1st Year Mathematics
Imperial College London

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Analysis I

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Syllabus

A rigorous treatment of the concept of a limit, as applied to sequences, series and functions.

Sequences

Real and complex sequences. Convergence, divergence and divergence to infinity. Sums and products of convergent sequences. The Sandwich Test. Sub-sequences, monotonic sequences, Bolzano-Weierstrass Theorem. Cauchy sequences and the general principle of convergence.

Series

Real and complex series. Convergent and absolutely convergent series. The Comparison Test for non-negative series and for absolutely convergent series. The Alternating Series Test. Rearranging absolutely convergent series. Radius of convergence of power series. The exponential series.

Continuity

Limits and continuity of real and complex functions. Left and right limits and continuity. Maxima and minima of real valued continuous functions on a closed interval. Inverse Function Theorem for strictly monotonic real functions on an interval.

Differentiation

An introduction to differentiability: definitions, examples, left and right derivative.

Appropriate books

K. G. Binmore, *Mathematical Analysis, A Straightforward Approach* (Cambridge University Press).

Contents

0 Preliminaries

Lecture 1 M1F stuff:

- $\forall$ - for any, **fix any**, for all, every...
- $\exists$ - there exists
- $\mathbb{N} - \{1, 2, 3, \dots\}$

Theorem 0.1: Triangle Inequality

(See Question Sheet 1)

$$|a + b| \leq |a| + |b|$$

Corollary 0.2.

$$\big||a| - |b|\big| \leq |a - b|$$

Proof.

$$\begin{aligned} |a - b| < \epsilon &\iff b - \epsilon < a < b + \epsilon \\ &\iff a \in (b - \epsilon, b + \epsilon) \\ &\iff b \in (a - \epsilon, a + \epsilon) \\ &\implies \big||a| - |b|\big| < \epsilon \end{aligned}$$

∎

Lots of other versions, see Question Sheet 1 - *don't try to memorise them!*

Clicker Question 0.3. Fix $a \in \mathbb{R}$. What does the statement

$$\forall \epsilon > 0, \ |x - a| < \epsilon \qquad (*)$$

mean for the number x?

Answer: $x = a$.

Proof. Assume $x \neq a$. Take $\epsilon := \frac{1}{2}|x - a| > 0$. Then $(*)$ does not hold. ∎

1 Sequences

A sequence $(a_n)_{n\geq 1}$ of real (or complex, etc.) numbers is an infinite list of numbers $a_1,\ a_2,\ a_3,\ldots$ all in $\mathbb{R}$ (or $\mathbb{C}$, etc.) Formally: Lecture 2

> **Definition.** A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$

Notation: We let $a_n \in \mathbb{R}$ denote $a(n)$ for $n \in \mathbb{N}$. The data $(a_n)_{n=1,2,\ldots}$ is equivalent to the function $a : \mathbb{N} \to \mathbb{R}$ because a function a is determined by its values a_n over all $n \in \mathbb{N}$.

We will denote a by $a_1,\ a_2,\ldots$ or $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n\geq 1}$ or even just (a_n).

Remark 1.1. a_i's could be repeated, so (a_n) is *not* equivalent to the set $\{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$. E.g. $(a_n) = 1,\ 0,\ 1,\ 0,\ldots$ is different from $(b_n) = 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ldots$

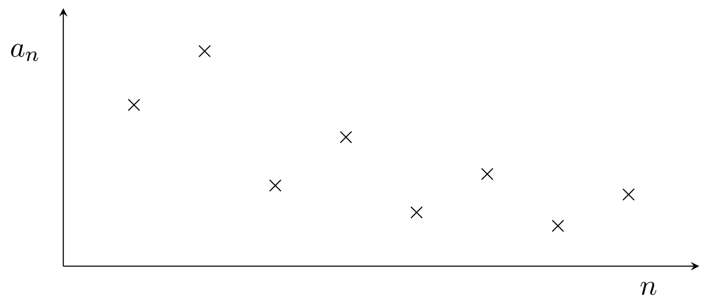
We can describe a sequence in may ways, e.g. formula for a_n as above $a_n = \frac{1-(-1)^n}{2}$, or a recursion e.g. $c_1 = 1 = c_2$, $c_n = c_{n-1} + c_{n-2}$ for $n \geq 3$, or a summation (see next section) e.g. $d_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Convergence of Sequences

We want to *rigorously* define $a_n \to a \in \mathbb{R}$, or "a_n converges to a as $n \to \infty$" or "a is the limit of (a_n)".

Idea: a_n should get closer and closer to a. Not necessarily monotonically, e.g.:

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{2n} & n \text{ even} \end{cases} \qquad \text{we want } a_n \to 0$$

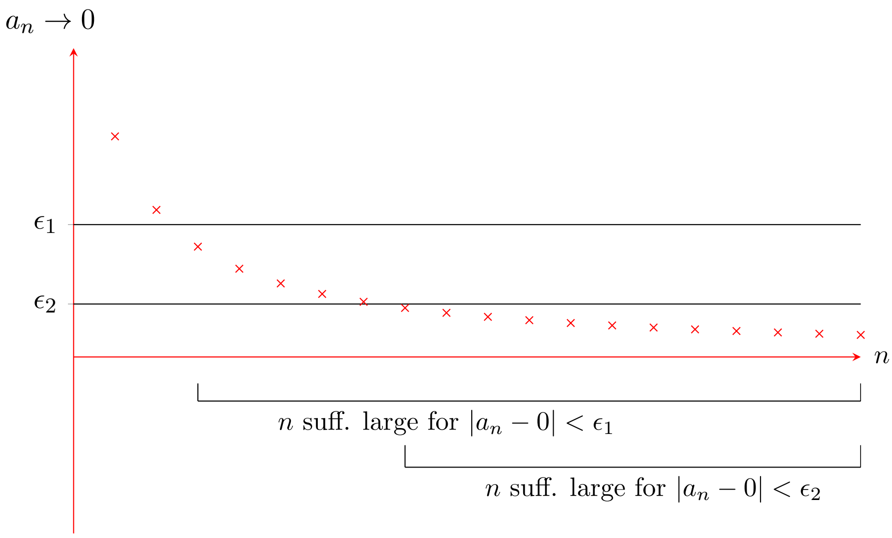


Also notice that $\frac{1}{n}$ gets closer and closer to -1! So we want to say instead that a_n gets *as close as we like to* a. We will measure this with $\epsilon > 0$. We phrase "a_n gets *arbitrarily* close to a" by "a_n gets to within ϵ of a for *any* $\epsilon > 0$".

> **Definition** (Mestel)**.** $u_n \to u$ if $\forall n$ sufficiently large, $|u_n - u|$ is *arbitrarily small.*
>
> Define a real number $b \in \mathbb{R}$ to be arbitrarily small if it is smaller than any $\epsilon > 0$ i.e. $\forall \epsilon > 0,\ |b| < \epsilon$.

Definition Mestel says that once n is large enough, $|u_n - u|$ is less than every $\epsilon > 0$, i..e it's zero, i.e. $u_n = u$. We want to *reverse* the order of specifying n and ϵ.

i.e. we want to say that to get *arbitrarily close to the limit a* (i.e. $|a_n - a| < \epsilon$), we need to go sufficiently far down the sequence. Then if I change $\epsilon > 0$ to be smaller, I simply go further down the sequence to get within ϵ of a.



There will not be a "n sufficiently large" that works for all ϵ at once! (unless $a_n = a$ eventually.)

But for *any* (fixed) $\epsilon > 0$ we want there to be an n sufficiently large such that $|a_n - a| < \epsilon$. So we change "$\exists n$ such that $\forall \epsilon$" to "$\forall \epsilon,\ \exists n$.". *This allows n to depend on ϵ.*

> **Definition** (Nestel)**.** $a_n \to a$ if $\forall \epsilon > 0,\ \exists n \in \mathbb{N}$ such that $|a_n - a| < \epsilon$.

e.g.

$$a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \quad \text{satisfies } a_n \to 0 \text{ according to Prof. Nestel.}$$

We want to modify this to say eventually $|a_n - a| < \epsilon$ *and it stays there!*

Lecture 3 **Ignore Mestel and Nestel's definition!**

> **Definition** (Convergence)**.** We say that $a_n \to a$ iff
>
> $$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that "} n \geq N \implies |a_n - a| < \epsilon\text{"}$$

This says that *however close* ($\forall \epsilon > 0$) I want to get to the limit a, there's a point in the sequence ($\exists N \in \mathbb{N}$) beyond which ($n \geq N$) my a_n is indeed that close to the limit a ($|a_n - a| < \epsilon$).

Remark 1.2. N depends on ϵ! $N = N(\epsilon)$

Equivalently:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that"} \forall n \geq N,\ |a_n - a| < \epsilon\text{"}$$

or equivalently

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon,\ \forall n \geq N_\epsilon$$

Clicker Question 1.3. Given a sequence of real numbers $(a_n)_{n\geq 1}$. Conisder

$$\boxed{\forall n \geq 1,\ \exists \epsilon > 0 \text{ such that } |a_n| < \epsilon}$$

This means?

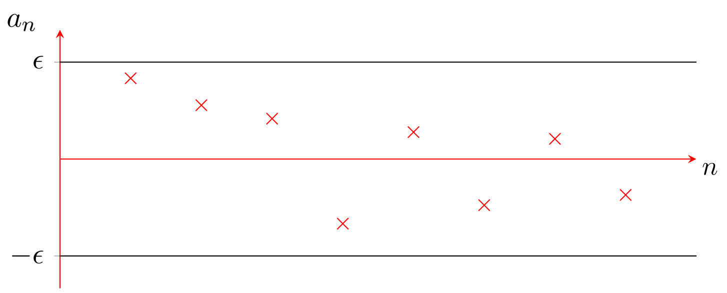
Answer: It always holds.

Proof. Fix any $n \in \mathbb{N}$. Take $\epsilon = |a_n| + 1$. ∎

What about

$$\boxed{\exists \epsilon > 0 \text{ such that } \forall n \geq 1,\ |a_n| < \epsilon}$$

Answer: (a_n) is bounded.

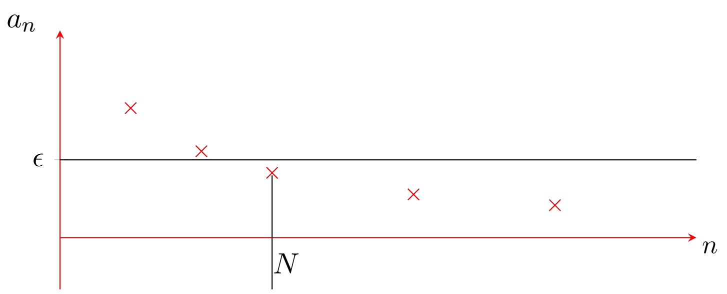


Proof. $\iff a_n \in (-\epsilon, \epsilon)\ \forall n \iff |a_n|$ is bounded by ϵ. ∎

Definition. If a_n does not converge to a for any $a \in \mathbb{R}$, we say that a_n *diverges.*

Example 1.4. I claim that $\frac{1}{n} \to 0$ as $n \to \infty$

Rough working: Fix $\epsilon > 0$. I want to find $N \in \mathbb{N}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N$.



Since $a_n = \frac{1}{n}$ is monotonic, it is *sufficient* to ensure that $\frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$ (This *implies* $\frac{1}{n} \leq \frac{1}{N} < \epsilon,\ \forall n \geq N$).

Proof. Fix $\epsilon > 0$. Pick any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. (This is the Archimedean axiom of $\mathbb{R}$. Notice N depends on ϵ!!). Then $n \geq N \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. ∎

Method to prove $a_n \to a$

(I) Fix $\epsilon > 0$

(II) Calculate $|a_n - a|$

(II′) Find a good estimate $|a_n - a| < b_n$

(III) Try to solve $a_n - a < b_n < \epsilon \quad (*)$

(IV) Find $N \in \mathbb{N}$ s.t. $(*)$ holds whenever $n \geq N$

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see examples below)

Example 1.5. $a_n = \dfrac{n+5}{n+1}$

Rough Working

$$|a_n - 1| = \left|\frac{n+5}{n+1} - 1\right| = \frac{4}{n+1}$$

This is $< \epsilon \iff n+1 > 4/\epsilon \iff n > 4/\epsilon$, so take $N \geq 4/\epsilon$.

Proof. Fix $\epsilon > 0$. Pick N such that $N \geq 4/\epsilon$. Then $\forall n \geq N$,

$$|a_n - 1| = \frac{4}{n+1} \leq \frac{4}{N+1} < \frac{4}{N} \leq \epsilon$$ ∎

Example 1.6. $a_n = \dfrac{n+2}{n-2} \to 1$

Rough Working

$$|a_n - 1| = \left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2}$$

We want $\frac{4}{n-2} < \epsilon$. We want implications in the $\impliedby$ direction (i.e. $\frac{4}{n-2} < \epsilon \impliedby n \geq N$) *not* $\implies$ direction. i.e. $\frac{4}{n-2} < \epsilon \implies \frac{4}{n} < \epsilon$.

But if we take $N = \frac{4}{\epsilon}$, we need the *opposite* implication, we *need* $\frac{4}{n-2} < \epsilon$. We *need* to estimate $\frac{4}{n-2} < b_n$, and then solve $b_n < \epsilon$. So we make denominator smaller.

To make $n-2$ smaller, make 2 bigger! e.g. $\frac{n}{2} > 2$ for $n > 4$. Then $\frac{4}{n-2} < \frac{4}{n-n/2} = \frac{8}{n}$

Also want $b_n = \frac{8}{n} < \epsilon \iff n > 8/\epsilon$. So take $N > \max(8/\epsilon, 4)$.

Proof. Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \max(8/\epsilon, 4)$. Then $n \geq N \implies n > 8/\epsilon$ (1) and $n > 4$ (2) $\implies$

$$\left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2} \underbrace{<}_{(2)} \frac{4}{n-n/2} = \frac{8}{n} \underbrace{<}_{(1)} \epsilon$$ ∎

Lecture 4

We can also define limits for *complex sequences.*

Definition. $a_n \in \mathbb{C}$, $\forall n \geq 1$. We say $a_n \to a \in \mathbb{C}$ iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon$$

(i.e. $\sqrt{\Re(a_n - a)^2 + \Im(a_n - a)^2} < \epsilon$)

This is equivalent (see problem sheet!) to $(\Re a_n) \to \mathfrak{a}$ *and* $(\Im a_n) \to \Im a$

Example 1.7. Prove $a_n = \dfrac{e^{in}}{n^3 - n^2 - 6} \to 0$ as $n \to \infty$

Rough Working

$$|a_n - a| = \left|\frac{e^{in}}{n^3 - n^2 - 6}\right| = \left|\frac{1}{n^3 - n^2 - 6}\right|$$

Estimate $\left|\dfrac{1}{n^3 - n^2 - 6}\right| < \dfrac{1}{c_n}$ by making c_n smaller than $n^3 - n^2 - 6$ (But not too small! We want $c_n \to \infty$). So let $c_n = n^3 -$ *something bigger than* $n^2 + 6$.

