

STAT4005: Time Series
Midterm Exam Solution

Ma Ting Tin, Martin

October 24, 2022

Let $\{a_t\} \sim WN(0, \sigma^2)$.

Question 1. *(10 marks) Consider the time series $\{Y_t\}$ satisfying*

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t.$$

Let $\alpha = 4$ and $\beta = 11/3$ be the two roots of its AR characteristic equation. Find the values of ψ_1, ψ_2 and ψ_3 if the process is expressed as

$$Y_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

Solution

The assumption means that we can rewrite the model as

$$\phi(x) = \frac{1}{\alpha\beta}(\alpha - x)(\beta - x) = \left(1 - \frac{1}{4}x\right)\left(1 - \frac{3}{11}x\right)$$

As $|\alpha|, |\beta| > 1$, the model is causal. We then find the MA-representation by

$$\begin{aligned}
Y_t &= \left(1 - \frac{1}{4}B\right)^{-1}\left(1 - \frac{3}{11}B\right)^{-1} a_t = \left(\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i B^i\right)\left(\sum_{j=0}^{\infty}\left(\frac{3}{11}\right)^j B^j\right) a_t \\
&= \left(1 + \frac{1}{4}B + \frac{1}{4^2}B^2 + \frac{1}{4^3}B^3 + \cdots\right)\left(1 + \left(\frac{3}{11}\right)B + \left(\frac{3}{11}\right)^2 B^2 + \left(\frac{3}{11}\right)^3 B^3 + \cdots\right) a_t
\end{aligned}$$

By comparing coefficient, we have

$$\begin{aligned}
\psi_1 &= \frac{1}{4} + \frac{3}{11} = \frac{23}{44}, \qquad \psi_2 = \left(\frac{1}{4}\right)\left(\frac{3}{11}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{3}{11}\right)^2 = \frac{397}{1936}, \\
\psi_3 &= \left(\frac{3}{11}\right)^3 + \left(\frac{1}{4}\right)\left(\frac{3}{11}\right)^2 + \left(\frac{1}{4}\right)^2\left(\frac{3}{11}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{6095}{85184}.
\end{aligned}$$

Question 2. *(20 marks) Suppose that* $X_t = 3 + a_t - 0.4a_{t-1} - 0.6a_{t-4}$

(a) *(2 marks) Is the process weakly stationary? Explain.*

(b) *(3 marks) Is the process invertible? Explain.*

(c) *(4 marks) Find its autocovariance function.*

(d) *(6 marks) Let* $\bar{X} = \sum_{t=1}^{10} X_t/10$. *Find* $\text{Var}(\bar{X})$.

(e) *(5 marks) Find its lag 1 and 2 PACF.*

Solution

(a) The model could be written as $(X_t - 3) = (1 - 0.4B - 0.6B^4)a_t$, i.e., $\{X_t - 3\} \sim \text{MA}(4)$. As the MA model is always weakly-stationary, $\{X_t - 3\}$ and hence $\{X_t\}$ is weakly-stationary.

(b) $\theta(x) = 1 - 0.4x - 0.6x^4 = (1 - x)(1 + 0.6x + 0.6x^2 + 0.6x^3)$ has unit root, hence not invertible. (Alternative: just evaluate $\theta(1) = 1 - 0.4 - 0.6 = 0$ and conclude θ has a unit root)

(c) We have $\gamma(0) = (1^2 + 0.4^2 + 0.6^2)\sigma^2 = 1.25\sigma^2$,

$$\begin{aligned}
\gamma(1) &= \text{Cov}(X_5, X_6) = \text{Cov}(a_5 - 0.4a_4 - 0.6a_1, a_6 - 0.4a_5 - 0.6a_2) = -0.4\sigma^2 \\
\gamma(3) &= \text{Cov}(X_5, X_8) = \text{Cov}(a_5 - 0.4a_4 - 0.6a_1, a_8 - 0.4a_7 - 0.6a_4) = 0.24\sigma^2 \\
\gamma(4) &= \text{Cov}(X_5, X_9) = \text{Cov}(a_5 - 0.4a_4 - 0.6a_1, a_9 - 0.4a_8 - 0.6a_5) = -0.6\sigma^2
\end{aligned}$$

Hence

$$\gamma(k) = \begin{cases} 1.52\sigma^2 & \text{, if } k = 0 \\ -0.4\sigma^2 & \text{, if } |k| = 1 \\ 0.24\sigma^2 & \text{, if } |k| = 3 \\ -0.6\sigma^2 & \text{, if } |k| = 4 \\ 0 & \text{, otherwise} \end{cases}$$

(d) $\text{Var}(\bar{(X)}) = \frac{1}{10^2}\Big(10\gamma(0) + 2(10-1)\gamma(1) + 2(10-3)\gamma(3) + 2(10-4)\gamma(4)\Big) = 0.0416\sigma^2.$

(e)
- Lag-1 PACF: $\phi_{11} = \rho(1) = \gamma(1)/\gamma(0) = -5/19$.
- Lag-2 PACF: Notice that $\rho(2) = 0$. Then

$$\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} 1 & -5/19 \\ -5/19 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -5/19 \\ 0 \end{pmatrix} = \begin{pmatrix} -95/336 \\ -25/336 \end{pmatrix}$$

Hence $\phi_{22} = -25/336$.

