STAT4005: Time Series
Mock Final Exam
Ma Ting Tin, Martin

Let $\{a_t\} \sim \text{WN}(0, \sigma^2)$ unless otherwise specified. You are reminded that **additional distribution assumptions on $\{a_t\}$ may be stated in certain parts of certain questions.**

(★★★) True and False

✎ **Exercise 1.** *State whether the following claims are true (T) or false (F).*

- *Briefly explain your answer. (A short and precise explanation is sufficient as long as the argument can reflect the understanding related to the question)*

- ***No credit will be given if you only answer true or false without proper explanation.***

(a) (2 marks) Assume that $X_t = T_t + a_t$, where $\{T_t\}$ is a deterministic trend.
Statement: there exists some p such that the p-th order differenced series $\{\Delta^p X_t\}$ is weakly stationary.

(b) (2 marks) For AR(p) model, i.e., $X_t = \sum_{k=1}^{p} \phi_p X_{t-k} + a_t$, approximates the quantity $\text{E}\{(\hat{\phi} - \phi)^T (\hat{\phi} - \phi)\} = \sum_{k=1}^{p} \text{MSE}(\hat{\phi}_k)$, where $\hat{\phi}_k$ is the MLE of ϕ_k.
Statement: if $p_1 < p_2$, the FPE of AR(p_1) always less than AR(p_2) model and we should choose AR(p_1) based on FPE.

(c) (2 marks) **Statement:** According to the following graphs, $\{X_t\}$ is likely to be an AR(3) model.
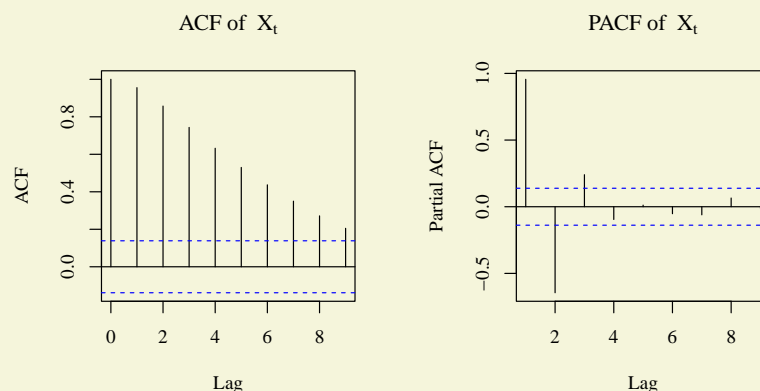
(d) (2 marks) Suppose that we want to choose a model among class of ARMA model. Define the criterion of the ARMA(p, q) model as $-2 \log L + 2(p - q + 1)$.
Statement: it is a sensible criterion for model selection.

(e) (2 marks) Consider AR(1) model $Y_t = \alpha Y_{t-1} + Z_t$, where $\{Z_t\} \overset{iid}{\sim} \text{N}(0, \sigma^2)$. Let $\hat{Y}_{n+h} = \text{E}(Y_{n+h}|Y_1, \ldots, Y_n)$ and $V_h = \text{Var}(e_n(h))$.
Statement: the 95% prediction interval of $Y_{n+1} + Y_{n+2}$ is given by

$$\left(Y_{n+1} + Y_{n+2} - 1.96(V_1 + V_2)^{1/2}, Y_{n+1} + Y_{n+2} + 1.96(V_1 + V_2)^{1/2}\right).$$

(f) (2 marks) Assume $X_t = \sigma_t \epsilon_t$ and $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$ ($0 < \alpha_0, \alpha_1 < 1$), where $\{\epsilon_t\} \overset{iid}{\sim} \text{N}(0, 1)$. Also assume $\text{E}(X_t^4) < \infty$.
Statement: $\{X_t\}$ is weakly stationary, while $\{X_t^2\}$ is not, in general.

Solution

(a) FALSE. Consider $T_t = e^t$, then $\Delta^p T_t$ is also an exponential function for all p, i.e. non-constant.

(b) FALSE. For example, when we fit $\{X_t\}$ by $X_t = \phi_1^{(1)} X_{t-1} + a_t$ or $X_t = \phi_1^{(2)} X_{t-1} + \phi_2^{(2)} X_{t-1} + a_t$. In general, $\phi_1^{(1)} X_{t-1} \neq \phi_1^{(2)} X_{t-1}$ and therefore the claim is improper.

