Intrinsic Finite Element Methods over Manifolds

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(i) Consider the Poisson Problem over a domain Ω:

$$-\nabla \cdot \nabla u = f, \quad u_{|\Gamma_D} = 0, \quad \vec{n} \cdot \nabla u_{|\Gamma_N} = 0.$$

For $f \in L^2(\Omega)$ we seek u in

$$\left\{ u \in H^1(\Omega) \mid \nabla u \in H(\mathrm{div}) \right\}$$

(ii) This is equivalent to seeking $u \in H^1_D(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x, \quad v \in H^1_D(\Omega).$$

In other words, the divergence is applied merely in the sense of distributions.

(i) We may impose inhomogeneous boundary conditions:

$$u_{|\Gamma_D} = g, \quad \vec{n} \cdot \nabla u_{|\Gamma_N} = h.$$

(ii) Let u_g satisfy $\operatorname{tr} u_g = g$ along Γ_D. We then seek $u_0 \in H^1_D(\Omega)$ such that for all $v \in H^1_D(\Omega)$ we have

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv - \nabla u_g \cdot \nabla v \, \mathrm{d}x + \int_{\partial\Omega} h\vec{n} \cdot \nabla v \, \mathrm{d}s.$$

(iii) Hence it makes sense to assume that the right-hand side is a linear functional over $H^1_D(\Omega)$, say,

$$F(v) = \int_\Omega fv - \nabla u_g \cdot \nabla v \, \mathrm{d}x + \int_{\partial\Omega} h\vec{n} \cdot \nabla v \, \mathrm{d}s.$$

(i) Let $A : \Omega \to \mathbb{R}^{n\times n}$ be a symmetric matrix field with eigenvalues uniformly bounded above and below.

(ii) We may generally consider the Poisson Problem with diffusion tensor: Find $u_0 \in H^1_D(\Omega)$ such that

$$\int_\Omega \nabla u \cdot A \nabla v \,\mathrm{d}x = F(v), \quad v \in H^1_D(\Omega).$$

(iii) Wellposedness follows (again) by functional analysis.

(i) **Original Problem:**
Find $u_0 \in H^1_D(\Omega)$ such that

$$\int_\Omega \nabla u \cdot A \nabla v \,\mathrm{d}x = F(v), \quad v \in H^1_D(\Omega).$$

(ii) **Galerkin Formulation:**
Let $V_h \subseteq H^1_D(\Omega)$ be a closed subspace.
Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot A \nabla v_h \,\mathrm{d}x = F(v_h), \quad v_h \in V_h.$$

(iii) Quasi-optimal approximation:

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v \in V_h} \|u - v\|_{H^1(\Omega)}.$$

Let $\mathcal{T}_h$ be a triangulation of the domain and

$$\mathcal{P}_{r,DC}(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) \mid v_{|T} \in \mathcal{P}_r(T),\, T \in \mathcal{T}_h \right\},$$
$$\mathcal{P}_{r,0}(\mathcal{T}_h) := H^1_D(\Omega) \cap \mathcal{P}_{r,DC}(\mathcal{T}_h).$$

The **finite element method** is the Galerkin method with

$$V_h = \mathcal{P}_{r,0}(\mathcal{T}_h).$$

Best approximation, concretized: if $u \in H^s(\Omega)$ with $s \geq 1$, then

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v \in V_h} \|u - v\|_{H^1(\Omega)} \leq C \sum_{T \in \mathcal{T}_h} h_T^{s-1} \|u\|_{H^s(T)}.$$

- The finite element method uses a triangulation to define the approximation space.
- Triangulations can be found only for domains with flat boundary.
- **How to extend FEM to domains with curved boundaries?**
- **More generally, how to extend FEM to manifolds?**

(i) Let Ω be the **physical** domain.

(ii) Let $\widetilde{\Omega}$ be the (polyhedral) **parametric** domain.

(iii) Suppose we have a homeomorphism

$$\Phi : \widetilde{\Omega} \to \Omega$$

such that Φ and Φ^{-1} feature regularity $W^{1,\infty}$.

(iv) In particular, their Jacobians are essentially bounded.
Restriction of Φ to any cell is a diffeomorphism in practice.

(i) **Physical Poisson Problem:** Find $u_0 \in H_D^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot A \nabla v \, \mathrm{d}x = F(v), \quad v \in H_D^1(\Omega).$$

(ii) **Parametric Poisson Problem:** Find $\widetilde{u}_0 \in H_D^1(\widetilde{\Omega})$ such that

$$\int_{\widetilde{\Omega}} \nabla \widetilde{u} \cdot \widetilde{A} \nabla \widetilde{v} \, \mathrm{d}\widetilde{x} = \widetilde{F}(\widetilde{v}), \quad \widetilde{v} \in H_D^1(\widetilde{\Omega}),$$

where we use

$$\widetilde{F}(\widetilde{v}) = F(\widetilde{v} \circ \Phi^{-1}),$$
$$\widetilde{A} = |\det \mathrm{D}\,\Phi| (\mathrm{D}\,\Phi^{-t} \circ \Phi)(A \circ \Phi)(\mathrm{D}\,\Phi^{-1} \circ \Phi).$$

(iii) The solutions are related by $u = \widetilde{u} \circ \Phi^{-1}$.

