

STAT4005 - Time Series
Tutorial 3 - Parametric Time Series Model
Ma Ting Tin, Martin
Week 5 & 6

3 Time Series Models

Recall that B is the backshifting operator in the sense that $B^k X_t = X_{t-k}$ for all $k \in \mathbb{N}$.

3.1 Identification of Models

Common Linear Time Series Model

Definition 1. *Let $\{Y_t\}$ be a time-series and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Then we say*

1. **(Autoregressive Model)** $\{Y_t\} \sim \mathrm{AR}(\boldsymbol{p})$ *if* $\phi(B)Y_t = Z_t$, *i.e.*

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + Z_t,$$

where $\phi(x) := 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$ *is the* ***AR characteristic polynomial***.

2. **(Moving Average Model)** $\{Y_t\} \sim \mathrm{MA}(\boldsymbol{q})$ *if* $Y_t = \theta(B)Z_t$, *i.e.*

$$Y_t = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q},$$

where $\theta(x) := 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$ *is the* ***MA characteristic polynomial***.

3. **(ARMA)** $\{Y_t\} \sim \mathrm{ARMA}(\boldsymbol{p}, \boldsymbol{q})$ *if* $\phi(B)Y_t = \theta(B)Z_t$, *i.e.*

$$Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q},$$

where (i). $\phi(1) \neq 0$, (ii). $\phi(\cdot)$ and $\theta(\cdot)$ have NO common root.

4. **(ARIMA)** $\{Y_t\} \sim \mathrm{ARIMA}(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{q})$ *if*

$$\phi(B)(1-B)^d Y_t = \theta(B)Z_t,$$

where (i). $\phi(1) \neq 0$, (ii). $\phi(x)(1-x)^d$ and $\theta(x)$ have NO common root.

5. **(SARIMA)** $\{Y_t\} \sim \mathrm{SARIMA}(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{q}) \times (\boldsymbol{P}, \boldsymbol{D}, \boldsymbol{Q})_{\boldsymbol{s}}$ *if*

$$\phi(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D Y_t = \theta(B)\Theta_Q(B^s)Z_t,$$

where $s > 1$,

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \cdots - \Phi_P B^{sP} \quad \text{and} \quad \Theta_Q(B^s) = 1 - \Theta_1 B^s - \cdots -_P B^{sQ},$$

(i). $\phi(1)$ and $\Phi_P(1)$ are non-zero, (ii). $\phi(x)\Phi_P(x^s)(1-x)^d(1-x^s)^D$ and $\theta(x)\Theta_Q(x^s)$ have NO common root.

Remark 1. *There are several remarks on the identification of models*

- *If there is common factor(s) in $\phi(\cdot)$ and $\theta(\cdot)$ [or in $\Phi_P(\cdot)$ and $\theta_Q(\cdot)$], i.e. say $\phi(x) = (1-c)\phi'(x)$ and $\theta(x) = (1-c)\theta'(x)$, we should cancel them out and consider the $\phi'(\cdot)$ and $\theta'(\cdot)$ as the true AR and MA characteristic polynomial instead.*
- SARIMA *model can be expressed as high-order* ARIMA *model. However, you are required to write a more informative representation for the assessment. See part b of exercise 1.*

The following flowchart summarizes the relationship among the models mentioned above.