Question 3. *(20 marks) Consider the time series*

$$Y_t = 0.4Y_{t-1} + a_t + 0.6a_{t-1}$$

(a) *(10 marks) Find its autocovariance function.*

(b) *(10 marks) Find ψ_i, $i = 0, 1, \ldots$ when Y_t is expressed as $a_t = \sum_{j=0}^{\infty} \psi_j Y_{t-j}$.*

Solution

(a) Rewrite the model as $(1 - 0.4B)Y_t = (1 + 0.6B)a_t$. As the AR root 2.5 is outside the unit circle, the model is causal and hence stationary. Noticing that

- $\text{Cov}(Y_t, a_t) = \text{Cov}(0.4Y_{t-1} + a_t + 0.6a_{t-1}, a_t) = \sigma^2$
- $\text{Cov}(Y_t, a_{t-1}) = \text{Cov}(0.4Y_{t-1} + a_t + 0.6a_{t-1}, a_{t-1}) = (0.4 + 0.6)\sigma^2 = \sigma^2$

Then consider the Yule-walker equation

- Taking covariance with Y_t:

$$\gamma(0) = 0.4\gamma(1) + \text{Cov}(Y_t, a_t + 0.6a_{t-1}) = 0.4\gamma(1) + 1.6\sigma^2.$$

- Taking covariance with Y_{t-1} :

$$\gamma(1) = 0.4\gamma(0) + \text{Cov}(Y_{t-1}, a_t + 0.6a_{t-1}) = 0.4\gamma(0) + 0.6\sigma^2.$$

On solving, $\gamma(0) = (46/21)\sigma^2$ and $\gamma(1) = (31/21)\sigma^2$.

- (For $k \geq 2$) Taking covariance with Y_{t-k} :

$$\gamma(k) = 0.4\gamma(k-1) + \text{Cov}(Y_{t-k}, a_t + 0.6a_{t-1}) = 0.4\gamma(k-1)$$

It follows that

$$\gamma(k) = \begin{cases} \frac{46}{21}\sigma^2 & , \text{ if } k = 0 \\ \frac{31}{21}\left(\frac{2}{5}\right)^{|k|-1}\sigma^2 & , \text{ if } |k| \neq 0 \end{cases}$$

(b) As the MA root $|-5/3| > 1$, the model is invertible. Then

$$a_t = (1 + 0.6B)^{-1}(1 - 0.4B)Y_t = \sum_{j=0}^{\infty}(-0.6B)^j(1 - 0.4B)Y_t$$

$$= \left[1 + \sum_{j=1}^{\infty}(-0.6)^j B^j - \sum_{j=0}^{\infty} 0.4(-0.6)^j B^{j+1}\right] Y_t = \left[1 + \sum_{j=1}^{\infty}(-0.6)^j B^j - \sum_{j=1}^{\infty} 0.4(-0.6)^{j-1} B^j\right] Y_t$$

$$= \left[1 + \sum_{j=1}^{\infty}\left[(-0.6)^j - 0.4(-0.6)^{j-1}\right] B^j\right] Y_t = \left[1 + \sum_{j=1}^{\infty}\left(\frac{5}{3}\right)(-0.6)^j B^j\right] Y_t.$$

Therefore, $\psi_0 = 1, \psi_i = 5(-0.6)^i/3$ for $i \geq 1$.

Question 4. *(15 marks) Identify the following as* ARIMA*models or* SARIMA*models (state the order of the model):*

(a) *(5 marks)* $Y_t - 4Y_{t-1} - Y_{t-2} = a_t - 0.7a_{t-6}$

(b) *(5 marks)* $20Y_t - 29Y_{t-1} + 10Y_{t-2} - Y_{t-3} = 8a_t - 6a_{t-1} + a_{t-2}$

(c) *(5 marks)* $Y_t - 0.7Y_{t-4} + 0.12Y_{t-8} = a_t - 0.5a_{t-1} - 0.3a_{t-4} + 0.15a_{t-5}$

Solution

(a) Rewrite the model as $(1-4B-B^2)Y_t = (1-0.7B^6)a_t$. As there is no common root and $\phi(1) = -4 \neq 0$, $\{Y_t\} \sim \text{SARIMA}(2,0,0) \times (0,0,1)_6$.

