Computable Poincaré–Friedrichs constants for the L^p de Rham complex over convex domains and domains with shellable triangulations

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Abstract

We construct potentials for the gradient, the curl, and the divergence operators, and more generally the exterior derivative, over domains with shellable triangulations. This class of triangulations in particular includes local patches (stars) in dimension two and three. We derive computable bounds on the operator norms of our potentials that only depend on the geometry. We thus obtain computable upper bounds for constants in Poincaré–Friedrichs inequalities as well as computable lower bounds for the eigenvalues of several vector Laplacians. As a result with independent standing, we also establish Poincaré–Friedrichs inequalities with computable constants for the L^p de Rham complex over convex bounded domains. This is achieved through a suitable construction of regularized Poincaré and Bogovskiĭ potential operators where we can bound their operator norms. Computational examples illustrate the theoretical findings.

Contents

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MV: Added table of contents: for us, to see the structure of and the main terminology used in the paper

1 Introduction

MV: Added subsections and reorganized.

Potentials for the differential operators of vector calculus and exterior calculus are of fundamental importance. The operator norms of these potentials are upper bounds for the Poincaré–Friedrichs constants, which quantify the fundamental stability features of a variety of partial differential equations. In numerical methods, estimates for the Poincaré–Friedrichs constants enter the stability and convergence theory and spectral bounds for stiffness matrices. Computable upper bounds on the Poincaré–Friedrichs constants also give computable lower bounds for the eigenvalues of the associated Laplacians. However, while potential operators and Poincaré–Friedrichs constants for the gradient have been subject to extensive study, quantifiable results regarding the curl and divergence operators or more generally the exterior derivative are largely unavailable.

Our main purpose here is to fill this gap. In the gradient case, we can go over with so-called face-connected triangulations. However, to proceed in the general exterior derivative case, we are brought to so-called shellable triangulations. Even though only contractible domains can ever admit a shellable triangulation, having computable upper bounds for such domains is an important stepping stone towards more general cases. Still, we cover most practical nonconvex and non-star-shaped triangulated domains, and as a particular example, local triangulations around a simplex within a larger triangulation (the so-called local patches or stars), which are shellable in two and three spaces dimensions. Over convex bounded domains, we derive Poincaré–Friedrichs inequalities with computable constants for the whole L^p de Rham complex.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be an connected open set.

MV: Here not necessarily bounded? But then we sometimes need $\Omega \subseteq \mathbb{R}^n$ of finite measure ... I would be fine with Ω bounded all the time.

Given any $1 \leq p \leq \infty$, we let $L^p(\Omega)$ denote the Lebesgue space over Ω with integrability exponent p. Then $\boldsymbol{L}^p(\Omega) := L^p(\Omega)^n$ is the corresponding Lebesgue space of vector-valued fields.

1.1 Potential for the gradient operator and the corresponding Poincaré–Friedrichs inequality

Let $W^{1,p}(\Omega)$ denote the first-order Sobolev space over Ω with integrability exponent p,

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) \mid \nabla u \in \boldsymbol{L}^p(\Omega)\}. \tag{1}$$

For a vector-valued field $\boldsymbol{f}$ that is a distributional gradient of an object from $W^{1,p}(\Omega)$, $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$, we are interested in *potentials*, i.e., scalar-valued functions $\Phi_{\mathrm{grad}}(\boldsymbol{f}) \in W^{1,p}(\Omega)$ such that $\nabla \Phi_{\mathrm{grad}}(\boldsymbol{f}) = \boldsymbol{f}$. Then,

$$\Phi_{\mathrm{grad}} : \nabla W^{1,p}(\Omega) \to L^p(\Omega)$$

is the *potential operator*. A particular example of our interest is

$$\Phi_{\mathrm{grad}}(\boldsymbol{f}) := \operatorname*{argmin}_{\substack{u \in W^{1,p}(\Omega) \\ \nabla u = \boldsymbol{f}}} \|u\|_{L^p(\Omega)}. \tag{2a}$$

Actually, in the present gradient setting and since Ω is connected, already the gradient constraint determines $\Phi_{\mathrm{grad}}(\boldsymbol{f})$ up to a constant, so that (2a) is a one-dimensional minimization problem.

We want to see whether the $L^p(\Omega)$ norm of the potential $\Phi_{\mathrm{grad}}(\boldsymbol{f})$ is bounded in terms of the $\boldsymbol{L}^p(\Omega)$ norm of the source field $\boldsymbol{f}$, i.e., we want to evaluate the *operator norm*

$$C_{\mathrm{grad},\Omega,p} := \max_{u \in W^{1,p}(\Omega) \setminus \mathbb{R}} \frac{\|\Phi_{\mathrm{grad}}(\nabla u)\|_{L^p(\Omega)}}{\|\nabla u\|_{\boldsymbol{L}^p(\Omega)}}. \tag{2b}$$

This operator norm is indeed finite and it equivalently turns out to be the best constant in the *Poincaré–Friedrichs inequality*: for every vector field $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$, there exists $u \in W^{1,p}(\Omega)$ such that $\nabla u = \boldsymbol{f}$ and

$$\|u\|_{L^p(\Omega)} \leq C_{\mathrm{grad},\Omega,p} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3a}$$

Actually, (3a) is further equivalent to the more conventional writing

$$\min_{c\in\mathbb{R}} \|u-c\|_{L^p(\Omega)} \le C_{\mathrm{grad},\Omega,p}\|\nabla u\|_{\boldsymbol{L}^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{3b}$$

We call $C_{\mathrm{grad},\Omega,p}$ the *Poincaré–Friedrichs constant* for the gradient operator with exponent p. A concise writing of all (2)–(3) is

$$\min_{\substack{u\in W^{1,p}(\Omega)\\ \nabla u=\boldsymbol{f}}} \|u\|_{L^p(\Omega)} \le C_{\mathrm{grad},\Omega,p}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4}$$

for all $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$.

Considerable research effort has gone into finding (or estimating) the optimal constants in the Poincaré–Friedrichs inequalities (4). Most research on optimal constants has addressed the case of convex domains. Notably, if the constants are required to depend on the convex domain only via its diameter, then the optimal constants $C_{\mathrm{grad},\Omega,p}$ for any $1 \le p < \infty$ are known explicitly [52, 7, 1, 22, 25]. When the domain is not convex but star-shaped with respect to a ball, then polynomial interpolation estimates already imply the Poincaré–Friedrichs inequality [10, 20]. Poincaré–Friedrichs inequalities actually hold over merely star-shaped open sets [38, Theorem 3.1]. We present a more profound discussion below in Section 2, where we also discuss the difference with the Poincaré inequality, where the mean value of u is fixed to zero, and address why we later employ linear potential operators to estimate the Poincaré–Friedrichs constant and not directly the minimizing but nonlinear (for $p \neq 2$) potential operators such as (2a).

MV: OK???

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1.2 Potentials for the curl and divergence operators and the corresponding Poincaré–Friedrichs inequalities

Consider the vector-valued spaces

$$\boldsymbol{W}^p(\mathrm{curl},\Omega) := \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \mid \mathrm{curl}\,\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)\}\,, \tag{5a}$$

$$\boldsymbol{W}^p(\mathrm{div},\Omega) := \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \mid \mathrm{div}\,\boldsymbol{u} \in L^p(\Omega)\}\,. \tag{5b}$$

The members of these spaces are those vector fields whose distributional curls and divergences are respectively in the Lebesgue spaces $\boldsymbol{L}^p$ or $L^p(\Omega)$. This only requires that certain sums of distributional partial derivatives are integrable, and hence these spaces are not classical Sobolev spaces of vector-valued fields.

Let a vector-valued field $\boldsymbol{f}$ that is a distributional curl of an object from $\boldsymbol{W}^p(\mathrm{curl},\Omega)$, $\boldsymbol{f} \in \mathrm{curl}\,\boldsymbol{W}^p(\mathrm{curl},\Omega)$, and scalar-valued field f that is a distributional divergence of an object from $\boldsymbol{W}^p(\mathrm{div},\Omega)$, $f \in \mathrm{div}\,\boldsymbol{W}^p(\mathrm{div},\Omega)$, be given. We are interested in constants for inequalities such as

$$\min_{\substack{\boldsymbol{u}\in \boldsymbol{W}^p(\mathrm{curl},\Omega)\\ \mathrm{curl}\,\boldsymbol{u}=\boldsymbol{f}}} \|\boldsymbol{u}\|_{L^p(\Omega)} \le C_{\mathrm{curl},\Omega,p}\|\boldsymbol{f}\|_{L^p(\Omega)}, \tag{6a}$$

$$\min_{\substack{\boldsymbol{u}\in \boldsymbol{W}^p(\mathrm{div},\Omega)\\ \mathrm{div}\,\boldsymbol{u}=f}} \|\boldsymbol{u}\|_{L^p(\Omega)} \le C_{\mathrm{div},\Omega,p}\|f\|_{L^p(\Omega)}, \tag{6b}$$

and how these constants can be estimated by the operator norms of *linear* potential operators

$$\Phi_{\mathrm{curl}} : \mathrm{curl}\,\boldsymbol{W}^p(\mathrm{curl},\Omega) \to \boldsymbol{W}^p(\mathrm{curl},\Omega),$$
$$\Phi_{\mathrm{div}} : \mathrm{div}\,\boldsymbol{W}^p(\mathrm{div},\Omega) \to \boldsymbol{W}^p(\mathrm{div},\Omega).$$

The fundamental difference with respect to Section 1.1 is that the curl and divergence have infinite-dimensional kernels. In Section 1.1, the kernel of the gradient is the one-dimensional space of constant functions. It is thus trivially complemented for all p, with a canonical choice of projection. By contrast, in the Banach space case $p \neq 2$, it is not immediately evident that the kernels of the curl and divergence operators are complemented, and a canonical projection only exists here in the Hilbert setting $p = 2$. Actually, the existence of a bounded linear potential operator is here a non-trivial fact even in the Hilbert space case $p = 2$.

Let for a moment the topology of Ω be trivial, let Ω be weakly Lipschitz, and consider the Hilbert case $p = 2$. Then, (6a) is related to the so-called Poincaré–Friedrichs–Weber inequality

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq \tilde{C}_{\mathrm{curl},\Omega,p} \|\operatorname{curl} \boldsymbol{u}\|_{L^2(\Omega)}, \tag{7}$$

for all $\boldsymbol{u} \in \boldsymbol{W}^2(\mathrm{curl}, \Omega) \cap \boldsymbol{W}^2(\mathrm{div}, \Omega)$ with either the tangential or the normal trace zero on $\partial\Omega$ and $\operatorname{div} \boldsymbol{u} = 0$. Yet another writing is

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq \tilde{C}_{\mathrm{curl},\Omega,p} \|\operatorname{curl} \boldsymbol{u}\|_{L^2(\Omega)}, \tag{8}$$

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq \tilde{C}_{\mathrm{div},\Omega,p} \|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega)} \tag{9}$$

for respectively all functions $\boldsymbol{u}$ from $\boldsymbol{W}^2(\mathrm{curl}, \Omega)$ such that $\int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{v}(x)\, dx = 0$ for all $\boldsymbol{v}$ from $\boldsymbol{W}^2(\mathrm{curl}, \Omega)$ with $\operatorname{curl} \boldsymbol{v} = 0$ and all functions $\boldsymbol{u}$ from $\boldsymbol{W}^2(\mathrm{div}, \Omega)$ such that $\int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{v}(x)\, dx = 0$ for all $\boldsymbol{v}$ from $\boldsymbol{W}^2(\mathrm{div}, \Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$. We refer to Friedrichs [26, equation (5)], Gaffney [27, equation (2)], Weber [60], [30, Lemmas 3.4 and 3.6], [24, Proposition 7.4], and the references therein. Still in the Hilbert space case $p = 2$, the Friedrichs inequality [11] over the Sobolev space with homogeneous Dirichlet boundary conditions on $\partial\Omega$ implies, via duality (6b). However, that easy observation seems restricted to $p = 2$.

Not much attention, however, seems to have been given to the study of computable constants in (6), even if the domain Ω is convex. Important results in the Hilbert case $p = 2$ can be found in [34, 51] that we discuss now in the broder setting of the exterior calculus formalism, but the Banach case $p \neq 2$ seems largely unexplored.

1.3 Potentials for the exterior derivative and the corresponding Poincaré–Friedrichs inequalities

Exterior calculus [32, 42] is used ubiquitously in the mathematical literature of physics and engineering and has found widespread adoption in the theoretical and numerical analysis for vector-valued partial differential equations [36, 33, 2, 3, 4, 19, 29, 5]. This formalism is independent of the space dimension and highlights extends the underlying geometric structures common to the gradient, curl, and divergence operators in three space dimensions. In this setting, the common writing of the Poincaré–Friedrichs inequalities we are interested in is just as (3a): for every $w \in W^p\Lambda^k(\Omega)$, there exists $u \in W^p\Lambda^k(\Omega)$ with $du = dw$ and such that (all the notation is in detail fixed in Section 5 below)

$$\|u\|_{L^p(\Omega} \leq C_{k,\Omega,p} \|dw\|_{L^p(\Omega)}. \tag{10}$$

MV: The roles of u and w are flipped in the various parts of the paper, see (10) in comparsion with Lemma 9.1. It would be great to unify.

In the Hilbert space case $p = 2$, Guerini and Savo [34] provide lower bounds for the spectrum of the Hodge–Laplace operator on convex domains with smooth boundary, thus giving upper bounds for the Poincaré–Friedrichs constant of the exterior derivative, and hence in particular for the curl and divergence operators. A comparison result shows that the Poincaré–Friedrichs constant for the gradient already estimates the corresponding constants for the curl and divergence operators. By duality, this also gives upper estimates of the Poincaré–Friedrichs constants for the gradient, curl, and divergence operators subject to Dirichlet, tangential, and normal boundary conditions, respectively, along the entire boundary. Computable constants for Lipschitz domains have recently been addressed in Pauly and Valdman [51], see also the references therein.

Unfortunately, no results as those of [34, 51] are known over (convex) (Lipschitz) domains and general Lebesgue exponents $1 \leq p \leq \infty$. A conceptual approach to obtain computable estimates for the Poincaré–Friedrichs constants for any $1 \leq p \leq \infty$ and when the domain is star-shaped with respect to a ball is to bound operator norms of regularized Poincaré operators such as those of Costabel and McIntosh [18] as mappings between Lebesgue spaces. Here, estimates for the higher-order seminorms of these potentials are available in [35], but estimates in Lebesgue norms have not been made explicit in the literature yet, to the best of our knowledge.

1.4 Numerically computed guaranteed bounds on Poincaré–Friedrichs constants seen as eigenvalues

The constant in (4) for $p = 2$ corresponds to a lower bound for the Laplace eigenvalues and quantifies the stability properties of the Laplacian on the domain Ω. Similarly, the constant in (6a) for $p = 2$ corresponds to a lower bound for the Maxwell eigenvalues and quantifies the stability properties of the Maxwell system on Ω. Thus, computable upper bounds on the Poincaré–Friedrichs constants also give computable lower bounds for the eigenvalues of the associated Laplacians and vice versa. This eigenvalues optics, upon employing a suitable numerical simulation over a general triangulation and some clever postprocessing estimates, enables to numerically compute guaranteed upper bounds on the Poincaré–Friedrichs constants, see [12, 45] for (4) and [28] for (6a). Our approaches, in contrast, are either analytical or combinatorial, and in any case we only employ the given geometry but no computer simulation.

1.5 Objectives and methodology

The main objective of this manuscript is the construction of linear potential operators with explicitly computable upper bounds on their operator norms (i.e., of computable bounds on the Poincaré–Friedrichs constants) for the gradient, curl, and divergence operators, and more broadly in the general case (10). We consider not necessarily convex subdomains and focus on local patches (stars) of simplicial elements within triangulated domains, but we will also devote an important effort specifically to convex domains.

1.5.1 Combinatorial constants over face-connected triangulated domains

Domains that admit finite triangulations allow for techniques using some form of passing through the triangulation and construction of the potentials step-by-step. For scalar-valued functions in the context of Section 1.1, Veeser and Verfürth [57] provide computable upper bounds in the case of the classical Sobolev space $W^{1,p}(\Omega)$ over triangulated domains, with focus on efficient estimates for vertex stars. Such approaches were also previously used in the context of finite volume methods [23], broken (weakly continuous) Sobolev spaces [58], or more recently in continuous–discrete comparison results [9, 21, 14, 59]. Triangle by triangle, the potential is constructed over increasing subdomains, matching along the interfacing intersections, until the entire domain is exhausted.

In the gradient case of Section 1.1, it is possible to perform a simple pass through the triangulation, where merely each subsequent simplex has to be connected to the previous ones by a $(n-1)$-dimensional face; face-connected triangulations are sufficient here. We always construct a gradient potential over the new simplex, and since the gradient kernel is trivial, formed merely by constants, it is enough to fix this constant using the value already known on the $(n-1)$-dimensional face.

1.5.2 Combinatorial constants over shellable triangulated domains

For vector-valued fields, new challenges arise because of the infinite-dimensional kernels. We, however, overcome these challenges and succeed in generalizing the sequential construction of the potential operator for the gradient to vector potentials, including in particular the curl and divergence. The basic inductive strategy remains the same. We start by constructing, say, a curl potential, over a single simplex. Having already defined a potential operator over a set of simplices that we term a subtriangulation, we then construct a potential over a simplex neighboring through some $(n-1)$-dimensional faces. However, the potential on the new simplex might here not have the same tangential trace along the interface between the new simplex and the previous ones. Actually, the passage through the triangulation has to be such that the intermediate subtriangulations are manifolds with boundary and the interface is a boundary submanifold of the intermediate subtriangulation and the new simplex. Then, we can define a well-posed mapping from the already visited simplices. Repeating this procedure exhausts the original triangulation and eventually provides the desired potential. These stronger conditions on the triangulations are exactly those of a so-called *shellable* triangulations. Shellability of simplicial complexes and, more generally, of polytopal complexes is a well-established notion in discrete geometry and combinatorics, see, e.g., Kozlov [39] or Ziegler [62]. With respect to our main interest, local patches (stars) in 2D and 3D triangulations are shellable.

1.5.3 Analytical constants over convex domains

We devote a considerable effort to convex domains for two reasons: first, they are evidently of independent interest, and second, we need them as a building block for the above-discussed recursive procedures on general (nonconvex and non-star-shaped) triangulated domains. Here, we follow closely the concepts of Costabel and McIntosh [18] who study regularized Poincaré and Bogovskiĭ-type potential operators. In comparison to their exposition, which allows for domains star-shaped with respect to a ball, the regularized potential operators we propose are simplified: we only study them over convex domains and use simpler (constant) weight functions. The resulting operator features generally lower regularity but its properties are sufficient for our purposes. Crucially, this allows us to easily estimate their operator norms.

1.5.4 Domains star-shaped with respect to a ball

Let us only briefly discuss why don't we treat local patches (stars) in triangulations as domains star-shaped with respect to a ball, which they are. This would enable the use of averaged Poincaré and Bogovskiĭ operators [18]. Unfortunately, estimates that rely on this geometric condition deteriorate when the aforementioned ball has radius much smaller than the domain diameter. This would not be as much a problem over local patches (stars) around interior subsimplices, where the size of the interior ball only depends on the shape regularity of the triangulation. Unfortunately, the interior ball can be arbitrarily small when the local patch is around a boundary simplex, even if the mesh has good shape regularity: this in particular occurs when the domain has sharp reentrant corners. Some illustrative limit cases include the slit domain [57] and the crossed bricks domain [44], which contain local finite element patches that are not star-shaped with respect to any ball.

1.5.5 Potentials subject to partial boundary conditions

In our proofs for triangulated domains, we actually also need to find potentials for the gradient, curl, and divergence operators, and in general the exterior derivative, over a simplex subject to partial boundary conditions. Again, we want to address the entire range $1 \leq p \leq \infty$ of Lebesgue exponents. We are not aware of explicit estimates in the literature and we know no regularized potential operators for these boundary conditions. Our basic idea here is to reuse the Poincaré–Friedrichs inequalities subject to boundary conditions on the entire boundary in order to prove Poincaré–Friedrichs inequalities over simplices and subject to partial boundary conditions. For example, given a divergence-free vector field over a tetrahedron with vanishing normal trace along three of the tetrahedron's faces, we want to find the unique vector field potential that is the preimage under the curl operator and which has vanishing tangential trace along the same three faces. We achieve this by an affine "symmetrization" map of the tetrahedron to a domain consisting of two tetrahedra, construction of an auxiliary problem subject to full boundary conditions, and then affine "folder" map back to the tetrahedron.

1.6 Main results

We treat specifically the gradient operator (the setting of Section 1.1) on triangulated domains that can be nonconvex and non-star-shaped but where the triangulations are face connected (though not necessarily shellable). Here, Theorems 4.2 and 4.5 are our main results. The computable Poincaré–Friedrichs constants obtained here depend on only a few parameters of the given triangulation: the length of any traversal from the root simplex, the ratios of the volumes of the adjacent simplices, the Poincaré–Friedrichs constants on each face patch, and the Poincaré–Friedrichs constant of the initial simplex. Here, the face patch Poincaré–Friedrichs constants are estimated in terms of shape regularity parameters of the triangulation; if the face patches of the triangulation are convex, then better estimates are possible.

MV: remark from after Theorems 4.2 and 4.5 moved here.

Our main result for the curl and divergence operators (Section 1.2) and more generally in the exterior calculus setting (Section 1.3) for triangulated domains are Theorems 9.4 and 9.5. They treat triangulated domains that can be nonconvex and non-star-shaped but where the triangulation is shellable. As discussed above, though this is slightly less generic than merely face-connected triangulations, this namely covers patches of simplices sharing a given vertex, edge, face, or n-simplex (stars) in two and three space dimensions, since these local patches are shellable. The dependencies of the obtained computable Poincaré–Friedrichs constants are similar to those of the gradient operator in Theorems 4.2 and 4.5.

Finally, our main result for convex domains in the exterior calculus setting (Section 1.3) is the construction of regularized potential operators for the exterior derivative with explicitly bounded operator norms. This is summarized in Theorem 6.1. The arising upper bounds for the Poincaré–Friedrichs constants are proportional to the domain diameter and are independent of the Lebesgue exponent $1 \leq p \leq \infty$, though the space dimension and the form degree enter~~s~~ the estimate~~s~~.

MV: The discussion from before Theorem 6.1 resumed here.

1.7 Organization of this manuscript

The remainder of this manuscript is structured as follows. We review Poincaré–Friedrichs constants for the gradient over convex domains in Section 2. We also discuss there the difference with the Poincaré inequality and motivate our interest in linear potential operators. We review basic notions of triangulations in Section 3. We develop computable upper bounds for the Poincaré–Friedrichs constants for the gradient over face-connected triangulated domains in Section 4. We pass to the Sobolev spaces in vector calculus and the calculus of differential forms in Section 5. Our regularized potential operators over convex sets are then introduced in Section 6, giving rise to to computable Poincaré–Friedrichs constants as their operator norms. We subsequently pass to a review of shellable triangulations of manifolds in Section 7. We construct an important geometric reflection operator in Section 8 and we finally provide computable upper bounds for Poincaré–Friedrichs constants for the exterior derivative over shellable triangulations in Section 9. We present numerical examples in Section 10 and conclude by some outlook in Section 11.

2 Nonlinear and linear potentials, the Poincaré–Friedrichs and Poincaré inequalities, and analytical Poincaré–Friedrichs constants in the gradient case

MV: Discussion reorganized.

In this section, we wish to explain why we employ linear potential operators to estimate the Poincaré–Friedrichs constants and we also address the difference of the Poincaré–Friedrichs inequality (3b) with the Poincaré inequality where the mean value of u is fixed to zero. We then review Poincaré–Friedrichs inequalities in the gradient case setting of Section 1.1, with the emphasis on know results on analytical upper bounds for the constants. We in particular take the opportunity to discuss convex domains as a special case. This survey serves as a building block in constructing computable constants over triangulated domains in a combinatorial way below but we believe it is also of independent interest.

2.1 Nonlinear and linear potentials

In the Banach setting $p \neq 2$, finding a norm-minimizing potential such as $\Phi_{\mathrm{grad}}(\boldsymbol{f})$ of (2a) for a given gradient $\boldsymbol{f}$ is a nonlinear operation. Its existence may actually not be guaranteed unless $1 < p < \infty$. However, any *linear* potential operator $\overline{\Phi}_{\mathrm{grad}} : \nabla W^{1,p}(\Omega) \to L^p(\Omega)$ satisfying $\nabla \overline{\Phi}_{\mathrm{grad}}(\boldsymbol{f}) = \boldsymbol{f}$ for any $\boldsymbol{f} \in \nabla W^{1,p}(\Omega)$ has the operator norm that gives the upper bound

$$C_{\mathrm{grad},\Omega,p} = \max_{u \in \cancel{\nabla} W^{1,p}(\Omega) \setminus \mathbb{R}} \frac{\|\Phi_{\mathrm{grad}}(\nabla u)\|_{L^p(\Omega)}}{\|\nabla u\|_{L^p(\Omega)}} \leq \max_{u \in \cancel{\nabla} W^{1,p}(\Omega) \setminus \mathbb{R}} \frac{\|\overline{\Phi}_{\mathrm{grad}}(\nabla u)\|_{L^p(\Omega)}}{\|\nabla u\|_{L^p(\Omega)}} =: \overline{C}_{\mathrm{grad},\Omega,p}. \tag{11}$$

This explains our interest in the construction of *linear potential operators*.

