Proof. This proof is similar to the proof of Theorem 4.5. By assumption, there exists a shelling $T_0, T_1, \dots, T_M$. We write $\Omega_m := \cup_{j=0}^{m} T_j$, $0 \leq m \leq M$, which is by Lemma 7.7 a triangulated n-dimensional submanifold with boundary.

MV: I think we should also write down the bound for $\|w\|_{L^p(\Omega)}$.

MV: It might also be good to synchronize the notations X_m and Ω_m. I would go for X_m, since this is a union of n-simplices (closed), whereas Ω_m has a connotation of a domain (open).

Let $u \in W^p\Lambda^k(\Omega)$. First, from Lemma 9.1, there exists $w_0 \in W^p\Lambda^k(T_0)$ satisfying $dw_0 = du$ over T_0 together with

$$\|w_0\|_{L^p(T_0)} \leq C_{\mathrm{PF},k,T_0,p} \|du\|_{L^p(T_0)}.$$

Suppose that $0 < m \leq M$ such that there exists $w_{m-1} \in W^p\Lambda^k(\Omega_{m-1})$ with $dw_{m-1} = du$ over Ω_{m-1}. By the definition of shellability in Section 7.2, T_m and Ω_{m-1} share the interface I_{m-1}, which is a collection of faces of T_m. It follows from Lemma 7.15 that adding T_m completes a star in $\mathcal{T}$ around the simplex $S_{m-1} \in \mathcal{T}$, where $U_{m-1} \subseteq \Omega_{m-1}$ is the complement of T_m in that newly completed star.

MV: already explained above in details.

We define $w'_m := \Xi^*_m(w_{m-1}|_{U_{m-1}}) \in W^p\Lambda^k(T_m)$, where $\Xi_m : T_m \to U_{m-1}$ is the reflection mapping as Ξ_1 in Proposition 8.3 and the pullback is defined in (32). By construction, $w'_m \in W^p\Lambda^k(T_m)$ with

$$\operatorname{tr}_{I_{m-1}} w_{m-1} = \operatorname{tr}_{I_{m-1}} w'_m.$$

Now, in generalization of (29) from the proof of Theorem 4.5 and just like in the two- and three-color maps in [21, Sections 5.3 and 6.3], we define $w''_m \in W^p\Lambda^k(T_m)$ by

$$w''_m := u - w'_m + \Xi^*_m((w_{m-1} - u)|_{U_{m-1}}) = u - \Xi^*_m(u|_{U_{m-1}}). \tag{37}$$

MV: Direct and simple definition. It namely avoids the checking of the compatibility of the boundary condition, which is a bit tedious and requires a rigorous apparatus.

These two equivalent writings immediately yield

$$dw''_m = du - dw'_m, \quad \operatorname{tr}_{I_{m-1}} w''_m = 0,$$

since namely $dw_{m-1} = du$ over Ω_{m-1}. ~~Indeed, such w''_m exists because~~

$$\cancel{\operatorname{tr}_{I_{m-1}}(du - dw'_m)} = \cancel{\operatorname{tr}_{I_{m-1}} du - \operatorname{tr}_{I_{m-1}} dw'_m}$$

$$\cancel{= d\operatorname{tr}_{I_{m-1}} u - d\operatorname{tr}_{I_{m-1}} w'_m = d\operatorname{tr}_{I_{m-1}} u - d\operatorname{tr}_{I_{m-1}} w_{m-1} = 0.}$$

Setting $w_m := w_{m-1}$ over Ω_{m-1} and $w_m := w'_m + w''_m$ over T_m, we find

$$dw_m = dw'_m + dw''_m = dw'_m + du - dw'_m = du,$$
$$\operatorname{tr}_{I_{m-1}} w_m = \operatorname{tr}_{I_{m-1}} w'_m = \operatorname{tr}_{I_{m-1}} w_{m-1}.$$

That means that $w_m \in W^p\Lambda^k(\Omega_m)$ with $dw_m = du$ over Ω_m.

We could also directly define

$$w|_{T_m} := u + \Xi^*_m((w_{m-1} - u)|_{U_{m-1}}),$$

just like in [21, equations (5.12) and (5.14)] or [14, equations (6.7) and (6.9)]

We want to estimate norms. By construction,

$$\|w_m\|_{L^p(T_m)} \leq \|w'_m\|_{L^p(T_m)} + \|w''_m\|_{L^p(T_m)}.$$

Due to Lemma 9.3, we can assume

$$\begin{aligned}\|w''_m\|_{L^p(T_m)} &\leq C_{\mathrm{PF},T_m,I_{m-1},k,p} \|du - dw'_m\|_{L^p(T_m)} \\ &\leq C_{\mathrm{PF},T_m,I_{m-1},k,p} \|du\|_{L^p(T_m)} + C_{\mathrm{PF},T_m,I_{m-1},k,p} \|dw'_m\|_{L^p(T_m)}.\end{aligned}$$