Scale-space and Image Restoration
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J. Sporring

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Scale-space operators

1. Consider a Gaussian kernel
$$G(x, y, \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}. \tag{1}$$
The convolution of a Gaussian with itself is also a Gaussian, i.e.,
$$G(x, y, \sigma) * G(x, y, \tau) = G(x, y, \sqrt{\sigma^2 + \tau^2}) \tag{2}$$
Make an image,
$$I(x, y) = G(x, y, \sigma), \tag{3}$$
for some fixed σ, and confirm the above by calculating images from its scale-space,
$$I(x, y, \tau) = I(x, y) * G(x, y, \tau), \tag{4}$$
using Matlab and, e.g., using the `scale` function written previously.

2. Consider the 2-dimensional scale normalized derivatives at scale τ,
$$I_{x^i y^j}(x, y, \tau) = \tau^{\gamma(i+j)} \frac{\partial^{i+j} I(x, y, \tau)}{\partial x^i \partial y^j}, \tag{5}$$
where $\gamma \in \mathbb{R}$ is a parameter of the scale normalization. Now consider the scale normalized image of the Laplacian,
$$H(x, y, \tau) = I_{xx}(x, y, \tau) + I_{yy}(x, y, \tau), \tag{6}$$
Using $\gamma = 1$, solve the following:
   (a) Calculate the analytical expression for $H(x, y, \tau)$.
   (b) Consider the point $(x, y) = (0, 0)$ and derive analytically the scale(s), τ, for which $H(0, 0, \tau)$ is extremal. Maple (or Mathematica) may be helpful. Characterize these extremal point(s) in terms of maximum, saddle, and minimum in (x, y, τ).
   (c) Confirm your result in Matlab.
   (d) The maxima and minima in (x, y, τ) of (6) is called blob detection. Detect the 20 largest maxima and minima in the `sunflower.tiff` image, and indicate each detected scale τ with a circle centered on the point of detection and with a radius of τ.

3. Consider a soft edge,
$$J(x, y) = \int_{-\infty}^{x} G(x', 0, \sigma)\, dx' \tag{7}$$
for some constant σ. Consider also its scale-space,
$$J(x, y, \tau) = J(x, y) * G(x, y, \tau), \tag{8}$$
and the scale-normalized spatial squared gradient magnitude operator
$$\|\nabla J(x, y, \tau)\|^2 = J_x^2(x, y, \tau) + J_y^2(x, y, \tau). \tag{9}$$
Using $\gamma = \frac{1}{2}$, solve the following:

(a) Write the closed form expression for $\|\nabla J\|^2$.

(b) Derive analytically the scale, τ, for which $\|\nabla J\|^2$ is maximal in the point $(x, y) = (0, 0)$. Is this a maximum in (x, y, τ)?

(c) Confirm your result in Matlab.

(d) The maxima and minima in (x, y, τ) of (9) is edge detection with scale-selection. Detect the 100 largest maxima and minima in the `hand.tiff` image, and indicate the point of detection and scale by circles.

Inverse filtering

4. Write a program that takes an image, a kernel and a realisation of a noise source, and which returns the linear, shift invariantly (LSI) degraded result.

5. Implement inverse filtering together with a band-pass filter to avoid numerical instabilities, and apply it to a LSI degraded image of `lena.tif`. Discuss this approach's ability to recover the original image.

6. Repeat the above exercise for the Wiener filter.

Extra assignment on inverse filtering

7. Consider the Laplacian as the linear operator and implement constrained deconvolution without using deconvreg. Evaluate the result on suitable test images.