

STAT4005 - Time Series
Mid-term Exam Solution
Isaac Leung & Martin Ma
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The solution is prepared jointly by Dr. Isaac Leung and Martin Ma.

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A Multiple Choice Questions (21 marks in Total)

Instruction

There are a total of six multiple choice questions. For each question,

- 3.5 marks will be given for a correct answer.
- 0 mark will be given for an empty or wrong answer.

The following is the solution to the multiple-choice questions:

Question	1	2	3	4	5	6
Answer	B	C	C	A	D	B

Question 1

Suppose that the time series $\{X_t\}$ has a seasonal effect with period d satisfying $\sum_{i=1}^{d} S_i = 0$. Which of the following statements must be correct?

(i). When using the least-square method, $\sum_{i=1}^{d} \hat{S}_i = 0$.

(ii). When using the filtering method, $\sum_{i=1}^{d} \hat{S}_i = 0$.

A. (i) and (ii).

B. **(i) only.**

C. (ii) only.

D. None of the above.

Explanation

- For least square method, $\hat{S}_i = \hat{\alpha}_i - \bar{\alpha}$ and hence $\sum_{i=1}^{d} \hat{S}_i = \sum_{i=1}^{d} \hat{\alpha}_i - d\bar{\alpha} = d\bar{\alpha} - d\bar{\alpha} = 0$.
- For the filtering method, $\sum_{i=1}^{d} \hat{S}_i \neq 0$ in general. You may refer to Exercise 7 of Tutorial 1 for a counter example. It holds if we further assume that $n_1 = n_2 = \cdots = n_d$.

Therefore, the answer is (B).

Email contact: martinmtt@link.cuhk.edu.hk ; Personal website: https://sites.google.com/view/martinmtt/home

Question 2

Let p be a positive integer and $\{a_r\}_{r=-s}^{s}$ be a filter. Which of the following statements must be correct?

(i). If a $(2p-1)$th order polynomial passes through $\{a_r\}_{r=-s}^{s}$, then a $(2p)$th order polynomial also passes through $\{a_r\}_{r=-s}^{s}$.

(ii). If a $(2p)$th order polynomial passes through $\{a_r\}_{r=-s}^{s}$, then a $(2p+1)$th order polynomial also passes through $\{a_r\}_{r=-s}^{s}$.

A. (i) and (ii).

B. (i) only.

C. **(ii) only.**

D. None of the above.

Explanation

Observe that by symmetricity of filter, if k is an odd number, $(-r)^k = -r^k$ and hence

$$\sum_{r=-s}^{s} a_r r^k = \sum_{r=-s}^{-1} a_r r^k + \sum_{r=1}^{s} a_r r^k = \sum_{r=1}^{s} a_r (-r)^k + \sum_{r=1}^{s} a_r r^k = 0.$$

Therefore, the answer is (C).

Question 3

Suppose that the time-series $\{X_t\}$ satisfies

$$X_t = 3 + 9t^2 + 20\mathbb{1}(t \text{ is odd}) + 23\mathbb{1}(t \text{ is even}) + Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Which of the following differenced series can eliminate both the trend and the seasonal component in $\{X_t\}$?

(i). $\{\Delta^2 X_t\}$ (Second-order differencing).

(ii). $\{\Delta(\Delta_4 X_t)\}$ (first-order differencing followed by seasonal differencing with $d = 4$).

A. (i) and (ii).

B. (i) only.

C. **(ii) only.**

D. None of the above.

Explanation

Write $X_t = T_t + S_t + N_t$ and recall that $S_1 + S_2 = 0$. You can verify that (i) can only remove the trend effect, while (ii) can remove both the trend and seasonal effect.

Email contact: martinmtt@link.cuhk.edu.hk ; Personal website: https://sites.google.com/view/martinmtt/home

Question 4

Let $\{X_t\}$ and $\{Y_t\}$ be independent and weakly-stationary time series (X_{t_1} and Y_{t_2} are independent for all t_1, t_2). Which of the following statements must be correct?

(i). $\{X_t + Y_t\}$ is a weakly-stationary time series.

(ii). $\{X_t Y_t\}$ is a weakly-stationary time series.

