

STAT4005 - Time Series
Tutorial 3 - Parametric Time Series Model
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Week 5 & 6

3 Time Series Models

Recall that B is the backshifting operator in the sense that $B^k X_t = X_{t-k}$ for all $k \in \mathbb{N}$.

3.1 Identification of Models

Common Linear Time Series Model

Definition 1. *Let $\{Y_t\}$ be a time-series and $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Then we say*

1. **(Autoregressive Model)** $\{Y_t\} \sim \mathrm{AR}(\boldsymbol{p})$ *if* $\phi(B)Y_t = Z_t$, *i.e.*

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + Z_t,$$

where $\phi(x) := 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$ *is the* ***AR characteristic polynomial****.*

2. **(Moving Average Model)** $\{Y_t\} \sim \mathrm{MA}(\boldsymbol{q})$ *if* $Y_t = \theta(B)Z_t$, *i.e.*

$$Y_t = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q},$$

where $\theta(x) := 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$ *is the* ***MA characteristic polynomial****.*

3. **(ARMA)** $\{Y_t\} \sim \mathrm{ARMA}(\boldsymbol{p}, \boldsymbol{q})$ *if* $\phi(B)Y_t = \theta(B)Z_t$, *i.e.*

$$Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q},$$

where (i). $\phi(1) \neq 0$, *(ii).* $\phi(\cdot)$ *and* $\theta(\cdot)$ *have NO common root.*

4. **(ARIMA)** $\{Y_t\} \sim \mathrm{ARIMA}(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{q})$ *if*

$$\phi(B)(1 - B)^d Y_t = \theta(B)Z_t,$$

where (i). $\phi(1) \neq 0$, *(ii).* $\phi(x)(1-x)^d$ *and* $\theta(x)$ *have NO common root.*

5. **(SARIMA)** $\{Y_t\} \sim \mathrm{SARIMA}(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{q}) \times (\boldsymbol{P}, \boldsymbol{D}, \boldsymbol{Q})_{\boldsymbol{s}}$ *if*

$$\phi(B)\Phi_P(B^s)(1 - B)^d(1 - B^s)^D Y_t = \theta(B)\Theta_Q(B^s)Z_t,$$

where $s > 1$,

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \cdots - \Phi_P B^{sP} \quad \text{and} \quad \Theta_Q(B^s) = 1 - \Theta_1 B^s - \cdots -_P B^{sQ},$$

(i). $\phi(1)$ *and* $\Phi_P(1)$ *are non-zero, (ii).* $\phi(x)\Phi_P(x^s)(1-x)^d(1-x^s)^D$ *and* $\theta(x)\Theta_Q(x^s)$ *have NO common root.*

Remark 1. *There are several remarks on the identification of models*

- *If there is common factor(s) in* $\phi(\cdot)$ *and* $\theta(\cdot)$ *[or in* $\Phi_P(\cdot)$ *and* $\theta_Q(\cdot)$*], i.e. say* $\phi(x) = (1-c)\phi'(x)$ *and* $\theta(x) = (1-c)\theta'(x)$*, we should cancel them out and consider the* $\phi'(\cdot)$ *and* $\theta'(\cdot)$ *as the true AR and MA characteristic polynomial instead.*

- SARIMA *model can be expressed as high-order* ARIMA *model. However, you are required to write a more informative representation for the assessment. See part b of exercise 1.*

The following flowchart summarizes the relationship among the models mentioned above.

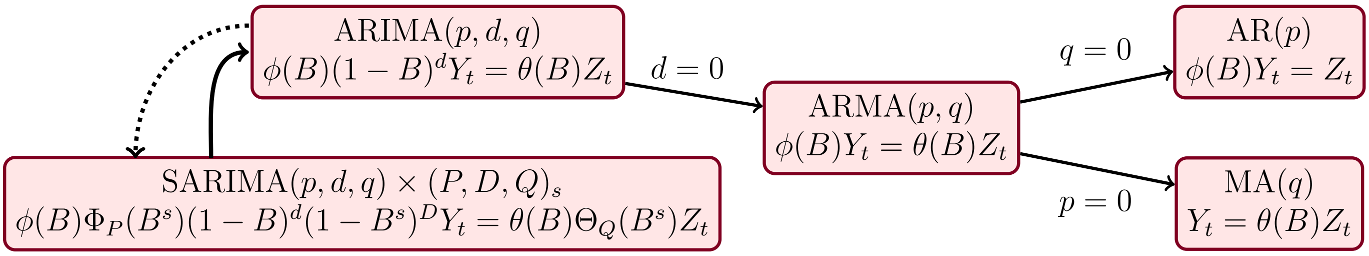

Figure 1: Relationship among models

General Guidance in Identification of Models

For the SARIMA model, we have to notice if there is a common pattern lag among $Y_t's$ or $Z_t's$ (eg: $Y_t - 4Y_{t-3} = Z_t + 2Z_{t-3} + 0.5Z_{t-6}$) within the model. If such a lag is not observed, it should be an ARIMA instead. The general strategy for identifying an ARIMA model is as follows:

