STAT4005 - Time Series
Tutorial 4 - Midterm Review
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Week 7

4 Midterm Review

4.1 Special Reminder/Hints

- If the question require you to **state** the answer, it is fine to write the final answer directly. If otherwise, you should **spell all details** that demonstrate your understanding of course material.
- "A if and only if B" means that both (i). "A implies B" and (ii). "B implies A" holds.
- SARIMA models are not necessarily uniquely identified, i.e., it is possible to have $\{Y_t\}$ that belongs to both classes of SARIMA$(p_1, d_1, q_1) \times (P_1, D_1, Q_1)_{d_1}$ and SARIMA$(p_2, d_2, q_2) \times (P_2, D_2, Q_2)_{d_2}$ with $(p_1, d_1, q_1, P_1, D_1, Q_1, d_1) \neq (p_2, d_2, q_2, P_2, D_2, Q_2, d_2)$. Both answer are acceptable in this case.
- In `R`, the MA coefficients are specified in different way with that in our course (opposite sign). In our course, whenever we state ARMA(p, q) model with AR coefficient $\phi_1, \ldots, \phi_p$ and MA coefficient $\theta_1, \ldots, \theta_q$, they are always referring to

$$Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = Z_t - \theta_1 Z_{t-1} - \cdots - \theta_q Z_{t-q}.$$

  but **NOT** $Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$. (It is the specification in `R`)
- For $Z \sim \mathrm{N}(0, \sigma^2)$, you can directly use the fact that $\mathrm{E}(Z^2) = \sigma^2$ and $\mathrm{E}(Z^4) = 3\sigma^4$.
- Try to attempt at least Exercise 3, 7, 13, 18 and 21.

It would be my last tutorial session in this semester ☺ Thanks for attending. The remaining session would be held by another TA, Kexin Yuan. Feel free to contact with me if you get any question.

4.2 Decomposition of Time-Series

Recall from Tutorial 01, a general time-series can be decomposed as

$$X_t = T_t + S_t + N_t,$$

which is in general **non-stationary**. It makes the analysis difficult; instead, we consider

$$X_t \overset{(\#)}{\Rightarrow} Y_t,$$

where $\# \in$ {Least-square, filtering, differencing}. The time series $\{Y_t\}$ is easier to deal with, where

- Least-square method and Filtering method: $\{Y_t\}_{t=1}^n = \{\hat{T}_t, \hat{S}_t, \hat{N}_t\}_{t \in \cdots}$;
- Differencing method: differenced series $\{\Delta_p \Delta^k X_t\} \approx \{\Delta_p \Delta^k N_t\}$ (hopefully).

❷ **Question 1.** *How to implement them and what are their* ***pros and cons****?*

❷ **Question 2.** *What is the characterization of trend, season, and noise component?*

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4.3 Stationarity & Dependence measure

If $\{Y_t\}$ is strictly stationary, then joint distribution of $(Y_{t_1}, \ldots, Y_{t_k})$ is the same with (Y_{t_1+h}, Y_{t_k+h}) for arbitrary set of index $(t_1, \ldots, t_k)$, where k is arbitrary positive integer. It implies the following identities hold:

$$\mathbb{P}(X_1 \leq 0, X_3 \leq 0) = \mathbb{P}(X_{1+h} \leq 0, X_{3+h} \leq 0) \quad \text{and} \quad \mathbb{P}(X_2 + X_3 > X_5) = \mathbb{P}(X_{2+h} + X_{3+h} > X_{5+h}).$$

If $\{Y_t\}$ is weakly stationary, i.e., $\mathrm{E}(Y_t) = \mu$ and $\mathrm{Cov}(Y_t, Y_{t+k})$ is free of t. Recall from Tutorial 02, we can study the serial dependence among $\{Y_t\}$ through

- ACVF: $\gamma(k) = \mathrm{Cov}(Y_t, Y_{t+k})$;
- PACF: $\phi_{kk} = \mathrm{Cov}(Y_t, Y_{t+k} | Y_{t+1}, \ldots, Y_{t+k-1})$.

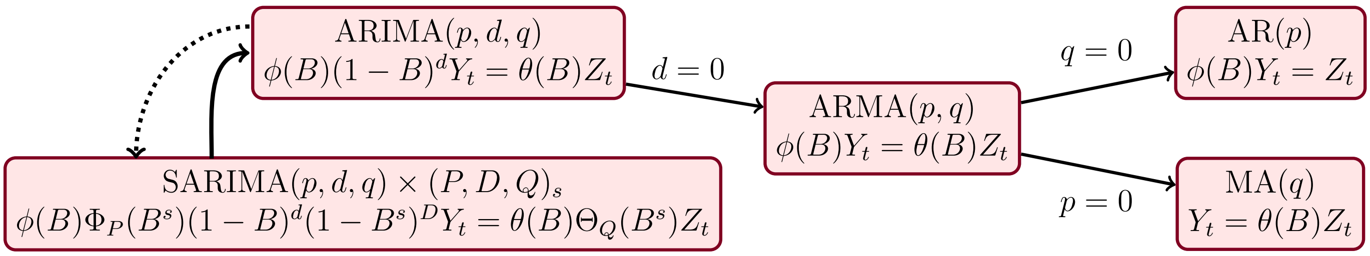
They could be estimated by the sample analog C_k and $\hat{\phi}_{kk}$ respectively.

For $\{X_t\}$ being identically and independently distributed,

1. **(ACF)** $\sqrt{n} r_k \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1)$ as $n \to \infty$.
2. **(PACF)** $\sqrt{n} \hat{\phi}_{kk} \xrightarrow{\mathrm{d}} \mathrm{N}(0, 1)$ as $n \to \infty$.

4.4 Linear Time Series models and its properties

Recall from Tutorial 03, we have studied the following parametric time series models:



ARIMA(p, d, q) model is always non-stationary for $d \geq 1$. For an ARMA(p, q) model,

1. **Stationary:** if none of roots of $\phi(x) = 0$ lies on unit circle.
2. **Causality:** if all roots of $\phi(x) = 0$ lies out of the unit circle.
3. **Invertibility:** if all roots of $\theta(x) = 0$ lies out of the unit circle.

4.5 Evaluation of ACVF

We have the MA representation (or AR representation) when the model is causal (or invertible), i.e.,

- MA representation: $Y_t = \phi(B)^{-1}\theta(B)Z_t = \sum_{k=0}^{\infty} \psi_k Z_{t-k}$;
- AR representation: $Z_t = \theta(B)^{-1}\phi(B)Y_t = \sum_{k=0}^{\infty} \psi'_k Y_{t-k}$.

In particular, given the MA representation, the ACVF of $\{Y_t\}$ is given by $\sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$. It makes the evaluation of the ACVF of MA models simple.

For a general stationary ARMA(p, q) model, we consider the Yule-walker equations, i.e.,

1. Consider the set of linear equations
$$\gamma(k) = \text{Cov}(Y_t, Y_{t-k}), \quad \text{for} \quad k = 0, \ldots, p$$
to solve for value of $\gamma(0), \ldots, \gamma(p)$.
2. For $k > p$, compute $\gamma(k) = \text{Cov}(Y_t, Y_{t-k})$ based on the value of $\gamma(0), \ldots, \gamma(p)$.

