Discrete Structures (MTH3105)

Lecture 2

Previous Lecture Summery

- Introduction to the Course
- Propositions
- Logical Connectives
- Truth Tables
- Compound propositions
- Translating English to logic and logic to English.

Today's Lecture

- Logical Equivalences.
- De Morgan's laws.
- Tautologies and Contradictions.
- Laws of Logic.
- Conditional propositions.

Logical Equivalence

Definition

Two proposition form are called logically equivalent if and only if they have identical truth values for each possible substitution of propositions for their proposition variable.

The logical equivalence of proposition forms **P** and **Q** is written

$$\mathbf{P \equiv Q}$$

Equivalence of Two Compound Propositions P and Q

1. Construct the truth table for P.
2. Construct the truth table for Q using the same proposition variables for identical component propositions.
3. Check each combination of truth values of the proposition variables to see whether the truth value of P is the same as the truth value of Q.

Equivalence Check

a. If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.

b. If in some row P has a different truth value from Q, then P and Q are not logically equivalent.

Example

- Prove that $\neg(\neg p) \equiv p$

Solution

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

As you can see the corresponding truth values of p and $\neg(\neg p)$ are same, hence equivalence is justified.

Example

Show that the proposition forms $\neg(p\wedge q)$ and $\neg p \wedge \neg q$ are NOT logically equivalent.

p	q	$\neg p$	$\neg q$	$(p\wedge q)$	$\neg(p\wedge q)$	$\neg p\wedge\neg q$
T	T	F	F	T	F	F
T	F	F	T	F	**T**	**F**
F	T	T	F	F	**T**	**F**
F	F	T	T	F	T	T

Here the corresponding truth values differ and hence equivalence does not hold

De Morgan’s laws

De Morgan’s laws state that:

The negation of an **and** proposition is logically equivalent to the **or** proposition in which each component is negated.

The negation of an **or** proposition is logically equivalent to the **and** proposition in which each component is negated.

Symbolically (De Morgan's Laws)

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$

2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Applying De-Morgan's Law

Question: Negate the following compound Propositions

1. John is six feet tall and he weighs at least 200 pounds.

2. The bus was late or Tom's watch was slow.

Solution

a) John is not six feet tall or he weighs less than 200 pounds.

b) The bus was not late and Tom's watch was not slow.

Inequalities and De Morgan's Laws

Question Use De Morgan's laws to write the negation of

$$-1 < x \leq 4$$

Solution: The given proposition is equivalent to

$$-1 < x \quad \text{and} \quad x \leq 4,$$

By De Morgan's laws, the negation is

$$-1 \geq x \quad \text{or} \quad x > 4.$$

Tautology and Contradiction

Definition A tautology is a proposition form that is always true regardless of the truth values of the individual propositions substituted for its proposition variables. A proposition whose form is a tautology is called a tautological proposition.

Definition A contradiction is a proposition form that is always false regardless of the truth values of the individual propositions substituted for its proposition variables. A proposition whose form is a contradiction is called a contradictory proposition.

Example

Show that the proposition form $p \vee \neg p$ is a tautology and the proposition form $p \wedge \neg p$ is a contradiction.

p	**¬p**	**p ∨¬p**	**p ∧¬p**
T	**F**	**T**	**F**
F	**T**	**T**	**F**

Exercise: If *t* is a tautology and *c* is contradiction, show that $p \vee t \equiv p$ and $p \wedge c \equiv c$?

Laws of Logic

1. Commutative laws

   $p \wedge q \equiv q \wedge p \quad ; \quad p \vee q \equiv q \vee p$

2. Associative laws

   $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \quad ; \quad p \vee (q \vee r) \equiv (p \vee q) \vee r$

3. Distributive laws

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Laws of Logic

4. Identity laws

   $p \wedge t \equiv p \quad ; p \vee c \equiv p$

5. Negation laws

   $p \vee \neg p \equiv t \;; p \wedge \neg p \equiv c$

6. Double negation law

   $\neg(\neg p) \equiv p$

7. Idempotent laws

   $p \wedge p \equiv p \;; \; p \vee p \equiv p$

Laws of Logic

8. Universal bound laws

   $p \vee t \equiv t$; $p \wedge c \equiv c$

9. Absorption laws

   $p \wedge (p \vee q) \equiv p$; $p \vee (p \wedge q) \equiv p$

10. Negation of *t* and *c*

   $\neg t \equiv c$; $\neg c \equiv t$

Exercise

Using laws of logic, show that

$\neg(\neg p \wedge q) \wedge (p \vee q) \equiv p.$

Solution

Take $\neg(\neg p \wedge q) \wedge (p \vee q)$

$\equiv (\neg(\neg p) \vee \neg q) \wedge (p \vee q),$ (by De Morgan's laws)