1. Rearrange the time series model such that it could be written in the form of $\phi_0(B)Y_t = \theta_0(B)Z_t$.
2. **(Identifying MA polynomial $\boldsymbol{\theta}$)** Cancel out common factors of $\phi_0(x)$ and $\theta_0(x)$ (if any) to obtain $\phi_1(x)$ and $\theta(x)$. Then we have $\phi_1(B)Y_t = \theta(B)Z_t$.
3. **(Identifying AR polynomial $\boldsymbol{\phi}$)** Write $\phi_1(x) = (1-x)^d\phi(x)$, where $\phi(1) \neq 0$.

Then the time series $\{Y_t\} \sim \text{ARIMA}(p, d, q)$, where $p = \deg(\phi)$ and $q = \deg(\theta)$.

(★★☆) Identification of Models

✎ **Exercise 1.** *Identify the time-series model below. Let $\{Y_t\}$ be a time-series and $\{Z_t\} \sim \text{WN}(0, \sigma^2)$.*

(a) $Y_t - 4Y_{t-1} + 5Y_{t-2} - 2Y_{t-3} = Z_t - \frac{8}{3}Z_{t-1} + \frac{4}{3}Z_{t-2}$.

(b) $Y_t - Y_{t-1} + 0.5Y_{t-4} - 0.5Y_{t-5} = Z_t + 0.25Z_{t-4} + 0.6Z_{t-8}$

Attempt

3.2 Causality and Invertibility of Models

Causality and Invertibility

Definition 2. *Let* $\{Y_t\}$ *be a time-series and* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$*. Then we say*

1. *A model is* ***causal*** *if there exists a sequence* $\{\psi_k\}_{k=0}^{\infty}$ *such that* $\sum_{k=0}^{\infty} |\psi_k| < \infty$ *with*

$$Y_t = \sum_{k=0}^{\infty} \psi_k Z_{t-k},$$

2. *A model is* ***invertible*** *if there exists a sequence* $\{\psi_k\}_{k=0}^{\infty}$ *such that* $\sum_{k=0}^{\infty} |\psi_k| < \infty$ *with*

$$Z_t = \sum_{k=0}^{\infty} \psi_k Y_{t-k},$$

Verification for Causality and Invertibility

Theorem 1. *An* $\mathrm{ARMA}(p, q)$ *process* $\phi(B)Y_t = \theta(B)Z_t$ *is*

1. ***Causal*** *if and only if roots of* $\phi(x) = 0$ *are ALL outside the unit circle.*
2. ***Invertible*** *if and only if roots of* $\theta(x) = 0$ *are ALL outside the unit circle.*

where the term "x out of the unit circle" means that $|x| > 1$*, and for all* $x \in \mathbb{C}$, $|x|^2 := \mathrm{Re}(x)^2 + \mathrm{Im}(x)^2$.

(★☆☆) Causality and Invertibility of Models

✎ **Exercise 2.** *Let* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$*. Show if the following time/series are causal or invertible.*

(a) $Y_t + 3Y_{t-1} = 2Z_t - Z_{t-1}$.

(b) $1.5Y_t - Y_{t-1} = 2Z_t + 3Z_{t-1}$.

(c) $Y_t = 4Y_{t-1} - 4Y_{t-2} + Z_t - 5Z_{t-1} + 6Z_{t-2}$.

Attempt

Consider the ARMA(p,q) process $\phi(B)Y_t = \theta(B)Z_t$. You might be asked to show whether a model is causal/invertible and write down the corresponding representation.

1. If a model is causal, then we can write the **MA/causal representation** as $Y_t = \phi(B)^{-1}\theta(B)Z_t$.
2. If a model is invertible, then we can write the **AR representation** as $Z_t = \theta(B)^{-1}\phi(B)Y_t$.

In order to evaluate $\phi(B)^{-1}$ or $\theta(B)^{-1}$ explicitly, recall for $|r| < 1$,

$$\frac{a}{1-r} = a\sum_{k=0}^{\infty} r^k$$

and by the fundamental theorem of algebra, we can write any degree-p polynomial $P(x)$ as $P(x) = c\prod_{k=1}^{p}(x-\xi_k)$ for some $c, \xi_1, \ldots, \xi_p \in \mathbb{C}$. Sometimes, the method of partial fraction can help further simplify the calculation. For example, see exercise 8.