(c) TRUE. AR(3) shows a sharp cutoff at lag 3 in the PACF plot.

(d) FALSE. For example, consider the ARMA(p_1, p_1) and ARMA(p_2, p_2) model with $p_2 > p_1$. Surely the likelihood under ARMA(p_1, p_1) is of lower value $(L_1 < L_2)$, then

$$-2\log(L_1) + 2(p_1 - p_1 + 1) - 2\log(L_1) => -2\log(L_2) = -2\log(L_2) + 2(p_2 - p_2 + 1),$$

i.e. this criterion always choose ARMA(p_2, p_2) model which have more parameter than ARMA(p_1, p_1), which is non-sense.

(e) FALSE. As $Y_{n+2} = \alpha Y_{n+1} + a_t$ is correlated with Y_{n+1}, the stated prediction interval regard $V_1 + V_2$ as the conditional variance, omitting the conditional covariance between Y_{n+1} and Y_{n+2}.

(f) FALSE. $E(X_t) = 0$, $\text{Var}(X_t) = \alpha_0/(1-\alpha_1)$ and $\text{Cov}(X_t, X_{t+h})$, i.e., $\{X_t\}$ is weakly stationary. As $\{X_t^2\} \sim \text{ARMA}(1,1)$ with

$$X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + v_t,$$

i.e. the AR root is $1/\alpha_1 > 1$ and hence $\{X_t^2\}$ is also weakly stationary.

❶ **Remark 1.** *These questions are designed to test the level of understanding of concepts and technical details related to time series. They are relatively difficult. No need to panic when you find them hard and you are suggested to complete the remaining parts before doing the True and False question in the examination.*

(★☆☆) Estimation

✎ **Exercise 2. (20 marks)** *Consider causal* AR(2) *Model* $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t$. *Given*

$$(Y_1, \ldots, Y_n) = (-1.5, 0.4, 1, 1.5, 0.6).$$

(a) *(4 marks) Evaluate sample ACVF* C_0, C_1 *and* C_2.

(b) *(7 marks) Find the Yule-Walker estimators of* ϕ_1 *and* ϕ_2. *Hence, find an estimator of* σ^2.

(c) *(6 marks) Suppose that we have*

$$\sqrt{n}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \xrightarrow{\text{d}} \text{N}_2(\mathbf{0}, \boldsymbol{\Sigma}) \quad and \quad \hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}.$$

Construct a 95% confidence interval for $\phi_1 - 2\phi_2$.

(d) *(3 marks) Consider testing of* $H_0 : \phi_1 - 2\phi_2 = 0$ *against* $H_1 : \phi_1 - 2\phi_2 \neq 0$, *should null* H_0 *be rejected under 95% confidence level?*

Solution

(a) Notice that $\bar{Y} = 0.4$. Recall that $C_k = n^{-1} \sum_{t=1}^{n-k} (Y_t - \bar{Y})(Y_{t+k} - \bar{Y})$, we have $C_0 = 1.044$, $C_1 = 0.176$ and $C_2 = -0.204$.

(b) Consider taking $\text{Cov}(Y_{t-k}, \cdot)$ for $k = 1, 2$ to both side of $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t$,

$$\begin{cases} \gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1) \\ \gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0) \end{cases}$$

Replacing $\gamma(k)$ be C_k gives

$$\begin{cases} 0.176 = 1.044\hat{\phi}_1 + 0.176\hat{\phi}_2 \\ -0.204 = 0.176\hat{\phi}_1 + 1.044\hat{\phi}_2 \end{cases}$$

On solving, we have $\hat{\phi}_1 = 0.2074186$ and $\hat{\phi}_2 = -0.2303694$. Then by causality of $\{Y_t\}$, taking $\text{Cov}(Y_t, \cdot)$ on both sides of $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t$ gives

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2 \quad \Rightarrow \quad \hat{\sigma}^2 = C_0 - \hat{\phi}_1 C_1 - \hat{\phi}_2 C_2 = 0.960499.$$

(c) It follows that $5\text{Var}(\hat{\phi}_1 - 2\hat{\phi}_2) \approx 0.2 + 2^2 \times 0.2 - 2 \times 2 \times 0.1 = 0.6$, i.e., $\text{Var}(\hat{\phi}_1 - 2\hat{\phi}_2) \approx 0.12$. Notice that $\hat{\phi}_1 - 2\hat{\phi}_2 = 0.6681574$. Hence 95% confidence interval of $\phi_1 - 2\phi_2$ is given by

$$\left(0.6681574 \pm 1.96 \times \sqrt{0.12}\right) = (-0.010, 1.347).$$

(d) By (c), we can see that 0 is included in the 95% confidence interval of $\phi_1 - 2\phi_2$. Therefore, H_0 is not rejected.