4.6 Revision Exercises — Chapter 1

(★★☆) Identifying the decomposed component

✎ **Exercise 1.** *Suppose that*
$$X_t = 26 + 9t + 20\sin\left(\frac{\pi t}{2}\right) + 2a_t + 2a_{t-1},$$
where $\{a_t\} \overset{iid}{\sim} \text{N}(3, 1)$*. Assume* $X_t = T_t + S_t + N_t$*. Write down the explicit form of* T_t*,* S_t *and* N_t*.*

Solution

Recall that for $Z \sim N(\mu, \sigma^2)$, we can rewrite it as $Z = \mu + \sigma\text{N}(0, 1)$. Therefore in this case, we can rewrite
$$\begin{aligned} X_t &= 26 + 9t + 20\sin\left(\frac{\pi t}{2}\right) + 2a_t + 2a_{t-1} = 26 + 9t + 20\sin\left(\frac{\pi t}{2}\right) + 2(3 + Z_t) + 2(3 + Z_{t-1}) \\ &= (38 + 9t) + 20\sin\left(\frac{\pi t}{2}\right) + (2Z_t + 2Z_{t-1}), \end{aligned}$$
where $\{Z_t\} \overset{iid}{\sim} \text{N}(0, 1)$. Therefore, we have $T_t = 38 + 9t$, $N_t = 2(Z_t + Z_{t-1})$ and
$$S_t = \begin{cases} 0 & \text{, if } t = 4d \text{ or } t = 4d + 2 \\ 20 & \text{, if } t = 4d + 1 \\ -20 & \text{, if } t = 4d + 3 \end{cases}$$
for some integer d.

✎ **Exercise 2** (label=()). *For each of the following time series X_t, identify the trend component T_t, the seasonal component S_t (together with the period d) and the noise component N_t.*

1. $X_t = 4005 + \cos(\pi t) + t + Z_t$, *where* $\{Z_t\} \overset{iid}{\sim} \mathrm{N}(0, 1)$.

2. $X_t = \mathbb{1}(t \textit{ is even}) + Z_t$, *where* $\{Z_t\} \overset{iid}{\sim} \chi^2_1$.

Hint: (i). N_t has to satisfy $\mathrm{E}(N_t) = 0$; (ii). $S_1, \ldots, S_d$ satisfy $\sum_{i=1}^d S_i = 0$ and $S_t = S_{t+d}$.

Solution

1. Notice that $\cos(x)$ is a periodic function with period 2π, i.e., $\cos(t) = \cos(t + 2\pi)$ for all t. Therefore, $S_t = \cos(\pi t)$ with period $d = 2$. To see this, $S_t + S_{t+1} = -1 + 1 = 0$ for all t. For the trend and noise component, it is clear that $T_t = 4005 + t$ and $N_t = Z_t$.

2. We can rewrite the equation as

$$\begin{aligned} X_t &= \mathbb{1}(t \text{ is even}) + Z_t \\ &= 1 + \mathbb{1}(t \text{ is even}) + (Z_t - 1) \\ &= 1.5 + 0.5\mathbb{1}(t \text{ is even}) - 0.5\mathbb{1}(t \text{ is odd}) + (Z_t - 1). \end{aligned}$$

   Therefore, we have $T_t = 1.5$ and $N_t = Z_t - 1$ (as to satisfy $\mathrm{E}N_t = 0$). Also, $S_t = 0.5\mathbb{1}(t \text{ is even}) - 0.5\mathbb{1}(t \text{ is odd})$ with $d = 2$, which also satisfies $S_t + S_{t+1} = 0.5 - 0.5 = 0$.

✎ **Exercise 3.** *Given observations*

$$(X_1, \ldots, X_8) = (4, 0, 0, 5, 2, 0, 2, 5).$$

Suppose there exists a seasonal component with period $d = 3$, suggest a sensible filter with length equals 3 that can handle the seasonal effect and find $\hat{T}_t, \hat{S}_t, \hat{N}_t$.

Solution

We use the filter $(1/3, 1/3, 1/3)$ to obtain the following result: As the period $d = 3$ is odd, we should set $(a_{-1}, a_0, a_1) = (1/3, 1/3, 1/3)$. Then we have

X_t	4	0	0	5	2	0	2	5
$\hat{T}_t$	NA	$4/3$	$5/3$	$7/3$	$7/3$	$4/3$	$7/3$	NA
D_t	NA	$-4/3$	$-5/3$	$8/3$	$-1/3$	$-4/3$	$-1/3$	NA

Therefore,

$$\bar{D} = \frac{(-4/3 - 5/3 + 8/3 + 1/3 - 4/3 - 1/3)}{6} = -\frac{7}{18}.$$

We then consider

X_t	4	0	0	5	2	0	2	5
D_t	NA	$-4/3$	$-5/3$	$8/3$	$-1/3$	$-4/3$	$-1/3$	NA
$D_t - \bar{D}$	NA	$-17/18$	$-23/18$	$55/18$	$1/18$	$-17/18$	$1/18$	NA

Notice that time $t = 1, 4, 5, 7$ belong to season 1 and $t = 2, 4, 6, 8$ belong to season 2. Hence

$$\hat{S}_1 = \left(\frac{55+1}{18}\right) \Big/ 2 = \frac{14}{9}, \quad \hat{S}_2 = \left(\frac{-17+1}{18}\right) \Big/ 2 = -\frac{4}{9} \quad \text{and} \quad \hat{S}_3 = \left(\frac{-23-17}{18}\right) \Big/ 2 = -\frac{10}{9}.$$

We can thus obtain

X_t	4	0	0	5	2	0	2	5
$\hat{T}_t$	NA	$4/3$	$5/3$	$7/3$	$7/3$	$4/3$	$7/3$	NA
$\hat{S}_t$	$14/9$	$-4/9$	$-10/9$	$14/9$	$-4/9$	$-10/9$	$14/9$	$-4/9$
$\hat{N}_t$	NA	$-8/9$	$-5/9$	$10/9$	$1/9$	$-2/9$	$-17/9$	NA

(★★☆) Tradeoff among filters

✎ **Exercise 4.** *Consider $X_t = T_t + S_t + N_t$, where $T_t = t^2$, the seasonal effect is of period $d = 3$ and $\{N_t\} \sim \mathrm{WN}(0, \sigma^2)$.*

(a) Consider the Spencer 15-point filter $(a_0, a_1, \ldots, a_7) = 320^{-1}(74, 67, 46, 21, 3, -5, -6, -3)$, Do you think this is an appropriate filter for filtering seasonal effect and why?

(b) Suggest a filter for filtering seasonal effect with length equal 3 and denote the estimated trend as $\hat{T}_t$. Calculate $\mathrm{E}(\hat{T}_t - T_t)$, interpret the result.

Solution

(a) Not sensible because in general $\sum_{r=-7}^{7} a_r S_{t+r} \neq 0$, i.e., the seasonal effect could not be filtered.