2.2 Poincaré–Friedrichs and Poincaré inequalities

In the setting of Section 1.1, when Ω is of finite measure, a natural (and unique, since the gradient potential constraint $\nabla\overline{\Phi}_{\mathrm{grad}}(\boldsymbol{f}) = \boldsymbol{f}$ determines $\overline{\Phi}_{\mathrm{grad}}(\boldsymbol{f})$ up to a constant) choice for a linear potential operator is the average-free potential operator $\overline{\Phi}_{\mathrm{grad}} : \nabla W^{1,p}(\Omega) \to L^p(\Omega)$ that is prescribed by

$$\overline{\Phi}_{\mathrm{grad}}(\nabla u) := u - u_\Omega, \quad \forall u \in W^{1,p}(\Omega),$$

where u_Ω denotes the average of u over Ω, that is,

$$u_\Omega := \mathrm{vol}(\Omega)^{-1} \int_\Omega u(x)\, dx.$$

This potential operator can also implicitly be defined by

$$\overline{\Phi}_{\mathrm{grad}}(\boldsymbol{f}) := \underset{\substack{u \in W^{1,p}(\Omega) \\ \int_\Omega u = 0}}{\operatorname{argmin}} \|\nabla u - \boldsymbol{f}\|_{L^p(\Omega)}, \quad \forall \boldsymbol{f} \in \nabla W^{1,p}(\Omega).$$

Its operator norm is the best constant $\overline{C}_{\mathrm{grad},\Omega,p} > 0$ in the *Poincaré inequality* (or also Poincaré–Wirtinger inequality)

$$\|u - u_\Omega\|_{L^p(\Omega)} \le \overline{C}_{\mathrm{grad},\Omega,p} \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{12}$$

Thus, rephrasing (11) in terms of (3b) and (12),

$$C_{\mathrm{grad},\Omega,p} \le \overline{C}_{\mathrm{grad},\Omega,p}.$$

Towards a converse inequality, let us first observe that the average of any $u \in W^{1,p}(\Omega)$ with $p < \infty$ satisfies the bound

$$\|u_\Omega\|^p_{L^p(\Omega)} = \int_\Omega \left(\mathrm{vol}(\Omega)^{-1} \int_\Omega |u(x)|\, dx \right)^p \le \int_\Omega \mathrm{vol}(\Omega)^{-1} \int_\Omega |u(x)|^p\, dx = \|u\|^p_{L^p(\Omega)}. \tag{13}$$

Here, we have used Hölder's or Jensen's inequality. In the case $p = \infty$, any $u \in L^\infty(\Omega)$ satisfies $\|u_\Omega\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$. We conclude that taking the average is a projection within Lebesgue spaces with unit norm. The triangle inequality now shows that

$$\|u - u_\Omega\|_{L^p(\Omega)} \le 2\|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega).$$

Thus, the Poincaré–Friedrichs inequality implies the Poincaré inequality with

$$C_{\mathrm{grad},\Omega,p} \le 2\overline{C}_{\mathrm{grad},\Omega,p}.$$

In the special case $p = 2$, since the average becomes the orthogonal projection, and this improves to $\|u - u_\Omega\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}$ for any $u \in L^2(\Omega)$. Hence,

$$\overline{C}_{\mathrm{grad},\Omega,2} = C_{\mathrm{grad},\Omega,2}.$$

From (13), Stern's generalized projection estimate [56, Theorem 4.1, Remark 5.1] implies improved estimate for all L^p spaces with $1 \le p \le \infty$ as

$$\|u - u_\Omega\|_{L^p(\Omega)} \le \min\left(2, 2^{|2/p-1|}\right) \|u\|_{L^p(\Omega)} = 2^{|2/p-1|}\|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega),$$

where we have also used $1 \le 2^{|2/p-1|} \le 2$ for $1 \le p \le \infty$. The upper bound is only attained in the limit cases $p = 1$ and $p = \infty$, and the lower bound is only attained if $p = 2$, where we reproduce the improved estimate for the Hilbert space case. We thus conclude

$$C_{\mathrm{grad},\Omega,p} \le 2^{|2/p-1|}\overline{C}_{\mathrm{grad},\Omega,p}. \tag{14}$$

In summary, our notion of Poincaré–Friedrichs constant is equivalent to the notion of Poincaré constant in the literature, up to a numerical factor that only depends on $1 \le p \le \infty$ and that is at most 2.

2.3 Analytical constants in Poincaré–Friedrichs inequalities over convex domains

We collect examples for Poincaré and Poincaré–Friedrichs inequalities for the important special case of convex domains. Over convex bounded domains, we have the Poincaré inequalities [52, 7, 1] or [20, Lemma 3.24]

$$\|u - u_\Omega\|_{L^1(\Omega)} \leq \frac{\delta(\Omega)}{2} \|\nabla u\|_{L^1(\Omega)}, \quad \forall u \in W^{1,1}(\Omega), \tag{15}$$

$$\|u - u_\Omega\|_{L^2(\Omega)} \leq \frac{\delta(\Omega)}{\pi} \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in W^{1,2}(\Omega), \tag{16}$$

where $\delta(\Omega)$ is the diameter of the domain Ω. These two estimates are the best possible Poincaré inequalities in the cases $p = 1$ and $p = 2$, respectively, in terms of the diameter alone. Upper bounds for the Poincaré constant over convex domains with $1 < p < \infty$ are known in the literature [17, Theorem 1.1, Theorem 1.2]:

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C_{\mathrm{P,CW},p} \delta(\Omega) \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \tag{17}$$

where we use an upper bound by Chua and Wheeden:

$$C_{\mathrm{P,CW},p} := \sup_{v \in C^\infty([0,1]) \setminus \mathbb{R}} \frac{\|v - v_{[0,1]}\|_{L^p([0,1])}}{\|\nabla v\|_{L^p([0,1])}} \leq \sqrt[p]{p} 2^{1-\frac{1}{p}} = 2 \left(\frac{p}{2}\right)^{\frac{1}{p}}.$$

Note that $C_{\mathrm{P,CW},p}$ is generally not optimal among the upper bounds that only depend on the domain diameter and the Lebesgue exponent. As discussed above, these Poincaré inequalities imply Poincaré–Friedrichs inequalities.

We know optimal Poincaré–Friedrichs constants over convex domains ([25, Theorem 1.1], [22, Theorem 1.1]): when $1 < p < \infty$, one can show that

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^p(\Omega)} \leq C_{\mathrm{PF,EFNT},p} \delta(\Omega) \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega), \tag{18}$$

where $C_{\mathrm{PF,EFNT},p}$ is the best possible constant that only depends on p and equals

$$C_{\mathrm{PF,EFNT},p} := \frac{p \sin(\pi/p)}{2\pi \sqrt[p]{p-1}}.$$

Note that the last inequalities from (14) imply, again when $1 < p < \infty$, the Poincaré inequalities

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq 2^{|1-\frac{2}{p}|} C_{\mathrm{PF,EFNT},p} \delta(\Omega) \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{19}$$

When $p = 1$, then the optimal Poincaré constant $1/2$ also bounds the Poincaré–Friedrichs constant:

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^1(\Omega)} \leq \frac{\delta(\Omega)}{2} \|\nabla u\|_{L^1(\Omega)}, \quad \forall u \in W^{1,1}(\Omega). \tag{20}$$

When $p = \infty$, since convex domains are Lipschitz domains, Rademacher's theorem leads to

$$\min_{c \in \mathbb{R}} \|u - c\|_{L^\infty(\Omega)} \leq \delta(\Omega) \|\nabla u\|_{L^\infty(\Omega)}, \quad \forall u \in W^{1,\infty}(\Omega). \tag{21}$$

Remark 2.1. *Any estimate for the Poincaré–Friedrichs constant implies an estimate for the Poincaré constant via* (14)*. Let us compare the estimate $C_{\mathrm{P,CW},p}$ for the Poincaré constant with the estimate obtained from the Poincaré–Friedrichs constant $C_{\mathrm{PF,EFNT},p}$. In the case $2 \leq p$,*

$$2^{1-\frac{2}{p}} C_{\mathrm{PF,EFNT},p} = \frac{2^{1-\frac{2}{p}}}{2} \frac{\sin(\pi/p)}{\pi/p} \frac{1}{\sqrt[p]{p-1}} \leq 4^{-\frac{1}{p}} \leq C_{\mathrm{P,CW},p}.$$

In the case $2 \geq p$,

$$2^{\frac{2}{p}-1} C_{\mathrm{PF,EFNT},p} = \frac{2^{\frac{2}{p}-1}}{2} \frac{\sin(\pi/p)}{\pi/p} \frac{1}{\sqrt[p]{p-1}} \leq \frac{2^{\frac{2}{p}-1}}{2} = 2^{\frac{2}{p}-2}$$
$$= 4^{\frac{1}{p}-1} \leq C_{\mathrm{P,CW},p}.$$

It follows that (19) *is generally a tighter estimate than* (17) *for* $1 < p < \infty$.

Remark 2.2. *The above Poincaré and Poincaré–Friedrichs constants are optimal for the class of convex domains, but individual convex domains may allow for better constants. We refer to [46, 13, 48] for discussions; for example, triangles allow the reduction of the constant by 20%.*

3 Basic notions of triangulations

We gather here basic notions and definitions concerning simplicial meshes.

3.1 A triangulation or a simplicial complex

A k-dimensional *simplex* T is the convex hull of $k+1$ affinely independent points $v_0, v_1, \dots, v_k \in \mathbb{R}^n$. We call these points the *vertices* of the simplex T. If S is a simplex whose vertices are also vertices of another simplex T, in which case $S \subseteq T$, then we call S a *subsimplex* of T and call T a *supersimplex* of S.

The strictly positive convex combinations of the vertices of the simplex constitute the *interior* of the simplex, and its remaining points constitute the *boundary* of the simplex.

A finite family of simplices $\mathcal{T}$ is a *simplicial complex* or *triangulation*, if it satisfies the following conditions: (i) $\mathcal{T}$ contains all the subsimplices of its members (ii) any non-empty intersection of two members of $\mathcal{T}$ is a common subsimplex of each of them. We say that a simplicial complex $\mathcal{T}$ has *dimension* n or is n*-dimensional* if each of its simplices is a subset of an n-dimensional member of that triangulation.[1] We also write $|\mathcal{T}|$ for the *underlying set* of the simplicial complex $\mathcal{T}$, which is defined as the union

MV: Don't we only use n-dimensional simplicial complexes? That would be a simplification of our presentation.

$$|\mathcal{T}| := \bigcup_{T \in \mathcal{T}} T.$$

We call a set *triangulable* if it is the underlying set of some triangulation.

Remark 3.1. *Our main interest in this manuscript are finite triangulations of compact sets. There is another reason why we insist on the triangulation being finite: if we do not require our simplicial complexes to be finite, then a reasonable definition of simplicial complex would need additional topological conditions. For example, if we modify our definition of simplicial complex to be infinite, then any subset of Euclidean space would give rise to an "infinite* 0*-dimensional complex". But this obviously does not reflect the topology. A more interesting such example is this: the Cantor set* $\mathcal{C} \subseteq [0,1]$ *is a compact set that underlies an "infinite* 0*-dimensional simplicial complex". If we place* $\mathcal{C}$ *on the x-axis of a 2D coordinate system and connect each member of* $\mathcal{C}$ *to* $(0,1) \in \mathbb{R}^2$ *via a straight line segment, then the resulting set is still compact, even path-connected, but has an "infinite* 1*-dimensional triangulation". However, none of these infinite families are infinite simplicial complexes in the sense of geometric topology [41].*

Given any simplex T, we write $\mathcal{S}^{\downarrow}(T)$ for the simplicial complex that contains all subsimplices of T. More specifically, $\mathcal{S}^{\downarrow}_k(T) \subseteq \mathcal{S}^{\downarrow}(T)$ denotes the set of k-dimensional subsimplices of T. Whenever $\mathcal{T}$ is a simplicial complex and $T \in \mathcal{T}$, we let $\mathcal{S}^{\uparrow}(\mathcal{T}, T)$ be the set of simplices in $\mathcal{T}$ that contain T. We write $\mathcal{S}^{\uparrow}_k(\mathcal{T}, T)$ for the set of k-dimensional simplices in $\mathcal{S}^{\uparrow}(\mathcal{T}, T)$. The set of k-dimensional simplices in $\mathcal{T}$ is denoted as $\mathcal{S}^{\downarrow}_k(\mathcal{T})$. Additionally, the notations $\mathcal{V}(\mathcal{T}) := \mathcal{S}^{\downarrow}_0(\mathcal{T})$ and $\mathcal{E}(\mathcal{T}) := \mathcal{S}^{\downarrow}_1(\mathcal{T})$ refer to the vertices and edges of

[1] Simplicial complexes that we call n-dimensional are called purely n-dimensional in the literature on polytopes (cf. [62]) and simply "simplicial meshes" in the finite element literature.

the simplicial complex, respectively. In practice, we not always distinguish between points and singleton simplices. Similarly, if $\mathcal{T}$ is an n-dimensional simplicial complex, we write $\mathcal{F}(\mathcal{T}) := \mathcal{S}^{\downarrow}_{n-1}(\mathcal{T})$ for the faces (that is, codimension one members) of this triangulation.[2]

MV: Do we really need all these notions?

When $\mathcal{T}$ is a triangulation and $T \in \mathcal{T}$, then $\mathrm{st}_{\mathcal{T}}(T)$ denotes the *local patch* or *star* of T, which is the simplicial subcomplex of $\mathcal{T}$ that contains all supersimplices of T and their subsimplices. We write $\partial\mathrm{st}_{\mathcal{T}}(T)$ for the subset of the local patch whose members do not contain T itself. Formally,

$$\mathrm{st}_{\mathcal{T}}(T) := \bigcup_{T' \in \mathcal{S}^{\uparrow}(\mathcal{T},T)} \mathcal{S}^{\downarrow}(T'), \qquad \partial\mathrm{st}_{\mathcal{T}}(T) := \bigcup_{\substack{T' \in \mathrm{st}_{\mathcal{T}}(T) \\ T \nsubseteq T'}} \mathcal{S}^{\downarrow}(T').$$

We write $U_T := |\,\mathrm{st}_{\mathcal{T}}(T)|$ for the underlying set of the local patch. A crucial observation is the following.

Lemma 3.2. *Let $\mathcal{T}$ be an n-dimensional simplicial complex and let $S, S' \in \mathcal{T}$. Then either $\mathrm{st}_{\mathcal{T}}(S) \cap \mathrm{st}_{\mathcal{T}}(S') = \emptyset$ or there exists $S'' \in \mathcal{T}$ such that $\mathrm{st}_{\mathcal{T}}(S) \cap \mathrm{st}_{\mathcal{T}}(S') = \mathrm{st}_{\mathcal{T}}(S'')$ with $S, S' \subseteq S''$.*

Proof. Let $T \in \mathcal{T}$ be n-dimensional. We have $T \in \mathrm{st}_{\mathcal{T}}(S)$ if and only if all vertices of S are vertices of T. We have $T \in \mathrm{st}_{\mathcal{T}}(S')$ if and only if all vertices of S' are vertices of T. Consequently, $T \in \mathrm{st}_{\mathcal{T}}(S) \cap \mathrm{st}_{\mathcal{T}}(S')$ if and only if $T \in \mathrm{st}_{\mathcal{T}}(S'')$, where $S'' \in \mathcal{T}$ is the convex closure of S and S'. □

We introduce a specific notion of connectivity. Given an n-dimensional simplicial complex $\mathcal{T}$, we call n-simplices $S, S' \in \mathcal{T}$ *face connected in* $\mathcal{T}$ if there exists a sequence $S_0 = S, S_1, \dots, S' = S_m \in \mathcal{T}$ such that $S_i \cap S_{i-1}$ is a face of both S_i and S_{i-1} for all $1 \leq i \leq m$. We call such a sequence a *face path* from S to S' in $\mathcal{T}$. Clearly, face connected in $\mathcal{T}$ is an equivalence relation among simplices. A *face-connected component* of $\mathcal{T}$ is an equivalence class under this equivalence relation, and we call $\mathcal{T}$ *face connected* if it has only one face-connected component.

3.2 Quantitative measures for the geometric quality of triangulations

We now shift gears and move from a combinatorial study of triangulations to an analytical study: we introduce quantitative measures for the geometric quality of triangulations. These indicate of the regularity of simplex shapes. Most of such *shape measures* in the literature are based on geometric features, but it will also be helpful to use indicators based on affine transformations associated with the mesh. All these quantities have in common that they can be computed from purely local information. While there are many equivalent shape measures, we use the ones that most immediately appear in our proofs.

MWL: The different geometric quantities in this section should really be just what is needed later. Any shape measure should be locally measurable and close to what appears in the right-hand sides of the main results. Any inequalities here should be as tight as possible.

We write $\delta(T)$ and $\mathrm{vol}(T)$ for the diameter and n-dimensional volume of any n-simplex T. Moreover, $h(T)$ refers to the smallest height of any of the vertices of the simplex T.

For the purpose of the usual scaling arguments, the *n-dimensional reference simplex* $\Delta^n \subseteq \mathbb{R}^n$ is the convex closure of the origin and the n canonical unit vectors.

Given a matrix $A \in \mathbb{R}^{n \times n}$, we let $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$ denote the descending sequence of its singular values, where $\sigma_{\max}(A) := \sigma_1(A)$ is the largest and $\sigma_{\min}(A) := \sigma_n(A)$ is the smallest. We recall that the ℓ^2 operator norm of A, $\|A\|_2$, agrees with $\sigma_{\max}(A)$, and, if A is invertible, then $\|A^{-1}\|_2 = \sigma_{\min}(A)^{-1}$.

Whenever T is any n-dimensional simplex T, we define the *geometric shape measure* $\kappa(T)$, the *aspect shape measure* $\kappa_{\mathrm{A}}(T)$, and the *algebraic shape measure* $\kappa_2(T)$ by

$$\kappa(T) := \frac{\delta(T)^n}{n!\,\mathrm{vol}(T)}, \qquad \kappa_{\mathrm{A}}(T) := \frac{\delta(T)}{h(T)}, \qquad \kappa_2(T) := \sup_{\varphi:\Delta^n \to T} \frac{\sigma_{\max}(\mathrm{Jac}\,\varphi)}{\sigma_{\min}(\mathrm{Jac}\,\varphi)}, \tag{22}$$

[2] Our use of the term *face* as is common in the finite element literature [10] and is synonymous with *facet* as used in the literature on polyhedral combinatorics [54]. Notably, this terminology differs from the uses *face* and *facet* in the theory of polyhedra [62].

where the last supremum is taken over all affine transformation from the reference n-simplex onto the n-simplex T.

When $\mathcal{T}$ is an n-dimensional simplicial complex, we naturally introduce

$$\kappa(\mathcal{T}) := \sup_{T \in \mathcal{S}_n^{\downarrow}(\mathcal{T})} \kappa(T), \quad \kappa_{\mathrm{A}}(\mathcal{T}) := \sup_{T \in \mathcal{S}_n^{\downarrow}(\mathcal{T})} \kappa_{\mathrm{A}}(T), \quad \kappa_2(\mathcal{T}) := \sup_{T \in \mathcal{S}_n^{\downarrow}(\mathcal{T})} \kappa_2(T). \tag{23}$$

We call these the geometric, aspect, and algebraic shape measure, respectively, of the triangulation.

We will use below the maximal ratio of volumes between face-neighboring n-simplices:

$$\rho(\mathcal{T}) := \sup_{\substack{T, T' \in \mathcal{S}_n^{\downarrow}(\mathcal{T}) \\ T \cap T' \in \mathcal{S}_{n-1}^{\downarrow}(\mathcal{T})}} \frac{\operatorname{vol}(T)}{\operatorname{vol}(T')}. \tag{24}$$

We will also compare the diameters of simplices that intersect at least in one point. For this purpose, we define

$$\theta(\mathcal{T}) := \sup_{\substack{T, T' \in \mathcal{S}_n^{\downarrow}(\mathcal{T}) \\ T \cap T' \neq \emptyset}} \frac{\delta(T)}{\delta(T')} \tag{25}$$

Finally, whenever S, T are two n-simplices that share a common face F, we let $\Xi_{S,T} : S \to T$ denote the affine diffeomorphism that preserves F. We then define

$$\xi(\mathcal{T}) := \sup_{\substack{T, T' \in \mathcal{S}_n^{\downarrow}(\mathcal{T}) \\ T \cap T' \in \mathcal{S}_{n-1}^{\downarrow}(\mathcal{T})}} \| \operatorname{Jac} \Xi_{S,T} \|_2 \tag{26}$$

to be the maximum of the operator norm of the Jacobian of any such diffeomorphism. This indicator quantifies how much reflection across the shared face distorts the geometry.

3.3 Relationships between geometric and algebraic entities in a single simplex

We now gather a few relationships between geometric and algebraic entities and compare the different shape measures in a single simplex.

Lemma 3.3. *Let T be an n-simplex and let $\varphi : \Delta^n \to T$ be an affine diffeomorphism from the reference n-simplex. Then*

$$\| \operatorname{Jac} \varphi \|_2 \leq C_{1,n} \delta(T), \qquad \| \operatorname{Jac} \varphi^{-1} \|_2 \leq C_{2,n} \kappa(T) \delta(T)^{-1},$$
$$\kappa_2(T) \leq C_{3,n} \kappa(T), \qquad \kappa_{\mathrm{A}}(T) \leq \kappa(T).$$

Here,

$$C_{1,n} \leq \sqrt{n}, \quad C_{2,n} \leq \left(\frac{n-1}{n} \right)^{\frac{2}{n-1}}, \quad C_{3,n} \leq C_{1,n} C_{2,n}.$$

MV: Exponent flipped???

Proof. We easily verify the first inequality. To prove the second inequality, let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ denote the singular values of the Jacobian $\operatorname{Jac} \varphi$ in descending order. The Jacobian determinant is the product of these singular values. By a result of Hong and Pan [37],

MV: Could we show how? Isn't $C_{1,n}$ just 1?

$$\sigma_n \geq \left(\frac{n-1}{n} \right)^{\frac{n-1}{2}} \frac{\det(\operatorname{Jac} \varphi)}{\delta(T)^{n-1}} = \left(\frac{n-1}{n} \right)^{\frac{n-1}{2}} \frac{n! \operatorname{vol}(T)}{\delta(T)^{n-1}}. \tag{27}$$

The second inequality follows by taking the reciprocal. The definition of the condition number now gives the third inequality. Finally, if F is the face of T opposite to the vertex with the smallest height, then

$$\frac{\delta(T)}{h(T)} = \frac{\delta(T)\operatorname{vol}(F)}{n\operatorname{vol}(T)} \leq \frac{\delta(T)^n}{n!\operatorname{vol}(T)}.$$

That shows the fourth and last of the inequalities. □

MV: ???

Remark 3.4. *Whenever T is an n-dimensional simplex, the ratio $\kappa(T)$ measures the "shape quality" of a simplex and is instance of a so-called* shape measure. *In our convention, the reference simplex has unit shape measure. For example, $\kappa(T) = 1$ whenever T is a line segment, an isosceles right-angled triangle, or the generalizations of these reference simplices to higher dimensions. $\kappa(T)$ is slightly bigger than one for equilateral simplices and blows up with distortion.*

The geometric ratio $\kappa(T)$ equals what is known as fatness *in differential geometry [15] and is precisely the reciprocal of the* fullness *discussed by Whitney [61], which also called* thickness *[50]. Alternative shape measures have been used throughout the literature of numerical analysis and computational geometry to quantify the quality of simplices. One example, commonly referenced in finite element literature, is the ratio of the diameter and the radius of the largest inscribed circle of a simplex.*

MV: Correct?

3.4 Relationships between geometric and algebraic entities in two neighboring simplices

We now continue in two simplices sharing a face.

Lemma 3.5. *Let T_1 and T_2 be two n-simplices that share a common face F. Then*

$$\delta(T_1) \leq \kappa(T_1)\delta(F), \qquad \frac{\operatorname{vol}(T_1)}{\operatorname{vol}(T_2)} \leq \kappa(T_1)\kappa(T_2).$$

If $\Xi : T_1 \to T_2$ is an affine diffeomorphism, then

$$\|\operatorname{Jac}\Xi\|_2 \leq C_{4,n}\kappa(T_2)\kappa(T_1).$$

Here, $C_{4,n} \leq C_{1,n}C_{2,n}$.

Proof. Let h_1 and h_2 be the heights of F in the simplices T_1 and T_2, respectively. By the volume formula for simplices, $\operatorname{vol}(T_1) = h_1 \operatorname{vol}(F)/n$ and $\operatorname{vol}(T_2) = h_2 \operatorname{vol}(F)/n$, and thus $\operatorname{vol}(T_1)/\operatorname{vol}(T_2) = h_1/h_2$. The diameter of F is at least as large as the height h_S of some other face~~t~~ S of T. Recursive application of the volume formula shows

$$\operatorname{vol}(T) = \frac{1}{n!}h_S h_2 \cdots h_n.$$

Now, as $h_S \leq \delta(F)$ and $h_2, \cdots, h_n \leq \delta(T)$,

$$\delta(T)\frac{n!\operatorname{vol}(T)}{\delta(T)^n} \leq h_S \leq \delta(F).$$

The first estimate follows.

On the other hand,

$$\begin{aligned} h_1 &\leq \delta(T_1) \leq \frac{\delta(T_1)^n}{n!\operatorname{vol}(T_1)}\delta(F), \\ h_2 &= n\frac{\operatorname{vol}(T_2)}{\operatorname{vol}(F)} \geq n(n-1)!\frac{\operatorname{vol}(T_2)}{\delta(T_2)^n}\delta(T_2) \geq n!\frac{\operatorname{vol}(T_2)}{\delta(T_2)^n}\delta(F). \end{aligned}$$

The second estimate follows.

Lastly, for $\Xi : T_1 \to T_2$ defined above, Lemma 3.3 and the first estimate imply

$$\begin{aligned}\|\operatorname{Jac}\Xi\|_2 &\leq C_{1,n}C_{2,n}\kappa(T_2)\delta(T_1)\delta(T_2)^{-1}\\ &\leq C_{1,n}C_{2,n}\kappa(T_2)\kappa(T_1)\delta(F)\delta(T_2)^{-1} \leq C_{1,n}C_{2,n}\kappa(T_2)\kappa(T_1).\end{aligned}$$

All the desired results thus follow. ∎

MV: How does the fist estimate follow?

Remark 3.6. *Lemma 3.5 fully controls the shape quantity $\rho(\mathcal{T})$ of (24) but not $\theta(\mathcal{T})$ of (25). As seen above, we can bound the ratio of diameters and volumes of simplices that share at least one face in terms of the shape measure. If we have an upper estimate for the number of simplices sharing a given vertex, we can also bound the ratios of diameters and volumes of simplices sharing only a single common vertex.*

MV: OK?

