Assignment 1

Advanced Algorithms and Datastructures

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1 Exercise 1: b-flow

A flow is a b-flow if its satisfies the following

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = b_v, \forall v \in V \quad (1)$$

$$0 \leq x_e \leq u_e, \forall e \in E \quad (2)$$

Below we have illustrated the b-flows for figure (a). We see that each node satisfies equation 1 and 2, giving us the b-flows

$$x_{(v_2 v_4)} = 2$$
$$x_{(v_5 v_1 4)} = 3$$
$$x_{(v_5 v_3 4)} = 4$$

which is illustrated in figure 2.

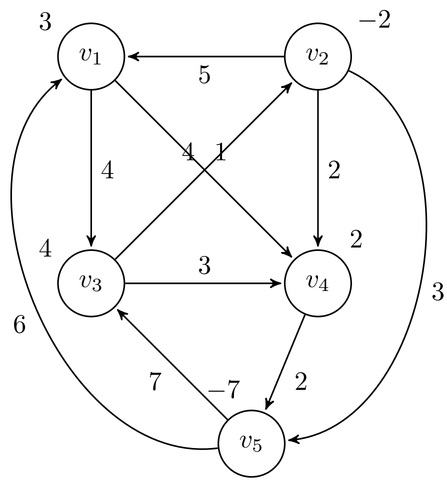

Figure 1: (a)

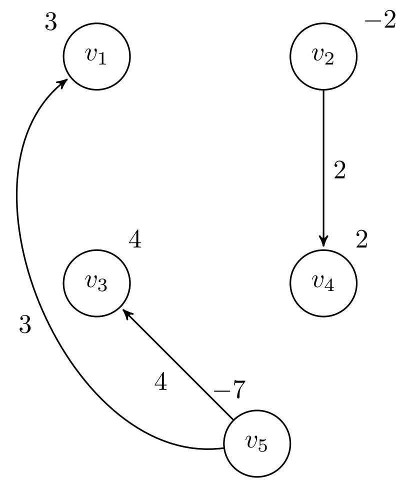

Figure 2: (a) b-flow

In figure (b) we can only satisfy equation 1 and 2 with some of the nodes. Due to the fact that vertex v_4 has no outgoing edges and we do not allow negative flows, we can not fulfil the demand of -2. Because of that, we have no b-flow here.

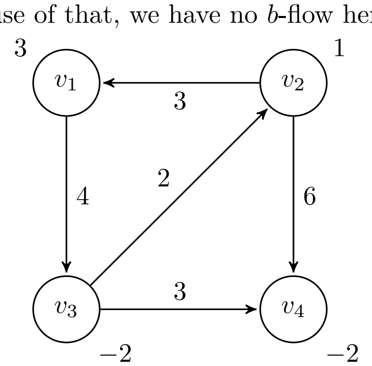

Figure 3: (b)

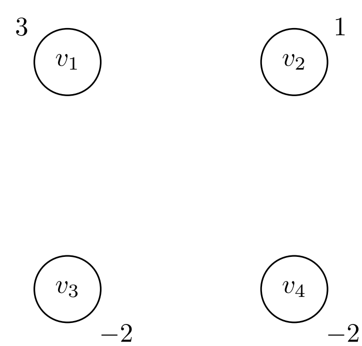

Figure 4: (b) b-flow

2 Exercise 2: An application of MCFP: rectilinear planar embedding

2.1 Exercise 2.1

The x_{vf} values for all vertices and and faces of [1, Figure 3] can be found in Table 1. The z_{fg} values

x_{vf}	a	b	c	d	e
v_1	0	1	1	0	0
v_2	0	0	1	1	0
v_3	1	0	1	1	1
v_4	0	0	0	-1	1
v_5	1	0	0	0	-1
v_6	1	1	0	1	1
v_7	0	0	0	0	0

Table 1: x_{vf}-values for all vertex/face combinations.

for the same graph can be found in Table 2. There are a total of 13 breakpoints in [1, Figure 3].

z_{fg}	a	b	c	d	e
a		0	0	0	0
b	2		1	1	0
c	1	1		0	0
d	0	1	0		2
e	4	0	0	0	

Table 2: z_{fg} values for all faces, values for empty sets are not displayed

This corresponds to the sum of all values of Table 2, as we would expect.

