Predicates and Quantifiers.

Predicates

Propositional logic, studied previously, cannot adequately express the meaning of all statements in mathematics and in natural language.

Examples?

$$\text{“}x \text{ is greater than 5.”}$$

$$\text{“}x \text{ is greater than } y\text{.”}$$

$$\text{“}n \text{ is a prime number.”}$$

$$\text{“}user \text{ is waiting.”}$$

Def. *A* predicate *is a proposition whose truth depends on the value of one or more variables.*

Predicates

For convenience, we can give every predicate a name:

$$P(x) = \text{"}x \text{ is greater than 5."}$$

$$Q(x, y) = \text{“}x \text{ is greater than } y\text{.”}$$

When the values of the variables are specified, the result is a simple proposition: Depending on the, the predicats are either true or false:

$$P(4) = F$$
$$P(10) = T$$

$$Q(2, 1) = T$$

How can we use predicates?

Let $C(x) =$ “x is playing chess.”

There are three persons in the room: $Mark$, $Paul$, and Tom.

$Mark$ and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Mark) = T$, $C(Paul) = T$, $C(Tom) = F$.

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: $Mark$, $Paul$, and Tom.

$Mark$ and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Mark) = T$, $C(Paul) = T$, $C(Tom) = F$.

Consider a proposition "Someone is playing chess in the room."
Is it true?

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: $Mark$, $Paul$, and Tom.

$Mark$ and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Mark) = T$, $C(Paul) = T$, $C(Tom) = F$.

Consider a proposition "Someone is playing chess in the room."
Is it true?

Yes, because, for example, $C(Mark) = T$.

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: $Mark$, $Paul$, and Tom.

$Mark$ and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Mark) = T$, $C(Paul) = T$, $C(Tom) = F$.

Another proposition "Everyone is playing chess in the room."

How can we use predicates?

Let $C(x) =$ "x is playing chess."

There are three persons in the room: $Mark$, $Paul$, and Tom.

$Mark$ and $Paul$ are playing chess, and Tom is sleeping.
Formally: $C(Mark) = T$, $C(Paul) = T$, $C(Tom) = F$.

Another proposition "Everyone is playing chess in the room."

It's false, because $C(Tom) = F$.

Always true or sometimes true ?

Let $P(x) =$ "$x^2 \geq 0$."

Always true

Let $P(x) =$ "$x^2 \geq 0$."

Always true:

"For all n, $P(n)$ is true."

"$P(n)$ is true for every n."

"For all x, $x^2 \geq 0$."

"$x^2 \geq 0$ for every x."

Always true

"For all x, $x^2 \geq 0$."

An assertion that a predicate is always true is called a *universal quantification*.

Always true or sometimes true?

Another predicate:

$$\text{Let } Q(x) : \text{“}5x^2 - 7 = 0.\text{”}$$

Always true or sometimes true?

Another predicate:

$$\text{Let } Q(x) : \text{“}5x^2 - 7 = 0.\text{”}$$

It’s true only when $x = \pm\sqrt{7/5}$.

Sometimes true

Let $Q(x)$: "$5x^2 - 7 = 0$."

It's true only when $x = \pm\sqrt{7/5}$.

Sometimes true:

"There exist an n such that $Q(n)$ is true."

"There exist an x such that $5x^2 - 7 = 0$."

"$Q(n)$ is true for some n."

"$5x^2 - 7 = 0$ for some x."

"$Q(n)$ is true for at least one n."

"$5x^2 - 7 = 0$ for at least one x."

Sometimes true

"Exists an x such that $5x^2 - 7 = 0$ is true."

An assertion that a predicate is true for some values of the variable is called

an *existential quantification*.

Sentences can be ambiguous

"If you can solve *any* problem we come up with, then you get an A for the course."

Is it a universal (for all), or an existential (for some) quantification?

Sentences can be ambiguous

The last sentence was ambiguous.
The right way to say it in math class:

Universal (always):

"You can solve *every* problem we come up with."

Existential (sometimes):

"You can solve *at least one* problem we come up with."

