Discrete Mathematics (MTH3105)

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Discrete vs Continuous

- Examples of discrete Data
  - Number of boys in the class.
  - Number of candles in a packet.
  - Number of suitcases lost by an airline.

- Examples of continuous Data
  - Height of a person.
  - Time in a race.
  - Distance traveled by a car.

What is discrete Mathematics?

- Discrete mathematics is the part of mathematics devoted to the study of discrete objects (Kenneth H. Rosen, 6th edition).
- Discrete mathematics is the study of mathematical structures that are fundamentally discrete rather than continuous (wikipedia).

Applications of discrete mathematics

- How can a circuit that adds two integers be designed?
- How many ways are there to choose a valid password on a computer?
- What is the shortest path between two cities using transportation system?
- How can I encrypt a message so that no unintended recipient can read it?
- How many valid internet addresses are there?
- How can a list of integers be sorted so that the integers are in increasing order?

Syllabus (Topics to be covered in this course)

- Logic
- Elementary Number Theory and Methods of Proof
- Set Theory
- Relations
- Sequences and Recursion
- Mathematical Induction
- Counting
- Relations and Equivalence Relations
- Graphs
- Trees

Reference Books

- Discrete Mathematics and its Applications
  (with Combinatorics and Graph Theory)
  6th Edition, The McGraw-Hill Companies, 2007,
  Kenneth H. Rosen.
- Discrete Mathematics with Applications
  2nd Edition, Thomson Learning, 1995,
  Susanna S. Epp.
- Discrete Mathematics for Computer Scientists
  2nd Edition, Addison-Wesley, 1999, John
  Truss.

Logic

- Propositional Logic
- Logic of Compound Statements
- Propositional Equivalences
- Conditional Statements
- Logical Equivalences
- Valid and Invalid Arguments
- Applications: Digital Logic Circuits
- Predicates and Quantifiers
- Logic of Quantified Statements

Logic

- Logic is the basis of all mathematical reasoning, and of all automated reasoning.
- It has practical applications to the design of computing machines, to
  - the specification of systems,
  - to artificial intelligence,
  - to computer programming,
  - to programming languages, and to other areas of computer science, as well as to many other fields of study.
- The rules of logic give precise meaning to mathematical statements.
- These rules are used to distinguish between valid and invalid mathematical arguments.
- These rules are used in the design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways

Propositional Logic

- **Propositional Logic**
  - is concerned with statements to which the truth values, “true” and “false”, can be assigned.
  - The purpose is to analyse these statements either individually or in a composite manner.

Propositional Logic

Proposition: A proposition (or Statement) is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Examples

1. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Paris is the capital of France.

This makes a declarative statement, and hence is a proposition. The proposition is TRUE (T).

Examples (Propositions Cont.)

2. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Can Ali come with you?.

This is a question not the declarative sentence and hence not a proposition.

Examples (Propositions Cont.)

3. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Take two aspirins.

This is an imperative sentence not the declarative sentence and therefore not a proposition.

Examples (Propositions Cont.)

4. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

$$x + 4 > 9.$$

Because this is true for certain values of x (such as $x = 6$) and false for other values of x (such as $x = 5$), it is not a proposition.

Examples (Propositions Cont.)

5. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

He is a college student.

Because truth or falsity of this proposition depend on the reference for the pronoun *he*. it is not a proposition.

Notations

- The small letters are commonly used to denote the propositional variables, that is, variables that represent propositions, such as, *p*, *q*, *r*, *s*, ….

- The truth value of a proposition is true, denoted by T or 1, if it is a true proposition and false, denoted by F or 0, if it is a false proposition.

Compound Propositions

- As we have seen in the previous examples, one or more propositions can be combined to form a single compound proposition.

- We formalize this by denoting propositions with letters such as p, q, r, s, and introducing several logical operators or logical connectives.

Compound Propositions

Producing new propositions from existing propositions.

Logical Operators or Connectives

1. Not $\neg$
2. And $\wedge$
3. Or $\vee$
4. Exclusive or $\oplus$
5. Implication $\rightarrow$
6. Biconditional $\leftrightarrow$

Truth tables can be used to show how these operators can combine propositions to compound propositions.

Compound Propositions

Negation of a proposition

Let p be a proposition. The negation of p, denoted by $\neg p$ *(also denoted by ~p)*, is the statement

"It is not the case that p".

The proposition $\neg p$ is read as "not p". The truth values of the negation of p, $\neg p$, is the opposite of the truth value of p.

Examples

1. Find the negation of the following proposition

p : Today is Friday.

The negation of is

$\neg\ p$: It is not the case that today is Friday.

This negation can be more simply expressed by

$\neg p$: Today is not Friday.

Examples

2. Write the negation of

“6 is negative”.

The negation is

“It is not the case that 6 is negative”.

or “6 is nonnegative”.

Truth Table (NOT)

- Unary Operator, Symbol: ¬

p	¬p
true	false
false	true

Conjunction (AND)

Definition

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "*p and q*".

The conjunction $p \wedge q$ is true when *p and q* are both true and is false otherwise.

Examples

1. Find the conjunction of the propositions p and q, where

p : Today is Friday.

q : It is raining today.

The conjunction is

$p \wedge q$: Today is Friday and it is raining today.

Truth Table (AND)

- Binary Operator, Symbol: $\wedge$

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

Disjunction (OR)

Definition

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "*p or q*".

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Examples

1. Find the disjunction of the propositions p and q, where

p : Today is Friday.

q : It is raining today.

The disjunction is

$p \vee q$: Today is Friday or it is raining today.

Truth Table (OR)

- Binary Operator, Symbol: $\lor$

p	q	$p \lor q$
true	true	true
true	false	true
false	true	true
false	false	false

Exclusive OR (XOR)

Definition

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition "$p \oplus q$".

