

# Exam notes for Advanced Algorithms and Datastructures 2014

Martin Jørgensen

June 5, 2014

## Dispositions

### Max-Flow

1. Define a Flow Network
  - Capacity Constraint
  - Flow Conservation
2. Define a Max Flow
3. How to have multiple source/sink networks
4. Introduce Residual Networks
5. Introduce Augmenting Paths
6. Cuts - In particular the min cut max flow
7. Introduce Ford-Fulkerson / Edmonds-Karp

# Notes

## Max-Flow

### Flow Network

A flow network  $G = (V, E)$  is a directed graph where each edge  $(u, v) \in E$  have a nonnegative capacity  $c(u, v) \geq 0$ . In addition, for any edge  $(u, v)$  there can be no antiparallel edge  $(v, u)$ .

Two vertices in the network have special characteristics the source  $s$  and sink  $t$ . We assume each vertex  $v \in V$  lies on some path from  $s$  to  $t$ , that is, for each vertex  $v \in V$ , the flow network contains a path  $s \rightsquigarrow v \rightsquigarrow t$ .

### Flow Definition

We have a flow network  $G = (V, E)$  with a source  $s$  and a sink  $t$ , the network has a capacity function  $c(u, v)$ . A flow is a real-valued function  $f : V \times V \rightarrow \mathbb{R}$  that satisfies the two following properties:

- **Capacity Constraint:**

For all  $u, v \in V$ , we require  $0 \leq f(u, v) \leq c(u, v)$

- **Flow Conservation:**

For all  $u \in V - \{s, t\}$  we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

When  $(u, v) \notin E$ , there can be no flow from  $u$  to  $v$ , and  $f(u, v) = 0$ . We call the nonnegative quantity  $f(u, v)$  the flow from vertex  $u$  to vertex  $v$ .

The value  $|f|$  of a flow  $f$  is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

that is the total flow out of the source minus the flow into the source. For an example of a flow see Figure 1

## Antiparallel Edges and Multiple Sources/Sinks

Since a flow network cannot contain anti-parallel edges, but we want to be able to represent them in our graph, we need a way to do so. This is done by inserting an additional node  $v'$  and let one of the edges go through this node instead, see Figure 2 for an example.

If a network have multiple sources or sinks, we can convert it to a single source/sink network by adding a supersource and supersink. An example of such conversion can be seen in Figure 3.

## Flow Examples

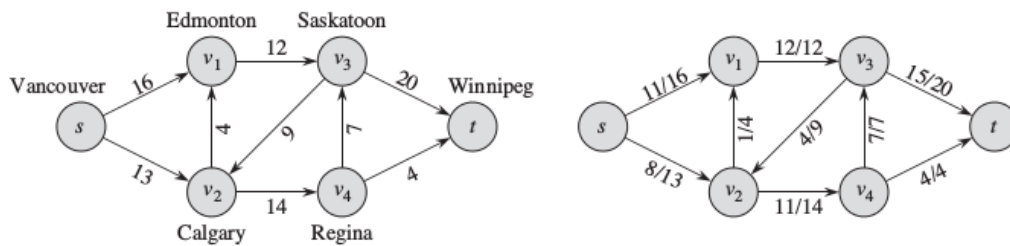


Figure 1: Example flow.

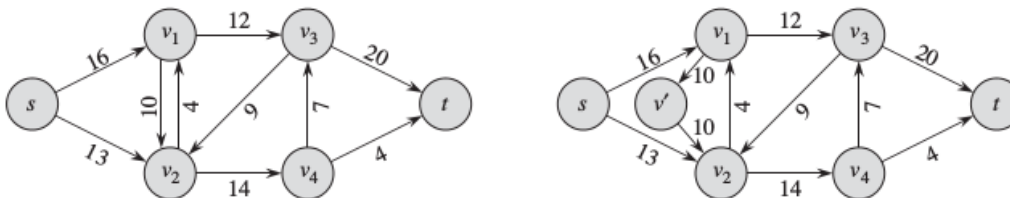


Figure 2: Conversion from antiparallel edges to proper flow.