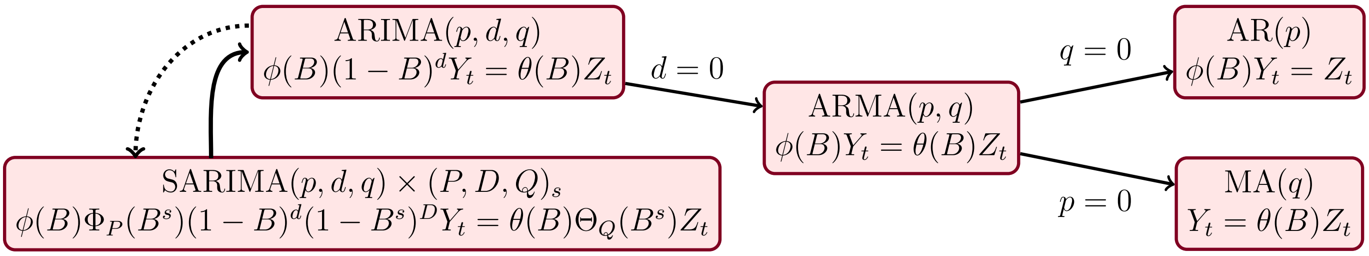

Figure 1: Relationship among models

General Guidance in Identification of Models

For the SARIMA model, we have to notice if there is a common pattern lag among $Y_t's$ or $Z_t's$ (eg: $Y_t - 4Y_{t-3} = Z_t + 2Z_{t-3} + 0.5Z_{t-6}$) within the model. If such a lag is not observed, it should be an ARIMA instead. The general strategy for identifying an ARIMA model is as follows:

1. Rearrange the time series model such that it could be written in the form of $\phi_0(B)Y_t = \theta_0(B)Z_t$.
2. **(Identifying MA polynomial $\boldsymbol{\theta}$)** Cancel out common factors of $\phi_0(x)$ and $\theta_0(x)$ (if any) to obtain $\phi_1(x)$ and $\theta(x)$. Then we have $\phi_1(B)Y_t = \theta(B)Z_t$.
3. **(Identifying AR polynomial $\boldsymbol{\phi}$)** Write $\phi_1(x) = (1-x)^d\phi(x)$, where $\phi(1)\neq 0$.

Then the time series $\{Y_t\}\sim$ ARIMA(p,d,q), where $p=\deg(\phi)$ and $q=\deg(\theta)$.

(★★☆) Identification of Models

✎ **Exercise 1.** *Identify the time-series model below. Let $\{Y_t\}$ be a time-series and $\{Z_t\}\sim \mathrm{WN}(0,\sigma^2)$.*

(a) $Y_t - 4Y_{t-1} + 5Y_{t-2} - 2Y_{t-3} = Z_t - \frac{8}{3}Z_{t-1} + \frac{4}{3}Z_{t-2}$.

(b) $Y_t - Y_{t-1} + 0.5Y_{t-4} - 0.5Y_{t-5} = Z_t + 0.25Z_{t-4} + 0.6Z_{t-8}$

Solution

(a) (i) The model could be rewritten as $(1-2B)(1-B)^2Y_t = (1-2B)(1-\frac{2}{3}B)Z_t$.

(ii) Cancelling out the common factor gives $(1-B)^2Y_t = (1-\frac{2}{3}B)Z_t$.

(iii) As $\phi(x)=1$ and $\theta(x) = 1-2x/3$, $\{Y_t\}\sim$ ARIMA$(0,2,1)$.

(b) The model could be written as

$$\underbrace{(1-B)}_{d=1}\underbrace{\overbrace{(1+0.5B^4)}^{\Phi_1(B^4)}}_{s=4}Y_t = \underbrace{\overbrace{(1+0.25B^4+0.6B^8)}^{\Theta_2(B^4)}}_{s=4}Z_t$$

Hence we have $\{Y_t\}\sim$ SARIMA$(0,1,0)\times(1,0,2)_4$, where $\Phi_1(x) = 1+0.5x$ and $\Theta_2(x) = 1+0.25x+0.6x^2$ with $s=4$.

❗ **Remark 2.** *We can also say $\{Y_t\}\sim$ ARIMA$(4,1,8)$. However, $\{Y_t\}\sim$ SARIMA$(0,1,0)\times(1,0,2)_4 \Rightarrow \{Y_t\}\sim$ ARIMA$(4,1,8)$ but the converse does not hold, so $\{Y_t\}\sim$ SARIMA$(0,1,0)\times(1,0,2)_4$ would be an more appropriate and informative answer.*

3.2 Causality and Invertibility of Models

Causality and Invertibility

Definition 2. *Let $\{Y_t\}$ be a time-series and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Then we say*

