STAT4005: Time Series
Midterm Examination
Solution
March 11, 2025

- If you have spotted any typo. Please inform Martin through martinmtt@link.cuhk.edu.hk.
- You are kindly invited to fill in the (anonymous) midterm evaluation form. (QR-code attached on the top-right corner)

Let $a_t \sim \text{WN}(0, \sigma^2)$.

1. **(20 marks)** Consider a time series $\{X_t\}_{t=1,2,\ldots}$ satisfying:

$$X_t = \sin(2\pi U t) \qquad U \sim \text{Unif}(0,1).$$

(a) Prove that $\{X_t\}$ is weakly stationary.
(Hint: $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\cos(A-B) = \cos A \cos B + \sin A \sin B$.)

(b) Prove that $\{X_t\}$ is not strictly stationary.
(Hint 1: For $0 < x \leq 2\pi$, $\sin(x) \leq 0$ if and only if $\pi \leq x \leq 2\pi$)
(Hint 2: Consider $P(X_1 \leq 0, X_3 \leq 0)$. It is also given that $P(X_2 \leq 0, X_4 \leq 0) = 0.25$)

Solution:

(a) First, notice that

$$\text{E}(X_t) = \int_0^1 \sin(2\pi u t)\,\mathrm{d}u = -\frac{1}{2\pi t}[\cos(2\pi u t)]\,|_{u=0}^{u=1} = 0. \qquad \text{(2 mark)}$$

Then, we compute

$$\text{Var}(X_t) = \text{E}(\sin^2(2\pi u t)) = \int_0^1 \frac{1-\cos(4\pi t u)}{2}\,\mathrm{d}u = \left[\frac{u}{2} - \frac{\sin(4\pi u t)}{8\pi t}\right] = \frac{1}{2}. \qquad \text{(3 mark)}$$

$$\begin{aligned}
\text{Cov}(X_t, X_{t+k}) = \mathbb{E}(X_t X_{t+k}) &= \int_0^1 \sin(2\pi u t)\sin(2\pi u(t+k))\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \cos(2\pi u k) - \cos[2\pi u(2t+k)]\,\mathrm{d}u \qquad \text{(2 mark)} \\
&= \frac{1}{2}\left(-\frac{\sin(2\pi u k)}{2\pi k} + \frac{\sin[2\pi u(2t+k)]}{2\pi(2t+k)}\right)\Bigg|_{u=0}^{u=1} \qquad \text{(2 mark)} \\
&= 0. \qquad \text{(1 mark)}
\end{aligned}$$

As $\text{E}(X_t)$ is constant and $\text{Cov}(X_t, X_{t+k})$ only depends on k, $\{X_t\}$ is weakly stationary. (1 mark)

(b) Notice that $X_1 \leq 0$ iff $\pi \leq 2\pi U \leq 2\pi$, i.e., $1/2 \leq U \leq 1$. (2 mark) Similarly, we have

$$X_3 = \sin(2\pi U \times 3) \leq 0 \qquad \Longleftrightarrow \qquad U \in [1/6, 1/3] \cup [1/2, 2/3] \cup [5/6, 1]. \qquad \text{(2 mark)}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(X_1 \leq 0, X_3 \leq 0) &= \mathbb{P}\left(U \in [1/2, 1] \cap \left([1/6, 1/3] \cup [1/2, 2/3] \cup [5/6, 1]\right)\right) \qquad \text{(2 mark)} \\
&= \mathbb{P}(U \in [1/2, 2/3] \cup [5/6, 1]) = \frac{1}{3}. \qquad \text{(2 mark)}
\end{aligned}$$

It follows that $\mathbb{P}(X_1 \leq 0, X_3 \leq 0) = 1/3 \neq 1/4 = \mathbb{P}(X_2 \leq 0, X_4 \leq 0)$, and hence $\{X_t\}$ is not strictly stationary. (1 mark)

Remark 1. *Alternative proof is also accepted as long as the argument is valid.*

- *(Part a) Many students forgot to check* $\mathrm{Var}(X_t)$*, but only computed* $\mathrm{Cov}(X_t, X_{t+k})$ *for* $k \neq 0$.
- *(Part b) It is incorrect to specify* U *as some fixed value and claim "*$(X_1, X_3) \neq (X_2, X_4)$ *and hence the process is not strictly stationary". It is because strict stationarity require the joint distribution being invariant to common time shift, but does not add any constraint on its particular realization. For example, consider* $X \sim \mathrm{N}(0,1)$*, then* $Y = -X \sim \mathrm{N}(0,1)$ *but of course* $X \neq Y$ *almost surely.*

- If only the final numeric answer is given, 1 mark each will be given.

