there exists $\mathring{w} \in W^p_0\Lambda^k(\widetilde{A})$ with $d\mathring{w} = d\tilde{u}$ satisfying the bounds

$$\|\mathring{w}\|_{L^p(\widetilde{A})} \leq C_{\mathrm{PF},\mathcal{B},k,\widetilde{A},p}\|\tilde{g}\|_{L^p(\widetilde{A})}. \tag{60}$$

Here,

$$\begin{aligned} C_{\mathrm{PF},\mathcal{B},k,\widetilde{A},p} &= C_{\mathcal{B}}(n,k+1)\operatorname{vol}_{n-1}(S_1(0))\frac{\delta(\widetilde{A})^n}{\operatorname{vol}(\widetilde{A})}\delta(\widetilde{A}) \\ &= C_{\mathcal{B}}(n,k+1)\operatorname{vol}_{n-1}(S_1(0))\frac{2^n}{2^{n-\ell}/n!}2 = C_{\mathcal{B}}(n,k+1)\operatorname{vol}_{n-1}(S_1(0)) \cdot n! 2^{\ell+1}. \end{aligned}$$

We let $\tilde{w} \in W^p\Lambda^k(\widetilde{A})$ be defined by averaging the $2^{n-\ell}$ reflections of $\mathring{w}$ along the coordinate axes $\ell+1$ through n. Thus,

$$\|\tilde{w}\|_{L^p(\Delta^n)} = 2^{-\frac{n-\ell}{p}}\|\tilde{w}\|_{L^p(\widetilde{A})}, \qquad \|\tilde{w}\|_{L^p(\widetilde{A})} \leq \|\mathring{w}\|_{L^p(\widetilde{A})}.$$

Since $\tilde{g}$ has got the same symmetries, $d\tilde{w} = \tilde{g}$.

We let $w \in W^p\Lambda^k(T)$ be defined by $w := \varphi^{-*}(\tilde{w}_{|\Delta^n})$. By construction, $dw = du$, and w has vanishing trace along $F_0 \cup \cdots \cup F_\ell$.

To complete the discussion, we recall by Proposition 5.3 that

$$\begin{aligned} \|w\|_{L^p(T)} &\leq |\det(\mathbf{J}\varphi)|^{\frac{1}{p}}\|\mathbf{J}\varphi^{-1}\|_2^k\|\tilde{w}\|_{L^p(\Delta^n)}, \\ \|\hat{g}\|_{L^p(\Delta^n)} &\leq |\det(\mathbf{J}\varphi^{-1})|^{\frac{1}{p}}\|\mathbf{J}\varphi\|_2^{k+1}\|du\|_{L^p(T)}. \end{aligned}$$

It remains to use Lemma 3.3, and the desired inequality is shown. □

Remark 9.2. *Under the assumptions of Lemma 9.1, the special case $p = 2$ allows to improve the preceding lemma. Specifically, we use a better estimate for the auxiliary constant in (60). This follows from the results in [48] as simplices are Lipschitz domains. The complete inequality then reads: there exists $w \in W^2\Lambda^k(T)$ such that $\operatorname{tr}_{T,\Gamma} w = 0$ and*

$$dw = du, \quad \|w\|_{L^p(\Omega)} \leq C_{\mathrm{PF},\Gamma,\ell,k,2}(T)\|du\|_{L^p(\Omega)},$$

where

$$C_{\mathrm{PF},\Gamma,\ell,k,p}(T) \leq \frac{2}{\pi}\kappa_{\mathrm{M}}(T)^{k-1}C_{1,n}(T)\delta(T).$$

We conjecture that the constant can be improved further, to be independent of the tetrahedron's eccentricity, and that the result can be extended to Lebesgue exponents $1 \leq p \leq \infty$.

We prepare some further notation for our two main results. Whenever $\mathcal{T}$ is an n-dimensional triangulation, we use the abbreviation

$$C_{\mathrm{PF},\Gamma,k,p}(\mathcal{T}) := \max_{\substack{T \in \mathcal{T} \\ \dim(T)=n \\ 0 \leq l \leq n}} C_{\mathrm{PF},\Gamma,l,k,p}(T).$$

Suppose that $\mathcal{T}$ is an n-dimensional triangulation and has a shelling $T_0, T_1, \dots, T_M$. For each $0 \leq m \leq M$, we write

$$X_m := T_0 \cup T_1 \cup \cdots \cup T_m,$$

which is by Lemma 7.5 a triangulated n-dimensional submanifold with boundary. For each $1 \leq m \leq M$, the interface of the new simplex to the previous intermediate triangulation is

$$\Gamma_m := T_m \cap X_{m-1}.$$

According to Lemma 7.16, there exists an interior simplex $S_m \in \mathcal{T}$ such that T_m is the last n-simplex in the shelling that belongs to $T_m \in \operatorname{st}_{\mathcal{T}}(S_m)$. In particular, already $\operatorname{st}_{\mathcal{T}}(S_m) \subseteq X_m$.

In addition to that, we introduce

$$A_{m-1} := \operatorname{st}_{\mathcal{T}}(S_m) \setminus T_m$$

for the complement $A_{m-1} \subseteq X_{m-1}$ of the simplex T_m in the star $\operatorname{st}_{\mathcal{T}}(S_m)$.

The two main results of this manuscript for shellable triangulations of domains, not necessarily convex or even star-shaped, is the following theorems. The first one is inspired by the recursive construction of gradient potentials in Theorem 4.4. The second one is specific to the fact that shellable triangulations are contractible.

