STAT4005: Time Series
Tutorial 8 - GARCH Model
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Week 12

8 GARCH Model

8.1 Introduction and Properties of GARCH Model

Define the filtration $\mathscr{F}_t$ as a set that contains all information concerning the time series up to time t.

Heteroskedasticity

Definition 1. *A sequence of random variables* $\{X_t\}$ *is*

1. ***Heteroskedastic****: if* $\operatorname{Var}(X_t)$ *varies as* t *change. Otherwise,* $\{X_t\}$ *is homoskedastic.*
2. ***Conditional Heteroskedastic****: if* $\operatorname{Var}(X_t|\mathscr{F}_{t-1})$ *varies as* t *changes.*

Remark 1. *In reality, the log-asset price data exhibit the property of volatility clustering, i.e.*

$$\operatorname{Var}(X_{t-1}) \textit{ is large (or small)} \quad \Rightarrow \quad \operatorname{Var}(X_t|X_{t-1}) \textit{ is large (or small).}$$

This means that a sensible model should induce conditional heteroskedasticity.

Homoskedasticity of Causal ARMA(p,q) Model

Consider $\{X_t\} \sim \operatorname{ARMA}(p,q)$ being causal, then we can write its causal representation by $X_t = Z_t + \sum_{k=1}^{\infty}\psi_k Z_{t-k}$, where $\{Z_t\} \sim \operatorname{WN}(0,\sigma^2)$ and $\sum_{k=1}^{\infty}|\psi_k| < \infty$. Then we have

$$\operatorname{Var}(X_t|\mathscr{F}_{t-1}) = \operatorname{Var}\left(Z_t + \sum_{k=1}^{\infty}\psi_k Z_{t-k}|\mathscr{F}_{t-1}\right) = \operatorname{Var}(Z_t|\mathscr{F}_{t-1}) = \operatorname{Var}(Z_t) = \sigma^2,$$

i.e., the causal ARMA(p,q) model is conditionally and unconditionally homoskedastic, which does NOT match with the properties of the log-asset price data.

It motivates us to consider a conditionally heteroskedastic model, which is known as the GARCH model.

GARCH Model (Generalized Auto-Regressive Conditional Heteroskedastic Model)

Definition 2. *We say* $\{X_t\} \sim \operatorname{GARCH}(p,q)$ *if*

$$X_t = \sigma_t\epsilon_t, \quad \{\epsilon_t\} \overset{iid}{\sim} \operatorname{N}(0,1); \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p}\beta_i\sigma_{t-i}^2 + \sum_{j=1}^{q}\alpha_j X_{t-j}^2$$

where $\alpha_i, \beta_j \geq 0$ *for all* i,j*. If* $p = 0$*, we say* $\{X_t\} \sim \operatorname{ARCH}(q)$*.*

Stationarity of GARCH

Theorem 1. *If* $\{X_t\} \sim \operatorname{GARCH}(p,q)$*, then* $\{X_t\}$ *is weakly stationary if and only if*

$$\sum_{j=1}^{q}\alpha_j + \sum_{i=1}^{p}\beta_i < 1.$$

Remark 2. *For* $\operatorname{GARCH}(1,1)$ *model with* $\alpha_1 + \beta_1 = 1$*, it is known as the I-GARCH(1,1) model.*

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Implication of Stationary GARCH Model

If $\{X_t\} \sim \text{GARCH}(p, q)$ is weakly-stationary, then

- (First Moment of $\{X_t\}$) $\text{E}(X_t) = \text{E}(\sigma_t \epsilon_t) = \text{E}(\sigma_t)\text{E}(\epsilon_t) = 0$.
- (Second Moment of $\{X_t\}$) $\text{E}(X_t^2) = \text{E}(\sigma_t^2 \epsilon_t^2) = \text{E}(\sigma_t^2)\text{E}(\epsilon_t^2) = \text{E}(\sigma_t^2)$.
- (First Moment of $\{\sigma_t^2\}$) $\text{E}(\sigma_t^2) = \text{E}(\sigma_t^2 \epsilon_t^2) = \text{E}(X_t^2) = \text{E}(X_{t+k}^2) = \text{E}(\sigma_{t+k}^2 \epsilon_{t+k}^2) = \text{E}(\sigma_{t+k}^2)$.