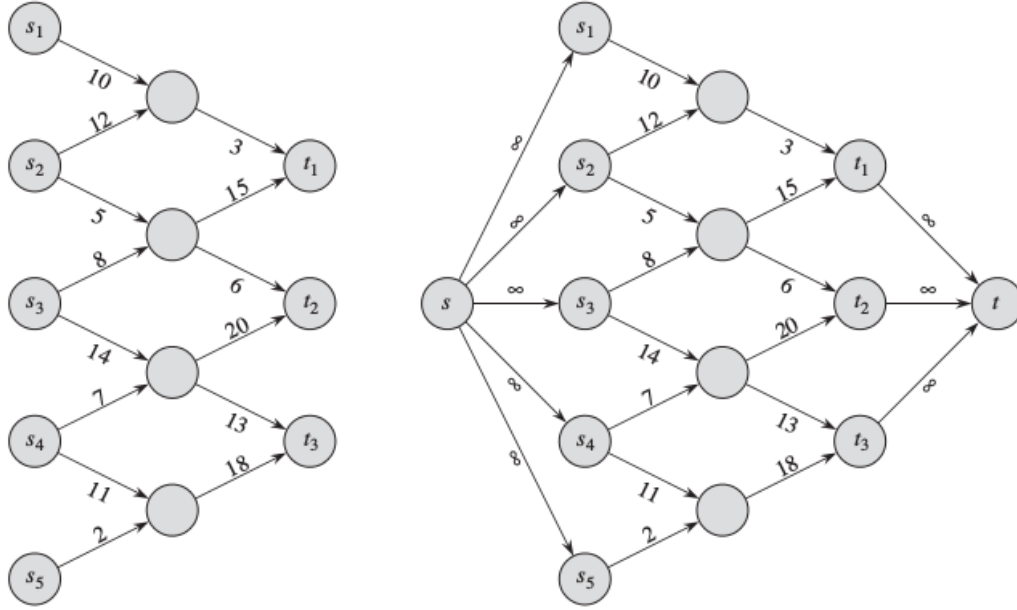


Figure 3: Example of a graph with multiple sources and sink, combined using a supersource and supersink.

### Residual Networks

Given a flow network  $G$  and a flow  $f$  the residual network  $G_f$  consists of edges and capacities that represent how we can change the flow on edges of  $G$ . Suppose we have a flow network  $G = (V, E)$  with source  $s$  and sink  $t$ . Let  $f$  be a flow in  $G$ , and consider a pair of vertices  $u, v \in V$ . We then define the residual capacity  $c_f(u, v)$  like this:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E, \\ f(v, u) & \text{if } (v, u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Given a flow network  $G = (V, E)$  and a flow  $f$ , the residual network of  $G$  induced by  $f$  is  $G_f(V, E_f)$ , where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

An example of a residual network can be seen in Figure 4.

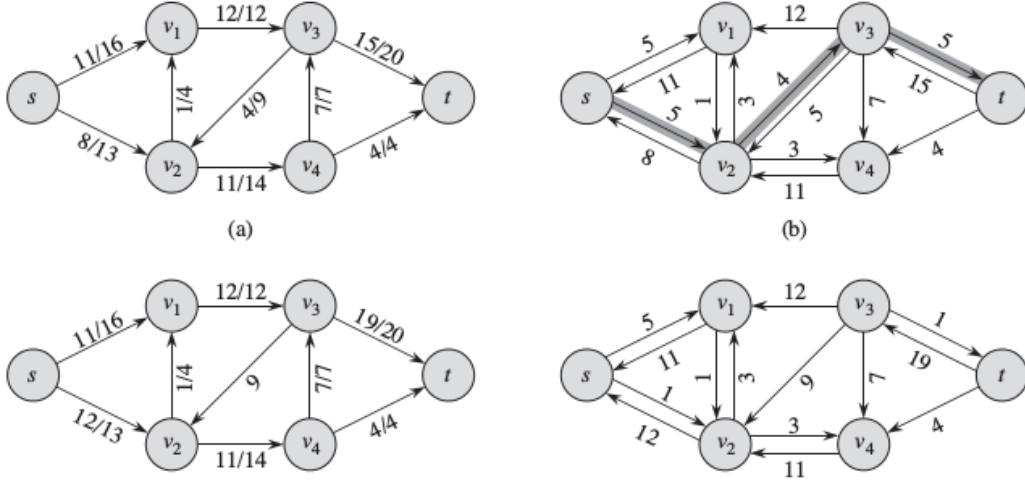


Figure 4: An example of a flow being augmented and showing the residual graph.

### Augmenting Flows

Augmenting paths are simply flows that can be added to other flows in order to increase the flow value through the network. Augmenting flows are described using the  $\uparrow$  operator like so:

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

an example of an applied augmenting path can be seen in Figure 4.