Mathematical notation

"For every x, it's true that $x + 1 > x$"

becomes

$$\forall x \, (x + 1 > x)$$

Mathematical notation

“For every x, $P(x)$”

becomes

$$\forall x\ (P(x))$$

Mathematical notation

“Exists x such that $x^2 = 4$”

becomes

$$\exists x \; (x^2 = 4)$$

Mathematical notation

"Exists x such that $P(x)$"

becomes

$$\exists x\ (P(x))$$

Mathematical notation

Universal:

$\forall x\ P(x)$ means that *for every* x, $P(x)$ is true.

Existential:

$\exists x\ P(x)$ means that there *exists* an x such that $P(x)$ is true.

We consider only those values of the variable x that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

Example of the universe of discourse

For the predicate $Odd(x)$ and $Even(x)$, the universe of discourse is the set of all integers:

$$\ldots, \text{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}, \ldots$$

$Odd(x)$ is true for -5, -3, -1, 1, 3, 5, etc.

$Even(x)$ is true for -4, -2, 0, 2, 4, etc.

We consider only those values of the variable x that belong to a given set called the *domain of discourse*, or the *universe of discourse*.

Translating sentences

$$\begin{aligned} W(c) &\quad : \quad c \text{ is white} \\ B(c) &\quad : \quad c \text{ is black} \\ R() &\quad : \quad c \text{ is red} \\ L(x,y) &\quad : \quad \text{cat } x \text{ likes cat } y \end{aligned}$$

We want to say that:

(a) All cats are either white, black, or red.

(b) There exist white, black, and red cats.

(c) Jonesey is liked by all cats.

(d) There is a cat that likes everyone.

To prove or disprove a quantification

Statement	When true?	When false?
$\forall x\ P(x)$	$P(x)$ is true for every x.	There is at least one *counterexample* x such that $P(x)$ is false.
$\exists x\ P(x)$	There is at least one x such that $P(x)$ is true.	$P(x)$ is false for every x.

Example

Let $P(x) =$ "$x^2 > 0$."

The universe of discourse are all integer numbers.

Is it true or false that $\forall x\ P(x)$?

Example

Let $P(x) =$ "$x^2 > 0$."

The universe of discourse are all integer numbers.

Is it true or false that $\forall x\ P(x)$?

To prove it, we must show that $P(x)$ is true for all integers.
To disprove, we have to find a counterexample for which it's false.

Example

Let $P(x) =$ "$x^2 > 0$."

The universe of discourse are all integer numbers.

Is it true or false that $\forall x\ P(x)$?

To prove it, we must show that $P(x)$ is true for all integers.
To disprove, we have to find a counterexample for which it's false.

Counterexample:
$P(x)$ is false for $x = 0$. So, the quantified statement $\forall x\ P(x)$ is false.

Scope, variable binding

$$\forall x\ (P(x) \wedge Q(x))$$

The *scope* of the quantifier is the expression to which it's applied. Here, the scope is $P(x) \wedge Q(x)$. Quantifiers *bind* variables inside their scope.

$\forall$ binds x in the logical expression $(P(x) \wedge Q(x))$.

Scope, variable binding

Predicates
Quantifiers
Scope, bound variables
Nested quantifiers
Distributivity
Negation
Examples

Why should we care?

$$\forall x\ (P(x) \wedge Q(x))\ \wedge\ \exists y\ (R(y))$$

is equivalent to

$$\forall x\ (\underline{P(x) \wedge Q(x)})\ \wedge\ \exists x\ (\underline{R(x)})$$

"Every dog has four legs and has a tail; and there exists a dog that barks."

Scope, variable binding

If all variables are bound by a quantifier or set equal to a particular value then a statement is a proposition:

$$\forall x\,(P(x) \wedge Q(z)) \;\wedge\; \exists y(R(y)), \text{ and it's given that } z = 3.1415.$$

x is bound by $\forall x$, y is bound by $\exists y$, and z is specified by the given equation.

So, this is a proposition.

