Internal Report

Parameters uncertainty impact on capital requirements

Roberto Baviera & Giulia Bianchi & Alessandro Canevisio
Politecnico di Milano

April 12, 2019

Draft - do not distribute

1 Introduction

The aim of this work is to quantify portfolio credit risk, subject to estimation risk, i.e. the risk arising from errors in model parameters when we can't rely on the assumption that the parameters of the model are known with certainty. To include estimation risk, we consider the "Bayesian approach" to model risk (see e.g. Sibbertsen *et al.* 2008) and we compute the parameters probability density functions (hereafter p.d.f.) following the two standard statistical inference procedures: a frequentist inference and a Bayesian inference.
The former one has been introduced in model risk for capital requirement by Löffler (2003), who analyses, via a Monte Carlo method, the impact on the α-quantile of two homogeneous reference portfolios, one BBB and another one B. The latter case has been presented by Tarashev (2010) and extends the classical Bayesian inference to the case of the capital requirement of a homogeneous reference portfolio; he also considers the strong hypothesis that a probability distribution - the conditional distribution of the observed data given the unknown parameters - attains the Cramer-Rao lower bound. In both papers the dataset is not available, for this reason we consider a dataset on US high-yield corporate bonds, present in Altman *et al.* (2005, Table 1, p.2210).
Having the p.d.f. of model parameters, it is possible to evaluate how much their uncertainty impacts capital requirements.

Four are the main research questions that we consider in this study. First, we want to obtain an estimate of the order of magnitude of the model risk impact to capital requirement due to the estimation risk; we also desire to show the accuracy of the Monte Carlo technique we use in some simple cases where a closed formula exists. Second, we wish to analyze the Cramer-Rao lower bound, showing whether it is a too strong hypothesis in model risk applied to capital requirements. Third, we desire to verify that the two statistical inferences yield similar results; this fact can be relevant, because the first inference requires a lower computational effort. Finally, it can be interesting to illustrate an example where the capital requirement is obtained via the Basel requirement for IRB.

The model considered for capital requirement is the Asymptotic Single Risk Factor (ASRF) model on a homogeneous portfolio. Thus, the parameters considered are the default probability (PD), the recovery rate ($\overline{\pi}$) and the correlation between obligors ($\overline{\rho}$). We also consider a small homogeneous portfolio (composed of $N_{ob} = 60$ obligors, as in Löffler (2003)); in both cases (the asymptotic and the small one), we include the uncertainty on one parameter at a time and on all parameters simultaneously. Moreover, we analyze the model risk in a case of interest from a practitioner point of view, where the correlation is a deterministic function of the default probability as in Basel requirements for IRB (cf. BIS 2005, p.13), and only the other two parameters are subject to estimation risk.

The rest of the report is divided as follows. In Section 2 we briefly recall the nominal and the alternative model we consider. In Section 3 we describe the estimation risk methodology accordingly to the frequentist and the Bayesian inference. In Section 4 we discuss the results. Section 5 concludes.

2 The nominal and the alternative model

The aim of this Section is to present the nominal and the alternative model. We consider a homogeneous portfolio, i.e. characterized by the same parameters for each obligor.
The nominal model considered, i.e. the one in absence of model risk, is the Vasicek (1987) model (a Gaussian single risk factor model for a homogeneous portfolio), in which the log-assets of a generic obligor i are modeled as:

$$\begin{aligned} &X_i = \sqrt{\rho} Z + \sqrt{1-\rho}\,\varepsilon_i, \quad i = 1, \ldots, N_{ob}, \\ &\text{where: } Z,\ \varepsilon_i \text{ are i.i.d. r.v. } \sim \mathbb{N}(0,1)\,. \end{aligned} \tag{1}$$

The variable Z is a common risk factor representing the market, ε_i is an idiosyncratic risk component different for each obligor, while ρ is the correlation parameter.
In the nominal model, a default of an obligor in one year happens with a probability PD and produces a Loss Given Default $LGD = 1 - \overline{\pi}$, where $\overline{\pi}$ is the average recovery rate; as correlation parameter we consider $\overline{\rho}$, the average correlation between obligors' assets. In particular, in the Vasicek model, a default happens when $X_i < \overline{d}$ where the threshold $\overline{d}$ is chosen s.t. $P(X_i < \overline{d}) = PD$; this implies $\overline{d} := \mathcal{N}^{-1}(PD)$, with $\mathcal{N}$ the cumulative distribution function (c.d.f.) of a st.n. distribution.
Thus, the parameters of the nominal model are:

- the (1 year) default probability PD,
- the Loss Given Default LGD,
- the average correlation $\overline{\rho}$.

We distinguish between a Large Homogeneous Portfolio (LHP) - i.e. an ASRF for a homogeneous portfolio -, and a Homogeneous Portfolio (HP), composed of N_{ob} obligors, in order to account for the impact of an imperfect granularity of the credit portfolio (see e.g. Gordy and Lütkebohmert 2007).
In this report nominal model parameters are obtained from the average of the corresponding observed quantities, the Default Rate (DR) and the Loss Given Default Rate (LGDR) defined as $\pi = 1 - LGDR$.