A. **(i) and (ii).**

B. (i) only.

C. (ii) only.

D. None of the above.

Explanation

- (i) is true. You can verify through the definition easily.
- $\mathrm{E}(X_t Y_t) = \mathrm{E}(X_t)\mathrm{E}(Y_t) = \mu_X \mu_Y$ is constant over time. Also,
$$\mathrm{Cov}(X_t Y_t, X_{t+k} Y_{t+k}) = \mathrm{E}(X_t Y_t X_{t+k} Y_{t+k}) - \mathrm{E}(X_t Y_t)\mathrm{E}(X_{t+k} Y_{t+k})$$
$$= \mathrm{E}(X_t X_{t+k})\mathrm{E}(Y_t Y_{t+k}) - \mu_X^2 \mu_Y^2 = (\gamma_X(k) + \mu_X^2)(\gamma_Y(k) + \mu_Y^2) - \mu_X^2 \mu_Y^2$$
only depends on k but not t. Hence, $\{X_t Y_t\}$ is a weakly-stationary time series.

Therefore, the answer is (A).

Question 5

Suppose that the time series $\{Y_t\}$ satisfies

$$(1 - 3B + 2B^2)(1 + 0.5B)Y_t = (1 + 2B - 3B^2)Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Which of the following options correctly describes $\{Y_t\}$?

A. Causal and invertible.

B. Causal, but not invertible.

C. Not causal, but invertible.

D. **Neither causal nor invertible.**

Explanation

The model can be simplified to $(1 - 2B)(1 + 0.5B)Y_t = (1 + 3B)Z_t$.

- AR-polynomial $\phi(x) = (1 - 2x)(1 + 0.5x)$ gives root $1/2$ and -2. As $1/2$ lies inside the unit circle, the model is not causal.
- MA-polynomial $\theta(x) = 1 + 3x$ gives root $-1/3$, which lies inside the unit circle, and hence the model is not invertible.

Therefore, the answer is (D).

Question 6

Let $\{Y_t\}$ be a weakly-stationary ARMA(1,1) model satisfying

$$Y_t - \phi Y_{t-1} = Z_t - \theta Z_{t-1} \quad (\phi \neq \theta;\ \phi \neq 0;\ \theta \neq 0),$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Which of the following statements is/are correct?

(i). If $|\phi| < 1$, then Y_t is uncorrelated with future noises Z_{t+k} for all $k \geq 1$.

(ii). If $|\theta| < 1$, then Z_t is uncorrelated with future observations Y_{t+k} for all $k \geq 1$.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None of the above.

Explanation

- $|\phi| < 1$ implies that $\{Y_t\}$ is causal and hence we can write $Y_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$. Then we have

$$\mathrm{Cov}(Y_t, Z_{t+k}) = \sum_{j=0}^{\infty} \psi_j \mathrm{Cov}(Z_{t-j}, Z_{t+k}) = 0$$

for all $k \geq 1$.

- $|\theta| < 1$ implies invertibility of the model. However, it does not imply that Z_t is uncorrelated with future observations. Subtituting an arbitrary value of θ, ϕ that satisfies $0 < |\theta|, |\phi| < 1$ and $\theta \neq \phi$ will serve as a counter example.

So, the final answer is (B).

B Long Questions (79 marks in Total)

- It is not necessary to answer the questions exactly the same way as in the solution to get full credit. Any sensible and logical attempt/argument will be acceptable.
- If you have any problem with the grade issue (after the grade is released), you can contact the grader, Dong Rong, via dongrongli@link.cuhk.edu.hk.

Instruction

- **Let $\{\mathbf{a_t}\} \sim \mathbf{WN}(\mathbf{0}, \sigma^\mathbf{2})$ unless otherwise specified.**
- You need to show your steps in detail to get full scores.

Question 1

(15 marks) Consider the data

$$(2.1,\ 3.9,\ 0.5,\ 2.8,\ 6.1,\ 8.2,\ 4.5,\ 6.9,\ 9.3,\ 11.9,\ 9.4,\ 11).$$

Suppose there is seasonal effect with period $d = 4$, find an estimate of the trend and seasonal component using the filtering method.