Theorem 9.3. *Let $\mathcal{T}$ be a shellable n-dimensional triangulation, and let the domain $\Omega \subseteq \mathbb{R}^n$ be the interior of the underlying set of $\mathcal{T}$. Let $T_0, T_1, \dots, T_M$ be a shelling of $\mathcal{T}$. Then for any $u \in W^p\Lambda^k(\Omega)$, where $1 \leq p \leq \infty$, there exists $w \in W^p\Lambda^k(\Omega)$ with $dw = du$ and such that the following estimates hold:*

$$\|w\|_{L^p(T_0)} \leq C_{\mathrm{PF},k,T_0,p}\|du\|_{L^p(T_0)},$$

and for each $T_m \in \mathcal{T}$, $1 \leq m \leq M$ we have the recursive estimate

$$\|w\|_{L^p(T_m)} \leq C_{\mathrm{PF},T_m,\Gamma_m,k,p}(\mathcal{T})\left(\|du\|_{L^p(T_m)} + C_{5,n,k+1,\ell,p}(\mathcal{T})\|du\|_{L^p(A_{m-1})}\right) + C_{5,n,k,\ell,p}(\mathcal{T})\|w\|_{L^p(A_{m-1})},$$

where $0 \leq \ell < n$ is such that T_m has $n - \ell$ faces in common with the previous simplices.

Proof. Let $u \in W^p\Lambda^k(\Omega)$. First, there exists $w_0 \in W^p\Lambda^k(T_0)$ satisfying $dw_0 = du$ over T_0 together with

$$\|w_0\|_{L^p(T_0)} \leq C_{\mathrm{PF},k,T_0,p}\|du\|_{L^p(T_0)}.$$

Suppose that $0 < m \leq M$ such that there exists $w_{m-1} \in W^p\Lambda^k(X_{m-1})$ with $dw_{m-1} = du$ over X_{m-1}. This is already true for $m = 1$. By assumption, T_m and X_{m-1} share the interface Γ_m, which is a collection of faces of T_m. In accordance to Lemma 7.16, adding T_m completes a star in $\mathcal{T}$ around some simplex S_m, and we let $A_{m-1} \subseteq X_{m-1}$ be the complement of T_m in that newly completed star. Write ℓ for the dimension of S_m.

We now define, as in [22, Equations (5.12) and (5.14)] or [16, Equations (6.7) and (6.9)], the field $\widetilde{w}_m \in W^p\Lambda^k(T_m)$ via

$$\widetilde{w}_m := u_{|T_m} + \Xi_1^* \left((w_{m-1} - u)_{|A_{m-1}} \right). \tag{61}$$

One verifies that

$$\begin{aligned} \operatorname{tr}_F \widetilde{w}_m &= \operatorname{tr}_F u_{|T_m} + \operatorname{tr}_F \Xi_1^* w_{m-1|A_{m-1}} - \operatorname{tr}_F \Xi_1^* u_{|A_{m-1}} \\ &= \operatorname{tr}_F u_{|T_m} + \operatorname{tr}_F \Xi_1^* w_{m-1|A_{m-1}} - \operatorname{tr}_F \Xi_1^* u_{|A_{m-1}} \\ &= \operatorname{tr}_F \Xi_1^* w_{m-1|A_{m-1}} \\ &= \operatorname{tr}_F w_{m-1|A_{m-1}}. \end{aligned}$$

Moreover,

$$\begin{aligned} d\widetilde{w}_m &= du_{|T_m} + d\Xi_1^* w_{m-1|A_{m-1}} - d\Xi_1^* u_{|A_{m-1}} \\ &= du_{|T_m} + \Xi_1^* dw_{m-1|A_{m-1}} - \Xi_1^* du_{|A_{m-1}} = du_{|T_m}. \end{aligned}$$

Setting $w_m := w_{m-1}$ over X_{m-1} and $w_m := \widetilde{w}_m + w''_m$ over T_m, we thus verify $w_m \in W^p\Lambda^k(X_m)$ with $dw_m = du$ over X_m.

We want to estimate norms. By construction,

$$\|w\|_{L^p(T_m)} \leq \|\Xi_1^* w_{|A_{m-1}}\|_{L^p(T_m)} + \|u - \Xi_1^* u_{|A_{m-1}}\|_{L^p(T_m)}.$$

Due to Lemma 9.1, which applies since $\operatorname{tr}_{\Gamma_m} \left(u_{|T_m} - \Xi_1^* u_{|A_{m-1}} \right) = 0$, we have

$$\begin{aligned} \|u - \Xi_1^* u_{|A_{m-1}}\|_{L^p(T_m)} &\leq C_{\mathrm{PF},T_m,\Gamma_{m-1},k,p} \left(\|du\|_{L^p(T_m)} + \|d\Xi_1^* u_{|A_{m-1}}\|_{L^p(T_m)} \right) \\ &\leq C_{\mathrm{PF},T_m,\Gamma_{m-1},k,p} \left(\|du\|_{L^p(T_m)} + \|\Xi_1^* du_{|A_{m-1}}\|_{L^p(T_m)} \right). \end{aligned}$$

Proposition 5.3 and Proposition 8.2 now show

$$\begin{aligned} \|\Xi_1^* w_{|A_{m-1}}\|_{L^p(T_m)} &\leq C_{5,n,k,\ell,p}(\mathcal{T}) \|w_{m-1}\|_{L^p(A_{m-1})}, \\ \|\Xi_1^* du_{|A_{m-1}}\|_{L^p(T_m)} &\leq C_{5,n,k+1,\ell,p}(\mathcal{T}) \|dw_{m-1}\|_{L^p(A_{m-1})}. \end{aligned}$$

We have assumed that $dw_{m-1} = du_{|X_{m-1}}$. The existence of $w \in W^p\Lambda^k(\Omega)$ satisfying the recursive estimate follows. □

We derive an estimate for the Poincaré–Friedrichs constant, in analogy to Theorem 4.5, though the generalization to the exterior derivative faces a more complicated recursive structure. For ease of discussion, we choose to present the result for a general class of recursive estimates. Of course, implementations should make use of the simplifications warranted by the specific recursive structure, such as the recursion presented in Theorem 9.3.