The aim of this research is to find a p.d.f. for each parameter to evaluate the impact of the uncertainty of each parameter on the capital requirement, in the frequentist and in the Bayesian inference.
The alternative model is the same as the nominal model but the PD, LGD and ρ are described by some r.v. Thus, in this case, the parameters of the model are:

- the default rate DR,
- the Loss Given Default rate $LGDR$,
- the correlation ρ,

whose average values are -respectively- PD, LGD and $\overline{\rho}$, the parameters of the nominal model.
In particular, within this report, we model instead of DR the r.v. $d := \mathcal{N}^{-1}(DR)$. For example, in the case of frequentist inference, we consider three independent r.v.s

- $d \sim \mathbb{N}(\hat{d}, \sigma_d^2)$,

- $\pi \sim \mathbb{N}(\overline{\pi}, \sigma_\pi^2)$,
- ρ modeled as a beta r.v. with mean $\overline{\rho}$.

In the alternative model, since the parameters are stochastic, in general there is no closed formula for αVaR. Hence, we use a Monte Carlo method, i.e. we simulate N_{sim} values for the unknown parameters (e.g. $N_{sim} = 10^6$) and we compute the Loss function for each simulation, finding a Loss distribution.
In both the nominal and the alternative model the Loss function is given by

$$Loss = EAD \cdot LGD \cdot m \ ,$$

where EAD is the Exposure At Default and m is the number of defaulted obligors. In this report the EAD is not considered a source of model risk. It is chosen as $1/N_{ob}$ because we are considering a homogeneous portfolio, obtaining in this way a capital requirement in percentage terms.
Given the Loss function we are able to compute the αVaR as the α^{th} quantile of the 1y-Loss distribution. We consider two values for the level α: $\alpha = 99\%$ (as it is considered in both Löffler (2003) and Tarashev (2010)) and $\alpha = 99.9\%$ as established by the Basel Committee (cf. BIS 2005, p.11).
What really matters in credit risk problems is not the αVaR, but the capital requirement αRC, defined as:

$$\alpha RC := \alpha VaR - \mathbb{E}[Loss] \ .$$

Let us stress that the expected loss $\mathbb{E}[Loss]$ is equal to $LGD \cdot PD$ in both the nominal and the alternative models for the HPs we are considering.

In particular we focus on the Add-on for the capital requirement, defined as the percentage increment of the αRC in the alternative model w.r.t. its nominal value.
We analyse two significant cases: one in which the three parameters are modelled as r.v. in line with existing literature (see e.g. Löffler 2003, Tarashev 2010) and another one where ρ is a deterministic function of DR as done by the Basel Committee for IRB models (cf. BIS 2005, p.13)

$$\rho(DR) = 0.12 \cdot \frac{1 - e^{-50 \cdot DR}}{1 - e^{-50}} + 0.24 \cdot \left(1 - \frac{1 - e^{-50 \cdot DR}}{1 - e^{-50}}\right) \ . \tag{2}$$

2.1 Large Homogeneous Portfolio

In the LHP case, the nominal model has a well known capital requirement

$$\alpha RC = (1 - \overline{\pi}) \cdot \mathcal{N}\left(\frac{\mathcal{N}^{-1}(PD) - \sqrt{\overline{\rho}}\mathcal{N}^{-1}(1-\alpha)}{\sqrt{1-\overline{\rho}}}\right) - (1 - \overline{\pi}) \cdot PD \ . \tag{3}$$

In a similar way it is possible to obtain the αRC in the alternative model. Given the parameters d, π and ρ, given the common risk factor Z, in the asymptotic case the fraction of defaults y is $p(Z)$, the default probability conditional on Z. Hence:

$$y|Z = p(Z) = \mathcal{N}\left(\frac{d - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right). \tag{4}$$

The loss becomes

$$Loss = (1-\pi) \cdot \mathcal{N}\left(\frac{d - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \tag{5}$$

and we get the αVaR in the LHP case for the alternative model via Monte Carlo simulation.
The case with an estimation risk due only to d can be computed analytically in a way similar to Tarashev (2010, eq. (10))

$$\alpha VaR = (1-\overline{\pi}) \cdot \mathcal{N}\left(\frac{\hat{d} - \sqrt{\overline{\rho} + \sigma_d^2}\,\mathcal{N}^{-1}(1-\alpha)}{\sqrt{1-\overline{\rho}}}\right), \tag{6}$$

where $\hat{d}$ and σ_d^2 are respectively the mean value and the variance of the distribution of d. In Section 4 we use this closed-form result in order to check the numerical accuracy.

Let us remind that we are modeling d instead of DR and $PD = \mathbb{E}[DR]$, so $\hat{d}$, the mean value of d, is related to $\bar{d}$ via a bias correction that can be easily computed, i.e.:

$$PD = \mathbb{E}[\mathcal{N}(d)] = \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \mathcal{N}(\hat{d} + \sigma_d x) \simeq \mathcal{N}(\hat{d}) - \frac{\sigma_d^2}{2} \cdot \frac{\hat{d}}{\sqrt{2\pi}} e^{-\hat{d}^2/2}. \tag{7}$$

2.2 Small portfolio

In the HP case, we consider a portfolio composed of $N_{ob} = 60$ obligors.
In order to compute the αRC, we follow a Monte Carlo approach, even for computing the number of defaulted obligors. In particular, for each scenario ($j = 1, ..., N_{sim}$), we simulate one value for Z, common to all obligors, and N_{ob} values of ε_i for each obligor ($i = 1, .., N_{ob}$). We compute the values of the log-assets X_i using (1) for each obligor and we count how many of these are below the default threshold d.
In the nominal case it is possible to obtain a closed formula, but it is not possible to obtain a simple formula in the alternative model, so we compute

all quantities with a Monte Carlo approach. In this case, the Loss in each scenario is

$$Loss_j = (1 - \pi_j) \cdot \frac{1}{N_{ob}} \sum_{i=1}^{N_{ob}} \mathbb{1}_{(X_i^{(j)} \leq d_j)}, \; j = 1, ..., N_{sim}. \tag{8}$$

As in the asymptotic case, once we have computed the Loss distribution we get the αVaR as its α^{th} quantile.