Therefore, $\text{E}(\boldsymbol{X_t^2}) = \text{E}(\boldsymbol{\sigma_t^2})$ **are constant over time**.

Then for $\{X_t\} \sim \text{GARCH}(p, q)$ being stationary, we have

Quantity	**Unconditional**	**Conditional** on $\mathscr{F}_{t-1}$
Mean	$\text{E}(X_t) = 0$	$\text{E}(X_t \mid \mathscr{F}_{t-1}) = 0$
Variance	$\text{Var}(X_t) = \dfrac{\alpha_0}{1 - \left(\sum_{i=1}^q \alpha_i + \sum_{j=1}^p \beta_j\right)}$	$\text{Var}(X_t \mid \mathscr{F}_{t-1}) = \sigma_t^2$
Covariance	$\text{Cov}(X_t, X_{t+h}) = 0$	$\text{Cov}(X_t, X_{t+h} \mid \mathscr{F}_{t-1}) = 0$

❗ **Remark 3.** *We make the following observations*

- $\text{Var}(X_t)$ *is constant over time, i.e. the model is unconditionally homoskedastic.*
- $\text{Var}(X_t \mid \mathscr{F}_{t-1})$ *is not constant over time, i.e. the model is conditionally heteroskedastic.*

Basic Properties of GARCH model

✎ **Exercise 1.** *Consider the following* GARCH *model:*

$$X_t = \sigma_t \epsilon_t; \quad \sigma_t^2 = 1.5 + 0.5\sigma_{t-1}^2 + 0.2X_{t-1}^2, \quad \{\epsilon_t\} \overset{iid}{\sim} \text{N}(0, 1)$$

(a) Is $\{X_t\}$ *stationary?*

(b) Evaluate $\text{E}(X_t)$, $\text{Var}(X_t)$, $\text{E}(\sigma_t^2)$ *and* $\text{Cov}(X_t, X_{t+k})$ *for* $k \neq 0$.

(c) Compare $\text{Var}(X_t)$ *and* $\text{Var}(X_t \mid \mathscr{F}_{t-1})$.

Solution

(a) As $\alpha_0, \alpha_1, \beta_1 \geq 0$ and $\alpha_1 + \beta_1 = 0.2 + 0.5 = 0.7 < 1$, $\{X_t\}$ is stationary.

(b) For $\{X_t\}$ being stationary, Both $\text{E}(X_t^2) = \text{E}(\sigma_t^2)$ is constant over-time and hence

$$\text{E}(\sigma_t^2) = 1.5 + 0.5\text{E}(\sigma_{t-1}^2) + 0.2\text{E}(X_{t-1}^2) = 1.5 + 0.5\text{E}(\sigma_t^2) + 0.2\text{E}(X_t^2) = 1.5 + 0.5\text{E}(\sigma_t^2) + 0.2\text{E}(\sigma_t^2).$$

Hence $\text{E}(X_t^2) = \text{E}(\sigma_t^2) = 5$. Also, $\text{E}(X_t) = \text{E}(\sigma_t \epsilon_t) = \text{E}(\sigma_t)\text{E}(\epsilon_t) = 0$. Then $\text{E}(X_t) = 0$, $\text{Var}(X_t) = \text{E}(X_t^2) - \text{E}(X_t)^2 = 5$. WLOG, assume that $k > 0$ (as $\{X_t\}$ is stationary). Then we have

$$\text{Cov}(X_t, X_{t+k}) = \text{E}(X_t X_{t+k}) = \text{E}(\sigma_t \sigma_{t+k} \epsilon_t \epsilon_{t+k}) = \text{E}(\sigma_t \sigma_{t+k} \epsilon_t)\text{E}(\epsilon_{t+k}) = 0.$$

(c) $\text{Var}(X_t \mid \mathscr{F}_{t-1}) = \sigma_t^2 = 1.5 + 0.5\sigma_{t-1}^2 + 0.2X_{t-1}^2 \neq 5 = \text{Var}(X_t)$ in general.