Take off $\frac{n^3}{2}$ to make the expression simple. For $n \geq 4$, we have $\frac{n^3}{2} > n^2 + 6$.

So for $n \geq 4$

$$\left|\frac{1}{n^3 - n^2 - 6}\right| < \frac{1}{n^3 - n^3/2} = \frac{2}{n^3}$$

and this is $< \epsilon$ for $n > \sqrt[3]{\frac{2}{\epsilon}}$.

Proof. $\forall \epsilon > 0$, choose $N \geq \max(4, \sqrt[3]{2/\epsilon})$. Then $\forall n \geq N$

$$|a_n - 0| = \left|\frac{1}{n^3 - n^2 - 6}\right| < \frac{1}{n^3 - n^3/2} = \frac{2}{n^3} \leq \frac{2}{N^3} \leq \epsilon$$ ∎

Example 1.8. Set $\delta = 10^{-1000000}$, $a_n = (-1)^n \cdot \delta$. Prove that a_n does not converge.

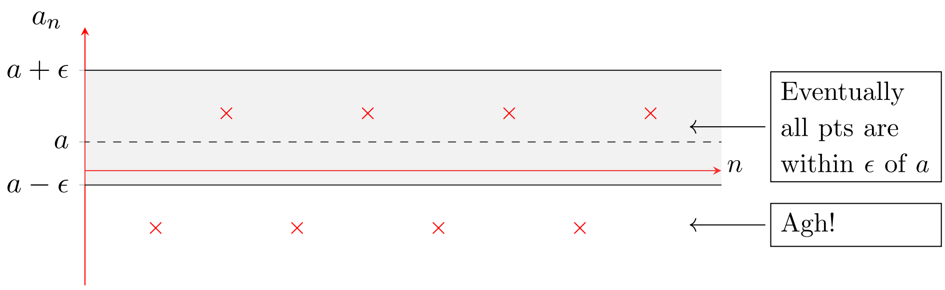
We want to show that the following is false:

$$\exists a \text{ s.t. } \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n - a| < \epsilon$$

i.e. we need to prove

$$\forall a,\ \exists \epsilon > 0 \text{ s.t. } \forall N \in \mathbb{N},\ \exists n \geq N \text{ s.t. } |a_n - a| \geq \epsilon$$

Rough: Assume for contadiction that $a_n \to a$, i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \implies$ $|a_n - a| < \epsilon$



For small enough $\epsilon > 0$, the fact that a is within ϵ of δ (a_{2n}) and $-\delta$ (a_{2n+1}) will be a contradiction.

Proof. Fix $a \in \mathbb{R}$. Take $\epsilon = \delta$ (or $\epsilon < \delta$ will do).

Then if $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \epsilon$ this implies

(i) $|a_{2N} - a| < \epsilon \iff a \in (\delta - \epsilon, \delta + \epsilon) \implies a > \delta - \epsilon = 0$

(ii) $|a_{2N+1} - a| < \epsilon \iff a \in (-\delta - \epsilon, -\delta + \epsilon) \implies a < -\delta + \epsilon = 0$, ※

(or use triangle inequality:

$$|\delta - (-\delta)| \leq |\delta - a| + |a - (-\delta)| < \epsilon + \epsilon \implies 2\delta < 2\epsilon = 2\delta \text{ ※})$$

So $a_n \not\to a$, but this holds $\forall a \in \mathbb{R}$, so a_n does not converge. ∎

Clicker Question 1.9. Fix $(a_n)_{n\geq 1}$, $a_n \in \mathbb{R}$. Then

$$\forall n,\ \exists \epsilon > 0 \text{ s.t. } |a_n| < \epsilon \text{ means?}$$

Answer: Nothing. This is always true. Take $\epsilon = |a_n| + 1$

Lecture 5

Theorem 1.10: Uniqueness of Limits

Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$

Idea: For n large, a_n should be close to a and to b. So a arbitrarily close to $b \implies a = b$.

Proof 1.

(i) $\forall \epsilon$, $\exists N_a$ s.t. $\forall n \geq N_a$, $|a_n - a| < \epsilon$

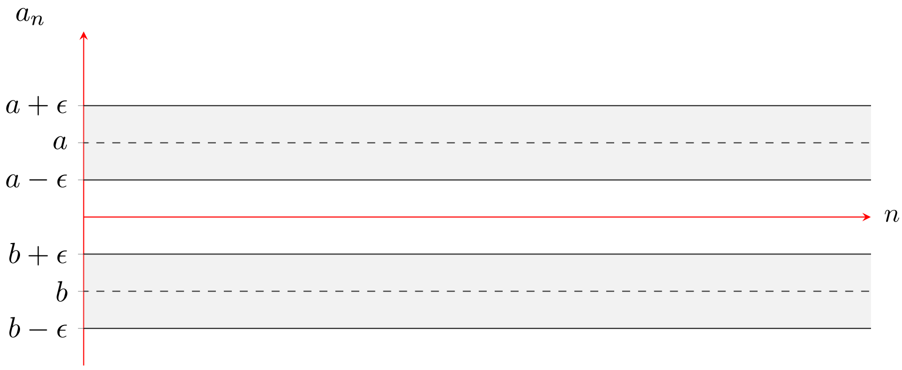
(ii) $\forall \epsilon$, $\exists N_b$ s.t. $\forall n \geq N_b$, $|a_n - b| < \epsilon$

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$, (i) and (ii) hold, so

$$|a - b| = |(a - a_n) + (a_n - b)| \leq |a - a_n| + |a_n - b| < 2\epsilon \implies |a - b| = 0! \quad ∎$$

(recall! if not, set $\epsilon = \frac{1}{2}|a - b| > 0$ to get a contradiction)

Proof 2. By contradiction. Assume $a \neq b$.



Eventually a_n is in *both* corridors. So if I choose ϵ sufficiently small so that corridors don't overlap to get a contradiction.

Set $\epsilon = \frac{|a-b|}{2} > 0$. Then $\exists N_a, N_b$ such that $\forall n \geq N_a, N_b$, we have

$$|a_n - a| < \epsilon \text{ and } |a_n - b| < \epsilon$$

w.l.o.g. $a > b$. Then $a_n > a - \epsilon$ and $a_n < b$

$$\begin{aligned} &\implies b + \epsilon > a - \epsilon \\ &\implies 2\epsilon > a - b = 2\epsilon \text{ ⨳} \end{aligned}$$

■

Clicker Question 1.11. Prove $\frac{1}{n-2} \to 0$. Student Answer: Fix $\epsilon > 0$.

(i) We want $|\frac{1}{n-2} - 0| = \frac{1}{n-2} < \epsilon$

(ii) $\implies n - 2 > 1/\epsilon$

(iii) $\implies n > 2 + 1/\epsilon$

(iv) $\implies n > 1/\epsilon$ $(*)$

(v) So take $N > 1/\epsilon$, then

(vi) $\forall n \geq N$, $n > 1/\epsilon$ which is $(*)$

(vii) So $\frac{1}{n-2} \to 0$

(viii) (This is correct)

Answer: (iv) is wrong.

Theorem 1.12: Algebra of Limits

$a_n \to a$ and $b_n \to b$ then:

(i) $a_n + b_n \to a + b$

(ii) $a_n b_n \to ab$

(iii) $\frac{a_n}{b_n} \to \frac{a}{b}$ $(b \neq 0)$

Proof of (i). Fix any $\epsilon > 0$. Then $\exists N_a \in \mathbb{N}$ such that $\forall n \geq N_a$, $|a_n - a| < \epsilon/2$ and $\exists N_b \in \mathbb{N}$ such that $\forall n \geq N_b$, $|b_n - b| < \epsilon/2$. Set $N = \max\{N_a, N_b\}$, so

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad \blacksquare \end{aligned}$$

Proof of (ii). Rough working:

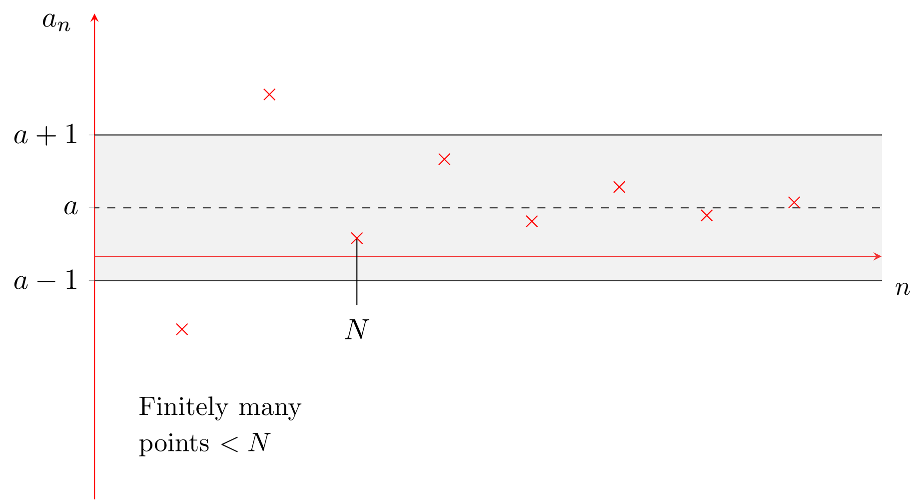
$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b - a_n b + a_n b_n| \\ &\leq |a_n - a||b| + |a_n||b_n - b| \end{aligned}$$

We can easily make $|a_n - a| < \epsilon/2$ if I take $|a_n - a| < \frac{\epsilon}{2|b|}$.

We need to show that $|a_n| < A$, so that I can take $|b_n - b| < \frac{\epsilon}{2A}$.

Lemma 1.13. *If $a_n \to a$, then (a_n) is bounded: $\exists A \in \mathbb{R}$ s.t. $|a_n| < A$, $\forall n$.*

Proof of Lemma.



Fix $\epsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < 1 \implies |a_n| < 1 + |a|$.

Then (a_n) is bounded by $\max\{a_1, a_2, \ldots, a_{N-1}, a + 1\}$. $\blacksquare$

Fix $\epsilon > 0$. Then $\exists N_a$ such that $\forall n \geq N_a$, $|a_n - a| < \dfrac{\epsilon}{2(|b| + 1)}$ (we add 1 in case $|b| = 0$) and $\exists N_b$ such that $\forall n \geq N_b$, $|b_n - b| < \dfrac{\epsilon}{2A}$.

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$

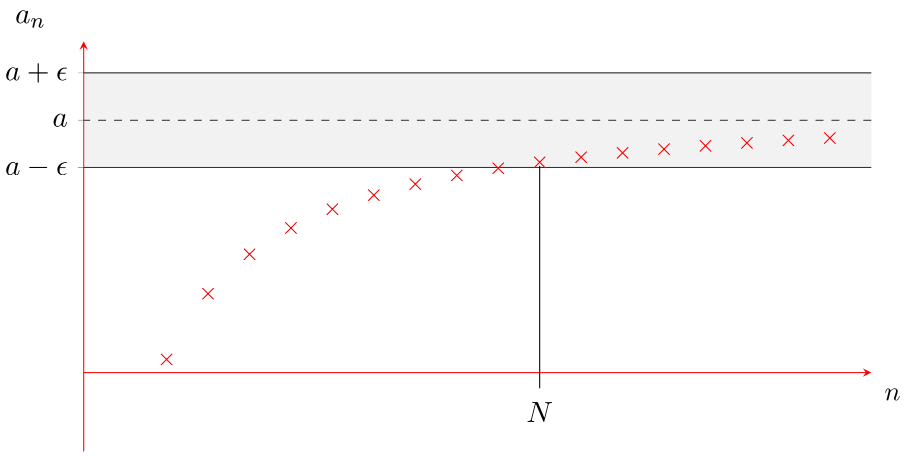
$$\begin{aligned} |a_n b_n - ab| &\leq |a_n - a||b_n| + |b_n - b||a| \\ &< \frac{\epsilon}{2} \frac{|b|}{|b| + 1} + A\frac{\epsilon}{2A} \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad \blacksquare \end{aligned}$$

See exercise sheet for proof of 1.12(iii).

Theorem 1.14

If (a_n) is bounded above *and* monotonically increasing then a_n is *convergent*.

Idea:



Eventually we get in the epsilon corridor (shaded area) because $a-\epsilon$ is *not* an upper bound. We stay in there because monotonic and bounded by a.

Proof. Fix $\epsilon > 0$. $a-\epsilon$ is *not* an upper bound for $\{a_n : n \in \mathbb{N}\}$ (because a is the *smallest* upper bound). So $\exists N \in \mathbb{N}$ such that $a_N > a - \epsilon$. Monotonic so $\forall n \geq N$ we have

$$a \geq a_n \geq a_N > a - \epsilon \implies |a_n - a| < \epsilon$$ ∎

Remark 1.15. Now it's easier to handle things like $a_n = \dfrac{n^2+5}{n^3-n+6}$. Lecture 6

Dividing by n^3, we get $a_n = \dfrac{1/n + 5/n^3}{1 - 1/n^2 + 6/n^3}$.

Use the fact that $1/n \to 0$ as $n \to \infty$ (Recall proof: $\forall \epsilon > 0$, let $N_\epsilon > 1/\epsilon$, then $n \geq N_\epsilon \implies n > 1/\epsilon \implies 1/n < \epsilon$), and the algebra of limits to deduce that

$$a_n \to \frac{0 + 5.0^3}{1 - 0^2 + 6.0^3} = 0.$$

Cauchy Sequences

Gives a way of proving convergence *without* knowing the limit.

> **Definition.** A sequence is Cauchy iff
>
> $$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N,\ |a_n - a_m| < \epsilon$$

Remark 1.16. $m, n \geq N$ are arbitrary. It is not enough to say that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a_{n+1}| < \epsilon$. See ex sheet.

Proposition 1.17. *If $a_n \to a$ then (a_n) is Cauchy.*

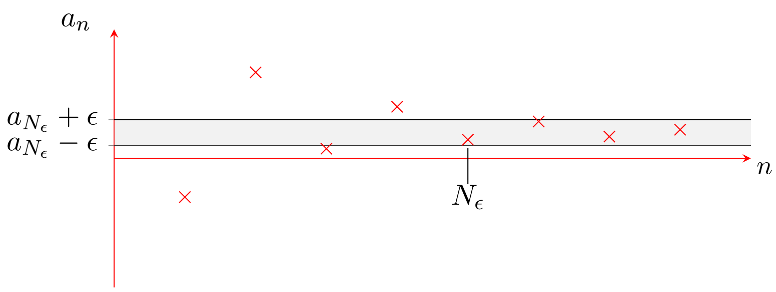
Proof. $a_n \to a \implies \forall \epsilon > 0, \exists N$ s.t. $n \geq N \implies |a_n - a| < \epsilon/2$ (1)

So $m \geq N \implies |a_m - a| < \epsilon/2$ (2). So

$$m, n \geq N \implies |a_n - a_m| \leq |a_n - a| + |a_m - a| < \underbrace{\epsilon/2}_{(1)} + \underbrace{\epsilon/2}_{(2)} = \epsilon$$ ∎

We want to prove converse: Cauchy $\implies$ Convergence.

