Affine Chabauty I

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30th June 2025

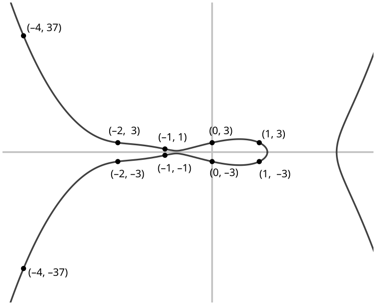
33èmes Journées Arithmétiques, Luxembourg

Diophantine equations

Motivating question: given $f(x,y) \in \mathbb{Z}[x,y]$, solve $f(x,y) = 0$ in $\mathbb{Z}$.

For example, what are the integer solutions to

$$y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9?$$



More generally, what are the solutions in $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{\ell} : \ell \in S]$ for a finite set of primes S?

Geometric formulation: an equation $f(x,y) = 0$ describes an affine curve $\mathcal{Y}$ in $\mathbb{A}^2_{\mathbb{Z}}$, the solutions in $\mathbb{Z}_S$ form the set $\mathcal{Y}(\mathbb{Z}_S)$ of S-integral points.

The Siegel–Mahler Theorem

Setup:

- $X/\mathbb{Q}$ smooth projective curve of genus g
- $D \subseteq X$ finite set of closed points ("cusps"), $n = \#D(\overline{\mathbb{Q}}) > 0$
- $Y = X \smallsetminus D$ affine curve
- $\mathcal{X}/\mathbb{Z}$ regular model of X
- $\mathcal{D}$ the closure of D in $\mathcal{X}$
- $\mathcal{Y} = \mathcal{X} \smallsetminus \mathcal{D}$ model of Y
- S finite set of primes, $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{\ell} : \ell \in S]$ ring of S-integers

Theorem (Siegel, Mahler)

If $2 - 2g - n < 0$ then $\#\mathcal{Y}(\mathbb{Z}_S) < \infty$.

Goal: determine $\mathcal{Y}(\mathbb{Z}_S)$ or bound $\#\mathcal{Y}(\mathbb{Z}_S)$ in practice

Chabauty–Coleman

The analogous problem for rational points on curves of genus ≥ 2 can often be solved by the Chabauty–Coleman method.

Mordell Conjecture (1922)

If $g \geq 2$ then $\#X(\mathbb{Q}) < \infty$.

- Chabauty (1941): proved finiteness if $r := \operatorname{rk} \operatorname{Jac}_X(\mathbb{Q}) < g$
- Faltings (1983): proved finiteness in general

Theorem (Coleman, 1985)

Let p be a prime of good reduction for X and fix a base point $P_0 \in X(\mathbb{Q})$. If $r < g$ then there exists a computable differential form $0 \neq \omega \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1)$ such that

$$X(\mathbb{Q}) \subseteq \left\{ P \in X(\mathbb{Q}_p) : \int_{P_0}^{P} \omega = 0 \right\} \subseteq X(\mathbb{Q}_p).$$

Affine Chabauty

The Chabauty–Coleman method produces a finite computable subset of $X(\mathbb{Q}_p)$ containing $X(\mathbb{Q})$. We develop a Chabauty–Coleman method for S-integral points on affine curves.

Differences:

- use logarithmic differential forms $\omega \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))$, i.e., simple poles at cusps are allowed
- partition S-integral points by (finitely many) reduction types

$$\mathcal{Y}(\mathbb{Z}_S) = \coprod_{\Sigma} \mathcal{Y}(\mathbb{Z}_S)_\Sigma$$

  and look at each $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ separately
- Jacobian is generalised by the generalised Jacobian

Main Theorem

Notation:

- $p \notin S$ prime of good reduction for $\mathcal{X}$
- $P_0 \in Y(\mathbb{Q})$ base point
- $n = n_1(D) + 2n_2(D)$ with $n_1(D) = \#D(\mathbb{R})$, $n_2(D) = \frac{1}{2}\#(D(\mathbb{C}) \smallsetminus D(\mathbb{R}))$

Theorem (Leonhardt–L., 2025+)

Assume the Affine Chabauty Condition (ACC)

$$r + \#S < g + \#|D| + n_2(D) - 1 .$$

Then for each reduction type Σ *there exists a computable log differential* $0 \neq \omega \in \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))$ *and constant* $c \in \mathbb{Q}_p$ *such that*

$$\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subseteq \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) : \int_{P_0}^{P} \omega = c \right\} \subseteq \mathcal{Y}(\mathbb{Z}_p).$$

Idea of proof

Generalised Jacobian J_Y of Y:

$$J_Y(\mathbb{Q}) = \mathrm{Div}^0(Y) \,/\, \left\{\mathrm{div}(f) : f \in k(X)^\times, f|_D = 1\right\}$$

Abel–Jacobi embedding $\mathsf{AJ}_{P_0} \colon Y \hookrightarrow J_Y$, $P \mapsto [P] - [P_0]$.