Theorem 9.4. *Let $\mathcal{T}$ be a shellable n-dimensional triangulation, and let the domain $\Omega \subseteq \mathbb{R}^n$ be the interior of the underlying set of $\mathcal{T}$. Let $T_0, T_1, \dots, T_M$ be a shelling of $\mathcal{T}$. Suppose that $u, w \in W^p\Lambda^k(\Omega)$, where $1 \leq p \leq \infty$, such that $dw = du$ and the following recursive estimate holds:*

$$\|w\|_{L^p(T_0)} \leq A_{0,0} \|du\|_{L^p(T_0)},$$

and for each $T_m \in \mathcal{T}$, $1 \leq m \leq M$ we have the recursive estimate

$$\|w\|_{L^p(T_m)} \leq \sum_{\ell=0}^{m} A_{m,\ell} \|du\|_{L^p(T_\ell)} + \sum_{\ell=0}^{m-1} B_{m,\ell} \|w\|_{L^p(T_\ell)}.$$

Then we have an inequality of the form

$$\|w\|_{L^p(\Omega)} \leq \left(\sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C_{m,\ell}^{q} \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \|du\|_{L^p(\Omega)},$$

where $q \in [1,\infty]$ *satisfies* $1 = 1/p + 1/q$ *and with obvious modifications if* $p = 1$ *or* $p = \infty$*. Here,*

$$C_{m,\ell} = \sum_{m=i_L > \cdots > i_1 \geq \ell} B_{i_L,i_{L-1}} \cdots B_{i_2,i_1} A_{i_1,\ell}.$$

Proof. Unwrapping the recursion, we obtain an estimate of the form

$$\|w\|_{L^p(T_m)} \leq \sum_{\ell=0}^{m} \underbrace{\left(\sum_{m=i_L > \cdots > i_1 \geq \ell} B_{i_L,i_{L-1}} \cdots B_{i_2,i_1} A_{i_1,\ell} \right)}_{=:C_{m,\ell}} \|du\|_{L^p(T_\ell)}$$

Here, $C_{m,\ell}$ denotes the coefficient of $\|du\|_{L^p(T_\ell)}$, possibly zero, as it appears in the unwrapped recursive estimate of $\|w\|_{L^p(T_m)}$. Once again, The global Poincaré–Friedrichs inequality follows via Hölder's inequality:

$$\|w\|_{L^p(\Omega)}^p \leq \sum_{m=0}^{M} \|w\|_{L^p(T_m)}^p \leq \sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C_{m,\ell}^q \right)^{\frac{p}{q}} \sum_{\ell'=0}^{M} \|du\|_{L^p(T_{\ell'})}^p \leq \left(\sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C_{m,\ell}^q \right)^{\frac{p}{q}} \right) \|du\|_{L^p(\Omega)}^p,$$

where $q \in [1,\infty]$ is as described above. The proof is complete. □

Remark 9.5. *Recall that the potential* w *for the gradient* du *satisfies the recursive estimate*

$$\begin{aligned} \|w\|_{L^p(T_m)} \leq\ & C_{\mathrm{PF},T_m,\Gamma_m,k,p}(\mathcal{T}) \|du\|_{L^p(T_m)} \\ & + C_{\mathrm{PF},T_m,\Gamma_m,k,p}(\mathcal{T}) C_{5,n,k+1,\ell_m,p}(\mathcal{T}) \|du\|_{L^p(A_{m-1})} \\ & + C_{5,n,k,\ell_m,p}(\mathcal{T}) \|w\|_{L^p(A_{m-1})}. \end{aligned}$$

Unwrapping the recursion for the norm over the m*-th simplex, we find an estimate of the form*

$$\|w\|_{L^p(T_m)} \leq \sum_{\ell=0}^{M} C_{m,\ell} \|du\|_{L^p(T_\ell)}. \tag{62}$$

Here, $C_{m,\ell}$ *is the coefficient of* $\|du\|_{L^p(T_\ell)}$*, possibly zero, as it appears in the unwrapped recursive estimate of* $\|w\|_{L^p(T_m)}$*. The global Poincaré–Friedrichs inequality follows via Hölder's inequality:*

$$\|w\|_{L^p(\Omega)}^p \leq \sum_{m=0}^{M} \|w\|_{L^p(T_m)}^p \leq \sum_{m=0}^{M} \left(\sum_{\ell'=0}^{M} C_{m,\ell'}^q \right)^{\frac{p}{q}} \sum_{\ell=0}^{M} \|du\|_{L^p(T_\ell)}^p \leq \left(\sum_{m=0}^{M} \left(\sum_{\ell=0}^{M} C_{m,\ell}^q \right)^{\frac{p}{q}} \right) \|du\|_{L^p(\Omega)}^p,$$

where $q \in [1,\infty]$ *satisfies* $1 = 1/p + 1/q$ *and with obvious modifications if* $p = 1$ *or* $p = \infty$*.*

Remark 9.6. *While our main interest in this manuscript, we point out an alternative application of Proposition 8.2, using the second class of transformations. Those transformations are a sequence of contractions. Here, we take inspiration from the topological observation that domains with shellable triangulations are contractible. More specifically, we have seen that every shellable triangulation can be transformed into a single simplex along a sequence of local bi-Lipschitz deformations, and their bi-Lipschitz constants are controlled by the shape regularity of the triangulation.*

Using successive pullbacks along those contractions, the original problem of finding a potential over the domain is reduced to finding a potential for some right-hand side over the first simplex of the shelling. The potential over that first simplex is then extended along successive reverse pullbacks to the whole domain.