(i) Find $\widetilde{u}_h \in \mathcal{P}_{r,0}(\mathcal{T})$ such that

$$\int_{\widetilde{\Omega}} \nabla \widetilde{u}_h \cdot \widetilde{A} \nabla \widetilde{v}_h \,\mathrm{d}\widetilde{x} = \widetilde{F}(\widetilde{v}_h), \quad v_h \in \mathcal{P}_{r,0}(\mathcal{T}).$$

We approximate the physical solution by $u_h = \widetilde{u}_h \circ \Phi^{-1}$.

(ii) Quasi-optimal error estimate:

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v_h \in \mathcal{P}_{r,0}(\mathcal{T})} \|u - v_h\|_{H^1(\Omega)} + \textit{Consistency}$$

(iii) **Problem:** We generally have no noteworthy global regularity of $\widetilde{u}$. Standard approximation estimates do not apply.

(iv) **Solution:** For reasonable choices of the coordinate transformation, we have piecewise regularity over each cell $T \in \mathcal{T}_h$. Generalized approximation results exploit this.

Let $\widetilde{u} \in H^1(\widetilde{\Omega})$.

Easy inequality:

$$\inf_{v_h \in \mathcal{P}_{r,DC}(\mathcal{T})} \|\widetilde{u} - v_h\|_{H^1(\widetilde{\Omega})} \leq \inf_{v_h \in \mathcal{P}_{r,0}(\mathcal{T})} \|\widetilde{u} - v_h\|_{H^1(\widetilde{\Omega})}. \qquad (1)$$

Very recent inequality:

$$\inf_{v_h \in \mathcal{P}_{r,0}(\mathcal{T})} \|\widetilde{u} - v_h\|_{H^1(\widetilde{\Omega})} \leq C \inf_{v_h \in \mathcal{P}_{r,DC}(\mathcal{T})} \|\widetilde{u} - v_h\|_{H^1(\widetilde{\Omega})}. \qquad (2)$$

Let $\widetilde{u} = u \circ \Phi$ be the solution of the parametric problem. Then

$$u_{|\Phi(T)} \in H^s(\Omega) \implies \widetilde{u}_{|T} \in H^s(\Omega).$$

Conclusion: despite the lack of global regularity, we get optimal convergence rates thanks to piecewise regularity and (2).

A very recent result:

Theorem

Let $\widetilde{u} \in H^1(\widetilde{\Omega})$ *and* $s \geq 1$ *with* $\widetilde{u}_{|T} \in H^s(T)$ *for each* $T \in \mathcal{T}$*. Then*

$$\inf_{v_h \in \mathcal{P}_{r,0}(\mathcal{T})} \|\widetilde{u} - v_h\|_{H^1(\widetilde{\Omega})} \leq C \sum_{T \in \mathcal{T}_h} h_T^{s-1} \|\widetilde{u}\|_{H^s(T)}.$$

A. Veeser, *Approximating Gradients with Continuous Piecewise Polynomial Functions*.

F. Camacho and A. Demlow, *L_2 and pointwise a posteriori error estimates for FEM for elliptic PDEs on surfaces*.

Areas of Applications

(i) **Domains:** alternative to isoparametric FEM

(ii) **Surface FEM**: see contribution by Camacho and Demlow.

(iii) **NE-FEM**: Nurbs-enhanced finite element methods.

R. Sevilla, S. Fernández-Méndez, and A. Huerta,
NURBS-Enhanced Finite Element Method (NEFEM).

(iv) **Parametric FEM**:

P. Zulian, T. Schneider, K. Hormann, and R. Krause,
Parametric finite elements with bijective mappings.

(v) **Instrinsic FEM over Manifolds**

Extension to Vector-Valued FEM

Goal: numerically solve the Hodge-Laplace equation over manifolds with intrinsic description.

Theorem

Let $\widetilde{u} \in H^s\Lambda^k(\widetilde{\Omega})$ *and* $s \geq 1$ *with* $\widetilde{u}_{|T} \in H^s(T)$ *for each* $T \in \mathcal{T}$*. Then*

$$\inf_{v_h \in \mathcal{P}_{r,0}\Lambda^k(\mathcal{T})} \|\widetilde{u} - v_h\|_{H^1\Lambda^k(\widetilde{\Omega})} \leq C \sum_{T \in \mathcal{T}_h} h_T^{s-1} \|\widetilde{u}\|_{H^s\Lambda^k(T)}.$$

The proof involves a Scott-Zhang-type interpolant for differential forms. *Ongoing work with E. Gawlik and M. Holst*.

Summary

(i) Best approximation error estimates with optimal convergence rates.

(ii) Consistency terms estimated by Strang's lemma and polynomial approximation estimates.

(iii) **Trade-off: simple geometry for non-simple coefficients.**

Thank you for your attention!

Remarks? Questions?