# Assignment 1

**Advanced Algorithms and Datastructures**

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# 1 Exercise 1: $b$-flow

A flow is a b-flow if its satisfies the following

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = b_v, \forall v \in V \tag{1}$$

$$0 \leq x_e \leq u_e, \forall e \in E \tag{2}$$

Below we have illustrated the $b$-flows for figure (a). We see that each node satisfies equation 1 and 2, giving us the $b$-flows

$$x_{(v_2 v_4)} = 2$$
$$x_{(v_5 v_1 4)} = 3$$
$$x_{(v_5 v_3 4)} = 4$$

which is illustrated in figure 2.

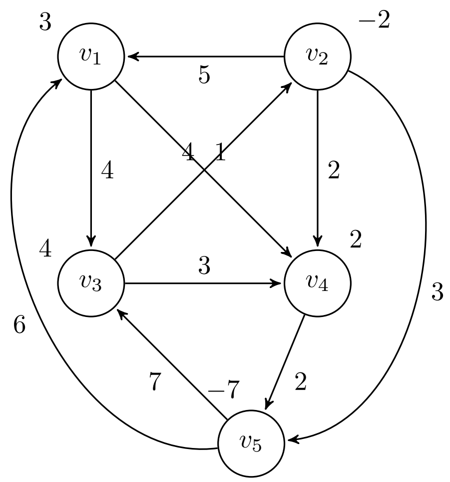



**Figure 1:** (a)

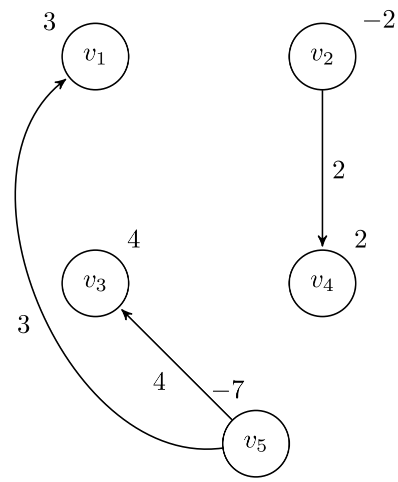



**Figure 2:** (a) $b$-flow

In figure (b) we can only satisfy equation 1 and 2 with some of the nodes. Due to the fact that vertex $v_4$ has no outgoing edges and we do not allow negative flows, we can not fulfil the demand of -2. Because of that, we have no $b$-flow here.

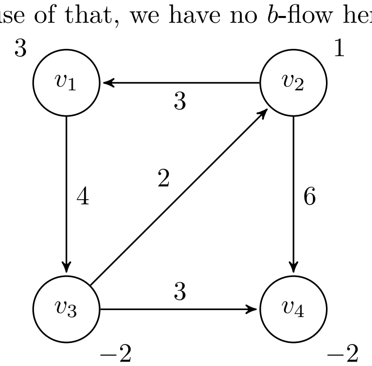



**Figure 3:** (b)

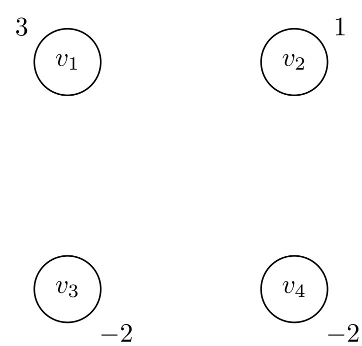



**Figure 4:** (b) $b$-flow

# 2 Exercise 2: An application of MCFP: rectilinear planar embedding

## 2.1 Exercise 2.1

The $x_{vf}$ values for all vertices and and faces of [1, Figure 3] can be found in Table 1. The $z_{fg}$ values

| $x_{vf}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $v_1$ | 0 | 1 | 1 | 0 | 0 |
| $v_2$ | 0 | 0 | 1 | 1 | 0 |
| $v_3$ | 1 | 0 | 1 | 1 | 1 |
| $v_4$ | 0 | 0 | 0 | -1 | 1 |
| $v_5$ | 1 | 0 | 0 | 0 | -1 |
| $v_6$ | 1 | 1 | 0 | 1 | 1 |
| $v_7$ | 0 | 0 | 0 | 0 | 0 |