(b) A sensible filter would be $(a_{-1}, a_0, a_1) = (1/3, 1/3, 1/3)$. Then we have

$$\hat{T}_t = \frac{1}{3}\sum_{r=-1}^{1} X_{t+r} = \frac{1}{3}\sum_{r=-1}^{1} T_{t+r} + \frac{1}{3}\sum_{r=-1}^{1} S_{t+r} + \frac{1}{3}\sum_{r=-1}^{1} N_{t+r} = T_t + \frac{2}{3} + \frac{1}{3}\sum_{r=-1}^{1} N_{t+r}$$

It follows that

$$\mathrm{E}(\hat{T}_t - T_t) = \frac{2}{3} + \frac{1}{3}\sum_{r=-1}^{1} \mathrm{E}(N_{t+r}) = \frac{2}{3} \neq 0.$$

If we filter the seasonal effect through the even-weighting filter, the $\hat{T}_t$ might be biased.

Recall that

$$S_m(X_t) = S_m(T_t) + S_m(S_t) + S_m(N_t),$$

we hope that

$$S_m(T_t) \approx T_t \qquad \text{and} \qquad S_m(S_t), S_m(N_t) \approx 0 \tag{1}$$

so that

$$S_m(X_t) \approx \hat{T}_t$$

is a sensible estimator of trend.

- In (a), $\{a_r\}$ is not even weighted filter: $S_m(S_t) \neq 0$ and $S_m(T_t) = T_t$.
- In (b), $\{a_r\}$ is an even weighted filter: $S_m(S_t) = 0$ and $S_m(T_t) \not\approx T_t$.

Indeed, (1) is impossible if **length of filter equal period** d. We observe that

- (even weighted filter) d (or $d+1$) is the minimal length of filter that can achieve $S_m(S_t) = 0$. However, it may not be able to handle even quadratic trend.

It is possible to achieve (1) if we are willing to sacrifice more data, i.e., use filter with length significantly greater than the period. See Exercise 5 of Tutorial 01.

(★☆☆) Basic properties of filter

✎ **Exercise 5.** *For the filter $\{a_r\}_{r=-2}^{2}$, it is known that $a_1 = 42/143$ and the quadratic trend can pass through it.*

(a) Find the value of a_{-2}, a_{-1}, a_0 and a_2.

(b) Can a cubic polynomial pass through this filter?

Solution

(a) By symmetricity,

$$a_{-1} = a_1 \quad \text{and} \quad a_{-2} = a_2.$$

By the normalized property,

$$1 = \sum_{r=-2}^{2} a_r = a_0 + 2a_1 + 2a_2 = a_0 + \frac{84}{143} + 2a_2.$$

As a quadratic trend can pass through $\{a_r\}_{r=-2}^{2}$,

$$0 = \sum_{r=-2}^{2} r^2 a_r = 0^2 \times a_0 + 2 \times (1)^2 \times a_1 + 2 \times 2^2 \times a_2 = \frac{84}{143} + 8a_2.$$

On solving, $a_0 = 80/143$, $a_{-1} = 42/143$ (symmetry) and $a_2 = a_{-2} = -21/286$.

(b) Upon checking, $\sum_{r=-2}^{2} a_r r^j = 0$ for $j = 1, 2, 3$. Therefore, a cubic trend can also pass through $\{a_r\}_{r=-2}^{2}$.

✎ **Exercise 6.** *Suppose the time-series X_t has seasonal component with period $d = 4$.*

(a) State a sensible filter that can eliminate the seasonal effect.

(b) Show that the estimated trend $\hat{T}_t = S_m(X_t)$ is robust to the form of S_t when using the filter proposed in (a), i.e., to show if

$$X_t^{(1)} = T_t + S_t^{(1)} + N_t, \qquad X_t^{(2)} = T_t + S_t^{(2)} + N_t,$$

then $S_m(X_t^{(1)}) = S_m(X_t^{(2)})$.

(c) Does quadratic trend pass through the filter proposed in (a)?

Solution

(a) $(1/8, 1/4, 1/4, 1/4, 1/8)$.

(b) Notice that $S_m(\cdot)$ is a linear operator. Therefore,

$$S_m(X_t^{(1)}) = S_m(T_t) + S_m(S_t^{(1)}) + S_m(N_t), \quad S_m(X_t^{(2)}) = S_m(T_t) + S_m(S_t^{(2)}) + S_m(N_t).$$

Therefore, it suffices to show $S_m(S_t) = 0$ whenever S_t are of period $d = 4$. It is easy to check as

$$\begin{aligned} S_m(S_t) &= \frac{1}{8}S_{t-2} + \frac{1}{4}S_{t-1} + \frac{1}{4}S_t + \frac{1}{4}S_{t+1} + \frac{1}{8}S_{t+2} \\ &= \frac{1}{8}S_{t-2+4} + \frac{1}{4}S_{t-1} + \frac{1}{4}S_t + \frac{1}{4}S_{t+1} + \frac{1}{8}S_{t+2} \\ &= \frac{1}{4}(S_{t-1} + S_t + S_{t+1} + S_{t+2}) = 0. \end{aligned}$$

(c) Upon checking, $\sum_{r=-2}^{2} a_r r^2 > 0$. Therefore, quadratic trend cannot pass through the filter.

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4.7 Revision Exercises — Chapter 2

(★☆☆) Non-linear time-series as a white noise process

✎ **Exercise 7.** *Suppose that*

$$X_t = a_t + a_{t-1}a_{t-2},$$

where $\{a_t\} \overset{iid}{\sim} \mathrm{N}(0,1)$. *Is* $\{X_t\}$ *a white-noise process?*

Solution

First, we have $\mathrm{E}(X_t) = \mathrm{E}(a_t) + \mathrm{E}(a_{t-1}a_{t-2}) = \mathrm{E}(a_t) + \mathrm{E}(a_{t-1})\mathrm{E}(a_{t-2}) = 0$. Also,

$$\begin{aligned}
\mathrm{Var}(X_t) &= \mathrm{E}\{(a_t + a_{t-1}a_{t-2})^2\} - \mathrm{E}(a_t + a_{t-1}a_{t-2})^2 = \mathrm{E}(a_t^2 + 2a_ta_{t-1}a_{t-2} + a_{t-1}^2a_{t-2}^2) \\
&= \mathrm{E}(a_t^2) + 2\mathrm{E}(a_t)\mathrm{E}(a_{t-1})\mathrm{E}(a_{t-2}) + \mathrm{E}(a_{t-1}^2)\mathrm{E}(a_{t-2}^2) = 1 + 0 + 1\cdot 1 = 2. \\
\mathrm{Cov}(X_t, X_{t+1}) &= \mathrm{Cov}(a_t + a_{t-1}a_{t-2}, a_{t+1} + a_ta_{t-1}) \\
&= \mathrm{Cov}(a_t, a_ta_{t-1}) + \mathrm{Cov}(a_{t-1}a_{t-2}, a_ta_{t-1}) \\
&= \mathrm{E}(a_t^2a_{t-1}) - \mathrm{E}(a_t)\mathrm{E}(a_ta_{t-1}) + \mathrm{E}(a_ta_{t-1}^2a_{t-2}) - \mathrm{E}(a_{t-1}a_{t-2})\mathrm{E}(a_ta_{t-1}) \\
&= \mathrm{E}(a_t^2)\mathrm{E}(a_{t-1}) - \mathrm{E}(a_t)\mathrm{E}(a_t)\mathrm{E}(a_{t-1}) + \mathrm{E}(a_t)\mathrm{E}(a_{t-1}^2a_{t-2}) - \mathrm{E}(a_{t-1})\mathrm{E}(a_{t-2})\mathrm{E}(a_t)\mathrm{E}(a_{t-1}) \\
&= 0. \\
\mathrm{Cov}(X_t, X_{t+2}) &= \mathrm{Cov}(a_t + a_{t-1}a_{t-2}, a_{t+2} + a_{t+1}a_t) = \mathrm{Cov}(a_t, a_{t+1}a_t) = \mathrm{E}(a_t^2a_{t+1}) - \mathrm{E}(a_{t+1})\mathrm{E}(a_t) = 0. \\
\mathrm{Cov}(X_t, X_{t+k}) &= 0 \qquad (k \geq 3)
\end{aligned}$$

As $\mathrm{Var}(X_t) < \infty$ is constant for all t and $\mathrm{Cov}(X_t, X_{t+k}) = 0$ whenever $k \neq 0$, $\{X_t\}$ is a white-noise process.