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ARMA Representation of GARCH Model

Theorem 2. *Consider* $\{X_t\} \sim \mathrm{GARCH}(p,q)$. *If*

1. $\{X_t\}$ *is stationary;*
2. $\mathrm{E}(\sigma_t^4)$ *exists and is constant over-time*

Then $\{X_t^2\} \sim \mathrm{ARMA}(m,p)$ $(m = \max\{p,q\})$ *with white noise* $\{v_t\}$ *and* $v_t = \sigma_t^2(\epsilon_t^2 - 1) = X_t^2 - \sigma_t^2$.

Remark 4. *The second condition is needed to ensure that* $\mathrm{Var}(v_t) < \infty$ *is constant. One sufficient condition is to assume* $\{\sigma_t^2\}$ *being weakly stationary.*

Proof of ARMA Representation

Exercise 2. *Given that* $\{X_t\} \sim \mathrm{GARCH}(p,q)$ *is stationary, then express* $\{X_t^2\}$ *as* ARMA *process and identify the associated white noise sequence.*

Solution

The white-noise sequence is given by $v_t = X_t^2 - \sigma_t^2 = \sigma_t^2(\epsilon_t^2 - 1)$. Then $\{X_t^2\} \sim \mathrm{ARMA}(m,q)$ as

$$X_t^2 = \sigma_t^2 + (X_t^2 - \sigma_t^2) = \alpha_0 + \sum_{j=1}^{q} \alpha_j X_{t-j}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + (X_t^2 - \sigma_t^2)$$

$$= \alpha_0 + \sum_{j=1}^{m} (\alpha_j + \beta_j) X_{t-j}^2 - \sum_{i=1}^{p} \beta_i (X_{t-i}^2 - \sigma_{t-i}^2) + v_t = \alpha_0 + \sum_{j=1}^{m} (\alpha_j + \beta_j) X_{t-j}^2 + v_t - \sum_{i=1}^{p} \beta_i v_{t-i},$$

where $\alpha_i := 0$ and $\beta_j := 0$ for $i \geq q, j \geq p$.

Remark 5. *Stationary GARCH exhibits some common features with the log-asset series in reality:*

- $\{X_t\}$ *is heavy-tailed. (Evaluate* $\mathrm{E}(X_t^4)$*)*
- $\{X_t\}$ *is serially uncorrelated; see Ex 1.*
- $\{X_t^2\}$ *is serially correlated; see Theorem 2.*
- $\{X_t\}$ *satisfies volatility cluster. (Definition)*

ARMA Representation of GARCH model

✎ **Exercise 3.** *(Continuation of Exercise 1) Consider the following* GARCH *model:*

$$X_t = \sigma_t \epsilon_t; \quad \sigma_t^2 = 1.5 + 0.5\sigma_{t-1}^2 + 0.2X_{t-1}^2, \quad \{\epsilon_t\} \overset{iid}{\sim} \mathrm{N}(0,1)$$

Further assume that $\mathrm{E}(\sigma_t^4) < \infty$ *is constant over time.*

(a) *Express* $\{X_t^2\}$ *as an ARMA model and identify the associated white noise sequence* $\{v_t\}$.

(b) *Evaluate* $\mathrm{E}(\sigma_t^4)$ *and* $\mathrm{Var}(v_t)$.

(c) *Show that* $\{X_t^2\}$ *is weakly stationary. Therefore, evaluate* $\mathrm{Cov}(X_t^2, X_{t+1}^2)$.