- 2 marks will be deducted if the student answers $\{Y_t\} \sim \text{ARIMA}(2,0,6)$ instead.

(b) Rewrite the model as $(B-5)(B-4)(1-B)Y_t = (B-4)(B-2)a_t$. As there is a common root and a unit root, $\{Y_t\} \sim \text{ARIMA}(1,1,1)$.

- 2 marks will be deducted if the student answers $\{Y_t\} \sim \text{ARIMA}(2,1,2)$ instead.
- 3 marks will be deducted if the student answers $\{Y_t\} \sim \text{ARIMA}(3,0,2)$ instead.

(c) Rewrite the model as $(1-0.4B^4)(1-0.3B^4)Y_t = (1-0.5B)(1-0.3B^4)a_t$. After canceling out the common root $(1-0.3B^4)$, we conclude $\{Y_t\} \sim \text{SARIMA}(0,0,1) \times (1,0,0)_4$.

- 2 marks will be deducted if the student answers $\{Y_t\} \sim \text{ARIMA}(8,0,5)$ instead.

Question 5. *(10 marks) Consider the time series* $\{X_t\}_{t=1}^n$ *satisfying*

$$X_t = T_t + S_t + N_t$$

where $N_t \sim \text{WN}(0, \sigma^2)$, T_t *stands for a polynomial trend in* t, S_t *stands for the seasonal effect, which satisfies* $S_{t+d} = S_t$ *and* $\sum_{i=1}^d S_i = 0$, *we also assume* $d = 2q+1$ *for some positive integer* q.
Further copying definition of $\hat{T}_t$ *and* $\hat{S}_t$ *from Chapter 1 slides as below, for* $t = q+1, q+2, ..., n-q$:

$$\hat{T}_t = (1/d)\sum_{r=-q}^{q} X_{t+r}$$

$$\hat{S}_i = \frac{\sum_{t\in\{i,i+d,i+2d,\ldots\}}[D_t - \bar{D}]}{n_i}, \qquad D_t = X_t - \hat{T}_t, \qquad \bar{D} = \frac{1}{n-2q}\sum_{t=q+1}^{n-q} D_t.$$

n_i *is the number of terms in the set* $\{i, i+d, i+2d, ...\}$

(a) *Show that* $\hat{S}_i = S_i - \left[\sum_{t\in\{i,i+d,i+2d,\ldots\}}(\hat{T}_t - T_t)/n_i - \sum_{t=q+1}^{n-q}(\hat{T}_t - T_t)/(n-2q)\right] + R_{t,n}$, *where* $R_{t,n}$ *is negligible term or converges to zero as* $n \to \infty$. *(Hint: Start by substituting* $D_t = S_t + N_t - (\hat{T}_t - T_t)$ *into the definition of* $\hat{S}_i$.*)*

(b) *Do you think that* $\hat{S}_i$ *is unbiased for every polynomial trend* T_t *and every* i*? (i.e.* $\mathbb{E}(\hat{S}_i) = S_i$*?) If yes, show it. If not, find a counter example by choosing some* T_t. *(Hint: Just try cubic* T_t *first with* $d = 3$, $i = 3$ *and* n *a multiple of* d, *then evaluate the bracketed part of expression in (a).)*