3 Estimation risk

In this section we analyse the different methods used to compute the p.d.f. for the unknown parameters. We study the Add-on generated by the uncertainty of each parameter one at a time and of all parameters together.

3.1 Frequentist inference

To compute the p.d.f. of the parameters (default rate, recovery rate and correlation) we consider all parameters uncorrelated and we simulate one value for each parameter independently from the others, as in Löffler (2003). As mentioned above, we model both π and d as Gaussian r.v. and ρ as beta distributed.
The hypothesis on π can be found in Löffler (2003). Reasoning by analogy also d is modeled as a Gaussian r.v. Both hypotheses are tested via a Shapiro-Wilk Gaussianity test.
To introduce the uncertainty in the correlation parameter, we suppose that the correlation is distributed as a beta,

$$\rho \sim Beta(\alpha_\rho, \beta_\rho), \quad \overline{\rho} = \frac{\alpha_\rho}{\alpha_\rho + \beta_\rho}, \quad \sigma_\rho^2 = \frac{\alpha_\rho \beta_\rho}{(\alpha_\rho + \beta_\rho)^2 (\alpha_\rho + \beta_\rho + 1)} \;. \tag{9}$$

that is a simple r.v. with values between 0 and 1. Regarding the beta parameters values, we use the ones in Tarashev and Zhu (2008, Table 2, p.152). Overlapping the histogram in Tarashev and Zhu (2008, Figure 2, p.152) and the density of the beta distribution with the parameters in Tarashev and Zhu (2008, Table 2, p.152) we get:

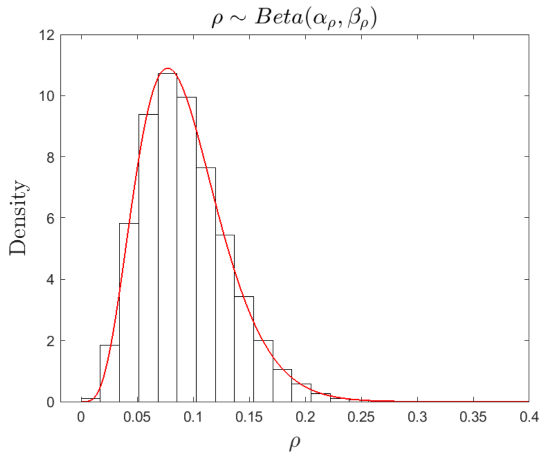


Figure 1: Correlation distribution. Plot of the distribution in Tarashev and Zhu (2008, Figure 2, p.152) and the beta distribution associated.

$\overline{\rho}$	0.0924
σ_ρ	0.0386
α_ρ	5.1083
β_ρ	50.1766

Table 1: Distribution parameters.

3.2 Bayesian inference

In this section we describe the Bayesian inference first presented in Tarashev (2010) to compute the parameters distributions in the model risk for capital requirements. A Bayesian inference consists in finding a posterior distribution of the unknown parameters given data starting from a prior distribution and a distribution for the observed data conditional on the unknown parameters.

Using Bayes' theorem (see e.g. Gelman *et al.* 2013, p.7, eq.(1.1)), the probability density function *a posteriori* is:

$$h(\mathbf{p}|\hat{\mathbf{p}}) = \frac{g(\mathbf{p})f(\hat{\mathbf{p}}|\mathbf{p})}{\int_{\mathbf{p}\in\Omega} g(\mathbf{p})f(\hat{\mathbf{p}}|\mathbf{p})d\mathbf{p}} \tag{10}$$

with $\mathbf{p}$ the vector of parameters, Ω the space where the parameters live, $\hat{\mathbf{p}}$ the observed parameters, g the prior distribution and f the distribution of data conditional on the unknown parameters.

In this framework, as in Tarashev (2010), the $LGDR$ is considered constant and equal to its average value LGD.

We consider ρ and d indipendently distributed. For the distributions of the observed data conditional on the unknown parameters we suppose:

- the mean value equal to the unknown parameter $\mathbf{p}$,
- the variance as the empirical one.

Moreover, following Tarashev (2010), we consider also the case where the correlation parameter ρ depends only on the observed correlation $\hat{\rho}$, whereas the default rate d depends both on the observed data $\hat{d}$ and the empirical correlation ρ. In this case, we suppose as in Tarashev (2010):

- the mean value equal to the unknown parameter $\mathbf{p}$,
- the variance as the Cramer-Rao lower bound for the unknown parameters, given by

$$\sigma^2_{CR,l}(\mathbf{p}) = -\frac{1}{T}\left(\mathbb{E}_{\mathbf{y}}\left[\frac{\partial^2 \ln L(\mathbf{y}|\mathbf{p})}{\partial \mathrm{p}_l^2}\right]\right)^{-1} \quad l = 1, \ldots, \#\text{ parameters}$$

  where $\ln L$ is the log-likelihood function for the observed data.