2. **(20 marks)** Show that $Q_t = a_t^2 a_{t-3}^2$ is weakly stationary, where $a_t \overset{iid}{\sim} \mathrm{N}(0,1)$.
(Hint: $\mathbb{E}(a_t^2) = 1$ and $\mathbb{E}(a_t^4) = 3$)
Solution:

- (Mean) $\mathrm{E}(Q_t) = \mathrm{E}(a_t^2 a_{t-3}^2) = \mathrm{E}(a_t^2)\mathrm{E}(a_{t-3}^2) = 1 \times 1 = 1.$ (4 mark)
- (lag–0)

$$\mathrm{E}(Q_t^2) = \mathrm{E}(a_t^4 a_{t-3}^4) = \mathrm{E}(a_t^4)\mathrm{E}(a_{t-3}^4) = 3 \times 3 = 9.$$
$$\Rightarrow \mathrm{Var}(Q_t) = \mathrm{E}(Q_t^2) - \mathrm{E}(Q_t)^2 = 9 - 1 = 8. \qquad \text{(5 mark)}$$

- (lag–3)

$$\mathrm{E}(Q_t Q_{t+3}) = \mathrm{E}(a_t^2 a_{t-3}^2 a_{t+3}^2 a_t^2) = \mathrm{E}(a_t^4 a_{t-3}^2 a_{t+3}^2) = \mathrm{E}(a_t^4)\mathrm{E}(a_{t-3}^2)\mathrm{E}(a_{t+3})^2 = 3 \times 1 \times 1 = 3.$$
$$\Rightarrow \mathrm{Cov}(Q_t, Q_{t+3}) = \mathrm{E}(Q_t Q_{t+3}) - \mathrm{E}(Q_t)\mathrm{E}(Q_{t+3}) = 3 - 1 = 2. \qquad \text{(5 mark)}$$

- (lag–k; $k \notin \{0, 3, -3\}$) In this case, $t, t+k, t-3, t+k-3$ are all distinct integers and hence

$$\mathrm{E}(Q_t Q_{t+k}) = \mathrm{E}(a_t^2 a_{t-3}^2 a_{t+k}^2 a_{t+k-3}^2) = \mathrm{E}(a_t^2)\mathrm{E}(a_{t-3}^2)\mathrm{E}(a_{t+k}^2)\mathrm{E}(a_{t+k-3}^2) = 1^4 = 1.$$
$$\Rightarrow \mathrm{Cov}(Q_t, Q_{t+k}) = \mathrm{E}(Q_t Q_{t+k}) - \mathrm{E}(Q_t)\mathrm{E}(Q_{t+k}) = 1 - 1 = 0. \qquad \text{(5 mark)}$$

As $\mathrm{E}(X_t)$ is constant and $\mathrm{Cov}(X_t, X_{t+k})$ only depends on k, $\{X_t\}$ is weakly-stationary. (1 mark)

Remark 2. *In evaluation of lag–k ACVF, it is fine to claim as* $(t, t-3)$ *and* $(t+k, t+k-3)$ *would not coincide when* $k \notin \{0, 3, -3\}$*, and hence* $\mathrm{Cov}(Q_t, Q_{t+k}) = 0$ *without doing any algebra.*

- *Many students forgot the term* $-\mathrm{E}(Q_t)\mathrm{E}(Q_{t+k})$ *in evaluation of* $\mathrm{Cov}(X_t, X_{t+k})$.
- *Expectation is a linear operator with following properties*

$$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y), \quad \mathrm{E}(XY) \overset{\perp\!\!\!\perp}{=} \mathrm{E}(X)\mathrm{E}(Y), \quad \textit{but}\,\mathrm{E}(XY) \neq \mathrm{E}(X) + \mathrm{E}(Y) \ \textit{in general.}$$

- If nothing but only definition of Q_t is written, 1 mark will be given.
- If general lag is not calculated, but only lag-1,2 are calculated, 3 marks will be given.
- If $\mathrm{E}(Q_t)$ omitted in evaluating covariance, only deduct 1 mark (in total)

3. Identify the following as an appropriate SARIMA model, determine whether the time series is causal and/or invertible as well:

   (a) **(6 marks)** $X_t = 0.8X_{t-1} - 0.15X_{t-2} + a_t - 0.3a_{t-1}$.