A drawing of a rectilinear layout for [1, Figure 2] can be seen in Figure 5.

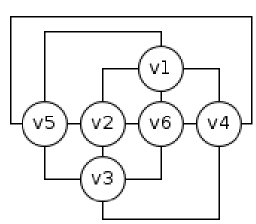

Figure 5: Graph in rectilinear form.

2.2 Exercise 2.2

Let B be the set of all boundary cycles and f_e the external boundary cycle. In the following, we make use of the fact that given two boundary cycles x and y, inner turns from x to y (z_{xy}) will be outer turns from y to x (z_{yx}). We do not verify the truth of this statement. The constraints can be expressed as:

$$\forall f \in B \setminus \{f_e\} : \sum_v x_{vf} + \sum_{b \in B \setminus \{f\}} z_{fb} - z_{bf} = 4 \tag{3}$$
$$\sum_v x_{vf_e} + \sum_{b \in B \setminus \{f_e\}} z_{f_e b} - z_{bf_e} = -4 \tag{4}$$

Boundary cycle a is the external boundary cycle, so Eq 4 must hold for a:

$$\begin{aligned}
&\sum_v x_{va} + \sum_{b \in B \setminus \{a\}} z_{ab} - z_{ba} \\
&= 3 + z_{ab} - z_{ba} + z_{ac} - z_{ca} + z_{ad} - z_{da} + z_{ae} - z_{ea} \\
&= 3 + 0 - 2 + 0 - 1 + 0 - 0 + 0 - 4 \\
&= 3 - 2 - 1 - 4 = -4
\end{aligned}$$

Boundary cycle e is an internal boundary cycle, so Eq 3 must hold for e specifically:

$$\begin{aligned}
&\sum_v x_{vf} + \sum_{b \in B \setminus f} z_{fb} - z_{bf} \\
&= 2 + z_{ea} - z_{ae} + z_{eb} - z_{be} + z_{ec} - z_{ce} + z_{ed} - z_{de} \\
&= 2 + 4 - 0 + 0 - 0 + 0 - 0 + 0 - 2 \\
&= 2 + 4 - 2 = 4
\end{aligned}$$

2.3 Exercise 2.3

The assumption is necesarry since only 4 edges can be connected to any single node, when the edges are limited to moving in two directions. From each node, an edge can go either up or down, or left or right.

To show that Equation 1 from [1] holds true we will divide it into the three different cases and show them seperetly.

- $\sum_f x_{vf} = 0$ if v has degree 2
  Any vertex v of degree 2 have 2 edges. This means that v is part of exactly 2 boundary cycles, lets call them f and g. This means that our sum can be defined as

  $$\sum_f x_{vf} = x_{vf} + x_{vg}.$$

  Since v is of second degree, there is only two possible “kinds” of configurations of the edges, either they form a 180 degree angle, or they form a 90 and a 270 degree angle. In the case of the 180 degree angles we have $x_{vf} = x_{vg} = 0$. The last case is the 90/270 degree case, this menas one of them will be an innerturn and one will be an outer turn. Because of this one cycle must have an x value of 1 and the other must have an x value of -1 giving a summation of 0, proving the initial statement..

- $\sum_f x_{vf} = 2$ if v has degree 3
  Having a degree of 3 means that the edge configuratin can be only one way, 2 edges will go either vertical or horizontal while the last edge is perpendicular to those. This configuration leaves 2 inner turns, and no outer turns. Naming the cycles f, g and h where f and g form inner turns with v, the sum can be written and calculated like this

  $$\sum_f x_{vf} = x_{vf} + x_{vg} + x_{vh} = 1 + 1 + 0 = 2$$

  which shows the second part of the initial sum is true.

