Examn notes for Advanced Algorithms and Datastructures 2014

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Dispositions

Max-Flow

1. Define a Flow Network
   - Capacity Constraint
   - Flow Conservation
2. Define a Max Flow
3. How to have multible source/sink networks
4. Introduce Ford-Fulkerson
5. Introduce Residual Networks
6. Introduce Augmenting Paths
7. Cuts - In particular the min cut max flow.

Notes

Max-Flow

Flow Network

A flow network $G = (V, E)$ is a directed graph where each edge $(u, v) \in E$ have a nonnegative capacity $c(u, v) \geq 0$. In addition, for any edge (u, v) there can be no antiparallel edge (v, u).

Two vertices in the network have special characteristics the source s and sink t. We assume each vertex $v \in V$ lies on some path from s to t, that is, for each vertex $v \in V$, the flow network contains a path $s \rightsquigarrow v \rightsquigarrow t$.

Flow Definition

We have a flow network $G = (V, E)$ with a source s and a sink t, the network has a capacity function $c(u, v)$. A flow is a real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the two following properties:

- **Capacity Constraint**:
  For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$
- **Flow Conservation**:
  For all $u \in V - \{s, t\}$ we require

  $$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

  When $(u, v) \notin E$, there can be no flow from u to v, and $f(u, v) = 0$. We call the nonnegative quantity $f(u, v)$ the flow from vertex u to vertex v. The value $|f|$ of a flow f is defined as

  $$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

  that is the total flow out of the source minus the flow into the source.

Antiparallel Edges and Multiple Sources/Sinks

Since a flow network cannot contain anti-parallel edges, but we want to be albe to represent them in our graph, we need a way to do so. This is done by inserting an additional node v' and let one of the edges go through this node instead, see Figure 2 for an example.

If a network have multiple sources or sinks, we can convert it to a single source/sink network by adding a supersource and supersink. An example of such conversion can be seen in Figure 3.

Flow Examples

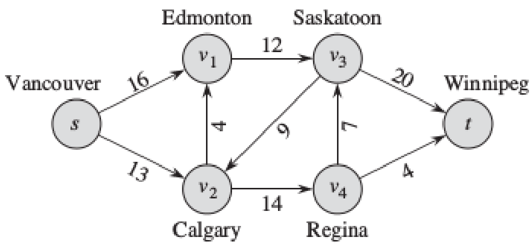

Figure 1: Example flow.

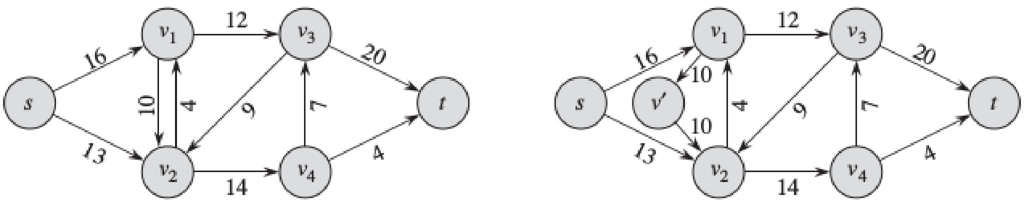

Figure 2: Conversion from antiparallel edges to proper flow.