(★★☆) Forecasting

✎ **Exercise 3.** *(20 marks)* *Consider $X_t = \Delta Y_t = Y_t - Y_{t-1}$, where $\{X_t\}$ satisfies*

$$X_t - \omega X_{t-1} = a_t + \omega a_{t-1} + 0.5\omega a_{t-2},$$

with $\{a_t\} \sim \text{WN}(0, \sigma^2)$. You are given n observations $(Y_1, Y_2, ..., Y_n)$.

(a) *(3 marks) Is the assumption made sufficient for the construction of the prediction interval of Y_{n+h} ($h \geq 1$) ? State the additional assumption needed if any.*

(b) *(8 marks) Assuming the additional assumption made in (a). Find the 1-Step and 2-Step forecasts in terms of $(Y_1, Y_2, \ldots, Y_n)$, $(a_1, a_2, \ldots, a_n)$ and ω.*

(c) *(9 marks) Assuming the additional assumption made in (a). Find the 95% prediction intervals of the above forecasts in terms of $(Y_1, Y_2, \ldots, Y_n)$, ω and σ^2.*

Solution

(a)
- We have to assume $\{a_t\} \overset{iid}{\sim} \text{N}(0, \sigma^2)$, i.e., the normality of the white-noise sequence.
- $\{X_t\}$ has to be causal and invertible.

(b) Noticing that as $X_t = Y_t - Y_{t-1}$, knowing $Y_1, \ldots, Y_n$ gives the value of $X_2, \ldots, X_n$.

$$\hat{X}_{n+1} = \text{E}(\omega X_n + a_{n+1} + \omega a_n + 0.5\omega a_{n-1}|Y_1, \ldots, Y_n) = \omega X_n + \mathbf{0} + \omega a_n + 0.5\omega a_{n-1}$$
$$= \omega(Y_n - Y_{n-1}) + \omega a_n + 0.5\omega a_{n-1}$$
$$\hat{X}_{n+2} = \text{E}(\omega X_{n+1} + a_{n+2} + \omega a_{n+1} + 0.5\omega a_n|Y_1, \ldots, Y_n) = \omega \hat{X}_{n+1} + \mathbf{0} + \omega(\mathbf{0}) + 0.5\omega a_n$$
$$= \omega\left(\omega(Y_n - Y_{n-1}) + \omega a_n + 0.5\omega a_{n-1}\right) + 0.5\omega a_n$$

Then we can write

$$\hat{Y}_{n+1} = Y_n + \hat{X}_{n+1} = Y_n + \omega(Y_n - Y_{n-1}) + \omega a_n + 0.5\omega a_{n-1}$$
$$= (1+\omega)Y_n - \omega Y_{n-1} + \omega a_n + 0.5\omega a_{n-1}$$
$$\hat{Y}_{n+2} = Y_n + \hat{X}_{n+1} + \hat{X}_{n+2}$$
$$= (1+\omega)Y_n - \omega Y_{n-1} + \omega a_n + 0.5\omega a_{n-1} + \omega\left(\omega(Y_n - Y_{n-1}) + \omega a_n + 0.5\omega a_{n-1}\right) + 0.5\omega a_n$$
$$= (1+\omega+\omega^2)Y_n - (\omega+\omega^2)Y_{n-1} + (1.5\omega+\omega^2)a_n + (0.5\omega+0.5\omega^2)a_{n-1}.$$

(c) We then find the forecasting error and its variance. We have

- $e_{n,X}(1) = X_{n+1} - \hat{X}_{n+1} = a_{n+1}$
- $e_{n,X}(2) = X_{n+2} - \hat{X}_{n+2} = \omega e_{n,X}(1) + a_{n+2} + \omega a_{n+1} = 2\omega a_{n+1} + a_{n+2}$.
- $e_{n,Y}(1) = e_{n,X}(1) = a_{n+1}$ and $e_{n,Y}(2) = e_{n,X}(1) + e_{n,X}(2) = (2\omega+1)a_{n+1} + a_{n+2}$