We need a candidate for the limit a



We will produce an auxiliary sequence which is *monotonic* (+ bounded) $\implies$ convergence. $b_n := \sup\{a_i : i \geq n\}$. Then picture shows that $b_{N_\epsilon} \in (a_{N_\epsilon} - \epsilon, a_{N_\epsilon} + \epsilon]$ and b_n's are monotonically *decreasing* because $b_{n+1} = \sup\{a_i : i \geq n+1\}$, a subset of $\{a_i : i \geq n\}$.

So b_ns converge to $\inf\{b_n : n \in \mathbb{N}\}$. We will show that a_n's converge to same number, a, using Cauchy condition.

Lemma 1.18. *(a_n) is Cauchy $\implies$ (a_n) is bounded*

Proof. Pick $\epsilon = 1$, then $\exists N$ such that $\forall n, m \geq N$, $|a_n - a_m| < 1$. In particular $|a_n| < 1 + |a_N|$ $\forall n \geq N$ (take $m = N$), so

$$|a_n| \leq \max\{|a_1|, |a_2|, \ldots |a_{N-1}|, 1 + |a_N|\} \ \forall N \in \mathbb{N}$$ ∎

Theorem 1.19

(a_n) is a Cauchy sequence of real numbers $\implies$ a_n convergent.

Corollary 1.20. *(a_n) Cauchy $\iff$ (a_n) convergent. (Ex: Show not true in $\mathbb{Q}$!)*

Proof. (a_n) Cauchy $\implies$ bounded. So we can define $b_n = \sup\{a_i : i \geq n\}$. Then define $a = \inf\{b_n : n \in \mathbb{N}\}$ and we prove that $a_n \to a$.

Fix $\epsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n, m \geq N \implies |a_n - a_m| < \epsilon/2 \iff a_n - \epsilon/2 < a_m < a_n + \epsilon/2$. Take supremum over all $m \geq i \geq N$

$$\begin{aligned}
&\implies a_n - \epsilon/2 < \sup\{a_m : m \geq i\} \leq a_n + \epsilon/2 \\
&\ \text{i.e. } a_n - \epsilon/2 < b_i \leq a_n + \epsilon/2 \\
&\implies a_n - \epsilon/2 \leq \underset{\substack{\| \\ a}}{\inf\{b_i : i \geq N\}} \leq a_n + \epsilon/2 \\
&\iff |a - a_n| \leq \epsilon/2 < \epsilon \quad \forall n \geq N.
\end{aligned}$$

(We used: $S \subseteq \mathbb{R}$ is bounded satisfying $x < M$ $\forall x \in S$. Then $\sup S \leq M$.) ∎

Lecture 7

Example 1.21. Prove that if $\left|a_{n+1}/a_n\right| \to L$, $L < 1$, then $a_n \to 0$

Idea: $a_N \approx c.L^n$ for $n >> 0$, $L < 1 \implies a_n \to 0$.

To turn this in to a proof, we want $\left|a_{n+1}/a_n\right|$ to be less than $\alpha < 1$! We can't take $\alpha = L$! We can take $\alpha = L + \epsilon$ (because $\left|a_{n+1}/a_n\right|$ is *not* equal to L; it just tends to it). So we need $L + \epsilon < 1$, so take $\epsilon = \frac{1-L}{2}$.

Proof. Fix $\epsilon = \frac{1-L}{2} > 0$ (because $L < 1$). $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon \implies \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon = L + \frac{1-L}{2} = \frac{1+L}{2} < 1$$

So inductively we find that

$$|a_{N+k}| \leq \frac{1+L}{2}|a_{N+k-1}| \leq \left(\frac{1+L}{2}\right)^2 |a_{N+k-2}| \leq \dots \leq \left(\frac{1+L}{2}\right)^k |a_N| \ (*)$$

[Ex sheet: $\alpha^k \to 0$ as $k \to \infty$ if $|\alpha| < 1$]

Applying this to $\alpha = \frac{1+L}{2} < 1$. $\exists M > 0$ s.t. $\forall m \geq M$

$$\left(\frac{1+L}{2}\right)^M < \frac{\epsilon}{1 + |a_N|}$$

(as before we add 1 in denominator in case $|a_N| = 0$)

So by $(*)$ we have $|a_{N+m}| < \dfrac{\epsilon|a_N|}{1+|a_N|} < \epsilon \ \forall m \geq M$. Rewriting this:

$$\forall n \geq N + M, \ |a_n| < \epsilon$$

∎

Subsequences

Definition. A *subsequence* of (a_n) is a new sequence $b_i = a_{n(i)}$ $\forall i \in \mathbb{N}$ where $n(1) < n(2) < \dots < n(i) < \dots$ $\forall i \implies n(i) \geq i$ (Ex: prove this by induction)

[Formally $n(i)$ is a function $\mathbb{N} \to \mathbb{N}$ with $i \mapsto n(i)$ which is strictly monotonically increasing.] "Just go down the sequence faster, missing some terms out"

Example 1.22. $a_n = (-1)^n$ has subsequences:

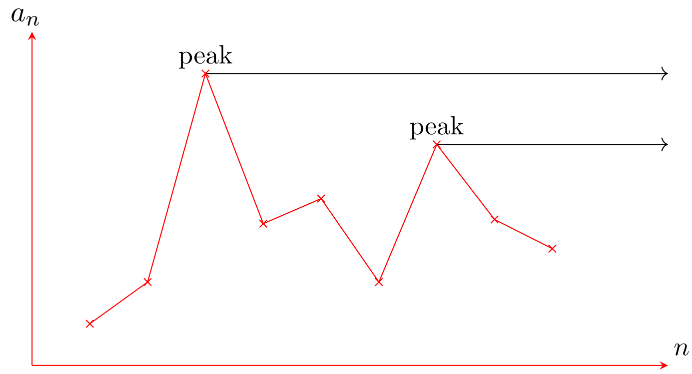
- $b_n = a_{2n}$, so $b_n = 1$ $\forall n \implies b_n \to 1$
- $c_n = a_{2n+1}$, so $c_n = -1$ $\forall n \implies c_n \to -1$
- $d_n = a_{3n}$, so $d_n = (-1)^n (= a_n!)$ doesn't converge.
- $e_n = a_{n+17}$, so $e_n = (-1)^{n+1} = -a_n$ doesn't converge.

Next we work up to

Theorem 1.23: Bolzano-Weierstrass

If (a_n) is a *bounded* sequence of real numbers then it has a *convergent subsequence*.

Cheap proof. Use "peak points" of (a_n)



We say that a_j is a *peak point* iff $a_k < a_j \ \forall k > j$. Either

(i) (a_n) has a finite no. of peak points

(ii) (a_n) has an infinite no. of peak points

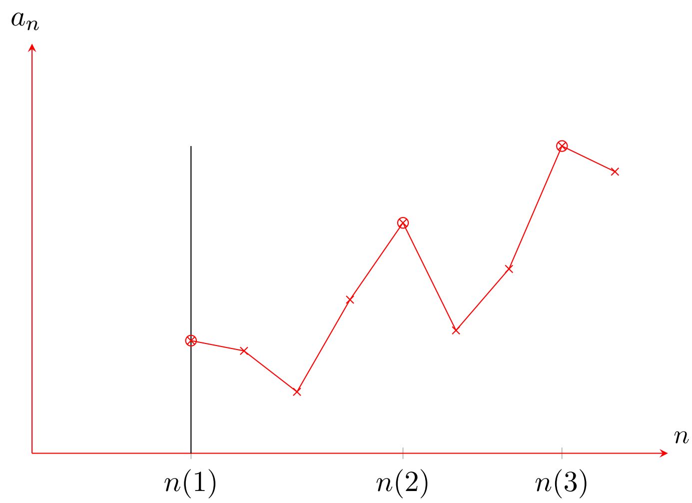
Case (i): Pick $n(1) \geq \max(j_1, \ldots, j_k)$ where $a_{j1}, \ldots, a_{jk}$ are the finite no. of peak points.

"Go beyond the (finitely many) peak points".

$a_{n(1)}$ is not a peak point $\implies \exists n(2) > n(1)$ s.t. $a_{n(2)} \geq a_{n(1)}$.

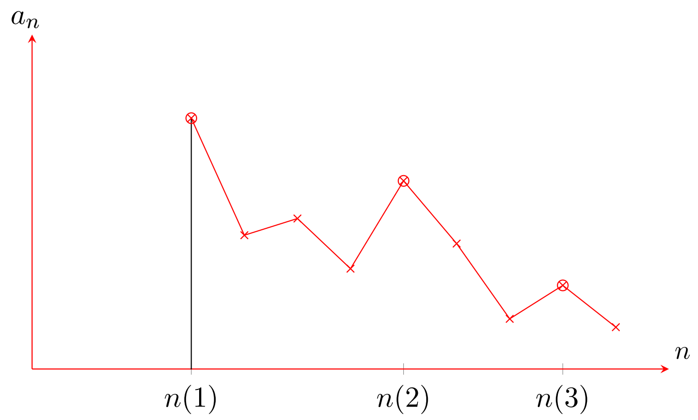
Similarly $a_{n(2)}$ not a peak point $\implies \exists n(3) > n(2)$ s.t. $a_{n(3)} \geq a_{n(2)}$.

Recursively no peak pints beyond $n(1) \implies$ we get $n(i) > n(i-1) > \cdots > n(1)$ s.t. $a_{n(i)} \geq a_{n(i-1)} \ \forall i$.



i.e. $a_{n(i)}$ is a monotonically increasing subsequence of a_n. $(a_n)_{n\geq 1}$ bounded $\implies$ $(a_{n(i)})_{i\geq 1}$ is bounded $\implies a_{n(i)}$ is convergent (to $\sup\{a_{n(i)} : i \in \mathbb{N}\}$.

Case (ii): $\exists$ infinitely many peak points. Call these peak points $a_{n(1)}, a_{n(2)}, \dots$ where $n(1) > n(2) > \dots$



$a_{n(i+1)} \leq a_{n(i)}$ because $n(i+1) > n(i)$ and $a_{n(i)}$ is a peak point $\implies$ $(a_{n(i)})_{i \geq 1}$ is monotonically decreasing and bounded $\implies$ convergent (to $\inf\{a_{n(i)} : i \in \mathbb{N}\}$. ∎

Proposition 1.24. *If $a_n \to a$ as $n \to \infty$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$* *Lecture 8*

Proof.
$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |a_n - a| < \epsilon\ (*)$$
But $\forall i \geq N$, then $n(i) \geq i \geq N \implies$ by $(*)$, $|a_{n(i)} - a| < \epsilon$. ∎

This gives us another proof that $(-1)^n$ is not convergent, because if $(-1)^n \to a$, then by Prop 1.24, $(-1)^{2n} \to a$ and $(-1)^{2n+1} \to a \implies a = 1$ and $a = 1$, ⨳

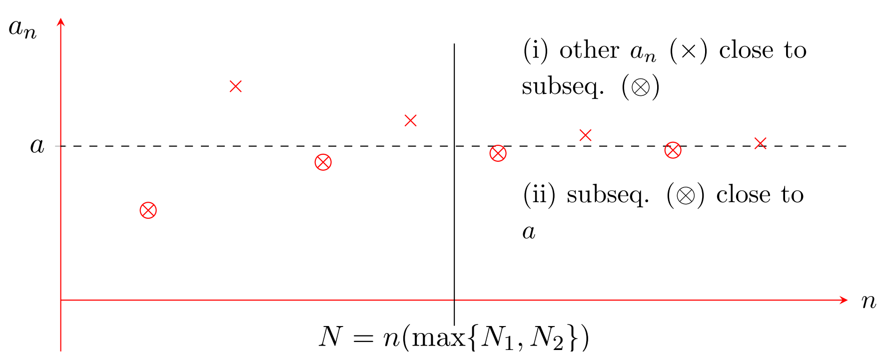
We also get another proof of "Cauchy $\implies$ convergence" using BW (Bolzano-Weierstrass). If a_n is Cauchy ($\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N\ |a_n - a_m| < \epsilon$), then a_n is convergent ($\exists a$ s.t. $a_n \to a$)

Proof. We know that a_n is bounded (by $\max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a_N| + 1)\}$. So by BW, $\exists$ a convergent subsequence $a_{n(i)}$, $i \geq 1$ s.t. $a_{n(i)} \to a$ as $i \to \infty$ for some $a \in \mathbb{R}$.

So fix $\epsilon > 0$. We have:

(i) $\exists N_1$ s.t. $\forall n, m \geq N_1$, $|a_n - a_m| < \epsilon$

(ii) $\exists N_2$ s.t. $\forall i \geq N_2$, $|a_{n(i)} - a| < \epsilon$



Set $N = n(\max\{N_1, N_2\}) \geq \max\{N_1, N_2\} \geq N_1$. Then $\forall n \geq N$ we have

$$\begin{aligned}|a_n - a| &= |(a_n - a_N) + (a_N - a)| \\ &\leq |a_n - a_N| + |a_N - a| \\ &< \epsilon + \epsilon = 2\epsilon\end{aligned}$$

∎

Aside: Fix $c > 0$. Then $a_n \to a$ iff

$$\boxed{\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ s.t. } n \geq N_\epsilon \implies |a_n - a| < c\epsilon (*)}$$

Ex: Show $\implies$

Proof $\impliedby$. Fix $\epsilon > 0$. Set $e' = \epsilon/c > 0$. Then $(*) \implies$

$$\exists N_\epsilon \in \mathbb{N} \text{ s.t. } n \geq N_\epsilon \implies |a_n - a| < c\epsilon' = \epsilon$$ ∎

Beware! Do not let c depend on ϵ (Nor N!), e.g. if we let $c = \frac{1}{\epsilon}$ then $(*)$ becomes $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N,\ |a_n - a| < 1$ and $a_n = \frac{1}{2} \forall n,\ a = 0$ satisfies this!

We can also go the other way round: Cauchy theorem $\implies$ BW.

Proof 2 of BW. Take a bounded sequence (a_n). We want to find a convergent subsequence.

Given $a_n \in [-R, R]\ \forall n$, repeatedly subdivide to make this interval smaller. So either

(i) $\exists$ infinite number of a_n's in $[-R, 0]$

(ii) $\exists$ infinite number of a_n's in $[0, R]$

Pick one of these intervals with inifnite number of a_n's; call it $[A_1, B_1]$, length $2R/2$.

Now subdivide again; call $[A_2, B_2]$ one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ with infinitely many a_n's in it with length $2R/2^2$ etc.