4 Combinatorial Poincaré–Friedrichs constants over triangulated domains: the gradient case

In this section, we develop a stepwise computable estimate of Poincaré–Friedrichs constants of triangulated domains. The following very classical procedure serves us as an inspiration: given a gradient vector field, we can reconstruct the scalar potential up to a constant by fixing a starting point and integrating the gradient vector field along lines emanating from that starting point. Over triangulated domains, we fix a starting triangle and perform this classical procedure. We then pass through the triangulation such that each subsequent simplex is connected to the previous ones by a face. We always construct a gradient potential over the new simplex and fix the constant using the value already known on the connecting face. This basic idea has appeared in the different variants before, for instance recently in [9, 21, 14, 59].

4.1 Poincaré–Friedrichs constants over face patches

We start by extending the results on Poincaré–Friedrichs constants over convex domains from Section 2.3 to face patches within simplicial triangulations in a way which captures the correct asymptotic behavior as p grows to infinity. Let us stress that face patches are not necessarily convex and remark that our result is in particular sharp when T_1 and T_2 have similar volumes and diameters.

Lemma 4.1. *Suppose that T_1, T_2 are two n-simplices from an n-dimensional triangulation $\mathcal{T}$ whose intersection is a common face $F := T_1 \cap T_2$. Write $U := T_1 \cup T_2$. Then*

$$\begin{aligned}\min_{c\in\mathbb{R}} \|u - c\|_{L^p(U)} &\leq 2C_{1,n}C_{\mathrm{PF,EFNT},p}\max\left(\frac{\operatorname{vol}(T_1)}{\operatorname{vol}(T_2)}, \frac{\operatorname{vol}(T_2)}{\operatorname{vol}(T_1)}\right)^{\frac{1}{p}}\\ &\qquad\cdot\max\left(\delta(T_1),\delta(T_2)\right)\|\nabla u\|_{L^p(U)}\\ &\leq \underbrace{2C_{1,n}C_{\mathrm{PF,EFNT},p}\rho(\mathcal{T})^{\frac{1}{p}}\max\left(\delta(T_1),\delta(T_2)\right)}_{C_{\mathrm{PF},T_1\cup T_2,p}}\|\nabla u\|_{L^p(U)}\end{aligned}$$

for any $u \in W^{1,p}(\Omega)$ with $p \in [1,\infty]$.

Proof. Let $\hat{T}_1$ be the reference n-simplex whose vertices $v_0 = 0, v_1 = e_1, \dots, v_{n-1} = e_{n-1}, v_n = e_n$ are the origin and the n canonical unit vectors. We let $\hat{T}_2$ be the n-simplex whose vertices are $v_0 = 0, v_1 = e_1, \dots, v_{n-1} = e_{n-1}, v_n' = -e_n$. Let $\hat{U} := \hat{T}_1 \cup \hat{T}_2$, which is (crucially) convex. We fix affine diffeomorphisms $\varphi_1 : \hat{T}_1 \to T_1$ and $\varphi_2 : \hat{T}_2 \to T_2$ which map the convex closure of $v_0, \dots, v_{n-1}$ onto the common face F. Note that these affine transformations define a bi-Lipschitz piecewise affine mapping $\varphi : \hat{U} \to U$.

Suppose that $u \in W^{1,p}(U)$. Then $\hat{u} := u \circ \varphi \in W^{1,p}(\hat{U})$. We observe

$$\|\nabla \hat{u}\|^p_{L^p(\hat{U})} = \|\nabla \hat{u}\|^p_{L^p(\hat{T}_1)} + \|\nabla \hat{u}\|^p_{L^p(\hat{T}_2)}, \tag{28a}$$

$$\|\nabla \hat{u}\|^p_{L^p(\hat{T}_1)} \leq |\det(\operatorname{Jac}\varphi_1^{-1})| \|\operatorname{Jac}\varphi_1\|_2^p \|\nabla u\|^p_{L^p(T_1)}, \tag{28b}$$

$$\|\nabla \hat{u}\|^p_{L^p(\hat{T}_2)} \leq |\det(\operatorname{Jac}\varphi_2^{-1})| \|\operatorname{Jac}\varphi_2\|_2^p \|\nabla u\|^p_{L^p(T_2)}. \tag{28c}$$

Hence

$$\|\nabla \hat{u}\|_{L^p(\hat{U})} \leq \max\left(|\det(\operatorname{Jac}\varphi_1)|^{-\frac{1}{p}} \|\operatorname{Jac}\varphi_1\|_2, |\det(\operatorname{Jac}\varphi_2)|^{-\frac{1}{p}} \|\operatorname{Jac}\varphi_2\|_2 \right) \|\nabla u\|_{L^p(U)}.$$

Notice that $\hat{U}$ has diameter 2. Thus, from (18), there exists $\hat{w} \in W^{1,p}(\hat{U})$ such that $\nabla \hat{w} = \nabla \hat{u}$ and

$$\|\hat{w}\|_{L^p(\hat{U})} \leq 2 C_{\mathrm{PF,EFNT},p} \|\nabla \hat{u}\|_{L^p(\hat{U})}.$$

Next, setting $w := \hat{w} \circ \varphi^{-1}$, we find $\nabla w = \nabla u$ and

$$\|w\|_{L^p(U)} \leq \max\left(|\det(\operatorname{Jac}\varphi_1)|, |\det(\operatorname{Jac}\varphi_2)| \right)^{\frac{1}{p}} \|\hat{w}\|_{L^p(\hat{U})}.$$

Abbreviate $D := \max(\delta(T_1), \delta(T_2))$. Also using $\|\operatorname{Jac}\varphi_i\|_2 \leq C_{1,n}\delta(T_i)$ from Lemma 3.3,

$$\begin{aligned}
\|w\|_{L^p(U)} &\leq 2 C_{1,n} C_{\mathrm{PF,EFNT},p} \max\left(|\det(\operatorname{Jac}\varphi_1)|, |\det(\operatorname{Jac}\varphi_2)| \right)^{\frac{1}{p}} \\
&\qquad \max\left(|\det(\operatorname{Jac}\varphi_1)|^{-\frac{1}{p}}, |\det(\operatorname{Jac}\varphi_2)|^{-\frac{1}{p}} \right) D \|\nabla u\|_{L^p(U)} \\
&= 2 C_{1,n} C_{\mathrm{PF,EFNT},p} \max\left(|\det(\operatorname{Jac}\varphi_1)|, |\det(\operatorname{Jac}\varphi_2)| \right)^{\frac{1}{p}} \\
&\qquad \min\left(|\det(\operatorname{Jac}\varphi_1)|, |\det(\operatorname{Jac}\varphi_2)| \right)^{-\frac{1}{p}} D \|\nabla u\|_{L^p(U)} \\
&= 2 C_{1,n} C_{\mathrm{PF,EFNT},p} \max\left(\frac{|\det(\operatorname{Jac}\varphi_1)|}{|\det(\operatorname{Jac}\varphi_2)|}, \frac{|\det(\operatorname{Jac}\varphi_2)|}{|\det(\operatorname{Jac}\varphi_1)|} \right)^{\frac{1}{p}} D \|\nabla u\|_{L^p(U)} \\
&= 2 C_{1,n} C_{\mathrm{PF,EFNT},p} \max\left(\frac{\operatorname{vol}(T_1)}{\operatorname{vol}(T_2)}, \frac{\operatorname{vol}(T_2)}{\operatorname{vol}(T_1)} \right)^{\frac{1}{p}} D \|\nabla u\|_{L^p(U)}.
\end{aligned}$$

In the last equality, we have also used $|\det(\operatorname{Jac}\varphi_i)| = n! \operatorname{vol}(T_i)$ that we have already employed in (27). This gives the desired result. □

4.2 Poincaré–Friedrichs constants over a face-connected triangulation

The following is our first computable estimate of Poincaré–Friedrichs constants over triangulated domains obtained in a stepwise, combinatorial way:

Theorem 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain with an n-dimensional face-connected finite triangulation $\mathcal{T}$. Let $T_0 \in \mathcal{T}$ be an n-simplex and for all n-simplices $T \in \mathcal{T}$, fix a face path $T_0, T_1, \dots, T_M = T$. Suppose $1 \leq p, q \leq \infty$ with $1 = 1/p + 1/q$. Then for any $u \in W^{1,p}(\Omega)$ there exists $w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$ and such that for all $T \in \mathcal{T}$, we have the following recursive estimate:*

$$\begin{aligned}
\|w\|_{L^p(T_M)} \leq \left(1 + \rho(\mathcal{T})^{\frac{q}{p}}\right)^{\frac{1}{q}} \sum_{l=1}^{M} \rho(\mathcal{T})^{\frac{M-l}{p}} C_{\mathrm{PF},T_l \cup T_{l-1},p} \|\nabla u\|_{L^p(T_l \cup T_{l-1})} \\
+ \rho(\mathcal{T})^{\frac{M-1}{p}} C_{\mathrm{PF},T_0,p} \|\nabla u\|_{L^p(T_0)}.
\end{aligned}$$

MV: I think we should also write down the bound for $\|w\|_{L^p(\Omega)}$.

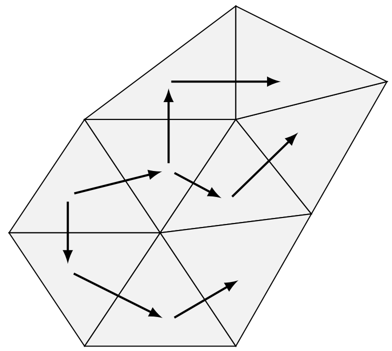


Figure 1: Face-connected triangulation of a domain. The arrows depict a tree in the face connection graph of the form used in Theorem 4.2. ~~The triangle in dark gray is the triangle being added in the current iteration, and the light gray triangles constitute the subdomain assembled so far.~~

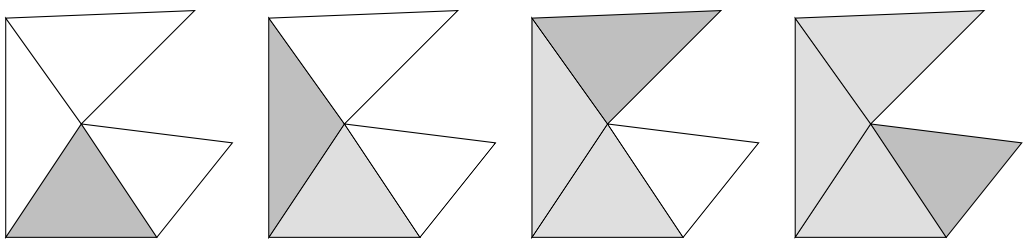


Figure 2: Exhausting a domain step by step as in Theorems 4.2 and 4.5. The dark gray triangle is the one being currently added, and the light gray triangles constitute the subdomain assembled so far.

Proof.

MV: Reworked and simplified the beginning of the proof.

Let $u \in W^{1,p}(\Omega)$. On the first simplex T_0, there exists $w_0 \in W^{1,p}(T_0)$ satisfying $\nabla w_0 = \nabla u$ over T_0 together with

$$\|w_0\|_{L^p(T_0)} \leq C_{\mathrm{PF},T_0,p} \|\nabla u\|_{L^p(T_0)}.$$

In particular, $c_0 := w_0 - u$ is a constant function. We then define

$$w := u + c_0$$

(note that $w|_{T_0} = w_0$). Clearly, $w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$. We verify that w fulfills the desired estimate.

Let $T \in \mathcal{T}$ be any n-simplex. By assumption, there exists a face path $T_0, T_1, \dots, T_M = T$, composed of n-simplices from $\mathcal{T}$. For any $1 \leq m \leq M$ we let $F_m := T_m \cap T_{m-1}$ be a face of dimension $n-1$ shared by the n-simplices T_m and T_{m-1}. Recall the notation $U_{F_m} := T_m \cup T_{m-1}$. We let $u_{F_m} \in W^{1,p}(U_{F_m})$ be such that $\nabla u_{F_m} = \nabla u$ over U_{F_m} and

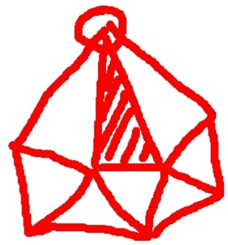
$$\|u_{F_m}\|_{L^p(U_{F_m})} \leq C_{\mathrm{PF},U_{F_m},p} \|\nabla u_{F_m}\|_{L^p(U_{F_m})},$$

where the computable constant $C_{\mathrm{PF},U_{F_m},p}$ is determined via Lemma 4.1. Since $\nabla(w|_{U_{F_m}}) = \nabla u_{F_m} = \nabla(u|_{U_{F_m}})$, we can define the constant

$$c_m := w|_{U_{F_m}} - u_{F_m}.$$

~~We finally write $\Omega_m := \cup_{j=0}^m T_m$, which is triangulated n-dimensional submanifold with boundary~~.

MV: False, Ω_m is not necessarily a triangulated n-dimensional submanifold with boundary, see the 7-triangles example standing for Ω_m (then the 8th one gives $\Omega_{m+1} = \Omega$). As discussed online, we fortunately do not need here a submanifold with boundary (but we will need it and have it for shellable triangulations.) Actually, here we do not need Ω_m at all.



We now assess the norm of the potential w. For any $1 \leq m \leq M$, we observe

$$\begin{aligned}
\|w\|_{L^p(T_m)} &\leq \|u_{F_m}\|_{L^p(T_m)} + \|c_m\|_{L^p(T_m)}, \\
\|c_m\|_{L^p(T_m)} &= \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|c_m\|_{L^p(T_{m-1})}, \\
\|c_m\|_{L^p(T_{m-1})} &\leq \|w\|_{L^p(T_{m-1})} + \|u_{F_m}\|_{L^p(T_{m-1})}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|w\|_{L^p(T_m)} \leq \|u_{F_m}\|_{L^p(T_m)} & \\
&+ \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|u_{F_m}\|_{L^p(T_{m-1})} + \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w\|_{L^p(T_{m-1})}.
\end{aligned}$$

We sum the two integrals of u_{F_m}. In the special case $p = 1$,

$$\|w\|_{L^1(T_m)} \leq \max\left(1, \frac{\operatorname{vol}(T_m)}{\operatorname{vol}(T_{m-1})}\right) \|u_{F_m}\|_{L^1(U_{F_m})} + \frac{\operatorname{vol}(T_m)}{\operatorname{vol}(T_{m-1})} \|w\|_{L^1(T_{m-1})}.$$

In the special case $p = \infty$,

$$\|w\|_{L^\infty(T_m)} \leq 2\|u_{F_m}\|_{L^\infty(U_{F_m})} + \|w\|_{L^\infty(T_{m-1})}.$$

When $1 < p < \infty$, recalling the complementary exponent $q = p/(p-1) \in [1, \infty]$, we observe using the Hölder inequality

$$\begin{aligned}
\|w\|_{L^p(T_m)} &\leq \left(1 + \frac{\operatorname{vol}(T_m)^{\frac{q}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{q}{p}}}\right)^{\frac{1}{q}} \left(\|u_{F_m}\|_{L^p(T_m)}^p + \|u_{F_m}\|_{L^p(T_{m-1})}^p\right)^{\frac{1}{p}} \\
&\quad + \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w\|_{L^p(T_{m-1})} \\
&= \left(1 + \frac{\operatorname{vol}(T_m)^{\frac{q}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{q}{p}}}\right)^{\frac{1}{q}} \|u_{F_m}\|_{L^p(U_{F_m})} + \frac{\operatorname{vol}(T_m)^{\frac{1}{p}}}{\operatorname{vol}(T_{m-1})^{\frac{1}{p}}} \|w\|_{L^p(T_{m-1})}.
\end{aligned}$$

Recursive application provides the following estimate:

$$\begin{aligned}
\|w\|_{L^p(T_m)} \leq \sum_{t=1}^{m} &\left(\prod_{t<l\leq m} \frac{\operatorname{vol}(T_l)^{\frac{1}{p}}}{\operatorname{vol}(T_{l-1})^{\frac{1}{p}}}\right) \left(1 + \frac{\operatorname{vol}(T_t)^{\frac{q}{p}}}{\operatorname{vol}(T_{t-1})^{\frac{q}{p}}}\right)^{\frac{1}{q}} \|u_{F_t}\|_{L^p(U_{F_t})} \\
&+ \left(\prod_{0<l\leq m} \frac{\operatorname{vol}(T_l)^{\frac{1}{p}}}{\operatorname{vol}(T_{l-1})^{\frac{1}{p}}}\right) \|u_{T_0}\|_{L^p(T_0)}.
\end{aligned}$$

We obtain the desired estimate. ☐

Remark 4.3. *The Poincaré–Friedrichs constant obtained from Theorem 4.2 overestimates the best one increasingly with increasing number of n-simplices in the triangulation $\mathcal{T}$. For this reason, we target its use to local patches (stars), in particular nonconvex boundary stars such as that of Figure 2. Clearly, the same building principle applies when we can obtain a nonoverlapping partition of $\overline{\Omega}$ into convex local patches of n-simplices U_m with a face-connected path. The proof proceeds verbatim, where we merely replace the simplices T_m by the convex local patches U_m. This may allow to a large decrease of the number of elements in the partition of $\overline{\Omega}$ and thus to a large decrease of the overestimation of the best Poincaré–Friedrichs constant.*

4.3 Poincaré–Friedrichs constants over an n-simplex with homogeneous Dirichlet condition on a face

We also want to assess an alternative route towards Poincaré–Friedrichs inequalities over triangulated domains. While the underlying idea remains the same, the construction of the potential is slightly different. As an auxiliary result with independent relevance, we compute an upper bound for the Poincaré–Friedrichs inequality when homogeneous boundary conditions are imposed on a single face of the boundary. We present this result with two different techniques. While it may not be obvious, the second estimate is tighter than the first estimate when using the optimal Poincaré–Friedrichs constants for convex domains.

Lemma 4.4. *Let T be an n-simplex with a face F. If $u \in W^{1,p}(T)$ such that $\operatorname{tr}_F u = 0$, then*

$$\|u\|_{L^p(T)} \leq C_{\mathrm{PF},T,F,p} \|\nabla u\|_{L^p(T)},$$

where $C_{\mathrm{PF},T,F,p} > 0$ is a constant such that

$$\begin{aligned}
C_{\mathrm{PF},T,F,p} &\leq C_{\mathrm{PF},T,p} + \left(C_{\mathrm{PF},T,p}^p + p\delta(T) C_{\mathrm{PF},T,p}^{p-1} \right)^{\frac{1}{p}}, \\
C_{\mathrm{PF},T,F,\infty} &\leq 2 C_{\mathrm{PF},T,\infty}, \\
C_{\mathrm{PF},T,F,p} &\leq \frac{1}{\sqrt[p]{p}} \delta(T).
\end{aligned}$$

MV: What is $\sqrt[\infty]{\infty}$?

Proof. There exists $w \in W^{1,p}(T)$ with $\nabla w = \nabla u$ and

$$\|w\|_{L^p(T)} \leq C_{\mathrm{PF},T,p} \|\nabla w\|_{L^p(T)}.$$

Then $w - u$ is constant, and thus $\gamma := \operatorname{tr}_F (w - u) = \operatorname{tr}_F w$. We use that

$$\|\gamma\|_{L^p(T)} = \gamma \operatorname{vol}(T)^{\frac{1}{p}} = \left(\frac{\operatorname{vol}(T)}{\operatorname{vol}(F)} \right)^{\frac{1}{p}} \gamma \operatorname{vol}(F)^{\frac{1}{p}} = \left(\frac{\operatorname{vol}(T)}{\operatorname{vol}(F)} \right)^{\frac{1}{p}} \|\gamma\|_{L^p(F)}.$$

Using a trace inequality [57, Lemma 2.8] when $1 \leq p < \infty$, we find

$$\begin{aligned}
\|\gamma\|_{L^p(F)}^p &\leq \frac{\operatorname{vol}(F)}{\operatorname{vol}(T)} \|w\|_{L^p(T)}^p + p\delta(T) \frac{\operatorname{vol}(F)}{\operatorname{vol}(T)} \|w\|_{L^p(T)}^{p-1} \|\nabla w\|_{L^p(T)} \\
&\cancel{\leq \left(\frac{\operatorname{vol}(F)}{\operatorname{vol}(T)} \right) \left(C_{\mathrm{PF},T,p}^p + p C_{\mathrm{PF},T,p}^{p-1} \right) \delta(T)^p \|\nabla w\|_{L^p(T)}^p} \\
&\leq \frac{\operatorname{vol}(F)}{\operatorname{vol}(T)} \left(C_{\mathrm{PF},T,p}^p + p\delta(T) C_{\mathrm{PF},T,p}^{p-1} \right) \|\nabla w\|_{L^p(T)}^p.
\end{aligned}$$

Recall that $u = w - \gamma$. The first inequality follows when $1 \leq p < \infty$. When $p = \infty$,

$$\|\gamma\|_{L^\infty(T)} = \|\gamma\|_{L^\infty(F)} = \|\operatorname{tr}_F w\|_{L^\infty(F)} \leq \|w\|_{L^\infty(T)}$$

and we conclude similarly by the triangle inequality.

The second inequality follows from Rademacher's theorem when $p = \infty$. Let assume that $1 \leq p < \infty$. Suppose that $u \in C^\infty(T)$ with support disjoint from F. Without loss of generality, the segment from to the midpoint of F to the opposing vertex lies on the first coordinate axis, and the minimal first coordinate among all the points of F equals 0. We write respectively u and $\boldsymbol{g}\cancel{\in \boldsymbol{L}^\infty(T)}$ for the trivial extension of u and ∇u outside of T. Using the fundamental theorem of calculus and Hölder's inequality,

$$\begin{aligned}
\int_\Omega |u(x)|^p \,\mathrm{d}x \, dx &\leq \int_{\mathbb{R}^{n-1}} \int_0^{\delta(T)} |u(x_1, \overline{x})|^p \, dx_1 \, d\overline{x} \\
&\leq \int_{\mathbb{R}^{n-1}} \int_0^{\delta(T)} \left| \int_0^{x_1} |\boldsymbol{g}(y, \overline{x})| \, dy \right|^p \, dx_1 \, d\overline{x} \\
&\leq \int_0^{\delta(T)} x_1^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^{x_1} |\boldsymbol{g}(y, \overline{x})|^p \, dy \, d\overline{x} \, dx_1 \\
&\leq \int_0^{\delta(T)} x_1^{p-1} \, dx_1 \int_\Omega |\boldsymbol{g}(y, \overline{x})|^p \, dy \, d\overline{x} \leq \frac{\delta(T)^p}{p} \int_\Omega |\nabla u(x)|^p \, dx.
\end{aligned}$$

That $\|u\|_{L^p(\Omega)} \leq \delta(T)/\sqrt[p]{p}\|\nabla u\|_{L^p(\Omega)}$ for all $W^{1,p}(T)$ whose members have vanishing trace along F follows from approximation via members of $u \in C^\infty(T)$ whose support is disjoint from F. Very briefly verify that density argument: There exists affine $\varphi : \hat{T} \to T$ mapping the convex closure of the n unit vectors (the reference element $\hat{T}$) onto the face F. We let $\hat{u} := u \circ \varphi$. Let $\hat{U}$ be the unit ball of the ℓ^1 metric, which contains $\hat{T}$. We let $\tilde{u}$ be the extension of $\hat{u}$ onto $\hat{U}$ by reflection across the coordinate axes. Then $\tilde{u} \in W_0^{1,p}(\hat{U})$, and $\tilde{u}$ is the limit of a sequence $u_m \in C_c^\infty(\hat{U})$. Now $u_m \circ \varphi^{-1} \in C^\infty(\hat{U})$ is the desired sequence that approximates u. ☐

4.4 Poincaré–Friedrichs constants over a face-connected triangulation

The following potential construction and upper bound for the Poincaré–Friedrichs constant serves as the blueprint for constructing potentials of the curl and divergence operators in later sections. It has the same building principle as the "discrete mean Poincaré inequality" of [23, Lemma 3.7].

MV: Why a second theorem? We should explain the difference of Theorem 4.5 wrt Theorem 4.2. Which is better and when?

Theorem 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain with an n-dimensional face-connected finite triangulation $\mathcal{T}$. Let $T_0 \in \mathcal{T}$ be an n-simplex and for all n-simplices $T \in \mathcal{T}$, fix a face path $T_0, T_1, \dots, T_M = T$. For any $1 \leq l \leq M$, let $F_l = T_l \cap T_{l-1}$. Then for any $u \in W^{1,p}(\Omega)$ there exists $w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$ and such that for all $T \in \mathcal{T}$, we have the following recursive estimate:*

$$\begin{aligned}
\|w\|_{L^p(T_M)} \leq \sum_{l=1}^{M} \rho(\mathcal{T})^{\frac{M-l}{p}} C_{\mathrm{PF},T_l,F_l,p} \left(\|\nabla u\|_{L^p(T_l)} + \rho(\mathcal{T})^{\frac{1}{p}} \xi(\mathcal{T}) \|\nabla u\|_{L^p(T_{l-1})} \right) \\
+ \rho(\mathcal{T})^{\frac{M}{p}} C_{\mathrm{PF},T_0,p} \|\nabla u\|_{L^p(T_0)}.
\end{aligned}$$

MV: Synchronized the announcement and notation with Theorem 4.2.

Proof.

MV: Reworked and simplified the proof.

We start as in Theorem 4.2. Let $u \in W^{1,p}(\Omega)$. On the first simplex T_0, there exists $w_0 \in W^{1,p}(T_0)$ satisfying $\nabla w_0 = \nabla u$ over T_0 together with

MV: I think we should also write down the bound for $\|w\|_{LP(\Omega)}$.

$$\|w_0\|_{L^p(T_0)} \leq C_{\mathrm{PF},T_0,p} \|\nabla u\|_{L^p(T_0)}.$$

In particular, $c_0 := w_0 - u$ is a constant function. We then define

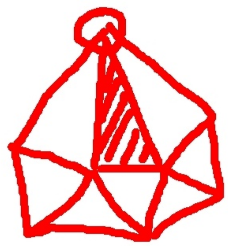
$$w := u + c_0$$

(note that $w|_{T_0} = w_0$). Clearly, $w \in W^{1,p}(\Omega)$ with $\nabla w = \nabla u$. We verify that w fulfills the desired estimate.