   (b) **(6 marks)** $X_t = X_{t-1} - 0.5X_{t-2} + a_t - a_{t-1}$

   (c) **(6 marks)** $X_t - 0.3X_{t-1} - 0.4X_{t-3} + 0.12X_{t-4} = Z_t - 0.5Z_{t-3}$

Solution:

(a) The model could be written as

$$(1-0.5B)(1-0.3B)X_t = (1-0.3B)a_t \quad \Rightarrow \quad \boxed{(1-0.5B)X_t = a_t}. \qquad \text{(1 mark)}$$

- $\{X_t\} \sim \text{AR}(1)$; (1 mark)
- root of AR polynomial is 2, i.e., it lies out of the unit circle and hence causal. (2 mark)
- $a_t = Y_t - 0.5Y_{t-1}$ itself is a MA representation with $\psi_0 = 1$, $\psi_1 = -0.5$ and $\psi_k = 0$ for $k = 3$. Therefore, $\{X_t\}$ is invertible. (2 mark)

- if ans ARMA$(2, 0, 1)$: 0.5 mark (specification) + 2 mark (causal) + 1 mark (invertible)
- if answer causal or invertible without reasoning, only 1 mark would be given.

(b) The model could be written as

$$\boxed{(1-B+0.5B^2)X_t = (1-B)a_t}. \qquad \text{(1 mark)}$$

- $\{X_t\} \sim \text{ARMA}(2, 1)$. (1 mark)
- The roots of AR polynomial are $1 \pm i$. As its modulus $|1 \pm i| = \sqrt{1^2 + 1^2} > 1$, all roots lies out of the unit circle $\{X_t\}$ is causal. (2 mark)
- The root of MA polynomial is 1, i.e., it lies on unit circle and thus $\{X_t\}$ is not invertible. (2 mark)

(c) The model could be written as

$$\boxed{(1-0.4B^3)(1-0.3B)X_t = (1-0.5B^3)a_t}. \qquad \text{(1 mark)}$$

- $\{X_t\} \sim \text{SARIMA}(1, 0, 0) \times (1, 0, 1)_3$.(2 mark)
- The roots of AR polynomial are $10/3$ and $(5/2)^{1/3}$ (together with another two complex root that satisfies $|x|^3 = 5/2$). All of them lies out of the unit circle, and hence $\{X_t\}$ is causal. (1.5 mark)
- The root of MA polynomial is $2^{1/3}$ (together with another two complex root that satisfies $|x|^3 = 2$). All of them lies out of the unit circle, and hence $\{X_t\}$ is invertible. (1.5 mark)

- If the model is stated as ARIMA$(4, 0, 3)$, 1 mark will be deducted.
- If answer causal or invertible without reasoning, only 1 mark would be given
- If the period $s = 3$ is not specified, 1 mark will be deducted.

Remark 3.
- *(Part a) It is allowed to just state "It is an AR model and hence invertible" without specifying value of ψ_k. Remember to cancel common factor.*
- *(Part b) It is incorrect to claim that "$\phi(x)$ has no real root and hence non-causal". Our theory also support imaginary roots.*
- *(Part c) It is not necessary to notice the imaginary root in order to gain full credit.*
- *(General) To check causality, we solve $\phi(x) = 0$ but NOT $\phi(B) = 0$ since B is the backshifting operator but not a numeric object. No mark will be deducted this time.*

4. **(20 marks)**

$$(a_{-2}, a_{-1}, a_0, a_1, a_2) = (1/9)(-1, 4, 3, 4, -1)$$

Let $S_m(X_t)$ be the resulting time series after applying a filter with above weights to a time series $\{X_t\}$, show that:

$$\text{If } X_t = at + bt^2 + ct^3 + S_t, \text{ then } S_m(X_t) = at + bt^2 + ct^3,$$

where S_t is the seasonal part with period 3.

Solution: First, notice that

$$S_m(X_t) = S_m(T_t) + S_m(S_t), \qquad T_t = at + bt^2 + ct^3,$$

it suffices to show $S_m(T_t) = T_t$ and $S_m(S_t) = 0$. (2 mark)

- ($S_m(T_t) = T_t$): Upon checking, we have

$$\sum_{r=-2}^{2} a_r r^j = 0, \qquad \text{for } j = 1, 2, 3. \qquad (5 \text{ mark})$$

  Therefore, cubic trend can pass through the filter, i.e., $S_m(T_t) = T_t$. (2 mark)

- ($S_m(S_t) = 0$): By the characterization of seasonal effect, we have

$$S_t = S_{t+3}, \qquad \text{for all } t. \qquad (2 \text{ mark})$$

  Therefore,

$$\begin{aligned} S_m(S_t) = \sum_{r=-2}^{2} a_r S_{t+r} &= \frac{1}{9}\Big(- S_{t-2} + 4S_{t-1} + 3S_t + 4S_{t+1} - S_{t+2}\Big) \qquad (4 \text{ mark}) \\ &= \frac{1}{9}\Big(- S_{t+1} + 4S_{t+2} + 3S_t + 4S_{t+1} - S_{t+2}\Big) \qquad (3 \text{ mark}) \\ &= \frac{1}{3}(S_t + S_{t+1} + S_{t+2}) = 0. \qquad (2 \text{ mark}) \end{aligned}$$

  The last equality holds as by characterization of seasonal effect, $\sum_{t=1}^{3} S_t = 0$.