We get a sequence of intervals $[A_n, B_n]$ of length $2R/2^n$ each containing an infinite number of a_ns which are nested: $[A_{k+1}, B_{k+1}] \subseteq [A_k, B_k]$

Now we use a *diagonal argument.* Let $b_i = a_{n(i)}$ be an elements of the sequence in $[A_i, B_i]$ s.t. $n(i) > n(i-1)$. (This is possible because $\exists$ infinite no. of elements of sequence in $[A_i, B_i]$.

Claim: $b_i = a_{n(i)}$ is convergent.

Fix $\epsilon > 0$. Take $N_\epsilon > \frac{2R}{\epsilon}$, so that $\frac{2R}{2^{N_\epsilon}} < \frac{2R}{N_\epsilon} < \epsilon$. Then $\forall i, j \geq N_\epsilon$ we have

$$|b_i - b_j| < \frac{2R}{2^{N_\epsilon}} < \epsilon$$

beacause $b_i, b_j \in [A_{N_\epsilon}, B_{N_\epsilon}] \implies (b_i)$ Cauchy $\implies$ convergent. ∎

2 Series

Lecture 9

Definition. An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \text{ or } a_1 + a_2 + \dots$$

where $(a_i)_{i \geq 1}$ is a sequence.

Convergence of Series

Definition. We say that the series $\sum a_n = A \in \mathbb{R}$ (or "converges to $A \in \mathbb{R}$") iff the sequence of partial sums $S_n := \sum_{i=1}^{n} a_i \in \mathbb{R}$ converges to $A \in \mathbb{R}$; $S_n \to A$ as $n \to \infty$.

$$\begin{tikzcd}
& \text{Sequence of partial sums } (s_n) \arrow[dl, leftrightarrow, "a_n = s_n - s_{n-1}"'] \arrow[dr, leftrightarrow, "s_n = \sum_{i=1}^{n} a_i"] & \\
\text{Sequence } (a_n) \arrow[rr, leftrightarrow, "\text{Equivalent Information}"] & & \text{Series } \sum a_n
\end{tikzcd}$$

Example 2.1. $a_n = x^n,\ n \geq 0$. Consider $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$.

Define $s_n = \sum_{i=0}^{n} x^i = 1 + x + \cdots + x^n$ then $xS_n = x + \cdots + x^n + x^{n+1} \implies S_n - xS_n = 1 - x^{n+1}$

$$\implies S_n = \begin{cases} \frac{1-x^{n+1}}{1-x} & x \neq 1 \\ n+1 & x = 1 \end{cases}$$

So for $|x| < 1$, we see that

$$S_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \to \frac{1}{1-x} \text{ as } n \to \infty$$

(Question Sheet 3: proves that $r^n \to 0$ if $|n| < 1$)

So we have proved that (s_n) is convergent and $\sum x^n = \frac{1}{1-x} \in \mathbb{R}$ for $|x| < 1$.

For $|x| \geq 1$, $a_n = x^n$ does not $\to 0$ as $n \to \infty$. So $\sum a_n = \sum x^n$ is *not* a real number (does not converge) by the following result:

Theorem 2.2

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$

Proof. $S_n - S_{n-1} = a_n$. If $S_n \to S$ then $S_{n-1} \to S$ (Ex). So by the algebra of limits a_n is convergent and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$. ∎

Proof from first principles. Fix $\epsilon > 0$. $s_n \to s$, so

$$\begin{aligned} &\exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |s_n - s| < \epsilon \\ \implies\ & |a_n| = |s_n - s_{n-1}| \\ &\leq |s_n - s| + |s_{n-1} - s| \\ &< \epsilon + \epsilon, \text{ for } n - 1 \geq N. \end{aligned}$$

So $\forall n \geq N+1$, $|a_n| < 2\epsilon$. ∎

Remark 2.3. Converse is *not* true. E.g. $a_n = \frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ is *not* convergent.

Example 2.4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent[†]

Proof. (Trick) First do $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and use $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) \\ &= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty \end{aligned}$$

$\implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent to 1.

Ao now compare the partial sums σ_n of $\sum \frac{1}{n^2}$ to those of $\sum \frac{1}{n(n+1)} = 1$

$$\begin{aligned} \sigma_n = \sum_{i=1}^{n} \frac{1}{i^2} &= 1 + \sum_{j=1}^{n-1} \frac{1}{(j+1)^2} \\ &\leq 1 + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} \\ &= 1 + s_{n-1} \end{aligned}$$

s_{n-1} is a bounded (by 1) monotonically increasing sequence (because $\frac{1}{n(n+1)} > 0$), convergent to 1. So $s_{n-1} < 1\ \forall n \implies \sigma_n < 2 \implies$ bounded above monotonic increasing sequence $\implies \sigma_n$ is convergent $\implies \sum \frac{1}{n^2}$ is convergent. ∎

[†] Famously to $\pi^2/6$ - see Basel Problem.

Similarly $\sum \frac{1}{n^k}$ is convergent for $k \geq 2$ because $\frac{1}{n^k} \leq \frac{1}{n^2}$. In fact $\zeta(k) = \sum \frac{1}{n^k}$ is convergent for $k \in (1, \infty)$... See later!

Theorem 2.5: Algebra of Limits for Sequences

If $\sum a_n = A \in \mathbb{R}$ and $\sum b_n = B \in \mathbb{R}$, then $\sum(\lambda a_n + \mu b_n) = \lambda A + \mu B \in \mathbb{R}$.

Put differently, if $\sum a_n$, $\sum b_n$ converge, then so does $\sum(\lambda a_n+\mu b_n)$ and it equals $\lambda \sum a_n+\mu \sum b_n$.

Proof. Partial sums (to n terms) of $\sum(\lambda a_n + \mu b_n)$ is

$$\sum_{i=1}^{n}(\lambda a_i + \mu b_i) = \lambda \sum_{i=1}^{n} \lambda a_i + \sum_{i=1}^{n} \mu b_i \to \lambda \sum_{i=1}^{\infty} a_n + \mu \sum_{i=1}^{\infty} \mu b_n$$

as $n \to \infty$ by the algebra of limits for sequences. So the partial sums converge. ∎

Lecture 10

Theorem 2.6: Comparison Test

If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (and $0 \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$)

Proof. Call the partial sums A_n, B_n respectively. Then

$$0 \le A_n \le B_n \le \sum_{i=1}^{\infty} b_i = \lim_{n\to\infty} B_n$$

So A_n is bounded and monotonically increasing $\implies$ convergent.

(Question Sheet 3 shows that if $A_n \le B_n$ and $A_n \to A$, $B_n \to B$, then $A \le B$) ∎

Proposition 2.7. *Suppose $a_n \ge 0$ $\forall n$. Then $\sum_{n=1}^{\infty} a_n$ converges iff $S_N = \sum_{n=1}^{N} a_N$ is bounded above and $\sum_{n=1}^{\infty} a_n$ diverges to ∞ (i.e. $S_n \to +\infty$ as $N \to \infty$) iff $S_N = \sum_{n=1}^{N} a_n$ is an unbounded sequence.*

Proof. $a_n \ge 0 \iff (S_n)$ is monotonic increasing. So (S_n) bounded $\iff$ convergent.

S_N unbounded $\iff \forall R > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \ge N,\ S_n > R \iff S_n \to +\infty$. ∎

Exercise: (Converse of Comparison Test) If $0 \le a_n \le b_n$ then $\sum a_n$ diverges to $\infty \implies \sum b_n$ diverges to ∞

Example 2.8. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha > 1$ is convergent.

Proof. (Trick!) Arrange the partial sum as follows:

$$\begin{aligned} 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots = 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) &+ \left(\frac{1}{4^\alpha} + \cdots + \frac{1}{7^\alpha}\right) \\ &+ \left(\frac{1}{8^\alpha} + \cdots + \frac{1}{15^\alpha}\right) \\ &+ \left(\frac{1}{16^\alpha} + \cdots + \frac{1}{31^\alpha}\right) \\ &+ \ldots \end{aligned}$$

Note that the kth bracketed term:

$$\left(\frac{1}{(2^k)^\alpha}+\cdots+\frac{1}{(2^{k+1}-1)^\alpha}\right)\le\frac{1}{2^{k\alpha}}+\cdots+\frac{1}{2^{k\alpha}}=\frac{2^k}{2^{k\alpha}}=\frac{1}{2^{k(\alpha-1)}}$$

So any partial sum is less than some finite sum of these bracketed terms, i.e. for some sufficiently large N:

$$S_N<\sum_{k=0}^{N}\frac{1}{2^{k(\alpha-1)}}=\frac{1-\frac{1}{2^{(N+1)(\alpha-1)}}}{1-\frac{1}{2^{(\alpha-1)}}}\le\frac{1}{1-\frac{1}{2^{\alpha-1}}}$$

because $\alpha>1$, so $\left|\frac{1}{2^{\alpha-1}}\right|<1$, so denominator >0.

So partial sums are bounded above $\implies$ convergent. ∎

Definition. Say that the series $\sum_{n=1}^{\infty}a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^{\infty}|a_n|$ is convergent

Example 2.9. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent, but it is convergent.

Rough Working. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\ldots$, the kth bracket $\frac{1}{2k-1}-\frac{1}{2k}=\frac{1}{2k(2k-1)}$. This is positive and $\le\frac{1}{2k(2k-2)}=\frac{1/4}{k(k-1)}$, seen earlier sum of these is convergent.

So cancellation between consecutive terms is enough to make series converge by comparison with $\sum\frac{1}{k(k-1)}$.

Proof. Fix $\epsilon>0$. Then use 2 things

(1) $\sum\frac{1}{2k(2k-1)}$ is convergent

(2) $\frac{(-1)^{n+1}}{n}\to 0$

By (1) $\exists N_1$ such that $\forall n\ge N_1$, $\sum_{n=1}^{\infty}\frac{1}{k(k-1)}<\epsilon$

By (2) $\exists N_2$ such that $\forall n\ge N_2$, $\left|\frac{(-1)^{n+1}}{n}\right|<\epsilon$

Set $N=\max(N_1,N_2)$. Then $\forall n\ge N$, we have:

$$S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\left(\frac{1}{2j-1}-\frac{1}{2j}\right)+\delta=\sum_{k=1}^{j}\frac{1}{2k(2k-1)}+\delta$$

$$\text{where } \delta=\begin{cases}\frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd.}\\ 0 & \text{if } n \text{ is even.}\end{cases}\qquad (j=\lfloor\tfrac{n}{2}\rfloor)\ j=\begin{cases}\frac{n-1}{2} & \text{if } n \text{ is odd.}\\ \frac{n}{2} & \text{if } n \text{ is even.}\end{cases}$$

$$\implies S_n=\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)}-\sum_{k=\lfloor\frac{n}{2}\rfloor+1}^{\infty}\frac{1}{2k(2k-1)}+\delta$$

$$\text{So } \left| S_n - \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \right| \le \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} \frac{1}{2k(2k-1)} + \frac{1}{n} < \epsilon + \epsilon$$

for all $n \ge 2N$ (so that $\lfloor \frac{n}{2} \rfloor + 1 > N$) ∎

Lecture 11

Theorem 2.10

If (a_n) is absolutely convergent, then it is convergent.

Proof. Let $S_n = \sum_{i=1}^{n} |a_i|$, $\sigma + n = \sum_{i=1}^{n} a_i$ be the partial sums.

We're assuming that S_n converges. Therefore S_n is Cauchy:

$$\forall \epsilon > 0 \; \exists N_\epsilon \text{ such that } n > m \ge N_\epsilon \implies |S_n - S_m| < \epsilon \iff |a_{m+1} + \dots + |a_n| < \epsilon$$

i.e. the terms in the tail of the series contribute little to the sum

$\implies |a_{m+1} + \dots + a_n| < \epsilon$ by the triangle inequality $\implies |\sigma_n - \sigma_m| < \epsilon \implies (\sigma_n)$ is Cauchy $\implies \sum a_i$ is convergent. ∎

Example 2.11. $\sum_{n=1}^{\infty} z_n$ is convergent for $|z| < 1$, divergent for $|z| \ge 1$

Proof. $\sum_{n=1}^{\infty} z_n$ is absolutely convergent because we showed that $\sum_{n=1}^{\infty} |z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$

For $|z| \ge 1$, the individual terms z^n have $|z^n| \ge 1$, so $z^n \not\to 0$, so $\sum z^n$ divergent. ∎

∗Re-arrangement of Series∗

This section was non-examinable in 2015

Beware. Do not rearrange series and sum them in a different order unless you can prove the result is the same.

Example 2.12. $\sum (-1)^{n+1} = 1 - 1 + 1 - 1 + \dots$

either this "$=$"$(1-1) + (1-1) + \dots = 0$

or this "$=$"$1 - (1-1) + (1-1) + \dots = 1$

A better (convergent) example

Example 2.13. $a_n := 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$

(See later for proof of result, it's the series for $\log(1+x) = x - \frac{x^2}{2} + \dots$ putting $n = 1$, which is on our radius of convergence!)

Reorder the sum as follows:

$$\begin{array}{cccccccc} 1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \dots \\ & -\frac{1}{2} & & -\frac{1}{4} & & -\frac{1}{6} & & \dots \end{array}$$

$$\begin{array}{cccccccc} =1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \dots \\ -\frac{1}{2}[& 1 & & +\frac{1}{2} & & +\frac{1}{3} & & \dots] \end{array}$$

Terms with even denominator appear only in bottom row $(\times -\frac{1}{2})$

Terms with odd denominator appear in the top row $(\times 1)$ + bottom row $\times -\frac{1}{2} \implies (\times \frac{1}{2})$ in total.

So $a = \frac{1}{2}[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots] \implies a = a/2$, ※ (But clearly $a \geq \frac{1}{2} > 0$)

This happened because when I reordered I went along the bottom row twice as fast as I went along the top row. Since the top and bottom row diverges to ∞, I'm computing $\infty - \infty$, and originally I did this like $(a+n) - n$ as $n \to \infty$. Now I'm doing it like $(a+n) - (n + \frac{a}{2})$ as $n \to \infty$.

In fact I can rearrange the sum to converge to anything I like.

Example 2.14. Rearrange $a_n = \dfrac{(-1)^{n+1}}{n} \to 42$.

We reorder the sum as follows

(i) Take only off terms $a_{2n+1} > 0$ until their sum is > 42. We can do this as $1 + \frac{1}{3} + \dots$ diverges to ∞!

(ii) Now take only even terms $a_{2n} < 0$ until sum gets < 42

(iii) Repeat (i) and (ii) to fade.

We can do each step because $\sum a_{2n+1}$ diverges to ∞ and $\sum a_{2n} \to -\infty$. We use all the terms eventually (so this is really a reordering of the whole sum)

Why? If not then we must eventually only take terms of one type (w.l.o.g. the even -ve terms) but these sum to $-\infty$, ※. At point they reach < 42 we switch back to odd +ve terms.