Let $T \in \mathcal{T}$ be any n-simplex. By assumption, there exists a face path $T_0, T_1, \dots, T_M = T$, composed of n-simplices from $\mathcal{T}$. For any $1 \leq m \leq M$, recall that $F_m := T_m \cap T_{m-1}$ is a face of dimension $n-1$ shared by the n-simplices T_m and T_{m-1}.

~~We write $\Omega_m := \cup_{j=0}^m T_m$, which is a triangulated n-dimensional submanifold with boundary.~~

MV: False, Ω_m is not necessarily a triangulated n-dimensional submanifold with boundary, see the 7-triangles example standing for Ω_m (then the 8th one gives $\Omega_{m+1} = \Omega$). As discussed online, we fortunately do not need here a submanifold with boundary (but we will need it and have it for shellable triangulations.) Actually, here we do not need Ω_m at all.



~~We use a recursive argument.~~ Suppose that $1 < m \leq M$. We define $w'_m := w|_{T_{m-1}} \circ \Xi \in W^{1,p}(T_m)$, where $\Xi : T_m \to T_{m-1}$ is the unique affine diffeomorphism that leaves F_m invariant. By construction, $w'_m \in W^{1,p}(T_m)$ with

$$\operatorname{tr}_{F_m} w'_m = \operatorname{tr}_{F_m} w|_{T_{m-1}}.$$

We now define $w''_m \in W^{1,p}(T_m)$ as

$$w''_m := u|_{T_m} - w'_m + c_0 = w|_{T_m} - w'_m = u|_{T_m} - u|_{T_{m-1}} \circ \Xi. \tag{29}$$

We crucially note that w''_m is trace free on F_m since

$$\operatorname{tr}_{F_m} w''_m = \operatorname{tr}_{F_m} (u|_{T_m} - w'_m + c_0) = \operatorname{tr}_{F_m} (u|_{T_{m-1}} - w|_{T_{m-1}} + c_0) = 0.$$

Lemma 4.4 gives, from the first expression in (29)

$$\|w''_m\|_{L^p(T_m)} \leq C_{\mathrm{PF},T_m,F_m,p} \left(\|\nabla u\|_{L^p(T_m)} + \|\nabla w'_m\|_{L^p(T_m)} \right).$$

We also employ (26) ~~Lemma 3.5~~ to find

$$\begin{aligned} \|\nabla w'_m\|_{L^p(T_m)} &\leq |\det(\operatorname{Jac}\Xi)|^{-\frac{1}{p}} \|\operatorname{Jac}\Xi\|_2 \|\nabla w\|_{L^p(T_{m-1})} \\ &\leq \rho(\mathcal{T})^{\frac{1}{p}} \xi(\mathcal{T}) \|\nabla w\|_{L^p(T_{m-1})} \\ &= \rho(\mathcal{T})^{\frac{1}{p}} \xi(\mathcal{T}) \|\nabla u\|_{L^p(T_{m-1})}. \end{aligned}$$

From the second expression in (29), we then finally have

MV: How $|\det(\operatorname{Jac}\Xi)|^{-\frac{1}{p}} \leq \rho(\mathcal{T})^{\frac{1}{p}}$ follows?

$$\begin{aligned} &\|w\|_{L^p(T_m)} \\ &\leq \|w'_m\|_{L^p(T_m)} + \|w''_m\|_{L^p(T_m)} \\ &\leq \rho(\mathcal{T})^{\frac{1}{p}} \|w\|_{L^p(T_{m-1})} + C_{\mathrm{PF},T_m,F_m,p} \left(\|\nabla u\|_{L^p(T_m)} + \rho(\mathcal{T})^{\frac{1}{p}} \xi(\mathcal{T}) \|\nabla u\|_{L^p(T_{m-1})} \right). \end{aligned}$$

Therefrom, the desired estimate follows. $\square$

5 Review of vector calculus and exterior calculus

We review in this section the Sobolev spaces of vector and exterior calculus and their transformation behavior. We refer the reader to Ern and Guermond [20] and Hiptmair [36] for background material on Sobolev vector analysis and to Greub [32] and Lee [42] for exterior algebra and exterior products. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set.

MV: Connected?? Weakly Lipschitz??

5.1 Vector calculus

MV: Slightly reorganized.

Recall the notations (1) and (5) from Section 1; actually, here we temporarily rather employ

$$W^p(\mathrm{grad}, \Omega) = W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \mathrm{grad}\, v \in \boldsymbol{L}^p(\Omega)\}.$$

We are interested in transformations of these Sobolev tensor fields from one domain onto another. Suppose that $\Omega, \Omega' \subset \mathbb{R}^3$ are open sets and suppose that $\phi : \Omega \to \Omega'$ is a bi-Lipschitz mapping. The gradient-, curl-, and divergence-conforming Piola transformations are respectively the mappings $\phi^{\mathrm{grad}} : L^p(\Omega') \to L^p(\Omega)$, $\phi^{\mathrm{curl}} : \boldsymbol{L}^p(\Omega') \to \boldsymbol{L}^p(\Omega)$, and $\phi^{\mathrm{div}} : \boldsymbol{L}^p(\Omega') \to \boldsymbol{L}^p(\Omega)$. We will also use $\phi^{\mathrm{b}} : L^p(\Omega') \to L^p(\Omega)$. We define them for any $v \in L^p(\Omega')$ and $\boldsymbol{w} \in \boldsymbol{L}^p(\Omega')$ by setting

$$\begin{aligned}
\phi^{\mathrm{grad}}(v) &= v \circ \phi, \\
\phi^{\mathrm{curl}}(\boldsymbol{w}) &= \mathrm{Jac}\,\phi^T(\boldsymbol{w} \circ \phi), \\
\phi^{\mathrm{div}}(\boldsymbol{w}) &= \mathrm{adj}(\mathrm{Jac}\,\phi)\,(\boldsymbol{w} \circ \phi)\,, \\
\phi^{\mathrm{b}}(v) &= \det(\mathrm{Jac}\,\phi)\,(v \circ \phi)\,,
\end{aligned}$$

and where $\mathrm{Jac}\,\phi$ is the Jacobian matrix of ϕ; see also [20, Definition 9.8]. These mappings are invertible. Bounds on the Lebesgue norms will follow from a more general result below.

MV: adj undefined

We use the commutativity relations

$$\mathrm{grad}\,\phi^{\mathrm{grad}}(v) = \phi^{\mathrm{curl}}(\mathrm{grad}\, v), \tag{31a}$$

$$\mathrm{curl}\,\phi^{\mathrm{curl}}(\boldsymbol{v}) = \phi^{\mathrm{div}}(\mathrm{curl}\,\boldsymbol{v}), \tag{31b}$$

$$\mathrm{div}\,\phi^{\mathrm{div}}(\boldsymbol{w}) = \phi^{\mathrm{b}}(\mathrm{div}\,\boldsymbol{w}), \tag{31c}$$

where $v \in W^p(\mathrm{grad}, \Omega)$, $\boldsymbol{v} \in \boldsymbol{W}^p(\mathrm{curl}, \Omega)$, and $\boldsymbol{w} \in \boldsymbol{W}^p(\mathrm{div}, \Omega)$. We summarize this as a commuting diagram:

$$\begin{tikzcd}
W^p(\mathrm{grad}, \Omega') \arrow[r, "\mathrm{grad}"] \arrow[d, "\phi^{\mathrm{grad}}"] & \boldsymbol{W}^p(\mathrm{curl}, \Omega') \arrow[r, "\mathrm{curl}"] \arrow[d, "\phi^{\mathrm{curl}}"] & \boldsymbol{W}^p(\mathrm{div}, \Omega') \arrow[r, "\mathrm{div}"] \arrow[d, "\phi^{\mathrm{div}}"] & L^p(\Omega') \arrow[d, "\phi^{\mathrm{b}}"] \\
W^p(\mathrm{grad}, \Omega) \arrow[r, "\mathrm{grad}"] & \boldsymbol{W}^p(\mathrm{curl}, \Omega) \arrow[r, "\mathrm{curl}"] & \boldsymbol{W}^p(\mathrm{div}, \Omega) \arrow[r, "\mathrm{div}"] & L^p(\Omega).
\end{tikzcd}$$

Remark 5.1. *The Piola transform goes into the opposite direction of the mapping $\phi : \Omega \to \Omega'$: scalar and vector fields over Ω' are transformed into scalar and vector fields over Ω. This definition is in accordance with the notion of pullback, which we will review shortly. One advantage of that definition is that it also makes sense whenever the transformation is not a diffeomorphism. The literature also knows the Piola transform in the direction of the original mapping.*

5.2 Exterior calculus

We now move to exterior calculus. We give a brief introduction into this topic, and refer to the literature for further background [42]. Let V be a real vector space. Given an integer $k \geq 0$, we let $\Lambda^k(V)$ denote the space of scalar-valued antisymmetric k-linear forms over V. Recall that any k-linear scalar-valued form u over V is called antisymmetric if

$$u(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)}) = \operatorname{sign}(\pi) u(v_1, v_2, \dots, v_k)$$

for any $v_1, v_2, \dots, v_k \in V$ and any permutation π of the indices $\{1, 2, \dots, k\}$. By definition, $\Lambda^1(V)$ is just the dual space of V, and $\Lambda^0(V)$ is the space of real numbers. Formally, we define $\Lambda^k(V)$ to be the zero vector space when $k < 0$.

The wedge product (or exterior product) of alternating multilinear forms is a fundamental operation in exterior algebra. Given two alternating multilinear forms $u_1 \in \Lambda^k(V)$ and $u_2 \in \Lambda^l(V)$, their wedge product $u_1 \wedge u_2$ is a member of $\Lambda^{k+l}(V)$ defined by the formula

$$(u_1 \wedge u_2)(v_1, v_2, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\pi} \operatorname{sgn}(\pi) u_1(v_{\pi(1)}, \dots, v_{\pi(k)}) u_2(v_{\pi(k+1)}, \dots, v_{\pi(k+l)}),$$

for any $v_1, v_2, \dots, v_{k+l} \in V$. Here, the sum runs over all permutations π of the index set $\{1, 2, \dots, k+l\}$. The exterior product is bilinear and associative, and satisfies

$$u_1 \wedge u_2 = (-1)^{kl} u_2 \wedge u_1, \quad \forall u_1 \in \Lambda^k(V), \quad \forall u_2 \in \Lambda^l(V).$$

We use the interior product, which is in some sense dual to the exterior product, in a special case. Given $v \in V$ and $u \in \Lambda^k(V)$, we define $v \lrcorner u \in \Lambda^{k-1}(V)$ via

$$(v \lrcorner u)(v_1, v_2, \dots, v_{k-1}) = u(v, v_1, v_2, \dots, v_{k-1}), \quad \forall v_1, v_2, \dots, v_{k-1} \in V.$$

We are interested only in the special case $V = \mathbb{R}^n$ of alternating forms over the n-dimensional Euclidean space. Here, it is customary to identify $\Lambda^k(V)$ with the space of antisymmetric tensors in k indices. Moreover, this particular setting comes with a canonical basis. We let $\{dx^1, dx^2, \dots, dx^n\}$ be the basis dual to the canonical unit vectors. This is a canonical basis of $\Lambda^1(\mathbb{R}^n)$. To define a canonical basis of $\Lambda^k(\mathbb{R}^n)$, we first introduce $\Sigma(k, n)$, the set of strictly ascending mappings $\sigma : \{1, \dots, k\} \to \{1, \dots, n\}$, and introduce the basic k-alternators

$$dx^\sigma := dx^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(k)}, \quad \forall \sigma \in \Sigma(k, n).$$

These define a basis of $\Lambda^k(\mathbb{R}^n)$. Note that $\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k}$. In particular, $\Lambda^k(\mathbb{R}^n)$ is the zero vector space whenever $k > n$.

We notice that the canonical scalar product on $\mathbb{R}^n$ gives rise to a scalar product on $\Lambda^1(\mathbb{R}^n)$, which induces a scalar product on $\Lambda^k(\mathbb{R}^n)$. The basic k-alternators are an orthonormal basis of $\Lambda^k(\mathbb{R}^n)$ with respect to that inner product.

5.3 Smooth differential forms

We write $C^\infty\Lambda^k(\Omega)$ for the space of smooth differential k-forms over $\Omega \subseteq \mathbb{R}^n$, which is the vector space of smooth mappings from Ω into $\Lambda^k(\mathbb{R}^n)$. The exterior derivative d is an operator that takes a k-form $\omega \in C^\infty\Lambda^k(\Omega)$ to a $(k+1)$-form $d\omega \in C^\infty\Lambda^{k+1}(\Omega)$. Every k-form $\omega \in C^\infty\Lambda^k(\Omega)$ can be written

$$\omega = \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma \, dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \dots \wedge dx^{\sigma(k)},$$

where $\omega_\sigma : \Omega \to \mathbb{R}$ are smooth functions. The exterior derivative $d\omega$ is defined by

$$d\omega = \sum_{\sigma \in \Sigma(k,n)} \sum_{j=1}^{n} \frac{\partial \omega_\sigma}{\partial x^j} \, dx^j \wedge dx^{\sigma(1)} \wedge dx^{\sigma(2)} \wedge \dots \wedge dx^{\sigma(k)}.$$

The exterior derivative is linear and nilpotent, which means $d(d\omega) = 0$ for any $\omega \in C^\infty\Lambda^k(\Omega)$. Moreover, it satisfies the Leibniz rule:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta, \qquad \forall \omega \in C^\infty\Lambda^k(\Omega), \qquad \forall \eta \in C^\infty\Lambda^l(\Omega).$$

5.4 Sobolev spaces of differential forms

Let us now turn our attention to Sobolev spaces of differential forms. Since the exterior product space $\Lambda^k(\mathbb{R}^n)$ carries a norm, induced from the Euclidean norm on $\mathbb{R}^n$, we can consider the pointwise norms of differential k-forms. We let $L^p\Lambda^k(\Omega)$ be the space of differential k-forms over Ω with locally integrable coefficients such that its pointwise norm is p-integrable. The exterior derivative is defined in the sense of distributions and we introduce

$$W^p\Lambda^k(\Omega) := \{u \in L^p\Lambda^k(\Omega) \mid du \in L^p\Lambda^{k+1}(\Omega)\}.$$

We observe that $u \in L^p\Lambda^k(\Omega)$ has weak exterior derivative $f \in L^p\Lambda^{k+1}(\Omega)$ if and only if for all $v \in C_c^\infty\Lambda^{n-k-1}(\Omega)$ we have the integration by parts formula

$$\int_\Omega dv \wedge u = (-1)^{k(n-k)+1} \int_\Omega v \wedge f.$$

~~We let $W_0^p\Lambda^k(\Omega)$ be the closed subspace of those $u \in W^p\Lambda^k(\Omega)$ for which the extension by zero $\tilde{u} : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$ is a member of $\tilde{u} \in W^p\Lambda^k(\mathbb{R}^n)$.~~

MV: repeated below.

Lastly, we are also interested in differential forms whose trace vanishes along only a part of the boundary. Suppose that $\Gamma \subseteq \partial\Omega$ is a relatively open subset of the boundary. We say that $u \in W^p\Lambda^k(\Omega)$ has vanishing trace along Γ if for all $x \in \Gamma$ there exists $r > 0$ such that for all $v \in C_c^\infty\Lambda^{n-k-1}(B_r(x))$ we have the integration by parts formula

$$\int_{B_r(x)} dv \wedge \tilde{u} = (-1)^{k(n-k)+1} \int_{B_r(x)} v \wedge \widetilde{du}.$$

If that condition is satisfied, we also write

$$\operatorname{tr}_\Gamma u = 0.$$

Accordingly, we write $\operatorname{tr}_\Gamma u = \operatorname{tr}_\Gamma u'$ for $\operatorname{tr}_\Gamma(u - u') = 0$ whenever $u, u' \in W^p\Lambda^k(\Omega)$. Lastly, we introduce the closed subspaces

$$W_0^p\Lambda^k(\Omega) := \{u \in W^p\Lambda^k(\Omega) \mid \operatorname{tr}_{\partial\Omega} u = 0\}.$$

The space $W_0^p\Lambda^k(\Omega)$ formalizes the idea of differential forms with vanishing boundary trace along the boundary $\partial\Omega$. We know that $dW_0^p\Lambda^k(\Omega) \subseteq W_0^p\Lambda^{k+1}(\Omega)$.

MV: Moved here.

5.5 Transformations by bi-Lipschitz mappings

We are interested in transformations of Sobolev tensor fields from one domain onto another. Suppose that $\Omega, \Omega' \subset \mathbb{R}^n$ are open sets and suppose that $\phi : \Omega \to \Omega'$ is a bi-Lipschitz mapping. The pullback of $u \in W^p\Lambda^k(\Omega')$ along ϕ is the differential form

$$\phi^* u_{|x}(v_1, v_2, \dots, v_k) := u_{|\phi(x)}(\operatorname{Jac}\phi_{|x} \cdot v_1, \operatorname{Jac}\phi_{|x} \cdot v_2, \dots, \operatorname{Jac}\phi_{|x} \cdot v_k) \tag{32}$$

for any $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ and any $x \in \Omega$. One can show that $\phi^* u \in W^p\Lambda^k(\Omega)$. We cite an explicit transformation estimate [44].

MV: Changed to "Proposition". Only the main results are termed "Theorem".

Proposition 5.2. *Let $\phi : \Omega \to \Omega'$ be a bi-Lipschitz mapping between open sets $\Omega, \Omega' \subseteq \mathbb{R}^n$. Let $p \in [1, \infty]$ and $u \in L^p\Lambda^k(\Omega)$. Then*

$$\begin{aligned} \|\phi^* u\|_{L^p\Lambda^k(\Omega)} &\leq \|\operatorname{Jac}\phi\|_{L^\infty(\Omega)}^k \|\det \operatorname{Jac}\phi^{-1}\|_{L^\infty(\Omega')}^{\frac{1}{p}} \|u\|_{L^p\Lambda^k(\Omega')} \\ \|d\phi^* u\|_{L^p\Lambda^k(\Omega)} &\leq \|\operatorname{Jac}\phi\|_{L^\infty(\Omega)}^k \|\operatorname{Jac}\phi^{-1}\|_{L^\infty(\Omega')}^{\frac{n}{p}} \|u\|_{L^p\Lambda^k(\Omega')}. \end{aligned} \tag{33}$$

MV: ??? What is the correct norm? $L^\infty(\Omega)$ or $L^2(\Omega)$ as in (28)??

Remark 5.3. *In three dimensions, the calculus of differential forms can be translated into classical vector calculus. This is expressed formally as the commuting diagram*

$$\begin{array}{ccccccc} C^\infty\Lambda^0(\Omega) & \xrightarrow{d} & C^\infty\Lambda^1(\Omega) & \xrightarrow{d} & C^\infty\Lambda^2(\Omega) & \xrightarrow{d} & C^\infty\Lambda^3(\Omega) \\ \downarrow{\scriptstyle\varpi^0} & & \downarrow{\scriptstyle\varpi^1} & & \downarrow{\scriptstyle\varpi^2} & & \downarrow{\scriptstyle\varpi^3} \\ C^\infty(\Omega) & \xrightarrow{\text{grad}} & C^\infty(\Omega)^3 & \xrightarrow{\text{curl}} & C^\infty(\Omega)^3 & \xrightarrow{\text{div}} & C^\infty(\Omega) \end{array},$$

where ϖ^0 and ϖ^3 are the identity mappings and where

$$\varpi^1\left(u_1 dx^1 + u_2 dx^2 + u_3 dx^3\right) = (u_1, u_2, u_3),$$
$$\varpi^2\left(u_{12} dx^1 \wedge dx^2 + u_{13} dx^1 \wedge dx^3 + u_{23} dx^2 \wedge dx^3\right) = (u_{23}, -u_{13}, u_{12}).$$

In two dimensions, the calculus of differential forms can be translated into 2D vector calculus in two different ways. To the authors' best knowledge, neither convention is dominant in the literature. We summarize the situation in the following commuting diagram:

$$\begin{array}{ccccc} C^\infty(\Omega) & \xrightarrow{\text{curl}} & C^\infty(\Omega)^2 & \xrightarrow{\text{div}} & C^\infty(\Omega) \\ \uparrow{\scriptstyle\varkappa^0} & & \uparrow{\scriptstyle\varkappa^1} & & \uparrow{\scriptstyle\varkappa^2} \\ C^\infty\Lambda^0(\Omega) & \xrightarrow{d} & C^\infty\Lambda^1(\Omega) & \xrightarrow{d} & C^\infty\Lambda^2(\Omega) \\ \downarrow{\scriptstyle\varpi^0} & & \downarrow{\scriptstyle\varpi^1} & & \downarrow{\scriptstyle\varpi^2} \\ C^\infty(\Omega) & \xrightarrow{\text{grad}} & C^\infty(\Omega)^2 & \xrightarrow{\text{rot}} & C^\infty(\Omega) \end{array}.$$

Here, $\varpi^1\left(u_1 dx^1 + u_2 dx^2\right) = (u_1, u_2)$ is the lower middle isomorphism. We introduce the rotation operator $J(x,y) = (y, -x)$ and define $\varkappa = J\varpi$ and $\operatorname{rot} = \operatorname{div} J$. The other vertical arrows are the identity. The utility of exterior calculus is that the operators of vector calculus can be translated into a common framework that does not depend on the dimension.

5.6 Some approximation properties

We finally review a few approximation properties. Let $\mathfrak{m}$ be a non-negative scalar function supported in the unit ball around the origin and with unit integral. Define $\mathfrak{m}_\epsilon(x) := \epsilon^{-n}\mathfrak{m}(x/\epsilon)$. We review the following approximation results.

Lemma 5.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $1 \leq p < \infty$. If $u \in L^p(\Omega)$, then the convolution $\mathfrak{m}_\epsilon \star u \to u$ in $L^p(\Omega)$ as $\epsilon \to 0$.*

Lemma 5.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $1 \leq p < \infty$. Smooth forms are dense in $W^p\Lambda^k(\Omega)$. If Ω is convex, then $C_c^\infty\Lambda^k(\Omega)$ is dense in $W_0^p\Lambda^k(\Omega)$.*

Proof. We notice $\mathfrak{m}_\epsilon \star u \in C^\infty\Lambda^k(\mathbb{R}^n)$. By the dominated convergence theorem and because u has a weak derivative, for any $v \in C_c^\infty\Lambda^{n-k-1}(\Omega)$

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathfrak{m}_\epsilon \star u) \wedge dv &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathfrak{m}(x-y) \wedge u(y) \wedge d_x v(x)\, dx\, dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} d_x\mathfrak{m}(x-y) \wedge u(y) \wedge v(x)\, dx\, dy \\ &= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} d_y\mathfrak{m}(x-y) \wedge u(y) \wedge v(x)\, dx\, dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathfrak{m}(x-y) \wedge d_y u(y) \wedge v(x)\, dx\, dy = \int_{\mathbb{R}^n} (\mathfrak{m}_\epsilon \star du) \wedge v. \end{aligned}$$

Hence $C^\infty\Lambda^k(\Omega)$ is dense in $W^p\Lambda^k(\Omega)$.

Next, suppose that Ω is convex. Without loss of generality, $0 \in \Omega$. Let $u \in W^p_0\Lambda^k(\Omega)$ and extend u trivially onto $\mathbb{R}^n$. Define $\varphi_t(x) = tx$ for $t > 1$. Then $\varphi_t^* u \in W^p_0\Lambda^k(t^{-1}\Omega)$ and $\varphi_t^* u$ converges to u as t decreases towards 1. Given any $t > 1$, by taking the convolution with $\mathfrak{m}_\epsilon$ for $\epsilon > 0$ small enough, we approximate $\varphi_t^* u$ through members of $C^\infty_c\Lambda^k(\Omega)$. The desired result follows. □

6 Regularized potential operators with analytical bounds on operator norms for exterior derivatives over convex sets

We now give bounds for Poincaré–Friedrichs constants for the exterior derivative over convex domains, in two special cases: either the L^p de Rham complex without boundary conditions, or the L^p de Rham with full boundary conditions. The corresponding potential operators are known as the regularized Poincaré and regularized Bogovskiĭ potentials in the literature. We build upon the discussion spearheaded by Costabel and McIntosh [18], who analyze them as pseudo-differential operators over domains star-shaped with respect to a ball. In comparison to their work, our discussion is more modest: we study potential operators merely over convex sets, and we are only interested in their operator norms between Lebesgue spaces. However, our purpose are explicit bounds for the operator norms, giving the Poincaré–Friedrichs constants. Let $\Omega \subseteq \mathbb{R}^n$ is a convex bounded open set with diameter $\delta(\Omega) > 0$.

MV: Discussion already in the introduction, or in Remark 6.2

MV: Weakly Lipschitz??

MV: Unifying the notation.

6.1 Regularized Poincaré and Bogovskiĭ operators

We begin with introducing the Costabel–McIntosh kernel. For any $\ell \in \{0, \dots, n\}$, we define the kernel $G_\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$G_\ell(x, y) = \int_1^\infty (t-1)^{n-\ell} t^{\ell-1} \operatorname{vol}(\Omega)^{-1} \chi_\Omega\left(y + t(x-y)\right) dt. \tag{34}$$

Given a differential form $u \in C^\infty_c(\mathbb{R}^n, \Lambda^\ell)$, where $1 \leq \ell \leq n$, we then define the integral operators

$$\mathfrak{P}_\ell u(x) = \int_\Omega G_{n-\ell+1}(y, x)\,(x-y) \lrcorner u(y)\, dy,$$
$$\mathfrak{B}_\ell u(x) = \int_\Omega G_\ell(x, y)\,(x-y) \lrcorner u(y)\, dy.$$

We call $\mathfrak{P}_\ell$ the *Poincaré operator* and $\mathfrak{B}_\ell$ the *Bogovskiĭ operator*.