(★☆☆) Differencing

✎ **Exercise 8.** *Consider* $X_t = T_t + S_t + N_t$, *where* $T_t = t^2$, *the seasonal effect is of period* $d = 3$ *and* $\{N_t\} \sim \mathrm{WN}(0,1)$.

(a) Let $W_t = \Delta_3 X_t$, *is* $\{W_t\}$ *stationary? If so, find its ACVF.*

(b) Show that $Y_t = \Delta W_t$, *is* $\{Y_t\}$ *stationary and find the associated ACVF* $\gamma_Y(\cdot)$.

Solution

(a) As $W_t = (T_t - T_{t-3}) + (S_t - S_{t-3}) + (N_t - N_{t-3}) = 6t - 9 + N_t - N_{t-3}$, we have $\mathrm{E}(W_t) = 6t - 9$ being non-constant overtime, i.e. $\{W_t\}$ is not stationary.

(b) $Y_t = \Delta W_t = 6 + N_t - N_{t-1} - N_{t-3} + N_{t-4}$. As $\{Y_t - 6\} \sim \mathrm{MA}(4)$, it is causal and therefore $\{Y_t\}$ is stationary. Its ACVF is given by

$$\gamma_Y(k) = \begin{cases} 4 & , \text{ if } |k| = 0 \\ -2 & , \text{ if } |k| = 1 \\ 1 & , \text{ if } |k| = 2 \\ -2 & , \text{ if } |k| = 3 \\ 1 & , \text{ if } |k| = 4 \\ 0 & , \text{ if } |k| \geq 5 \end{cases} = \begin{cases} 4 & , \text{ if } |k| = 0 \\ -2 & , \text{ if } |k| = 1 \text{ or } 3 \\ 1 & , \text{ if } |k| = 2 \text{ or } 4 \\ 0 & , \text{ if } |k| \geq 5 \end{cases}$$

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(★★☆) Stationarity of non-linear time-series

✎ **Exercise 9.** *Consider a time series* $\{X_t\}_{t=1,2,\dots}$ *satisfying:*

$$X_t = \cos(tU), \qquad U \sim \mathrm{Unif}(-\pi, \pi)$$

(a) Find lag k ACVF (for k=1,2,....) for $\{X_t\}_{t=1,2,\dots}$.

(b) Hence show that the process is weakly stationary.

Solution

(a) Notice that

$$\mathrm{E}(X_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(tu)\ \mathrm{d}u = \frac{\sin(tu)|_{-\pi}^{\pi}}{2\pi t} = 0.$$

Its lag k ACVF (k=1,2,...) equals

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+k}) &= \mathbb{E}(X_t X_{t+k}) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(tu)\cos((t+k)u)\ \mathrm{d}u \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \bigg[\cos[(2t+k)u] + \cos[ku]\bigg]\ \mathrm{d}u \\
&= \frac{1}{4\pi}\left[\frac{\sin[(2t+k)u]}{2t+k}\bigg|_{-\pi}^{\pi} + \frac{\sin(ku)}{k}\bigg|_{-\pi}^{\pi}\right] \\
&= 0
\end{aligned}$$

Note that $2t+k$ is non-zero, else $k = -2t$ and $\mathrm{Cov}(X_t, X_{t+k}) = \mathrm{Cov}(X_t, X_{-t})$ is not considered as this involves X_t with negative time point. Also, the above argument gives the same result when we consider $\mathrm{Cov}(X_t, X_{t-k})$ instead.

(b) As we have shown, $\mathrm{E}(X_t)$ is constant over time and $\mathrm{Cov}(X_t, X_{t+k})$ is free of t for $k \neq 0$. It suffices to show $\mathrm{Var}(X_t)$ is free of t. We have

$$\mathrm{Var}(X_t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2(tu)\ \mathrm{d}u = \frac{1}{4\pi}\left[\frac{\sin(2tu)}{2t}\bigg|_{-\pi}^{\pi} + 2\pi\right] = 1/2.$$

As $\mathrm{Var}(X_t)$ is free of t, $\{X_t\}$ is weakly-stationary.

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(★☆☆) Evaluation of ACVF & PACF

Exercise 10. *Consider* $X_t = 2 + 3a_t - 4a_{t-1} + 5a_{t-2}$*, where* $\{a_t\} \sim \mathrm{WN}(0,1)$*.*

(a) Evaluate the ACVF $\gamma(\cdot)$ *and ACF* $\rho(\cdot)$ *of* $\{X_t\}$*.*

(b) Compute the lag-2 PACF ϕ_{22} *of* $\{X_t\}$*.*

(c) Compute $\mathrm{Var}(X_1 + 9X_2 + 2X_4)$*.*

Solution

(a) The ACVF $\gamma(\cdot)$ is given by

$$\gamma(k) = \begin{cases} 3^2 + (-4)^2 + 5^2 & , k = 0 \\ 3 \cdot (-4) + (-4) \cdot 5 & , k = 1 \\ 3 \cdot 5 & , k = 2 \\ 0 & , k \geq 3 \end{cases}$$
$$= \begin{cases} 50 & , k = 0 \\ -32 & , k = 1 \\ 15 & , k = 2 \\ 0 & , k \geq 3 \end{cases}$$

While the ACF $\rho(\cdot)$ is given by

$$\rho(k) = \gamma(k)/\gamma(0) = \begin{cases} 1 & , k = 0 \\ -16/25 & , k = 1 \\ 3/10 & , k = 2 \\ 0 & , k \geq 3 \end{cases}$$

(b) It could be solved through

$$\begin{pmatrix} \phi_{21} \\ \phi_{22} \end{pmatrix} = \begin{pmatrix} \rho(0) & \rho(1) \\ \rho(1) & \rho(0) \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix} = \begin{pmatrix} -280/369 \\ -137/738 \end{pmatrix}$$

Hence $\phi_{22} = -137/738$.

(c) The variance can be computed as follow

$$\begin{aligned} \mathrm{Var}(X_1 + 9X_2 + 2X_4) &= \mathrm{Cov}(X_1 + 9X_2 + 2X_4, X_1 + 9X_2 + 2X_4) \\ &= (1^2 + 9^2 + 2^2)\gamma(0) + (2 \cdot 1 \cdot 9)\gamma(1) + (2 \cdot 9 \cdot 2)\gamma(2) \\ &= 4264. \end{aligned}$$

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(★☆☆) Miscellaneous

✎ **Exercise 11.** *Suppose* $X_t = t + \sin(\pi t/2) + N_t$, *where* $\{N_t\}$ *is mean-zero weakly stationary with ACVF* $\gamma_N(\cdot)$.