**Lemma 26.1** Let  $G = (V, E)$  be a flow network with source  $s$  and sink  $t$ , and let  $f$  be a flow in  $G$ . Let  $G_f$  be the residual network of  $G$  induced by  $f$ , and let  $f'$  be a flow in  $G_f$ . Then the function  $f \uparrow f'$  is a flow in  $G$  with value  $|f \uparrow f'| = |f| + |f'|$ .

### Augmenting Paths

Given a network  $G = (V, E)$  and a flow  $f$ , an augmenting path is a simple path from  $s$  to  $t$  in the residual network  $G_f$ . The shaded path in Figure 4(b) is an augmenting path. We can increase the flow on each edge in the augmenting

path  $p$  by an amount equal to the residual capacity of  $p$  given by

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

That is, the smallest amount of spare capacity on any edge in  $p$ .

**Lemma 26.2** let  $G = (V, E)$  be a flownetwork, let  $f$  be a flow in  $G$ , and let  $p$  be an augmenting path in  $G_f$ . Define a function  $f_p : V \times V \rightarrow \mathbb{R}$  by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then,  $f_p$  is flow in  $G_f$  with value  $|f_p| = c_f(p) > 0$ .

**Corollary 26.3** let  $G = (V, E)$  be a flownetwork, let  $f$  be a flow in  $G$ , and let  $p$  be an augmenting path in  $G_f$ . Let  $f_p$  be defined as in Lemma 26.2, and suppose that we augment  $f$  by  $f_p$ . Then the function  $f \uparrow f'$  is a flow in  $G$  with value  $|f \uparrow f'| = |f| + |f_p| > |f|$ .

### Cuts in flow networks

A cut  $(S, T)$  of a flow network  $G = (V, E)$  is a partition of  $V$  into  $S$  and  $T = V - S$  such that  $s \in S$  and  $t \in T$ . If  $f$  is a flow, then the net flow  $f(S, T)$  across the cut  $(S, T)$  is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u).$$

The capacity of the cut  $(S, T)$  is

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v).$$

A **minimum cut** of a network is a cut whose capacity is minimum over all cuts of the network. Note that for capacity we count only edges going from  $S$  to  $T$  while for flow we count edges going both directions. Lemma 26.4 shows that for a given flow  $f$ , the net flow across any cut is the same and it equals  $|f|$ , the value of the flow.

**Lemma 26.4** Let  $f$  be a flow in a flow network  $G = (V, E)$  with source  $s$  and sink  $t$ , and let  $(S, T)$  be any cut of  $G$ . Then the net flow across  $(S, T)$  is  $f(S, T) = |f|$

**Corollary 26.5** The value of any flow  $f$  in a flow network  $G$  is bounded from above by the capacity of any cut of  $G$ .

**Max-flow min-cut theorem:**

**Theorem 26.6** If  $f$  is a flow in a flow network  $G = (V, E)$  with source  $s$  and sink  $t$ , then the following conditions are equivalent:

1.  $f$  is a maximum flow in  $G$ .
2. The residual network  $G_f$  contains no augmenting paths.
3.  $|f| = c(S, T)$  for some cut  $(S, T)$  of  $G$ .

**Proof** (1)  $\implies$  (2): If we assume  $f$  is a maximum flow but there still is an augmenting path  $p$  in the residual graph  $G_f$ , then by Corollary 26.3, the flow found by  $f \uparrow f_p$  is a flow with value strictly greater than  $|f|$ , contradicting the assumption that  $f$  is a max flow.

(2)  $\implies$  (3): Suppose  $G_f$  has no augmenting paths, that is, that  $G_f$  contains no paths from  $s$  to  $t$ . We define

$$S = \{v \in V : \text{there exists a path from } s \text{ to } v \text{ in } G_f\}$$

and  $T = V - S$ . The partition  $(S, T)$  is a cut: we have  $s \in S$  trivially, and  $t \notin S$  because there is no path from  $s$  to  $t$  in  $G_f$ .

We now consider a pair of vertices  $u \in S$  and  $v \in T$ . If  $(u, v) \in E$ , we must have  $f(u, v) = c(u, v)$  otherwise  $(u, v) \in E_f$  which would place  $v$  in  $S$ .

Finish this off. See page 723-724

(3)  $\implies$  (1):

Finish this off. See page 723-724

## **Ford-Fulkerson**

FORD-FULERSON-METHOD( $G, s, t$ )

- 1 initialize flow  $f$  to 0
- 2 **while** there exists an augmenting path  $p$  in the residual network  $G_f$
- 3     augment flow  $f$  along  $p$
- 4 **return**  $f$