STAT4005 - Time Series
Tutorial 2 - Stationarity & dependence measure
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Week 4

2 Stationarity and dependence measure

2.1 Stationary Time Series

Time series is a study of a set of possibly dependent data $\{X_t\}$. A time series can be characterized by its **finite-dimensional distributions** $\mathscr{F}$, where $\mathscr{F}$ contains all the joint distribution of time series at any finite set of time points, i.e., $F_t \in \mathscr{F}$ if it could be written in the form of

$$F_{\boldsymbol{t}}(\boldsymbol{x}) = \mathbb{P}(X_{t_1} \leq x_1, \ldots, X_{t_k} \leq x_k),$$

where $\boldsymbol{t} = (t_1, \ldots, t_k)$, $1 \leq t_1 < t_2 < \cdots < t_k < \infty$ for all $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$.

Many existing theories do not work for complicated models. We restrict our current study of interest to the class of stationary time series models.

Stationary Time Series

Definition 1. *A Time Series $\{X_t\}$ is said to be **strictly stationary** if for all $n \in \mathbb{N}^+, \boldsymbol{t} = (t_1, \cdots, t_n)$ with $1 \leq t_1 < \ldots < t_n < \infty$, $\boldsymbol{x} = (x_1, \ldots, x_n), h \in \mathbb{R}$,*

$$F_{\boldsymbol{t}}(\boldsymbol{x}) = \mathbb{P}(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n) = \mathbb{P}(X_{t_1+h} \leq x_1, \ldots, X_{t_n+h} \leq x_n) = F_{\boldsymbol{t}+h}(\boldsymbol{x})$$

Definition 2. *A Time series $\{X_t\}$ is said to be **weakly stationary** if the mean is constant over time and the covariance between X_i, X_j depends only on its time lag $|i-j|$, i.e.,*

1. $\mathrm{E}(X_t) = \mu$ *for some* $\mu \in \mathbb{R}$ *for all* t.
2. $\mathrm{Cov}(X_t, X_{t+k}) = \gamma(k)$ *for all* $t, k \in \mathbb{N}^+$.

*In this case, $\gamma(\cdot)$ is said to be the **auto-covariance function (ACVF)** of $\{X_t\}$.*

Remark 1. *Unless mentioned otherwise, the term "stationary" always refer to "weakly stationary".*

Implication between strict and weakly stationarity

Theorem 1. *Let $\{X_t\}$ be a time series. Then*

1. *Strictly stationary* $+\ \mathrm{E}(X_t^2) < \infty \Rightarrow$ *Weakly stationary.*
2. *Weakly stationary* $+\ X_t \sim N(\mu, \sigma^2)$ *for some* $\mu \in \mathbb{R},\ \sigma \in \mathbb{R}^+ \Rightarrow$ *Strictly stationary.*
3. *(General case) Weakly stationary* $\not\Rightarrow$ *Strictly stationary.*

Remark 2. *For $X \sim t_{df}$, $\mathrm{E}(X^k) < \infty$ if and only if $k < df$.*

White-Noise

Definition 3. *A time series $\{a_t\}$ is called **White Noise** sequence, denoted by $\mathrm{WN}(0, \sigma^2)$ if*

1. $\mathrm{E}(a_t) = 0$ *and* $\mathrm{Var}(a_t) = \sigma^2$ *for all* t.
2. $\mathrm{Cov}(a_t, a_{t+k}) = 0$ *for all* $k \neq 0$.

Remark 3. *White noise is a weakly stationary process (may NOT be strictly stationary).*

(★★☆) Stationarity of Time Series

Exercise 1. *Consider the time series $\{X_t\}$ defined by*

$$X_t = \beta_0 + \beta_1 t + a_t,$$

where $\beta_1 \neq 0$ and $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) Show if $\{X_t\}$ is a weakly stationary process.

(b) Show that $\{Y_t\}$ is a weakly stationary process, where $Y_t := \Delta X_t = X_t - X_{t-1}$.

(c) Show that it is not necessary for $\{Y_t\}$ to be strictly stationary.

Solution

(a) As $\mathrm{E}(X_t) = \beta_0 + \beta_1 t + \mathrm{E}(a_t) = \beta_0 + \beta_1 t$ and $\beta_1 \neq 0$, $\mathrm{E}(X_t)$ is not constant overtime. Hence $\{X_t\}$ is NOT weakly stationary.