Using the Poincaré–Friedrichs constant over the first simplex and pullback estimates (Proposition 5.3), one obtains the following estimate.

Let $\mathcal{T}$ be a shellable n-dimensional triangulation, and let the domain $\Omega \subseteq \mathbb{R}^n$ be the interior of the underlying set of $\mathcal{T}$. Let $T_0, T_1, \dots, T_M$ be a shelling of $\mathcal{T}$ and let $0 \leq \ell_m < n$, for $1 \leq m \leq M$, be such that T_m has $n - \ell_m$ faces in common with the previous simplices. Then for any $u \in W^p\Lambda^k(\Omega)$, where $1 \leq p \leq \infty$, there exists $w \in W^p\Lambda^k(\Omega)$ with $dw = du$ such that

$$\|w\|_{L^p(\Omega)} \leq \left(\prod_{m=1}^{M} C_{8,n,k+1,\ell_m,p}(\mathcal{T}) C_{7,n,k,\ell_m,p}(\mathcal{T}) \right) C_{\mathrm{PF},k,T_0,p} \|du\|_{L^p(\Omega)}.$$

However, in our numerical examples (see Section 10), the estimate performs significantly worse than the estimate derived in Theorem 9.3, which is why we choose to not deploy this method in our computations.

10 Numerical examples

We wish to assess the practical quality of our upper bounds, with exclusive focus on the Hilbert space case $p = 2$. To that end, we compare our estimates for the Poincaré–Friedrichs constants with the exact constants of a few examples domains in dimension 2 and 3. In lieu of the exact values, finite element eigenvalue computations on a refined mesh provide reference values as a proxy for the exact value. We study the

Theorem 9.3 gives estimates for the gradient, curl and divergence Poincaré–Friedrichs constants. In addition, Theorem 4.4 provides anotherr upper bound for the Poincaré constant. We remark that the Poincaré–Friedrichs constant for the divergence can always be estimated by less than the diameter of the domain (see Lemma 4.1), so our recursive estimates are of no practical interest in that case.

10.1 Software and algorithms

All computations have been implemented with an in-house C++ code developed specifically for implementing finite element exterior calculus. The reference Poincaré constants are the inverse square roots of reference eigenvalues. We compute the latter by solving finite element eigenvalue problems that involve mixed formulations of the vector Laplacian over sufficiently refined meshes.

Computing the upper bound for the Poincaré constant of Theorem 4.5 requires to find paths between simplices. One quickly sees that a spanning tree of the face-connection graph minimizes the estimate. We choose a depth-first search that attempts to minimize the resulting constant. While this does not take much time for small two-dimensional examples, the computation takes considerably for our three-dimensional examples.

Computing the upper bound for the Poincaré–Friedrichs constant of Theorem 4.5 requires the construction of shellings. Finding a shelling that optimizes a geometric target quantity is a challenging problem in computational geometry and theoretical computer science. While a brute-force enumeration of shellings is still feasible for small two-dimensional triangulations, it becomes unfeasible in dimension three due to the combinatorial explosion of the number of possible shellings. Instead, we perform a greedy backtracking search: Starting with a single simplex, we successively add simplices to the partial shellings until a complete shelling is found. This backtracking algorithm does generally not find the optimal shelling until all shellings have been checked. However, enumerating and checking all shellings seems practically infeasible.

Therefore, we need a practical heuristic to find shellings that induce small constants. Indeed, our computations show that the Poincaré–Friedrichs estimates depend on the shelling and can differ by orders of magnitudes. Our implementation starts with a fixed simplex and generates up to 10 shellings. We repeat this for all possible initial simplices and pick the optimal shelling among those found. We use a greedy strategy: Among the possible candidate simplices, we prioritize those that introduce the smallest possible geometric constant. We moreover prioritize the completion of face stars since the induced constants are easier to control via Lemma 3.4. In other words, we defer the completion of non-face stars in order to not introduce large factors early into the recursion.

For tight estimates, we must compute the relevant geometric parameters, such as heights, individually for each simplex and use individualized estimates for each geometric transformation. The estimates that

	grad	curl	div
$\Gamma = F_0 \cup F_1 \cup F_2 \cup F_3$	0.0862501765	0.1453729386	0.2601720480
$\Gamma = F_1 \cup F_2 \cup F_3$	0.1093817645	0.1829680131	0.3493931507
$\Gamma = F_2 \cup F_3$	0.1450219664	0.2428248005	0.1845090722
$\Gamma = F_3$	0.2057601732	0.1527746636	0.1223402289
$\Gamma = \emptyset$	0.2631059409	0.1458215887	0.0874066554

Table 1: Reference L^2 Poincaré–Friedrichs constants for the gradient, curl, and divergence operators over the reference tetrahedron, where boundary conditions are imposed along the boundary part Γ, computed via lowest-order finite element methods on a refined mesh. Notably, the top-left should and the bottom-right should asymptotically coincide, as should the bottom-left and the top-right. Lemma 9.1 and Remark 9.2 give the upper bound $2/\pi \cdot \delta(\Delta^3) \approx 0.90031631615$.

we use are slightly tighter than the ones stated in Theorems 4.4 and 9.3, as we directly implement the calculations used in the proofs.

10.2 Estimates for partial boundary conditions

The Poincaré–Friedrichs constants over simplices enter our estimates. They are subject to boundary conditions that hold along a few faces. Since those constants appear so frequently, we prefer to have tight upper estimates for them.

Specifically, and subject to different boundary conditions We first assess the Poincaré–Friedrichs constant over the reference simplex where boundary conditions hold along a few faces. This is of interest because the constant appears repeatedly within our main result.