We show that the integrals in the definition of $\mathfrak{P}_\ell$ and $\mathfrak{B}_\ell$ actually exist. In order to analyze the properties of the potential operators, we first rewrite the Costabel–McIntosh kernel G_ℓ. Letting $x, y \in \mathbb{R}^n$ with $x \neq y$, we find

$$
\begin{aligned}
G_\ell(x,y) &= \int_0^\infty t^{n-\ell}(t+1)^{\ell-1}\operatorname{vol}(\Omega)^{-1}\chi_\Omega\left(x+t(x-y)\right)\,dt \\
&= \int_0^\infty \sum_{k=0}^{\ell-1}\tbinom{\ell-1}{k} t^{n-\ell+k}\operatorname{vol}(\Omega)^{-1}\chi_\Omega\left(x+t(x-y)\right)\,dt \\
&= \int_0^\infty \sum_{k=0}^{\ell-1}\tbinom{\ell-1}{\ell-1-k} t^{n-\ell+\ell-1-k}\operatorname{vol}(\Omega)^{-1}\chi_\Omega\left(x+t(x-y)\right)\,dt \\
&= \int_0^\infty \sum_{k=0}^{\ell-1}\tbinom{\ell-1}{k} t^{n-k-1}\operatorname{vol}(\Omega)^{-1}\chi_\Omega\left(x+t(x-y)\right)\,dt \\
&= \sum_{k=0}^{\ell-1}\tbinom{\ell-1}{k}\int_0^\infty t^{n-k-1}\operatorname{vol}(\Omega)^{-1}\chi_\Omega\left(x+t(x-y)\right)\,dt \\
&= \operatorname{vol}(\Omega)^{-1}\sum_{k=0}^{\ell-1}\tbinom{\ell-1}{k}\,|x-y|^{k-n}\int_0^\infty r^{n-k-1}\chi_\Omega\left(x+r\frac{x-y}{|x-y|}\right)\,dr.
\end{aligned}
$$

If $x,y\in\Omega$, then we can restrict the inner integrals to the range $0\leq r\leq\delta(\Omega)$, which gives

$$
\begin{aligned}
G_\ell(x,y) &= \operatorname{vol}(\Omega)^{-1}\sum_{k=0}^{\ell-1}\tbinom{\ell-1}{k}|x-y|^{k-n}\int_0^{\delta(\Omega)} r^{n-k-1}\,dr \\
&= \operatorname{vol}(\Omega)^{-1}\sum_{k=0}^{\ell-1}\tbinom{\ell-1}{k}|x-y|^{k-n}\frac{\delta(\Omega)^{n-k}}{n-k}.
\end{aligned}
$$

We are now in a position to show that the potentials are bounded with respect to Lebesgue norms.

6.2 An operator norm bound with respect to the Lebesgue norm: the Poincaré case

We begin with the Poincaré operator. Let $B_{\delta(\Omega)}(0)$ be the n-dimensional ball centered at the origin. Suppose that $u\in L^\infty\Lambda^\ell(\Omega)$ with $1\leq\ell\leq n$. We pointwise estimate $\mathfrak{P}_\ell u(x)$ for any $x\in\Omega$ by the result of a convolution of a locally integrable function with u:

$$
\begin{aligned}
|\mathfrak{P}_\ell u(x)| &= \left|\int_\Omega G_{n-\ell+1}(x,y)\,(x-y)\lrcorner u(y)\,dy\right| \\
&\leq \int_\Omega \operatorname{vol}(\Omega)^{-1}\sum_{k=0}^{n-\ell}\tbinom{n-\ell}{k}\frac{\delta(\Omega)^{n-k}}{n-k}|x-y|^{k+1-n}\chi_{B_{\delta(\Omega)}(0)}(x-y)|u(y)|\,dy.
\end{aligned}
$$

We recall the radial integrals, where S_1 stands for MV: ???

$$
\begin{aligned}
\int_{B_{\delta(\Omega)}(0)}|z|^{k+1-n}\,dz &= \operatorname{vol}_{n-1}(S_1)\int_0^{\delta(\Omega)} r^{k+1-n}r^{n-1}\,dr \\
&= \operatorname{vol}_{n-1}(S_1)\int_0^{\delta(\Omega)} r^k\,dr = \operatorname{vol}_{n-1}(S_1)\frac{\delta(\Omega)^{k+1}}{k+1}.
\end{aligned}
$$

We compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{k=0}^{n-\ell} \tbinom{n-\ell}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \chi_{B_{\delta(\Omega)}(0)}(z) |z|^{k+1-n} \, dz \\
&= \sum_{k=0}^{n-\ell} \tbinom{n-\ell}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \int_{B_{\delta(\Omega)}(0)} |z|^{k+1-n} \, dz \\
&= \operatorname{vol}_{n-1}(S_1) \sum_{k=0}^{n-\ell} \tbinom{n-\ell}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \frac{\delta(\Omega)^{k+1}}{k+1} = \operatorname{vol}_{n-1}(S_1) \delta(\Omega)^{n+1} \underbrace{\sum_{k=0}^{n-\ell} \frac{\binom{n-\ell}{k}}{(n-k)(k+1)}}_{=:A_{\mathfrak{P}}(n,\ell) \leq 2^{n-\ell}} .
\end{aligned} \tag{35}$$

Here, $A_{\mathfrak{P}}(n,\ell)$ is a numerical constant that depends only on n and ℓ and which is bounded by $2^{n-\ell}$. In particular, the integral $\mathfrak{P}_\ell u(x)$ is absolutely convergent for any choice of $x \in \Omega$. So the convolution of u is taken against an integrable function. Young's convolution inequality now implies:

$$\begin{aligned}
\|\mathfrak{P}_\ell u\|_{L^p(\Omega)} &\leq \operatorname{vol}_{n-1}(S_1) A_{\mathfrak{P}}(n,\ell) \frac{\delta(\Omega)^n}{\operatorname{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)} \\
&\leq n A_{\mathfrak{P}}(n,\ell) \frac{\operatorname{vol}(B_{\delta(\Omega)}(0))}{\operatorname{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)}.
\end{aligned}$$

We have assumed so far that $u \in L^\infty\Lambda^\ell(\Omega)$. Since that space is dense in the Lebesgue spaces, a density argument establishes the following: for any $1 \leq p \leq \infty$ we have a bounded linear operator

$$\mathfrak{P}_\ell : L^p\Lambda^\ell(\Omega) \to L^p\Lambda^{\ell-1}(\mathbb{R}^n).$$

6.3 An operator norm bound with respect to the Lebesgue norm: the Bogovskiĭ case

We analyze the Bogovskiĭ potential operator with similar means. Suppose that $u \in L^\infty\Lambda^\ell(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq \overline{\Omega}$ and that $x \in \mathbb{R}^n$. First, if $x \notin \Omega$, then the convexity of Ω implies that $y + t(x-y) \notin \Omega$ for all $t > 1$. Hence $G(x,y) = 0$ and therefore $\mathfrak{B}_\ell u(x) = 0$ in that case. Consider now the case $x \in \overline{\Omega}$. We estimate $\mathfrak{B}_\ell u(x)$ pointwise by

$$\begin{aligned}
|\mathfrak{B}_\ell u(x)| &= \left| \int_\Omega G_\ell(x,y)\, (x-y) \lrcorner u(y) \, dy \right| \\
&\leq \int_\Omega \operatorname{vol}(\Omega)^{-1} \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} \frac{\delta(\Omega)^{n-k}}{n-k} |x-y|^{k+1-n} |u(y)| \, dy \\
&\leq \int_{\mathbb{R}^n} \operatorname{vol}(\Omega)^{-1} \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \chi_{B_{\delta(\Omega)}(0)}(x-y) |x-y|^{k+1-n} |u(y)| \, dy.
\end{aligned}$$

Using once more the radial integrals discussed above, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \chi_{B_{\delta(\Omega)}(0)}(z) |z|^{k+1-n} \, dz \\
&= \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \int_{B_{\delta(\Omega)}(0)} |z|^{k+1-n} \, dz \\
&= \operatorname{vol}_{n-1}(S_1) \sum_{k=0}^{\ell-1} \tbinom{\ell-1}{k} \frac{\delta(\Omega)^{n-k}}{n-k} \frac{\delta(\Omega)^{k+1}}{k+1} = \operatorname{vol}_{n-1}(S_1) \delta(\Omega)^{n+1} \underbrace{\sum_{k=0}^{\ell-1} \frac{\binom{\ell-1}{k}}{(n-k)(k+1)}}_{=:A_{\mathfrak{B}}(n,\ell) \leq 2^{\ell-1}} .
\end{aligned} \tag{36}$$

Here, $A_{\mathfrak{B}}(n,\ell)$ is a numerical constant that depends only on n and ℓ and which is bounded by $2^{\ell-1}$. In particular, the integral $\mathfrak{B}_\ell u(x)$ is absolutely convergent for any choice of $x \in \mathbb{R}^n$. So the convolution of u is taken against an integrable function. Young's convolution inequality now implies:

$$\begin{aligned}
\|\mathfrak{B}_\ell u\|_{L^p(\Omega)} &\leq \mathrm{vol}_{n-1}(S_1) A(n,\ell) \frac{\delta(\Omega)^n}{\mathrm{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)} \\
&\leq n A_{\mathfrak{B}}(n,\ell) \frac{\mathrm{vol}(B_{\delta(\Omega)}(0))}{\mathrm{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)}.
\end{aligned}$$

We have assumed so far that u is essentially bounded. Since that space is dense in the Lebesgue spaces, a density argument yields: for any $1 \leq p \leq \infty$ we have a bounded linear operator

$$\mathfrak{B}_\ell : L^p\Lambda^\ell(\Omega) \to L^p\Lambda^{\ell-1}(\mathbb{R}^n).$$

Moreover, $\operatorname{supp} \mathfrak{B}_\ell u \subseteq \overline{\Omega}$, that is, the reconstructed potential has support contained within $\overline{\Omega}$.

6.4 Additional properties

Additional properties of these operators become apparent after a change of variables. We write down the full definition of these operators and perform two substitutions. For the Poincaré operator, we substitute $a = x + t(y-x)$ and then we substitute $s = (t-1)/t$, leading to

$$\begin{aligned}
\mathfrak{P}_\ell u(x) &= \int_\Omega \int_1^\infty (t-1)^{\ell-1} t^{n-\ell} \chi_\Omega\left(x + t(y-x)\right) (x-y) \lrcorner u(y) \, dt \, dy \\
&= \int_{\mathbb{R}^n} \chi_\Omega(a) \, (x-a) \lrcorner \int_0^1 t^{\ell-1} u\left(a + t(x-a)\right) \, dt \, da.
\end{aligned}$$

For the Bogoskiĭ operator, we substitute $a = y + t(x-y)$ and then we substitute $s = t/(t-1)$, leading to

$$\begin{aligned}
\mathfrak{B}_\ell u(x) &= \int_\Omega \int_1^\infty (t-1)^{n-\ell} t^{\ell-1} \chi_\Omega\left(y + t(x-y)\right) (x-y) \lrcorner u(y) \, dt \, dy \\
&= -\int_{\mathbb{R}^n} \chi_\Omega(a) \, (x-a) \lrcorner \int_1^\infty t^{\ell-1} u\left(a + t(x-a)\right) \, dt \, da.
\end{aligned}$$

Given $a \in \Omega$, we introduce the potential operators

$$\begin{aligned}
\mathfrak{P}_{\ell,a} u(x) &:= (x-a) \lrcorner \int_0^1 t^{\ell-1} u\left(a + t(x-a)\right) \, dt, \\
\mathfrak{B}_{\ell,a} u(x) &:= -(x-a) \lrcorner \int_1^\infty t^{\ell-1} u\left(a + t(x-a)\right) \, dt.
\end{aligned}$$

By definition,

$$\mathfrak{P}_\ell u(x) = \int_\Omega \mathfrak{P}_{\ell,a} u(x) \, da, \quad \mathfrak{B}_\ell u(x) = \int_\Omega \mathfrak{B}_{\ell,a} u(x) \, da.$$

We study the interaction of these potential operator with the exterior derivative in more detail. That discussion is classical and establishes the exactness of several de Rham complexes. We recapitulate these arguments since our variants of the regularized potential operators are not yet included in the published literature.

Suppose that $u \in C^\infty\Lambda^\ell(\mathbb{R}^n)$. We rewrite the Poincaré potential,

$$\begin{aligned}
\mathfrak{P}_{\ell,a}u(x) &= (x-a)\lrcorner \int_0^1 t^{\ell-1} u\left(a+t(x-a)\right)\, dt\, da \\
&= (x-a)\lrcorner \sum_{\sigma\in\Sigma(k,n)} \int_0^1 t^{\ell-1} u_\sigma\left(a+t(x-a)\right) dx^\sigma\, dt \\
&= \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_0^1 t^{\ell-1} u_\sigma\left(a+t(x-a)\right)(-1)^{i-1}(x-a)_{\sigma(i)}\, dx^{\sigma-\sigma(i)}\, dt,
\end{aligned}$$

and compute its exterior derivative:

$$\begin{aligned}
d\mathfrak{P}_{\ell,a}u(x) = \ell &\sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_0^1 t^{\ell-1} u_\sigma\left(a+t(x-a)\right)\, dx^{\sigma-\sigma(i)}\, dt \\
&+ \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} \int_0^1 t^{\ell} \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right)(-1)^{i-1}(x-a)_{\sigma(i)}\, dx^j \wedge dx^{\sigma-\sigma(i)}\, dt.
\end{aligned}$$

We write the exterior derivative of u as

$$du(x) = \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x)\, dx^j \wedge dx^\sigma\, dt,$$

and apply the Poincaré potential operator to this result, which gives

$$\begin{aligned}
\mathfrak{P}_{\ell+1,a}du(x) &= (x-a)\lrcorner \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_0^1 t^{\ell} \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right)\, dt\, dx^j \wedge dx^\sigma \\
&= \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_0^1 t^{\ell} \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right)\, dt (x-a)_j\, dx^\sigma \\
&\quad - \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} (-1)^{i-1} \int_0^1 t^{\ell} \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right)\, dt\, (x-a)_{\sigma(i)}\, dx^j \wedge dx^{\sigma-\sigma(i)}.
\end{aligned}$$

We add the exterior derivative of the potential and the potential of the exterior derivative. Taking into account cancellations, this gives the identity

$$
\begin{aligned}
& d\mathfrak{P}_{\ell,a}u(x) + \mathfrak{P}_{\ell+1,a}du(x) \\
&= \ell \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_0^1 t^{\ell-1} u_\sigma \left(a + t(x-a)\right) \, dx^{\sigma-\sigma(i)} \, dt \\
&\quad + \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} \int_0^1 t^\ell \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)} \, dx^j \, dx^{\sigma-\sigma(i)} \, dt \\
&\quad + \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_0^1 t^\ell \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt \, (x-a)_j \, dx^\sigma \\
&\quad - \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} (-1)^{i-1} \int_0^1 t^\ell \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, dt \, (x-a)_{\sigma(i)} \, dx^j \wedge dx^{\sigma-\sigma(i)} \\
&= \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_0^1 \ell t^{\ell-1} u_\sigma \left(a + t(x-a)\right) \, dx^{\sigma-\sigma(i)} \, dt \\
&\quad + \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_0^1 t^\ell \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) \, (x-a)_j \, dt \, dx^\sigma \\
&= \sum_{\sigma\in\Sigma(k,n)} \int_0^1 \frac{\partial}{\partial t} \left(t^\ell u_\sigma \left(a + t(x-a)\right)\right) \, dt \, dx^\sigma \\
&= \sum_{\sigma\in\Sigma(k,n)} u_\sigma(x) \, dx^\sigma = u(x).
\end{aligned}
$$

In summary, after taking the average over $a \in \Omega$:

$$
u(x) = d\mathfrak{P}_\ell u(x) + \mathfrak{P}_{\ell+1} du(x).
$$

In particular, if $du = 0$, then $d\mathfrak{P}_\ell u = u$.

The discussion for the Bogovskiĭ operator is almost analogous. Suppose that $u \in C^\infty\Lambda^\ell(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq \overline{\Omega}$. We rewrite the Bogovskiĭ potential,

$$
\begin{aligned}
\mathfrak{B}_{\ell,a}u(x) &= (x-a) \lrcorner \int_1^\infty t^{\ell-1} u \left(a + t(x-a)\right) \, dt \, da \\
&= (x-a) \lrcorner \sum_{\sigma\in\Sigma(k,n)} \int_1^\infty t^{\ell-1} u_\sigma \left(a + t(x-a)\right) \, dx^\sigma \, dt \\
&= \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_1^\infty t^{\ell-1} u_\sigma \left(a + t(x-a)\right) (-1)^{i-1} \, (x-a)_{\sigma(i)} \, dx^{\sigma-\sigma(i)} \, dt,
\end{aligned}
$$

and compute its exterior derivative:

$$
\begin{aligned}
d\mathfrak{B}_{\ell,a}u(x) = \ell & \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_1^\infty t^{\ell-1} u_\sigma \left(a + t(x-a)\right) \, dx^{\sigma-\sigma(i)} \, dt \\
&+ \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j} \left(a + t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)} \, dx^j \wedge dx^{\sigma-\sigma(i)} \, dt.
\end{aligned}
$$

We write the exterior derivative of u as

$$du(x) = \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x)\, dx^j \wedge dx^\sigma \, dt,$$

and apply the Bogovskiĭ potential operator to this result, which gives

$$\begin{aligned}
\mathfrak{B}_{\ell+1,a} du(x) &= (x-a) \lrcorner \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) dt\, dx^j \wedge dx^\sigma \\
&= \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) dt\, (x-a)_j \, dx^\sigma \\
&\quad - \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} (-1)^{i-1} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) dt (x-a)_{\sigma(i)}\, dx^j \wedge dx^{\sigma-\sigma(i)}.
\end{aligned}$$

We add the exterior derivative of the potential and the potential of the exterior derivative. Taking into account cancellations, this gives the identity

$$\begin{aligned}
& d\mathfrak{B}_{\ell,a} u(x) + \mathfrak{B}_{\ell+1,a} du(x) \\
&= \ell \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_1^\infty t^{\ell-1} u_\sigma\left(a+t(x-a)\right) dx^{\sigma-\sigma(i)}\, dt \\
&\quad + \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) (-1)^{i-1} (x-a)_{\sigma(i)}\, dx^j\, dx^{\sigma-\sigma(i)}\, dt \\
&\quad + \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) dt\, (x-a)_j \, dx^\sigma \\
&\quad - \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k}\sum_{j=1}^{n} (-1)^{i-1} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) dt\, (x-a)_{\sigma(i)}\, dx^j \wedge dx^{\sigma-\sigma(i)} \\
&= \sum_{\sigma\in\Sigma(k,n)} \sum_{i=1}^{k} \int_1^\infty \ell t^{\ell-1} u_\sigma\left(a+t(x-a)\right) dx^{\sigma-\sigma(i)}\, dt \\
&\quad + \sum_{\sigma\in\Sigma(k,n)} \sum_{j=1}^{n} \int_1^\infty t^\ell \frac{\partial u_\sigma}{\partial x_j}\left(a+t(x-a)\right) (x-a)_j \, dt\, dx^\sigma \\
&= \sum_{\sigma\in\Sigma(k,n)} \int_1^\infty \frac{\partial}{\partial t}\left(t^\ell u_\sigma\left(a+t(x-a)\right)\right) dt\, dx^\sigma \\
&= \sum_{\sigma\in\Sigma(k,n)} u_\sigma(x) dx^\sigma = u(x).
\end{aligned}$$

In summary, after taking the average over $a \in \Omega$:

$$u(x) = d\mathfrak{B}_\ell u(x) + \mathfrak{B}_{\ell+1} du(x).$$

In particular, if $du = 0$, then $d\mathfrak{B}_\ell u = u$.

6.5 Operator norms as bounds for the Poincaré–Friedrichs constants

We now state the main results of this section. Recall that $\delta(\Omega) > 0$ is the diameter of Ω and that $B_{\delta(\Omega)}(0)$ is the n-dimensional ball centered at the origin. The following upper bounds for the Poincaré–Friedrichs constants are proportional to the domain diameter and are independent of the Lebesgue exponent $1 \leq p \leq \infty$, though the space dimension n and the form degree ℓ enters the estimates, namely through definitions (35) and (36) of respectively $A_{\mathfrak{P}}(n,\ell)$ and $A_{\mathfrak{B}}(n,\ell)$.

Theorem 6.1. *Let $\Omega \subseteq \mathbb{R}^n$ be and convex bounded open set and let $1 \leq p < \infty$. We have bounded operators*

MV: Weakly Lipschitz??

$$\mathfrak{P}_\ell : L^p\Lambda^\ell(\Omega) \to W^p\Lambda^{\ell-1}(\Omega), \qquad \mathfrak{B}_\ell : L^p\Lambda^\ell(\Omega) \to W^p_0\Lambda^{\ell-1}(\Omega).$$

They satisfy the operator norm bounds

$$\|\mathfrak{P}_\ell u\|_{L^p(\Omega)} \leq n A_{\mathfrak{P}}(n,\ell) \frac{\mathrm{vol}(B_{\delta(\Omega)}(0))}{\mathrm{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)},$$
$$\|\mathfrak{B}_\ell u\|_{L^p(\Omega)} \leq n A_{\mathfrak{B}}(n,\ell) \frac{\mathrm{vol}(B_{\delta(\Omega)}(0))}{\mathrm{vol}(\Omega)} \delta(\Omega) \|u\|_{L^p(\Omega)}.$$

If $u \in W^p\Lambda^\ell(\Omega)$ with $du = 0$, then $\mathfrak{P}_\ell u \in W^p\Lambda^{\ell-1}(\Omega)$ with $u = d\mathfrak{P}_\ell u$. If $u \in W^p_0\Lambda^\ell(\Omega)$ with $du = 0$, then $\mathfrak{B}_\ell u \in W^p_0\Lambda^{\ell-1}(\Omega)$ with $u = d\mathfrak{B}_\ell u$.

MV: Do $\mathfrak{P}_\ell$ and $\mathfrak{B}_\ell$ still map piecewise polynomials to piecewise polynomials? Like Raviart–Thomas and Nédélec spaces to Raviart–Thomas and Nédélec spaces??

Proof. The potential operators are linear and the operator norm bounds are satisfied over the dense subspace $C^\infty_c\Lambda^\ell(\Omega)$.

Consider now $u \in W^p\Lambda^\ell(\Omega)$ with $du = 0$. We write $w := \mathfrak{P}_\ell u$. There exists a sequence $u_i \in C^\infty\Lambda^\ell(\overline{\Omega})$ that converges to u in $W^p\Lambda^\ell(\Omega)$. For any test form $v \in C^\infty_c\Lambda^{n-\ell-1}(\Omega)$, we verify

$$\begin{aligned}\int_\Omega v \wedge u_i &= \int_\Omega v \wedge \mathfrak{P}_{\ell+1} du_i + \int_\Omega v \wedge d\mathfrak{P}_\ell u_i \\ &= \int_\Omega v \wedge \mathfrak{P}_{\ell+1} du_i + (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathfrak{P}_\ell u_i.\end{aligned}$$

By the continuity of bounded linear functionals, we find

$$\int_\Omega v \wedge u = (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathfrak{P}_\ell u_i.$$

By definition, $w \in W^p\Lambda^{\ell-1}(\Omega)$ with $dw = u$.

Analogously, suppose that $u \in W^p_0\Lambda^\ell(\Omega)$ with $du = 0$. We write $w := \mathfrak{B}_\ell u$. There exists a sequence $u_i \in C^\infty_c\Lambda^\ell(\overline{\Omega})$ that converges to u in $W^p\Lambda^\ell(\Omega)$. For any test form $v \in C^\infty\Lambda^{n-\ell-1}(\Omega)$, which is the restriction of some member of $C^\infty\Lambda^{n-\ell-1}(\mathbb{R}^n)$, it holds that

$$\begin{aligned}\int_\Omega v \wedge u_i &= \int_\Omega v \wedge \mathfrak{B}_{\ell+1} du_i + \int_\Omega v \wedge d\mathfrak{B}_\ell u_i \\ &= \int_\Omega v \wedge \mathfrak{B}_{\ell+1} du_i + (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathfrak{B}_\ell u_i.\end{aligned}$$

By the continuity of bounded linear functionals, we find

$$\int_\Omega v \wedge u = (-1)^{k(n-k)+1} \int_\Omega dv \wedge \mathfrak{B}_\ell u_i.$$

By definition, $w \in W^p_0\Lambda^{\ell-1}(\Omega)$ with $dw = u$. This completes the proof. □

Remark 6.2. *The classical Poincaré operator is known for its role in proving the exactness of the smooth de Rham complex over star-shaped domains [42]. To the best of our knowledge, the work of Costabel and McIntosh [18] was the first to propose a regularized Poincaré operator (regularized via averaging). These potentials enable the analysis of de Rham complexes without boundary conditions. The Bogovskiĭ-type operators were first studied for the divergence operator and are a staple in the mathematics of hydrodynamics [8]. Their generalization to differential forms was again first discussed by Costabel and McIntosh [18]. In some sense dual to the regularized Poincaré operators, they establish the exactness of the L^p de Rham complex with full boundary conditions.*

Costabel and McIntosh regularize the potentials by smoothly averaging over pivot points within an interior ball, which is why they can study domains star-shaped with respect to a ball. Their operators are pseudodifferential operators of negative order, because their averaging uses a smooth weight; this proves boundedness of the operators between a variety of function spaces. Explicit bounds for the higher-order seminorms of these pseudodifferential operators have been recently contributed by Guzman and Salgado [35]. Instead, we average over the entire domain, which requires a convex geometry, and we are only interested in boundedness in the Lebesgue p norms. This has not been established yet, to the best of our knowledge. Crucially, this exactly allows us to establish computable bounds on the Poincaré–Friedrichs constants. MV: moved.

7 Shellable triangulations of manifolds

We return to the theory of triangulations, as our main objective requires some further concepts. We are interested in simplicial complexes that triangulate manifolds. Our particular interest are manifold triangulations that are *shellable*. Such simplicial complexes are constructed by successively adding simplices in a particularly well-behaved manner. Local patches (stars) within triangulations of dimension two and three are examples of such shellable complexes. The monographs by Kozlov [39] and Ziegler [62] are our main references for this section. We also refer to Lee's monograph [41] for any further background on manifolds.