**Table 1:** $x_{vf}$-values for all vertex/face combinations.

for the same graph can be found in Table 2. There are a total of 13 breakpoints in [1, Figure 3].

| $z_{fg}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | | 0 | 0 | 0 | 0 |
| $b$ | 2 | | 1 | 1 | 0 |
| $c$ | 1 | 1 | | 0 | 0 |
| $d$ | 0 | 1 | 0 | | 2 |
| $e$ | 4 | 0 | 0 | 0 | |

**Table 2:** $z_{fg}$ values for all faces, values for empty sets are not displayed

This corresponds to the sum of all values of Table 2, as we would expect.

A drawing of a rectilinear layout for [1, Figure 2] can be seen in Figure 5.

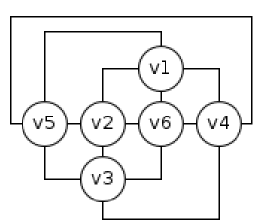



**Figure 5:** Graph in rectilinear form.

## 2.2 Exercise 2.2

Let $B$ be the set of all boundary cycles and $f_e$ the external boundary cycle. In the following, we make use of the fact that given two boundary cycles $x$ and $y$, inner turns from x to y ($z_{xy}$) will be outer turns from y to x ($z_{yx}$). We do not verify the truth of this statement. The constraints can be expressed as:

$$\forall f \in B \setminus \{f_e\} : \sum_v x_{vf} + \sum_{b \in B\setminus\{f\}} z_{fb} - z_{bf} = 4 \tag{3}$$

$$\sum_v x_{vf_e} + \sum_{b \in B\setminus\{f_e\}} z_{f_e b} - z_{bf_e} = -4 \tag{4}$$

Boundary cycle $a$ is the external boundary cycle, so Eq 4 must hold for $a$:

$$\begin{aligned}
&\sum_v x_{va} + \sum_{b \in B\setminus\{a\}} z_{ab} - z_{ba} \\
&= 3 + z_{ab} - z_{ba} + z_{ac} - z_{ca} + z_{ad} - z_{da} + z_{ae} - z_{ea} \\
&= 3 + 0 - 2 + 0 - 1 + 0 - 0 + 0 - 4 \\
&= 3 - 2 - 1 - 4 = -4
\end{aligned}$$

Boundary cycle $e$ is an internal boundary cycle, so Eq 3 must hold for $e$ specifically:

$$\begin{aligned}
&\sum_v x_{vf} + \sum_{b \in B\setminus f} z_{fb} - z_{bf} \\
&= 2 + z_{ea} - z_{ae} + z_{eb} - z_{be} + z_{ec} - z_{ce} + z_{ed} - z_{de} \\
&= 2 + 4 - 0 + 0 - 0 + 0 - 0 + 0 - 2 \\
&= 2 + 4 - 2 = 4
\end{aligned}$$

## 2.3 Exercise 2.3

The assumption is necesarry since only 4 edges can be connected to any single node, when the edges are limited to moving in two directions. From each node, an edge can go either up or down, or left or right.

To show that Equation 1 from [1] holds true we will divide it into the three different cases and show them seperetly.

- $\sum_f x_{vf} = 0$ if $v$ has degree 2
  Any vertex $v$ of degree 2 have 2 edges. This means that $v$ is part of exactly 2 boundary cycles, lets call them $f$ and $g$. This means that our sum can be defined as

  $$\sum_f x_{vf} = x_{vf} + x_{vg}.$$

  Since $v$ is of second degree, there is only two possible “kinds” of configurations of the edges, either they form a 180 degree angle, or they form a 90 and a 270 degree angle. In the case of the 180 degree angles we have $x_{vf} = x_{vg} = 0$. The last case is the 90/270 degree case, this menas one of them will be an innerturn and one will be an outer turn. Because of this one cycle must have an $x$ value of 1 and the other must have an $x$ value of $-1$ giving a summation of 0, proving the initial statement..