Whenever T is a simplex, the Poincaré–Friedrichs constants have upper bounds of the form $C\delta(T)$ for some numerical constant $C > 0$. Lemma 9.1 and the subequent remark (using $p = 2$) have established that $C = 2/\pi \approx 0.45015815807$. In the special case of the gradient potential, where the Poincaré constant can be characterized variationally, the Poincaré constant without boundary conditions is already an upper bound for the Poincaré constant with boundary conditions, and one can choose $C = 1/\pi \approx 0.31830988618$. This conforms to the well-known fact that the eigenvalues of the scalar Laplacian with mixed boundary conditions are between the Neumann and Dirichlet eigenvalues. Finally, the factor $1/\pi$ in the previous two estimates has a known improvement when the simplex is a triangle: then we can replace the factor $1/\pi$ by $1/j_{1,1} \approx 0.2609803592$, where $j_{1,1}$ is the zero of a Bessel function; see [42].

For the purpose of comparison, we compute reference Poincaré–Friedrichs constants over the reference tetrahedron via a higher-order finite element method over a mesh after three iterations of uniform refinement. These data are tabulated in Table 1.

We overestimate by a factor of at most six. We take this as an indication that the estimate is within feasible range, at least on simplices with good shape regularity. We expect the bound to deteriorate as the eccentricity of the tetrahedron increases. However, the results indicate room for improvement. Future estimates should tighten the gap towards the actual computed values, including for ill-shaped simplices.

10.3 Two-dimensional examples

We consider the following example domains in two dimensions: the unit square Ω_Q, the L-shaped domain Ω_L, and the slit domain Ω_S:

$$\Omega_Q = (0,1)^2, \qquad \Omega_L = (-1,1)^2 \setminus (0,1)^2,$$
$$\Omega_S = (-1,1)^2 \setminus ([0,1) \times \{0\}).$$

We consider the triangulations:

- $\mathcal{T}_Q$: square triangulation with two triangles
- $\mathcal{T}_L$: L-shaped domain triangulation with four triangles
- $\mathcal{T}_S$: Slit domain triangulation, five triangles

	grad ref	grad est	grad ratio	grad est	grad ratio	div ref	div est	div ratio
$\mathcal{T}_Q$	0.318	0.904	2.842	0.904	2.842	0.225	1.339	5.953
$\mathcal{T}_L$	0.822	2.381	2.896	1.421	1.729	0.322	2.505	7.781
$\mathcal{T}_{L'}$	0.822	2.391	2.909	2.391	2.909	0.322	4.053	12.587
$\mathcal{T}_{S,5}$	0.978	2.752	2.814	1.779	1.819	0.346	3.717	10.744
$\mathcal{T}_{S'}$	0.978	3.131	3.202	3.131	3.202	0.346	5.453	15.762
$\mathcal{T}_{S''}$	0.978	2.761	2.824	2.761	2.824	0.346	4.662	13.476

Table 2: Estimates for L^2 Poincaré–Friedrichs constants over various triangulated 2D domains. Reference values for the gradient and divergence (2nd and 7th column) computed with finite element methods together with estimates and ratios: Theorem 4.4 (3rd and 4th column), using Theorem 9.3 with $k = 0$ (5th and 6th column), and using Theorem 9.3 with $k = 0$ (8th and 9th column). Note that the divergence constant can always be estimated using Lemma 4.1.

- $\mathcal{T}_{S'}$: Slit domain triangulation, 8 triangles, all of which touch the origin.
- $\mathcal{T}_{S''}$: Slit domain triangulation, 8 triangles, four of which touch the origin.

Only two differential operators appear in the two-dimensional de Rham complexes, and their Poincaré–Friedrichs constants are the inverse square roots of the smallest non-zero Dirichlet and Neumann Laplacians. Standard finite element eigenvalue computations on a sufficiently refined mesh (four steps of uniform refinement) provide reference values for these.

We compare the reference Poincaré constants with the gradient potential estimate in Theorem 4.4. Moreover, we compare the reference Poincaré–Friedrichs constants with the bounds computed in Theorem 9.3. Here, we take into account Lemma 3.4 and Theorem 8.2.

The shelling-based estimate for the Poincaré constant of the gradient potential outperforms the estimate in Section 4 in our examples where the triangles are not all congruent to each other ($\mathcal{T}_L$ and $\mathcal{T}_{S,5}$). That is a result of our implementation, where the face-based reflections are not necessarily onto, which allows for some improved estimates.

10.4 Three-dimensional examples

We consider the following example domains in two dimensions: the unit cube Ω_C, the Fichera corner domain Ω_F, and the crossed bricks domain Ω_B:

$$\begin{aligned}
\Omega_C &= (0,1)^3, \qquad \Omega_F = (-1,1)^3 \setminus [0,1)^3, \\
\Omega_B &= ((-1,0) \times (-1,0) \times (-1,1)) \\
&\quad \cup ((-1,0) \times (-1,1) \times (-1,0)) \\
&\quad \cup ((-1,1) \times (-1,0) \times (-1,0)).
\end{aligned}$$

We consider the triangulations:

- $\mathcal{T}_{C,5}$: cube triangulation with five tetrahedra
- $\mathcal{T}_{C,K}$: Kuhn triangulation of the cube, consisting of six tetrahedra
- $\mathcal{T}_{B,5}$: crossed bricks, four copies of $\mathcal{T}_{C,5}$
- $\mathcal{T}_{B,K}$: crossed bricks, four copies of $\mathcal{T}_{C,K}$
- $\mathcal{T}_F$: Fichera corner, 24 simplices

The reference Poincaré–Friedrichs constants for the gradient, curl, and divergence operators are found via standard finite element eigenvalue computations, using the lowest-order finite element de Rham complex over a sufficiently refined mesh (four steps of uniform refinement). Again, we compare these reference values with estimates obtained Theorem 9.3, and in the special case of the gradient, with Theorem 4.4.