Hence $\text{Var}(e_{n,Y}(1)|Y_1, \ldots, Y_n) = \sigma^2$ and $\text{Var}(e_{n,Y}(2)|Y_1, \ldots, Y_n) = [(2\omega+1)^2 + 1]\sigma^2$. Therefore, the 95% confidence interval is given by

$$(\hat{Y}_{n+1} - 1.96\sigma, \hat{Y}_{n+1} + 1.96\sigma) \quad \text{and} \quad \left(\hat{Y}_{n+2} \pm 1.96[(2\omega+1)^2 + 1]^{1/2}\sigma\right)$$

(★☆☆) Estimation and Forecasting

✎ **Exercise 4.** *(20 marks) Consider an invertible* $\mathrm{ARMA}(1,1)$ *model* $Y_t = \phi Y_{t-1} + a_t - \theta a_{t-1}$,

(a) *(12 marks) Given* $Y_1 = 0$, $Y_2 = 5$, $Y_3 = 3$ *and* $Y_4 = 2$. *Write down the objective function to be minimized in order to obtain the conditional least-square estimator of* ϕ *and* θ.

(b) *(8 marks) Suppose* $\phi = 0.4$ *and* $\theta = -0.6$, *find the 1-st and 2-nd step forecast* $\hat{Y}_5$ *and* $\hat{Y}_6$.

Solution

(a) Assume $a_t = Y_t = 0$ for $t \le 0$. As $a_t = Y_t - \phi Y_{t-1} + \theta a_{t-1}$, we have

$$a_1 = Y_1 - \phi Y_0 + \theta a_0 = 0$$
$$a_2 = Y_2 - \phi Y_1 + \theta a_1 = 5$$
$$a_3 = Y_3 - \phi Y_2 + \theta a_2 = 3 - 5\phi + 5\theta$$
$$a_4 = Y_4 - \phi Y_3 + \theta a_3 = 2 - 3\phi + \theta(3 - 5\phi + 5\theta)$$

It follows that

$$S_*(\phi, \theta) = \sum_{t=1}^{4} a_t^2 = 5^2 + (3 - 5\phi + 5\theta)^2 + [2 - 3\phi + \theta(3 - 5\phi + 5\theta)]^2.$$

(b) We have $a_3 = 3 - 5\phi + 5\theta = -2$ and $a_4 = 2 - 3\phi + \theta a_3 = 2$. Then

- $\hat{Y}_5 = \mathrm{E}(\phi Y_4 + a_5 - \theta a_4 | Y_1, \ldots, Y_4) = \phi Y_4 - \theta a_4 = 2$
- $\hat{Y}_6 = \mathrm{E}(\phi Y_5 + a_6 - \theta a_5 | Y_1, \ldots, Y_4) = \phi \hat{Y}_5 = 0.8$

(★★☆) GARCH Model

✎ **Exercise 5.** *(20 marks) Consider the stationary GARCH(1,1) model*

$$X_t = \epsilon_t \sigma_t, \quad \epsilon_t \stackrel{iid}{\sim} N(0,1), \qquad \sigma_t^2 = 0.5 + 0.1 X_{t-1}^2 + 0.2\sigma_{t-1}^2$$

Also assume that $E(\sigma_t^4) < \infty$ is constant over-time.

(a) *(4 marks) Express $\{X_t^2\}$ as an ARMA model. (In this exercise, you can directly assume that the sequence $\{v_t = X_t^2 - \sigma_t^2\}$ is white noise.)*

(b) *(10 marks) Evaluate $\mathrm{Var}(v_t)$.*

(c) *(6 marks) Find the value of $\mathrm{Cov}(X_t^2, X_{t+1}^2)$.*

Solution

(a) The white-noise sequence is given by $v_t = X_t^2 - \sigma_t^2 = \sigma_t^2(\epsilon_t^2 - 1)$. Then

$$\begin{aligned}X_t^2 &= \sigma_t^2 + (X_t^2 - \sigma_t^2) = 0.5 + 0.1X_{t-1}^2 + 0.2\sigma_{t-1}^2 + (X_t^2 - \sigma_t^2) \\ &= 0.5 + (0.1 + 0.2)X_{t-1}^2 - 0.2(X_{t-1}^2 - \sigma_{t-1}^2) + v_t = 0.5 + 0.3X_{t-1}^2 + v_t - 0.2v_{t-1}.\end{aligned}$$

Hence $\{X_t^2\} \sim \mathrm{ARMA}(1,1)$ with white noise $\{v_t\}$.