1. *A model is **causal** if there exists a sequence $\{\psi_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} |\psi_k| < \infty$ with*

$$Y_t = \sum_{k=0}^{\infty} \psi_k Z_{t-k},$$

2. *A model is **invertible** if there exists a sequence $\{\psi_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty} |\psi_k| < \infty$ with*

$$Z_t = \sum_{k=0}^{\infty} \psi_k Y_{t-k},$$

Verification for Causality and Invertibility

Theorem 1. *An $\mathrm{ARMA}(p, q)$ process $\phi(B)Y_t = \theta(B)Z_t$ is*

1. ***Causal** if and only if roots of $\phi(x) = 0$ are ALL outside the unit circle.*
2. ***Invertible** if and only if roots of $\theta(x) = 0$ are ALL outside the unit circle.*

where the term "x out of the unit circle" means that $|x| > 1$, and for all $x \in \mathbb{C}$, $|x|^2 := \mathrm{Re}(x)^2 + \mathrm{Im}(x)^2$.

(★☆☆) Causality and Invertibility of Models

✎ **Exercise 2.** *Let $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Show if the following time/series are causal or invertible.*

(a) $Y_t + 3Y_{t-1} = 2Z_t - Z_{t-1}$.

(b) $1.5Y_t - Y_{t-1} = 2Z_t + 3Z_{t-1}$.

(c) $Y_t = 4Y_{t-1} - 4Y_{t-2} + Z_t - 5Z_{t-1} + 6Z_{t-2}$.

Solution

(a) Rewrite the model as $(1 + 3B)Y_t = (2 - B)Z_t$. Then

- AR polynomial: $\phi(x) = 1 + 3x$ have root $-1/3$ within unit circle, hence not causal.
- MA polynomial: $\theta(x) = 1 + 3x$ have root 2 outside unit circle, hence invertible.

(b) Rewrite the model as $(1.5 - B)Y_t = (2 + 3B)Z_t$. Then

- AR polynomial: $\phi(x) = 1.5 - x$ have root 1.5 outside unit circle, hence causal.
- MA polynomial: $\theta(x) = 2 + 3x$ have root $-2/3$ within unit circle, hence not invertible.

(c) Rewrite the model as $(1-2B)(1-2B)Y_t = (1-2B)(1-3B)Z_t$ and simplify it to $(1-2B)Y_t = (1 - 3B)Z_t$. Then

- AR polynomial: $\phi(x) = 1 - 2x$ have root $1/2$ within unit circle, hence not causal.
- MA polynomial: $\theta(x) = 1 - 3x$ have root $1/3$ within unit circle, hence not invertible.

Consider the ARMA(p, q) process $\phi(B)Y_t = \theta(B)Z_t$. You might be asked to show whether a model is causal/invertible and write down the corresponding representation.

1. If a model is causal, then we can write the **MA/causal representation** as $Y_t = \phi(B)^{-1}\theta(B)Z_t$.
2. If a model is invertible, then we can write the **AR representation** as $Z_t = \theta(B)^{-1}\phi(B)Y_t$.

In order to evaluate $\phi(B)^{-1}$ or $\theta(B)^{-1}$ explicitly, recall for $|r| < 1$,

$$\frac{a}{1-r} = a\sum_{k=0}^{\infty} r^k$$

and by the fundamental theorem of algebra, we can write any degree-p polynomial $P(x)$ as $P(x) = c\prod_{k=1}^{p}(x - \xi_k)$ for some $c, \xi_1, \ldots, \xi_p \in \mathbb{C}$. Sometimes, the method of partial fraction can help further simplify the calculation. For example, see exercise 8.

(★★☆) AR and MA representation

Exercise 3. ***(Continuation of Exercise 2)*** *Let* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. *Find the AR and MA representation (if possible) of the following model*

(a) $Y_t + 3Y_{t-1} = 2Z_t - Z_{t-1}$.

(b) $1.5Y_t - Y_{t-1} = 2Z_t + 3Z_{t-1}$.