3.2.1 Correlation

As already mentioned, we suppose that ρ depends only on its observed value $\hat{\rho}$. The distributions *a priori* and for the observed data conditional on the unknown parameter are respectively (see Tarashev 2010, Appendix B, p.2076):

$$\rho \sim Unif(0,1) \qquad \rightarrow g(\rho) = \mathbb{1}_{(0,1)}(\rho) \tag{11}$$

$$\hat{\rho}|\rho \sim Beta(\alpha^B_\rho, \beta^B_\rho) \rightarrow f(\hat{\rho}|\rho) = \frac{\Gamma(\alpha^B_\rho + \beta^B_\rho)}{\Gamma(\alpha^B_\rho)\Gamma(\beta^B_\rho)}\hat{\rho}^{\alpha^B_\rho - 1}(1-\hat{\rho})^{\beta^B_\rho - 1}. \tag{12}$$

The *a priori* distribution is modelled flat in the interval $(0,1)$: it is called *uninformative prior*, because it does not provide any additional information on the distribution of the parameter. The choice of a beta distribution for the unknown parameter given data is due to the fact that its values are between 0 and 1; we notice that this is the same choice done also in the frequentist inference. We choose the parameters α^B_ρ and β^B_ρ in such a way that the condition

$$\mathbb{E}[\hat{\rho}|\rho] = \rho \tag{13}$$

holds and $var[\hat{\rho}|\rho]$ is the empirical variance.

In Figure 2 a comparison between the posterior distribution and a beta distribution with the same parameters is provided. We show that the posterior density for ρ is very close to a beta distribution with the same mean and the same variance. It is also similar to the distribution obtained in the frequentist inference.

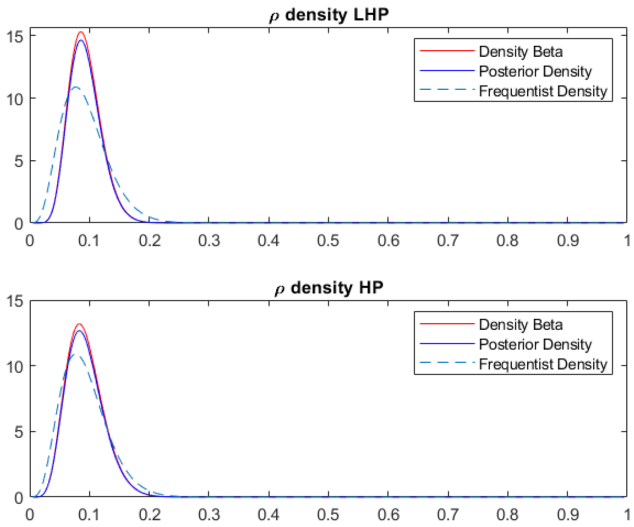


Figure 2: Comparison between the posterior density for ρ (blue line) and a beta distribution with the same mean and the same variance (red line) in the LHP and HP cases. The dotted line represents the beta distribution corresponding to the frequentist inference.

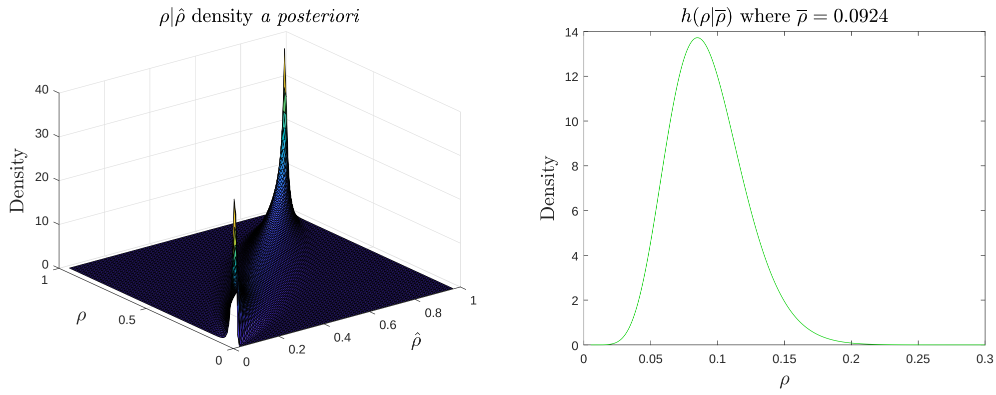
Moreover, we also consider the case where the parameters α_ρ^B e β_ρ^B are chosen in such a way that,

$$\mathbb{E}[\hat{\rho}|\rho] = \rho, \quad var[\hat{\rho}|\rho] = \sigma^2_{CR,\rho}(\rho), \tag{14}$$

where $\sigma^2_{CR,\rho}(\rho)$ represents the Cramer Rao lower bound for ρ, i.e. the minimum variance for the estimator $\hat{\rho}$, and in particular it has the following expression (see Tarashev 2010, Appendix C, p.2076)

$$\sigma^2_{CR,\rho}(\rho) = \frac{2(1-\rho)^2(1+(N_{ob}-1)\rho)^2}{N_{ob}(N_{ob}-1)T}. \tag{15}$$

The proof of this formula is provided in the appendix.
Since this is a 1-dimensional problem, we can compute numerically the distributions *a posteriori* given all possible values of the observed data $\hat{\rho}$ obtaining the surface in Figure 3.a. In particular, using (10), (11) and (12) and considering the same expected value for ρ as in Tarashev and Zhu (2008, Table 2, p.152), we get the distribution for $\rho|\overline{\rho}$ in Figure 3.b.