(b) As $\alpha_1 + \beta_1 = 0.1 + 0.2 = 0.3 < 1$, $\{X_t\}$ is stationary. Then both $E(X_t^2)$ and $E(\sigma_t^2)$ is constant over-time and hence

$$E(\sigma_t^2) = 0.5 + 0.2E(\sigma_{t-1}^2) + 0.1E(X_{t-1}^2) = 0.5 + 0.2E(\sigma_t^2) + 0.1E(X_t^2) = 0.5 + 0.2E(\sigma_t^2) + 0.1E(\sigma_t^2).$$

Then we have $E(\sigma_t^2) = 5/7$. Noticing that $E(\epsilon_t^2) = 1$ and $E(\epsilon_t^4) = 3$. Then

$$\mathrm{Var}(v_t) = E\{(X_t^2 - \sigma_t^2)^2\} - E(X_t - \sigma_t)^2 = E(\epsilon_t^4\sigma_t^4 - 2\epsilon_t^2\sigma_t^4 + \sigma_t^4) = 2E(\sigma_t^4).$$

Then we have

$$\begin{aligned}E(\sigma_t^4) &= E\{(0.5 + 0.2\sigma_{t-1}^2 + 0.1X_{t-1}^2)^2\} \\ &= 0.5^2 + 0.2^2E(\sigma_{t-1}^4) + 0.1^2E(X_{t-1}^4) + 2(0.2)(0.1)E(\sigma_{t-1}^2X_{t-1}^2) \\ &\quad + 2(0.5)(0.2)E(\sigma_{t-1}^2) + 2(0.5)(0.1)E(X_{t-1}^2) \\ &= 0.25 + 0.04E(\sigma_t^4) + 0.01E(\epsilon_{t-1}^4\sigma_{t-1}^4) + 0.04E(\epsilon_{t-1}^2\sigma_{t-1}^4) + 0.2E(\sigma_{t-1}^2) + 0.1E(\epsilon_{t-1}^2\sigma_{t-1}^2) \\ &= 0.25 + 0.04E(\sigma_t^4) + 0.01(3)E(\sigma_t^4) + 0.04E(\sigma_t^4) + 0.2(5/7) + 0.1(5/7) = \frac{13}{28} + 0.11E(\sigma_t^4)\end{aligned}$$

It follows that $E(\sigma_t^4) = 325/623$ and hence $\mathrm{Var}(v_t) = 2E(\sigma_t^4) = 650/623$.

(c) The AR polynomial $1 - 0.3x$ associated to $\{X_t^2\}$ gives root outside unit circle, it is causal.

- $\mathrm{Cov}(X_t^2, v_t) = \mathrm{Cov}(0.5 + 0.3X_{t-1}^2 + v_t - 0.2v_{t-1}, v_t) = \mathrm{Var}(v_t)$.
- $\mathrm{Cov}(X_t^2, v_{t-1}) = \mathrm{Cov}(0.5 + 0.3X_{t-1}^2 + v_t - 0.2v_{t-1}, v_{t-1}) = 0.1\mathrm{Var}(v_t)$.
- $\mathrm{Cov}(X_t^2, \cdot) : \gamma(0) = 0.3\gamma(1) + \mathrm{Var}(v_t) - 0.2 \times 0.1\mathrm{Var}(v_t) = 0.3\gamma(1) + 0.98\mathrm{Var}(v_t)$.
- $\mathrm{Cov}(X_{t-1}^2, \cdot) : \gamma(1) = 0.3\gamma(0) - 0.2\mathrm{Var}(v_t)$.

and hence $\gamma(0) = 92\mathrm{Var}(v_t)/91 = 4600/4361$ and

$$\mathrm{Cov}(X_t^2, X_{t+1}^2) = \gamma(1) = 0.3\gamma(0) - 0.2\mathrm{Var}(v_t) = 470/4361.$$

(★★☆) Non-linear Time-Series

✎ **Exercise 6.** *(8 marks) Consider a causal and weakly stationary time series $\{Y_t\}$ defined by*

$$Y_t = \phi|Y_{t-2}|a_t + \theta a_{t-1},$$

where $\{a_t\} \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$ and $0 < \phi, \sigma^2 < 1$. Assume that $\alpha = \mathrm{E}(|Y_t|)$ is a known constant over time.