	grad ref	grad est	grad ratio	grad est	grad ratio
$\mathcal{T}_{C,5}$	0.317	4.317	13.581	3.246	10.214
$\mathcal{T}_{C,K}$	0.317	3.571	11.23	11.034	34.722
$\mathcal{T}_{B,5}$	1.022	7.698	7.5285	25.972	25.397
$\mathcal{T}_{B,K}$	1.022	10.106	9.8824	53.763	52.573
$\mathcal{T}_{F}$	0.711	10.622	14.927	268.071	377.033

Table 3: Estimates for L^2 Poincaré–Friedrichs constants of the gradient over various triangulated 3D domains. Reference values for the gradient (2nd column) computed with finite element methods together with estimates and ratios: Theorem 4.4 (3rd and 4th column), and using Theorem 9.3 with $k = 0$ (5th and 6th column).

	curl ref	curl est	curl ratio	div ref	div est	div ratio
$\mathcal{T}_{C,5}$	0.225	141.148	627.310506212	0.183	3.391	18.453
$\mathcal{T}_{C,K}$	0.225	12.157	54.0308334194	0.183	25.899	140.920
$\mathcal{T}_{B,5}$	0.331	113.084	341.056871148	0.233	152.886	655.993
$\mathcal{T}_{B,K}$	0.331	162.687	490.655711308	0.233	512.273	2198.026
$\mathcal{T}_{F}$	0.554	25752.342	46403.3023359	0.310	8958.467	28835.266

Table 4: Estimates for L^2 Poincaré–Friedrichs constants over various triangulated 3D domains. Reference values for the curl and divergence (2nd and 5th column) computed with finite element methods together with estimates and ratios that rely on Theorem 9.3 with $k = 0$ (3rd and 4th column) and with $k = 1$ (5th and 6th column). Note that the divergence constant can always be estimated using Lemma 4.1.

11 Outlook

The most important contribution of this manuscript is the computation of upper bounds of Poincaré–Friedrichs constants for the curl and divergence operators over local patches in low dimensions (i.e., $n = 2$ or $n = 3$). However, our upper bounds are hardly the last word on estimating the Poincaré–Friedrichs constants over local patches (stars). There are several avenues for further research and improvement.

We use local Poincaré–Friedrichs inequalities over single simplices, subject to boundary conditions along some faces. As mentioned earlier, the Poincaré constant over triangles admits a sharper upper bound than for general convex domains. We conjecture that tetrahedra permit a similar improvement.

The gradient Poincaré–Friedrichs constant already dominates the Poincaré–Friedrichs constants of the L^2 de Rham complex without boundary conditions (or with full boundary conditions) over convex domains [36, 48]. We believe that this generalizes to the entire range of Lebesgue exponents $1 \leq p \leq \infty$ Such estimates would likely improve on the Poincaré–Friedrichs constants derived via the regularized Poincaré and Bogovskiĭ operators.

Our method uses the extension of differential forms onto a simplex from the complement of that simplex within a local star. In the present manuscript, this extension is via pullback along a bi-Lipschitz mapping, but different techniques, such as via trace and extension operators, could lead to better estimates.

There is little research on practical algorithms that construct shellings. Checking whether a general triangulation has a shelling is computationally infeasible. While the literature knows a few theoretical results, there is interest in algorithms that attempt to shell practically relevant triangulations while optimizing geometric target heuristics.

We expect the present results to generalize to shellable polytopal complexes. Apart from the intrinsic interest in polytopal simplicial complexes, this extension is also relevant for any simplicial triangulation: As a rule of thumb, the estimates deteriorate as the number of simplices increases. We might improve our estimates by lumping simplices into a few polytopal subdomains for which the Poincaré–Friedrichs constants are controllable. For example, the crossed bricks domain easily partitions into four quadrilaterals.

Lastly, estimating Poincaré–Friedrichs constants over domains and manifolds with shellable triangulations is restricted to topological balls and spheres. In particular, the underlying set of any shellable triangulations is contractible. We expect forthcoming works to utilize our estimates over local patches as subcomponents in computing upper bounds for Poincaré–Friedrichs constants over simplicial triangulations of general n-dimensional manifolds.

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References

[1] Acosta, G., and Durán, R. G. An optimal Poincaré inequality in L^1 for convex domains. *Proc. Amer. Math. Soc.* **132** (2004), 195–202. https://doi.org/10.1090/S0002-9939-03-07004-7.

[2] Akbulut, S., and McCarthy, J. D. *Casson's Invariant for Oriented Homology Three-Spheres: An Exposition.(MN-36)*, vol. **36**. Princeton University Press, 2014.

[3] Arnold, D. N., Falk, R. S., and Winther, R. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.* **15** (2006), 1–155. https://doi.org/10.1017/S0962492906210018.

[4] Arnold, D. N., Falk, R. S., and Winther, R. Geometric decompositions and local bases for spaces of finite element differential forms. *Comput. Methods Appl. Mech. Engrg.* **198** (2009), 1660–1672. https://doi.org/10.1016/j.cma.2008.12.017.

[5] Arnold, D. N., Falk, R. S., and Winther, R. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.)* **47** (2010), 281–354. https://doi.org/10.1090/S0273-0979-10-01278-4.

[6] Arnold, D. N., and Hu, K. Complexes from complexes. *Found. Comput. Math.* **21** (2021), 1739–1774. https://doi.org/10.1007/s10208-021-09498-9.

[7] Bagchi, B., and Datta, B. Combinatorial triangulations of homology spheres. *Discrete Math.* **305** (2005), 1–17. https://doi.org/10.1016/j.disc.2005.06.026.

[8] Bebendorf, M. A note on the Poincaré inequality for convex domains. *Z. Anal. Anwendungen* **22** (2003), 751–756. http://dx.doi.org/10.4171/ZAA/1170.