Finally proof that the reordered sum converges to 42

$$a_n \to 0 \text{ so } \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n| < \epsilon\ (*)$$

So now we go to a point in the reordering where we have used all a_i up to N and then further to the point where the partial sum crosses 42. At this point, $(*)$ holds, so I'm within ϵ of 42. from this point on the sum is always within ϵ of 42 by design and by $(*)$.

$$\implies |s_n - 42| < \epsilon \text{ from this point on} \quad \blacksquare$$

Lecture 12 More generally if (a_n) is a sequence whose terms tend to zero, $a_n \to 0$ and such that:

- $\sum\limits_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n$ diverges $(\to \infty)$
- $\sum\limits_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n$ diverges $(\to -\infty)$

then I can rearrange the series $\sum a_n$ (1) to make it converge to *any* number I like $\in \mathbb{R}$ or (2) to make it diverge to ∞ or (3) to $-\infty$.

For (1), the Algorithm is same as for $\sum \frac{(-1)^n}{n}$

(i) Pick +ve terms until partial sums are $>$ my fixed real number, a

(ii) Now pick -ve terms until partial sum is $< a$

(iii) Go back to (i) and repeat.

If however $a_n \to 0$ and

$\bullet \sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n \to \infty \quad \bullet \sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n$ converges

Then however I rearrange $\sum a_n$ it will always diverge to $+\infty$

Similarly if $a_n \to 0$ and

$\bullet \sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n$ converges $\quad \bullet \sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n \to -\infty$

$\implies \sum a_n$ diverges to $-\infty$ (however rearranged)

Final case: $a_n \to 0$ and

$\bullet \sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n$ converges $\quad \bullet \sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n$ converges

This is the *good case* where *however* you rearrange, $\sum a_n$ is *absolutely convergent* to the same limit, $\sum_{a_n \geq 0} a_n + \sum_{a_n < 0} a_n$. We will prove this next time.

Remark 2.15. Rearrange partial sums only. $a+b = b+a$ is fine. Infinite sums are tricky!

> **Definition** (Rearrangement of a Sequence)**.** If $M : \mathbb{N} \to \mathbb{N}$ is a bijection (i.e. a reordering!) then define $b_m := a_{M(m)}$. Then $(b_m)_{m \geq 1}$ is a rearrangement of (a_n).

e.g. if $M(1), M(2), M(3), M(4), \ldots$ is $5, 1, 6, 2, \ldots$ then $b_1, b_2, b_3, b_4, \ldots$ is $a_5, a_1, a_6, a_2, \ldots$.

Theorem 2.16

Suppose that $\sum a_n$ is absolutely convergent. Then

(1) $\sum_{a_n \geq 0} a_n$ is convergent to A (say)

(2) $\sum_{a_n < 0} a_n$ is convergent to B (say)

(3) $\sum a_n = A + B$

(4) $\sum b_m = A + B$ where (b_m) is any rearrangement of (a_n)

Proof. Key Idea: $\sum |a_n|$ is convergent so has a small "tail", so by the triangle inequality $\sum a_n$ has an even smaller tail so should converge. Lecture 13

But what to? No idea, so we use the Cauchy criterion!

(1) $s_n = \sum_{i=1}^{n} a_i$, $\sigma_n = \sum_{i=1}^{n} |a_i|$. σ_n convergent $\implies$ σ_n is Cauchy.

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, \; |\sigma_n - \sigma_m| < \epsilon$$

w.l.o.g. $n \geq m$, this says

$$\sum_{i=m+1}^{n} |a_i| < \epsilon \implies \left| \sum_{i=m+1}^{m} a_i \right| < \epsilon \iff |s_n - s_m| < \epsilon$$

So (s_n) is Cauchy $\implies$ s_n is convergent.

(2) $\sum_{a_n \geq 0} a_n$ is also convergent because the partial sums are monotonic increasing, bounded above by $\sum |a_n|$. Similarly $\sum_{a_n < 0} a_n$ is decreasing, $\geq -\sum |a_n|$, so also cvgt.

(3) Let $A = \sum_{a_n \geq 0} a_n$ and $B = \sum_{a_n < 0} a_n$. Then $\forall \epsilon > 0$

$$\exists N_1 \text{ s.t. } n \geq N_1 \implies \left| \sum_{a_n \geq 0}^{\text{first } n \text{ terms}} -A \right| < \epsilon$$

$$\exists N_2 \text{ s.t. } n \geq N_2 \implies \left| \sum_{a_n < 0}^{\text{first } n \text{ terms}} -B \right| < \epsilon$$

Let N be $\max(I, J)$ where I is the N_ith $a_i \geq 0$ (the N_ith positive term) and a_J the N_Jth -ve term. Then $\forall n \geq N$

$$\left| \sum_{i=1}^{n} -(A+B) \right| \leq \left| \sum_{a_i \geq 0}^{n} a_i - A \right| + \left| \sum_{a_i < 0}^{n} a_i - B \right| < \epsilon + \epsilon = 2\epsilon$$

So $\sum_{i=1}^{n} \to A + B$ as $n \to \infty$.

(4) Finally (b_m) is a rearrangement of (a_n). We want to show that $\sum b_m$ converges to $A + B$ as well.

Pick $M \in \mathbb{N}$ such that $b_1, b_2, \dots, b_M$ contains all of $P_1, P_2, \dots, P_I$ and $N_1, N_2, \dots, N_J$ where P_i is the ith $a_i \geq 0$ and N_J is the jth $a_j < 0$.

[i.e. we're far enough down the rearranged series to have included all significant $a_i \geq 0$ and $a_i < 0$ which sum to $< \epsilon$ by (1) and (2)]

Then $\forall m \geq M$ we have

$$\begin{aligned}
\left| \sum_{i=1}^{m} b_i - (A+B) \right| &\leq \left| \sum_{b_i \geq 0}^{m} b_i - A \right| + \left| \sum_{b_i < 0}^{m} b_i - B \right| \\
&\leq \left| \sum_{a_k \geq 0}^{I} a_k + \delta - A \right| + \left| \sum_{a_k < 0}^{J} a_k + \delta' - B \right| \\
&< \epsilon + \epsilon = 2\epsilon
\end{aligned}$$

(where δ = sum of $a_k \geq 0$ with $k > I$ and δ' = sum of $a_k < 0$ with $k > J$) ∎

Tests for convergence

We already met the first test:

Theorem 2.5: Comparison I

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (and $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$)

Recall proof from earlier: $s_n = \sum a_i$ is monotonic increasing and bounded above by $\sum b_i \in \mathbb{R}$.

Theorem 2.18: Comparison II - Sandwich Test

Suppose $c_m \leq a_n \leq b_n$ and $\sum c_n$, $\sum b_n$ are both convergent. Then $\sum a_n$ is convergent.

Proof. Use Cauchy. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$

$$\left| \sum_{i=m+1}^{n} b_i \right| < \epsilon, \quad \left| \sum_{i=m+1}^{n} c_i \right| < \epsilon$$

since the partial sums of b_i, c_i are Cauchy. Therefore

$$-\epsilon < \sum_{i=m+1}^{n} c_i \leq \sum_{i=m+1}^{n} a_i \leq \sum_{i=m+1}^{n} b_i < \epsilon$$

$$\implies \left| \sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i \right| < \epsilon \implies \left(\sum_{i=1}^{n} a_i \right) \text{ is Cauchy.}$$ ∎

Lecture 14

Theorem 2.19: Comparison III

If $\frac{a_n}{b_n} \to l \in \mathbb{R}$ then $\sum b_n$ absolutely convergent $\implies \sum a_n$ is absolutely convergent.

Proof. Pick $\epsilon = 1$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$\left| \frac{a_n}{b_n} - l \right| < 1 \implies \left| \frac{a_n}{b_n} \right| < |l| + 1 \implies |a_n| < (|l| + 1)|b_n|$$

So now by the comparison test $\sum_{n \geq N} |b_n|$ convergent $\implies \sum_{n \geq N} |a_n|$ convergent $\implies \sum_{n \geq 1} |a_n|$ convergent. ∎

We have used the obvious fact that if $\sum_{n \geq N} c_n$ is convergent then $\sum_{n \geq 1} c_n$ is also convergent (and vice-versa). Exercise: prove this!

Theorem 2.20: Alternating Series Test.

Given an alternating sequence a_n where $a_{2n} \geq 0,\ a_{2n+1} \leq 0\ \forall n$. Then $|a_n|$ monotonic decreasing to $0 \implies \sum a_n$ convergent

Proof. Write $a_n = (-1)^n b_n,\ b_n \geq 0\ \forall n$. Consider the partial sums $S_n = \sum_{i=1}^{n}(-1)^n b_n$.

Observe that:

(1) $S_i \leq S_{2n}\ \forall i \geq 2n$

(2) $S_i \geq S_{2n+1}\ \forall i \geq 2n+1$

Since if $i = 2j$ is even, then

$$\begin{aligned} S_{2j} &= S_{2n} + a_{2n+1} + \cdots + a_{2j} \\ &= S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} \leq S_2 n \end{aligned}$$

If $i = 2j+1$ is odd, then similarly:

$$S_{2j} = S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} - b_{2j+1} \leq S_2 n$$

So now $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ |b_n| < \epsilon$. So $\forall n, m \geq 2n$, we have:

$$\begin{aligned} S_{2N+1} \leq S_n&,\ S_m \leq S_{2N} \\ \text{So } |S_n - S_m| &\leq |S_{2N+1} - S_{2N}| \\ &= b_{2n+1} < \epsilon \end{aligned}$$

∎

Theorem 2.21: Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Proof. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon \implies |a_{n+1}| < (r+\epsilon)|a_n|$$

Set $\alpha := r + \epsilon = \frac{1+r}{2} < 1$.

Inductively

$$|a_{N+m}| < \alpha |a_{N+m-1}| < \cdots < \alpha^m |a_N|$$

So $\forall k \geq N$

$$|a_k| < \alpha^{k-N}|a_N| = C\alpha^k$$

Then

$$C\sum_{k=N}^{n} \alpha^k = \frac{C(\alpha^N - \alpha^n)}{1-\alpha} \to \frac{C'}{1-\alpha} \text{ as } n \to \infty, \text{ since } \alpha < 1$$

So by the comparison test $\sum_{k \geq N} |a_k|$ is convergent $\implies \sum_{k\geq 1} |a_k|$ is convergent ∎

The point is that the ratio test, when it applies, says that $a_n \approx r^n$ i.e. decays exponentially. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.

Example 2.22. $a_n = \dfrac{100^n(\cos n\theta + i \sin n\theta}{n!} = \dfrac{(100e^{i\theta})^n}{n!}$

Then
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(100e^{i\theta})^{n+1}/(n+1)!}{(100e^{i\theta})^n/n!} = \frac{100}{n+1} \to 0$$
So by the ratio test, $\sum a_n$ is absolutely convergent $\implies \sum a_n$ is convergent.

Lecture 15

Theorem 2.23: Root Test

If $\lim_{n\to\infty} |a_n|^{1/n} = r < 1$, then $\sum a_n$ is absolutely convergent.

Proof. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$
$$\left| |a_n|^{1/n} - r \right| < \epsilon \implies |a_n|^{1/n} < r + \epsilon$$
Set $\alpha := r + \epsilon = \frac{1+r}{2} < 1$, so that $|a_n| < \alpha^n$. Then
$$\sum_{k=1}^{n} \alpha^k = \frac{\alpha(1-\alpha^n)}{1-\alpha} \to \frac{\alpha}{1-\alpha} \text{ as } n \to \infty \text{ since } \alpha < 1$$
So by the comparison test $\sum_{k\geq 1} |a_k|$ is convergent. ∎

Power Series

Theorem 2.24: Radius of Convergence

Consider the series $\sum a_n z^n$ $(*)$, $z, a_n \in \mathbb{C}$.

Then $\exists R \in [0, \infty]$ such that $|z| < R \implies (*)$ is aboslutely convergent, $|z| > R \implies$ $(*)$ divergent

Proof. Define $R = \sup S = \{|z| : a_n z^n \to 0\}$ or $R = \infty$ if the set is unbounded. (1) Suppose $|z| < R$. $|z|$ not an upperbound for $S \implies \exists w$ such that $|w| > |z|$ and $a_n w^n \to 0$. Then
$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq A \left|\frac{z}{w}\right|^n$$
Since $\left|\frac{z}{w}\right| < 1 \implies \sum |a_n z^n|$ cvgt. Similarly $|z| > R \implies \sum |a_n z^n|$ divergent.

(2) Suppose $|z| > R$. Then $a_n z^n \not\to 0$ as $n \to \infty \implies \sum a_n z^n$ does not converge. ∎

Clicker Question 2.25. What is the radius of convergence for $\sum \frac{z^n}{n}$?

Answer: $R = 1$, in fact the series

(i) $\sum z^n$

(ii) $\sum \frac{z^n}{n}$

(iii) $\sum \frac{z^n}{n^2}$

all have this R.

Proof. The ratio test gives $\left|\dfrac{z^{n+1}}{z^n} \cdot f(n)\right|$ where f is a rational function of n of degree 0. $= |zf(n)| \to |z|$ as $n \to \infty$. So convergent for $|z| < 1$ and divergent for $|z| > 1$. ∎

But notice different behaviours on $|z| = 1$.

(i) Never converges on $|z| = 1$ as $z^n \not\to 0$

(ii) Convergent for some $|z| = 1$ (in fact $z \neq 1$), divergent for others

(iii) Also convergent $\forall z$ with $|z| = 1$ (comparison with $\sum \frac{1}{n^2}$)

Products of Series

Consider

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\ &\text{“}{=}\text{''}\ a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \\ &= \sum_{n=0}^{\infty} c_n z^n\end{aligned}$$

where $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b + 0, \dots c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$.

So we set $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ and ask when is the product $\sum a_n z^n \sum b_n z^n$ equal to $\sum c_n z^n$? We can also do this without the z^n's:

Definition. Given series $\sum a_n$, $\sum b_n$, their *Cauchy Product* is the series $\sum c_n$ where $c_n := \sum_{i=0}^{n} a_i b_{n-i}$.

Lecture 16

Theorem 2.26: Cauchy Product

If $\sum a_n, \sum b_n$ are absolutely convergent, then $\sum c_n$ is absolutely convergent to $(\sum a_n) \cdot (\sum b_n)$

Proof. See handout on blackboard. Non-examinable.

Corollary 2.27. *If $\sum A_n z^n$ and $\sum B_n z^n$ have radius of convergence R_A and R_B respectively, then $\sum c_n z^n$ has radius of convergence $R_C \geq \min\{R_A, R_B\}$.*

Proof. By the previous theorem, for $|z| < \min\{R_A, R_B\}$ $(*)$ we have $\sum A_n z^n$ and $\sum B_n z^n$ absolutely convergent $\implies \sum c_n z^n$ absolutely convergent to their product.