- $\sum_f x_{vf} = 2$ if $v$ has degree 3
  Having a degree of 3 means that the edge configuratin can be only one way, 2 edges will go either vertical or horizontal while the last edge is perpendicular to those. This configuration leaves 2 inner turns, and no outer turns. Naming the cycles $f, g$ and $h$ where $f$ and $g$ form inner turns with $v$, the sum can be written and calculated like this

$$\sum_f x_{vf} = x_{vf} + x_{vg} + x_{vh} = 1 + 1 + 0 = 2$$

  which shows the second part of the initial sum is true.

- $\sum_f x_{vf} = 4$ if $v$ has degree 4
  If $v$ is uf degree 4, there is exactly 4 edges connected to $v$ and $v$ must be part of 4 cycles. In order to have space for these 4 edges all edges must be inner turns, the sum can then be written out as so

$$\sum_f x_{vf} = 1 + 1 + 1 + 1 = 4$$

This shows that Equation 1 from [1] holds in all three cases for all vertices.

## 2.4 Exercise 2.4

The objective function $\sum_{f \in B} \sum_{g \in B \setminus \{f\}} z_{fg} + z_{gf}$ expresses the total number of breakpoints, which we wish to minimize.

$$\begin{aligned}
&\text{minimize} \quad \sum_{f,g \in B | g \neq f} z_{fg} \\
&\text{subject to} \\
&\quad \forall f \in B \setminus \{f_e\} : \sum_v x_{vf} + \sum_{b \in B \setminus \{f\}} z_{fb} - z_{bf} && = 4 \\
&\quad \sum_v x_{vf_e} + \sum_{b \in B \setminus \{f_e\}} z_{f_e b} - z_{bf_e} && = -4 \\
&\quad \sum_f x_{vf} && = \begin{cases} 0 & \text{if } v \text{ has degree 2} \\ 2 & \text{if } v \text{ has degree 3} \\ 4 & \text{if } v \text{ has degree 4} \end{cases} \\
&\quad z_{fg}, z_{gf} && \geq 0
\end{aligned}$$

where the sum $\sum_g$ is over all boundary cycles $g$ containing vertex $u$.

### 2.5 Exercise 2.5

We limit the capacity, demands and costs of the resulting MCFP to be integers. Real-valued amounts of breakpoints does not make sense.

Consider a rectilinear graph $G = (V, E)$ and an MCFP $G' = (V', E')$.

$V'$ contains a vertex for every face $f \in G$, and a vertex for every vertex $v \in V$. $E'$ contains an edge $e_{fg}$ for every pair of faces $f, g$ that share at least one vertex, and an edge $e_{vf}$ for every vertex $v$ that was part of $f$ in $G$. The intuition here is that faces and vertices both become vertices in the MCFP, and edges exist from vertices to faces and from faces to faces.

The demands of the MCFP are derived from the constraints previously established, and are as follows:

$$b_v = 4 \text{ if } v \text{ was an inner face}$$
$$b_v = -4 \text{ if } v \text{ was the external face}$$
$$b_v = \begin{cases} 0 & \text{if } v \in V \text{ and } v \text{ has degree 2} \\ 2 & \text{if } v \in V \text{ and } v \text{ has degree 3} \\ 4 & \text{if } v \in V \text{ and } v \text{ has degree 4} \end{cases}$$

Between two vertices that were faces in $G$, e.g. $v_f$ and $v_g$, the capacity is $\infty^+$ and the cost is 1. Between a vertex that was a face in $G$ and a vertex from $V$, e.g. $v_f$ and $v_v$, the capacity is 1 and the cost is 0.