Scope, variable binding

If a variable is not bound, it's called *free*.

$$\forall x\,\big(A(y) \wedge B(x)\big) \;\wedge\; \exists y\,\big(C(x,y) \to D(y)\big).$$

Scope, variable binding

If a variable is not bound, it's called *free*.

$$\underline{\forall x}\,\big(A(y) \wedge B(\underline{x})\big) \;\wedge\; \underline{\exists y}\,\big(C(x, \underline{y}) \to D(\underline{y})\big).$$

Only the first x is bound by $\forall x$.

This is not a proposition.

Nested quantifiers

Quantifiers are nested if one is within the scope of the other:

$$\forall x\, \big(\exists y\, (x + y = 0)\big)$$

It reads as follows

"For every x exists y such that $x + y = 0$."

We usually drop the external parentheses. Equivalent expression:

$$\forall x\, \exists y\, (x + y = 0)$$

Nested quantifiers

A few more examples (you know this formulas):

$$\forall x \; \forall y \; \forall z \; \big(x + (y + z) = (x + y) + z\big)$$

$$\forall x \; \forall y \; \big(x + y = y + x\big)$$

$$\forall w \; \forall x \; \forall y \; \forall z \left((y \neq 0 \wedge w \neq 0) \rightarrow \frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw}\right)$$

Order of quantifiers

Does the order of quantifiers matter?

$$\forall x \left(\exists y \, (x + y = 0) \right)$$

"For every x exists y such that $x + y = 0$."

$$\exists y \left(\forall x \, (x + y = 0) \right)$$

"Exists y such that for every x: $x + y = 0$."

Order of quantifiers

Does the order of quantifiers matter?

$$\forall x \left(\exists y \, (x + y = 0)\right)$$

"For every x exists y such that $x + y = 0$."

$$\exists y \left(\forall x \, (x + y = 0)\right)$$

"Exists y such that for every x: $x + y = 0$."

The meaning of the two expressions is different. You cannot swap nested $\forall x$ and $\exists y$.

Order of quantifiers

However, we can swap two (or more) nested quantifiers of the same kind:

$$\forall x\ \forall y\ \left(x + y = y + x\right)$$
$$\forall y\ \forall x\ \left(x + y = y + x\right)$$

$$\exists x\ \exists y\ \left(x^y = 4\right)$$
$$\exists y\ \exists x\ \left(x^y = 4\right)$$

Order of quantifiers

These two statements are equivalent:

$$\forall w\, \exists x\, \forall y\, \forall z\, \big(P(w,x,y,z)\big)$$

$$\forall w\, \exists x\, \underline{\forall z\, \forall y}\, \big(P(w,x,y,z)\big)$$

But, these two statements are not equivalent:

$$\forall w\, \exists x\, \forall y\, \underline{\forall z}\, \big(P(w,x,y,z)\big)$$

$$\forall w\, \underline{\forall z}\, \exists x\, \forall y\, \big(P(w,x,y,z)\big)$$

Order of quantifiers

"For every even integer n greater than 2, there exist prime numbers p and q such that $n = p + q$."

(Prime numbers are integers > 1, divisible only by itself and 1. They are 2, 3, 5, 7, 11, 13, 17, $\ldots$)

The universe of discourse:

n is an *even* integer, $n > 2$.

p and q are *prime* numbers.

$$\forall n \; \exists p \; \exists q \; (n = p + q)$$

Order of quantifiers

Swapping the order of different kinds of quantifiers (existential or universal) changes the meaning of a proposition.

$$\forall n\,(\exists p\,(\exists q\,(n = p + q)))$$

$$\begin{aligned} 4 &= 2 + 2 \\ 6 &= 3 + 3 \\ 8 &= 3 + 5 \\ &\dots \end{aligned}$$

$$\exists p\,(\exists q\,(\forall n\,(n = p + q)))$$

$$\begin{aligned} 4 &\stackrel{?}{=} p + q \\ 6 &\stackrel{?}{=} p + q \\ &\dots \end{aligned}$$

Distributivity

$$\forall x \; (P(x) \wedge Q(x)) \; \equiv \; (\forall x \; P(x)) \wedge (\forall x \; Q(x))$$

we can distribute a universal quantifier $\forall$ over a conjunction.

$$\exists x \; (P(x) \vee Q(x)) \; \equiv \; (\exists x \; P(x)) \vee (\exists x \; Q(x))$$

and we can distribute an existential quantifier $\exists$ over a disjunction.

