

STAT4005: Time Series
Tutorial 9 - Non-stationarity
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Week 12

9 Non-stationarity

Recall that a time-series $\{Y_t\}$ is weakly-stationary if

1. $\mu = \mathrm{E}(Y_t)$ is constant; and
2. $\gamma(k) = \mathrm{Cov}(Y_t, Y_{t+k})$ depends only on k but not t.

Consider its contrapositive statement, i.e., $\{Y_t\}$ is not weakly-stationary if

1. **(Non-stationarity in mean)** $\mathrm{E}(Y_t)$ is non-constant; or
2. **(Non-stationarity in variance)** For some k_0, $\mathrm{Cov}(Y_t, Y_{t+k_0})$ varies for different value of t.

9.1 Non-stationarity in mean

Trend-stationary process

Definition 1. *A time-series $\{Y_t\}$ is trend-stationary if it could be expressed as*

$$Y_t = T_t + N_t,$$

where T_t is the trend function, N_t is a weakly-stationary process.

Consider the model

$$Y_t = \alpha Y_{t-1} + T_t + N_t,$$

- When $\alpha = 1$, the process is non-stationary (i.e., having a unit root);
- When $\alpha < 1$, the process is trend-stationary.

However, it is difficult to obtain a feasible test due to the complicated structure of $\{N_t\}$. Therefore, we restrict N_t as the white-noise sequence, denoted by Z_t, i.e., to consider testing of

$$H_0 : \alpha = 1 \qquad \text{against} \qquad H_1 : \alpha < 1,$$

under the model

$$Y_t = \alpha Y_{t-1} + T_t + Z_t,$$

some common examples of T_t is given in the following table:

Case	1	2	3
T_t	0	β_0	$\beta_0 + \beta_1 t$

Those tests for detecting the existence of unit roots are known as the unit-root test. In this course, we focus on Case 1 and introduce the Dickey-Fuller test as one of the unit-root tests.

Dickey-Fuller Test

Theorem 1. *Given that* $Y_t = \alpha Y_{t-1} + Z_t$, *consider testing*

$$H_0 : \alpha = 1 \qquad \text{against} \qquad H_1 : \alpha < 1,$$

Under H_0, *the Dickey-Fuller test statistics*

$$T = n(\hat{\alpha} - 1) \xrightarrow{\mathrm{d}} \mathbb{T} = \frac{\int_0^1 W(t)dW(t)}{\int_0^1 W^2(t)dt}, \quad \text{where} \quad \hat{\alpha} = \frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=2}^{n} Y_{t-1}^2},$$

and $\{W(t)\}$ *is a Brownian motion. Then* H_0 *is rejected at* $100(1-\alpha)\%$ *confidence when* T *is lower than the* $100\alpha\%$ *quantile of* $\mathbb{T}$.

Remark 1. *The following are some remarks on the Dickey-Fuller test:*

- *In this course, unit-root test always refer to the Dickey-Fuller test;*
- *There is a more general version of Dickey-Fuller test, known as the augmented Dickey–Fuller test. The Dickey-Fuller test is equivalent to the augmented Dickey–Fuller test with lag-order* $p = 0$;
- *If* T_t *is chosen as other function, see Table* **??**, *the critical value and the limiting distribution of the test will also be different.*

Application of Dickey-Fuller test

Exercise 1. *Let* Y_t *be the log-price of HSBC Holdings plc on the* t*th day in 2023. Given the following R-command:*

```
library(tseries) ; library(quantmod)
x=getSymbols("0005.HK", from="2023-01-01",
             to="2023-12-31", src="yahoo", auto.assign=FALSE)
y=as.numeric(log(x[!is.na(x[,6]),6]))
adf.test(y, k=0)

    Augmented Dickey-Fuller Test

data:  y
Dickey-Fuller = -3.7238, Lag order = 0, p-value = 0.02342
alternative hypothesis: stationary
```

1. *State the null and alternative hypothesis of unit-root test.*
2. *Interpret the result obtained from implementing the R-command.*

Solution

1. Assuming that $Y_t = \beta_0 + \alpha Y_{t-1} + Z_t$, we are testing

$$H_0 : \alpha = 1 \quad \text{against} \quad H_1 : \alpha < 1.$$

2. Based on the R-output, as p-value is less than 0.05, there is significant evidence to claim that $\alpha < 1$, i.e., the model is weakly-stationary.

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9.2 Non-stationarity in variance

There are many possibilities for a process exhibiting non-stationarity in variance. The following is one of the common setting:

$$Y_t = T_t + N_t, \quad \text{where} \quad \mathrm{Var}(N_t) = h^2(T_t)\sigma^2$$

for some function $h(\cdot)$. It could be tackled by using the variance stabilizing transformation.

Variance Stabilizing Transform

Given that $X \sim [\mu, h^2(\mu)]$, the variance stabilizing transformation of X is given by

$$g(x) = k\int_0^x \frac{1}{h(\mu)}d\mu \quad (k > 0).$$

Noticing that by Taylor-expansion,

$$g(X) \approx g(\mu) + g'(\mu)(X-\mu) \quad \Rightarrow \quad \mathrm{Var}\{g(X)\} \approx \{g'(\mu)\}^2 \cdot \mathrm{Var}(X) = \frac{k^2}{\{h'(\mu)\}^2} \cdot h^2(\mu)\sigma^2 = k^2\sigma^2.$$

In short, the task is simplified as follows.

$$\boxed{\{Y_t\} \text{ (non-constant variance)}} \overset{\text{Variance-stabilizing}}{\Rightarrow} \boxed{\{h(Y_t)\} \text{ (constant variance)}}$$

Remark 2. *The following are some remarks to the variance-stabilizing approach:*

- *The setting in this subsection only focus on the non-stationarity in variance, but not covariance. It means that there is no guarantee that the transformed series is a stationary process.*
- *In reality, the form of $h(\cdot)$ is commonly unknown. In this case, the Box-cox transformation may serve as a practical candidate for a variance-stabilizing function.*

Variance-stabilizing transform

Exercise 2. *Assume that $Y_t = T_t + N_t$, where $\mathrm{Var}(N_t) = T_t^4\sigma^2$. Find a variance stabilizing transformation function for $\{Y_t\}$ such that $\mathrm{Var}\{g(Y_t)\} = \sigma^2$ for all t.*

Solution

As $\mathrm{Var}(N_t) = T_t^4\sigma^2$, we have $h(T_t) = T_t^2$. Therefore, a variance stabilizing transform is given by

$$g(y) = \int_0^y \mu^{-2}d\mu = -\frac{1}{2y}.$$