Flow corresponds to the amount of breakpoints, which we wish to minimize. That is, we wish to find an assignment of real numbers $x_e$ to each edge $e \in E'$, such that

$$\forall v \in V' : \sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = b_v,$$
$$\forall e \in E' : 0 \leq x_e \leq u_e$$

# 3 Exercise 3: Reduction to MCFP

In this assignment there is 4 cases that should be taken into account with regards to the $l$ and $u$ capacities. These cases can be seen in Figure 7.

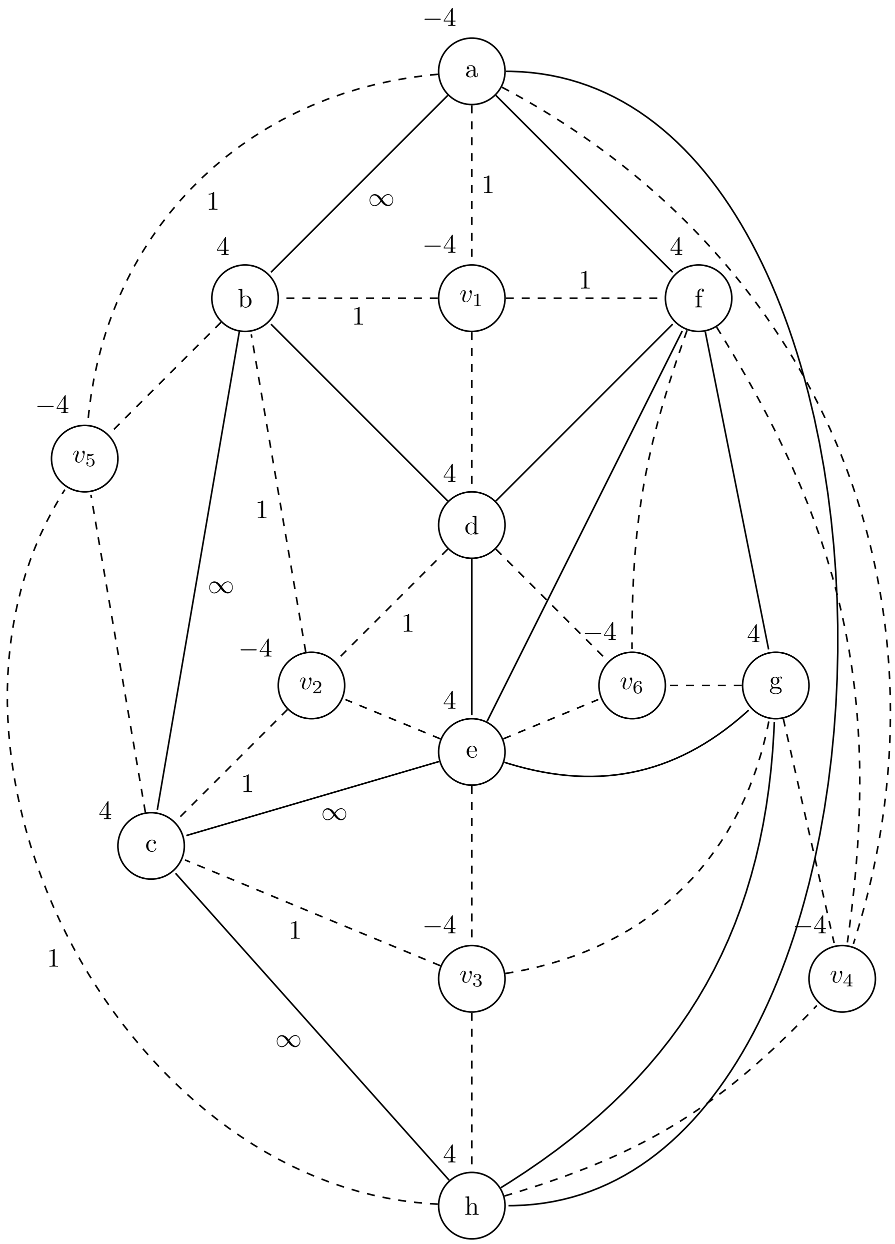
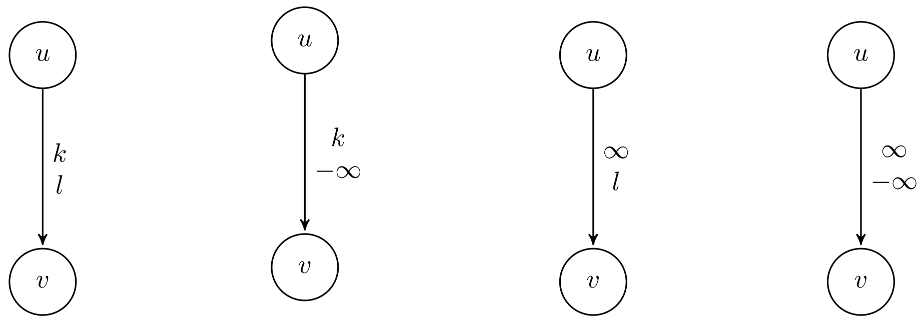