Solution

(a) This model is only invertible, where the AR representation is given by

$$\begin{aligned}
Z_t &= (2-B)^{-1}(1+3B)Y_t = 0.5\left(\sum_{k=0}^{\infty} 0.5^k B^k\right)(1+3B)Y_t \\
&= 0.5\left(\sum_{k=0}^{\infty} 0.5^k B^k + 3\sum_{k=0}^{\infty} 0.5^k B^{k+1}\right)Y_t = \left[0.5 + 0.5\left(\sum_{k=1}^{\infty} 0.5^k B^k + 3\sum_{k=1}^{\infty} 0.5^{k-1} B^k\right)\right]Y_t \\
&= \left[0.5 + \left(\sum_{k=1}^{\infty}[0.5^{k+1} + 3(0.5)^k]B^k\right)\right]Y_t = \left[0.5 + \sum_{k=1}^{\infty} 3.5(0.5)^k B^k\right]Y_t
\end{aligned}$$

Hence $Z_t = \sum_{k=0}^{\infty}\psi_k Y_{t-k}$, where $\psi_0 = 0.5$ and $\psi_k = 3.5(0.5)^k$ for $k \geq 1$.

(b) This model is only causal, where the MA representation is given by

$$\begin{aligned}
Y_t &= (1.5-B)^{-1}(2+3B)Y_t = \frac{2}{3}\left(\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k B^k\right)(2+3B)Z_t \\
&= \frac{2}{3}\left(\sum_{k=0}^{\infty} 2\left(\frac{2}{3}\right)^k B^k + \sum_{k=0}^{\infty} 3\left(\frac{2}{3}\right)^k B^{k+1}\right)Z_t \\
&= \frac{2}{3}\left[\frac{4}{3} + \left(\sum_{k=1}^{\infty}\left[2\left(\frac{2}{3}\right)^k + 3\left(\frac{2}{3}\right)^{k-1}\right]B^k\right)\right]Z_t = \left[\frac{8}{9} + \sum_{k=1}^{\infty}\frac{13}{3}\left(\frac{2}{3}\right)^k B^k\right]Z_t
\end{aligned}$$

Hence $Y_t = \sum_{k=0}^{\infty}\psi_k Z_{t-k}$, where $\psi_0 = 8/9$ and $\psi_k = (13/3)(2/3)^k$ for $k \geq 1$.

Takeaway 1. *For a sequence* $\{a_n\}$, $\sum_{k=L}^{U} a_k = \sum_{k=L-s}^{U-s} a_{k+s}$.

3.3 Stationarity of Models

In this subsection, we will answer the following questions

1. How to check the stationarity of a given time-series model?
2. If a time-series is stationary, how to find its ACVF $\gamma(\cdot)$

We then answer the first question.

Stationarity of Models

Theorem 2. *Consider a time series $\{Y_t\}$.*

1. *If $\{Y_t\} \sim \text{ARIMA}(p, d, q)$ with $d \geq 1$, then $\{Y_t\}$ is NOT weakly-stationary.*
2. *If $\{Y_t\} \sim \text{ARMA}(p, q)$, then $\{Y_t\}$ is weakly-stationary if NO root of $\phi(x) = 0$ is **on** the unit circle, i.e. if $\phi(x) = 0$, $|x| \neq 1$.*

❶ **Remark 3.** *There are several implications due to the theorem above.*

- $\{Y_t\} \sim \text{MA}(q) \Rightarrow$ *Causal* $\Rightarrow$ *Weakly-Stationary.*
- *For $\{Y_t\} \sim \text{ARMA}(p, q)$ being stationary,* $\text{E}(Y_t) = 0$.

(★☆☆) Verification of Stationarity

✎ **Exercise 4.** *Show whether the following time-series is weakly-stationary.*

(a) $Y_t + Y_{t-2} = 3Z_t - Z_{t-1}$.

(b) $Y_t - Y_{t-1} = Z_t - 1.5Z_{t-1}$.

(c) $Y_t = 0.5Y_{t-1} + 0.5Y_{t-2} + Z_t - 1.2Z_{t-1} + 0.2Z_{t-2}$.