(★★☆) AR and MA representation

✎ **Exercise 3.** ***(Continuation of Exercise 2)*** *Let* $\{Z_t\} \sim \mathrm{WN}(0,\sigma^2)$. *Find the AR and MA representation (if possible) of the following model*

(a) $Y_t + 3Y_{t-1} = 2Z_t - Z_{t-1}$.

(b) $1.5Y_t - Y_{t-1} = 2Z_t + 3Z_{t-1}$.

Attempt

Takeaway 1. *For a sequence* $\{a_n\}$, $\sum_{k=L}^{U} a_k = \sum_{k=L-s}^{U-s} a_{k+s}$

3.3 Stationarity of Models

In this subsection, we will answer the following questions

1. How to check the stationarity of a given time-series model?
2. If a time-series is stationary, how to find its ACVF $\gamma(\cdot)$

We then answer the first question.

Stationarity of Models

Theorem 2. *Consider a time series* $\{Y_t\}$.

1. *If* $\{Y_t\} \sim \text{ARIMA}(p, d, q)$ *with* $d \geq 1$, *then* $\{Y_t\}$ *is NOT weakly-stationary.*
2. *If* $\{Y_t\} \sim \text{ARMA}(p, q)$, *then* $\{Y_t\}$ *is weakly-stationary if NO root of* $\phi(x) = 0$ *is* ***on*** *the unit circle, i.e. if* $\phi(x) = 0$, $|x| \neq 1$.

Remark 2. *There are several implications due to the theorem above.*

- $\{Y_t\} \sim \text{MA}(q) \Rightarrow$ *Causal* $\Rightarrow$ *Weakly-Stationary.*
- *For* $\{Y_t\} \sim \text{ARMA}(p, q)$ *being stationary,* $\text{E}(Y_t) = 0$.

(★☆☆) Verification of Stationarity

Exercise 4. *Show whether the following time-series is weakly-stationary.*

(a) $Y_t + Y_{t-2} = 3Z_t - Z_{t-1}$.

(b) $Y_t - Y_{t-1} = Z_t - 1.5Z_{t-1}$.

(c) $Y_t = 0.5Y_{t-1} + 0.5Y_{t-2} + Z_t - 1.2Z_{t-1} + 0.2Z_{t-2}$.

Attempt

We then discuss the evaluation of ACVF $\gamma(\cdot)$. As the ARIMA(p, d, q) model with $d \geq 1$ is always non-stationary, we restrict our focus of study to be the ARMA(p, q) model. There are two common ways to complete the task. The first approach requires finding the MA representation of a time-series.

Evaluation of ACVF - Method (I) - MA representation

Theorem 3. *For **causal** ARMA process $\phi(B)Y_t = \theta(B)Z_t$, we can write the MA representation $Y_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$ with $\sum_{i=0}^{\infty} |\psi_j| < \infty$ and ACVF*

$$\gamma(k) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}.$$

Remark 3. *For **stationary** ARMA process (NOT necessarily causal), we can represent Y_t by $Y_t = \sum_{i=-\infty}^{\infty} \psi_i Z_{t-i}$ with $\sum_{i=-\infty}^{\infty} |\psi_j| < \infty$ and ACVF $\gamma(k) = \sigma^2 \sum_{i=-\infty}^{\infty} \psi_i \psi_{i+k}$.*

(★☆☆) Application of Method I

Exercise 5. *Consider the process $Y_t = 0.8Y_{t-1} + Z_t$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.*

(a) Is the process $\{Y_t\}$ weakly stationary?

(b) Find the mean and ACVF $\gamma(\cdot)$ of $\{Y_t\}$ by Theorem 3.

Attempt

Evaluation of ACVF - Method (II) - Yule-Walker Equations

Theorem 4. *The general procedure is as follow: For* ***stationary*** ARMA(p,q) *model*

1. *Consider the set of linear equations (i.e. the Yule-Walker Equations)*

$$\gamma(k) = \mathrm{Cov}(Y_t, Y_{t-k}), \quad \textit{for} \quad k = 0, \ldots, p$$

   to solve for value of $\gamma(0), \ldots, \gamma(p)$.

2. *For* $k > p$, *compute* $\gamma(k) = \mathrm{Cov}(Y_t, Y_{t-k})$ *based on value of* $\gamma(0), \ldots, \gamma(p)$.

