Parameter Uncertainty for Credit Risk in the Vasicek Model

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The Vasicek Model

The Vasicek model for credit risk is inferred from the hypothesis that a firm would defaults if its firm value $V_i(T) < d$, where

$$V_i = \sqrt{\rho} Z + \sqrt{1-\rho}\epsilon_i$$

where

- $Z \sim \mathcal{N}(0,1)$ is the common risk factor
- $\epsilon_i \sim \mathcal{N}(0,1)$ i.i.d are idiosyncratic risk factors
- ρ is the correlation between firms values

Nominal model

HP case

$$Loss = (1 - RR) \cdot 1/N_{ob} \cdot m$$

$$\alpha VaR = \alpha\text{-}quantile(Loss)$$

LHP case

$$\mathbb{E}[Loss] = DR \cdot (1 - RR)$$

$$\alpha VaR = (1 - RR)\,\mathcal{N}\left(\frac{\mathcal{N}^{-1}(DR) - \sqrt{\rho}\,\mathcal{N}^{-1}(1-\alpha)}{\sqrt{1-\rho}}\right)$$

Regulatory Capital:

$$\alpha RC := \alpha VaR - \mathbb{E}[Loss]$$

Remark (Deterministic parameters)

DR, RR, ρ are parameters obtained from the average of the corresponding observed quantities

Altenative model

Inferential analysis

$\left\{ \begin{array}{l} DR \\ RR \text{ as r.v.} \\ \rho \end{array} \right. \longrightarrow$ Find a distribution:
- Classical inference
- Bayesian inference

$\longrightarrow$ Sampling new data

RC computation

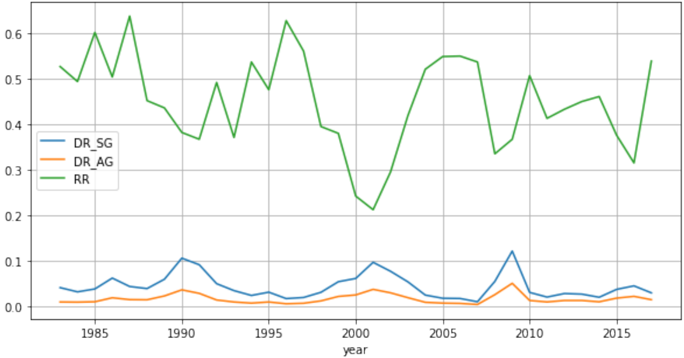
Monte Carlo simulation

$\downarrow$

LHP Case: $$Loss_j = (1 - RR_j) \cdot \mathcal{N}\left(\frac{\mathcal{N}^{-1}(DR_j) - \sqrt{\rho} Z_j}{\sqrt{1-\rho}}\right) \quad for\ j = 1 \ldots N_{sim}$$

HP Case: $$Loss_j = (1 - RR_j) \cdot \frac{1}{N_{ob}} \sum_{i=1}^{N_{ob}} 1_{(V_i^j \leq d_j)} \quad for\ j = 1 \ldots N_{sim}$$

Data-set



Correlation ρ is inferred from publicly traded firms in the US from asset returns in the month of July 2006.

Classical statistical inference

Frequentistic inference parameters distribution assumption:

- $d = \mathcal{N}^{-1}(DR) \sim \mathcal{N}(\hat{d}, \sigma_d)$
- $RR \sim \mathcal{N}(\overline{RR}, \sigma_{RR})$
- $\rho \sim Beta(\alpha_\rho, \beta_\rho)$ vs $\rho_{Basel}(DR, k)|_{k=50}$

Gaussianity of d and RR verified with Shapiro-Wilk test

Remark (Default barrier)

We have two ways of choosing the mean of the distribution of d:

- *sample mean of $\{\mathcal{N}^{-1}(DR_i)\}_i$ ($\hat{d} = -1.7733$)*
- *$\hat{d}$ such that $\mathbb{E}[\mathcal{N}(d)] = \frac{1}{N}\sum_{i=1}^{N} DR_i$ ($d = -1.7731$)*

Bayesian Inference

Bayesian Estimation

Data: $\hat{p}$
Prior: $g(p)$
Likelihood: $f(\hat{p}|p)$

$$\left.\begin{array}{l}\text{Data: } \hat{p}\\ \text{Prior: } g(p)\\ \text{Likelihood: } f(\hat{p}|p)\end{array}\right\} \longrightarrow h(p|\hat{p}) \propto g(p)f(\hat{p}|p)$$