- $\sum_f x_{vf} = 4$ if v has degree 4
  If v is uf degree 4, there is exactly 4 edges connected to v and v must be part of 4 cycles. In order to have space for these 4 edges all edges must be inner turns, the sum can then be written out as so

  $$\sum_f x_{vf} = 1 + 1 + 1 + 1 = 4$$

This shows that Equation 1 from [1] holds in all three cases for all vertices.

2.4 Exercise 2.4

The objective function $\sum_{f \in B} \sum_{g \in B \setminus \{f\}} z_{fg} + z_{gf}$ expresses the total number of breakpoints, which we wish to minimize.

$$\text{minimize} \quad \sum_{f,g \in B | g \neq f} z_{fg}$$

subject to

$$\begin{aligned}
&\forall f \in B \setminus \{f_e\} : \sum_v x_{vf} + \sum_{b \in B \setminus \{f\}} z_{fb} - z_{bf} && = 4 \\
&\sum_v x_{vf_e} + \sum_{b \in B \setminus \{f_e\}} z_{f_e b} - z_{b f_e} && = -4 \\
&\sum_f x_{vf} && = \begin{cases} 0 & \text{if } v \text{ has degree 2} \\ 2 & \text{if } v \text{ has degree 3} \\ 4 & \text{if } v \text{ has degree 4} \end{cases} \\
&z_{fg}, z_{gf} && \geq 0
\end{aligned}$$

where the sum $\sum_g$ is over all boundary cycles g containing vertex u.

2.5 Exercise 2.5

We limit the capacity, demands and costs of the resulting MCFP to be integers. Real-valued amounts of breakpoints does not make sense.

Consider a rectilinear graph $G = (V, E)$ and an MCFP $G' = (V', E')$.

V' contains a vertex for every face $f \in G$, and a vertex for every vertex $v \in V$. E' contains an edge e_{fg} for every pair of faces f, g that share at least one vertex, and an edge e_{vf} for every vertex v that was part of f in G. The intuition here is that faces and vertices both become vertices in the MCFP, and edges exist from vertices to faces and from faces to faces.

The demands of the MCFP are derived from the constraints previously established, and are as follows:

$$b_v = 4 \text{ if } v \text{ was an inner face}$$
$$b_v = -4 \text{ if } v \text{ was the external face}$$
$$b_v = \begin{cases} 0 & \text{if } v \in V \text{ and } v \text{ has degree } 2 \\ 2 & \text{if } v \in V \text{ and } v \text{ has degree } 3 \\ 4 & \text{if } v \in V \text{ and } v \text{ has degree } 4 \end{cases}$$

Between two vertices that were faces in G, e.g. v_f and v_g, the capacity is ∞^+ and the cost is 1. Between a vertex that was a face in G and a vertex from V, e.g. v_f and v_v, the capacity is 1 and the cost is 0.

Flow corresponds to the amount of breakpoints, which we wish to minimize. That is, we wish to find an assignment of real numbers x_e to each edge $e \in E'$, such that

$$\forall v \in V' : \sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = b_v,$$
$$\forall e \in E' : 0 \leq x_e \leq u_e$$

3 Exercise 3: Reduction to MCFP

Unfortunately we did not manage to break this exercise, even though we have tried for several hours. If we could get some help with it, we would appreciate it very much, and we would be happy to solve it for the re-handin.

3.1 Exercise 3.1

3.2 Exercise 3.2

3.3 Exercise 3.3

3.4 Exercise 3.4

3.5 Exercise 3.5 (Optional)

References

[1] Noy Rotbart and Christian Wulff-Nilsen. Minimum-cost flow, advanced algirithms 2014 assignment 1, 2014.