Solution

(a) The white-noise sequence is given by $v_t = X_t^2 - \sigma_t^2 = \sigma_t^2(\epsilon_t^2 - 1)$. Then

$$\begin{aligned} X_t^2 &= \sigma_t^2 + (X_t^2 - \sigma_t^2) = 1.5 + 0.2X_{t-1}^2 + 0.5\sigma_{t-1}^2 + (X_t^2 - \sigma_t^2) \\ &= 1.5 + (0.5 + 0.2)X_{t-1}^2 - 0.5(X_{t-1}^2 - \sigma_{t-1}^2) + v_t \\ &= 1.5 + 0.7X_{t-1}^2 + v_t - 0.5v_{t-1} \end{aligned}$$

Hence $\{X_t^2\} \sim \mathrm{ARMA}(1,1)$ with white noise $\{v_t\}$.

(b) Noticing that $\mathrm{E}(\epsilon_t^2) = 1$ and $\mathrm{E}(\epsilon_t^4) = 3$. Then

$$\mathrm{Var}(v_t) = \mathrm{E}\{(X_t^2 - \sigma_t^2)^2\} - \mathrm{E}(X_t - \sigma_t)^2 = \mathrm{E}(\epsilon_t^4\sigma_t^4 - 2\epsilon_t^2\sigma_t^4 + \sigma_t^4) = 2\mathrm{E}(\sigma_t^4).$$

Recall from Exercise 1 that $\mathrm{E}(\sigma_t^2) = \mathrm{E}(X_t^2) = 5$. Then

$$\begin{aligned} &\mathrm{E}(\sigma_t^4) = \mathrm{E}\{(1.5 + 0.5\sigma_{t-1}^2 + 0.2X_{t-1}^2)^2\} \\ =&1.5^2 + 0.5^2\mathrm{E}(\sigma_{t-1}^4) + 0.2^2\mathrm{E}(X_{t-1}^4) + 2(0.5)(0.2)\mathrm{E}(\sigma_{t-1}^2X_{t-1}^2) \\ &+ 2(1.5)(0.5)\mathrm{E}(\sigma_{t-1}^2) + 2(1.5)(0.2)\mathrm{E}(X_{t-1}^2) \\ =&2.25 + 0.25\mathrm{E}(\sigma_t^4) + 0.04\mathrm{E}(\epsilon_{t-1}^4\sigma_{t-1}^4) + 0.2\mathrm{E}(\epsilon_{t-1}^2\sigma_{t-1}^4) + 1.5\mathrm{E}(\sigma_{t-1}^2) + 0.6\mathrm{E}(\epsilon_{t-1}^2\sigma_{t-1}^2) \\ =&2.25 + 0.25\mathrm{E}(\sigma_t^4) + 0.04(3)\mathrm{E}(\sigma_t^4) + 0.2\mathrm{E}(\sigma_t^4) + 1.5(5) + 0.6(5) = 12.75 + 0.57\mathrm{E}(\sigma_t^4) \end{aligned}$$

It follows that $\mathrm{E}(\sigma_t^4) = 1275/43$ and hence $\mathrm{Var}(v_t) = 2\mathrm{E}(\sigma_t^4) = 2550/43$.

(c) Recall that $X_t^2 = 1.5 + 0.7X_{t-1}^2 + v_t - 0.5v_{t-1}$. As the AR polynomial $\phi(x) = 1 - 0.7x$ gives root $10/7 > 1$, it is causal and therefore weakly stationary. Notice that

- $\mathrm{Cov}(X_t^2, v_t) = \mathrm{Cov}(1.5 + 0.7X_{t-1}^2 + v_t - 0.5v_{t-1}, v_t) = \mathrm{Var}(v_t)$.
- $\mathrm{Cov}(X_t^2, v_{t-1}) = \mathrm{Cov}(1.5 + 0.7X_{t-1}^2 + v_t - 0.5v_{t-1}, v_{t-1}) = 0.2\mathrm{Var}(v_t)$.