(a) Is $\{X_t\}$ *weakly-stationary? If so, find its mean and ACVF.*

(b) Let $Y_t = \Delta^{d_1}(\Delta_{d_2})X_t$, *where* $d_1, d_2 \geq 0$ *(could be 0). State (without proof) the minimal value of* d_1 *and* d_2 *such that* $\{Y_t\}$ *is weakly stationary.*

(c) Evaluate $\mathrm{E}(Y_t)$ *and the ACVF* $\gamma_Y(\cdot)$ *of* $\{Y_t\}$ *based on the value of* d_1, d_2 *found in (b). The answer could be in terms of* $\gamma_N(\cdot)$.

Solution

(a) As $\mathrm{E}(X_2) = 2 \neq 6 = \mathrm{E}(X_6)$, $\{X_t\}$ is not weakly-stationary.

(b) $d_1 = 0$ and $d_2 = 4$.

(c) Taking $d_1 = 0$ and $d_2 = 4$ gives

$$\begin{aligned} Y_t &= \left(t + \sin\left(\frac{\pi t}{2}\right) + N_t\right) - \left((t-4) + \sin\left(\frac{\pi(t-4)}{2}\right) + N_{t-4}\right) \\ &= 4 + N_t - N_{t-4}. \end{aligned}$$

Therefore, we have

$$\mathrm{E}(Y_t) = 4 + \mathrm{E}(N_t) - \mathrm{E}(N_{t-4}) = 4 + \mathrm{E}(N_t) - \mathrm{E}(N_t) = 4$$

and

$$\begin{aligned} \gamma_Y(k) &= \mathrm{Cov}(N_t - N_{t-4}, N_{t+k} - N_{t-4+k}) \\ &= 2\gamma_N(k) - \gamma_N(k-4) - \gamma(k+4). \end{aligned}$$

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(★★☆) True and False Questions

✎ **Exercise 12.** *Let $\{X_t\}$ be a weakly stationary time-series.* ***STATE*** *(circle the answer without proof) whether the following statements are correct.*

(a) If the lag-2 ACVF $\gamma(2) = 0$, then the lag-2 PACF $\phi_{22} = 0$. ***(T/F)***

(b) If the lag-2 PACF $\phi_{22} = 0$, then the lag-2 ACVF $\gamma(2) = 0$. ***(T/F)***

(c) The time-series $\{X_t^2\}$ is also weakly-stationary. ***(T/F)***

(d) The time-series $\{\Delta X_t\}$ is also weakly-stationary. ***(T/F)***

Solution

(a) **F**. Counter example: $X_t = a_t + a_{t-1}$.

(b) **F**. Counter example: $X_t = 0.5X_{t-1} + a_t$.

(c) **F**. Counter example: $X_t \overset{iid}{\sim} t_3$, then $\mathrm{E}(X_t^2)$ exists, but $\mathrm{E}(X_t^4)$ does not exist.

(d) **T**. Trivial.

4.8 Revision Exercises — Chapter 3

(★☆☆) Identification of models

✎ **Exercise 13.** *Identify the following as ARIMA models or SARIMA models and state the order of the model (Suppose the factorization has been done).*

(a) $(1 - 0.5B^3)(1 + 2B + 0.5B^2)Y_t = (1 + 0.3B^3 - B^6)Z_t$.

(b) $(1 - B)^2(1 - 0.2B)(1 + 0.3B)(1 + B^3 + B^6)Y_t = (1 + 0.3B)(1 + 0.5B)Z_t$.

(c) $(1 - 2B)(1 + B^2)Y_t = (1 + 0.3B)Z_t$.

Solution

(a) $\{Y_t\} \sim \mathrm{SARIMA}(2, 0, 0) \times (1, 0, 2)_3$.

(b) The model could be simplified to $(1 - B)^2(1 - 0.2B)(1 + B^3 + B^6)Y_t = (1 + 0.5B)Z_t$ and hence $\{Y_t\} \sim \mathrm{SARIMA}(1, 2, 1) \times (2, 0, 0)_3$.

(c) $\{Y_t\} \sim \mathrm{SARIMA}(1, 0, 1) \times (1, 0, 0)_2$.

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(★★☆) Identification of models – II

✎ **Exercise 14.** *Identify the following as specific SARIMA models:*

(a) $X_t = 1.5X_{t-1} - 0.5X_{t-2} + a_t - 0.3a_{t-1} + 0.6a_{t-2}$.

(b) $X_t = 3X_{t-1} - 3X_{t-2} + X_{t-3} + a_t + 0.1a_{t-1}$.

(c) $X_t - 0.6X_{t-1} - X_{t-3} + 0.6X_{t-4} = a_t - 0.5a_{t-1} - 0.4a_{t-3} + 0.2a_{t-4}$

Solution

(a) $(1 - 0.5B)(1 - B)X_t = (1 - 0.3B + 0.6B^2)a_t$
$\hookrightarrow$ ARIMA$(1, 1, 2)$.

(b) $(1 - B)^3 X_t = (1 + 0.1B)a_t$
$\hookrightarrow$ ARIMA$(0, 3, 1)$.

(c)

$$\begin{aligned} [1 - 0.6B - B^3(1 - 0.6B)]X_t &= [1 - 0.5B - 0.4B^3(1 - 0.5B)]a_t \\ (1 - B^3)(1 - 0.6B)X_t &= (1 - 0.4B^3)(1 - 0.5B) \end{aligned}$$

$\hookrightarrow$ SARIMA$(1, 0, 1) \times (0, 1, 1)_3$.

(★☆☆) Quick Concept Check

✎ **Exercise 15.** *Consider a general model $P_1(B)Y_t = P_2(B)Z_t$, where P_1 and P_2 are polynomials and there is no common root between them, $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$.*

(a) State the condition on the roots of $P_1(B)$ such that the model is causal.

(b) State the condition on the roots of $P_2(B)$ such that the model is invertible.

(c) If $P_1(1) \neq 0$ and $P_2(1) = 0$, comment on the causality, invertibility, and stationarity of the model?

Solution

(a) All the roots of $P_1(B)$ must be outside the unit circle.

(b) All the roots of $P_2(B)$ must be outside the unit circle.

(c) The model is NOT invertible. However, we cannot comment on causality and stationarity based only on P_2.

(★★☆) Yule-Walker Equation – I

✎ **Exercise 16.** *Consider the process*

$$X_t = 0.5X_{t-1} - 0.06X_{t-2} + a_t\,,$$

where $\{a_t\} \sim \mathrm{WN}(0, 1)$.

(a) Identify the order of the ARIMA model for $\{X_t\}$.

(b) Find the roots of the AR characteristic polynomial.

(c) Justify whether $\{X_t\}$ *is weakly stationary and causal? If so, evaluate* $\gamma(0)$, $\gamma(1)$ *and* $\gamma(2)$.