(a) (b)

Figure 3: Probability density function *a posteriori* for ρ.

3.2.2 Default rate

Having clear the behaviour of ρ, we are able to compute the distribution *a posteriori* for the unknown parameter DR. Tarashev (2010) supposes i) the distribution *a posteriori* for d to be Gaussian (see Appendix B, p.2076) and ii) the density *a priori* for the default rates to be uninformative, i.e. uniformly distributed

$$DR \sim U(0,1), \text{ which implies that } d \sim \mathbb{N}(0,1)\,. \tag{16}$$

In order to fulfill the first hypothesis, the distribution of data given the two unknown parameters is chosen as

$$\hat{d}^B|(d,\rho) \sim \mathbb{N}(d, \sigma_d^2)\,. \tag{17}$$

We consider two cases for the variance σ_d^2, where σ_d^2 is equal either to the empirical variance or to the Cramer-Rao lower bound $\sigma^2_{CR,d}(\hat{d}^B, \rho)$. A closed formula for the Cramer-Rao lower bound for d is available only in the LHP case (see Appendix B for details), while in the HP case it has been computed numerically.
Then, we can easily compute the density *a posteriori* for d given $\{\hat{d}^B, \rho\}$ as in Gelman *et al.* (2013, p.46, eq.(2.10))

$$\begin{aligned} d|(\hat{d}^B, \rho) &\sim \mathbb{N}(\mu_{post}, \sigma^2_{post}) \\ \mu_{post} &= \frac{\hat{d}^B}{1+\sigma_d^2} \\ \sigma^2_{post} &= \frac{\sigma_d^2}{1+\sigma_d^2}. \end{aligned} \tag{18}$$

4 Results

In this section we briefly describe the dataset we consider, we analyze the impact on capital requirements stemming from parameter uncertainty and we compare the results obtained with both inferences.

4.1 The Dataset

As already stated in the introduction, the datasets used by Löffler (2003) and by Tarashev (2010) are not available.
In this report we consider a dataset on US high-yield corporate bonds, present in Altman *et al.* (2005, Table 1, p.2210). In particular, in this dataset there is an annual value for the years 1982-2001 ($T = 20$ years) for the following quantities:

- Default rate DR (%)
- Recovery rate π (%)
- Issued bond volume (in USD Million)
- Defaulted bond volume (in USD Million)

4.2 Large homogeneous portfolio

We start our analysis comparing the methodologies applied to a LHP.

4.2.1 Nominal case

First, we compute the regulatory capital in the nominal case. The αRC in the nominal case is obtained via formula (3) considering the mean values for each parameter.

	99%	99.9%
αRC	0.0508	0.0837

Table 2: Capital requirements in the nominal case LHP.

	99%	99.9%
αRC	0.0698	0.1194

Table 3: Capital requirements in the nominal case LHP with Basel correlation.

The capital requirements in the nominal model, with the three parameters estimated, are shown in Table 2; they are the same for both the frequentist and the Bayesian inferences. Capital requirements in the Basel case can be found in Table 3; they are higher due to the higher correlation values in equation (2) compared to the estimated values.

4.2.2 Estimation risk

In this subsection we report the results obtained considering the uncertainty for one parameter at a time and for all parameters at the same time, following the methodologies explained in Section 3 for the two main inferences studied (i.e. frequentist and Bayesian).

Tables 4 and 5 show the results obtained considering either one parameter at time or more parameters simultaneously. It is interesting to observe that, in line with the literature, the greatest contribution is due to the uncertainty in DR. In the Bayesian inference we consider the estimated variance of the data for the variance of the distribution of data conditional on the unknown parameters. Comparing these two Tables we notice that the results obtained in the frequentist and the Bayesian inference look quite similar: these tables justify the preference for an analysis according to the classical inference.

Frequentist	99%	99.9%
d	75.86%	89.35%
π	7.66%	11.14%
ρ	4.31%	16.48%
d,ρ	76.96%	96.22%
All	84.56%	110.49%

Table 4: Add-on% with frequentist inference

Bayesian	99%	99.9%
d	68.70%	79.22%
ρ	1.39%	13.14%
d,ρ	67.54%	83.87%

Table 5: Add-on% with Bayesian inference and empirical standard deviation

In Table 6 we consider the Bayesian technique described in Tarashev (2010), where the variance of the distribution of data conditional on the unknown parameters is chosen with its lower bound, i.e. the minimum possible variance known as Cramer-Rao (CR) lower bound.

Bayesian (CR)	99%	99.9%
d	4.52%	4.79%
ρ	2.52%	8.96%
d, ρ	7.61%	8.72%

Table 6: Add-on% with Bayesian inference and Cramer-Rao lower bound

4.3 Homogeneous Portfolio

We conclude our analysis considering a portfolio which is still homogeneous but is referred to $N_{ob} = 60$ obligors.

