

STAT4005: Time Series
Mock Final Exam
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Let $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$ unless otherwise specified. You are reminded that **additional distribution assumptions on $\{a_t\}$ may be stated in certain parts of certain questions.**

(★★★) True and False

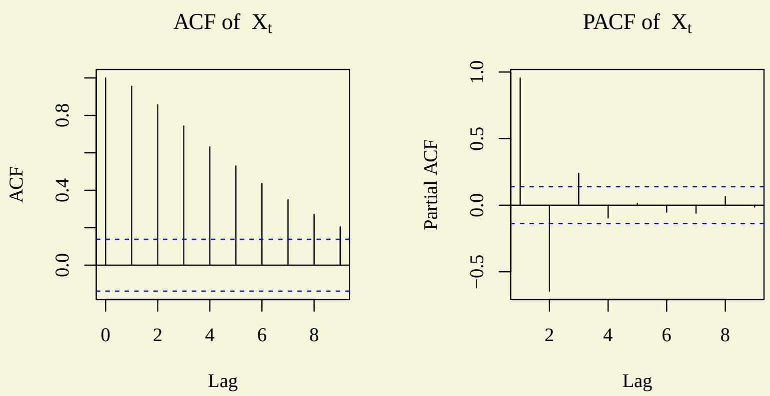
✎ **Exercise 1.** *State whether the following claims are true (T) or false (F).*

- *Briefly explain your answer. (A short and precise explanation is sufficient as long as the argument can reflect the understanding related to the question)*
- ***No credit will be given if you only answer true or false without proper explanation.***

(a) (2 marks) Assume that $X_t = T_t + a_t$, where $\{T_t\}$ is a deterministic trend.
Statement: there exists some p such that the p-th order differenced series $\{\Delta^p X_t\}$ is weakly stationary.

(b) (2 marks) For AR(p) model, i.e., $X_t = \sum_{k=1}^{p} \phi_p X_{t-k} + a_t$, approximates the quantity $\mathrm{E}\{(\hat{\phi} - \phi)^T(\hat{\phi} - \phi)\} = \sum_{k=1}^{p} \mathrm{MSE}(\hat{\phi}_k)$, where $\hat{\phi}_k$ is the MLE of ϕ_k.
Statement: if $p_1 < p_2$, the FPE of AR(p_1) always less than AR(p_2) model and we should choose AR(p_1) based on FPE.

(c) (2 marks) **Statement:** According to the following graphs, $\{X_t\}$ is likely to be an AR(3) model.



(d) (2 marks) Suppose that we want to choose a model among class of ARMA model. Define the criterion of the ARMA(p, q) model as $-2 \log L + 2(p - q + 1)$.
Statement: it is a sensible criterion for model selection.

(e) (2 marks) Consider AR(1) model $Y_t = \alpha Y_{t-1} + Z_t$, where $\{Z_t\} \overset{iid}{\sim} \mathrm{N}(0, \sigma^2)$. Let $\hat{Y}_{n+h} = \mathrm{E}(Y_{n+h}|Y_1, \ldots, Y_n)$ and $V_h = \mathrm{Var}(e_n(h))$.
Statement: the 95% prediction interval of $Y_{n+1} + Y_{n+2}$ is given by

$$\left(Y_{n+1} + Y_{n+2} - 1.96(V_1 + V_2)^{1/2}, Y_{n+1} + Y_{n+2} + 1.96(V_1 + V_2)^{1/2}\right).$$

(f) (2 marks) Assume $X_t = \sigma_t \epsilon_t$ and $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$ $(0 < \alpha_0, \alpha_1 < 1)$, where $\{\epsilon_t\} \overset{iid}{\sim} \mathrm{N}(0, 1)$. Also assume $\mathrm{E}(X_t^4) < \infty$.
Statement: $\{X_t\}$ is weakly stationary, while $\{X_t^2\}$ is not, in general.

(★☆☆) Estimation

✎ **Exercise 2.** ***(20 marks)*** *Consider causal* AR(2) *Model* $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + a_t$. *Given*

$$(Y_1, \ldots, Y_n) = (-1.5, 0.4, 1, 1.5, 0.6).$$

(a) (4 marks) Evaluate sample ACVF C_0, C_1 *and* C_2.

(b) (7 marks) Find the Yule-Walker estimators of ϕ_1 *and* ϕ_2. *Hence, find an estimator of* σ^2.