(a) *(2 marks) Evaluate $\mathrm{E}(Y_t)$ and $\mathrm{E}(Y_t^2)$.*

(b) *(3 marks) Find the ACVF $\gamma(\cdot)$ of Y_t in terms of ϕ, θ, σ^2 and α.*

(c) *(1 marks) Which of the following methods can be applied to estimate ϕ and θ? (I) Method of moments, (II) Least Square Estimation, OR (III) Maximum likelihood estimation.*

(d) *(2 marks) Given observation $(Y_1, \ldots, Y_n)$, outline and describe an estimation algorithm to find $\hat{\phi}$, $\hat{\theta}$ and $\hat{\sigma}^2$.*

Solution

(a) Notice that a_t is independent with Y_{t-2} by the causality and normality assumption. Then

$$\mathrm{E}(Y_t) = \phi\mathrm{E}(|Y_{t-2}|a_t) + \theta\mathrm{E}(a_{t-1}) = \phi\mathrm{E}(|Y_{t-2}|)\mathrm{E}(a_t) = 0$$

Similarly, a_t, a_{t-1} and Y_{t-2} are independent. Then

$$\begin{aligned} \mathrm{E}(Y_t^2) &= \mathrm{E}\{(\phi|Y_{t-2}|a_t + \theta a_{t-1})^2\} = \phi^2\mathrm{E}(a_t^2 Y_{t-2}^2) + \theta^2\mathrm{E}(a_{t-1}^2) + 2\phi\theta\mathrm{E}(a_t a_{t-1}|Y_{t-2}|) \\ &= \phi^2\mathrm{E}(a_t^2)\mathrm{E}(Y_{t-2}^2) + \theta^2\sigma^2 + \phi^2\theta\mathrm{E}(a_t)\mathrm{E}(a_{t-1})\mathrm{E}(|Y_{t-2}|) \\ &= \phi^2\sigma^2\mathrm{E}(Y_t^2) + \theta^2\sigma^2. \end{aligned}$$

and hence $\mathrm{E}(Y_t^2) = \theta^2\sigma^2/(1 - \phi^2\sigma^2)$.

(b) For $k \neq 0$, we have

$$\begin{aligned} \mathrm{E}(Y_t Y_{t+k}) &= \mathrm{E}\Big\{(\phi|Y_{t-2}|a_t + \theta a_{t-1})(\phi|Y_{t+k-2}|a_{t+k} + \theta a_{t+k-1})\Big\} \\ &= \phi^2\mathrm{E}(|Y_t Y_{t+k-2}|a_t a_{t+k}) + \theta^2\mathrm{E}(a_{t-1}a_{t+k-1}) + \phi\theta\mathrm{E}(|Y_{t-2}|a_t a_{t+k-1}) + \phi\theta\mathrm{E}(|Y_{t+k-2}|a_{t+k}a_{t-1}) \\ &= 0 + 0 + \phi\theta\alpha\sigma^2\mathbb{1}(|k| = 1). \end{aligned}$$

It follows that

$$\gamma(k) = \begin{cases} \dfrac{\theta^2\sigma^2}{1-\phi^2\sigma^2} & \text{if } k = 0 \\ \phi\theta\alpha\sigma^2 & \text{if } |k| = 1 \\ 0 & \text{if } |k| \geq 2 \end{cases}$$

(c) Least-square method. For method of moment, there are three unknown to be estimate while there are only two non-trivial ACVF. For MLE, it is hard to write the joint density.

(d) The procedure is as follow:

    i Assume $Y_t = a_t = 0$ for $t \leq 0$. Write $a_t = (Y_t - \theta a_{t-1})/(\phi|Y_{t-2}|)$ for $t = 1, \ldots, n$.

    ii Find the minimizer of $\sum_{t=1}^n a_t^2$ and denote them by $\hat{\phi}$ and $\hat{\theta}$.

    iii Define $\hat{a}_t = (Y_t - \hat{\theta}\hat{a}_{t-1})/(\hat{\phi}|Y_{t-2}|)$ for $t = 1, \ldots, n$. Let $\hat{\sigma}^2$ be the estimated variance of $(\hat{a}_1, \ldots, \hat{a}_n)$. [Or $\hat{\sigma}^2 = (n-2)^{-1}\sum_{t=1}^n a_t^2$. Other sensible form is also acceptable]