Remark 4. *Alternatively, you may directly work on $S_m(X_t)$ and handle the tedious algebra. In that case, the mark will be deducted severely if the steps are too jumpy.*

- If only state $S_m(X_t) = \sum_{r=-2}^{2} a_r X_{t+r}$, only 2 marks will be given.
- For checking $S_m(T_t) = T_t$, if the summation is verified only for $j = 3$ but not $j = 1, 2, 3$, only 2.5 mark is given.

5. Let c be a real constant. Consider the time series $\{Y_t\}$ satisfying

$$Y_t - 0.8Y_{t-1} + 0.15Y_{t-2} + c = a_t - 0.5a_{t-1}.$$

In this question, we further assume that $\{a_t\} \sim \mathrm{WN}(0,1)$, also we have $\mathrm{E}(Y_t) = 2$.

(a) **(4 marks)** Find c.

(b) **(20 marks)** Find the autocovariance function of $\{Y_t\}$ up to lag 4.

(c) **(8 marks)** Find $\mathrm{E}(Y_1 + Y_2 + Y_3 + Y_4)^2$.

Solution:

(a) Taking expectation to both side of the equation yield

$$0.35\mathrm{E}(Y_t) + c = \mathrm{E}(Y_t) - 0.8\mathrm{E}(Y_{t-1}) + 0.15\mathrm{E}(Y_{t-2}) + c = 0. \qquad \text{(3 mark)}$$

Therefore, $c = -7/10$. (1 mark)

(b) First, we simplify the model as

$$X_t = 0.8X_{t-1} - 0.15X_{t-2} + a_t - 0.5a_{t-1},$$

where $X_t = Y_t - 2$. It is clear that $\gamma_X = \gamma_Y$. (remark: this step is not necessary)

- (Causality) The AR polynomial has roots 10/3 and 2 and thus $\{X_t\}$ is causal. (2 mark)
- (Preliminary 1)

$$\mathrm{Cov}(X_t, a_t) = \mathrm{Cov}(0.8X_{t-1} - 0.15X_{t-2} + a_t - 0.5a_{t-1}, a_t) = \mathrm{Cov}(a_t, a_t) = 1. \qquad \text{(2 mark)}$$

- (Preliminary 2)

$$\begin{aligned}\mathrm{Cov}(X_t, a_{t-1}) &= \mathrm{Cov}(0.8X_{t-1} - 0.15X_{t-2} + a_t - 0.5a_{t-1}, a_{t-1}) \\ &= 0.8\mathrm{Cov}(X_{t-1}, a_{t-1}) - 0.5\mathrm{Cov}(a_{t-1}, a_{t-1}) \\ &= 0.8 \times 1 - 0.5 \times 1 = 0.3. \qquad \text{(3 mark)}\end{aligned}$$

- (YW1) Taking covariance with X_t yields

$$\begin{aligned}\gamma(0) &= 0.8\gamma(1) - 0.15\gamma(2) + 1 - 0.5 \times 0.3 \\ &= 0.8\gamma(1) - 0.15\gamma(2) + 0.85. \qquad \text{(2 mark)}\end{aligned}$$

- (YW2) Taking covariance with X_{t-1} yields

$$\gamma(1) = 0.8\gamma(0) - 0.15\gamma(1) - 0.5. \qquad \text{(2 mark)}$$

- (YW3) Taking covariance with X_{t-2} yields

$$\gamma(2) = 0.8\gamma(1) - 0.15\gamma(0). \qquad \text{(2 mark)}$$

Solving (YW1), (YW2) and (YW3) yield

$$\gamma(0) = \frac{100}{91} \approx 1.0989, \quad \gamma(1) = \frac{30}{91} \approx 0.3297, \quad \text{and} \quad \gamma(2) = \frac{9}{91} \approx 0.0989. \qquad \text{(3 mark)}$$