(b) $Y_t = X_t - X_{t-1} = (\beta_0 + \beta_1 t + a_t) - \big(\beta_0 + \beta_1(t-1) + a_{t-1}\big) = \beta_1 + (a_t - a_{t-1})$. Then,

- $\mathrm{E}(Y_t) = \beta_1 + \mathrm{E}(a_t - a_{t-1}) = \beta_1$, which is constant overtime.
- As $\mathrm{Cov}(a_i, a_j) = \sigma^2 \mathbb{1}(i = j)$, we have

$$\begin{aligned}\mathrm{Cov}(X_t, X_{t+k}) =& \mathrm{Cov}(a_t - a_{t-1}, a_{t+k} - a_{t+k-1}) \\ \overset{(\star)}{=}& \begin{cases} \mathrm{Cov}(\boldsymbol{a_t} - \boldsymbol{a_{t-1}}, \boldsymbol{a_t} - \boldsymbol{a_{t-1}}) & , \text{ if } k = 0 \\ \mathrm{Cov}(\boldsymbol{a_t} - a_{t-1}, a_{t+1} - \boldsymbol{a_t}) & , \text{ if } k = 1 \\ \mathrm{Cov}(a_t - a_{t-1}, a_{t+k} - a_{t+k-1}) & , \text{ if } k \geq 2 \end{cases} = \begin{cases} 2\sigma^2 & , \text{ if } k = 0 \\ -\sigma^2 & , \text{ if } k = 1 \\ 0 & , \text{ if } k \geq 2 \end{cases}\end{aligned}$$

It follows that $\{Y_t\}$ is weakly stationary with mean 0, and its ACVF $\gamma(\cdot)$ is given by

$$\gamma(k) = \begin{cases} 2\sigma^2 & , \text{ if } k = 0 \\ -\sigma^2 & , \text{ if } |k| = 1 \\ 0 & , \text{ if } |k| \geq 2 \end{cases}$$

(c) Let $a_1 \sim \sigma(\mathrm{Exp}(1) - 1)$ and $a_2, a_3, \cdots \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$, then $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$. Then $Y_3 = \beta_1 + (a_3 - a_2)$ is normally distributed but $Y_2 = \beta_1 + (a_2 - a_1)$ is not, i.e., Y_2 and Y_3 have different marginal distributions. Hence $\{Y_t\}$ may not be strictly stationary.

Remark 4. *Notice that in equality $(\star)$, we only showed that*

$$\mathrm{Cov}(X_t, X_{t+k}) \textit{ depends only on } k \textit{ but not on } t, \textbf{\textit{ for }} k \geq 0.$$

Even though we did not directly compute $\mathrm{Cov}(X_t, X_{t+k})$ for case $k < 0$, substituting $t' = t - k$ into the above statement yields

$$\mathrm{Cov}(X_t, X_{t-k}) = \mathrm{Cov}(X_{t-k}, X_t) \textit{ depends only on } k \textit{ but not } t, \textit{ for } k \geq 0.$$

Therefore, we only need to check whether $\mathrm{Cov}(X_t, X_{t+k})$ depends only on k but not on t, for $k \geq 0$.

Takeaway 1. *We should remove the trend and seasonal effects of X_t so that we can obtain a relatively more stationary process $Y_t = \Delta X_t$ for further analysis and model its dependence.*

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2.2 Measures of Serial Dependence

We answer the following questions in this section

1. $\gamma(\cdot)$ is NOT the only measure for serial dependence. What are the other possible candidates?
2. How to estimate $\gamma(\cdot)$ and the other potential measures of serial dependence?

For simplicity, assume that all the time series $\{X_t\}$ mentioned in this subsection are weakly stationary.

Measure of Serial Dependence I - Autocovariance

Definition 4. *ACVF and ACF*

- **Autocovariance function (ACVF)**: $\gamma(k) := \mathrm{Cov}(X_t, X_{t+k})$
- **Autocorrelation function (ACF)**: $\rho(k) = \mathrm{Corr}(X_t, X_{t+k}) = \gamma(k)/\gamma(0)$.

Definition 5. *The estimator of ACVF and ACF*

- **Sample ACVF**:
$$C_k := \frac{1}{n}\sum_{t=1}^{n-k}(X_t - \bar{X})(X_{t+k} - \bar{X}).$$
- **Sample ACF**: $r_k = C_k/C_0$

Remark 5. *There are several remarks on the ACVF and ACF.*

1. *Both ACVF and ACF are symmetric in k, i.e., $\gamma(k) = \gamma(-k)$ and $\rho(k) = \rho(-k)$ for all $k \in \mathbb{Z}$.*
2. *For the ACVF sample C_h, the denominator is n instead of $n-h$. It guarantees $\widehat{\mathrm{Var}}(\boldsymbol{a^T X}) \geq 0$, for all $\boldsymbol{a}^T = (a_1, \dots, a_n)^T$ and $\boldsymbol{X} = (X_1, \dots, X_n)^T$*
3. *$\rho(k)$ is a measure of **linear dependence** among X_t and X_{t+k}. Explicitly,*
$$X_{t+k} = \beta_k X_t + Z_{t+k} \quad \Rightarrow \quad \beta_k = \rho(k),$$
where Z_{t+k} is some residual satisfying $X_t \perp\!\!\!\perp Z_{t+k}$.