$\equiv (p \vee \neg q) \wedge (p \vee q),$ (by double negative law)

$\equiv p \vee (\neg q \wedge q),$ (by distributive law)

contd…

$\equiv p \vee (q \wedge \neg q)$, (by the commutative law)

$\equiv p \vee c$, (by the negation law)

$\equiv p$, (by the identity law)

Skill in simplifying proposition forms is useful in constructing logically efficient computer programs and in designing digital circuits.

Lecture Summary

- Logical Equivalence
- Equivalence Check
- Tautologies and Contradictions
- Laws of Logic
- Simplification of Compound Propositions

Another Example

Prove that $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \equiv \neg r \wedge (p \vee \neg q)$

$\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))]$

$\equiv \neg r \wedge \neg(q \wedge (\neg r \rightarrow \neg p))$,	De Morgan's law
$\equiv \neg r \wedge \neg(q \wedge (\neg\neg r \vee \neg p))$,	Conditional rewritten as disjunction
$\equiv \neg r \wedge \neg(q \wedge (r \vee \neg p))$,	Double negation law
$\equiv \neg r \wedge (\neg q \vee \neg(r \vee \neg p))$,	De Morgan's law
$\equiv \neg r \wedge (\neg q \vee (\neg r \wedge p))$,	De Morgan's law, double negation
$\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge (\neg r \wedge p))$,	Distributive law
$\equiv (\neg r \wedge \neg q) \vee ((\neg r \wedge \neg r) \wedge p)$,	Associative law
$\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge p)$,	Idempotent law
$\equiv \neg r \wedge (\neg q \vee p)$,	Distributive law
$\equiv \neg r \wedge (p \vee \neg q)$,	Commutative law

Conditional propositions

Definition

If **p** and **q** are propositions, the **conditional of q by p** is **if p then q** or **p implies q** and is denoted by $p \rightarrow q$.

It is false when p is true and q is false otherwise it is true.

Examples

If you work hard then you will succeed.

If sara lives in Islamabad, then she lives in Pakistan.

Implication (if - then)

- Binary Operator, Symbol: $\rightarrow$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Interpreting Conditional Statements

Examples

"The online user is sent a notification of a link error if the network link is down".

The statement is equivalent to

"If the network link is down, then the online user is sent a notification of a link error."

Using

p : The network link is down,
q : the online user is sent a notification of a link error.

The statement becomes (q if p)

$$p \rightarrow q.$$

Examples

"When you study the theory, you understand the material".

The statement is equivalent to (using if for when)

"If you study the theory, then you understand the material."

Using

p : you study the theory,
q : you understand the material.

The statement becomes (when p, q)

$$p \rightarrow q.$$

Examples

"*Studying the theory is sufficient for solving the exercise*".

The statement is equivalent to

"If you study the theory, then you can solve the exercise."

Using

p : you study the theory,
q : you can solve the exercise.

The statement becomes (p is sufficient for q)

$$p \rightarrow q.$$

Activity

- Show that

$$p \rightarrow q \equiv \neg p \vee q$$

This shows that a conditional proposition is simple a proposition form that uses **a not and an or**.

- Show that

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

This means that negation of '**if p then q**' is logically equivalent to '**p and not q**'.

Solution

p	q	$p \to q$	$\neg p \vee q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	T	T	T	T
F	F	T	T	T	T

From the above table it is obvious that conditional proposition is equivalent to a "not or proposition" and that its negation is not of the form 'if then'.

Negations of some Conditionals

Proposition: If my car is in the repair shop, then I cannot get the class.

Negation: My car is in the repair shop and I can get the class.

Proposition: If Sara lives in Athens, then she lives in Greece.

Negation: Sara lives in Athens and she does not live in Greece.

Converse, contrapositive and inverse

- **Inverse** – An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is “If p, then q”, the inverse will be “If not p, then not q”.

- **Converse** – The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is “If p, then q”, the converse will be “If q, then p”.

- **Contra-positive** – The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is “If p, then q”, the contra-positive will be “If not q, then not p”.