Solution

(a) We have $\phi(x) = 1 + x^2$, i.e. AR roots are i and $-i$, while $|i| = 1$ lies on the unit circle. Hence $\{Y_t\}$ is NOT stationary.

(b) As $\{Y_t\} \sim \text{ARIMA}(0, 1, 1)$, $\{Y_t\}$ is NOT stationary. (it is wrong to write $\phi(x) = 1 - B$!).

(c) The model is equivalent to $(1 + 0.5B)(1 - B)Y_t = (1 - 0.2B)(1 - B)Z_t$, simplifying the common factor gives $(1 + 0.5B)Y_t = (1 - 0.2B)Z_t$. Then we have $\phi(x) = 1 + 0.5x$ with AR root 2 outside the circle, hence $\{Y_t\}$ is weakly-stationary.

We then discuss the evaluation of ACVF $\gamma(\cdot)$. As the ARIMA(p,d,q) model with $d \geq 1$ is always non-stationary, we restrict our focus of study to be the ARMA(p,q) model. There are two common ways to complete the task. The first approach requires finding the MA representation of a time-series.

Evaluation of ACVF - Method (I) - MA representation

Theorem 3. *For* ***causal*** ARMA *process* $\phi(B)Y_t = \theta(B)Z_t$, *we can write the MA representation* $Y_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$ *with* $\sum_{i=0}^{\infty} |\psi_j| < \infty$ *and ACVF*

$$\gamma(k) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}.$$

❗ **Remark 4.** *For* ***stationary*** ARMA *process (NOT necessarily causal), we can represent* Y_t *by* $Y_t = \sum_{i=-\infty}^{\infty} \psi_i Z_{t-i}$ *with* $\sum_{i=-\infty}^{\infty} |\psi_j| < \infty$ *and ACVF* $\gamma(k) = \sigma^2 \sum_{i=-\infty}^{\infty} \psi_i \psi_{i+k}$.

(★☆☆) Application of Method I

✎ **Exercise 5.** *Consider the process* $Y_t = 0.8Y_{t-1} + Z_t$, *where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) Is the process $\{Y_t\}$ *weakly stationary?*

(b) Find the mean and ACVF $\gamma(\cdot)$ *of* $\{Y_t\}$ *by Theorem 3.*

Solution

(a) Rewrite the model as $(1 - 0.8B)Y_t = Z_t$, as the AR root 1.25 lies out of the unit circle, it is causal and also stationary.

(b) We first find the MA representation of the model.

$$Y_t = (1 - 0.8B)^{-1} Z_t = \sum_{i=0}^{\infty} 0.8^i B^i Z_t = \sum_{i=0}^{\infty} 0.8^i Z_{t-i}.$$

Hence $Y_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$, where $\psi_i = 0.8^i$ for all $i \in \mathbb{N}_0$. As $\sum_{i=0}^{\infty} |\psi_i| < \infty$, we have

$$\mathrm{E}(Y_t) = \mathrm{E}\left(\sum_{i=0}^{\infty} 0.8^i Z_{t-i}\right) = \sum_{i=0}^{\infty} 0.8^i \mathrm{E}(Z_{t-i}) = 0$$

Alternatively, you can obtain the same result by using the second bullet of remark 3. Then by Theorem 3, the ACVF is given by

$$\gamma(k) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} = \sigma^2 \sum_{i=0}^{\infty} (0.8)^i (0.8)^{i+k} = \sigma^2 (0.8)^k \sum_{i=0}^{\infty} (0.8)^{2i} = \sigma^2 \frac{0.8^k}{1 - 0.8^2} = \frac{25}{9}\left(\frac{4}{5}\right)^k \sigma^2.$$

for all $k \in \mathbb{N}$, i.e.,

$$\gamma(k) = \sigma^2 \frac{0.8^{|k|}}{1 - 0.8^2} = \frac{25}{9}\left(\frac{4}{5}\right)^{|k|} \sigma^2.$$

Evaluation of ACVF - Method (II) - Yule-Walker Equations

Theorem 4. *The general procedure is as follow: For* ***stationary*** $\mathrm{ARMA}(p, q)$ *model*

1. *Consider the set of linear equations (i.e. the Yule-Walker Equations)*

$$\gamma(k) = \mathrm{Cov}(Y_t, Y_{t-k}), \quad \textit{for} \quad k = 0, \ldots, p$$

to solve for value of $\gamma(0), \ldots, \gamma(p)$.