Bayesian Prediction

$$p(x|\hat{p}) = \int_{\mathbb{R}} f(x|p)h(p|\hat{p})dp$$

Model: $p(x|\hat{p})$

Correlation

- Prior distribution: $g(\rho) = \mathbb{1}_{[0,1]}(\rho)$
- Likelihood $f(\hat{\rho}|\rho)$ is a beta distribution with $\alpha(\rho)$ and $\beta(\rho)$ s.t.

$$\rho = \frac{\alpha}{\alpha + \beta} \qquad \sigma^2 = \frac{\alpha\beta}{(1 + \alpha + \beta)(\alpha + \beta)^2}$$

Hence the posterior can be written as

$$h(\rho|\hat{\rho}) \propto \mathbb{1}_{[0,1]}(\rho) Beta(\hat{\rho}|\alpha(\rho), \beta(\rho))$$

We recover the class conditional density as

$$p(x|\hat{\rho}) = \int_{\mathbb{R}} Beta(x|\rho, \sigma_\rho) h(\rho|\hat{\rho}) d\rho$$

Correlation

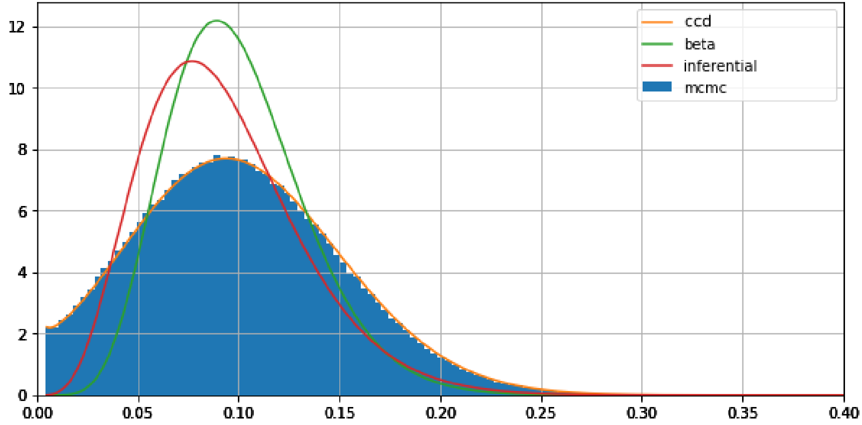


Figure: Distribution of the parameter ρ according to different approaches.

Default Barrier

- Prior distribution: $d \sim \mathcal{N}(0,1)$ as $DR \sim \mathcal{U}[0,1]$
- Likelihood: $f(\{d_i\}_i | d, \sigma_d) = \prod_{i=1}^{N} \phi(d_i | d, \sigma_d)$ where σ_d is the sample standard deviation of $\{d_i\}_i = \{\mathcal{N}^{-1}(PD_i)\}_i$.

Since the prior and the likelihood are conjugated, we know that also the posterior $h(d | \{d_i\}_i) \sim \mathcal{N}\left(\frac{\sum d_i}{N+\sigma_d^2}, \frac{\sigma_d^2}{N+\sigma_d^2}\right)$

Remark

In [Baviera et al.] was used just $\hat{d}$ and the variance of the mean of σ_d^2/N for a normal i.i.d. sample. Therefore the posterior $h \sim \mathcal{N}\left(\frac{\hat{d}}{1+\sigma_d^2/N}, \frac{\sigma_d^2}{N+\sigma_d^2}\right)$

Cramer Rao lower bound (HP case)

We might also want to use the Cramer-Rao lower bound for variance of unbiased estimators for the parameter, this would be a function of the unknown parameter p. It is defined as:

$$\sigma^2_{CR}(p) = -\frac{1}{T} \left(\mathbb{E}_y \left[\frac{\partial^2 LL(y|p)}{\partial p^2} \right] \right)^{-1}$$

where $LL(\cdot|p)$ is the log-likelihood function for the observed data and T the length of time series.