Solution

(a) The AR characteristic polynomial is $\phi(x) = 1 - 0.5x + 0.06x^2$. Therefore, it is AR(2) model.

(b) The roots of $\phi(x) = 0$ are given by $x_1 = 5, x_2 = 10/3$.

(c) As the roots x_1 and x_2 lies out of the unit circle, $\{X_t\}$ is weakly stationary and causal. The lag-k ACVF $(k = 0, 1, 2)$ can be solved through the Yule-Walker equation

$$\begin{aligned}\gamma(0) &= 0.5\gamma(1) - 0.06\gamma(2) + 1,\\ \gamma(1) &= 0.5\gamma(0) - 0.06\gamma(1),\\ \gamma(2) &= 0.5\gamma(1) - 0.06\gamma(0).\end{aligned}$$

Upon solving, we have

$$(\gamma_0, \gamma_1, \gamma_2) = \left(\frac{33125}{25662}, \frac{15625}{25662}, \frac{5825}{25662}\right) \approx (1.2908, 0.6089, 0.2270).$$

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(★★☆) Yule-Walker Equation – II

✎ **Exercise 17.** *Find ACVF $\gamma(k)$, k=0,1,2,3,.... of the process*

$$X_t = 0.4X_{t-3} + a_t,$$

where $\{a_t\} \sim \mathrm{WN}(0,1)$.

Solution

First, noticing that AR polynomial has root 5/2 and hence the process $\{X_t\}$ is causal. Consider the Yule-Walker equation:

$$\gamma(0) = 0.4\gamma(3) + \sigma^2 \tag{2}$$
$$\gamma(1) = 0.4\gamma(2) \tag{3}$$
$$\gamma(2) = 0.4\gamma(1) \tag{4}$$
$$\gamma(3) = 0.4\gamma(0) \tag{5}$$
$$\gamma(k) = 0.4\gamma(k-3), \qquad (k \geq 4) \tag{6}$$

Solving (1) and (4) yield $\gamma_0 = \sigma^2/0.84$, combining with (5) yield

$$\gamma(3k) = 0.4^k\gamma(0) = \frac{0.4^k}{0.84}\sigma^2,\ \ k = 1, 2, \cdots$$

Solving (2) and (3) yields $\gamma(1) = \gamma(2) = 0$, combining with (5) yield

$$\gamma(3k-2) = \gamma(3k-1) = 0,\ \ k = 1, 2, \cdots .$$

In general, we can write

$$\gamma(k) = \frac{0.4^{k/3}}{0.84}\sigma^2 \mathbb{1}(k \text{ is multiple of } 3).$$

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(★★☆) Yule-Walker Equation – III

✎ **Exercise 18.** *Consider the ARMA(2,1) model*

$$X_t = 0.6X_{t-1} - 0.09X_{t-2} + a_t - 0.2a_{t-1},$$

where $\{a_t\} \sim \mathrm{WN}(0,1)$.

(a) Find the AR representation of $\{X_t\}$.

(b) Find the ACVF $\gamma(k)$ *of* $\{X_t\}$. *(You are only required to find the value of* $\gamma(k)$ *for* $k = 0, 1, 2$. *For* $k \geq 3$, *try to represent them through a recursive equation without its analytical form)*

Solution

(a) $(1-0.3B)^2 X_t = (1-0.2B)a_t$
$\hookrightarrow a_t = (1-0.3B)^2(\sum_{i=0}^{\infty}(0.2)^i B^i)X_t = X_t - 0.4X_{t-1} + \sum_{i=2}^{\infty}(0.25)(0.2)^i X_{t-i}$

(b) From (a), the AR polynomial has root 10/3, i.e., the process is causal, we thus have

$$\begin{cases} E[a_t X_t] &= E[a_t^2] = 1 \\ E[a_{t-1}X_t] &= 0.6E[a_{t-1}X_{t-1}] - 0.2E[a_{t-1}^2] = 0.4 \end{cases}$$

$$\hookrightarrow \begin{cases} \gamma_0 = 0.6\gamma_1 - 0.09\gamma_2 + 0.92 \\ \gamma_1 = 0.6\gamma_0 - 0.09\gamma_1 - 0.2 \\ \gamma_2 = 0.6\gamma_1 - 0.09\gamma_0 \\ \gamma_k = 0.6\gamma_{k-1} - 0.09\gamma_{k-2},\ k \geq 3 \end{cases}$$

$$\hookrightarrow \begin{cases} \gamma_0 = \frac{893600}{753571} = 1.1858 \\ \gamma_1 = \frac{353620}{753571} = 0.4693 \\ \gamma_2 = \frac{131748}{753571} = 0.1748 \\ \gamma_k = 0.6\gamma_{k-1} - 0.09\gamma_{k-2},\ k \geq 3 \end{cases}$$

Note also that by causality, $E[a_t X_{t-k}] = 0$ for $k > 0$.

(★☆☆) AR & MA representation

✎ **Exercise 19.** *Find the AR and MA representation of the process*

$$X_t = 0.6X_{t-1} + a_t + 0.2a_{t-1},$$

where $\{a_t\} \sim \mathrm{WN}(0, 1)$.

Solution

Rewrite the model as

$$(1 - 0.6B)X_t \quad = \quad (1 + 0.2B)a_t.$$

Then we can represent

$$\begin{aligned}
\textbf{(AR representation):} \quad a_t &= X_t(1 - 0.6B)(\sum_{i=0}^{\infty}(-0.2)^i B^i) \\
&= X_t + \sum_{i=1}^{\infty}\left(4\right)(-0.2)^i X_{t-i}. \\
\textbf{(MA representation):} \quad X_t &= a_t(1 + 0.2B)(\sum_{i=0}^{\infty}(0.6)^i B^i) \\
&= a_t + \sum_{i=1}^{\infty}\left(\frac{4}{3}\right)(0.6)^i a_{t-i}.
\end{aligned}$$

(★☆☆) Comprehensive Exercise – I

✎ **Exercise 20.** *Consider the process* $Y_t - Y_{t-1} - 0.25Y_{t-2} + 0.25Y_{t-3} = Z_t + 1.2Z_{t-1} + 0.35Z_{t-2}$.

(a) *Identify the order of this ARIMA model.*

(b) *Is the model invertible? If yes, find* ψ_0, ψ_1 *and* ψ_2 *such that* $Z_t = \sum_{i=0}^{\infty} \psi_i Y_{t-i}$.

Solution

(a) The model could be written as $(1-B)(1-0.5B)(1+0.5B)Y_t = (1+0.7B)(1+0.5B)Z_t$ and therefore should be simplified to $(1-B)(1-0.5B)Y_t = (1+0.7B)Z_t$, i.e. $\{Y_t\} \sim$ ARIMA$(1,1,1)$.

(b) As the MA polynomial $\theta(x) = 1 + 0.7x$ have root $-10/7$ lies outside the unit circle, the model is invertible. We have

$$\begin{aligned}
Z_t &= (1+0.7B)^{-1}(1-1.5B+0.5B^2)Y_t \\
&= \left(\sum_{k=0}^{\infty}(-0.7)^k B^k\right)(1-1.5B+0.5B^2)Y_t \\
&= \left(1 - 0.7B + 0.49B^2 + \sum_{k=3}^{\infty}(-0.7)^k B^k\right)(1-1.5B+0.5B^2)Y_t \\
&= \Big[1 + \big(-0.7-1.5\big)B + \big(0.49+0.5+(-0.7)(-1.5)\big)B^2 + \cdots\Big]Y_t \\
&= Y_t - 2.2Y_{t-1} + 2.04Y_{t-2} + \cdots
\end{aligned}$$

Hence $\psi_0 = 1, \psi_1 = -2.2, \psi_2 = 2.04$.