4.3.1 Nominal case

As in the asymptotic portfolio, firstly we need to compute the capital requirements in the nominal case in order to get the impact of parameters uncertainty. Via a Monte Carlo method, we get the nominal αRC in Table 7.

	99%	99.9%
αRC	0.0672	0.1061

Table 7: Capital requirements in the nominal case HP

	99%	99.9%
αRC	0.0867	0.1449

Table 8: Capital requirements in the nominal case HP with Basel correlation

As in the LHP case, the nominal case is the same for both the inferences.

4.3.2 Estimation Risk

Following the methodologies explained in Section 3, using Monte Carlo simulations for all the unknown parameters, we obtain the following values for the mean and the standard deviation of parameters d and ρ in the frequentist and in the Bayesian one using the empirical variance and the Cramer-Rao lower bound.

d	Mean	Std. dev.
Frequentist	-1.9228	0.3550
Bayesian σ_d^2	-1.9107	0.3345
Bayesian $\sigma_{CR,d}^2$	-1.8208	0.0988

Table 9: Parameters of the distribution of d

ρ	Mean	Std. dev.
Frequentist	0.0924	0.0386
Bayesian σ_ρ^2	0.0891	0.0383
Bayesian $\sigma_{CR,\rho}^2$	0.0928	0.0312

Table 10: Parameters of the distribution of ρ

The results obtained for the percentage Add-On are the following:

Frequentist	99%	99.9%
d	57.96%	64.21%
π	8.13%	13.19%
ρ	0.01%	9.17%
d,ρ	57.96%	73.39%
All	60.02%	83.69%

Table 11: Add-on% with frequentist inference

Bayesian	99%	99.9%
d	43.49%	64.23%
ρ	0.002%	9.17%
d,ρ	43.50%	64.23%

Table 12: Add-on% with Bayesian inference and empirical standard deviation

Frequentist (Basel)	99%	99.9%
d	33.71%	33.56%
π	2.43%	5.71%
d, π	36.85%	45.32%

Table 13: Add-on% with frequentist inference and Basel correlation

Bayesian (CR)	99%	99.9%
d	0.006%	9.17%
ρ	0.008%	9.18%
d, ρ	6.81%	9.30%

Table 14: Add-on% with Bayesian inference and Cramer-Rao lower bound

Notation

Symbol	Description
$\overline{\bullet}$	Average value of the parameter $\bullet$
$\hat{\bullet}$	Observed parameter $\bullet$ in the Bayesian inference
$\mathbb{1}$	Indicator function
α	Confidence level
α_ρ, β_ρ	Parameters of the beta distribution for ρ in the frequentist inference
α_ρ^B, β_ρ^B	Parameters of the beta distribution for ρ in the Bayesian inference
d	Default threshold, defined as $\mathcal{N}^{-1}(DR)$
DR	Default rate
EAD	Exposure at default
$\varepsilon_\bullet$	Idiosyncratic risk component, modeled as a st.n. r.v.
$\mathbb{E}[\bullet]$	Expected value
Γ	Gamma function
LGD	Loss given default, estimated as the average Loss given default rate
$LGDR$	Loss given default rate
m	Number of defaults in the reference portfolio
$\mathcal{N}(X)$	Cumulative distribution function of the st.n. r.v. X
$\mathbb{N}(\mu, \sigma^2)$	Gaussian distribution with mean μ and variance σ^2
N_{ob}	Number of obligors in the reference portfolio
N_{sim}	Number of simulations in Monte Carlo method
$\mathbf{p}$	Vector of parameters
$p(Z)$	Conditional probability of default given Z
PD	Probability of default, estimated as the average default rate
π	Recovery rate
ρ	Correlation parameter
$\sigma_\bullet^2$	Variance of the $\bullet$ parameter
$\sigma_{CR,\bullet}^2$	Cramer-Rao lower bound for the variance of the $\bullet$ parameter
T	Length of time series
$X_\bullet$	Log-asset for an obligor
y	Fraction of default in the reference portfolio
Z	Market risk variable, modeled as a st.n. r.v.

Shorthands

ASRF	Asymptotic Single Risk Factor model
c.d.f.	cumulative distribution function
CR	Cramer-Rao lower bound
e.g.	for example (Latin: exempli gratia)
HP	Homogeneous Portfolio
i.e.	that is (Latin: id est)
i.i.d.	independent identically distributed
IRB	Internal Rating Based model
LHP	Large Homogeneous Portfolio
p.d.f.	probability density function
r.v.	random variable
s.t.	such that
st.n.	standard normal
w.r.t.	with respect to

A Cramer-Rao lower bound for ρ

To compute the Cramer-Rao lower bound for ρ we use the log-likelihood of N_{ob} jointly Gaussian r.v., each one with mean μ, standard deviation σ and correlation ρ. We define the log-likelihood $\ell(\underline{X}|\psi^*) = LL^{ar}(\{X_i\}_{i=1}^{N_{ob}}|\psi^*)$, (with $\psi^* = [\mu^*, \sigma^*, \rho]$ vector of the unknown parameters), so the Fisher information matrix is:

$$I(\psi^*) = -\mathbb{E}\left[\left. \frac{\partial^2 LL^{ar}(\{X_i\}_{i=1}^{N_{ob}}|\psi)}{\partial \psi^2} \right|_{\psi=\psi^*} \right].$$