Taking covariance with X_{t-k} $(k \geq 3)$ yields

$$\gamma(k) = 0.8\gamma(k-1) - 0.15\gamma(k-2). \qquad \text{(2 mark)}$$

Hence,

$$\begin{aligned}\gamma(3) &= 0.8\gamma(2) - 0.15\gamma(1) = \frac{27}{910} \approx 0.02967. \qquad \text{(1 mark)} \\ \gamma(4) &= 0.8\gamma(3) - 0.15\gamma(2) = \frac{81}{9100} \approx 0.008901. \qquad \text{(1 mark)}\end{aligned}$$

If (YW1) and (YW2) only missed those $\mathrm{E}(a_t X_{t-k})$ terms, no mark deducted (already deducted in preliminary)

(c) **(Method 1)** Noticing that $\{\mathrm{E}(Y_1+Y_2+Y_3+Y_4)\}^2 = (4\times 2)^2 = 64$ (2 mark) and

$$\begin{aligned}\mathrm{Var}(Y_1+Y_2+Y_3+Y_4) &= \mathrm{Cov}\left(\sum_{i=1}^{4} Y_i, \sum_{j=1}^{4} Y_j\right) \qquad \text{(1 mark)}\\ &= 4\gamma(0)+2\times 3\gamma(1)+2\times 2\gamma(2)+2\gamma(3) \qquad \text{(2 mark)}\\ &= \frac{239}{35} \approx 6.8286. \qquad \text{(1 mark)}\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathrm{E}(Y_1+Y_2+Y_3+Y_4)^2 &= \mathrm{Var}(Y_1+Y_2+Y_3+Y_4)+\{\mathrm{E}(Y_1+Y_2+Y_3+Y_4)\}^2 \qquad \text{(1 mark)}\\ &= \frac{2479}{35} \approx 70.8286. \qquad \text{(1 mark)}\end{aligned}$$

(Method 2) Noticing that

$$\begin{aligned}&\{\mathrm{E}(Y_1+Y_2+Y_3+Y_4)\}^2\\ =&\mathrm{E}(Y_1^2+Y_2^2+Y_3^2+Y_4^2)+2\mathrm{E}(Y_1Y_2+Y_1Y_3+Y_1Y_4+Y_2Y_3+Y_2Y_4+Y_3Y_4), \qquad \text{(1 mark)}\end{aligned}$$

where

$$\begin{aligned}\mathrm{E}(Y_1^2+Y_2^2+Y_3^2+Y_4^2) &= 4\{\gamma(0)+\mathrm{E}(Y_1)^2\} \qquad \text{(2 mark)}\\ &= 4\left(\frac{464}{91}\right) = \frac{1856}{91} \approx 20.3956, \qquad \text{(1 mark)}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{E}(Y_1Y_2+Y_1Y_3+Y_1Y_4+Y_2Y_3+Y_2Y_4+Y_3Y_4)\\ =&3\gamma(1)+2\gamma(2)+\gamma(3)+6\mathrm{E}(Y_1)^2 \qquad \text{(2 mark)}\\ =&\frac{22947}{910} \approx 25.2165. \qquad \text{(1 mark)}\end{aligned}$$

Therefore,

$$\mathrm{E}(Y_1^2+Y_2^2+Y_3^2+Y_4^2) = \frac{1856}{91}+2\left(\frac{22947}{910}\right) = \frac{2479}{35}. \qquad \text{(1 mark)}$$

- If $\mathrm{E}(Y_1)^2$ is omitted in the calculation, 2 marks will be deducted.

Remark 5.
- *(Part b) It is not necessary to introduce the intermediate process $\{X_t\}$. It is introduced to make the steps look clearer.*
- *(Part b) It is possible to cancel the common factor and simplify the model to be $(1-0.3B)X_t = a_t$ and find the ACVF by*
  - *Finding the MA representation and use the typical formula; or*
  - *Yule-walker equation.*
- *(Part c) Accept alternative approach.*

In all questions, unless the intermediate step is important, i.e., it shows whether you understand the concept clearly, you will still get its credit if your next step is correct. For example, in Q5c, you can directly write

$$\mathrm{Var}(Y_1+Y_2+Y_3+Y_4) = 4\gamma(0)+2\times 3\gamma(1)+2\times 2\gamma(2)+2\gamma(3)$$

and 3 mark will be given without stating $\mathrm{Var}(Y_1+Y_2+Y_3+Y_4) = \mathrm{Cov}\left(\sum_{i=1}^4 Y_i, \sum_{j=1}^4 Y_j\right)$. The purpose of introducing those relatively trivial intermediate steps is to ensure that the student can gain partial credit even if the final answer is incorrect.

Some important intermediate step could not be skipped. For example, in Q5(b), you have to show that the process is causal to ensure the validity of the evaluation of $\mathrm{Cov}(X_t, a_t)$ and $\mathrm{Cov}(X_t, a_{t-1})$.