Solution

Notice that $n = 12$, $d = 4$ and $q = 2$. The estimate of the trend and the seasonal component is available for $t = q + 1, \ldots, n - q$, i.e., from 3 to 10:

$$\widehat{T}_t = \frac{1}{4}\left(\frac{1}{2}X_{t-2} + X_{t-1} + X_t + X_{t+1} + \frac{1}{2}X_{t+2}\right).$$

We have $\widehat{T}_{3:10} = (2.8250,\ 3.8625,\ 4.9000,\ 5.9125,\ 6.8250,\ 7.6875,\ 8.7625,\ 9.8875)$,

$$\widehat{S}_i = \frac{\sum_{t \text{ belongs to season } i}(D_t - \bar{D})}{n_i}, \quad D_t = X_t - \widehat{T}_t, \quad \bar{D} = \frac{1}{n_d}\sum_{t=q+1}^{n-q} D_t\,,$$

Then, we obtain

$$\begin{aligned}
\bar{D} &= -0.0578125; \\
D_{3:10} &= (-2.3250,\ -1.0625,\ 1.2000,\ 2.2875,\ -2.3250,\ -0.7875,\ 0.5375,\ 2.0125); \\
\widehat{S}_{1:4} &= (0.9265625,\ 2.2078125,\ -2.2671875,\ -0.8671875).
\end{aligned}$$

❗ **Remark 1.** *You can also answer this question by listing a table as in Exercise 4 of Tutorial 1.*

Question 2

(15 marks) Let $X_t = a_t^2 a_{t-2}^2 - 1$. In this question, we further assume that $\{a_t\} \sim \mathrm{N}(0,1)$ independently. Justify whether $\{X_t\}$ is weakly stationary or not? If so, evaluate the autocorrelation function of $\{X_t\}$. (*Hint: For $Z \sim \mathrm{N}(0,1)$, $\mathrm{E}(Z^4) = 3$.*)

Solution

First, we have$\mathrm{E}(X_t) = \mathrm{E}(a_t^2 a_{t-2}^2) - 1 = \mathrm{E}(a_t^2)\mathrm{E}(a_{t-2}^2) - 1 = 1 \times 1 - 1 = 0$, i.e., $\mathrm{E}(X_t)$ is constant over time. Also, we have

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t-k}) &= \mathrm{E}(X_t X_{t-k}) = \mathrm{E}(a_t^2 a_{t-2}^2 - 1)(a_{t-k}^2 a_{t-2-k}^2 - 1) \\
&= \mathrm{E}(a_t^2 a_{t-2}^2 a_{t-k}^2 a_{t-2-k}^2) + 1 - 2\mathrm{E}(a_t^2 a_{t-2}^2).
\end{aligned}$$

$$\mathrm{Cov}(X_t, X_{t-k}) = \begin{cases} \mathrm{E}(a_t^4 a_{t-2}^4) + 1 - 2\mathrm{E}(a_t^2 a_{t-2}^2) = 3(3) + 1 - 2(1) = 8 & , \text{ if } k = 0; \\ \mathrm{E}(a_t^4 a_{t-2}^2 a_{t+2}^2) + 1 - 2\mathrm{E}(a_t^2 a_{t-2}^2)3(1)(1) + 1 - 2(1) = 2 & , \text{ if } |k| = 2; \\ \mathrm{E}(a_t^2)^4 + 1 - 2\mathrm{E}(a_t^2 a_{t-2}^2) = 1 + 1 - 2 = 0 & , \text{ if otherwise.} \end{cases}$$

Since $\mathrm{Cov}(X_t, X_{t-k})$ depends on k but not t, $\{X_t\}$ is weakly stationary. The autocorrelation function of $\{X_t\}$ is thus given by

$$\rho(k) = \begin{cases} 1 & , \text{ if } k = 0; \\ 1/4 & , \text{ if } |k| = 2; \\ 0 & , \text{ if otherwise.} \end{cases}$$

Email contact: martinmtt@link.cuhk.edu.hk ; Personal website: https://sites.google.com/view/martinmtt/home

Question 3

(15 marks) Consider the stationary time series $\{Y_t\}$ satisfying

$$Y_t - \frac{23}{44}Y_{t-1} + \frac{3}{44}Y_{t-2} - \frac{24}{44} = a_t.$$

Find the values of μ, ψ_1, ψ_2 and ψ_3 if the process is expressed in the form of

$$Y_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots .$$

(*Hint: 4 and 11/3 are roots of the equation* $\phi(x) = 1 - (23/44)x + (3/44)x^2 = 0$.)