**Figure 6:** The MCFP we were tasked with drawing.




**a:** Case where both $u_e$ and $l_e$ are finite constants.

**b:** Case where both $u_e$ are a finite constant and $l_e$ is negative infinite.

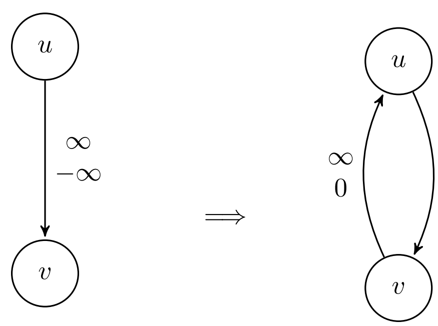
**c:** Case where $u_e$ is infinite and $l_e$ is a constant.

**d:** Case where $u_e$ are infinite and $l_e$ is negative infinite.

**Figure 7:** The different cases for vertices and edges.

## 3.1 Exercise 3.1

The only case we need to handle here is case 4, as seen in Figure 6d. If we have a graph $G = (V, E)$ where $V = \{u, v\}$ and $E = \{(u, v)\}$, with $l_{(u,v)} = -\infty$ and $u_{(u,v)} = \infty$ we can view the negative $l$ capacity as an edges ability to carry flow in the reverse direction. But inserting an anti-parallel edge $(v, u)$ with $u_{(v,u)} = \infty$, this will make the $l$ value for both edges equal to 0. Figure 8 shows the example graph before and after this operation.

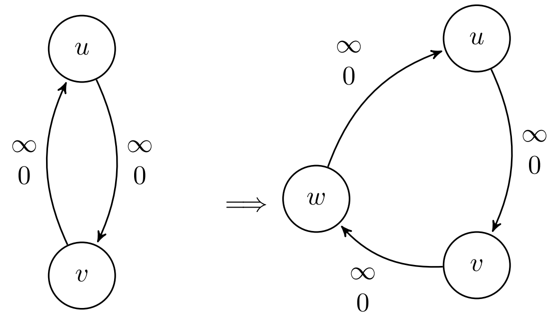



**a:** The initial graph $G$ with two vertices and on edge.

**b:** The resulting graph $G$ with 2 vertices and 2 edges.

**Figure 8:** The result of the first part of the operation.

Since we cannot have anti-parallel edges we will insert an extra vertex and connect one of the edges to this vertex and add a new edge. The new vertex $w$ shall have a demand $b = 0$ so as to not consume or produce any additional flow. The edge $(v, u)$ shall be changed to $(v, w)$ and a new edge with $(w, u)$ shall be introduced with the same capacities but with cost $c(w, u) = 0$. Figure 9 Illustrates this example.