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(★★☆) Comprehensive Exercise – II

Exercise 21. *Consider the process*

$$X_t = a_t - a_{t-1} + 0.25a_{t-2},$$

where $\{a_t\} \sim \mathrm{WN}(0, 1)$.

(a) Identify the order of the ARIMA model for $\{X_t\}$.

(b) Justify whether $\{X_t\}$ *is causal or not.*

(c) Justify whether $\{X_t\}$ *is invertible or not.*

(d) Find the ACVF $\gamma(k)$ *and ACF* $\rho(k)$ *of* $\{X_t\}$ *for* $k = 0, 1, 2, 3, \ldots$.

(e) Find the values of ψ_k, $k = 0, 1, 2, 3, \ldots$ *if the process is written as*

$$a_t = \sum_{k=0}^{\infty} \psi_k X_{t-k}\ .$$

Solution

(a) MA(2) or ARIMA(0,0,2).

(b) Yes, as X_t is a finite linear combination of past innovations $a_{t-k}\,(k \geq 0)$

Remark 1. *(Formal argument) we can write it as* $X_t = \sum_{i=0}^{\infty} \psi_i a_{t-i}$, *where*

$$\psi_0 = 0, \quad \psi_1 = -1, \quad \psi_2 = 0.25, \quad \text{and} \quad \psi_i = 0 \text{ for } i \geq 3.$$

Therefore, $\sum_{i=0}^{\infty} |\psi_i| < \infty$ *and thus the model is invertible.*

(c) Yes, it is invertible, as the roots of the MA characteristic polynomial $\theta(x) = 1 - x + 0.25x^2$ are 2 and 2 (repeated roots), which are both out of the unit circle.

(d) Recall that whenever $X_t = \sum_{i=0}^{\infty} \psi_i a_{t-i}$ (MA representation), we have $\gamma(k) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$, where $\sigma^2 = \mathrm{Var}(X_t)$. In our case,

$$\psi_0 = 0, \quad \psi_1 = -1, \quad \psi_2 = 0.25, \quad \text{and} \quad \sigma^2 = 1.$$

Therefore, the ACVF is computed by

$$\begin{aligned}
\gamma(0) &= 1^2 + (-1)^2 + 0.25^2 = 33/16, \\
\gamma(1) &= 1(-1) + (-1) \times 0.25 = -5/4, \\
\gamma(2) &= 1 \times 0.25 = 1/4, \\
\gamma(k) &= 0 \qquad (k \geq 3),
\end{aligned}$$

and hence

$$\rho(k) = \gamma(k)/\gamma(0) = \begin{cases} 1 & \text{, if } k = 0 \\ -20/33 & \text{, if } k = 1 \\ 4/33 & \text{, if } k = 2 \\ 0 & \text{, if } k \geq 3. \end{cases}$$

Solution

(e) First, rewrite the model as

$$X_t = (1 - B + 0.25B^2)a_t = (1 - 0.5B)^2 a_t.$$

Inverting $(1 - 0.5B)^2$ to the another side of the equality yield

$$a_t = \left(\sum_{i=0}^{\infty} 0.5^i B^i\right)^2 X_t = \left(\sum_{i=0}^{\infty} d^i\right)^2 X_t,$$

where $d := 0.5B$. Then, we have

$$\begin{aligned}
\left(\sum_{i=0}^{\infty} d^i\right)^2 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d^{i+j} \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} d^k \mathbb{1}(k \geq i) \\
&\overset{(\star)}{=} \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} d^k \\
&= \sum_{k=0}^{\infty}\sum_{i=0}^{k} d^k \\
&= \sum_{k=0}^{\infty} (k+1) d^k
\end{aligned}$$

Therefore,

$$a_t = \sum_{k=0}^{\infty} (k+1) 0.5^k X_{t-k}$$

and hence $\psi_k = (k+1)0.5^k$, $k = 0, 1, 2, \ldots$

❗ **Remark 2.** *To see* $(\star)$ *holds, try to draw a figure with* i *and* k *as the* x *and* y *axis respectively. Both expression are summing over the same set of value of* (i, k) *and hence equals.*

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(★★☆) Comprehensive Exercise (III)

Let $\alpha = 2$ and $\beta = 3$ be the two roots of the AR characteristic equation of the process

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t,$$

where $\{a_t\} \sim \mathrm{WN}(0, 1)$.

(a) Find the value of ϕ_1 and ϕ_2.

(b) Find the values of ψ_1,ψ_2 and ψ_{10} if the process is re-written as

$$X_t = \sum_{i=0}^{\infty} \psi_i a_{t-i}.$$

Solution

(a) $(1 - \phi_1 B - \phi_2 B^2)X_t = a_t$, we have

- sum of roots $= 5 = -\phi_1/\phi_2$
- product of roots $= 6 = -1/\phi_2$

Solving gives us $\phi_2 = -1/6$ and $\phi_1 = 5/6$.

(b) **(Method 1)**

$$\begin{aligned} X_t &= \frac{1}{(1-\frac{1}{3}B)(1-\frac{1}{2}B)} a_t \\ &= \sum_{i=0}^{\infty} \left(\frac{1}{3}B\right)^i \sum_{j=0}^{\infty} \left(\frac{1}{2}B\right)^j a_t \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^i \left(\frac{1}{2}\right)^j a_{t-i-j} \end{aligned}$$

For $j = 1, 2, ..., 10$, we have

$$\psi_j = \sum_{i=0}^{j} \left(\frac{1}{3}\right)^i \left(\frac{1}{2}\right)^{j-i}$$

This gives us $\psi_1 = 5/6$, $\psi_2 = 19/36$ and $\psi_{10} = 175099/6046617$.

(Method 2) Alternatively, you can consider the partial fraction approach, i.e., to consider

$$\frac{1}{(1-\frac{1}{3}x)(1-\frac{1}{2}x)} = \frac{C}{1-\frac{1}{3}x} + \frac{D}{1-\frac{1}{2}x} \quad \Rightarrow \quad \frac{1}{(1-\frac{1}{3}x)(1-\frac{1}{2}x)} = \frac{(C+D)-(C/2+D/3)x}{(1-\frac{1}{3}x)(1-\frac{1}{2}x)},$$

i.e., to solve $C + D = 1$ and $C/2 + D/3 = 0$ and we obtain $(C, D) = (-2, 3)$.

$$\begin{aligned} X_t &= \frac{1}{(1-\frac{1}{3}B)(1-\frac{1}{2}B)} a_t \\ &= \left(-\frac{2}{1-\frac{1}{3}B} + \frac{3}{1-\frac{1}{2}B}\right) a_t \\ &= \sum_{k=0}^{\infty} \left(-\frac{2}{3^k} + \frac{3}{2^k}\right) a_{t-k}. \end{aligned}$$

Therefore, $\psi_k = -2/3^k + 3/2^k$ and we have $\psi_1 = 5/6$, $\psi_2 = 19/36$ and $\psi_{10} = 175099/6046617$.