The *exclusive or*, $p \oplus q$, is true when exactly one of p and q is true and is false otherwise.

Examples

1. Find the *exclusive or* of the propositions p and q, where

p : Atif will pass the course MTH3105.

q : Atif will fail the course MTH3105.

The *exclusive or* is

$p \oplus q$: Atif will pass or fail the course MTH3105.

Truth Table (XOR)

- Binary Operator, Symbol: ⊕

p	q	$p \oplus q$
true	true	false
true	false	true
false	true	true
false	false	false

Examples (OR vs XOR)

The following proposition uses the (English) connective "or". Determine from the context whether "or" is intended to be used in the inclusive or exclusive sense.

1. "Nabeel has one or two brothers".

A person cannot have both one and two brothers.
Therefore, "or" is used in the exclusive sense.

Examples (OR vs XOR)

2. To register for BSC you must have passed the qualifying exam or be listed as an Math major.

Presumably, if you have passed the qualifying exam and are also listed as an Math major, you can still register for BCS. Therefore, “or” is inclusive.

Composite Statements

Statements and operators can be combined in any way to form new statements.

p	q	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
true	true	false	false	false
true	false	false	true	true
false	true	true	false	true
false	false	true	true	true

Translating English to Logic

I did not buy a lottery ticket this week or I bought a lottery ticket and won the million dollar on Friday.

Let p and q be two propositions

p: I bought a lottery ticket this week.

q: I won the million dollar on Friday.

In logic form

$$\neg p \vee (p \wedge q)$$

Translating English Sentences

Example

- p = "It is below freezing"
- q = "It is snowing"

It is below freezing and it is snowing	$p \wedge q$
It is below freezing but not snowing	$p \wedge \neg q$
It is not below freezing and it is not snowing	$\neg p \wedge \neg q$
It is either snowing or below freezing (or both)	$p \vee q$
If it is below freezing, it is also snowing	$p \rightarrow q$

Logical Operators

Example 1

- Let p,q,r be the following propositions:

P="it is raining", q="the sun is shining", r="there are clouds in the sky."

1. Translate the following into logical notation , using p,q,r and the logical connectives.

a) It is raining and the sun is shining.

b) If it is raining then there are clouds in the sky.

c) If it is not raining, then the sun is not shining and there are clouds in the sky.

d) If there are no clouds in the sky, then the sun is shining.

Logical operators

Examples

2. Translate the following into English sentences.

a) $(p \wedge q) \rightarrow r$

b) $(p \rightarrow r) \rightarrow q$

c) $\neg(p \vee q) \wedge r$

3. Give the truth values of the propositions for (a) and (b).

Precedence of operators

- Just as in algebra, operators have precedence
  - 4+3*2 = 4+(3*2), not (4+3)*2
- Precedence order (from highest to lowest):

  $\neg \ \wedge \ \vee \ \rightarrow \ \leftrightarrow$

  - The first three are the most important
- This means that $p \vee q \wedge \neg r \rightarrow s \leftrightarrow t$ yields: $(p \vee (q \wedge (\neg r)) \rightarrow s) \leftrightarrow (t)$
- Not is *always* performed before any other operation

Conditional Statements

Implication

Definition: Let p and q be propositions. The *conditional statement* $p \rightarrow q$, is the proposition "*If* p*, then* q".

The *conditional statement* $p \rightarrow q$ is false when p is true and q is false and is true otherwise.

where p is called hypothesis, antecedent or premise.

q is called conclusion or consequence

Implication (if - then)

- Binary Operator, Symbol: $\rightarrow$

P	Q	$P \rightarrow Q$
true	true	true
true	false	false
false	true	true
false	false	true

Implication (if - then)

- $p \rightarrow q$ = “If today is Friday, then today is my birthday”

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication (if - then)

Example

- Let p = "I am elected" and q = "I will lower taxes"
- $p \rightarrow q$ = "If I am elected, then I will lower taxes"
- Consider all possibilities

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Note that if p is false, then the conditional is true regardless of whether q is true or false

Implication (if - then)

If p then q $\qquad p \longrightarrow q$

p is called the hypothesis; q is called the conclusion

The department says: "If your GPA is 4.0, then you don't need to pay tuition fee."

When is the above sentence false?

- It is false when your GPA is 4.0 but you still have to pay tuition fee.
- But it is not false if your GPA is below 4.0.

Another example: "If there is typhoon T8 today, then there is no class."

When is the above sentence false?

If-Then as Or

$$p \rightarrow q \equiv ?$$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Idea 1: Look at the false rows, negate and take the **"and"**.

$$\neg(P \wedge \neg Q)$$
$$\equiv \neg P \vee Q$$

- If you don't give me all your money, then I will kill you.
- Either you give me all your money or I will kill you (or both).

- If you talk to her, then you can never talk to me.
- Either you don't talk to her or you can never talk to me (or both).

Negation of If-Then

$$\neg(p \rightarrow q) \equiv ?$$

- If your GPA is 4.0, then you don't need to pay tuition fee.
- Your term GPA is 4.0 and you still need to pay tuition fee.

- If my computer is not working, then I cannot finish my homework.
- My computer is not working but I can finish my homework.

$$\begin{aligned} &\neg(P \rightarrow Q) \\ \equiv\ &\neg(\neg P \vee Q) && \boxed{\text{previous slide}} \\ \equiv\ &\neg\neg P \wedge \neg Q && \boxed{\text{DeMorgan}} \\ \equiv\ &P \wedge \neg Q \end{aligned}$$

Graph Theory

Euler Paths and Circuits

In order to minimize cost to the city, how should weekly garbage collection routes be designed for Detroit's 350,000 households?