STAT4005: Time Series
Final Exam (True and False)
Ma Ting Tin, Martin

True and False Question

State whether the following claims are true (T) or false (F).

- Briefly explain your answer. (*Short and precise explanation* is sufficient as long as the argument can reflect the understanding related to the question)
- **No credit will be given if one only answer true or false without proper explanation** to the answer.

Let $a_t \sim \mathrm{WN}(0, \sigma^2)$ unless otherwise specified. You are reminded that **additional distribution assumptions on a_t may be stated in certain parts of certain questions.**

(a) (2 marks) Suppose $\{Y_t\}$ is weakly-stationary. Given $Y_1, \ldots, Y_n$, Andrew claims that it is impossible to have the sample ACVF with value $C_0 = 0.5$, $C_1 = 0.4$ and $C_2 = -0.4$. *(Hint: consider the transformed series $X_t = Y_t - Y_{t-1} + Y_{t-2}$)*

(b) (2 marks) Given $Y_t = -1.5Y_{t-1} + Y_{t-2} + a_t$, i.e. an AR(2) model. Suppose σ^2 is known. Louis claims that the value of lag-k ACVF of $\{Y_t\}$ for $k = 1, 2, 3$, i.e. $\gamma(0), \gamma(1)$ and $\gamma(2)$ could be found through solving the following system of linear equation

$$\begin{cases} \gamma(0) &= -1.5\gamma(1) + \gamma(2) + \sigma^2; \\ \gamma(1) &= -1.5\gamma(0) + \gamma(1); \\ \gamma(2) &= -1.5\gamma(1) + \gamma(0). \end{cases}$$

(c) (2 marks) Assume $Y_t = a_t - \theta a_{t-1}$, where $\theta = -2$ and σ^2 are unknown. Consider

$$\gamma(k) = \begin{cases} (1+\theta^2)\sigma^2 & , \text{ if } k = 0 \\ -\theta\sigma^2 & , \text{ if } |k| = 1 \end{cases} \quad \Rightarrow \quad \rho(1) = -\frac{\theta}{1+\theta^2} \quad \Rightarrow \quad \rho(1)\theta^2 + \theta + \rho(1) = 0.$$

Elvis claims that the estimator $\hat{\theta} = (-1 + \sqrt{1 - 4r_1^2})/(2r_1) \approx \theta$ when the sample size is large, where r_1 is the lag-1 sample ACF.

(d) (2 marks) Consider the model $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + a_t$ under constraints $\phi_1 = 1$ and $\phi_2 + \phi_3 = 0$. Brian claims that the least-square method is applicable for finding an estimator of ϕ_2 and ϕ_3.

(e) (2 marks) Suppose $\{Y_t\}$ satisfies

$$Y_t - 1.5Y_{t-1} + 0.5Y_{t-2} = a_t - 0.3a_{t-1} - 0.7a_{t-2},$$

where $\{a_t\} \overset{iid}{\sim} \mathrm{N}(0, 1)$. Given value of $Y_1, \ldots, Y_n$, Harry claims that the Box-and-Jenkins Approach can be used to construct a 95% prediction interval for Y_{n+1}.

(f) (2 marks) Consider $X_t = \sigma_t \epsilon_t$ with $\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j X_{t-j}^2$ and $\{\epsilon_t\} \overset{iid}{\sim} \mathbb{N}(0, 1)$, Winky claims that even $\{X_t^2\}$ is weakly-stationary, $\{\sigma_t^2\}$ may NOT be weakly-stationary.

Solution

(a) TRUE. Notice that

$$\widehat{\mathrm{Var}}(X_t) = \widehat{\mathrm{Cov}}(Y_t - Y_{t-1} + Y_{t-2}, Y_t - Y_{t-1} + Y_{t-2}) = 3C_0 - 4C_1 + C_2 = -0.5 < 0.$$

Takeaway 1. *The weakly-stationary assumption add constraint on decay of the ACVF. It is NOT the case that arbitrary function can be a legitimate ACVF.*

(b) FALSE. Rewrite the model as

$$(0.5 + B)(2 - B)Y_t = (1 + 1.5B - B^2)Y_t = a_t,$$

i.e. the AR polynomial have root -0.5, which is within the unit circle and hence non-causal. For non-causal model, $\mathrm{Cov}(Y_t, Z_{t+k}) \neq 0$ in general for $k > 0$ and hence

$$\mathrm{Cov}(Y_t, a_t) = \mathrm{Cov}(-1.5Y_{t-1} + Y_{t-2} + a_t, a_t) \neq \sigma^2.$$

Indeed from line 2, $\gamma(1) = -1.5\gamma(0) + \gamma(1)$ implies $\mathrm{Var}(Y_t) = \gamma(0) = 0$, which is impossible.