Then we have

- $\mathrm{Cov}(X_t^2, \cdot) : \gamma(0) = 0.7\gamma(1) + \mathrm{Var}(v_t) - 0.5 \times 0.2\mathrm{Var}(v_t) = 0.7\gamma(1) + 0.9\mathrm{Var}(v_t)$.
- $\mathrm{Cov}(X_{t-1}^2, \cdot) : \gamma(1) = 0.7\gamma(0) - 0.5\mathrm{Var}(v_t)$.

and hence $\gamma(0) = 55\mathrm{Var}(v_t)/51 = 2750/43$ and

$$\mathrm{Cov}(X_t^2, X_{t+1}^2) = \gamma(1) = 13\mathrm{Var}(v_t)/51 = 650/43.$$

8.2 Estimation and Testing for GARCH Model

In general, we may want to check whether the model is conditional heteroskedastic or not before proceeding.

Lagrange Multiplier Test

Theorem 3. *For $\{X_t\} \sim \mathrm{ARCH}(p)$. Consider the regression model $X_t^2 = \alpha_0 + \sum_{i=1}^p \alpha_i X_{t-i}^2$ and denote the least-square estimator by $\hat{\alpha}_i$ for $i = 0, \ldots, p$. Define*

$$\hat{X}_t^2 = \hat{\alpha}_0 + \sum_{i=1}^p \hat{\alpha}_i X_{t-i}^2.$$

Then under $H_0 : \alpha_1 = \cdots = \alpha_p = 0$ (i.e., conditional homoskedastic), then $T = nR^2 \xrightarrow{\mathrm{d}} \chi_p^2$, where R^2 is the coefficient of determination of the regression model and H_0 is rejected when $T > \chi_{0.95,p}^2$.

We then consider the maximum likelihood estimation of the GARCH model.

Likelihood Estimation for GARCH Model

Theorem 4. *For $\{X_t\} \sim \mathrm{GARCH}(p, q)$, define the (conditional) log-likelihood as $\ell(\boldsymbol{\theta}|X_1, \ldots, X_n)$, where $\boldsymbol{\theta} = (\alpha_0, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$. Then*

1. *Assume $\sigma_t^2 = X_t^2 = 0$ for all $t \leq 0$.*
2. *Evaluate $\sigma_t^2 = \alpha_0 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2 + \sum_{j=1}^q \alpha_j X_{t-j}^2$ for $t = 1, \ldots, n$.*
3. *Write $\ell(\boldsymbol{\theta}|X_{1:n}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^n \log(\sigma_t^2) - \frac{1}{2}\sum_{t=1}^n \frac{x_t^2}{\sigma_t^2}$*

The MLE is thus given by $\hat{\boldsymbol{\theta}} = \arg\max \ell(\boldsymbol{\theta}|X_{1:n})$.

Also, we would like to do model selection. However, there is no general pattern for the ACF and ACVF plots associated with $\mathrm{GARCH}(p, q)$. Therefore, we should use AIC and BIC for model selection.

Information Criterion for GARCH Model

Definition 3. *Let $\{X_t\} \sim \mathrm{GARCH}(p, q)$ with $X_1, \ldots, X_n$ be observed. Define L as the likelihood for $(X_1, \ldots, X_n)$ evaluated under the MLE. Then*

1. AIC*: $-2\log L + 2(p + q + 1)$.*
2. BIC*: $-2\log L + (p + q + 1)\log n$.*

After choosing the desired $\mathrm{GARCH}(p, q)$ model by AIC or BIC, the residual analysis should be carried out to check the goodness of fit of the model. Again, we consider the portmanteau test (Ljung-Box Test)

Ljung-Box Test for GARCH Model

Theorem 5. *Let $\{\hat{\epsilon}_t\}$ be the estimated residual of the fitted $\mathrm{GARCH}(p, q)$ model for $\{X_t\}$. Let r_j be the ACF sample of $\{\hat{\epsilon}_t\}$. Then under $H_0 : \mathrm{Cov}(\epsilon_t, \epsilon_{t+k}) = 0$ whenever $|k| \leq h$, the Portmanteau Test (Ljung-Box Test).*

$$Q(h) = n(n+2)\sum_{j=1}^h \frac{r_j^2}{n-j} \xrightarrow{\mathrm{d}} \chi_{h-p-q-1}^2.$$

Then H_0 is rejected when $Q(h) \geq \chi_{0.95,h-p-q-1}^2$.