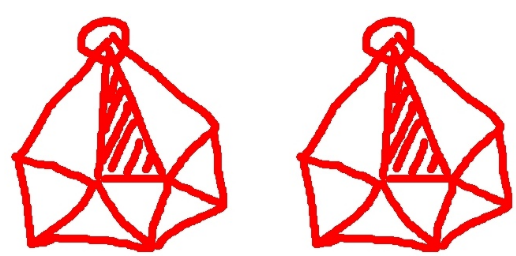


Figure 3: A manifold triangulation (left) and not a manifold triangulation (right)

7.1 Triangulations of manifolds

We define an n-dimensional simplicial complex to be a *manifold triangulation* if the underlying set $|\mathcal{T}|$ is an n-dimensional manifold with boundary. We recall that this means that for every $x \in |\mathcal{T}|$ there exists an open neighborhood $U(x) \subseteq |\mathcal{T}|$ and an embedding $\phi : U(x) \to \mathbb{R}^n$ such that $\phi(0) = 0$ and ϕ is an isomorphism either onto the open unit ball $\mathcal{B} = \{x \in \mathbb{R}^n \mid |x| < 1\}$ or onto the half-ball $\{x \in \mathcal{B} \mid x_1 \geq 0\}$. In the former case, x is called an *interior point*, and in the latter case x is called a *boundary point*. Any simplicial complex that triangulates an n-dimensional manifold must be n-dimensional. An example of a manifold triangulation and an example which is not a manifold triangulation are given in Figure 7.

The following special cases receive particular interest: an *n-ball triangulation* is any triangulation of a topological (closed) n-ball~~, and we sometimes call this a *n-disk* triangulation~~. An *n-sphere triangulation* is any triangulation of a topological n-sphere.

We know that any manifold $\mathcal{M}$ has got a topological boundary $\partial\mathcal{M}$, possibly empty. If $\mathcal{M}$ is n-dimensional, then the $\partial\mathcal{M}$ is a topological manifold without boundary of dimension $n-1$. We gather a few helpful observations on how these notions relate to triangulations. While the reader might deem them obvious, we nevertheless include proofs.

Lemma 7.1. *Let $\mathcal{T}$ be a finite n-dimensional simplicial complex whose underlying set is a manifold $\mathcal{M}$.*

- *Any face $F \in \mathcal{F}(\mathcal{T})$ is not contained in the boundary if and only if it is contained in exactly two n-simplices.*
- *Any face $F \in \mathcal{F}(\mathcal{T})$ is contained in the boundary if and only if it is contained in exactly one n-simplex.*
- *The simplices contained in the boundary constitute a triangulation of the boundary.*

Proof. We prove these statements in several steps.

1. Let $\mathring{\mathcal{M}} := \mathcal{M} \setminus \partial\mathcal{M}$ denote the interior of the manifold. We will use the following fact:[3] if $S \in \mathcal{T}$ has an inner point that lies on $\partial\mathcal{M}$, then all inner points of S are on $\partial\mathcal{M}$. Since the boundary $\partial\mathcal{M}$ is closed, every $S \in \mathcal{T}$ is either a subset of the boundary or all its inner points lie in the interior $\mathring{\mathcal{M}}$ of the manifold.

2. We recall an auxiliary result. Suppose that Y is a topological space homeomorphic to a sphere of dimension m and that $X \subseteq Y$ is homeomorphic to a sphere of dimension $m-1$, where $m \geq 1$. As a consequence of the Jordan–Brouwer separation theorem [49, Corollary IV.5.24] [47, Corollary VIII.6.4], we know that $Y \setminus X$ has got two connected components.

3. Let $F \in \mathcal{F}(T)$ and let $z_F \in F$ be its barycenter. Since $\mathcal{T}$ is finite, we let $\mathring{B}_F$ be an open neighborhood around z_F so small that it only intersects those n-simplices of $\mathcal{T}$ that already contain z_F and no faces other than F. Suppose there are distinct n-simplices $T_1, T_2, \dots, T_K$ that contain z_F. The intersection of any two of them is F, but their interiors are disjoint because otherwise they would coincide.

   If z_F is an interior point of $\mathcal{M}$, then we it follows by our assumptions that $\mathring{B}_F$ is homeomorphic to an open n-ball and $\partial\mathring{B}_F$ is homeomorphic to a sphere of dimension $n-1$. Consider $X = F \cap \partial\mathring{B}_F$. If $n = 1$, then X is empty and $\partial\mathring{B}_F$ has K distinct connected components. If $n > 1$, then X is homeomorphic to a sphere of dimension $n-2$ and again $\partial\mathring{B}_F \setminus X$ has K distinct connected components. But by the auxiliary result above, $K = 2$. We conclude that F is contained in two n-simplices of $\mathcal{T}$.

   Consider the case that z_F lies on the boundary of $\mathcal{M}$ and suppose that F is contained in K distinct n-simplices of $\mathcal{T}$. By adding at least one dimension, we can double[4] the manifold $\mathcal{M}$ along the boundary and obtain the doubled manifold $\mathcal{M}'$. Similarly, we can construct a doubling of the triangulation $\mathcal{T}'$ such that F is contained in exactly $2K$ distinct n-simplices of $\mathcal{T}'$. We know that $\mathcal{M}'$ is a manifold without boundary, and hence F is an interior simplex of $\mathcal{T}'$. This implies $K = 1$. So any boundary face can only be contained in one single n-simplex.

4. Clearly, the simplices of $\mathcal{T}$ contained in the boundary constitute a simplicial complex. Every $x \in \mathcal{M}$ is an inner point of some simplex $S \in \mathcal{T}$. If $x \in \partial\mathcal{M}$ is a boundary point, then S must be a boundary simplex, so the boundary simplices triangulate all of $\partial\mathcal{M}$.

All desired results are thus proven. ☐

Suppose that $\mathcal{T}$ is an n-dimensional simplicial complex that triangulates a manifold. Those simplices of the manifold triangulation that are subsets of the boundary of the underlying manifold are called *boundary simplices*. All other simplices of the manifold triangulation are called *inner simplices*. We have seen that the boundary simplices of a manifold triangulation constitute a triangulation of the manifolds boundary. We call this simplicial complex the *boundary complex*. It has dimension $n-1$.

[3]To see this, one easily constructs a continuous deformation of $\mathcal{M}$ into itself to move any chosen point on S to any other chosen point on S.

[4]The reader is referred to Lee's textbook [42] for more background and the technicalities.

We continue with a few more observations about manifold triangulations at the intersection of topology and combinatorics. These include the topology of local patches (stars), which is a surprisingly non-trivial topic. We say that an n-dimensional manifold triangulation $\mathcal{T}$ is *locally vertex combinatorial* if the star of every simplex $S \in \mathcal{T}$ is a triangulated n-dimensional ball that contains S. We gather some results that are hard to find in the literature.

Lemma 7.2. *Let $\mathcal{T}$ be a finite n-dimensional simplicial complex whose underlying space is a manifold $\mathcal{M}$. Suppose that $1 \leq n \leq 3$. Then the following holds:*

- *If $S \in \mathcal{T}$ is an inner simplex, then $\operatorname{st}_{\mathcal{T}}(S)$ is a simplicial n-ball with S as an inner simplex and $\partial\operatorname{st}_{\mathcal{T}}(S)$ is a simplicial $(n-1)$-sphere.*
- *If $S \in \mathcal{T}$ is a boundary subsimplex, then $\operatorname{st}_{\mathcal{T}}(S)$ is a simplicial n-ball with S as a boundary simplex, and $\partial\operatorname{st}_{\mathcal{T}}(S)$ is a simplicial $(n-1)$-ball.*

Proof. The lemma is obvious if $n = 1$, so we assume $n \geq 2$ in what follows. We prove these statements in several steps. The reader is assumed to have some background in topology.

1. Let S be any simplex with vertices $v_0, v_1, \dots, v_k$, with midpoint z_S, and dimension k. Let $\mathcal{S} := \operatorname{st}_{\mathcal{T}}(S)$ be its star. Each l-dimensional simplex $T \in \mathcal{S}$ that contains S has vertices $v_0, v_1, \dots, v_k, v^S_{k+1}, \dots, v^S_l$. For any such simplex, we introduce a decomposition $T_0, \dots, T_k$, where each T_i has vertices

   MV: barycenter???

   MV: OK?

$$v_0, \dots, v_{i-1}, z_S, v_{i+1}, \dots, v_k, v^S_{k+1}, \dots, v^S_l.$$

   The collection $\mathcal{S}'$ of these simplices and their subsimplices constitute a simplicial complex that triangulates the same underlying set as $\mathcal{S}$. Moreover, $\mathcal{S}' = \operatorname{st}_{\mathcal{S}'}(z_S)$. In particular, z_S is a boundary vertex of $\mathcal{S}'$ if and only if S is a boundary simplex of $\mathcal{S}$. So it remains to study the topology of vertex stars.

2. Suppose that $2 \leq n \leq 3$ and that $\mathcal{M}$ is a manifold without boundary. Under these assumptions, as explained in the proof of Theorem 1 in [55], the set $\partial\operatorname{st}(V)$ is a triangulation of a sphere of dimension $n-1$ for any inner vertex V. There exists a homeomorphism from the closed cone of $|\partial\operatorname{st}(V)|$ onto the star $|\operatorname{st}_{\mathcal{T}}(V)|$. But then that closed cone and hence $|\operatorname{st}_{\mathcal{T}}(V)|$ are homeomorphic to an n-dimensional ball.

3. If $2 \leq n \leq 3$ and $\mathcal{M}$ has a non-empty boundary, then we use an approach as in the proof of Lemma 7.1: we let $\mathcal{M}'$ denote the doubling of the manifold and $\mathcal{T}'$ be the doubling of the triangulation $\mathcal{T}$. Let $V \in \mathcal{T}$ be a vertex. If V is an inner vertex of $\mathcal{T}$, then $\partial\operatorname{st}(V) \subseteq \mathcal{T} \subseteq \mathcal{T}'$ triangulates a sphere of dimension $n-1$ and $\operatorname{st}_{\mathcal{T}}(V) \subseteq \mathcal{T} \subseteq \mathcal{T}'$ triangulates a ball of dimension n, as discussed above. If V is a boundary vertex of $\mathcal{T}$, then it is an inner vertex of $\mathcal{T}'$, and so $\partial\operatorname{st}_{\mathcal{T}}(V) \subseteq \mathcal{T}'$ triangulates a sphere of dimension $n-1$ and $\operatorname{st}_{\mathcal{T}}(V) \subseteq \mathcal{T}'$ triangulates a ball of dimension n. We also know that $\partial\operatorname{st}_{\partial\mathcal{T}}(V) \subseteq \partial\mathcal{T}$ triangulates a sphere of dimension $n-2$ and $\operatorname{st}_{\partial\mathcal{T}}(V) \subseteq \partial\mathcal{T}$ triangulates a ball of dimension $n-1$. The embedding of $\partial\operatorname{st}_{\partial\mathcal{T}}(V) \subseteq \partial\mathcal{T}$ is homeomorphic to the standard embedding of the $(n-2)$-dimensional unit sphere into the $(n-1)$-dimensional unit sphere. It follows that $\partial\operatorname{st}_{\mathcal{T}}(V)$ triangulates a topological ball of dimension $n-1$. Since the closed cone of $|\partial\operatorname{st}(V)|$ is homeomorphic to the star $|\operatorname{st}_{\mathcal{T}}(V)|$, we conclude that $\operatorname{st}_{\mathcal{T}}(V)$ triangulates an n-dimensional ball.

All relevant results are proven. ☐

Lemma 7.3. *Let $\mathcal{T}$ be a finite n-dimensional simplicial complex whose underlying space is a manifold $\mathcal{M}$. If the underlying space of $\mathcal{T}$ is connected, then $\mathcal{T}$ is face connected.*

Proof. We first show that each vertex star is face connected via a short induction argument. Clearly, any simplicial 1-ball and simplicial 1-sphere are face connected. Now, if $n \geq 1$, then any vertex star in any $(n+1)$-dimensional manifold triangulation is already face connected if all simplicial n-balls and n-spheres are face connected. The induction argument implies that each vertex star in $\mathcal{T}$ is face connected.

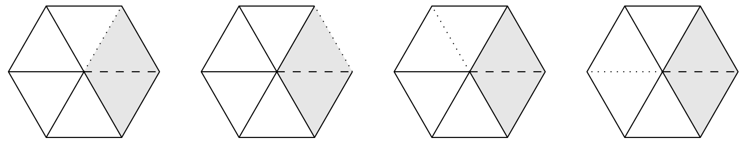


Figure 4: Examples and counterexamples for building a manifold-like 1-patching of a 2-dimensional manifold-like triangulation. (1) We have already added the edge patch along the dashed line and we add the edge patch along the dotted line. (2) The new edge patch may already be a subset of the subtriangulation (3) An inadmissible choice because the new edge patch has no overlap with current subtriangulation, though the next subtriangulation would be manifold-like (4) An inadmissible choice because the new edge patch has no overlap with current subtriangulation and the next subtriangulation would not be manifold-like.

If the underlying space $|\mathcal{T}|$ is connected, then we easily see that the union of the 1-simplices is path-connected. Given n-simplices $S, S' \in \mathcal{T}$, we can thus choose a sequence of $\hat{S}_0 = S, \hat{S}_1, \dots, \hat{S}_m = S' \in \mathcal{T}$ such that $\hat{S}_i \cap \hat{S}_{i-1} \neq \emptyset$ for all $1 \leq i \leq m$, and so each two consecutive simplices in that sequence have at least one vertex in common. As each vertex star is face connected, we can thus assume without loss of generality that the sequence $S_0 = S, S_1, \dots, S' = S_m \in \mathcal{T}$ is such that $S_i \cap S_{i-1}$ is a face of both S_i and S_{i-1} for all $1 \leq i \leq m$. This just means that $\mathcal{T}$ is face connected. □

Remark 7.4. *Triangulations with the property that all vertex stars are homeomorphic to a ball are also called* combinatorial *[6, Section 1]. All manifolds of dimension up to three admit smooth structures and smooth manifolds admit combinatorial triangulations. There are triangulations of manifolds in dimension higher than* 3 *where not every vertex star is homeomorphic to a ball.*

Clearly, not every simplicial complex is the triangulation of some (embedded) topological manifold with or without boundary. When the dimension is at least five, then there is no computer algorithm that, given a finite simplicial complex as input, decides whether the input is the triangulation of some fixed manifold [16]. Going further, it has been shown that deciding whether a simplicial complex is the triangulation of a manifold cannot be decided by any computer algorithm [53]. We therefore are not in pursuit of any easy combinatorial property that indicates whether a simplicial complex (without any further specific assumptions) triangulates a manifold.

Conversely, not all topological manifolds, even if compact, can be described as a triangulation. Such manifolds appear in dimension four and higher: one example is the E8 manifold, which is a compact simply-connected topological 4*-manifold that is homeomorphic to the underlying set of any simplicial complex. The construction of such counterexamples typically involves an infinite procedure.*

MV: reference?

7.2 Shellable simplicial complexes

The notions of shelling and shellable triangulation have been discussed widely in combinatorial topology and polytopal theory. Formally, a triangulation is shellable if its full-dimensional simplices can be enumerated such that each simplex intersects the union of the previously listed simplices in a codimension one triangulation of a manifold. This forces the intermediate triangulations to be particularly well-shaped. We build upon the notion of shelling as introduced in [62, Definition 8.1], where our definition of shelling is equivalent to the notion of the shellings of simplicial complexes, see also [62, Remark 8.3].

MV: Shortened, narrowed, removed repetitions.

Suppose that $\mathcal{T}$ is an n-dimensional simplicial complex and we have an enumeration of the n-simplices $T_0, T_1, T_2, \dots \in \mathcal{S}_n^{\downarrow}(\mathcal{T})$. For any enumeration, we call

$$I_m := \big(T_0 \cup T_1 \cup \dots \cup T_m\big) \cap T_{m+1}$$

be the m-th *interface set*. We call the enumeration a *shelling* if each interface set I_m is a triangulated manifold of dimension $n-1$.

Remark 7.5. *We interpret a shelling as the construction of a triangulation by successively attaching simplices such that the intermediate triangulations are well-behaved. Conversely, the reverse enumeration describes a successive decomposition of the triangulation, hence the name "shelling".*

Remark 7.6. *Whether a simplicial complex is shellable can be checked, in principle, simply by trying out all the possible enumerations. That we cannot do much better than this is captured in the result that testing for shellability is NP-complete [31]: this complexity result is even true if we merely consider simplicial complexes of dimension two embedded in some Euclidean space.*

The reason of our interest in shellable simplicial complexes is that they can be constructed by a successive adhesion of simplices to intermediate complexes where these intermediate complexes are simplicial balls ~~or a simplicial spheres~~, as shows the following important consequence.

MV: OK? "Intermediate" excludes the last one.

Lemma 7.7. *Let $\mathcal{T}$ be an n-dimensional simplicial complex with a shelling $T_0, T_1, T_2, \dots, T_M$, such that each simplex of dimension $n-1$ is contained in at most two simplices. Then*

$$X_m := T_0 \cup T_1 \cup \cdots \cup T_m$$

is a triangulated manifold with boundary for all $0 \leq m \leq M$. In particular, X_m is a topological n-ball when $m < M$, and X_M is either a topological n-ball or topological n-sphere.

Proof. We prove this claim by induction. Certainly, if $\mathcal{T}$ contains only one single n-simplex, then it is a shellable triangulation of a topological n-ball. Suppose that $X_m := T_0 \cup T_1 \cup \cdots \cup T_m$ is a topological n-ball and that T_{m+1} is the next n-simplex in the shelling. By definition, $I_m := X_m \cap T_{m+1}$ is a submanifold of ∂T_{m+1}, and it is triangulated by some faces of T_{m+1} and their subsimplices.

Let F be such a face. By assumption, F must be contained in exactly one n-simplex of $T_0, T_1, \dots, T_m$, and F is in the boundary of X_m. We conclude that I_m triangulates a submanifold of the boundary of X_m.

On the one hand, if I_m is the entire boundary of T_{m+1}, then it must also be the boundary of the topological n-ball X_m. Hence $X_m \cup T_{m+1}$ must be a topological n-sphere and thus a manifold without boundary, which requires $m = M$. On the other hand, if I_m is a proper subset of the entire boundary of T_{m+1}, then $X_m \cup T_{m+1}$ is still a topological n-ball. ☐

We now collect important examples of shellable triangulations. Essentially, in two space dimensions, interesting triangulations shellable, but starting from three space dimensions, nonshellable situations can arise. Our main interest are local patches (stars) within triangulations: these are shellable up to three space dimensions, but not necessarily beyond.

Example 7.8. *Any simplex T (trivially) has a shelling. The boundary complex $\partial\mathcal{T}(T)$ has a shelling: in fact, any enumeration of the boundary faces of T constitutes a shelling, cf. [62, Example 8.2.(iii)].*

Example 7.9. *The standard triangulation of the 3-dimensional cube by six tetrahedra, the Kuhn triangulation [40], is shellable, as are its higher-dimensional generalizations.*[5]

Example 7.10. *There exists a non-shellable triangulation of a tetrahedron and of a cube in $n = 3$, see [62, Example 8.9].*

MV: 3D counterexamples directly here and straight to the point.

Lemma 7.11. *Any simplicial 2-ball is shellable. Any simplicial 2-sphere is shellable.*

Proof. First, let $\mathcal{S}$ be any triangulation of a 2-sphere. By removing any triangle $S \in \mathcal{S}$, we obtain a triangulation $\mathcal{T}$ of a 2-ball. Any shelling of that ball can be extended to a shelling of $\mathcal{S}$ by re-inserting the first triangle S. So it remains to show that any triangulation $\mathcal{T}$ of the two-dimensional ball is shellable. We will construct the shelling in reverse.

[5] We remark that Kuhn attributes this triangulation to Lefschetz [43].

Write $M = |\mathcal{T}|$. There is nothing to show if $\mathcal{T}$ contains only one triangle. We call a triangle $T \in \mathcal{T}$ *good in* $\mathcal{T}$ if it intersects the boundary ∂M in a non-empty union of edges. Hence a triangle is good in $\mathcal{T}$ if its intersection with ∂M is either one, two, or three edges, and a triangle is not good in $\mathcal{T}$ if that intersection is either empty, only some of its vertices, or a vertex and the opposite edge. We show by an induction argument over the number of triangles that every triangulation of a 2-ball that contains at least two triangles also contains at least two good triangles.

Clearly, this is the case if the triangulation of the 2-ball contains two triangles. Now suppose the induction claim is true when the triangulation includes at most N triangles, and assume that $\mathcal{T}$ includes $N+1$ triangles. There are at least two triangles with an edge on the boundary. Suppose that $\mathcal{T}$ does not have at least two triangles that are good in $\mathcal{T}$. Then there exists a triangle T' that intersects ∂M in one edge and its opposite vertex. Removing T' splits the manifold into two face-connected components, each of which is a topological 2-ball. By the induction assumption, each of those components contains at least two triangles that are good in the respective component. So each component has at least one triangle that is good in $\mathcal{T}$. Hence $\mathcal{T}$ contains two good triangles, which completes the induction step.

We conclude that whenever $\mathcal{T}$ triangulates a 2-ball, it contains a good triangle T. If T has three edges in the boundary, then $T = M$ and we are trivially done. If T intersects with the boundary in exactly one or two edges, then $\overline{M \setminus T}$ is still a topological 2-ball. The triangulation $\mathcal{T}'$ that is obtained by removing T is a triangulation of some 2-ball that intersects T only at either two or one edges. Any shelling of $\mathcal{T}'$ can in this way be extended to a shelling of $\mathcal{T}$, and the proof is complete. □

Lemma 7.12. *Let $\mathcal{T}$ be a n-dimensional shellable triangulation and $V \in \mathcal{V}(\mathcal{T})$ be a vertex. Then $\operatorname{st}_{\mathcal{T}}(V)$ is shellable.*

Proof. This is Lemma 8.7 in [62]. □

Lemma 7.13. *Let $\mathcal{T}$ be a 3-dimensional manifold triangulation and $S \in \mathcal{T}$. Then $\operatorname{st}_{\mathcal{T}}(S)$ is shellable.*

Proof. For S a tetrahedron, this is Example 7.8. The statement is also clear if S is an inner or boundary face of $\mathcal{T}$, where we only need to enumerate respectively one or two tetrahedra. The statement is still easily verified if S is an inner or boundary edge of $\mathcal{T}$: one chooses a starting tetrahedron (with a boundary face) and rotates around the edge in a fixed direction to create a suitable enumeration.

When S is an inner vertex, then the faces (triangles) of $\operatorname{st}_{\mathcal{T}}(S)$ that do not contain V constitute a simplicial 2-sphere. Similarly, when S is a boundary vertex, then the faces (triangles) of $\operatorname{st}_{\mathcal{T}}(S)$ that do not contain V constitute a simplicial 2-ball. Both these 2-dimensional complexes are shellable by Lemma 7.11, and any such shelling immediately yields a shelling of $\operatorname{st}_{\mathcal{T}}(S)$ since there is a one-to-one correspondance between the tetrahedra in $\mathcal{T}$ and the triangles.[6] □

MV: but not Lemma 7.12, because this first needs the triangulation $\mathcal{T}$ itself to be shellable, right?

Remark 7.14. *Not all triangulable sets admit a triangulation that is shellable. Moreover, even if a set admits a shellable triangulation, not all of its triangulations are be shellable. For example, if we extend the non-shellable triangulation of a tetrahedron from [62, Example 8.9] to a triangulation of a hypertetrahedron by suspending it from a new point $v_\star$, then the resulting new triangulation is non-shellable and coincides with the patch around $v_\star$. This demonstrates that patches around boundary simplices are not necessarily shellable when the space dimension n is larger than three.*

MV: Moved up a bit.

A major structural feature of shellable simplicial complexes is that each time an n-simplex is added, stars around lower-dimensional simplices gets completed.

Lemma 7.15. *Suppose that an n-dimensional manifold triangulation $\mathcal{T}$ has a shelling $T_0, T_1, T_2, \dots, T_M$. For $0 \leq m < M$, write*

$$X_m := T_0 \cup T_1 \cup \dots \cup T_m, \qquad I_m := X_m \cap T_{m+1}.$$

~~Whenever~~ Then I_m is a union of k different facets of T, $1 \leq k \leq n$, and the intersection of those facets is an interior simplex $S_m \in \mathcal{T}$ of dimension $n-k$ that satisfies

MV: this is a property, not an assumption, right?

$$\operatorname{st}_{X_{m+1}}(S_m) = \operatorname{st}_{\mathcal{T}}(S_m).$$

[6]For S an inner vertex, [21, Lemma B.1] also yields the claim.

Proof. We know I_m is a triangulated submanifold of the boundary of T, and so it must be a collection of k facets of T, $1 \leq k \leq n$ ($k = n+1$ can happen for the last enumerated simplex $m = M$ when $\mathcal{T}$ is an n-sphere). I_m also constitutes a local patch (star) of $(n-1)$-dimensional simplices around some simplex S_m of dimension $n-k$ in I_m. By definition, S_m is be a boundary simplex of X_m, and it is an interior simplex of X_{m+1}. But then S_m cannot be a subsimplex of any of the simplices $T_{m+1}, \dots, T_M$, which means that $\mathrm{st}_{X_{m+1}}(S_m) = \mathrm{st}_{\mathcal{T}}(S_m)$. □

8 Reflections and deformations on shellable stars

This section is devoted to geometric operations that are crucially important for our main result in Section 9 below. Suppose that we have a shellable simplicial complex. Every time we add a simplex according to the shelling sequence, we complete the local patch (star) of some simplex in the triangulation (see Lemma 7.15). We will need a homeomorphism going from the new simplex onto its complement within the newly completed star, such as the two- and three-color maps in [21, Sections 5.3 and 6.3] and the symmetrization maps in [14, Section 7.6]. This homeomorphism is required to preserve the interface and to be bi-Lipschitz. We are particularly interested in the norms of its Jacobian. We interpret this as a nonlinear reflection. It will be used subsequently in generalizing the discussion in Section 4 to the setting of differential forms.