In fact $|c_n z^n| \to 0$ so $|z| < R_c$. So by $(*)$, $R_c \geq \min\{R_A, R_B\}$. ∎

Example 2.28. $\sum z^n$ has $R_A = 1$, $1 - z$ has $R_B = \infty$ So their cauchy product $\sum c_n z^n$ has $R_c \geq 1$.

Ex: Check $c_0 = 1, c_n = 0\ \forall n \geq 1$, so in fact $R_c = \infty$.

But we only know that $\sum c_n z^n = 1 = (\sum z^n)(1 - z)$ when $|z| < 1 = \min\{R_A, R_B\}$.

Exponential Power Series

Definition (Exponential Series)**.**

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!},\ z \in \mathbb{C}$$

Ratio test: $|a_{n+1}/a_n| = \frac{z}{n+1} \to 0$ as $n \to \infty\ \forall z \in \mathbb{C} \implies E(z)$ is absolutely convergent $\forall z \in \mathbb{C}$.

Proposition 2.29. $E(z)E(w) = E(z + w)$

Proof. By Cauchy product theorem

$$E(z)E(w) = \sum_{n=0}^{\infty} c_n$$

where $c_n = \sum_{i=0}^{n} \dfrac{z^i}{i!}\dfrac{w^{n-i}}{(n-i)!} \implies c_n = \dfrac{(z+w)^n}{n!}$. ∎

Corollary 2.30. $E(z) \neq 0$ *and* $\dfrac{1}{E(z)} = E(-z)$

Proof. $E(z)E(-z) = E(0) = 1$. ∎

Definition. $e := E(1) = \sum \frac{1}{n!} \in (-0, \infty)$

Corollary 2.31. $E(n) = e^n$ *for* $n \in \mathbb{N}$

Proof. $E(n) = E(1 + (n - 1)) = E(1)E(n - 1) = \cdots = (E(1))^n$. ∎

Proposition 2.32. $E(q) = e^q$ *for* $q \in \mathbb{Q}$ *(recall rational powers of* $a \in \mathbb{R}$ *were defined in M1F)*

Proof. Suppose $q > 0$; write $q = \frac{m}{n}$, $m, n \in \mathbb{N}$. Then

$$E(q) = E(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}}) = E(\tfrac{1}{n})^m$$

But
$$E(\frac{1}{n})^n = E(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}) = E(1) = e$$

$$\implies E(\frac{1}{n}) = e^{1/n} \text{ and } E(q) = E(\frac{1}{n})^m = e^{m/n} = e^q$$

If $q = \frac{-m}{n}$ then $E(q) = 1/E(m/n) = \frac{1}{e^{m/n}} = e^{-m/n} = e^q$. ∎

So we know that $E(x) = e^x\ \forall x \in \mathbb{Q}$. Later we define $e^x\ \forall x \in \mathbb{R}$ by *continuity* and we will show $E(x)$ is also continuous and so $E(x) = e^x\ \forall x \in \mathbb{R}$.

Lecture 17

Some useful properties of $E(x)$:

(i) $x \geq 0 \implies E(x) \geq 1$ and $x > 0 \implies E(x) > 1$ (obvious from series)

(ii) $E(x) > 0\ \forall x \in \mathbb{R}$

(iii) $E(x)$ is strictly increasing for $x \in \mathbb{R}$: $x < y \implies E(y) = E(x)E(y-x) > E(x).1$

(iv) $|x| < 1$ then $|E(x) - 1| < \frac{|x|}{1-|x|}$

(v) $\mathbb{R} \ni x \mapsto E(x)$ is a continuous bijection onto $(0, \infty)$. (proven later)

(vi) So we can define $\log : (0, \infty) \to \mathbb{R}$ as inverse of E, i.e. $y = \log x$ defined by $\iff x = e^y$ with the usual log properties

We can also define a^x for $a \in (0, \infty), x \in \mathbb{R}$ by $a^x = E(x \log a)$

Exercise: If $x \in \mathbb{Q}$ this agrees with Corti's definition.

And trig functions $\cos\theta = \Re E(i\theta)$, $\sin\theta = \Im E(i\theta)$ etc.

Exercise: $E(i\theta + i\phi) = E(i\theta)E(i\phi)$ implies what?

3 Continuity

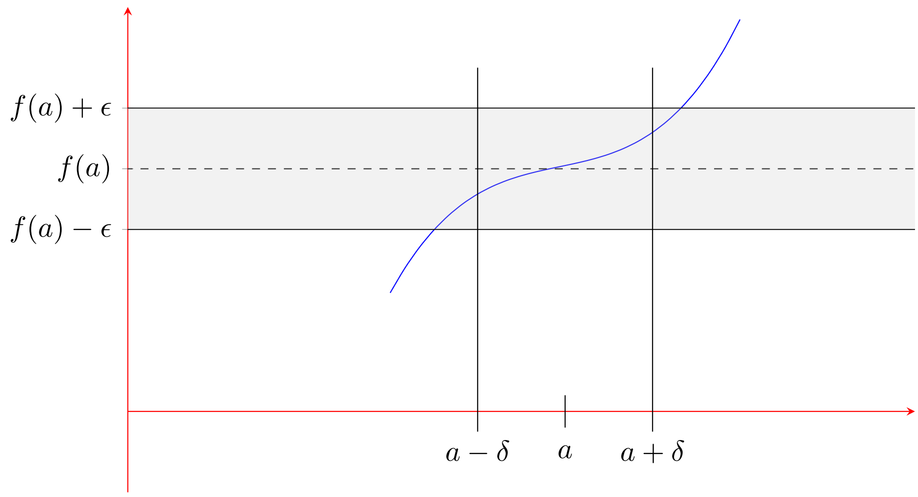
Continuity and Limits

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *continuous at* $a \in \mathbb{R}$ if and only if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

So δ depends on a, ϵ. "Once x is close to a, then $f(x)$ is close to $f(a)$".

More precisely: "However close (i.e. within ϵ) I want $f(x)$ to be to $f(a)$, I can arrange it by taking x close (i.e. within δ) to a".



Equivalently: $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon\ \forall x$ with $|x - a| < \delta$

Or: $\forall \epsilon > 0,\ \exists \delta > 0$ such that $f(a - \delta, a + \delta) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$

Where $S \subseteq R$ then $f(S)$ is the set $\{f(x) : x \in S\}$

Or: $\forall \epsilon,\ \exists \delta > 0$ such that $f^{-1}(f(a) - \epsilon, f(a) + \epsilon) \supseteq (a - \delta, a + \delta)$

Where $f : A \to B \subset T$ then $f^{-1}(T) = \{a \in A : f(a) \in T\}$ [Don't need f^{-1} to exist !!]

Example 3.1.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then f is not continuous at $x = 0$

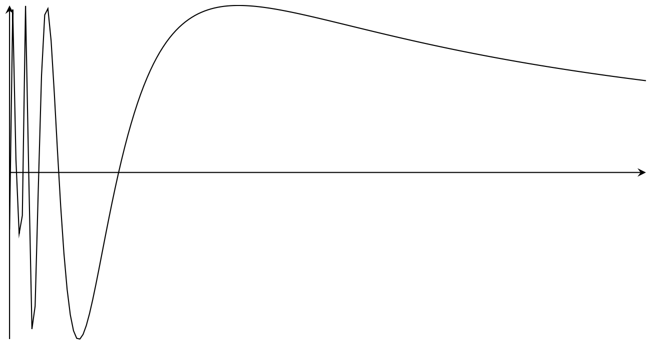
Proof. Take $\epsilon = 1$ (or $0 < \epsilon < 1$). Then if f is continuous at $x = 0$ we know that $\exists \delta > 0$ such that $|f(x) - f(0)| < 1\ \forall x \in (0 - \delta, 0 + \delta)$ $(*)$. In particular, take $x = \delta/2$ to find that $|1 - 0| < 1$ by $(*)$. ∎

"Jump discontinuity" is another type of discontinuity

Example 3.2.

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ r & x = 0 \end{cases}$$

Then f is discontinuous at $x = 0$ (for any r).



Idea of proof: If f is continuous at $x = 0$, then $f(x) \in (r - \epsilon, r + \epsilon)$ is close to $f(0) = r$ for $x \in (-\delta, \delta)$. In particular, $f(x)$ and $f(y)$ are close to each other (within 2ϵ). But $f(x)$ could be $+1$ and $f(y)$ could be -1, ⨳.

Proof. Fix $\epsilon \in (0, 1]$. If f is continuous at 0, then $\exists \delta > 0$ such that $|f(x) - f(0)| < \epsilon \ \forall x \in (\delta, \delta)$. In particular, $\forall x, y \in (-\delta, \delta), |f(x) - f(y)| < 2\epsilon \leq 2$, by the triangle inequality.

Now choose $n \in \mathbb{N}$, $n > \frac{1}{\delta}$. Then take $x = \frac{1}{(4n+1)\pi/2} \in (0, \delta)$, $y = \frac{1}{(4n+3)\pi/2} \in (0, \delta)$. Then

$$|\sin(1/x) - \sin(1/y)| = |1 - (-1)| = 2 \text{ ⨳}$$

∎

Lecture 18

Example 3.3. $f : \mathbb{R} \to \mathbb{R}$, $f = mx + c$ is continuous at a, $\forall a \in \mathbb{R}$.

Rough working: We want

$$\begin{aligned} |f(x) - f(a)| < \epsilon &\iff |(mx + c) - (ma + c)| < \epsilon \\ &\iff |mx(-a)| < \epsilon \\ &\iff |x - a| < \frac{\epsilon}{|m|} \text{ if } m \neq 0 \\ &\impliedby |x - a| < \frac{\epsilon}{|m| + 1} \end{aligned}$$

So set $\delta := \epsilon/(1 + |m|)$. Then $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$

Proof. Set $\delta := \frac{\epsilon}{1+|m|} > 0$. Then when $|x - a| < \delta$ we have

$$\begin{aligned} |(mx + c) - (ma + c)| &= |f(x) - f(a)| \\ &= |m||x - a| \\ &< |m|\delta = \epsilon\frac{|m|}{|m| + 1} < \epsilon \end{aligned}$$

∎

Example 3.4. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ Proposition: f continuous on $\mathbb{R}$ (i.e. at a, $\forall a \in \mathbb{R}$)

Rough working:

$$|f(x) - f(a)| = |x^2 - a^2| = |x + a||x - a|$$

we want this to be $< \epsilon$, i.e. $|x - a| < \frac{\epsilon}{|x+a|} * (*)$

But we can't let δ depend on x!!

Problem: If $|x - a| < \frac{\epsilon}{R} \forall R > 0$, then $|x - a| = 0$.

Solution: I only care about x close to a; within 1 say.

So, so long as I choose $\delta \leq 1$, then I know that

$$|x - a| < \delta \implies |x + a| \leq |x - a| + 2|a| \leq 1 + 2|a|$$

So now $|x - a| < \dfrac{\epsilon}{1 + 2|a|} \implies (*)$

So to ensure both conditions we set $\delta = \min\{1, \epsilon/(1 + 2|a|)\}$

Proof. Fix $\epsilon > 0$, $a \in \mathbb{R}$. Set $\delta = \min\{1, \frac{\epsilon}{1+2|a|}\}$. Then $|x - a| < \delta \implies$

(i) $|x - a| < 1 \implies |x + a| < 1 + 2|a|$

(ii) $|x - a| < \dfrac{\epsilon}{1 + 2|a|}$

$$\implies |x^2 - a^2| = |x - a||x + a| < \frac{\epsilon}{1 + 2|a|} \cdot (1 + 2|a|) = \epsilon$$

∎

Clicker Question 3.5. Fix $a, b \in \mathbb{R}$. Then $x < a \implies x < b$ tells us?

Answer: $a \geq b$.

Prove that $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is discontinuous at $x = 0$

Student answer:

(i) Suppose f is cts at 0

(ii) Then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

(iii) $|x| < \delta \implies |f(x) - f(0)| = |1/x| < \epsilon$

(iv) $\implies |1/(x/2)| = |2/x| < 2\epsilon$

(v) But $|x| < \delta \implies |x/2| < \delta$ so

(vi) should get that $|f(x/2) - f(0)| = |1/(x/2)| < \epsilon$

(vii) This contradicts $(*)$

(viii) So f is not continuous at 0

Answer: (vii) is the problem. (vi) $\implies$ (iv) doesn't contradict (iv).

Lecture 19 Notice the definition of continuity makes sense whenever I have a notion of distance. e.g. in $\mathbb{R}^n$ use $|\vec{x}-\vec{y}| := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

> **Definition.** $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ iff
> $$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } |\vec{x}-\vec{a}| < \delta \implies |f(\vec{x}) - f(\vec{a})| < \epsilon$$

Notation: The ϵ-ball around $\vec{a} \in \mathbb{R}^n$ is $B_\epsilon(\vec{a}) := \{\vec{x} \in \mathbb{R}^n | \vec{x} - \vec{a}| < \epsilon\}$

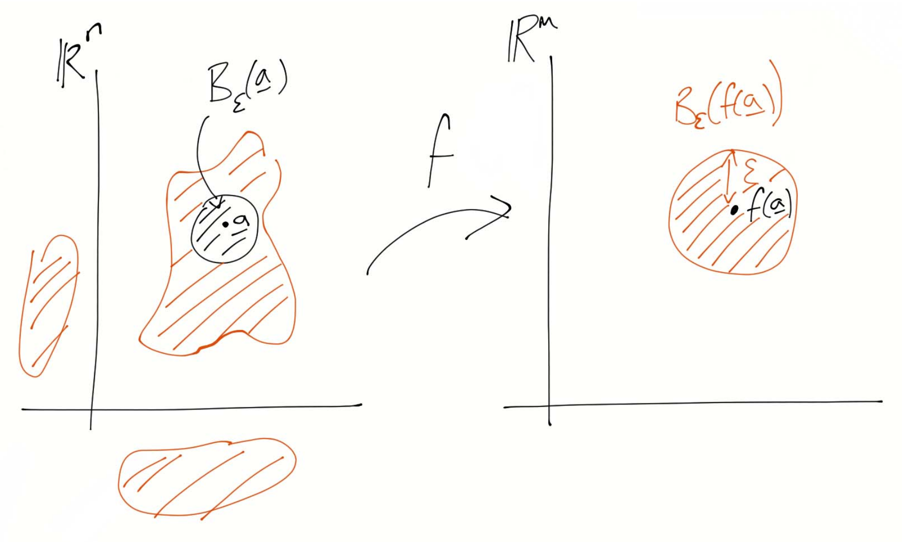
So if $n = 1$, $B_\epsilon(a) = (a-\epsilon, a+\epsilon) \subseteq \mathbb{R}$.