[9] Bogovskiĭ, M. E. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. *Dokl. Akad. Nauk SSSR* **248** (1979), 1037–1040.

[10] Braess, D. *Finite Elemente. Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie*, 5th revised ed. ed. Springer-Lehrb. Mastercl. Berlin: Springer Spektrum, 2013.

[11] Braess, D., Pillwein, V., and Schöberl, J. Equilibrated residual error estimates are p-robust. *Comput. Methods Appl. Mech. Engrg.* **198** (2009), 1189–1197. http://dx.doi.org/10.1016/j.cma.2008.12.010.

[12] Brenner, S. C., and Scott, L. R. *The mathematical theory of finite element methods*, third ed., vol. **15** of *Texts in Applied Mathematics*. Springer, New York, 2008. http://dx.doi.org/10.1007/978-0-387-75934-0.

[13] Burenkov, V. I. *Sobolev spaces on domains*, vol. **137** of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1998. https://doi.org/10.1007/978-3-663-11374-4.

[14] Carstensen, C., and Gedicke, J. Guaranteed lower bounds for eigenvalues. *Math. Comp.* **83** (2014), 2605–2629. http://dx.doi.org/10.1090/S0025-5718-2014-02833-0.

[15] Carstensen, C., Gedicke, J., and Rim, D. Explicit error estimates for Courant, Crouzeix-Raviart and Raviart-Thomas finite element methods. *J. Comput. Math.* **30** (2012), 337–353. http://dx.doi.org/10.4208/jcm.1108-m3677.

[16] Chaumont-Frelet, T., and Vohralík, M. Constrained and unconstrained stable discrete minimizations for p-robust local reconstructions in vertex patches in the de Rham complex. *Found. Comput. Math.* (2024). DOI 10.1007/s10208-024-09674-7, https://hal.inria.fr/hal-03749682.

[17] Chernavsky, A. V., and Leksine, V. P. Unrecognizability of manifolds. *Ann. Pure Appl. Logic* **141** (2006), 325–335. https://doi.org/10.1016/j.apal.2005.12.011.

[18] Chua, S.-K., and Wheeden, R. L. Estimates of best constants for weighted Poincaré inequalities on convex domains. *Proc. London Math. Soc. (3)* **93** (2006), 197–226. http://dx.doi.org/10.1017/S0024611506015826.

[19] Costabel, M., and McIntosh, A. On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.* **265** (2010), 297–320. http://dx.doi.org/10.1007/s00209-009-0517-8.

[20] Demlow, A., and Hirani, A. N. A posteriori error estimates for finite element exterior calculus: the de Rham complex. *Found. Comput. Math.* **14** (2014), 1337–1371. https://doi.org/10.1007/s10208-014-9203-2.

[21] Ern, A., and Guermond, J.-L. *Finite Elements I. Approximation and Interpolation*, vol. **72** of *Texts in Applied Mathematics*. Springer International Publishing, Springer Nature Switzerland AG, 2021. https://doi-org/10.1007/978-3-030-56341-7.

[22] Ern, A., and Vohralík, M. Stable broken H^1 and $\boldsymbol{H}(\mathrm{div})$ polynomial extensions for polynomial-degree-robust potential and flux reconstruction in three space dimensions. *Math. Comp.* **89** (2020), 551–594. http://dx.doi.org/10.1090/mcom/3482.

[23] Esposito, L., Nitsch, C., and Trombetti, C. Best constants in Poincaré inequalities for convex domains. *J. Convex Anal.* **20** (2013), 253–264. https://www.heldermann.de/JCA/JCA20/JCA201/jca20016.htm.

[24] Eymard, R., Gallouët, T., and Herbin, R. Finite volume methods. In *Handbook of Numerical Analysis, Vol. VII*. North-Holland, Amsterdam, 2000, pp. 713–1020.

[25] Fernandes, P., and Gilardi, G. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Methods Appl. Sci.* **7** (1997), 957–991. https://doi.org/10.1142/S0218202597000487.

[26] Ferone, V., Nitsch, C., and Trombetti, C. A remark on optimal weighted Poincaré inequalities for convex domains. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **23** (2012), 467–475. https://doi.org/10.4171/RLM/640.

[27] Friedrichs, K. O. Differential forms on Riemannian manifolds. *Comm. Pure Appl. Math.* **8** (1955), 551–590. https://doi.org/10.1002/cpa.3160080408.

[28] Gaffney, M. P. Hilbert space methods in the theory of harmonic integrals. *Trans. Amer. Math. Soc.* **78** (1955), 426–444. https://doi.org/10.2307/1993072.

[29] Gallistl, D., and Olkhovskiy, V. Computational lower bounds of the Maxwell eigenvalues. *SIAM J. Numer. Anal.* **61** (2023), 539–561. https://doi.org/10.1137/21M1461447.

[30] Gawlik, E., Holst, M. J., and Licht, M. W. Local finite element approximation of Sobolev differential forms. *ESAIM Math. Model. Numer. Anal.* **55** (2021), 2075–2099. https://doi.org/10.1051/m2an/2021034.

[31] Girault, V., and Raviart, P.-A. *Finite element methods for Navier-Stokes equations*, vol. **5** of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

[32] Goaoc, X., Paták, P., Patáková, Z., Tancer, M., and Wagner, U. Shellability is NP-complete. *J. ACM* **66** (2019), Art. 21, 18. https://doi.org/10.1145/3314024.

[33] Gol'dshtein, V., Mitrea, I., and Mitrea, M. Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds. *J. Math. Sci., New York* **172** (2011), 347–400.

[34] Greub, W. H. *Multilinear algebra*. Die Grundlehren der mathematischen Wissenschaften, Band 136. Springer-Verlag New York, Inc., New York, 1967.