Another common measure of serial dependence is the Partial Autocorrelation function (PACF).

Measure of Serial Dependence II - Partial Autocorrelation

Definition 6. *The **lag k Partial Autocorrelation function (PACF)** of a weakly stationary time series $\{X_t\}$ is defined by*
$$\phi_{kk} := \mathrm{Corr}(X_t, X_{t+k} | X_{t+1}, \dots, X_{t+k-1}).$$

Evaluation of PACF and its estimation

Theorem 2. *Let $\{X_t\}$ be weakly stationary with ACF $\{\rho(h)\}$. Then the lag $\boldsymbol{k}$ PACF ϕ_{kk} could be solved through*
$$\begin{pmatrix} \phi_{k1} \\ \vdots \\ \phi_{kk} \end{pmatrix} = \begin{pmatrix} \rho(0) & \cdots & \rho(k-1) \\ \vdots & \ddots & \vdots \\ \rho(k-1) & \cdots & \rho(0) \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \vdots \\ \rho(k) \end{pmatrix}$$
and its estimator $\hat{\phi}_{kk}$ could be solved through
$$\begin{pmatrix} \hat{\phi}_{k1} \\ \vdots \\ \hat{\phi}_{kk} \end{pmatrix} = \begin{pmatrix} r_0 & \cdots & r_{k-1} \\ \vdots & \ddots & \vdots \\ r_{k-1} & \cdots & r_0 \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}$$

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Remark 6. *Partial autocorrelation is a measure the* ***additional*** *linear dependence among X_t and X_{t+k}* ***after accounting for*** *the linear independence among $(X_{t+1}, \ldots, X_{t+k-1})$. Explicitly,*

$$X_{t+k} = \underbrace{\left(\phi_{k1}X_{t+k-1} + \phi_{k2}X_{t+k-2} + \cdots + \phi_{k,k-1}X_{t+1}\right)}_{\text{Contributions by RVs between } X_t \text{ and } X_{t+k}} + \underbrace{\phi_{kk}}_{\text{Contribution by } X_t} X_t + Z_{k+1},$$

where Z_{t+k} is some residual satisfying $X_t \perp\!\!\!\perp Z_{t+k}$.

(★★☆) Evaluation of ACF and PACF

Exercise 2. *Suppose that $X_t = 1 + 2a_{t-1} - 3a_{t+1}$, where $\{a_t\} \sim \mathrm{WN}(0,1)$.*

(a) Show if $\{X_t\}$ is weakly stationary. If so, find its ACVF $\gamma(\cdot)$ and ACF $\rho(\cdot)$.

(b) Find the value of $\mathrm{Var}(4X_1 - 5X_2 + X_4)$.

(c) Find the lag 2 PACF ϕ_{22} of $\{X_t\}$.

Solution

(a) Notice that $\mathrm{E}(X_t) = 1 + 2\mathrm{E}(a_t) - 3\mathrm{E}(a_{t-1}) = 1$ is constant overtime.

$$\begin{aligned} &\mathrm{Cov}(X_t, X_{t+k}) = \mathrm{Cov}(2a_{t-1} - 3a_{t+1}, 2a_{t+k-1} - 3a_{t+k+1}) \\ =&4\mathrm{Cov}(a_{t-1}, a_{t+k-1}) - 6\mathrm{Cov}(a_{t-1}, a_{t+k+1}) - 6\mathrm{Cov}(a_{t+1}, a_{t+k-1}) + 9\mathrm{Cov}(a_{t+1}, a_{t+k+1}) \\ =&\begin{cases} 13 & , \text{ if } k = 0 \\ -6 & , \text{ if } k = 2 \\ 0 & , \text{ if otherwise} \end{cases} \end{aligned}$$

only depends on k. Hence $\{X_t\}$ is weakly stationary. The ACVF and ACF are given by

$$\gamma(k) = \begin{cases} 13 & , \text{ if } k = 0 \\ -6 & , \text{ if } |k| = 2 \\ 0 & , \text{ if otherwise} \end{cases} \quad \text{and} \quad \rho(k) = \begin{cases} 1 & , \text{ if } k = 0 \\ -6/13 & , \text{ if } |k| = 2 \\ 0 & , \text{ if otherwise} \end{cases}$$