Cramer Rao lower bound (HP case)

- For ρ there is a closed formula:

$$\sigma^2_{CR,\rho}(\rho) = \frac{2(1-\rho)^2(1+(N_{ob}-1)\rho)^2}{N_{ob}(N_{ob}-1)T}$$

- For d, it is computed numerically using as likelihood:

$$P(X=m) = \int_{-\infty}^{+\infty} \binom{N_{ob}}{m} p(y)^m (1-p(y))^{N_{ob}-m} \phi(y)\, dy$$

where m is the number of defaults and $p(y) = \mathcal{N}\left(\frac{d-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)$.

Cramer Rao lower bound (HP case)

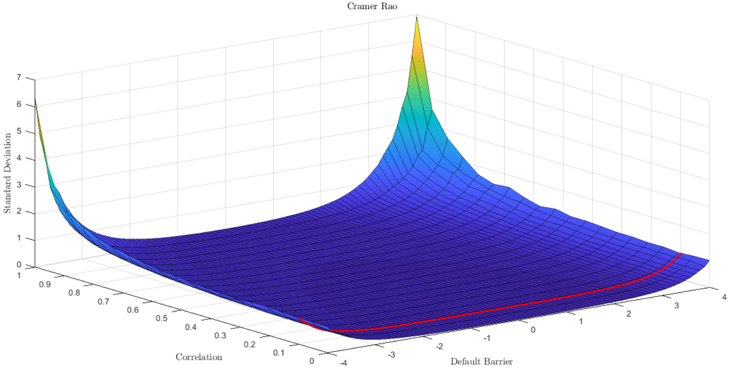


Figure: Surface for the standard deviation (Cramer Rao)

Frequentist inference

Add-on	99%	99,90%
d	47,33%	54,28%
$d^{[closed]}$	47,25%	53,20%
RR	8,41%	12,04%
ρ	4,16%	16,60%
d, ρ	49,11%	62,54%
All	57,30%	75,08%

Table: Add-on for LHP with frequentist inference (Speculative Grade)

Add-on	99%	99,90%
d	29,33%	36,97%
RR	7,93%	10,58%
ρ	0,01%	9,23%
d, ρ	29,34%	46,22%
All	39,30%	58,81%

Table: Add-on for HP with frequentist inference (Speculative Grade)

Bayesian Inference

Add-on	**99%**	**99,90%**
d	49,52%	56,00%
ρ	14,65%	31,22%
All	59,34%	76,99%

Table: Add-on for LHP with Bayesian inference (Speculative Grade)

Add-on	**99%**	**99,90%**
d	28,97%	36,76%
ρ	14,66%	18,47%
All	43,66%	55,25%

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d	28,66%	36,52%
ρ	14,63%	18,47%
All	43,29%	54,96%

Table: Add-on for HP with Bayesian inference fixing variance equal to empiric one (table on the left) and Cramer Rao lower bound (table on the right)

In the 1980, Global Sensitivity Analysis (GSA) has been developed, which aims to identify what inputs are the most influential in the model.

$$f(X_1, \cdots, X_n) = Y$$

Importance of fixing X_i is high when $Var[Y|X_i = x_i^*]$ is low. But the "true" value x_i^* is unknown $\implies$ use $\mathbb{E}[Var[Y|X_i]]$.

$$Var[Y] = Var[\mathbb{E}[Y|X_i]] + \mathbb{E}[Var[Y|X_i]]$$

$$\implies S_1^{(i)} = \frac{Var[\mathbb{E}[Y|X_i]]}{V[Y]} \text{(Sobol index)}$$

Sensitivity analysis

	DR	RR	ρ
S_1	36.84	7.99	23.16
NS_1	53.76	12.25	33.99

Table: Sobol indices for LHP (%)

	DR	RR	ρ
S_1	51.24	12.08	30.26
NS_1	54.75	12.91	32.33

Table: Sobol indices for LHP (%). Independent parameters

Conclusions

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Conclusions

- Default Rate is the most sensitive parameter in the model and provides the most variance to the output
- Frequentist and Bayesian analysis have produced very similar results, but with very different degrees of complexity.
- It would be unfair to say that banks are for sure under-capitalized, since our add-on was computed with respect to the mean of a many years worth of data.
- Most important parameter in the model is the Default Rate! $\Longrightarrow$ more capital/larger team/most defined model is needed to estimate it.