Takeaway 2. *For non-causal (but stationary) model, the observation might be correlated with future noise, causing troublesome in applying Yule-Walker model. MA-representation can be considered instead for finding the ACVF.*

(c) FALSE. As $n \to \infty$, $r_1 \to \rho(1) = -\theta/(1+\theta^2) = -(-2)/\{1 + (-2)^2\} = 2/5$. Hence

$$\hat{\theta} \to \frac{-1 + \sqrt{1 - 4\rho(1)^2}}{2\rho(1)} = -0.5 \neq 2 = \theta.$$

Takeaway 3. *Noticing that the equation $\rho(1)\theta^2 + \theta + \rho(1)$ is quadratic, i.e. there exists two possible value of θ, namely θ_1 and θ_2 that gives the same ACVF $\gamma(k)$. It means that those parameter are non-identifiable through the ACVF. Indeed, it is one of the reason for people to concern invertible model. If you assume the model to be invertible, then the model parameter would be identifiable. For example, in this question, if invertibility is assumed, then you should consider $\hat{\theta} = (-1 + \sqrt{1 - 4r_1^2})/(2r_1) \to -0.5$. While you should reject*

$$\widetilde{\theta} = \frac{-1 - \sqrt{1 - 4\rho(1)^2}}{2\rho(1)} \to 2,$$

which yield an non-invertible model.

Solution (Continued)

(d) TRUE. For $\phi_1 = 1$ and $\phi_2 = -\phi_3$, you can rewrite the model as

$$Y_t - Y_{t-1} = \phi_2(Y_{t-2} - Y_{t-3}) + a_t \quad \Rightarrow \quad X_t = \phi_2 X_{t-2} + a_t, \quad X_t = Y_t - Y_{t-1}$$

You can hence apply least-square method to $\{X_t\}$ to obtain $\hat{\phi}_2$ and $\hat{\phi}_3 = -\hat{\phi}_2$.

Takeaway 4. *Differencing method can be applied together with regression.*

(e) TRUE. Rewrite the model as $(1-B)(1-0.5B)Y_t = (1-B)(1+0.7B)a_t$. Cancelling out the common factor gives

$$(1-0.5B)Y_t = (1+0.7B)a_t,$$

as both AR and MA polynomial roots are out of unit-circle, $\{Y_t\}$ is causal and invertible. Hence Box-and-Jenkins Approach is applicable to $\{Y_t\}$.

Takeaway 5. *Box-and-Jenkins method is only applicable to* $\{Y_t\} \sim \text{ARIMA}(p,d,q)$ *model which satisfies* $\{\Delta^d Y_t\}$ *being causal and invertible.*

(f) FALSE. Denote $\gamma(k) = \text{Cov}(X_t, X_{t+k})$ (it is valid due to stationarity of X_t^2). For $\{X_t^2\}$ being stationary, $\text{E}(X_t^2) = \text{E}(\sigma_t^2 \epsilon_t^2) = \text{E}(\sigma_t^2)$ is constant over time and Then we can write

$$\text{Cov}(\sigma_t^2, \sigma_{t+k}^2) = \text{Cov}\left(\sum_{i=1}^{q} \alpha_i X_{t-i}^2, \sum_{j=1}^{q} \alpha_j X_{t+k-j}^2\right)$$
$$= \sum_{i=1}^{q}\sum_{j=1}^{q} \alpha_i \alpha_j \text{Cov}(X_{t-i}^2, X_{t+k-j}^2) = \sum_{i=1}^{q}\sum_{j=1}^{q} \alpha_i \alpha_j \gamma(|k+i-j|),$$

which is free of t and only depends on k. Hence $\{\sigma_t^2\}$ have to be weakly-stationary.