(b) According to part (a), we have

$$\begin{aligned} &\mathrm{Var}(4X_1 - 5X_2 + X_4) = \mathrm{Cov}(4X_1 - 5X_2 + X_4, 4X_1 - 5X_2 + X_4) \\ =&(4^2 + (-5)^2 + 1^2)\gamma(0) + (-5 \times 1 + 1 \times (-5))\gamma(2) = 606. \end{aligned}$$

(c) It could be solved through

$$\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} \rho(0) & \rho(1) \\ \rho(1) & \rho(0) \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix} = \begin{pmatrix} -78/133 \\ -36/133 \end{pmatrix}$$

Hence $\phi_{22} = -36/133$.

Remark 7. *In both Exercise 1 and 2, we need to verify the stationarity of time series according to Definition 2. Otherwise, it is illegitimate to define the ACVF γ. The stationarity can be verified easily through some tricks (to be covered in next tutorial).*

2.3 Time-Dependence Test

In general. we want to test whether time dependence exists for some pre-specified lag $k \geq 1$.

Time-Dependence Test

Theorem 3. *Let $\{X_t\}$ be identically and independently distributed. Then*

1. **(ACF)** $\sqrt{n}r_k \xrightarrow{d} \mathrm{N}(0,1)$ as $n \to \infty$.
2. **(PACF)** $\sqrt{n}\hat{\phi}_{kk} \xrightarrow{d} \mathrm{N}(0,1)$ as $n \to \infty$.

Remark 8. *The assumption can be generalized by just assuming (1) X_t and X_{t+k} are uncorrelated (2) X_t^2 and X_{t+k}^2 are uncorrelated for all t. (refer to the proof in Chapter 2)*

2.4 Additional Exercise

(★☆☆) Stationarity of Time Series II

✎ **Exercise 3.** *Consider the time series $\{X_t\}$ defined by*

$$X_t = \delta + a_{t-1} + 2a_t + a_{t+1},$$

where $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) What should the value of δ be such that $\{X_t\}$ becomes weakly stationary?

(b) Find the ACVF of $\{X_t\}$ when $\delta = 0$.

Quick Solution

$\mathrm{E}(X_t) = \delta$ is constant over time. And the covariance structure does NOT depend on the value of δ.

$$\mathrm{Cov}(X_t, X_{t+k}) = \begin{cases} 6\sigma^2 & \text{, if } k = 0 \\ 4\sigma^2 & \text{, if } |k| = 1 \\ \sigma^2 & \text{, if } |k| = 2 \\ 0 & \text{, otherwise} \end{cases}$$

Hence, δ could be arbitrary real number.

(★☆☆) Computation of Estimators

✎ **Exercise 4.** *Given that $\boldsymbol{x} = (7, 4, 6, 5, 3, 9)$. Evaluate the sample ACVF C_0 and C_2, sample ACF r_2 and sample PACF $\hat{\psi}_{22}$.*

Quick Solution

$C_0 = 3.888889$, $C2 = -0.2592593$, $r_2 = -0.06666667$ and $\phi_{22} = -1.071429$.

(★★☆) Uniqueness of ACVF

✎ **Exercise 5.** *Given two weakly stationary time series $\{X_t\}$ and $\{Y_t\}$ defined by*

$$X_t = a_t + \frac{5}{2}a_{t-1} - \frac{3}{2}a_{t-2} \quad \textit{and} \quad Y_t = 3a_t - \frac{1}{2}a_{t-1} - \frac{1}{2}a_{t-2},$$

where $\{a_t\} \sim \mathrm{WN}(0,1)$.

(a) Find the ACF of $\{X_t\}$ and $\{Y_t\}$, denoted by $\gamma_X(\cdot)$ and $\gamma_Y(\cdot)$.

(b) Compare $\gamma_X(\cdot)$ and $\gamma_Y(\cdot)$. Comment on it.

Solution

The answer is given by

$$\gamma_X(k) = \gamma_Y(k) = \begin{cases} 9.5 & , \text{ if } k = 0 \\ -1.25 & , \text{ if } |k| = 1 \\ -1.5 & , \text{ if } |k| = 2 \\ 0 & , \text{ otherwise} \end{cases}$$

Notice that $\gamma_X = \gamma_Y$ while they are different time series, i.e. ACVF does NOT fully characterize the time series.