2. *For* $k > p$, *compute* $\gamma(k) = \mathrm{Cov}(Y_t, Y_{t-k})$ *based on value of* $\gamma(0), \ldots, \gamma(p)$.

Lemma 1. *The following are the Yule-Walker Equations of some particular models.*

1. *For* $\{Y_t\} \sim \mathrm{AR}(p)$ *being causal with* $Z_t \sim \mathrm{WN}(0, \sigma^2)$, *the Yule-Walker Equations are*

$$\begin{cases} \gamma(0) &= \phi_1\gamma(1) + \phi_2\gamma(2) + \cdots + \phi_p\gamma(p) + \sigma^2 \\ \gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) + \cdots + \phi_p\gamma(p-1) \\ \gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \cdots + \phi_p\gamma(p-2) \\ \quad\vdots &= \quad\vdots \\ \gamma(p) &= \phi_1\gamma(p-1) + \phi_2\gamma(p-2) + \cdots + \phi_p\gamma(0). \end{cases}$$

and for $k > p$, $\gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p)$.

2. *For* $\{Y_t\} \sim \mathrm{ARMA}(p, q)$, *i.e.* $\phi(B)Y_t = \theta(B)Z_t$ *with* $Z_t \sim \mathrm{WN}(0, \sigma^2)$, *the Yule-Walker Equations are*

$$\begin{cases} \gamma(0) &= \phi_1\gamma(1) + \phi_2\gamma(2) + \cdots + \phi_p\gamma(p) + \mathrm{Cov}(Y_t, \theta(B)Z_t) \\ \gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) + \cdots + \phi_p\gamma(p-1) + \mathrm{Cov}(Y_{t-1}, \theta(B)Z_t) \\ \gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \cdots + \phi_p\gamma(p-2) + \mathrm{Cov}(Y_{t-2}, \theta(B)Z_t) \\ \quad\vdots &= \quad\vdots \\ \gamma(p) &= \phi_1\gamma(p-1) + \phi_2\gamma(p-2) + \cdots + \phi_p\gamma(0) + \mathrm{Cov}(Y_{t-p}, \theta(B)Z_t). \end{cases}$$

and for $k > p$, $\gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p)$.

Remark 5. *The causality assumption is important for deriving the Yule-Walker equations. Recall that causality is equivalent to require* $Y_t \perp\!\!\!\perp Z_{t+k}$ *for all* $k > 0$.

- *For the first equation in the Yule-Walker equation of causal* $\mathrm{AR}(p)$ *model, the term* σ^2 *arises as*
$$\mathrm{Cov}(Z_t, Y_t) = \mathrm{Cov}(Z_t, \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + Z_t) = \mathrm{Cov}(Z_t, Z_t) = \sigma^2.$$
However for non-causal model, $\mathrm{Cov}(Y_{t-k}, Z_t) \neq 0$ *in general for* $k \geq 1$.

- *The Yule-Walker equation derived for* $\mathrm{ARMA}(p, q)$ *does not require causality in general. However, the evaluation of the term* $\mathrm{Cov}(Y_{t-k}, \theta(B)Z_t)$ *would be complicated.*

Hence, please ensure whether the model is causal before applying the method of Yule-Walker equation.