In addition to that, this endeavor will produce another geometric tool that enables an additional estimate of Poincaré–Friedrichs constants: we construct a bi-Lipschitz deformation which maps the entire star into the complement of the newly completed star. Again, we give explicit estimates of the Jacobian.

We begin with the following observation. Suppose that T is any n-dimensional simplex with some proper subsimplex F. We find a hyperplane of codimension one that intersects T only at the F and such that T is on only one side of the hyperplane.

Lemma 8.1. *Let $v_0, v_1, \dots, v_n \in \mathbb{R}^n$ be the vertices of a simplex T. Let F, F' be the two subsimplices of T with respective vertices $v_0, \dots, v_k$ and $v_{k+1}, \dots, v_n$. Then T lies on one side of the affine subspace*

$$A := v_0 + \mathrm{lin}\{v_1 - v_0, \dots, v_k - v_0\} + \mathrm{lin}\{v_{k+1} - v_n, \dots, v_{n-1} - v_n\}.$$

Moreover, $F = T \cap A$.

Proof. Evidently, the linear spans

$$A_0 := \mathrm{lin}\{v_1 - v_0, \dots, v_k - v_0\}, \qquad A_1 := \mathrm{lin}\{v_{k+1} - v_n, \dots, v_{n-1} - v_n\}$$

intersect only at the origin, since otherwise the vertices of T are not affinely independent. So A has dimension $n-1$. Obviously, $F \subset A \cap T$. Let us now suppose that $x \in A \cap T$. For some real coefficients $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{R}$, it holds that

$$\begin{aligned} x &= v_0 + \sum_{i=1}^{k} \lambda_i (v_i - v_0) + \sum_{i=k+1}^{n-1} \lambda_i (v_n - v_i) \\ &= \left(1 - \sum_{i=1}^{k} \lambda_i \right) v_0 + \sum_{i=1}^{k} \lambda_i v_i + \left(\sum_{i=k+1}^{n-1} \lambda_i \right) v_n + \sum_{i=k+1}^{n-1} (-\lambda_i) v_i. \end{aligned}$$

We know that this is the unique expression of x as non-negative linear combination of the vertices of T. Hence the $\lambda_{k+1}, \dots, \lambda_{n-1}$ are non-positive. But if $x \notin F$, then at least one of them must be negative, and thus their sum must be negative, which is a contradiction. Hence the $\lambda_{k+1}, \dots, \lambda_{n-1}$ are zero, which implies $x \in F$. The desired statement follows. □

Next, we observe that stars around subsimplices can by represented by vertex stars after a local subdivision of the mesh. The shape regularity is controlled in that process.

Lemma 8.2. *Let T be an n-dimensional simplex with a $(n-1)$-dimensional subsimplex S, let z_S be the barycenter of S, and let T' be one of the n-dimensional simplices obtained by splitting T at z_S. Then* $\mathrm{vol}(T') = \mathrm{vol}(T)/n$.

Proof. The proof is obvious. □

MV: Some illustrations would be great.

We now construct the main result of this section, namely the desired bi-Lipschitz homeomorphism. It can be imagined as the reflection across the interface between the selected simplex and the remainder of the star. The construction is completed in several steps, and the Lipschitz estimate reflects the shape regularity.

MV: Changed to "Proposition". Only the main results are termed "Theorem".

Proposition 8.3. *Let $\mathcal{T}$ be an n-dimensional triangulation of an n-dimensional domain. Let $S \in \mathcal{T}$ be a simplex of dimension $k < n$, let $T \in \mathrm{st}_{\mathcal{T}}(S)$ be of dimension n, and let*

$$U := \overline{|\,\mathrm{st}_{\mathcal{T}}(S)| \setminus T}, \qquad \Gamma_1 := U \cap T, \qquad \Gamma_2 := \overline{\partial T \setminus \partial U}.$$

1. *There exists a bi-Lipschitz mapping*

$$\Xi_1 : T \to U$$

which is the identity along $\Gamma_1 = T \cap U$ and which satisfies the upper bounds

$$\|\operatorname{Jac}\Xi_1\|_2 \leq C_{5,n,k}(\mathcal{T}), \quad \|\operatorname{Jac}\Xi_1^{-1}\|_2 \leq C_{6,n,k}(\mathcal{T}),$$
$$\|\det\operatorname{Jac}\Xi_1\|_2 \leq C^{\det}_{5,n,k}(\mathcal{T}), \quad \|\det\operatorname{Jac}\Xi_1^{-1}\|_2 \leq C^{\det}_{6,n,k}(\mathcal{T}),$$

where the constants on the right-hand sides are as stated in the proof.

2. *There exists a bi-Lipschitz mapping*

$$\Xi_2 : |\mathrm{st}_{\mathcal{T}}(S)| \to U$$

which is the identity along $\partial U \setminus \partial T$ and which satisfies the upper bounds

$$\|\operatorname{Jac}\Xi_2\|_2 \leq C_{7,n,k}(\mathcal{T}), \quad \|\operatorname{Jac}\Xi_2^{-1}\|_2 \leq C_{8,n,k}(\mathcal{T}),$$
$$\|\det\operatorname{Jac}\Xi_2\|_2 \leq C^{\det}_{7,n,k}(\mathcal{T}), \quad \|\det\operatorname{Jac}\Xi_2^{-1}\|_2 \leq C^{\det}_{8,n,k}(\mathcal{T}),$$

where the constants on the right-hand sides are as stated in the proof.

Proof. We derive the estimate in several steps. In what follows, we use the notation $\hat{z}$ for the normalization of any vector $z \in \mathbb{R}^n$.

- Without loss of generality, the barycenter z_S of S is the origin. We fix the subsimplex $S' \subseteq T$ that is complementary to S. Note that Γ_2 is the union of exactly those faces of T that contain S', whereas $\Gamma_1 = U \cap T$ is the union of exactly those faces of T that contain S.

- We let z'_S be the midpoint of S'. We first define a new triangulation $\mathcal{S}$ by the barycentric refinement of the complementary simplex S'. All n-simplices in $\mathcal{S}$ contain S and z'_S.

  We define another simplicial complex $\mathcal{S}^c$ as follows: whenever $K \in \mathcal{S}$ is an n-simplex, we replace the vertex z'_S by the new vertex $-z'_S$ at the opposite position, giving us a simplex K'.

  We let $\mathcal{S}'$ be the union of $\mathcal{S}$ and $\mathcal{S}^c$. By construction, all these n-simplices contain S as a subsimplex, and they are disjoint except for their proper subsimplices. Hence $\mathcal{S}'$ is indeed a triangulation. Moreover, S is an inner subsimplex of that triangulation. In particular, $\mathcal{S}'$ is its own star around S.

- In what follows, we also use $\mathcal{R}'$, the simplicial complex obtained from $\mathcal{S}'$ by barycentric subdivision of S. Then $\mathcal{R}'$ is the star around the barycenter z_S, and all n-simplices in $\mathcal{R}'$ contain z_S. We let $\mathcal{R}$ and $\mathcal{R}^c$ be the corresponding refinements of $\mathcal{S}$ and $\mathcal{S}^c$, respectively.

  The last simplicial complex that we introduce is called $\mathcal{K}$, and it is the simplicial complex obtained from $\mathrm{st}_{\mathcal{T}}(S)$ via barycentric refinement of S.

- We introduce a new mapping $\Theta : |\mathcal{R}| \to |\mathcal{R}^c|$ as follows. Let $K \in \mathcal{R}$ be an n-simplex and $K' \in \mathcal{R}^c$ be constructed from K. We let $A_K : \Delta^n \to K$ and $A_{K'} : \Delta^n \to K'$ be affine reference transformations that agree on the vertices common to K and K'. We then define

$$\Theta_{|K} := A_{K'} \circ A_K^{-1}.$$

  Notice that $A_{K'}$ and A_K have the same singular values. Thus

$$\|\operatorname{Jac}\Theta_{|K}\|_2 \leq \|A_{K'}\|_2 \|A_K^{-1}\|_2 \leq \kappa(A_K),$$
$$\|\operatorname{Jac}\Theta_{|K}^{-1}\|_2 \leq \|A_{K'}^{-1}\|_2 \|A_K\|_2 \leq \kappa(A_K).$$

  By construction, the mapping preserves volumes:

$$|\det(\operatorname{Jac}\Theta_{|K})| = |\det(\operatorname{Jac}^{-1}\Theta_{|K})| = 1.$$

  We want to characterize the singular values of this transform. We $h_z \in \mathbb{R}^n$ be the height vector of $z_{S'}$ inside the simplex K. We can write

$$\Theta_{|K}(x) = x - 2\frac{\langle \hat{h}_z, x\rangle}{\langle \hat{h}_z, \hat{z}_{S'}\rangle}\hat{z}_{S'}.$$

  Indeed, we check that the right-hand side equals x whenever x lies in the plane orthogonal to h_z, and that it equals $-z_{S'}$ when $x = z_{S'}$. If we orthogonally decompose $z_{S'} = h_z + b_z$, then

$$\Theta_{|K}(h_z) = h_z - 2\frac{\langle h_z, h_z\rangle}{\langle h_z, z_{S'}\rangle}z_{S'} = h_z - 2\frac{\langle h_z, z_{S'}\rangle}{\langle h_z, z_{S'}\rangle}z_{S'} = h_z - 2z_{S'} = -h_z - 2b_z.$$

  Let β be the angle between $z_{S'}$ and h_z. Then $\|h_z\|_2 = \cos(\beta)\|z_{S'}\|_2$ and $\|b_z\|_2 = \sin(\beta)\|z_{S'}\|_2$. We study the singular values of the matrix

$$B := \begin{pmatrix} -1 & 0 \\ -2\|b_z\|_2/\|h_z\|_2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2\tan(\beta) & 1 \end{pmatrix}.$$

  We compute the eigenvalues of the symmetric matrix B^*B. Its square roots are the desired singular values of B:

$$\sigma_{\max} = \sqrt{2\tan(\beta)^2 + 1 + 2\tan(\beta)\sqrt{\tan(\beta)^2 + 1}} = \sqrt{1 + \tan^2(\beta)} + \tan(\beta),$$
$$\sigma_{\min} = \sqrt{2\tan(\beta)^2 + 1 - 2\tan(\beta)\sqrt{\tan(\beta)^2 + 1}} = \sqrt{1 + \tan^2(\beta)} - \tan(\beta).$$

  They are monotonously increasing and decreasing, respectively, in $\tan(\beta)$. These are also the maximal and minimal singular values of $\operatorname{Jac}\Theta_{|K}$, whereas all its other singular values equal 1.

  By the definition of the tangent, $\tan(\beta) \leq \|b_z\|_2/\|h_z\|_2$, and we know $\|b_z\|_2 \leq \|z_{S'}\|_2$. Let $F_z \subseteq K$ be the face opposite to $z_{S'}$, which is also a face of T. Then $(n-k)h_z$ is the height vector of F_z in T. Thus

$$\frac{\|b_z\|_2}{\|h_z\|_2} = \frac{\|b_z\|_2}{(n-k)^{-1}\|(n-k)h_z\|_2} \leq (n-k)\kappa_{\mathrm{A}}(T).$$

  This establishes bounds on the singular values of the transformation. We write

$$B_T := \sqrt{1 + (n-k)^2\kappa_{\mathrm{A}}(T)^2} + (n-k)\kappa_{\mathrm{A}}(T).$$

- We introduce another mapping $\Phi : |\mathcal{R}'| \to |\mathcal{R}^c|$ as follows. Consider any n-simplex $K \in \mathcal{R}$ and let $K^c \in \mathcal{R}^c$ be its image under Φ. We construct a bi-Lipschitz mapping

$$\Phi_K : K \cup K^c \to K^c$$

and analyze its properties. The construction will be such that the union of Φ_K for all n-simplices $K \in \mathcal{R}$ will define the required mapping $\Phi : |\mathcal{R}'| \to |\mathcal{R}^c|$, which will be bi-Lipschitz and the identity along $\partial\,|\mathcal{R}'| \setminus \partial\,|\mathcal{R}|$.

We write h_z for the height of $z_{S'}$ within K. Given $x \in K \cup K^c$, we have a unique representation

$$x = t z_S + x^{\perp}, \quad t \in [-1, 1], \quad x^{\perp} \in K \cap K^c.$$

Notice that

$$t = \frac{\langle h_z, x\rangle}{\langle h_z, z_{S'}\rangle}.$$

Based on that observation, we define

$$\Phi_K(x) := x - \frac{1}{2}\left(\frac{\langle h_z, x\rangle}{\langle h_z, z_{S'}\rangle} - 1 \right) z_{S'}.$$

We readily verify that this transformation is a bi-Lipschitz mapping from $K \cup K^c$ onto K^c that satisfies the desired mapping properties. It remains to analyze its Jacobian to get explicit estimates.

We have an orthogonal decomposition $h_z + b_z = z_{S'}$. With that,

$$\begin{aligned}
\operatorname{Jac}\Phi_K(x) &= \mathrm{Id} - \frac{1}{2\langle h_z, z_{S'}\rangle} z_{S'} \otimes h_z^t \\
&= \mathrm{Id} - \frac{1}{2\langle h_z, z_{S'}\rangle} h_z \otimes h_z^t - \frac{1}{2\langle h_z, z_{S'}\rangle} b_z \otimes h_z^t \\
&= \mathrm{Id} - \frac{1}{2\langle h_z, h_z\rangle} h_z \otimes h_z^t - \frac{1}{2\langle h_z, z_{S'}\rangle} b_z \otimes h_z^t \\
&= \mathrm{Id} - \frac{1}{2}\hat{h}_z \otimes \hat{h}_z^t - \frac{\|b_z\|_2}{2\|h_z\|_2}\hat{b}_z \otimes \hat{h}_z^t.
\end{aligned}$$

We write β for the angle between h_z and $z_{S'}$. Hence $\tan(\beta) = \|b_z\|_2/\|h_z\|_2$. It remains to study the singular values of the matrix

$$C = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ -\tan(\beta) & 2 \end{pmatrix}.$$

By building the symmetric matrix $C^t C$ and computing its eigenvalues, we obtain

$$\begin{aligned}
\sigma_{\max} &= \sqrt{\frac{\tan(\beta)^2 + 5}{8} + \frac{1}{8}\sqrt{\tan(\beta)^4 + 10\tan(\beta)^2 + 9}} = \frac{1}{4}\sqrt{9 + \tan(\beta)^2} + \frac{1}{4}\sqrt{1 + \tan(\beta)^2}, \\
\sigma_{\min} &= \sqrt{\frac{\tan(\beta)^2 + 5}{8} - \frac{1}{8}\sqrt{\tan(\beta)^4 + 10\tan(\beta)^2 + 9}} = \frac{1}{4}\sqrt{9 + \tan(\beta)^2} - \frac{1}{4}\sqrt{1 + \tan(\beta)^2}.
\end{aligned}$$

These are also the maximal and minimal eigenvalues of the Jacobian $\operatorname{Jac}\Phi_K$, the remaining eigenvalues being equal to 1. These are monotonously increasing and decreasing, respectively, in $\tan(\beta)$. Moreover, $\sigma_{\max}\sigma_{\min} = 1/2$. We now recall that the height of h_z in $K \in \mathcal{R}$ equals $(k+1)^{-1}$ multiplied with the height of some vertex of S within T. Similar as above, we use the upper bound

$$\tan(\beta) = \frac{\|b_z\|_2}{\|h_z\|_2} \leq (k+1)\kappa_{\mathrm{A}}(T).$$

We conclude that

$$
\begin{gathered}
\| \operatorname{Jac} \Phi_K \|_2 \leq \frac{1}{4} \sqrt{9 + (k+1)^2 \kappa_{\mathrm{A}}(T)^2} + \frac{1}{4} \sqrt{1 + (k+1)^2 \kappa_{\mathrm{A}}(T)^2}, \\
\| \operatorname{Jac} \Phi_K^{-1} \|_2 \leq \frac{1}{2} \sqrt{9 + (k+1)^2 \kappa_{\mathrm{A}}(T)^2} + \frac{1}{2} \sqrt{1 + (k+1)^2 \kappa_{\mathrm{A}}(T)^2}, \\
\det \operatorname{Jac} \Phi = \frac{1}{2}.
\end{gathered}
$$

This finishes the discussion of the transformation Φ.

- We prove another auxiliary result. Let $x, h_1, h_2 \in \mathbb{R}^n$ be non-zero. Define

$$
F(x) = \frac{\|h_2\|_2^2}{\langle h_2, x \rangle} \frac{\langle h_1, x \rangle}{\|h_1\|_2^2} x.
$$

This is defined away from the hyperplane orthogonal to h_2. This is a radial mapping that maps each point on the hyperplane defined by the normal vector h_1 onto the colinear point that lies on the hyperplane defined by the normal vector h_2. Indeed, if $x \in \mathbb{R}^n$ with $\langle h_1, x \rangle = \|h_1\|_2^2$, then $\langle F(x), h_2 \rangle = \|h_2\|_2^2$.

We identify the Lipschitz properties of this mapping by computing its Jacobian. We write $\alpha_1, \alpha_2 \geq 0$ for the two angles between h_1 and h_2, respectively, and the vector x. In what follows, $y \in \mathbb{R}^n$.

$$
\begin{aligned}
\operatorname{Jac} F(x) y &= \frac{\langle h_2, h_2 \rangle}{\|h_2\|_2 \|x\|_2 \cos(\alpha_2)} \left(y - \frac{\langle h_2, y \rangle}{\langle h_2, x \rangle} x \right) \frac{\langle x, h_1 \rangle}{\langle h_1, h_1 \rangle} + \frac{\|h_2\|_2^2}{\langle h_2, x \rangle} \frac{\langle h_1, y \rangle}{\|h_1\|_2^2} x \\
&= \frac{\langle \hat{h}_2, h_2 \rangle}{\cos(\alpha_2)} \left(y - \frac{\langle \hat{h}_2, y \rangle}{\langle \hat{h}_2, \hat{x} \rangle} \hat{x} \right) \frac{\langle \hat{x}, \hat{h}_1 \rangle}{\langle h_1, \hat{h}_1 \rangle} + \frac{\|h_2\|_2}{\langle \hat{h}_2, \hat{x} \rangle} \frac{\langle \hat{h}_1, y \rangle}{\|h_1\|_2} \hat{x} \\
&= \frac{\|h_2\|_2}{\cos(\alpha_2)} \left(y - \frac{\langle \hat{h}_2, y \rangle}{\cos(\alpha_2)} \hat{x} \right) \frac{\cos(\alpha_1)}{\|h_1\|_2} + \frac{\|h_2\|_2}{\langle \hat{h}_2, \hat{x} \rangle} \frac{\langle \hat{h}_1, y \rangle}{\|h_1\|_2} \hat{x} \\
&= \frac{\|h_2\|_2}{\|h_1\|_2} \frac{\cos(\alpha_1)}{\cos(\alpha_2)} \left(y - \frac{\langle \hat{h}_2, y \rangle}{\cos(\alpha_2)} \hat{x} + \frac{\langle \hat{h}_1, y \rangle}{\cos(\alpha_1)} \hat{x} \right).
\end{aligned}
$$

We define $a := \cos(\alpha_1)^{-1} \hat{h}_1 - \cos(\alpha_2)^{-1} \hat{h}_2$. Notice that $a \perp \hat{x}$ by definition. The matrix $\mathrm{Id} + \hat{x} \otimes a^t$ is the identity over the orthogonal complement of the linear span of a and x. Over that two-dimensional subspace, it can be represented by a triangular matrix. Similar as above, its largest and the smallest singular values $\sigma_{max} \geq 1 \geq \sigma_{\min}$ are

$$
\sigma_{\max} = \sqrt{1 + \frac{\|a\|_2^2}{4}} + \frac{\|a\|_2}{2}, \qquad \sigma_{\min} = \sqrt{1 + \frac{\|a\|_2^2}{4}} - \frac{\|a\|_2}{2}.
$$

Notice that these are strictly monotonously increasing or decreasing, respectively, in $\|a\|_2$. We notice the obvious upper bound

$$
\|a\|_2 \leq 2 \max_{1 \leq i \leq 2} \left(\frac{\hat{h}_i}{\langle \hat{x}, \hat{h}_i \rangle} \right).
$$

- We introduce a bi-Lipschitz mapping

$$
\Psi : |\mathcal{R}'| \to |\mathcal{K}|
$$

in the following manner. Let $G \in \mathcal{R}'$ be an n-dimensional simplex and let $x \in G$ be non-zero. There exists a simplex $K \in \mathcal{K}$ that intersects the ray $\mathbb{R}_0^+ \cdot x$. We let h_G and h_K be the normals of the origin within

G and K, respectively. They define respective hyperplanes H_G and H_K, and we write $F_G = G \cap H_G$ and $F_K = K \cap H_K$. We notice that x has a sharp angle to both h_G and h_K. We define

$$\Psi(x) = \frac{\|h_K\|_2^2}{\langle h_K, x\rangle} \frac{\langle h_G, x\rangle}{\|h_G\|_2^2} x.$$

One easily verifies that this defines a continuous mapping Ψ. By the same line of thought, we see that it is invertible with inverse satisfying

$$\Psi^{-1}(z) = \frac{\|h_G\|_2^2}{\langle h_G, z\rangle} \frac{\langle h_K, z\rangle}{\|h_K\|_2^2} z, \qquad z = \Psi(x).$$

Our previous observations enable estimates for the Jacobians. We have the operator norm and determinant formulas

$$\|\operatorname{Jac}\Psi(x)\|_2 = \frac{\|h_K\|_2}{\|h_G\|_2} \frac{\cos(\alpha_G)}{\cos(\alpha_K)} \left(\sqrt{1 + \frac{\|a\|_2^2}{4}} + \frac{\|a\|_2}{2} \right),$$

$$\|\operatorname{Jac}\Psi(z)^{-1}\|_2 = \frac{\|h_G\|_2}{\|h_K\|_2} \frac{\cos(\alpha_K)}{\cos(\alpha_G)} \left(\sqrt{1 + \frac{\|a\|_2^2}{4}} - \frac{\|a\|_2}{2} \right)^{-1},$$

$$\det \operatorname{Jac}\Psi(x) = \left(\frac{\|h_K\|_2}{\|h_G\|_2} \frac{\cos(\alpha_G)}{\cos(\alpha_K)} \right)^n, \qquad \det \operatorname{Jac}\Psi^{-1}(x) = \left(\frac{\|h_G\|_2}{\|h_K\|_2} \frac{\cos(\alpha_K)}{\cos(\alpha_G)} \right)^n,$$

where α_K and α_G are the angles of x with the vectors h_G and h_K, respectively, and where

$$a := \cos(\alpha_G)^{-1} \hat{h}_G - \cos(\alpha_K)^{-1} \hat{h}_K.$$

We let $x_G \in F_G$ and $x_K \in F_K$ be the intersection points of the ray $\mathbb{R} \cdot x$ with the respective faces. By the definition of the cosine,

$$\frac{\|h_K\|_2}{\|h_G\|_2} \frac{\cos(\alpha_G)}{\cos(\alpha_K)} \leq \frac{\|x_K\|_2}{\|h_G\|_2} \leq \frac{\delta(K)}{\|h_G\|_2}, \qquad \frac{\|h_G\|_2}{\|h_K\|_2} \frac{\cos(\alpha_K)}{\cos(\alpha_G)} \leq \frac{\|x_G\|_2}{\|h_K\|_2} \leq \frac{\delta(G)}{\|h_K\|_2}.$$

The diameters of the simplices in $\mathcal{K}$ are already bounded by the maximum diameter in $\operatorname{st}_{\mathcal{T}}(S)$. The height h_K is obtained from a height vector of a vertex in $\operatorname{st}_{\mathcal{T}}(S)$ by scaling with the factor $(k+1)^{-1}$. Every height of the origin in some simplex of $\mathcal{R}'$ is obtained from a height vector of the origin in some simplex in $\mathcal{S}'$ by scaling with $(k+1)^{-1}$. Each of the latter height vectors has length bounded from below by the height of some vertex in S within a simplex of $\mathcal{S}$, multiplied by B_T. The height of any vertex of S in some simplex of $\mathcal{S}$ is the same as the height h_T of that vertex in the original simplex T. In order to estimate $\|x_G\|_2$, recall that $x = \lambda(-z_{S'}) + (1-\lambda) z_x$ for some $\lambda \in [0,1]$ and $z_x \in G$ being in the face of G that is opposite to z_x. The preimage of x under Θ_G is $\lambda z_{S'} + (1-\lambda) z_x$. It follows that $\|x_G\|_2 \leq \delta(T)$.

We know that whenever T is a simplex with some face F and h_F is corresponding height vector, then

$$\frac{\delta(T)}{\|h_F\|_2} \leq \kappa_{\mathrm{A}}(T).$$

We conclude that

$$\frac{\delta(K)}{\|h_G\|_2} \leq (k+1) B_T \frac{\delta(K)}{\|h_T\|_2} \leq (k+1) B_T \theta(\mathcal{T}) \frac{\delta(T)}{\|h_T\|_2} \leq (k+1) B_T \theta(\mathcal{T}) \kappa_{\mathrm{A}}(T),$$

$$\frac{\|x_G\|_2}{\|h_K\|_2} \leq (k+1) \sup_{K \in \operatorname{st}_{\mathcal{T}}(S)} \frac{\delta(T)}{\|h_K\|_2} \leq (k+1)\theta(\mathcal{T}) \sup_{K \in \operatorname{st}_{\mathcal{T}}(S)} \frac{\delta(K)}{\|h_K\|_2} \leq (k+1)\theta(\mathcal{T}) \sup_{K \in \operatorname{st}_{\mathcal{T}}(S)} \kappa_{\mathrm{A}}(K).$$

$$\frac{\delta(K)}{\|h_K\|_2} \leq (k+1) \sup_{K \in \mathrm{st}_{\mathcal{T}}(S)} \kappa_{\mathrm{A}}(K), \qquad \frac{\|x_G\|_2}{\|h_G\|_2} \leq (k+1) B_T \kappa_{\mathrm{A}}(G).$$

The vector a is the difference between $\hat{h}_G$ and $\hat{h}_K$ after both are rescaled such that they lie on the affine hyperplane with normal vector $\hat{x}$. That difference has length at most $\tan(\alpha_G) + \tan(\alpha_K)$. We calculate

$$\begin{aligned} \|a\|_2 = \left\| \frac{\|x_G\|_2}{\|h_G\|_2} \hat{h}_G - \frac{\|x_K\|_2}{\|h_K\|_2} \hat{h}_K \right\|_2 &\leq \frac{\|x_G\|_2}{\|h_G\|_2} + \frac{\|x_K\|_2}{\|h_K\|_2} \\ &\leq (k+1) \left(1 + B_T\right) \sup_{K \in \mathrm{st}_{\mathcal{T}}(S)} \kappa_{\mathrm{A}}(K). \end{aligned}$$

We have shown that Ψ and Ψ^{-1} are piecewise Lipschitz with respect to some essentially disjoint closed partitions over their respective domains. It remains to show both are Lipschitz, which we illustrate for Ψ, the argument for Ψ^{-1} being analogous. Rademacher's theorem then shows that both have essentially bounded weak Jacobians.