(c) (6 marks) Suppose that we have

$$\sqrt{n}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \xrightarrow{\text{d}} \text{N}_2(\mathbf{0}, \boldsymbol{\Sigma}) \quad \textit{and} \quad \hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}.$$

Construct a 95% confidence interval for $\phi_1 - 2\phi_2$.

(d) (3 marks) Consider testing of $H_0 : \phi_1 - 2\phi_2 = 0$ *against* $H_1 : \phi_1 - 2\phi_2 \neq 0$, *should null* H_0 *be rejected under 95% confidence level?*

(★★☆) Forecasting

✎ **Exercise 3.** ***(20 marks)*** *Consider* $X_t = \Delta Y_t = Y_t - Y_{t-1}$, *where* $\{X_t\}$ *satisfies*

$$X_t - \omega X_{t-1} = a_t + \omega a_{t-1} + 0.5\omega a_{t-2},$$

with $\{a_t\} \sim \text{WN}(0, \sigma^2)$. *You are given* n *observations* $(Y_1, Y_2, ..., Y_n)$.

(a) (3 marks) Is the assumption made sufficient for the construction of the prediction interval of Y_{n+h} $(h \geq 1)$ *? State the additional assumption needed if any.*

(b) (8 marks) Assuming the additional assumption made in (a). Find the 1-Step and 2-Step forecasts in terms of $(Y_1, Y_2, \ldots, Y_n)$, $(a_1, a_2, \ldots, a_n)$ *and* ω.

(c) (9 marks) Assuming the additional assumption made in (a). Find the 95% prediction intervals of the above forecasts in terms of $(Y_1, Y_2, \ldots, Y_n)$, ω *and* σ^2.

(★☆☆) Estimation and Forecasting

✎ **Exercise 4.** ***(20 marks)*** *Consider an invertible* ARMA(1, 1) *model* $Y_t = \phi Y_{t-1} + a_t - \theta a_{t-1}$,

(a) (12 marks) Given $Y_1 = 0$, $Y_2 = 5$, $Y_3 = 3$ *and* $Y_4 = 2$. *Write down the objective function to be minimized in order to obtain the conditional least-square estimator of* ϕ *and* θ.

(b) (8 marks) Suppose $\phi = 0.4$ *and* $\theta = -0.6$, *find the 1-st and 2-nd step forecast* $\hat{Y}_5$ *and* $\hat{Y}_6$.

(★★☆) GARCH Model

✎ **Exercise 5.** ***(20 marks)*** *Consider the stationary GARCH(1,1) model*

$$X_t = \epsilon_t \sigma_t, \;\; \epsilon_t \overset{iid}{\sim} \mathrm{N}(0,1) ------(*)$$

$$\sigma_t^2 = 0.5 + 0.1X_{t-1}^2 + 0.2\sigma_{t-1}^2 ------(**)$$

Also assume that $\mathrm{E}(\sigma_t^4) < \infty$ *is constant over-time.*

(a) *(4 marks) Express* $\{X_t^2\}$ *as an ARMA model. (In this exercise, you can directly assume that the sequence* $\{v_t = X_t^2 - \sigma_t^2\}$ *is white noise.)*

(b) *(10 marks) Evaluate* $\mathrm{Var}(v_t)$.

(c) *(6 marks) Find the value of* $\mathrm{Cov}(X_t^2, X_{t+1}^2)$.

(★★☆) Non-linear Time-Series

✎ **Exercise 6.** ***(8 marks)*** *Consider a causal and weakly stationary time series* $\{Y_t\}$ *defined by*

$$Y_t = \phi|Y_{t-2}|a_t + \theta a_{t-1},$$

where $\{a_t\} \overset{iid}{\sim} \mathrm{N}(0,\sigma^2)$ *and* $0 < \phi, \sigma^2 < 1$. *Assume that* $\alpha = \mathrm{E}(|Y_t|)$ *is a known constant over time.*

(a) *(2 marks) Evaluate* $\mathrm{E}(Y_t)$ *and* $\mathrm{E}(Y_t^2)$.

(b) *(3 marks) Find the ACVF* $\gamma(\cdot)$ *of* Y_t *in terms of* ϕ, θ, σ^2 *and* α.

(c) *(1 marks) Which of the following methods can be applied to estimate* ϕ *and* θ*? (I) Method of moments, (II) Least Square Estimation, OR (III) Maximum likelihood estimation.*

(d) *(2 marks) Given observation* $(Y_1, \ldots, Y_n)$*, outline and describe an estimation algorithm to find* $\hat{\phi}$*,* $\hat{\theta}$ *and* $\hat{\sigma}^2$.