Lemma 1. *The following are the Yule-Walker Equations of some particular models.*

1. *For* $\{Y_t\} \sim \mathrm{AR}(p)$ *being causal with* $Z_t \sim \mathrm{WN}(0, \sigma^2)$, *the Yule-Walker Equations are*

$$\begin{cases} \gamma(0) &= \phi_1\gamma(1) + \phi_2\gamma(2) + \cdots + \phi_p\gamma(p) + \sigma^2 \\ \gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) + \cdots + \phi_p\gamma(p-1) \\ \gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \cdots + \phi_p\gamma(p-2) \\ \quad\vdots &= \quad\vdots \\ \gamma(p) &= \phi_1\gamma(p-1) + \phi_2\gamma(p-2) + \cdots + \phi_p\gamma(0). \end{cases}$$

   and for $k > p$, $\gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p)$.

2. *For* $\{Y_t\} \sim \mathrm{ARMA}(p,q)$, *i.e.* $\phi(B)Y_t = \theta(B)Z_t$ *with* $Z_t \sim \mathrm{WN}(0, \sigma^2)$, *the Yule-Walker Equations are*

$$\begin{cases} \gamma(0) &= \phi_1\gamma(1) + \phi_2\gamma(2) + \cdots + \phi_p\gamma(p) + \mathrm{Cov}(Y_t, \theta(B)Z_t) \\ \gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) + \cdots + \phi_p\gamma(p-1) + \mathrm{Cov}(Y_{t-1}, \theta(B)Z_t) \\ \gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \cdots + \phi_p\gamma(p-2) + \mathrm{Cov}(Y_{t-2}, \theta(B)Z_t) \\ \quad\vdots &= \quad\vdots \\ \gamma(p) &= \phi_1\gamma(p-1) + \phi_2\gamma(p-2) + \cdots + \phi_p\gamma(0) + \mathrm{Cov}(Y_{t-p}, \theta(B)Z_t). \end{cases}$$

   and for $k > p$, $\gamma(k) = \phi_1\gamma(k-1) + \cdots + \phi_p\gamma(k-p)$.

Remark 4. *The causality assumption is important for deriving the Yule-Walker equations. Recall that causality is equivalent to require* $Y_t \perp\!\!\!\perp Z_{t+k}$ *for all* $k > 0$.

- *For the first equation in the Yule-Walker equation of causal* AR(p) *model, the term* σ^2 *arises as*

$$\mathrm{Cov}(Z_t, Y_t) = \mathrm{Cov}(Z_t, \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + Z_t) = \mathrm{Cov}(Z_t, Z_t) = \sigma^2.$$

  However for non-causal model, $\mathrm{Cov}(Y_{t-k}, Z_t) \neq 0$ *in general for* $k \geq 1$.

- *The Yule-Walker equation derived for* ARMA(p,q) *does not require causality in general. However, the evaluation of the term* $\mathrm{Cov}(Y_{t-k}, \theta(B)Z_t)$ *would be complicated.*

Hence, please ensure whether the model is causal before applying the method of Yule-Walker equation.

(★★☆) Application of Method II

✎ **Exercise 6.** ***(Continuation of Exercise 5)*** *Consider the process* $Y_t = 0.8Y_{t-1} + Z_t$*, where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$*. Find the mean and ACVF* $\gamma(\cdot)$ *of* $\{Y_t\}$ *by Theorem 4.*

Attempt

3.4 Chapter Summary

In this subsection, we make a brief summary to this chapter

Model	Causality	Invertibility	Stationarity	ACVF
SARIMA	✘	$\lvert\text{MA root}\rvert > 1$	✘	\
ARIMA	✘	$\lvert\text{MA root}\rvert > 1$	✘	\
ARMA	$\lvert\text{AR root}\rvert > 1$	$\lvert\text{MA root}\rvert > 1$	$\lvert\text{AR root}\rvert \neq 1$	Both OK
AR	$\lvert\text{AR root}\rvert > 1$	✔	$\lvert\text{AR root}\rvert \neq 1$	Yule-Walker
MA	YES	$\lvert\text{MA root}\rvert > 1$	✔	MA-representation

3.5 Additional Exercises

(★★☆) ACF of AR(2) Model

✎ **Exercise 7.** *Consider causal AR(2) model* $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t$*, where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$*.*

(a) Find $\rho(1)$ *and* $\rho(2)$ *in terms of* ϕ_1 *and*ϕ_2*.*

(b) Express $\gamma(0)$ *in terms of* ϕ_1, ϕ_2 *and* σ^2*.*

Attempt

(★★★) Comprehensive Exercise

✎ **Exercise 8.** *Consider* $Y_t - 3.5Y_{t-1} + 3Y_{t-2} = 2Z_t - Z_{t-1}$, *where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.

(a) *Is the model causal or invertible? If yes, find its corresponding MA and AR representation.*

(b) *Is the model stationary?*

(c) *Write* $Y_t = \sum_{i=-\infty}^{\infty} \psi_i Z_{t-i}$. *Find the ACVF of* $\gamma(\cdot)$. *(Optional!)*

Attempt