Since we have three unknown parameters, the Cramer-Rao lower bound for ρ is the (3,3) element of the matrix $I^{-1}(\psi^*)/T$.
The p.d.f. of N_{ob} jointly Gaussian r.v. is:

$$f(x|\mu, \Sigma) = (2\pi)^{-\frac{N_{ob}}{2}} \sqrt{\det \Sigma} \cdot \exp\left\{ -\frac{1}{2}(x-\mu)' \cdot \Sigma^{-1} \cdot (x-\mu) \right\}. \tag{19}$$

In our case the covariance matrix Σ has dimension $N_{ob} \times N_{ob}$, and it's like:

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix}$$

To go on with computations, we need an explicit formula for the determinant, indeed:

$$\det \Sigma = (1-\rho)^{N_{ob}}(1+(N_{ob}-1)\rho)\sigma^{2N_{ob}}$$

The elements of the inverse matrix are:

$$\Sigma_{ij}^{-1} = \frac{1+(N_{ob}-2)\rho}{(1-\rho)(1+(N_{ob}-1)\rho)\sigma^2}, \quad i=j$$
$$\Sigma_{ij}^{-1} = -\frac{\rho}{(1-\rho)(1+(N_{ob}-1)\rho)\sigma^2}, \; i \neq j$$

Using the just found expressions in (19) and applying the natural logarithm, we get:

$$
\begin{aligned}
\ell(\underline{X}|\psi) =& -\frac{1}{2}\left[(N_{ob}-1)\ln(1-\rho)+\ln(\sigma^2)+\ln(1+(N_{ob}-1)\rho)\right] \\
& -\frac{1}{2\sigma^2}\frac{1+(N_{ob}-2)\rho}{(1-\rho)(1+(N_{ob}-1)\rho)}\sum_{i=1}^{N_{ob}}(X_i-\mu)^2-\frac{N_{ob}}{2}\ln(2\pi) \\
& +\frac{1}{2\sigma^2}\frac{\rho}{(1-\rho)(1+(N_{ob}-1)\rho)}\sum_{\substack{i,j=1\\ i\neq j}}^{N_{ob}}(X_i-\mu)(X_j-\mu)
\end{aligned}
$$

Since the Fisher information matrix is:

$$
I(\psi^*) = -\mathbb{E}\left[\left.\begin{bmatrix}
\frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu^2} & \frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu\partial\sigma^2} & \frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu\partial\rho} \\
\frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu\partial\sigma^2} & \frac{\partial^2\ell(\underline{X},\psi)}{\partial(\sigma^2)^2} & \frac{\partial^2\ell(\underline{X},\psi)}{\partial\sigma^2\partial\rho} \\
\frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu\partial\rho} & \frac{\partial^2\ell(\underline{X},\psi)}{\partial\sigma^2\partial\rho} & \frac{\partial^2\ell(\underline{X},\psi)}{\partial\rho^2}
\end{bmatrix}\right|_{\psi=\psi^*}\right],
$$

then, applying the expectation to each element of the matrix, the (3,3) element of the inverse is:

$$
I^{-1}_{(3,3)}(\psi^*) = -\frac{\mathbb{E}\left[\left.\frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu^2}\right|_{\psi=\psi^*}\right]\mathbb{E}\left[\left.\frac{\partial^2\ell(\underline{X},\psi)}{\partial(\sigma^2)^2}\right|_{\psi=\psi^*}\right]-\mathbb{E}\left[\left.\frac{\partial^2\ell(\underline{X},\psi)}{\partial\mu\partial\sigma^2}\right|_{\psi=\psi^*}\right]^2}{\det(I(\psi^*))}. \tag{20}
$$

Starting from the first order derivatives, we get:

$$
\begin{aligned}
\frac{\partial\ell(\underline{X},\psi)}{\partial\mu} =& \frac{1}{2\sigma^2}\frac{\rho}{(1-\rho)(1+(N_{ob}-1)\rho)}\left[2N_{ob}(N_{ob}-1)\mu-\sum_{\substack{i,j=1\\ i\neq j}}^{N_{ob}}(X_i+X_j)\right] \\
& +\frac{1}{\sigma^2}\frac{1+(N_{ob}-2)\rho}{(1-\rho)(1+(N_{ob}-1)\rho)}\left[\sum_{i=1}^{N_{ob}}X_i-N_{ob}\mu\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\ell(\underline{X},\psi)}{\partial\sigma^2} =& -\frac{N}{2\sigma^2}+\frac{1}{2\sigma^4}\frac{1+(N_{ob}-2)\rho}{(1-\rho)(1+(N_{ob}-1)\rho)}\sum_{i=1}^{N_{ob}}(X_i-\mu)^2 \\
& -\frac{1}{2\sigma^4}\frac{\rho}{(1-\rho)(1+(N_{ob}-1)\rho)}\sum_{\substack{i,j=1\\ i\neq j}}^{N_{ob}}(X_i-\mu)(X_j-\mu)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell(\underline{X}, \psi)}{\partial \rho} =& \frac{1}{2\sigma^2} \frac{1+(N_{ob}-1)\rho^2}{(1-\rho)^2(1+(N_{ob}-1)\rho)^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N_{ob}} (X_i-\mu)(X_j-\mu) \\
& - \frac{1}{2\sigma^2} \frac{(N_{ob}-1)\rho(2+(N_{ob}-2)\rho)}{(1-\rho)^2(1+(N_{ob}-1)\rho)^2} \sum_{i=1}^{N_{ob}} (X_i-\mu)^2 \\
& + \frac{N_{ob}-1}{2(1-\rho)} - \frac{N_{ob}-1}{2(1+(N_{ob}-1)\rho)}
\end{aligned}
$$