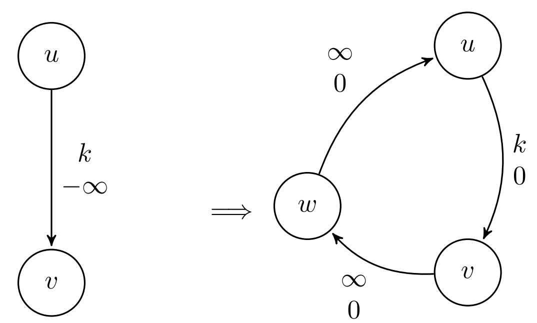

**a:** The graph $G$ after the first operation.

**b:** The final graph $G$ at the end of the procedure.

**Figure 9:** The result of the entire operation.

## 3.2 Exercise 3.2

For this part we need only consider case 2 as seen in Figure 6b since we have already solved this for case 4 in the previous question. This problem is solvable using the exact same method as above, only the $u_e$ values are different. The reduction can be seen in Figure 10




**a:** The graph $G$ in the beginning.

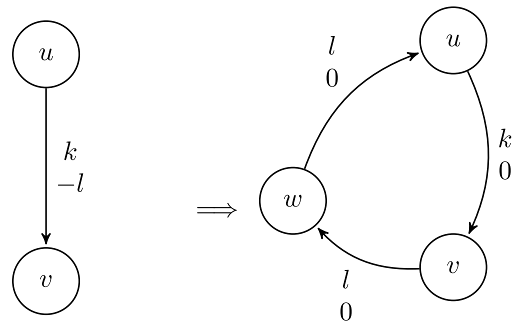
**b:** The final graph $G$ at the end of the procedure.

**Figure 10:** The result of the entire operation.

## 3.3 Exercise 3.3

In the two previous questions we did this for case 2 and 4 (Figure 6b and 6d). We will now pay attention to case 1 and 3 (Figure 6a and 6c).

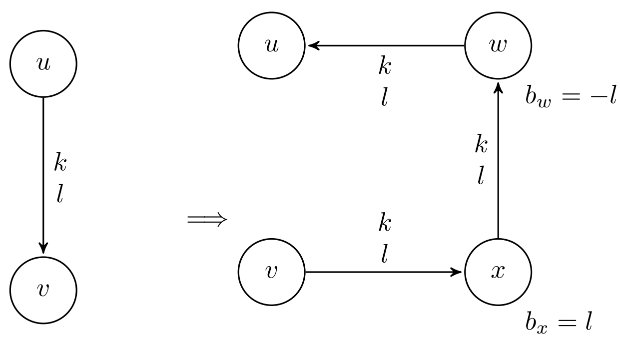
The cases correspond to a constant $l$ being either negative, or positive.




**a:** The graph $G$ in the beginning.

**b:** The final graph $G$ at the end of the procedure.

**Figure 11:** The result of the entire operation.




**a:** The graph $G$ in the beginning.

**b:** The final graph $G$ at the end of the procedure.

**Figure 12:** The result of the entire operation.

## 3.4 Exercise 3.4

We begin with $|E|$ edges. $I_1$, $I_2$, $I_3$ all add at most two edges per edge, meaning at most $\mathcal{O}(2E)$ additonal edges. This is in the worst case where all edges fall into one of the four cases in Figure 7. We assume that the four cases cover all possible cases. Transforming from $I_0$ to $I_3$ is thus adding, at most, $\mathcal{O}(2E) + \mathcal{O}(2E) + \mathcal{O}(2E) = \mathcal{O}(E)$.

## 3.5 Exercise 3.5 (Optional)

# References

[1] Noy Rotbart and Christian Wulff-Nilsen. Minimum-cost flow, advanced algirithms 2014 assignment 1, 2014.