Using this we can rewrite our definition of continuity:

> **Definition.** $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{a} \in \mathbb{R}^n$ iff
> $$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } f(B_\delta(\vec{a})) \subseteq B_\epsilon(f(\vec{a}))$$

So every point within δ of $\vec{a}$ gets mapped by f to within ϵ of $f(\vec{a})$, equivalently

$$\boxed{\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } B_\epsilon(\vec{a}) \subseteq f^{-1}(B_\epsilon(f(\vec{a})))}$$

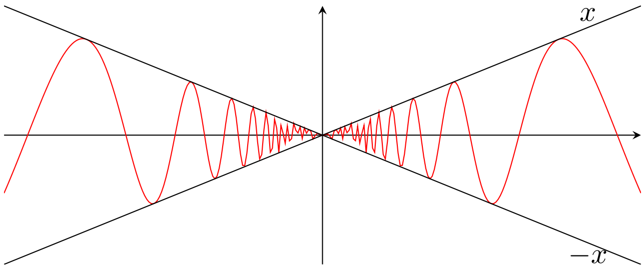


$f^{-1}(B_\epsilon(f(\vec{a}))) := \{\vec{x} \in \mathbb{R}^n : f(\vec{x}) \in B_\epsilon(f(\vec{a}))\}$

Continuity at $\vec{a}$ says that $\vec{a}$ is in the "interior" of $f^{-1}(B_\epsilon(f(\vec{a})))$, i.e. $\exists$ a small ball $B_\delta(\vec{a})$ around it which is also in $f^{-1}(B_\epsilon(f(\vec{a})))$.

So continuity at $\vec{a}$ $\iff$ If $\vec{x}$ moves a tiny bit around $\vec{a}$ then $f(\vec{x})$ moves a tiny bit around $f(\vec{a})$.

Example 3.6. $f(x) = \begin{cases} x \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$



Proposition 3.7. *f is continuous at 0*

Proof. Fix $\epsilon > 0$. Then

$$|f(x) - f(0) = |x \sin \frac{1}{x}| \leq |x|$$

Take $\delta = \epsilon$. Then $|x| < \delta \implies |x| < \epsilon \implies |f(x) - f(0)| < \epsilon$ ∎

Proposition 3.8. *$E : \mathbb{C} \to \mathbb{C}$ defined by $E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is continuous (i.e. continuous at a, $\forall a \in \mathbb{C}$)*

[Ex: from this show that $x \mapsto \sin x$ is continuous on $\mathbb{R}$]

Rough working:

$$\begin{aligned} |E(z) - E(a)| &= |E(a)(E(z-a) - E(0))| \\ &= |E(a)||E(z-a) - 1| \\ &\leq |E(a)| \cdot \frac{|z-a|}{1 - |z-a|} \end{aligned}$$

for $|z - a| < 1$ (see earlier lecture)

$$\begin{aligned} \text{We want this to be } < \epsilon &\iff |z-a| < \frac{\epsilon}{|E(a)|}(1 - |z-a|) \\ &\iff (1 + \epsilon/|E(a)|)|z-a| < \frac{\epsilon}{|E(a)|} \\ &\iff |z-a| < \epsilon/|E(a)|/(1 + \epsilon/|E(a)|) \end{aligned}$$

Proof. Fix $\epsilon > 0$. Set $\delta = \dfrac{\epsilon}{|E(a)| + \epsilon}$ $(*)$

Then we calculate that

$$\begin{aligned} |E(z) - E(a)| &\leq |E(a)| \frac{|z-a|}{1 - |z-a|} \\ &< |E(a)| \cdot \frac{\delta}{1 - \delta} \end{aligned}$$

for all z with $|z - a| < \delta$. But by $(*)$, $\dfrac{\delta}{1-\delta} = \dfrac{\epsilon}{|E(a)|}$.

So $|z - a| < \delta \implies |E(z) - E(a)| < \epsilon$. ∎

Theorem 3.9

$f, g : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R} \implies (f+g), f \cdot g$ are cts at a.

Proof. Fix $\epsilon > 0$.

$$\exists \delta_1 > 0 \text{ such that } |x-a| < \delta_1 \implies |f(x) - f(a)| < \epsilon$$

and

$$\exists \delta_2 > 0 \text{ such that } |x-a| < \delta_1 \implies |g(x) - g(a)| < \epsilon$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x$ such that $|x-a| < \delta :$

$$|(f+g)(x) - (f+g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < 2\epsilon$$

For (2): Similarly

$$|f(x)g(x) - f(a)g(a)| \leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)| \ (*)$$

We need a bound on $|g(x)|$. We cannot bound $g(x)$ $\forall x$! But near a, $g(x)$ is close to $g(a)$, so we can bound $g(x)$ near a

Take $\epsilon = 1$

$$\exists \delta_1 > 0 \text{ s.t. } |x-a| < \delta_1 \implies |g(x) - g(a)| < 1 \implies |g(x)| < 1 + |g(a)| \ (A)$$

Now fix any $\epsilon > 0$. Then

$$\exists \delta_2 > 0 \text{ s.t. } |x-a| < \delta_2 \implies |f(x) - f(a)| < \epsilon/1 + |g(a)| \ (B)$$

(to cancel $|g(x)| < 1 + |g(a)|$ in $(*)$)

$$\exists \delta_3 > 0 \text{ s.t. } |x-a| < \delta_3 \implies |g(x) - g(a)| < \frac{\epsilon}{1 + |f(a)|} \ (C)$$

(to cancel $|f(a)|$ in $(*)$)

Set $\delta := \min\{\delta_1, \delta_2, \delta_3\}$. Then $|x-a| < \delta \implies (A), (B), (C)$ all hold.

Substitute into $(*)$ to find

$$\begin{aligned} |f(x)g(x) - f(a)g(a)| &< 1 + |g(a)| \frac{\epsilon}{1 + |g(a)|} + |f(a)| \frac{\epsilon}{1 + |f(a)|} \\ &\leq \epsilon + \epsilon = 2\epsilon \end{aligned}$$

∎

Theorem 3.10

$f : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ cts at $f(a) \in \mathbb{R}$, then $g \circ f$ cts at a

Idea of Proof: We want $g(f(x))$ to be close (within ϵ) to $g(f(a))$.

But g is continuous at $f(a)$! So sufficient for $f(x)$ to be close (within δ_g to $f(a)$. But f is continuous at a! so we can arrange this (by taking $\epsilon = \delta_g$ by taking x to be close (within δ_g) to a.

Proof. Fix $\epsilon > 0$. g is continuous at $f(a)$, so Lecture 20

$$\exists \delta > 0 \text{ s.t. } |g - f(a)| < \delta \implies |g(y) - g(f(a))| < \epsilon$$

Also f is continuous at a, so

$$\exists \eta > 0 \text{ s.t. } |x - a| < \eta \implies |f(x) - f(a)| < \delta$$

Hence $|x - a| < \eta \implies |f(x) - f(a)| < \delta \implies |g(f(x)) - g(f(a))| < \epsilon$. ∎

Corollary 3.11. $a^x := E(x \log a),\ a > 0$ *is continuous* $\forall x \in \mathbb{R}$

Proof. It is a composition $\mathbb{R} \xrightarrow[x \mapsto x\log a]{} \mathbb{R} \xrightarrow[y \mapsto E(y)]{} \mathbb{R}$ of two functions. ∎

Exercise: Show $\sin 1/x$ is continuous for $\mathbb{R}\backslash\{0\} \to \mathbb{R}$. (i.e. show $1/x$ is continuous from first principles, $\sin x$ is continuous using continuity of $E(x)$ and compose!)

Example 3.12. Suppose $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous. Then $1/f$ is continuous.

Proof. Pick $a \in \mathbb{R}$. Show $1/f(x)$ is continuous at a:

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| = \frac{1}{|f(x)f(a)|}|f(x) - f(a)|\ (*)$$

We need to bound $f(x)$ below! Need $|f(x)| >$ some $\eta > 0 \iff \frac{1}{|f(x)|} < \frac{1}{\eta}$

We can't, but we can near a! $f(a) \neq 0$, so take $\epsilon' = |f(a)|/2 > 0$. Then

$$\exists \delta' \text{ s.t. } |x - a| < \delta' \implies |f(x) - f(a)| < \epsilon' = \frac{|f(a)|}{2}$$

$$\implies |f(x)| > |f(a)| - \epsilon = \frac{|f(a)|}{2}$$

$$\text{So by } (*)\text{, we have } \left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| < \frac{1}{|f(a)|/2 \cdot |f(a)|}|f(x) - f(a)|$$
$$= \frac{2}{|f(a)|^2}|f(x) - f(a)|$$

Fix $\epsilon > 0$. Set $\epsilon'' = \min\left(\frac{|f(a)|}{2}, \frac{\epsilon}{2}|f(a)|^2\right) > 0$

Then $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon''$ (by continuity of f at a) $\implies (1)|f(x)| > |f(a)| - \epsilon'' \geq |f(a)| - |f(a)|/2$ and $(2)|f(x) - f(a)| < \frac{\epsilon}{2}|f(a)|^2$. So

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| = \frac{1}{|f(x)||f(a)|}|f(x) - f(a)|$$
$$< \frac{1}{|f(a)|/2 \cdot |f(a)|} \cdot \frac{\epsilon}{2}|f(a)|^2 = \epsilon$$ ∎

Theorem 3.13

$f : \mathbb{R} \to \mathbb{R}$ is cts at $a \in \mathbb{R}$ iff $\forall$ sequences $x_n \to a$, $f(x_n) \to f(a)$

In one direction this is somewhat easy: if $x_n \to a$ and f is continuous at a, then $f(x_n)$ gets close to $f(a0$ as x_n gets close to $a \implies f(x_n) \to f(a)$.

The converse is *much harder*. If I want to see if f is continuous, I can test with a sequence $x_n \to a$ to see if $f(x_n)$ if close to $f(a)$ when n is large. But x_n's doesn't cover all x's! But if I use *all* sequences $x_n \to a$ then I do cover all x and get a theorem.

Proof. If f is cts at a, fix $\epsilon > 0$. $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. Now $x_n \to a$, so $\exists N \in \mathbb{N}$ such that $n \geq N \implies |x_n - a| < \delta \implies |f(x_n) - f(a)| < \epsilon$.

Suppose f is not cts at $a \in \mathbb{R}$ for contradiction.

Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in (a - \delta, a + \delta)$ such that $|f(x) - f(a)| \geq \epsilon$.

Choose $\delta = \frac{1}{n}$. $\exists x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ such that $|f(x_n) - f(a)| \geq \epsilon$.

So $|x_n - a| < \frac{1}{n}$ $\forall n \implies x_n \to a$. But $f(x_n) \not\Rightarrow f(a)$, $\times$. ∎

Example 3.14. $f(x) = \begin{cases} \sin 1/x & x \neq 0 \\ 0 = 0 \end{cases}$

This is *not* continuous at 0. But if we take $x_n \to 0$, then $f(x_n) = \sin(n\pi) = 0$ $\forall n$, so $f(x_n) \to f(0)$. so this sequence does not defect.

Have to choose a different sequence e.g. $x_n = \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \ldots$, gives $\sin \frac{1}{x_n} = (-1)^{n+1} \not\to f(0) \implies f$ discontinuous at 0.

To get this problem of sequences not covering the whole of an interval $(a - \delta, a + \delta)$ (so having to consider all sequences at once - nasty), we can let x run through all of $\mathbb{R}$ with the following definition:

Lecture 21

Definition. $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$.

We say that $f(x) \to b$ as $x \to a$ (or "$\lim_{x \to a} f(x) = b$") iff

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - b| < \epsilon$$

"x close to a (but not equal!!) $\implies f(x)$ close to b"

Example 3.15. $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$

Then $\lim_{x \to 0} f(x) = 0$

e.g. We can talk about $\lim_{x \to 0} f(x)$ for $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$.

Theorem 3.16

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff $f(x) \to f(a)$ as $x \to a$

Proof. f is continuous at $a \in \mathbb{R}$ says (1):

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Whereas $f(x) \to f(a)$ as $x \to a$ says (2):

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

So (1) $\implies$ (2).

Suppose (2). Then I get (1) except for when $|x - a| = 0$. But when $|x - a| = 0$, then $x = a$, so $f(x) = f(a)$, so $|f(x) - f(a)| < \epsilon$, so I still get (1). ∎

Can extend the definition of continuity to functions defined on subsets of $\mathbb{R}$ or $\mathbb{R}^n$ e.g.

Definition. $f : S \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^n$, is continuous at $\vec{a} \in S$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } (0 < |\vec{x} - \vec{a}| < \delta \text{ and } x \in S) \implies |f(\vec{x}) - f(\vec{a})| < \epsilon$$

Example 3.17. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$

This is discontinuous. But $f|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ is continuous.

Related to this is one-sided continuity:

Definition. $f : \mathbb{R} \to \mathbb{R}$ is *right continuous* at $a \in \mathbb{R}$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } x \in [a, a + \delta) \implies |f(x) - f(a)| < \epsilon$$

Exercise: f is right continuous at $a \in \mathbb{R} \iff f|_{[a,\infty)} : [a, \infty) \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$.

Exercise: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f$ is both right and left continuous at $a \in \mathbb{R}$

Definition. $f(x) \to b$ as $x \to a_+$ "as x tends to a from above" means

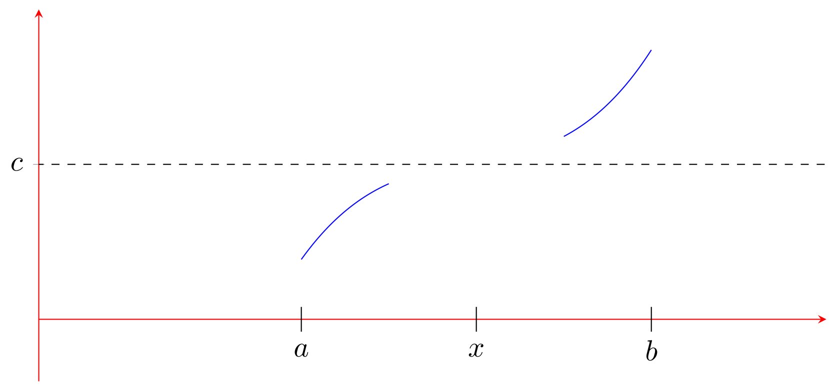
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } x \in (a, a + \delta) \implies |f(x) - f(a)| < \epsilon$$

Exercise: Just as before find that f is right continuous at $a \iff f(x) \to f(a)$ as $x \to a_+$

Intermediate Value Theorem

Theorem 3.18: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ cts, $c \in (f(a), f(b))$, then $\exists x \in [a, b]$ such that $f(x) = c$



If f is continuous it must cross the line $y = c$ at some point $x \in [a, b]$.