(★★☆) Application of Method II

✎ **Exercise 6.** ***(Continuation of Exercise 5)*** *Consider the process $Y_t = 0.8Y_{t-1} + Z_t$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Find the mean and ACVF $\gamma(\cdot)$ of $\{Y_t\}$ by Theorem 4.*

Solution

From Exercise 5, we know $\{Y_t\}$ is stationary. Hence

$$\mathrm{E}(Y_t) = 0.8\mathrm{E}(Y_{t-1}) + \mathrm{E}(Z_t) = 0.8\mathrm{E}(Y_t) \quad \Rightarrow \quad \mathrm{E}(Y_t) = 0.$$

Noticing that the model is indeed causal AR(1) model. Then consider the Yule-Walker equation:

$$\begin{aligned}\gamma(0) &= \mathrm{Cov}(Y_t, Y_t) = \mathrm{Cov}(0.8Y_{t-1} + Z_t, Y_t) = 0.8\gamma(1) + \sigma^2 \\ \gamma(1) &= \mathrm{Cov}(Y_t, Y_{t-1}) = \mathrm{Cov}(0.8Y_{t-1} + Z_t, Y_{t-1}) = 0.8\gamma(0)\end{aligned}$$

On solving, we have $\gamma(0) = (9/25)\sigma^2$ and $\gamma(1) = (4/5)\gamma(0) = (36/125)\sigma^2$. For $k > 1$, we have $\gamma(k) = 0.8\gamma(k-1)$. Hence in general, we have

$$\gamma(k) = \frac{25}{9}\left(\frac{4}{5}\right)^k \sigma^2.$$

for all $k \in \mathbb{N}$, which is consistent with the solution in Exercise 5.

3.4 Chapter Summary

In this subsection, we make a brief summary to this chapter

Model	**Causality**	**Invertibility**	**Stationarity**	**ACVF**
SARIMA	✘	$\lvert$MA root$\rvert > 1$	✘	\
ARIMA	✘	$\lvert$MA root$\rvert > 1$	✘	\
ARMA	$\lvert$AR root$\rvert > 1$	$\lvert$MA root$\rvert > 1$	$\lvert$AR root$\rvert \neq 1$	Both OK
AR	$\lvert$AR root$\rvert > 1$	✔	$\lvert$AR root$\rvert \neq 1$	Yule-Walker
MA	YES	$\lvert$MA root$\rvert > 1$	✔	MA-representation

3.5 Additional Exercises

(★★☆) ACF of AR(2) Model

✎ **Exercise 7.** *Consider causal AR(2) model* $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t$*, where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) Find $\rho(1)$ *and* $\rho(2)$ *in terms of* ϕ_1 *and*ϕ_2.

(b) Express $\gamma(0)$ *in terms of* ϕ_1, ϕ_2 *and* σ^2.

Solution

(a) Recall by causality assumption, $\mathrm{Cov}(Y_t, Z_{t+k}) = 0$ for all $k \geq 1$. Hence

- $\mathrm{Cov}(Z_t, Y_t) = \mathrm{Cov}(Z_t, \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t) = \mathrm{Cov}(Z_t, Z_t) = \sigma^2$.
- $\mathrm{Cov}(Z_t, Y_{t-k}) = 0$

Then we consider taking the following operation to both side of the AR(2) model:

- $\mathrm{Cov}(\cdot, Y_{t-1}) : \gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1) + \mathrm{Cov}(Z_t, Y_{t-1}) = \phi_1\gamma(0) + \phi_2\gamma(1)$
- $\mathrm{Cov}(\cdot, Y_{t-2}) : \gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0) + \mathrm{Cov}(Z_t, Y_{t-2}) = \phi_1\gamma(1) + \phi_2\gamma(0)$

Divide both sides by $\gamma(0)$ gives

$$\rho(1) = \phi_1 + \phi_2\rho(1) \quad \Rightarrow \quad \rho(1) = \frac{\phi_1}{1-\phi_2}$$

and

$$\rho(2) = \phi_1\rho(1) + \phi_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2$$

(b) Consider $\gamma(k) = \rho(k)\gamma(0)$, taking covariance with Y_t to the AR(2) model gives

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma^2 = \phi_1\rho(1)\gamma(0) + \phi_2\rho(2)\gamma(0) + \sigma^2$$

It follows that

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1\rho(1) - \phi_2\rho(2)} = \frac{(1-\phi_2)\sigma^2}{1 - \phi_2 - \phi_1^2 - \phi_2^2 - \phi_1^2\phi_2 + \phi_2^3}$$