Solution

(a) Noticing that
$$\frac{1}{n-2q}\left|\sum_{t=q+1}^{n-q} S_t\right| \leq \frac{1}{n-2q}\sum_{i=1}^{d}|S_i| = o(1) = o_p(1) \tag{1}$$
and by law of large number,
$$\frac{1}{n-2q}\left|\sum_{t=q+1}^{n-q} N_t\right| = o_p(1) \quad \text{and} \quad \frac{1}{n_i}\left|\sum_{t\in\{i,i+d,i+2d,\ldots\}} N_t\right| = o_p(1). \tag{2}$$
As $D_t = X_t - \hat{T}_t = S_t + N_t - (\hat{T}_t - T_t)$, together with equation (1) and (2) gives
$$\begin{aligned}\bar{D} &= \frac{1}{n-2q}\sum_{t=q+1}^{n-q} D_t = \frac{1}{n-2q}\sum_{t=q+1}^{n-q} S_t + \frac{1}{n-2q}\sum_{t=q+1}^{n-q} N_t - \frac{1}{n-2q}\sum_{t=q+1}^{n-q}(\hat{T}_t - T_t)\\ &= o_p(1) + o_p(1) - \frac{1}{n-2q}\sum_{t=q+1}^{n-q}(\hat{T}_t - T_t) = -\frac{1}{n-2q}\sum_{t=q+1}^{n-q}(\hat{T}_t - T_t) + o_p(1).\end{aligned}$$
Also, we have $S_i = S_i + kd$ for all integer k. Hence
$$\begin{aligned}\frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}} D_t &= \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}} S_t + \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}} N_t - \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}}(\hat{T}_t - T_t)\\ &= \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}} S_i + o_p(1) - \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}}(\hat{T}_t - T_t)\\ &= S_i - \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}}(\hat{T}_t - T_t) + o_p(1).\end{aligned}$$
Noticing $\frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}}\bar{D} = \frac{1}{n_i}(n_i)\bar{D} = \bar{D}$, it follows that
$$\begin{aligned}\hat{S}_i &= \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}}[D_t - \bar{D}]\\ &= S_i - \frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}}(\hat{T}_t - T_t) + o_p(1) - \left(-\frac{1}{n-2q}\sum_{t=q+1}^{n-q}(\hat{T}_t - T_t) + o_p(1)\right)\\ &= S_i - \left[\frac{1}{n_i}\sum_{t\in\{i,i+d,i+2d,\ldots\}} D_t - \frac{1}{n-2q}\sum_{t=q+1}^{n-q}(\hat{T}_t - T_t)\right] + o_p(1).\end{aligned}$$

(b) Upon checking, consider $d = 3$ with filter $(a_{-1}, a_0, a_1) = (1/3, 1/3, 1/3)$ and let $T_t = t^3$, $n = 3a$ (a is an integer) and $i = 3$. Notice that $\mathrm{E}(\hat{T}_t - T_t) = 2t$ in this case. Therefore by (a),
$$\begin{aligned}\mathrm{E}(\hat{S}_3) &= S_3 - \left[\frac{1}{n_3}\sum_{t\in\{3,3+d,3+2d,\ldots\}} 2t - \frac{1}{n-2q}\sum_{t=q+1}^{n-q} 2t\right] + o_p(1)\\ &= S_3 - \left[\sum_{t=1}^{a} 2(3t)/a - \sum_{t=2}^{3a-1}(2t)/(3a-2)\right] + o_p(1)\\ &= S_3 - [3(a+1) - (3a+1)] + o_p(1) = S_3 - 2 + o_p(1) \neq S_3\end{aligned}$$
in general.

Question 6. *(5 marks) Consider a time series $\{X_t\}_{t=1,2,...}$ satisfying:*

$$X_t = \cos(tU), \qquad U \sim \text{Unif}(-\pi, \pi)$$

(a) Find lag k ACVF (for k=1,2,....) for $\{X_t\}_{t=1,2,...}$.

(b) Hence show that the process is weakly stationary.

Solution

(a) Notice that

$$\text{E}(X_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(tu)\ \text{d}u = \frac{\sin(tu)|_{-\pi}^{\pi}}{2\pi t} = 0.$$

Its lag k ACVF (k=1,2,...) equals

$$\begin{aligned}
\text{Cov}(X_t, X_{t+k}) &= \mathbb{E}(X_t X_{t+k}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(tu)\cos((t+k)u)\ \text{d}u \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \Big[\cos[(2t+k)u] + \cos[ku]\Big]\ \text{d}u \\
&= \frac{1}{4\pi}\left[\frac{\sin[(2t+k)u]}{2t+k}\Big|_{-\pi}^{\pi} + \frac{\sin(ku)}{k}\Big|_{-\pi}^{\pi}\right] \\
&= 0
\end{aligned}$$

Note that $2t+k$ is non-zero, else $k = -2t$ and $\text{Cov}(X_t, X_{t+k}) = \text{Cov}(X_t, X_{-t})$ is not considered as this involves X_t with negative time point. Also, the above argument gives the same result when we consider $\text{Cov}(X_t, X_{t-k})$ instead.

(b) As we have shown, $\text{E}(X_t)$ is constant over time and $\text{Cov}(X_t, X_{t+k})$ is free of t for $k \neq 0$. It suffices to show $\text{Var}(X_t)$ is free of t. We have

$$\text{Var}(X_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(tu)\ \text{d}u = \frac{1}{4\pi}\left[\frac{\sin(2tu)}{2t}\Big|_{-\pi}^{\pi} + 2\pi\right] = 1/2.$$

As $\text{Var}(X_t)$ is free of t, $\{X_t\}$ is weakly-stationary.

Remark 1. *This is only a draft of the mid-term solution and you should just take it as a reference. You do not need to present or answer exactly the same way as the solution in order to get full credit. We retain the right to change the marking scheme to the midterm exam.*