Contraposition

Definition

The contrapositive of a conditional proposition of the form 'if p then q' is '**if ¬q then ¬p**'. Symbolically, the contrapositive of $\mathbf{p \rightarrow q}$ is $\mathbf{\neg q \rightarrow \neg p}$.

A conditional proposition is logically equivalent to its contrapositive.

Example

If today is Sunday, then tomorrow is Monday.

Contrapositive:

If tomorrow is not Monday, then today is not Sunday.

Converse and inverse of the Conditional

Suppose a conditional proposition of the form 'If p then q' is given.

1. The converse is 'if q then p'.
2. The inverse is 'if ⌐p then ⌐q'.

Symbolically,

The converse of $p \rightarrow q$ is $q \rightarrow p$,

And

The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

Converse, contrapositive and inverse

- **Example**: "If it rains, then they cancel school." Here, *"It rains"* is the hypothesis, p and *"They cancel school"* is the conclusion, q
- **Inverse** : To form the inverse of the conditional statement, take the negation of both the hypothesis and the conclusion.
  - The inverse of *"If it rains, then they cancel school"* is *"If it does not rain, then they do not cancel school."*
- **Converse**: The converse of *"If it rains, then they cancel school"* is *"If they cancel school, then it rains."*
- **Contra-positive**:To form the contrapositive of the conditional statement, interchange the hypothesis and the conclusion of the inverse statement. The contrapositive of *"If it rains, then they cancel school"* is *"If they do not cancel school, then it does not rain."*

Contrapositive

The contrapositive of "if p then q" is "if ~q then ~p".

Statement: If you are a CS year 2 student,
then you are taking MTH 3105.

Contrapositive: If you are not taking MTH 3105,
then you are not a CS year 2 student.

Statement: If x^2 is an even number,
then x is an even number.

Contrapositive: If x is an odd number,
then x^2 is an odd number.

Fact: A conditional statement is logically equivalent to its contrapositive.

Proofs

Statement: If P, then Q

Contrapositive: If ¬Q, then ¬P.

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg Q \to \neg P$$

The Biconditional

Definition Given proposition variables p and q, the biconditional of p and q is **p if and only if q** and is denoted $p \leftrightarrow q$.

It is true if both p and q have the same truth values and is false if p and q have opposite truth values.

The words if and only if are sometime abbreviated **iff.**

Example This computer program is correct **iff** it produces the correct answer for all possible sets of input data.

Biconditional (if and only if)

- Binary Operator, Symbol: $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
true	true	true
true	false	false
false	true	false
false	false	true

Composite Statements

- Statements and operators can be combined in any way to form new statements.

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
true	true	false	false	false
true	false	false	true	true
false	true	true	false	true
false	false	true	true	true

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Equivalent Statements

P	Q	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
true	true	false	false	true
true	false	true	true	true
false	true	true	true	true
false	false	true	true	true

- Two statements are called logically equivalent if and only if (iff) they have identical truth tables

  - The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, because $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

Interpreting Necessary and sufficient conditions

“If a number is divisible by 10, then it is divisible by 2”.

The clause introduced by *If A number is divisible by 10”* is called the hypothesis. It is what we are given, or what we may assume.

The clause introduced by *then It is divisible by 2* is called the conclusion. It is the statement that "follows" from the hypothesis.

When the If-then sentence is *true*, we say that the hypothesis is a sufficient condition for the conclusion. Thus it is sufficient to know that a number is divisible by 10, in order to conclude that it is divisible by 2.

The conclusion is then called a necessary condition of that hypothesis. For, if a number is divisible by 10, it *necessarily* follows that it will be divisible by 2.

Interpreting Necessary and sufficient conditions

Example: Consider the proposition

‘**if John is eligible to vote then he is at least 18 year old**’.

The truth of the condition ‘**John is eligible to vote**’ is sufficient to ensure the truth of the condition ‘**John is at least 18 year old**’.

In addition, the condition ‘**John is at least 18 year old**’ is necessary for the condition ‘**John is eligible to vote**’ to be true. If John were younger than 18, then he would be not eligible to vote.

Necessary and Sufficient Conditions

Let r and s are two propositions

r is a sufficient condition for **s** means 'if r then s'.

r is a necessary condition for **s** means 'if not r then not s'

r is necessary and sufficient condition for **s** means 'r if and only if s'

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