[35] Gross, P. W., and Kotiuga, P. R. *Electromagnetic theory and computation: a topological approach*, vol. **48** of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, 2004. https://doi.org/10.1017/CBO9780511756337.

[36] Guerini, P., and Savo, A. Eigenvalue and gap estimates for the Laplacian acting on p-forms. *Trans. Amer. Math. Soc.* **356** (2004), 319–344. https://doi.org/10.1090/S0002-9947-03-03336-1.

[37] Guzmán, J., and Salgado, A. J. Estimation of the continuity constants for Bogovskiĭ and regularized Poincaré integral operators. *J. Math. Anal. Appl.* **502** (2021), Paper No. 125246, 36. https://doi.org/10.1016/j.jmaa.2021.125246.

[38] Hiptmair, R. Finite elements in computational electromagnetism. *Acta Numer.* **11** (2002), 237–339. https://doi.org/10.1017/S0962492902000041.

[39] Hurri, R. *Poincaré domains in* $\mathbb{R}^n$. Ph.d. dissertation, University of Jyväskylä, 1988.

[40] Kozlov, D. *Combinatorial algebraic topology*, vol. **21** of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2008. https://doi.org/10.1007/978-3-540-71962-5.

[41] Kuhn, H. W. Some combinatorial lemmas in topology. *IBM J. Res. Develop.* **4** (1960), 508–524. https://doi.org/10.1147/rd.45.0518.

[42] Laugesen, R. S., and Siudeja, B. A. Minimizing neumann fundamental tones of triangles: an optimal poincaré inequality. *Journal of Differential Equations* **249** (2010), 118–135.

[43] Lee, J. M. *Introduction to topological manifolds*, second ed., vol. **202** of *Graduate Texts in Mathematics*. Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-7940-7.

[44] Lee, J. M. *Introduction to smooth manifolds*, second ed., vol. **218** of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[45] Lefschetz, S. *Introduction to Topology*. Princeton Mathematical Series, vol. 11. Princeton University Press, Princeton, NJ, 1949.

[46] Licht, M. W. Smoothed projections and mixed boundary conditions. *Math. Comp.* **88** (2019), 607–635. https://doi.org/10.1090/mcom/3330.

[47] Licht, M. W. On basis constructions in finite element exterior calculus. *Adv. Comput. Math.* **48** (2022), 36. Id/No 14.

[48] Licht, M. W. On the geometry of star domains and the spectra of hodge-laplace operators under domain approximation. *arXiv preprint arXiv:2411.01655* (2024).

[49] Liu, X. A framework of verified eigenvalue bounds for self-adjoint differential operators. *Appl. Math. Comput.* **267** (2015), 341–355. http://dx.doi.org/10.1016/j.amc.2015.03.048.

[50] Liu, X., and Kikuchi, F. Analysis and estimation of error constants for P_0 and P_1 interpolations over triangular finite elements. *J. Math. Sci. Univ. Tokyo* **17** (2010), 27–78. http://www.ms.u-tokyo.ac.jp/journal/abstract/jms170102.html.

[51] Massey, W. S. *A basic course in algebraic topology*, vol. **127** of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[52] Matculevich, S., and Repin, S. Explicit constants in Poincaré-type inequalities for simplicial domains and application to a posteriori estimates. *Comput. Methods Appl. Math.* **16** (2016), 277–298. https://doi.org/10.1515/cmam-2015-0037.

[53] Mayer, K. H. *Algebraische Topologie.* Birkhäuser Verlag, Basel, 1989. https://doi.org/10.1007/978-3-0348-9269-8.

[54] Moise, E. E. *Geometric Topology in Dimensions 2 and 3.* Springer New York, 1977. http://dx.doi.org/10.1007/978-1-4612-9906-6.

[55] Payne, L. E., and Weinberger, H. F. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.* **5** (1960), 286–292.

[56] Poonen, B. Undecidable problems: a sampler. In *Interpreting Gödel.* Cambridge Univ. Press, Cambridge, 2014, pp. 211–241.

[57] Schrijver, A. A Wiley-Interscience Publication. *Theory of linear and integer programming.* Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Ltd., Chichester, 1986.

[58] Siebenmann, L., and Sullivan, D. *On complexes that are Lipschitz manifolds.* Academic Press, New York-London, 1979, pp. 503–525.

[59] Stern, A. L^p change of variables inequalities on manifolds. *Math. Inequal. Appl.* **16** (2013), 55–67. https://doi.org/10.7153/mia-16-04.

[60] Stern, A. Banach space projections and Petrov-Galerkin estimates. *Numer. Math.* **130** (2015), 125–133. https://doi.org/10.1007/s00211-014-0658-5.

[61] Veeser, A., and Verfürth, R. Poincaré constants for finite element stars. *IMA J. Numer. Anal.* **32** (2012), 30–47. http://dx.doi.org/10.1093/imanum/drr011.

[62] Vohralík, M. On the discrete Poincaré–Friedrichs inequalities for nonconforming approximations of the Sobolev space H^1. *Numer. Funct. Anal. Optim.* **26** (2005), 925–952. http://dx.doi.org/10.1080/01630560500444533.

[63] Vohralík, M. p-robust equivalence of global continuous and local discontinuous approximation, a p-stable local projector, and optimal elementwise hp approximation estimates in H^1. HAL Preprint 04436063, submitted for publication, https://hal.inria.fr/hal-04436063, 2024.

[64] Weber, C. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.* **2** (1980), 12–25. https://doi.org/10.1002/mma.1670020103.

[65] Xu, J., and Zikatanov, L. Some observations on Babuška and Brezzi theories. *Numer. Math.* **94** (2003), 195–202. https://doi.org/10.1007/s002110100308.

[66] Ziegler, G. M. *Lectures on polytopes*, vol. **152** of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. http://dx.doi.org/10.1007/978-1-4613-8431-1.