2.5 ✄ Comparison between ACF and PACF

In this subsection, we discuss and compare the intuition behind ACF and PACF.

- Recall in Remark 5, ACF quantifies the effect of X_t on X_{t+k} solely without considering $(X_{t+1}, \ldots, X_{t+k-1})$, i.e.
$$X_{t+k} = \beta^{(k)} X_t + Z_{t+k} \quad \Rightarrow \quad \beta^{(k)} = \rho(k),$$
- For PACF, it quantifies the effect of X_t on X_{t+k} that $(X_{t+1}, \ldots, X_{t+k-1})$ could NOT explain, i.e., consider the following model
$$X_{t+k} = \Big(\phi_{k1} X_{t+k-1} + \phi_{k2} X_{t+k-2} + \cdots + \phi_{k,k-1} X_{t+1}\Big) X_t + \phi_{kk} X_t + Z_{k+1},$$

❗ **Remark 9.** *we assume that neither of those models is true during analysis. It is just the case that **if** those models hold, the truth model parameter would be ACF and PACF. respectively.*

The following graph is available for better understanding.

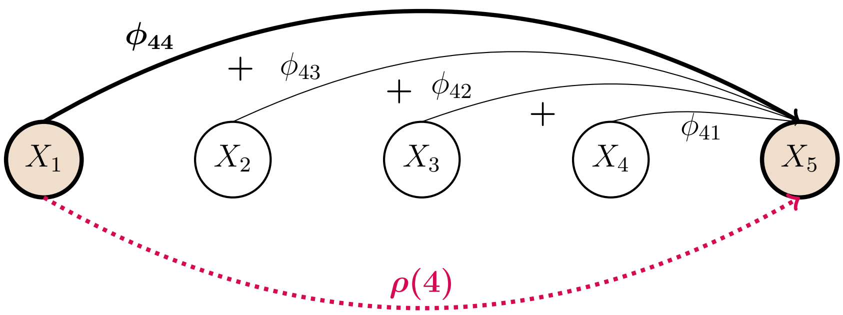

Figure 1: Graphical illustration of ACF and PACF

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1. The **red dotted line** is analogous to the ACF.
2. The **thick black line** is analogous to the PACF.

2.6 ✄ Simulation-Studies

This subsection aims to provide examples of utilizing R-programming for simulation. However, those content are NOT going to be tested and are totally optional. The interested can read this subsection. Recall that in Exercise 2, we consider

$$X_t = 1 + 2a_{t-1} - 3a_{t+1}, \quad \text{where} \quad \{a_t\} \sim \mathrm{WN}(0, 1).$$

It has been shown that $\{X_t\}$ is weakly stationary with (i). mean $\mu = 1$; (ii). ACVF $\gamma(k) = 13\mathbb{1}(k = 0) - 6\mathbb{1}(|k| = 2)$ and (iii). lag-2 PACF $\phi_{22} = 0.4615385$. We want to verify them through simulation.

Law of Large Number

Theorem 4. *Let $U_1, \ldots, U_n \overset{iid}{\sim} F$ for some distribution F with $\mathrm{E}(U_1^2) < \infty$, we have*

$$\bar{U}_n := \frac{1}{n} \sum_{i=1}^{n} U_i \overset{\text{a.s.}}{\to} \mathrm{E}(U_1).$$

The above theorem gives a theoretical guarantee and motivation for carrying out the Monte-Carlo simulation. We carry out the simulation step by step.

```
set.seed(4005) #Fix the seed so that you can replicated the same result
n = 1000000 #Set the sample size
a = rnorm(n+2) # Generate the white-noise sequence
X = 1 + 2*a[1:n] -3*a[3:(n+2)] # Generate X_t
round(var(X)*acf(X,plot=FALSE)$acf[1:5],3) # Calculate the ACVF
[1] 13.028  0.005 -6.018 -0.007  0.009
lm(X[3:n]~X[2:(n-1)]+X[1:(n-2)])$coefficient[3] # Calculate the PACF
[1] -0.4619412
```

The R-output is consistent with our theoretical solution. There is some commonly-used command in the R-code above:

1. `rnorm(n)`: generate n iid normal random variables.
2. `acf(x)`: report the sample acf of $\{X_t\}$.
3. `lm(Y~X+Z)`: report the details concerning the regression model $Y = \beta_0 + \beta_1 X + \beta_2 Z$.

❓ **Question 1.** *Is Monte-Carlo simulation always valid? Or is its result always reliable?*

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