Affine Chabauty diagram:

$$\begin{tikzcd}
\mathcal{Y}(\mathbb{Z}_S)_\Sigma \arrow[r, hook] \arrow[d, "\mathsf{AJ}_{P_0}"] & \mathcal{Y}(\mathbb{Z}_S) \arrow[r, hook] \arrow[d, "\mathsf{AJ}_{P_0}"] & \mathcal{Y}(\mathbb{Z}_p) \arrow[d, "\mathsf{AJ}_{P_0}"] \arrow[dr, "\int_{P_0}"] & \\
\mathsf{Sel}(P_0, \Sigma) \arrow[r, hook] & J_Y(\mathbb{Q}) \arrow[r, hook] & J_Y(\mathbb{Q}_p) \arrow[r, "\log_{J_Y}"] & \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^\vee
\end{tikzcd}$$

Key insight: $J_Y(\mathbb{Q})$ is not finitely generated but the Abel–Jacobi image of $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ is contained in a subset $\mathsf{Sel}(P_0, \Sigma)$, a translate of a f. g. subgroup of rank

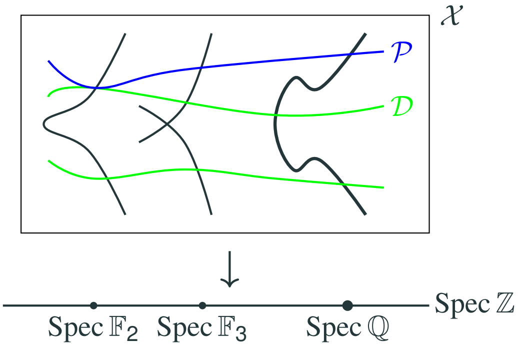
$$\mathrm{rk}\,\mathsf{Sel}(P_0, \Sigma) = r + n_1(D) + n_2(D) - \#|D| + \#S \overset{\text{(ACC)}}{<} g + n - 1 = \dim_{\mathbb{Q}_p} \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))$$

Arithmetic intersection theory

The Abel–Jacobi image of $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ in $J_Y(\mathbb{Q})$ is constrained using intersection theory on the arithmetic surface $\mathcal{X}$.

We construct the *D-intersection map*

$$\sigma_\ell \colon J_Y(\mathbb{Q}) \to Z_0(\mathcal{D}_{\mathbb{F}_\ell})/[\mathcal{D}_{\mathbb{F}_\ell}] \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$F \mapsto \sum_{x \in |\mathcal{D}_{\mathbb{F}_\ell}|} i_x(\Psi_\ell(F), \mathcal{D})[x]$$



such that $\sigma_\ell(\mathrm{AJ}_{P_0}(P))$ depends only on:

- the component of $\mathcal{X}_{\mathbb{F}_\ell}$ onto which P reduces
- intersection multiplicities $i_x(\mathcal{P}, \mathcal{D})$ with the boundary divisor at $x \in |\mathcal{X}_{\mathbb{F}_\ell}|$ ($\ell \in S$)

Here, $\Psi_\ell(F) =$ horizontal extension $\mathcal{F}$ + a vertical $\mathbb{Q}$-divisor $\Phi_\ell(F)$

Selmer sets

The reduction type $\Sigma = (\Sigma_\ell)_\ell$ prescribes for each ℓ the component of $\mathcal{X}_{\mathbb{F}_\ell}$ or (if $\ell \in S$) the cusp onto which the point reduces. We get

$$\sigma_\ell(\mathrm{AJ}_{P_0}(\mathcal{Y}(\mathbb{Z}_S)_\Sigma)) \subseteq \mathfrak{S}_\ell(P_0, \Sigma) \text{ of rank } 0\ (\ell \notin S) \text{ or } 1\ (\ell \in S).$$

Define the Selmer set

$$\mathrm{Sel}(P_0, \Sigma) := \{F \in J_Y(\mathbb{Q}) \mid \forall \ell : \sigma_\ell(F) \in \mathfrak{S}_\ell(P_0, \Sigma)\}$$

then $\mathrm{Sel}(P_0, \Sigma)$ contains the Abel–Jacobi image of $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$ and is a translate of a subgroup of small finite rank.

The end

Theorem (LL)

The integral points of the rank 2, genus 2 curve

$$y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9?$$

are $(-1, \pm 1),\ (0, \pm 3),\ (1, \pm 3),\ (-2, \pm 3),\ (-4, \pm 37)$.

Proof.

Use the Affine Chabauty method with $p = 5$, find $0 \neq \omega \in \mathrm{H}^0(X_{\mathbb{Q}_5}, \Omega^1(D))$ with

$$\mathcal{Y}(\mathbb{Z}) \subseteq \left\{ P \in \mathcal{Y}(\mathbb{Z}_5) : \int_{(-1,1)}^{P} \omega = 0 \right\},$$

and check that the RHS only contains the listed points. $\square$

Thank you!