Solution

From lecture notes we know that $\{Y_t\}$ is a stationary AR(2) model with non-zero mean. Let $X_t = Y_t - \mathrm{E}(Y_t) = Y_t - \mu$, by comparing the constant terms of the following 2 equations

$$\begin{aligned} Y_t - \frac{23}{44}Y_{t-1} + \frac{3}{44}Y_{t-2} - \frac{24}{44} &= a_t; && (1) \\ (Y_t - \mu) - \frac{23}{44}(Y_{t-1} - \mu) + \frac{3}{44}(Y_{t-2} - \mu) &= a_t. && (2) \end{aligned}$$

We get

$$-\frac{24}{44} = -\mu + \frac{23}{44}\mu - \frac{3}{44}\mu.$$

It follows that $\mu = 1$. By letting $X_t = Y_t - 1$, we will obtain a mean zero stationary AR(2) model satisfying:

$$X_t - \frac{23}{44}X_{t-1} + \frac{3}{44}X_{t-2} - \frac{24}{44} = a_t.$$

The AR characteristic polynomial can be written as:

$$\phi(x) = 1 - \frac{23}{44}x + \frac{3}{44}x^2 = \left(1 - \frac{1}{4}x\right)\left(1 - \frac{3}{11}x\right).$$

As $|\alpha|, |\beta| > 1$, the model is causal. We then find the MA-representation by

$$\begin{aligned} Y_t - 1 &= \left(1 - \frac{1}{4}B\right)^{-1}\left(1 - \frac{3}{11}B\right)^{-1} a_t = \left(\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i B^i\right)\left(\sum_{j=0}^{\infty}\left(\frac{3}{11}\right)^j B^j\right) a_t \\ &= \left(1 + \frac{1}{4}B + \frac{1}{4^2}B^2 + \frac{1}{4^3}B^3 + \cdots\right)\left(1 + \left(\frac{3}{11}\right)B + \left(\frac{3}{11}\right)^2 B^2 + \left(\frac{3}{11}\right)^3 B^3 + \cdots\right) a_t. \end{aligned}$$

By comparing coefficient, we have

$$\begin{aligned} \psi_1 &= \frac{1}{4} + \frac{3}{11} = \frac{23}{44}, \qquad \psi_2 = \left(\frac{1}{4}\right)\left(\frac{3}{11}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{3}{11}\right)^2 = \frac{397}{1936}, \\ \psi_3 &= \left(\frac{3}{11}\right)^3 + \left(\frac{1}{4}\right)\left(\frac{3}{11}\right)^2 + \left(\frac{1}{4}\right)^2\left(\frac{3}{11}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{6095}{85184}. \end{aligned}$$

Question 4

Consider the time series $\{Y_t\}$ satisfying

$$Y_t - 0.3Y_{t-1} = a_t - 0.9a_{t-1} + 0.2a_{t-2}.$$

In this question, we further assume that $\{a_t\} \sim \mathrm{WN}(0, 1)$:

(a) **(4 marks)** Justify whether $\{Y_t\}$ is (i). causal; (ii). invertible.

(b) **(18 marks)** Find the autocovariance function of $\{Y_t\}$ up to lag 4.

Solution

(a) First, we have $\phi(x) = 1 - 0.3x$ with root $= 10/3 > 1$ and we have $\theta(B) = 1 - 0.9x + 0.2x^2 = (1 - 0.4x)(1 - 0.5x)$ with roots $= 2.5$ or 2 both > 1. Therefore, $\{Y_t\}$ is both causal and invertible.