Estimation and Testing for GARCH Model

✎ **Exercise 4.** *Given* $X_1 = 0$, $X_2 = 4$, $X_3 = 2$ *and* $X_4 = 3$. *Consider fitting* $\{X_t\}$ *by* GARCH(1,1) *model, i.e.,*

$$X_t = \sigma_t \epsilon_t, \quad \{\epsilon_t\} \overset{iid}{\sim} \mathrm{N}(0,1); \quad \sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 X_{t-1}^2.$$

(a) *Consider the regression model* $X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$ *and the following* **R**-*code:*

```
x = c(0,4,2,3) ; n = length(x)
fit_LS = lm(x[2:4]^2~x[1:3]^2)
(R2 = summary(fit_LS)$r.squared)
[1] 0.9908257
(T = n*R2)
[1] 3.963303
```

Given $\chi^2_{0.95,1} = 3.84$ *and* $\chi^2_{0.05,2} = 5.99$, *test whether there is significant GARCH effect in* $\{X_t\}$.

(b) *Write down the log-likelihood of* (X_1, X_2, X_3, X_4) *in terms of* $(\alpha_0, \alpha_1, \beta_1)$.

(c) *Consider the following* **R**-*code:*

```
library("tseries")
fit = garch(x,order = c(1,1)) # Fit the data by GARCH(1,1) model
(round(fit$coef,2))
  a0    a1    b1
2.77 0.00  0.74
```

Suppose $\hat{\sigma}_4^2 = 2$. *Find the prediction interval for* X_5.

Solution

(a) $T = nR^2 = 3.96 > 3.84 = \chi^2_{0.05,1}$. Hence we conclude that there is a significant GARCH effect.

(b) Let $\sigma_t^2 = X_t^2 = 0$ for $t \leq 0$. Then

$$\begin{aligned}
\sigma_1^2 &= \alpha_0 + \beta_1 \sigma_0^2 + \alpha_1 X_0^2 = \alpha_0 \\
\sigma_2^2 &= \alpha_0 + \beta_1 \sigma_1^2 + \alpha_1 X_1^2 = \alpha_0 + \beta_1 \alpha_0 \\
\sigma_3^2 &= \alpha_0 + \beta_1 \sigma_2^2 + \alpha_1 X_2^2 = \alpha_0 + \beta_1(\alpha_0 + \beta_1 \alpha_0) + 16\alpha_1 \\
\sigma_4^2 &= \alpha_0 + \beta_1 \sigma_3^2 + \alpha_1 X_3^2 = \alpha_0 + \beta_1[\alpha_0 + \beta_1(\alpha_0 + \beta_1 \alpha_0) + 16\alpha_1] + 9\alpha_1
\end{aligned}$$

Hence

$$\ell(\alpha_0, \alpha_1, \beta_1 | X_{1:3}) = -\frac{3}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{4}\log(\sigma_t^2) - \frac{1}{2}\sum_{t=1}^{4}\frac{X_t^2}{\sigma_t^2}$$

can be written in terms of $(\alpha_0, \alpha_1, \beta_1)$.

(c) As $\hat{\sigma}_5^2 = \hat{\alpha}_0 + \hat{\beta}_1 \hat{\sigma}_4^2 + \hat{\alpha}_1 X_4^2 = 2.77 + 0.74 \times 2 = 4.25$, then $\mathrm{Var}(X_5 | \mathscr{F}_4) \approx 4.25$ and $\mathrm{E}(X_5 | \mathscr{F}_4) = 0$, the 95% prediction interval for X_5 is given by

$$(-1.96 \times 4.25^{1/2}, 1.96 \times 4.25^{1/2}) = (-8.33, 8.33).$$

8.3 Additional Exercise

ARCH(∞) Representation

✎ **Exercise 5.** *Consider* GARCH$(1,1)$ *model*

$$X_t = \sigma_t \epsilon_t, \quad \{\epsilon_t\} \overset{iid}{\sim} \mathrm{N}(0,1); \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

with $\alpha_1 + \beta_1 < 1$. *Express the* GARCH$(1,1)$ *model as an* ARCH(∞) *model.*