We **cannot** distribute $\forall$ over a disjunction, or $\exists$ over a conjunction.

Negating quantifiers

"It is not the case that everyone likes to snowboard."

$$\neg(\forall x\, S(x))$$

"There exists someone who does not like to snowboard."

$$\exists x\, (\neg S(x))$$

Negating quantifiers

“It is not the case that everyone likes to snowboard.”

$$\neg(\forall x\ S(x))$$

“There exists someone who does not like to snowboard.”

$$\exists x\ (\neg S(x))$$

To negate “$\forall$”:

$$\neg(\forall x\ S(x)) \equiv \exists x\ (\neg S(x))$$

Negating quantifiers

"There does not exist anyone who likes skiing over magma."

$$\neg(\exists x \; M(x))$$

"Everyone dislikes skiing over magma."

$$\forall x \; (\neg M(x))$$

Negating quantifiers

"There does not exist anyone who likes skiing over magma."

$$\neg(\exists x\ M(x))$$

"Everyone dislikes skiing over magma."

$$\forall x\ (\neg M(x))$$

To negate "∃":

$$\neg(\exists x\ M(x)) \equiv \forall x\ (\neg M(x))$$

Negating quantifiers

When negating more complex expressions with quantifiers, you "flip" the quantifier, and negate the expression to which the quantifier was applied.

$$\neg\Big(\forall z\ \Big(\exists y\ \big(\forall x\ (P(x) \land Q(y,z))\big)\Big)\Big)$$

$$\begin{aligned}
&= \exists z\ \neg\Big(\exists y\ \big(\forall x\ (P(x) \land Q(y,z))\big)\Big) \\
&= \exists z\ \forall y\ \neg\big(\forall x\ (P(x) \land Q(y,z))\big) \\
&= \exists z\ \forall y\ \exists x\ \neg(P(x) \land Q(y,z))
\end{aligned}$$

$$= \exists z\ \Big(\forall y\ \Big(\exists x\ (\neg P(x) \lor \neg Q(y,z))\Big)\Big)$$

Example

“Everyone at Hunter College is smart.”

$$\forall x\,\big(AtHunter(x) \wedge Smart(x)\big)$$

$$\forall x\,\big(AtHunter(x) \rightarrow Smart(x)\big)$$

Example

“Everyone at Hunter College is smart.”

$$\forall x\,\big(AtHunter(x) \land Smart(x)\big)$$ **Wrong!**

“Everyone is at Hunter College and is smart. No one is elsewhere.”

$$\forall x\,\big(AtHunter(x) \rightarrow Smart(x)\big)$$

Example

"Someone at City College is smart."

$$\exists x \left(AtCCNY(x) \wedge Smart(x) \right)$$

$$\exists x \left(AtCCNY(x) \rightarrow Smart(x) \right)$$

Example

"Someone at City College is smart."

$$\exists x \left(AtCCNY(x) \wedge Smart(x) \right)$$

$$\exists x \left(AtCCNY(x) \rightarrow Smart(x) \right) \textbf{Wrong!}$$

"There is someone, who is smart if he(she) is at City College."
It is true if there is anyone who is not at City College, say in Boston.

Example

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

Predicates:
$L(x) = x$ is a lion.
$C(x) = x$ drinks coffee.
$F(x) = x$ is fierce.

Example

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

Predicates:
$L(x) = x$ is a lion.
$C(x) = x$ drinks coffee.
$F(x) = x$ is fierce.

$$\forall x\ (L(x) \to F(x))$$

$$\exists x\ (L(x) \wedge \neg C(x))$$

$$\exists x\ (F(x) \wedge \neg C(x))$$