(★★★) Comprehensive Exercise

✎ **Exercise 8.** *Consider* $Y_t - 3.5Y_{t-1} + 3Y_{t-2} = 2Z_t - Z_{t-1}$, *where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) *Is the model causal or invertible? If yes, find its corresponding MA and AR representation.*

(b) *Is the model stationary?*

(c) *Write* $Y_t = \sum_{i=-\infty}^{\infty} \psi_i Z_{t-i}$. *Find the ACVF of* $\gamma(\cdot)$. *(Optional!)*

Solution

(a) Rewrite the model as $(1-1.5B)(1-2B)Y_t = (2-B)Z_t$. Then

- $\phi(x) = (1-1.5x)(1-2x)$ have AR roots $2/3$ and $1/2$, which are inside the unit circle. Hence NOT causal.
- $\theta(x) = 2 - x$ have MA root 2, which is outside the unit circle . Hence invertible.

The AR representation is given by

$$\begin{aligned}
Z_t &= 0.5\left(\sum_{k=0}^{\infty}(0.5)^k B^k\right)(1-3.5B+3B^2)Y_t \\
&= 0.5\left(\sum_{k=0}^{\infty}(0.5)^k B^k - \sum_{k=0}^{\infty}3.5(0.5)^k B^{k+1} + \sum_{k=0}^{\infty}3(0.5)^k B^{k+2}\right)Y_t \\
&= 0.5\left(\left[1+0.5B+\sum_{k=2}^{\infty}(0.5)^k B^k\right] - \left[3.5B+\sum_{k=1}^{\infty}3.5(0.5)^k B^{k+1}\right] + \sum_{k=0}^{\infty}3(0.5)^k B^{k+2}\right)Y_t \\
&= 0.5\left(1-3B+\sum_{k=2}^{\infty}(0.5)^k B^k + \sum_{k=2}^{\infty}3.5(0.5)^{k-1}B^k + \sum_{k=2}^{\infty}3(0.5)^{k-2}B^k\right)Y_t \\
&= \left(0.5-1.5B+10\sum_{k=2}^{\infty}(0.5)^k B^k\right)Y_t.
\end{aligned}$$

(b) As the AR root $2/3$ and $1/2$ do not lies on the unit circle, the process is stationary.

(c) We only give a drafted solution here. You may try to work out the remaining steps. Recall

$$3B^2\left(1-\frac{1}{1.5B}\right)\left(1-\frac{1}{2B}\right)Y_t = (1-1.5B)(1-2B)Y_t = (2-B)Z_t.$$

It follows that

$$\begin{aligned}
3Y_{t-2} &= \left(1-\frac{1}{1.5B}\right)^{-1}\left(1-\frac{1}{2B}\right)^{-1}(2-B)Z_t = \left(\frac{4}{1-\frac{2}{3}B^{-1}} + \frac{3}{1-\frac{1}{2}B^{-1}}\right)(2-B)Z_t \\
&= \sum_{k=0}^{\infty}\left[4\left(\frac{2}{3}\right)^k + 3\left(\frac{1}{2}\right)^k\right]B^{-k}(2-B) = -7Z_{t-1} + \sum_{k=0}^{\infty}\left[\left(\frac{32}{3}\right)\left(\frac{2}{3}\right)^k + \left(\frac{15}{2}\right)\left(\frac{1}{2}\right)^k\right]Z_{t+k}.
\end{aligned}$$

Hence

$$Y_t = -\frac{7}{3}Z_{t+1} + \sum_{k=0}^{\infty}\left[\left(\frac{32}{9}\right)\left(\frac{2}{3}\right)^k + \left(\frac{5}{2}\right)\left(\frac{1}{2}\right)^k\right]Z_{t+k+2}.$$

You may attempt to evaluate the ACVF through Theorem 3.

Email contact: martinmtt@link.cuhk.edu.hk ; Personal website: https://sites.google.com/view/martinmtt/home