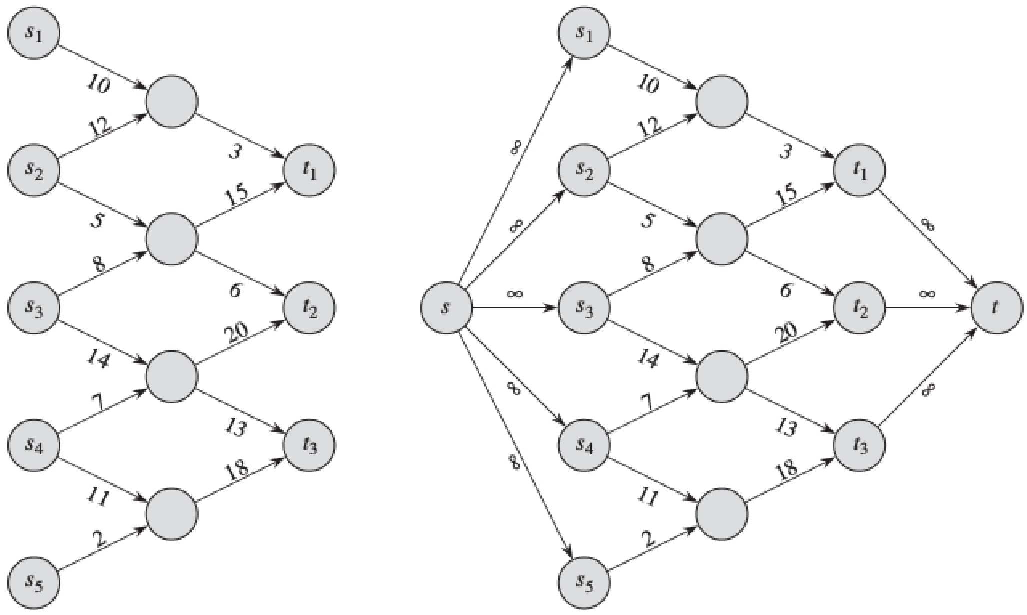

Figure 3: Example of a graphi with multiple sources and sink, combined using a supersource and supersink.

Residual Networks

Given a flow network G and a flow f the residual network G_f consists of edges and capacities that represent how we can change the flow on edges of G. Suppose we have a flow network $G = (V, E)$ with source s and sink t. LEt f be a flow in G, and consider a pair of vertices $u, v \in V$. We then define the residual flow like this:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

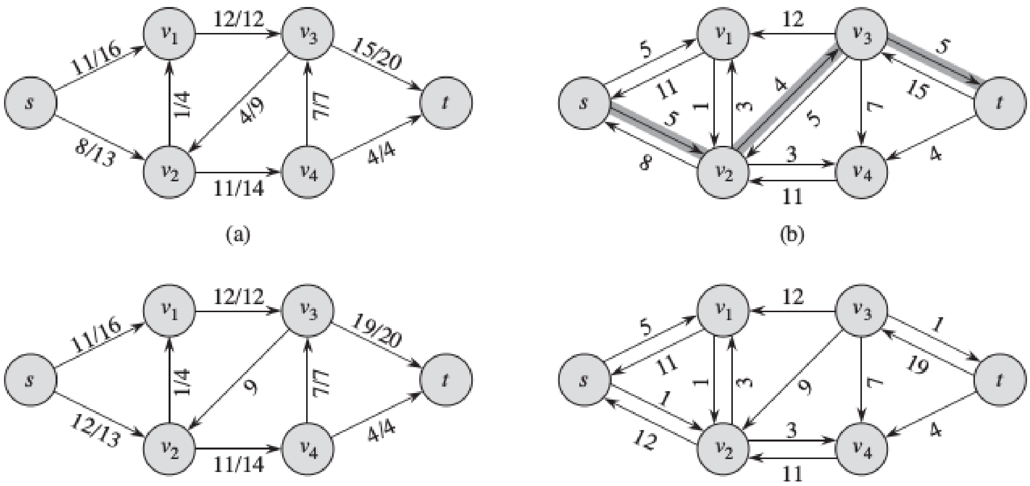

Figure 4: An example of a flow being augmented and showing the residual graph.

An example of a residual network can be seen in Figure 4.

Augmenting Paths

Augmenting paths are simply flows that can be added to other flows in order to increase the flow value through the network. Augmenting flows are described using the $\uparrow$ operator like so:

$$(f \uparrow f')(u,v) = \begin{cases} f(v,u) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

an example of an applied augmenting path can bee seen in Figure 4.

Finish this up! Lemma 26.1 coming up.

Ford-Fulkerson

FORD-FULERSON-METHOD(G, s, t)

initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f