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(★★★) Comprehensive Exercise – IV

✎ **Exercise 22.** *Consider* $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t + \psi Z_{t-1}$, *where* $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. *Suppose*

- *The roots of the AR characteristic polynomial are* $\alpha = 2.5$ *and* $\beta = -1.25$.
- *The root of the MA characteristic polynomial is* 2.

(a) *Find the value of* ϕ_1, ϕ_2 *and* ψ.

(b) *Is the model causal? If yes, find its MA representation.* ***[remark: the trick involving partial fraction would not be tested]***

(c) *Is the model invertible? If yes, find its AR representation.*

(d) *Is the model stationary? If yes, find the ACVF* $\gamma(\cdot)$.

Solution

(a) The AR and MA polynomial is given by $\phi(x) = (1-\alpha^{-1}x)(1-\beta^{-1}x)$ and $\theta(x) = 1 - 0.5x$. Therefore, the model could be written as

$$(1 + 0.4B - 0.32B^2)Y_t = (1-0.4B)(1+0.8x)Y_t = (1-0.5B)Z_t,$$

i.e. $\phi_1 = -0.4, \phi_2 = 0.32, \psi = -0.5$.

(b) The AR root α, β lies outside the unit circle and is therefore causal. We first notice that

$$\frac{1-0.5x}{(1-0.4x)(1+0.8x)} = \frac{C}{1-0.4x} + \frac{D}{1+0.8x} = \frac{(C+D) + (0.8C - 0.4D)}{(1-0.4x)(1+0.8x)}$$

Solving equations $C + D = 1$ and $0.8C - 0.4D = -0.5$ gives $C = -1/12$ and $D = 13/12$. Then the MA representation is given by

$$\begin{aligned} Y_t &= \frac{1-0.5B}{(1-0.4B)(1+0.8B)} Z_t = \left(\frac{-1/12}{1-0.4B} + \frac{13/12}{1+0.8B}\right) Z_t \\ &= \left(-\frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k B^k + \frac{13}{12}\sum_{k=0}^{\infty}\left(-\frac{4}{5}\right)^k B^k\right) Z_t = \sum_{k=0}^{\infty}\left[-\frac{1}{12}\left(\frac{2}{5}\right)^k + \frac{13}{12}\left(-\frac{4}{5}\right)^k\right] Z_{t-k}. \end{aligned}$$

(c) The MA root 2 lies outside the unit circle, and hence is invertible. Then we have

$$\begin{aligned} Z_t =& (1-0.5B)^{-1}(1 + 0.4B - 0.32B^2)Y_t = \left(\sum_{k=0}^{\infty}(0.5)^k B^k\right)(1 + 0.4B - 0.32B^2)Y_t \\ =& \left(\sum_{k=0}^{\infty}(0.5)^k B^k + \sum_{k=0}^{\infty} 0.4(0.5)^k B^{k+1} - \sum_{k=0}^{\infty} 0.32(0.5)^k B^{k+2}\right) Y_t \\ =& \left(1 + \Big[0.5 + 0.4\Big] B + \sum_{k=2}^{\infty}(0.5)^k B^k + \sum_{k=1}^{\infty} 0.4(0.5)^k B^{k+1} - \sum_{k=0}^{\infty} 0.32(0.5)^k B^{k+2}\right) Y_t \\ =& Y_t + 0.9Y_{t-1} + \sum_{k=2}^{\infty}\left[(0.5)^k + 0.4(0.5)^{k-1} - 0.32(0.5)^{k-2}\right] Y_{t-k} \\ =& Y_t + 0.9Y_{t-1} + \sum_{k=2}^{\infty} 0.13(0.5)^{k-2} Y_{t-k}. \end{aligned}$$

Solution (Continued)

(d) Recall the model given by $Y_t = -0.4Y_{t-1} + 0.32Y_{t-2} + Z_t - 0.5Z_{t-1}$. Noticing that $\{Y_t\}$ is causal

$$\begin{aligned}\text{Cov}(Y_t, Z_t) &= \text{Cov}(-0.4Y_{t-1} + 0.32Y_{t-2} + Z_t - 0.5Z_{t-1}, Z_t) = \sigma^2 \\ \text{Cov}(Y_t, Z_{t-1}) &= \text{Cov}(-0.4Y_{t-1} + 0.32Y_{t-2} + Z_t - 0.5Z_{t-1}, Z_{t-1}) \\ &= -0.4\text{Cov}(Y_{t-1}, Z_{t-1}) - 0.5\text{Cov}(Z_{t-1}, Z_{t-1}) = -0.9\sigma^2\end{aligned}$$

and $\text{Cov}(Y_t, Z_{t+k}) = 0$ for all $k > 0$ by causality of $\{Y_t\}$. Then we have

$$\begin{aligned}\text{Cov}(\cdot, Y_t) :& \gamma(0) = -0.4\gamma(1) + 0.32\gamma(2) + \sigma^2 - 0.5(-0.9)\sigma^2 = -0.4\gamma(1) + 0.32\gamma(2) + 1.45\sigma^2 \\ \text{Cov}(\cdot, Y_{t-1}) :& \gamma(1) = -0.4\gamma(0) + 0.32\gamma(1) - 0.5\sigma^2 \\ \text{Cov}(\cdot, Y_{t-2}) :& \gamma(2) = -0.4\gamma(1) + 0.32\gamma(0)\end{aligned}$$

On solving, $\gamma(0) = \frac{1328125}{424116}\sigma^2$ and $\gamma(1) = -\frac{16075}{6237}\sigma^2$. Then consider for $k \geq 2$,

$$\text{Cov}(\cdot, Y_{t-k}) : \gamma(k) = -0.4\gamma(k-1) + 0.32\gamma(k-2)$$

Hence the ACVF is given by

$$\gamma(k) = \begin{cases} 3.13151\sigma^2 & , \text{ if } k = 0 \\ -2.57736\sigma^2 & , \text{ if } |k| = 1 \\ -0.4\gamma(|k|-1) + 0.32\gamma(|k|-2) & , \text{ if } |k| \geq 2 \end{cases}$$

Notice that we can further simplify the recursion (you are NOT required to solve the recursion in the examination). By considering the following linear recurrence relation with initial conditions:

$$\begin{cases} \gamma(0) &= \frac{1328125}{424116}\sigma^2 \; ; \; \gamma(1) = -\frac{16075}{6237}\sigma^2 \\ \gamma(k) &= -0.4\gamma(k-1) + 0.32\gamma(k-2) \quad \text{for } k \geq 2 \end{cases}$$

the general solution of $\gamma(k)$ for $k \in \mathbb{Z}$ would be given by

$$\gamma(k) = -\frac{745}{12474}\left(\frac{2}{5}\right)^{|k|} + \frac{79615}{24948}\left(-\frac{4}{5}\right)^{|k|}.$$

Remark 3. *In order to solve the recursion in 22(d) to find the explicit formula for $\gamma(k)$, you can study the general solution to the* ***linear recurrence relations****. Refer to here for a detailed discussion. (Optional)*