Solution

As $\beta_1 < \alpha_1 + \beta_1 < 1$, we have $(1-\beta_1 B)\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$ and hence

$$\begin{aligned}\sigma_t^2 = (1-\beta_1 B)^{-1}\alpha_0 + \alpha_1 X_{t-1}^2 &= \frac{\alpha_0}{1-\beta_1} + \alpha_1 \sum_{k=0}^{\infty} \beta_1^k B^k X_{t-1}^2 \\ &= \frac{\alpha_0}{1-\beta_1} + \alpha_1 \sum_{k=1}^{\infty} \beta_1^{k-1} X_{t-k}^2.\end{aligned}$$

Moments of ARCH(1)

✎ **Exercise 6.** *Assume* $\{X_t\} \sim$ ARCH(1) *satisfies*

$$X_t = \sigma_t \epsilon_t, \quad \{\epsilon_t\} \overset{iid}{\sim} \mathrm{N}(0,1); \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$$

with $\alpha_1 < 1$. *Assume that* $\{\sigma_t^2\}$ *is weakly stationary. Evaluate* $\mathrm{E}(X_t^k)$ *for* $k = 1,2,3,4$.

Solution

- $\mathrm{E}(X_t) = \mathrm{E}(\sigma_t\epsilon_t) = \mathrm{E}(\sigma_t)\mathrm{E}(\epsilon_t) = 0$.
- $\mathrm{E}(X_t^2) = \mathrm{E}(\sigma_t^2\epsilon_t^2) = \mathrm{E}(\sigma_t^2)\mathrm{E}(\epsilon_t^2) = \alpha_0 + \alpha_1\mathrm{E}(X_{t-1}^2) = \alpha_0 + \alpha_1\mathrm{E}(X_t^2) \Rightarrow \mathrm{E}(X_t^2) = \alpha_0/(1-\alpha_1)$.
- $\mathrm{E}(X_t^3) = \mathrm{E}(\sigma_t^3\epsilon_t^3) = \mathrm{E}(\sigma_t^3)\mathrm{E}(\epsilon_t^3) = 0$.
- Recall that $\mathrm{E}(\epsilon_t^4) = 3$. Then

$$\begin{aligned}\mathrm{E}(X_t^4) = \mathrm{E}(\sigma_t^4\epsilon_t^4) = \mathrm{E}(\sigma_t^4)\mathrm{E}(\epsilon_t^4) &= 3\mathrm{E}\{(\alpha_0 + \alpha_1 X_{t-1}^2)^2\} \\ &= 3\Big(\alpha_0^2 + 2\alpha_0\alpha_1\mathrm{E}(X_{t-1}^2) + \alpha_1^2\mathrm{E}(X_{t-1}^4)\Big) \\ &= 3\Big(\alpha_0^2 + \frac{2\alpha_0^2\alpha_1}{1-\alpha_1} + \alpha_1^2\mathrm{E}(X_{t-1}^4)\Big).\end{aligned}$$

  Hence

$$(1-3\alpha_1^2)\mathrm{E}(X_t^4) = 3\frac{\alpha_0^2 - \alpha_0^2\alpha_1 + 2\alpha_0^2\alpha_1}{1-\alpha_1}.$$

  It follows that

$$\mathrm{E}(X_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

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8.4 ✄ Interpretation of GARCH Coefficients

We try to understand more about the GARCH model. For simplicity, consider GARCH$(1,1)$ model, i.e.