We show that every function that is piecewise Lipschitz with respect to the triangulation $\mathcal{K}$ must be Lipschitz over all of $|\mathcal{K}|$. We use a technique similar to Lemma 7.12. After application of a piecewise linear Lipschitz mapping, we can assume that $|\mathcal{K}|$ is convex without loss of generality. Any function piecewise Lipschitz with respect to $\mathcal{K}$ must be Lipschitz over $|\mathcal{K}|$.

MV: ???

MWL: not enough

- We define a mapping $\Xi_1 : T \to U = \overline{\mathrm{st}_{\mathcal{T}}(S) \setminus T}$ by setting

  $$\Xi_1 := \Psi \circ \Theta.$$

  It preserves the interface from T onto U. The combination of the previous estimates for Ψ and Θ, as well as the upper bound $\sqrt{s^2+t^2} \leq s + t$ for any $s, t \geq 0$, provide

  $$\begin{aligned} \|\operatorname{Jac}\Xi_1\|_2 &\leq C_{5,n,k}(\mathcal{T}) := (k+1)\theta(\mathcal{T})\kappa_{\mathrm{A}}(\mathcal{T}) B_T^2 \Big(1 + (k+1)\left(1+B_T\right)\kappa_{\mathrm{A}}(\mathcal{T})\Big), \\ \|\operatorname{Jac}\Xi_1^{-1}\|_2 &\leq C_{6,n,k}(\mathcal{T}) := (k+1)\theta(\mathcal{T})\kappa_{\mathrm{A}}(\mathcal{T}) B_T \Big(1 + (k+1)\left(1+B_T\right)\kappa_{\mathrm{A}}(\mathcal{T})\Big). \end{aligned}$$

  We get the determinant estimates

  $$\begin{aligned} |\det(\Xi_1)| &\leq C^{\det}_{5,n,k}(\mathcal{T}) := (k+1)^n \theta(\mathcal{T})^n B_T^n \kappa_{\mathrm{A}}(\mathcal{T})^n, \\ |\det(\Xi_1^{-1})| &\leq C^{\det}_{6,n,k}(\mathcal{T}) := (k+1)^n \theta(\mathcal{T})^n \kappa_{\mathrm{A}}(\mathcal{T})^n. \end{aligned}$$

  This completes the desired estimates for Ξ_1.

- We define another mapping $\Xi_2 : |\mathrm{st}_{\mathcal{T}}(S)| \to U$ by setting

  $$\Xi_2 := \Psi \circ \Phi.$$

  It is bi-Lipschitz and preserves $\partial\,|\mathrm{st}_{\mathcal{T}}(S)| \setminus \partial T$. The combination of the previous estimates for Ψ and Φ now lead to

  $$\begin{aligned} \|\operatorname{Jac}\Xi_2\|_2 &\leq C_{7,n,k}(\mathcal{T}) := \frac{1}{4}\left(4 + (k+1)\kappa_{\mathrm{A}}(T)\right)(k+1)\kappa_{\mathrm{A}}(\mathcal{T}) B_T \Big(1 + (k+1)\left(1+B_T\right)\kappa_{\mathrm{A}}(\mathcal{T})\Big), \\ \|\operatorname{Jac}\Xi_2^{-1}\|_2 &\leq C_{8,n,k}(\mathcal{T}) := \frac{1}{2}\left(4 + (k+1)\kappa_{\mathrm{A}}(T)\right)(k+1)\kappa_{\mathrm{A}}(\mathcal{T}) \Big(1 + (k+1)\left(1+B_T\right)\kappa_{\mathrm{A}}(\mathcal{T})\Big). \end{aligned}$$

  We get the determinant estimate

  $$\begin{aligned} |\det(\Xi_2)| &\leq C^{\det}_{7,n,k}(\mathcal{T}) := \frac{1}{2}(k+1)^n \theta(\mathcal{T})^n B_T^n \kappa_{\mathrm{A}}(\mathcal{T})^n, \\ |\det(\Xi_2^{-1})| &\leq C^{\det}_{8,n,k}(\mathcal{T}) := 2(k+1)^n \theta(\mathcal{T})^n \kappa_{\mathrm{A}}(\mathcal{T})^n. \end{aligned}$$

  This completes the desired estimates for Ξ_2.

The proof is complete. $\square$

Remark 8.4. *We notice that the mappings Ξ_1 and Ξ_2 above are not only bi-Lipschitz, but in fact also piecewise affine, where piecewise refers to an essentially non-overlapping decomposition of their respective domains into convex polytopes. While the reflection and deformation mappings in Proposition 8.3 serve our purpose, it might be possible to improve the analysis or construction and lower the Jacobian estimates.*

9 Combinatorial Poincaré–Friedrichs constants over triangulated domains: the exterior derivative case

We are now in the position to estimate Poincaré–Friedrichs constants for the exterior derivative over domains with shellable triangulations; this in particular includes the curl and divergence operators in three space dimensions. The estimates require local Poincaré–Friedrichs constants over simplices, with or without boundary conditions.

9.1 Poincaré–Friedrichs constants over an n-simplex

The analysis of the Poincaré potential operators in Section 6 implies the following estimate for Poincaré–Friedrichs constants over convex domains.

Lemma 9.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a convex bounded open set. Let $0 \leq k < n$. Then for each $u \in W^p\Lambda^k(\Omega)$ there exists $w \in W^p\Lambda^k(\Omega)$ with $dw = du$ and*

MV: We need weakly Lipschitz, right?

$$\|w\|_{L^p(\Omega)} \leq \underbrace{n A_{\mathfrak{P}}(n, k+1) \operatorname{vol}_n(B_1(0)) \frac{\delta(\Omega)^n}{\operatorname{vol}(\Omega)} \delta(\Omega)}_{=:C_{\mathrm{PF},k,\Omega,p}} \|du\|_{L^p(\Omega)}.$$

Proof. This follows from Theorem 6.1 using $\operatorname{vol}(B_{\delta(\Omega)}(0)) = \operatorname{vol}_n(B_1(0))\delta(\Omega)^n$. $\square$

These constants are generally not optimal. For example, when $p = 2$ and when only the divergence is considered, we have the following improved estimate.

Lemma 9.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then for each $\boldsymbol{u} \in \boldsymbol{H}(\operatorname{div}, \Omega)$ there exists $\boldsymbol{w} \in \boldsymbol{H}(\operatorname{div}, \Omega)$ with $\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{w}$ and*

MV: We need weakly Lipschitz, right?

$$\|\boldsymbol{w}\|_{L^2(\Omega)} \leq \delta(\Omega) \|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega)}.$$

Proof. This is a reduction to the Friedrichs inequality. Let $W^{1,2}_0(\Omega)$ be the closure of the smooth functions with support in Ω in the Hilbert space $W^{1,2}(\Omega)$. Then $\nabla : W^{1,2}_0(\Omega) \subseteq L^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ is a closed densely-defined linear operator with smallest singular value bounded from below by $\delta(\Omega)^{-1}$, according to the Friedrichs inequality. The adjoint is the closed densely-defined linear operator $\boldsymbol{W}^2(\operatorname{div}, \Omega) \subseteq \boldsymbol{L}^2(\Omega) \to L^2(\Omega)$, which has the same smallest singular value. $\square$

9.2 Poincaré–Friedrichs constants over an n-simplex with homogeneous Dirichlet condition on some faces

We also need a Poincaré–Friedrichs inequality for differential forms over a simplex but subject to homogeneous boundary conditions along a collection of faces.

Lemma 9.3. *Let T be an n-simplex and let $F_0, \dots, F_l$ be $l+1$ different faces of T. Suppose that $u \in W^p\Lambda^k(T)$ such that $\operatorname{tr}_{F_i} u = 0$ for $0 \leq i \leq l$. Then there exists $w \in W^p\Lambda^k(T)$ such that $\operatorname{tr}_{F_i} w = 0$ for $0 \leq i \leq l$ and*

$$dw = du, \quad \|w\|_{L^p(\Omega)} \leq C_{\mathrm{PF},\Gamma,l,k,p}(T)\delta(T)^k \|du\|_{L^p(\Omega)}.$$

MV: There is indeed $\delta(T)^k$, with the power k? Nothing like this appears in Lemma 9.1. Moreover, so far, all the constants $C_{\mathrm{PF},T,p}$, $C_{\mathrm{PF},T_1\cup T_2,p}$ in Lemma 4.1, $C_{\mathrm{PF},T,F,p}$ in Lemma 4.4, or $C_{\mathrm{PF},k,\Omega,p}$ in Lemma 9.1 had the diameter $\delta(T)$ inside, only the constants in Section 2.3 have it outside. Finally, Ω in the indices is missing, whereas l is superfluous. We should unify.

Here, $C_{\mathrm{PF},\Gamma,l,k,p}(T) > 0$ is a constant such that

$$C_{\mathrm{PF},\Gamma,l,k,p}(T) \leq 2^{\frac{n}{p}} 2^{n+1} n A_{\mathfrak{B}}(n,k+1) \frac{\mathrm{vol}_n(B_1(0))}{(l+1)/n!} \kappa_2(T)^{k-1} C_{1,n}\cancel{(T)}.$$

MV: ???

Proof. There exists $\varphi : \hat{T} \to T$ mapping the convex closure of the n unit vectors onto the face F_0. We can also assume that the face of $\hat{T}$ orthogonal to the i-th coordinate axis is mapped onto the face F_i. In what follows, we let $\tilde{U}$ be the domain obtained from reflecting $\hat{T}$ along the coordinate axes $l+1$ through n. We see that $\mathrm{vol}(\tilde{U}) = (l+1)/n!$

We let $\hat{u} := \varphi^* u$ and define $\hat{g} \in L^p(\hat{T})$ via $\hat{g} := d\varphi^* u$. Then $d\hat{u} = \hat{g}$. We let $\tilde{u}$ be the extension of $\hat{u}$ onto $\tilde{U}$ by reflection across the coordinate axes. We define $\tilde{g} = d\tilde{u}$ and see that $\tilde{g}$ is the extension of $\hat{g}$ onto $\tilde{U}$ by reflection across the coordinate axes. By construction, $\tilde{u} \in W^p_0\Lambda^k(\tilde{U})$.

The bounds for the potential operators imply the existence of $\tilde{w} \in W^p_0\Lambda^k(\tilde{U})$ with $d\tilde{w} = d\tilde{u}$ and

$$\begin{aligned}
\|\tilde{w}\|_{L^p(\tilde{U})} &\leq n A_{\mathfrak{B}}(n,k+1) \frac{\mathrm{vol}_n(B_2(0))}{\mathrm{vol}(\tilde{U})} 2 \|\tilde{g}\|_{L^p(\tilde{U})} \\
&\leq 2^{n+1} n A_{\mathfrak{B}}(n,k+1) \frac{\mathrm{vol}_n(B_1(0))}{(l+1)/n!} \|\tilde{g}\|_{L^p(\tilde{U})}.
\end{aligned}$$

We let $w \in W^p\Lambda^k(T)$ be defined by $w := \varphi^{-*}\tilde{w}$. By construction, $dw = du$, and w has vanishing trace along the faces $F_0, \dots, F_l$.

In addition to that, by Proposition 5.2:

$$\begin{aligned}
\|w\|_{L^p(T)} &\leq |\det(\mathrm{Jac}\,\varphi)|^{\frac{1}{p}} \|\,\mathrm{Jac}\,\varphi^{-1}\|^{k-1} \|\tilde{w}\|_{L^p(\hat{T})} \\
&\leq |\det(\mathrm{Jac}\,\varphi)|^{\frac{1}{p}} \|\,\mathrm{Jac}\,\varphi^{-1}\|^{k-1} \|\tilde{w}\|_{L^p(\tilde{U})}, \\
\|\tilde{g}\|_{L^p(\tilde{U})} &\leq 2^{\frac{n}{p}} \|\hat{g}\|_{L^p(\hat{T})} \leq 2^{\frac{n}{p}} |\det(\mathrm{Jac}\,\varphi^{-1})|^{\frac{1}{p}} \|\,\mathrm{Jac}\,\varphi\|^{k} \|du\|_{L^p(T)}.
\end{aligned}$$

The second inequality follows. □

MV: the norms of the Jacobians have to be fixed.

MV: Please a few more details.

Whenever $\mathcal{T}$ is an n-dimensional triangulation, we use the abbreviation

$$C_{\mathrm{PF},\Gamma,k,p}(\mathcal{T}) := \max_{\substack{T\in\mathcal{T}\ \dim(T)=n \\ 0\leq l\leq n}} C_{\mathrm{PF},\Gamma,l,k,p}(T).$$

9.3 Poincaré–Friedrichs constants over a shellable triangulation

Suppose that an n-dimensional triangulation $\mathcal{T}$ has a shelling $T_0, T_1, \dots, T_M$. For each $1 \leq m \leq M$, there exists a simplex $S_{m-1} \in \mathcal{T}$ such that T_m is the last n-simplex in the shelling that belongs to $\cancel{T_m \in}\, \mathrm{st}_{\mathcal{T}}(S_{m-1})$. This is precisely $S_m \in \mathcal{T}$ from Lemma 7.15 (there the last simplex is called T_{m+1}).

MV: It might be good to synchronize the indices between the shellability definitions in Section 7.2 and Theorems 4.2, 4.5, 9.4, 9.5.

We then let U_{m-1} be the closure of $\mathrm{st}_{\mathcal{T}}(S_{m-1}) \setminus T_m$, and we call it the complement of T_m in the star $\mathrm{st}_{\mathcal{T}}(S_{m-1})$. The main result of this manuscript for nonconvex and non-star-shaped triangulated domains is the following theorem.

Theorem 9.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain with an n-dimensional shellable triangulation $\mathcal{T}$. Let $T_0, T_1, \dots, T_M$ be a shelling. Then for any $u \in W^p\Lambda^k(\Omega)$ there exists $w \in W^p\Lambda^k(\Omega)$ with $dw = du$ and such that for each*

$T_m \in \mathcal{T}$ *in the shelling,* $1 \leq m \leq M$, *we have the following recursive estimate:*

$$\|w\|_{L^p(T_m)} \leq C_{\mathrm{PF},T_m,I_{m-1},k,p}(\mathcal{T})\delta(\mathcal{T})^k \left(\|du\|_{L^p(T_m)} + C_{5,n,k}(\mathcal{T})^{k+1} C^{\det}_{6,n,k}(\mathcal{T})^{\frac{1}{p}} \|du\|_{L^p(U_{m-1})}\right)$$
$$+ C_{5,n,k}(\mathcal{T})^k C^{\det}_{6,n,k}(\mathcal{T})^{\frac{1}{p}} \|w\|_{L^p(U_{m-1})}.$$

MWL: The notation U_m should be settled in the definition of shellings.

MV: I think we should also write down the bound for $\|w\|_{L^p(\Omega)}$.

Proof. This proof is similar to the proof of Theorem 4.5. By assumption, there exists a shelling $T_0, T_1, \dots, T_M$. We write $\Omega_m := \cup_{j=0}^m T_j$, $0 \leq m \leq M$, which is by Lemma 7.7 a triangulated n-dimensional submanifold with boundary.

MV: It might also be good to synchronize the notations X_m and Ω_m. I would go for X_m, since this is a union of n-simplices (closed), whereas Ω_m has a connotation of a domain (open).

For $0 < m \leq M$, by the definition of shellability in Section 7.2, T_m and Ω_{m-1} share the interface I_{m-1}, which is a collection of faces of T_m. It follows from Lemma 7.15 that adding T_m completes a star in $\mathcal{T}$ around the simplex $S_{m-1} \in \mathcal{T}$, where $U_{m-1} \subseteq \Omega_{m-1}$ is the complement of T_m in that newly completed star.

MV: Short, as already explained above in details.

We will recursively construct $w \in W^p\Lambda^k(\Omega)$ with $dw = du$, satisfying the desired bound. Let $u \in W^p\Lambda^k(\Omega)$. First, on T_0, from Lemma 9.1, there exists $w_0 \in W^p\Lambda^k(T_0)$ satisfying $dw_0 = du$ over T_0 together with

$$\|w_0\|_{L^p(T_0)} \leq C_{\mathrm{PF},k,T_0,p}\|du\|_{L^p(T_0)}.$$

We define $w|_{T_0} := w_0$. Suppose now that $0 < m \leq M$ such that $w|_{\Omega_{m-1}} \in W^p\Lambda^k(\Omega_{m-1})$ with $dw = du$ over Ω_{m-1}. This is true for $m = 1$. We now define, as in [21, equations (5.12) and (5.14)] or [14, equations (6.7) and (6.9)],

$$w|_{T_m} := u|_{T_m} + \Xi_m^*((w-u)|_{U_{m-1}}), \tag{37}$$

MV: Direct and simple definition. Only one object. Namely avoids the checking of the compatibility of the boundary condition, which is a bit tedious and requires a rigorous apparatus.

where $\Xi_m : T_m \to U_{m-1}$ is the reflection mapping as Ξ_1 in Proposition 8.3 and the pullback is defined in (32). Clearly, $w|_{T_m} \in W^p\Lambda^k(T_m)$. Moreover,

$$\mathrm{tr}_{I_{m-1}} w|_{T_m} = \mathrm{tr}_{I_{m-1}} w|_{\Omega_{m-1}},$$

since u is trace continuous over I_{m-1} and Ξ_m is the identity along I_{m-1}, and

$$dw = du \text{ over } T_m,$$

since $dw = du$ over Ω_{m-1} and Ξ_m^* preserves this. Thus

$$w|_{\Omega_m} \in W^p\Lambda^k(\Omega_m) \text{ with } dw_m = du \text{ over } \Omega_m.$$

We want to estimate norms. By construction,

$$\|w\|_{L^p(T_m)} \leq \|\Xi_m^*(w|_{U_{m-1}})\|_{L^p(T_m)} + \|u - \Xi_m^*(u|_{U_{m-1}})\|_{L^p(T_m)}.$$

Due to Lemma 9.3, since $\mathrm{tr}_{I_{m-1}}(u|_{T_m} - \Xi_m^*(u|_{U_{m-1}})) = 0$, we have

$$\|u - \Xi_m^*(u|_{U_{m-1}})\|_{L^p(T_m)} \leq C_{\mathrm{PF},T_m,I_{m-1},k,p}(\|du\|_{L^p(T_m)} + \|d(\Xi_m^*(u|_{U_{m-1}}))\|_{L^p(T_m)}).$$

We also use Proposition 5.2 to find

MV: $\delta(\mathcal{T})^k$???

$$\|\Xi_m^*(w|_{U_{m-1}})\|_{L^p(T_m)} \leq |\det(\mathrm{Jac}\,\Xi_m)|^{-\frac{1}{p}} \|\mathrm{Jac}\,\Xi_m\|^k \|w\|_{L^p(U_{m-1})},$$
$$\|d(\Xi_m^*(w|_{U_{m-1}}))\|_{L^p(T_m)} \leq |\det(\mathrm{Jac}\,\Xi_m)|^{-\frac{1}{p}} \|\mathrm{Jac}\,\Xi_m\|^{k+1} \|du\|_{L^p(U_{m-1})},$$

where we have employed that $dw = du$ over Ω_{m-1}. Using the properties of the reflection mapping in Proposition 8.3, the desired recursive estimate follows. ∎

MV: Which norms of the Jacobians and their determinants?

MV: Why a second theorem? Difference of Theorem 9.5 wrt Theorem 9.4? Which is better and when?

Theorem 9.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain with an n-dimensional shellable triangulation $\mathcal{T}$. Let $T_0, T_1, \dots, T_M$ be a shelling. Then for any $u \in W^p\Lambda^k(\Omega)$ there exists $w \in W^p\Lambda^k(\Omega)$ with $dw = du$ and such that for each $T_m \in \mathcal{T}$ in the shelling, $1 \leq m \leq M$, we have the following recursive estimate:*

$$\|w\|_{L^p(T_m)} \leq C_{\mathrm{PF},T,k,p}(T_0) \prod_{i=0}^{m} C_{???} \|du\|_{L^p(\Omega)}.$$

Proof. We prove this via induction. The statement is clearly satisfied when $m = 0$ because then we only consider a single simplex. Let us assume that the statement is true for some integer $m - 1 \geq 0$.

We write $\Omega_{m-1} := \cup_{j=0}^{m-1} T_j$, which is a triangulated n-dimensional submanifold with boundary. By assumption, T_m and Ω_{m-1} share the interface $I_{m-1} \subset \partial T_m$, which is a collection of faces of T_m. In accordance to Lemma 7.15, adding T_m completes a star in $\mathcal{T}$ around some simplex $S \subseteq T$, and we let $U_{m-1} \subseteq \Omega_{m-1}$ be the complement of T_m in that newly completed star.

We use the bi-Lipschitz mapping $\Xi_2 : \mathrm{st}_{\mathcal{T}}(S) \to U_{m-1}$ as discussed in Proposition 8.3, which acts as the identity along $\partial U_{m-1} \setminus \partial T$. By assumption, there exists $w_{m-1} \in W^p\Lambda^k(\Omega_{m-1})$ such that $dw_{m-1} = \Xi_2^{-1} du$ and

$$\|w_{m-1}\|_{L^p(T_m)} \leq C_{\mathrm{PF},T,k,p}(T_0) \prod_{i=0}^{m} C_{???} \|\Xi_2^{-*} du\|_{L^p(\Omega)}.$$

Because pullback and exterior derivative commute, we have $w := \Xi_2^* w_{m-1}$ satisfying $dw = du$. Moreover, the desired estimate follow from Proposition 8.3. The proof is complete. ☐ MV: I do not get this.

10 Numerical examples

We assess the "efficiency" of our upper bounds with a few numerical examples in dimension 2D and 3D, with focus on the Hilbert space case $p = 2$. Unless the Poincaré–Friedrichs constants are known explicitly, we use eigenvalue estimates on a refined mesh as a proxy for the exact value. We compare the (approximations of) the exact values with the upper bounds obtained via Theorem 9.4.

10.1 Estimates for partial boundary conditions

We first assess the constant in Lemma 9.3, which relies on estimates for the Bogovskiĭ operator over an auxiliary reference domain. Our experiments over triangles and tetrahedra indicate we overestimate by several magnitudes. We conjecture that over convex domains Ω, the Poincaré–Friedrichs constant of the gradient $\mathrm{grad} : W^{1,p}(\Omega) \to \boldsymbol{L}^p(\Omega)$ is an upper estimate for other Poincaré–Friedrichs constants in de Rham complex. Numerical examples indicate that this holds for $p = 2$.

10.2 Two-dimensional examples

We consider the following example domains in two dimensions: the unit square Ω_Q, the L-shaped domain Ω_L, and the slit domain Ω_S:

$$\Omega_Q = (-1,1)^2, \qquad \Omega_L = (-1,1)^2 \setminus (0,1)^2,$$
$$\Omega_S = (-1,1)^2 \setminus ([0,1) \times \{0\}).$$

10.3 Three-dimensional examples

We consider the following example domains in two dimensions: the unit cube Ω_C, the Fichera corner domain Ω_F, and the crossed bricks domain Ω_B:

$$\Omega_C = (-1,1)^3, \qquad \Omega_F = (-1,1)^3 \setminus [0,1)^3,$$
$$\begin{aligned} \Omega_B = & \left((-1,0)\times(-1,0)\times(-1,1)\right) \\ & \cup \left((-1,0)\times(-1,1)\times(-1,0)\right) \\ & \cup \left((-1,1)\times(-1,0)\times(-1,0)\right). \end{aligned}$$

11 Outlook

There are several avenues for further research going further from the results established in this manuscript. MV: Added this section at the end and moved here the corresponding contents from the introduction

Our upper bounds are hardly the last word on estimating the Poincaré–Friedrichs constants over local patches (stars), where there are at least two points where the present techniques can be improved further. First, we utilize local Poincaré–Friedrichs constants over single simplices, subject to various boundary conditions, but these local estimates over convex ~~domains~~simplices already leave room for improvement. Second, MV: ??? our construction needs to extend differential forms onto a simplex from the complement of that simplex inside a star. In the present manuscript, this is achieved via pullback along a bi-Lipschitz mapping. Better transformations or more careful estimates are likely possible, as are completely different approaches. For example, it is likely that further research on trace and extension operators on triangulations will enable better estimates.

Related to the first of the aforementioned opportunities for improvement, we conjecture that the Guerini–Savo estimates, hitherto only known for the exterior derivative as a mapping between Hilbert spaces and over convex domains with smooth boundary, could be extended to the entire range of Lebesgue exponents $1 \leq p \leq \infty$ and to convex Lipschitz domains. We believe such estimates would yield considerably better bounds on the Poincaré–Friedrichs constants than the regularized potential operators in the sprit of Costabel and McIntosh [18] that we employ now. On a related note, explicit estimates for the operator norms of regularized potential operators with different degrees of smoothing are of general interest.

Concerning the domains considered, estimating Poincaré–Friedrichs constants over domains and manifolds with shellable triangulations is fundamentally restricted to ~~spaces~~ domains with simplicial homology groups MV: ??? of balls and spheres. In particular, domains with shellable triangulations are contractible. We believe that in the forthcoming works, these estimates will become subcomponents in computing upper bounds for Poincaré–Friedrichs constants over simplicial triangulations of general n-dimensional manifolds.

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