Proceeding with the computation of second order derivatives and expectations, we have:

$$
\mathbb{E}\left[\left. \frac{\partial^2 \ell(\underline{X}, \psi)}{\partial \mu^2} \right|_{\psi=\psi^*} \right] = \frac{N_{ob}[\rho(N_{ob}-1)-(1+(N_{ob}-2)\rho)]}{\sigma^2(1-\rho)(1+(N_{ob}-1)\rho)}
$$

$$
\mathbb{E}\left[\left. \frac{\partial^2 \ell(\underline{X}, \psi)}{\partial \mu \partial \sigma^2} \right|_{\psi=\psi^*} \right] = \mathbb{E}\left[\left. \frac{\partial^2 \ell(\underline{X}, \psi)}{\partial \mu \partial \rho} \right|_{\psi=\psi^*} \right] = 0
$$

$$
\mathbb{E}\left[\left. \frac{\partial^2 \ell(\underline{X}, \psi)}{\partial (\sigma^2)^2} \right|_{\psi=\psi^*} \right] = \frac{N_{ob}}{2\sigma^4} + \frac{N_{ob}[(N_{ob}-1)\rho^2-(1+(N_{ob}-2)\rho)]}{\sigma^4(1-\rho)(1+(N_{ob}-1)\rho)}
$$

$$
\mathbb{E}\left[\left. \frac{\partial^2 \ell(\underline{X}, \psi)}{\partial \sigma^2 \partial \rho} \right|_{\psi=\psi^*} \right] = \frac{N_{ob}(N_{ob}-1)\rho[1+(N_{ob}-2)\rho-(N_{ob}-1)\rho^2]}{2\sigma^2(1-\rho)^2(1+(N_{ob}-1)\rho)^2}
$$

$$
\begin{aligned}
\mathbb{E}\left[\left. \frac{\partial^2 \ell(\underline{X}, \psi)}{\partial \rho^2} \right|_{\psi=\psi^*} \right] =& \frac{N_{ob}(N_{ob}-1)\rho[2+3\rho(N_{ob}-1)+(N_{ob}-1)^2\rho^3]}{(1-\rho)^3(1+(N_{ob}-1)\rho)^3} \\
& - \frac{N_{ob}[N_{ob}^2(3-4\rho)\rho^2+N_{ob}^3\rho^3-(1-\rho)^2(1+2\rho)]}{(1-\rho)^3(1+(N_{ob}-1)\rho)^3} \\
& - \frac{N_{ob}^2(N_{ob}-1)\rho}{(1-\rho)^3(1+(N_{ob}-1)\rho)^3} + \frac{(N_{ob}-1)^2}{(1+(N_{ob}-1)\rho)^2} \\
& + \frac{N_{ob}-1}{2(1-\rho)^2} - \frac{N_{ob}^2(1-6\rho^2+5\rho^3)}{(1-\rho)^3(1+(N_{ob}-1)\rho)^3}
\end{aligned}
$$

Substituting all these expressions in (20), we get the closed formula for the Cramer-Rao lower bound for ρ:

$$
\sigma^2_{CR,\rho}(\rho) = \frac{2(1-\rho)^2(1+(N_{ob}-1)\rho)^2}{N_{ob}(N_{ob}-1)T}. \tag{21}
$$

References

Altman, E.I., Brady, B., Resti, A., and Sironi, A., 2005. The link between default and recovery rates: Theory, empirical evidence, and implications, *The Journal of Business*, 78 (6), 2203–2228.

BIS, 2005. An explanatory note on the Basel II IRB risk weight functions, *Bank for International Settlements*.

Gelman, A., Stern, H.S., Carlin, J.B., Dunson, D.B., Vehtari, A., and Rubin, D.B., 2013. *Bayesian data analysis*, Chapman and Hall/CRC.

Gordy, M.B. and Lütkebohmert, E., 2007. Granularity adjustment for basel ii, Tech. rep., Bundesbank Series 2 Discussion Paper No. 2007,01.

Löffler, G., 2003. The effects of estimation error on measures of portfolio credit risk, *Journal of Banking & Finance*, 27 (8), 1427–1453.

Sibbertsen, P., Stahl, G., and Luedtke, C., 2008. Measuring model risk, Tech. rep., Discussion papers of the School of Economics and Management of the Hanover Leibniz.

Tarashev, N., 2010. Measuring portfolio credit risk correctly: Why parameter uncertainty matters, *Journal of Banking & Finance*, 34 (9), 2065–2076.

Tarashev, N. and Zhu, H., 2008. Specification and calibration errors in measures of portfolio credit risk: The case of the asrf model, *International Journal of Central Banking*, 4 (2), 129–173.

Vasicek, O., 1987. Probability of loss on loan portfolio, *KMV corporation*.