$$X_t = \sigma_t \epsilon_t, \quad \{\epsilon_t\} \overset{iid}{\sim} \mathrm{N}(0,1); \quad \sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2$$

with $0 < \alpha + \beta < 1$. Then we have

$$V_L \coloneqq \mathrm{Var}(X_t) = \frac{\omega}{1-\alpha-\beta}$$

being constant for all t and hence

$$\sigma_t^2 = (1-\alpha-\beta)V_L + \alpha X_{t-1}^2 + \beta\sigma_{t-1}^2 \quad \Rightarrow \quad \sigma_t^2 - V_L = \alpha(X_{t-1}^2 - V_L) + \beta(\sigma_{t-1}^2 - V_L).$$

Define $\mathscr{F}_t$ as the set that contains all the information about $(\sigma_t^2, \sigma_{t-1}^2, \sigma_{t-2}^2, \ldots)$. (Formally speaking, $\mathscr{F}_t$ is the sigma-algebra generated by $\{\sigma_k^2\}_{k\le t}$) Then as $\mathrm{E}(X_t^2|\mathscr{F}_t) = \mathrm{E}(\sigma_t^2\epsilon_t^2|\mathscr{F}_t) = \sigma_t^2\mathrm{E}(\epsilon_t^2|\mathscr{F}_t) = \sigma_t^2$,

$$\begin{aligned}
\mathrm{E}(\sigma_t^2 - V_L|\mathscr{F}_{t-1}) &= \alpha\Big(\mathrm{E}(X_{t-1}^2|\mathscr{F}_{t-1}) - V_L\Big) + \beta(\sigma_{t-1}^2 - V_L) \\
&= (\alpha+\beta)(\sigma_{t-1}^2 - V_L) \\
&= (\alpha+\beta)\mathrm{E}(\sigma_{t-1}^2 - V_L|\mathscr{F}_{t-2}) \\
&= \cdots = (\alpha+\beta)^k \mathrm{E}(\sigma_{t-k}^2 - V_L|\mathscr{F}_{t-k-1})
\end{aligned}$$

It follows that

$$\mathrm{E}(\sigma_t^2|\mathscr{F}_{t-1}) = V_L + (\alpha+\beta)^k\mathrm{E}(\sigma_{t-k}^2 - V_L|\mathscr{F}_{t-k-1})$$

and hence

$$\lim_{t\to\infty} \mathrm{E}(\sigma_t^2|\mathscr{F}_{t-1}) = V_L.$$

Recall that $\mathrm{E}(\sigma_t^2) = \mathrm{E}(X_t^2) = V_L$ while $\mathrm{E}(\sigma_t^2|\mathscr{F}_{t-1}) \neq V_L$ in general. We then conclude that even though the GARCH model is conditionally heteroskedastic, the conditional variance would still converge to the unconditional variance. Therefore, the quantity V_L is interpreted as **Long-Run Variance** in the GARCH model. A similar result also holds for the general stationary GARCH(p,q) model.

8.5 ✄ Extension of GARCH Model

The following are some well-known extensions of the GARCH model.

Extension of GARCH

Definition 4. *Assume $X_t = \sigma_t\epsilon_t$, where $\epsilon_t \overset{iid}{\sim} \mathrm{N}(0,1)$.*

1. GARCH(1,1): $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1\sigma_{t-1}^2$.
2. GJR-GARCH(1,1): $\sigma_t^{2\delta} = \alpha_0 + \alpha_1(|X_{t-1}| - \gamma X_{t-1}^2)^\delta + \beta_1\sigma_{t-1}^{2\delta}$.
3. EGARCH(1,1): $\log(\sigma_t^2) = \alpha_0 + \alpha_1|X_{t-1}/\sigma_{t-1}| + \gamma(X_{t-1}/\sigma_{t-1}) + \beta_1\log(\sigma_{t-1}^2)$.
4. TGARCH(1,1): $\sigma_t^2 = \alpha_0 + \alpha_1\mathcal{X}_{t-1}^2 + \lambda X_{t-1}^2\mathbb{1}(X_{t-1} < 0) + \beta_1\sigma_{t-1}^2$.

❗ **Remark 6.** *Those models have their own aspects and concern in modeling the stochastic volatility process. You can search for their properties if you are interested.*