(b) Multiplying Y_t to both sides of the model equation and taking the expectation gives

$$\gamma(0) - 0.3\gamma(1) = \mathrm{E}(a_t Y_t) - 0.9\mathrm{E}(a_{t-1}Y_t) + 0.2\mathrm{E}(a_{t-2}Y_t), \tag{3}$$

where

$$\begin{aligned}
\mathrm{E}(a_t Y_t) &= \mathrm{E}\{a_t(0.3Y_{t-1} + a_t - 0.9a_{t-1} + 0.2a_{t-2})\} \\
&= \mathrm{E}(a_t^2) = 1. \\
\mathrm{E}(a_{t-1} Y_t) &= \mathrm{E}\{a_{t-1}(0.3Y_{t-1} + a_t - 0.9a_{t-1} + 0.2a_{t-2})\} \\
&= 0.3\mathrm{E}(a_{t-1}Y_{t-1}) - 0.9\mathrm{E}(a_{t-1}^2) = 0.3(1) - 0.9(1) = -0.6. \\
\mathrm{E}(a_{t-2} Y_t) &= \mathrm{E}\{a_{t-2}(0.3Y_{t-1} + a_t - 0.9a_{t-1} + 0.2a_{t-2})\} \\
&= 0.3\mathrm{E}(a_{t-2}Y_{t-1}) + 0.2\mathrm{E}(a_{t-2}^2) = 0.3(-0.6) + 0.2(1) = 0.02. \qquad (4)
\end{aligned}$$

Therefore combining (3) and (4), we get

$$\gamma(0) - 0.3\gamma(1) = 1 - 0.9(-0.6) + 0.2(0.02) = 1.544. \tag{5}$$

Next, multiply Y_{t-1} to both sides of the model equation and take E, we have

$$\begin{aligned}
\gamma(1) - 0.3\gamma(0) &= \mathrm{E}(a_t Y_{t-1}) - 0.9\mathrm{E}(a_{t-1}Y_{t-1}) + 0.2\mathrm{E}(a_{t-2}Y_{t-1}) \\
&= 0 - 0.9(1) + 0.2(-0.6) = -1.02. \qquad (6)
\end{aligned}$$

Then, multiply Y_{t-2} to both sides of the model equation and take E, we have

$$\begin{aligned}
\gamma(2) - 0.3\gamma(1) &= \mathrm{E}(a_t Y_{t-2}) - 0.9\mathrm{E}(a_{t-1}Y_{t-2}) + 0.2\mathrm{E}(a_{t-2}Y_{t-2}) \\
&= 0 - 0.9(0) + 0.2(1) = 0.2. \qquad (7)
\end{aligned}$$

Finally, for $k > 2$, multiply Y_{t-k} to both sides of the model equation and take E, we have

$$\gamma(k) - 0.3\gamma(k-1) = 0. \tag{8}$$

Solving (5) and (6) gives $\gamma(0)$ and $\gamma(1)$ equals to 1.3604396 and -0.6118681 respectively.
Substituting the obtained $\gamma(0)$ and $\gamma(1)$ into (7), we have $\gamma(2) = 0.01643957$.
Finally iteratively substituting the most recent $\gamma(k-1)$ obtained to (8), we obtain $\gamma(3) = 0.004931871$ and $\gamma(4) = 0.001479561$.

Email contact: martinmtt@link.cuhk.edu.hk ; Personal website: https://sites.google.com/view/martinmtt/home

Question 5

Identify the following as an appropriate ARIMA or SARIMA model:

(a) **(6 marks)** $Y_t + 0.7Y_{t-1} + 0.12Y_{t-2} = a_t + 0.3a_{t-3} + 0.02a_{t-6}$.

(b) **(6 marks)** $Y_t - Y_{t-1} - Y_{t-2} + Y_{t-3} = a_t - 0.5a_{t-1}$.

Solution

(a) The model can be written as

$$(1 + 0.3B)(1 + 0.4B)Y_t = (1 + 0.1B^3)(1 + 0.2B^3)a_t.$$

Hence, it is a SARIMA$(2, 0, 0) \times (0, 0, 2)_3$ model.

- *3 marks will be given if it is stated as a* ARMA$(2, 6)$ *model.*
- *No marks will be given otherwise.*

(b) The model can be written as

$$(1 - B)(1 - B^2)Y_t = (1 - 0.5B)a_t.$$

Hence, it is a SARIMA$(0, 1, 1) \times (0, 1, 0)_2$ model.

- *4 marks will be given if it is stated as a* ARIMA$(1, 2, 1)$ *model.*
- *3 marks will be given if it is stated as a* ARIMA$(2, 1, 1)$ *model.*
- *2 mark will be given if it is stated as a* ARMA$(3, 1)$ *model.*
- *No marks will be given otherwise.*