Corollary 3.19. *Any odd degree polynomial over* $\mathbb{R}$ *has a root* $\in \mathbb{R}$

Proof. w.l.o.g. $p(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$

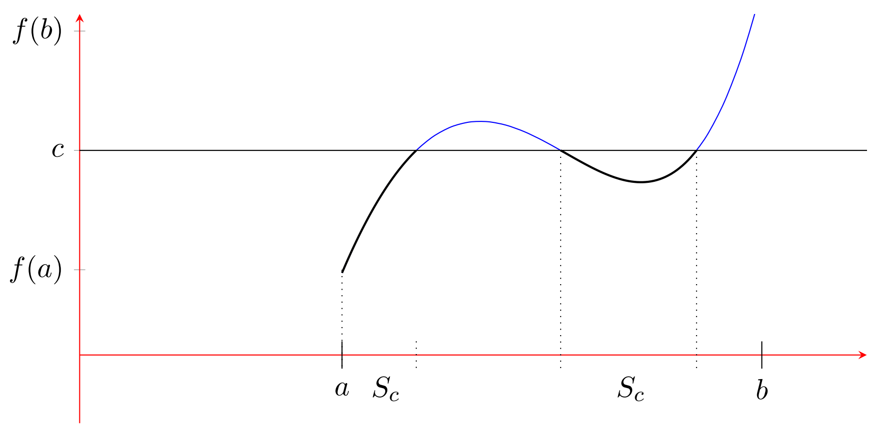
If we write this as $p(x) = x^{2n+1}(1 + \frac{a_{2n}}{x} + \cdots + \frac{a_0}{x^{2n+1}})$ then we see that $p(x) < 0$ for $x << 0$, and $p(x) > 0$ for $x >> 0$.

So we can find $a, b \in \mathbb{R}$ such that $p(a) < 0$, $p(b) > 0$.

So we apply IVT to $p|_{[a,b]} : [a, b] \to \mathbb{R}$ with $c = 0$ to find an $x \in [a, b]$ with $p(x) = c = 0$. ∎

We used the facts (proved in earlier lectures) that $mx + c$ is continuous and the product/sum of continuous functions are also continuous $\implies p(x)$ is continuous.

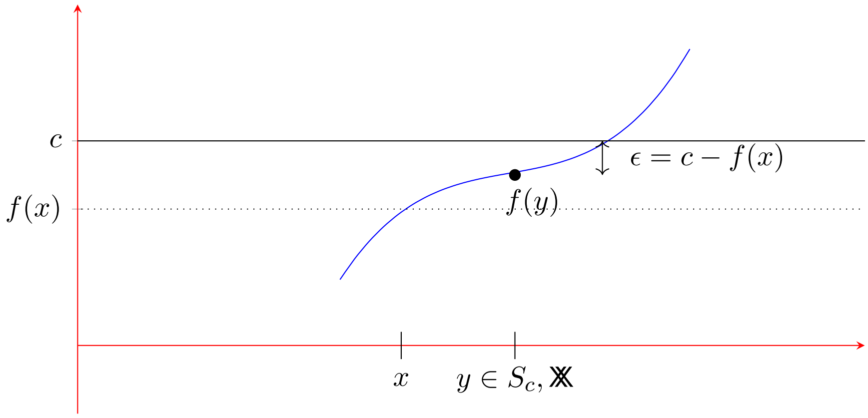
Lecture 22 *Proof of IVT.*



Consider $S_c = \{y \in [a, b] : f(y) \leq c\}$. Define $x := \sup S_c$ ($S_c \neq \emptyset$ since $a \in S_c$ and bounded above by b so sup exists)

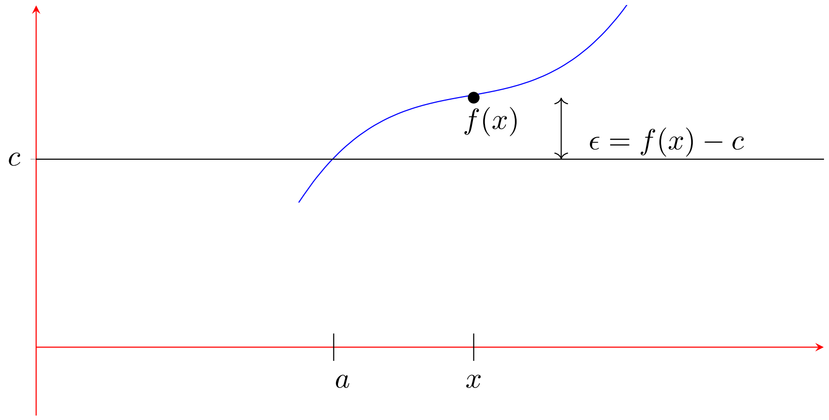
Claim: $f(x) = c$. *Proof:*

(i) Suppose $f(x) < c$.



Take $\epsilon = c - f(x) > 0$. f is cts at x, so $\exists \delta > 0$ such that $\forall y \in (x, x+\delta) \cap [a,b]$, $|f(y) - f(x)| < \epsilon$. Hence $f(y) < f(x) + \epsilon = c$. So $y \in S_c \implies x \neq \sup S_c$.

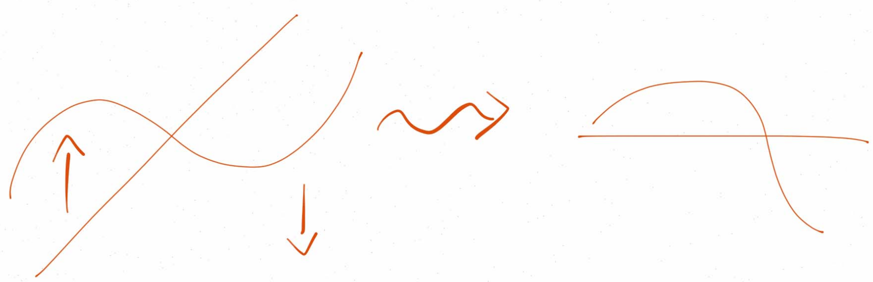
(ii) Suppose $f(x) > c$.



Take $\epsilon = f(x) - c > 0$. f is cts at x, so $\exists \delta > 0$ such that $\forall y \in (x-\delta, x) \cap [a,b]$, $|f(y) - f(x)| < \epsilon$. Hence $f(y) > f(x) - \epsilon = c \implies x - \delta$ is an upperbound for S_c, so $x \neq \sup S_c$. ∎

Proposition 3.20. *Suppose $f : [0,1] \to [0,1]$ is continuous. Then it has a fixed point (i.e. $\exists x \in [0,1]$ s.t. $f(x) = x$)*

Idea of proof: Rotate picture to make it look like IVT.



Proof. Set $g(x) = f(x) - x$, $g : [0,1] \to [0,1]$ is continuous.

So $g(0) = f(0) - 0 \geq 0$, $g(1) = f(1) - 1 \leq 0$

So by IVT $\exists x \in [0,1]$ s.t. $g(x) = 0 \iff f(x) = x$ ∎

So if during the lecture you watch last weeks lecture on Panopto, using pause, fast-forward, rewind, play (but no jumping!) then at some point you will be watching a time in the lecture which equals the time now. (No matter where you start or end.)

Definition. $S \subseteq \mathbb{R}^n$,$f : S \to \mathbb{R}$. Then we say that f is bounded above if $\exists M \in \mathbb{R}$ s.t. $f(\vec{x}) \leq M \ \forall \vec{x} \in S$.

Similar for bounded below, bounded is both.

Example 3.21. $f(x) = \frac{1}{x} : (0,1] \to \mathbb{R}$ is not bounded above

Proof. Suppose $\frac{1}{x} \leq M \ \forall x \in (0,1]$ (Then $M > 0!$).

Then take $x = \min\{\frac{1}{2m}, 1) \implies x \leq 1/2m \implies 1/x \geq 2M > M$, ※. ∎

Also $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f : [0,1] \to \mathbb{R}$ is also unbounded. Note that f is not continuous at 0!

So $\begin{cases} \text{discontinuous functions can be unbounded} \\ \text{continuous functions can be unbounded on non-closed intervals} \end{cases}$

But..

Theorem 3.22

$f : [a,b] \to \mathbb{R}$ cts $\implies$ f is bounded.

Ex: Give a function $f : [a,b] \cap \mathbb{Q} \to \mathbb{R}$ which is continuous and unbounded.

Lecture 23 *Proof.* Suppose not. Then $\forall N \in \mathbb{N}$, N is not an upperbound, so $\exists x_N \in [a,b]$ such that $|f(x_n)| > N$.

By BW Theorem, exists convergent subsequence, $y_i := x_{N(i)}, y_i \to y \in [a,b]$. With $|f(y_i)| = |f(x_{N(i)})| > N(i) \geq i$ $(*)$.

Fix $\epsilon = 1$, then

$$\exists \delta > 0 \text{ such that } \forall x \in (y - \delta, y + \delta) : |f(x) - f(y)| < 1 \implies |f(x)| < |f(y)| + 1.$$

Since $y_i \to y$,

$$\exists N \text{ such that } \forall n \geq N \ |y_n - y| < \delta \implies y_n \in (y - \delta, y + \delta) \implies |f(y_n)| < |f(y)| + 1.$$

By $(*)$, $n \leq |f(y_n)| < |f(y)| + 1 \ \forall n \geq N$, not true by the Archimedean Axiom ※. ∎

Slicker Proof. Suppose not. Then $\forall N \in \mathbb{N}$, N is not an unpperbound, so $\exists x_N \in [a, b]$ such that $|f(x_n)| > N$.

By BW Theorem, exists cvgt subsequence, $y_i := x_{N(i)}, y_i \to y \in [a, b]$. With $|f(y_i)| = |f(x_{N(i)})| > N(i) \geq i$ $(*)$. f is cts at $y \implies f(y_i) \to f(y)$, contradicting $(*)$. ∎

Extreme Value Theorem

Theorem 3.23: Extreme Value Theorem

$f : [a, b] \to \mathbb{R}$ cts $\implies f$ bounded and attains its bounds.

So max $f(x)$ exists (not just sup)

Proof. By boundedness theorem, $\exists \sup_{x \in [a,b]} f(x) = s$. Suppose for contradiction $\nexists c \in [a, b]$ such that $f(x) = s$.

2 proofs:

(1) Then $s - f(x) > 0$ $\forall x \in [a, b]$, so $g(x) = \frac{1}{s - f(x)} : [a, b] \to \mathbb{R}$ is well defined and cts. So $g(x)$ is bounded by $M > 0 \implies \frac{1}{s-f(x)} \leq M \implies f(x) \leq s - \frac{1}{M}$, so $s \neq \sup f(x)$, ⨳.

(2) From M1F $\exists$ a sequence $x_n \in [a, b]$ such that $f(x) \to \sup_{x \in [a,b]} f(x) = s$. BW Theorem $\implies$ exists subsequence $y_i := x_{N(i)}$ such that $y_i \to c \in [a, b]$. f is cts $\implies f(y_i) \to f(x)$. Since $f(y_i) \to s$, by uniqueness of limits, $f(c) = s$. ∎

Combining IVT + EVT we get

Theorem 3.24

$f : [a, b] \to \mathbb{R}$ is continuous then $\exists c, d \in [a, b]$ s.t. $\mathrm{im} f = f[a, b]$ is the interval $[f(c), f(d)]$.

Proof. EVT $\implies \exists c, d$ s.t. $f[a, b] \subseteq [f(c), f(d)]$ $(*)$

Given any $y \in [f(c), f(d)]$ the IVT $\implies \exists x$ between c and d s.t. $f(x) = y$, so $(*)$ is onto. ∎

Inverse Function Theorem

Proposition 3.25. *If $f : [a, b] \to \mathbb{R}$ is continuous and strictly increasing ($x > y \implies f(x) > f(y)$), then f is a bijection $[a, b] \to [f(a), f(b)]$*

Proof. $f(a)$ is a minimum of $f[a, b]$ because $x > a \implies f(x) > f(a)$. $f(b)$ is maximum. So by previous result $f[a, b] = [f(a), f(b)]$. We just need too show that f is injective:

If $x \neq y$, w.l.o.g. $x \neq y$ then $x < y \implies f(x) < f(y) \implies f(x) \neq f(y)$. So f is injective. ∎

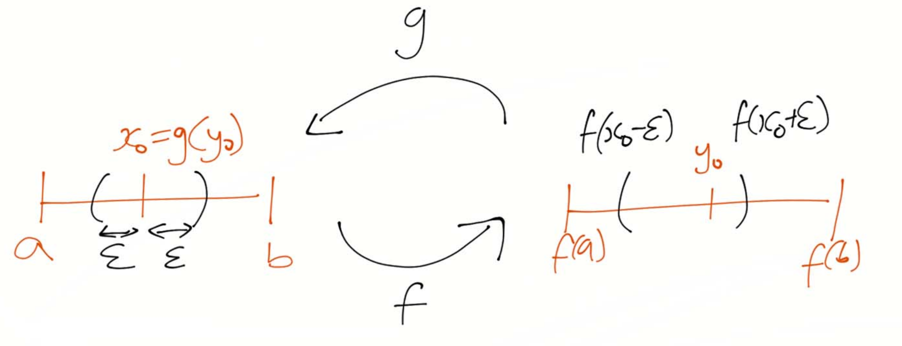
So $\exists$ inverse $g := f^{-1} : [f(a), f(b)] \to [a, b]$

Proposition 3.26. *g is continuous (and also strictly increasing - Ex!)*

Proof. Fix $\epsilon > 0$ and $y_0 \in [f(a), f(b)]$.

$$\begin{aligned}\text{Set } \delta :&= \min(f(g(y_0)+\epsilon) - f(g(y_0)), f(g(y_0)) - f(g(y_0)-\epsilon)) \\ &= \min(f(x_0+\epsilon) - y_0, y_0 - f(x-\epsilon)), \text{ where } x_0 = g(y_0)\end{aligned}$$

Picture:



In this definition we use the convention that if $x_0 - \epsilon < a$ then by $f(x_)) - \epsilon$ I mean $f(a)$ if $x_0 + \epsilon > b$ then $f(x_0 + \epsilon)$ means $f(b)$.

(Equivalently I've extended f to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(x) = \begin{cases} f(a) & x \le a \\ f(x) & x \in [a,b] \\ f(b) & x \ge b \end{cases}$)

So δ was chosen s.t. $(y_0 - \delta, y_0 + \delta) \subseteq (f(x_0 - \epsilon), f(x + \epsilon))$, so $y \in (y_0 - \delta, y_0 + \delta) \cap [a, b]$ then $f(x_0 - \epsilon) < y < f(x_0 + \epsilon)$

Apply $g \implies x_0 - \epsilon < g(y) < x_0 + \epsilon$. Recall $x_0 = g(y_0) \implies |g(y) - g(y_0)| < \epsilon$. ∎

Corollary 3.27. $\sqrt{x} : [0, \infty) \to [0, \infty)$, $x^{1/n} : [0, \infty) \to [0, \infty)$, $n \in \mathbb{N}$ *are continuous.*

Simpler exposition: Fix $f : \mathbb{R} \to \mathbb{R}$ bijective and continuous. Before we prove f^{-1} is continuous we prove

Lecture 24 **Lemma 3.28.** *$f : \mathbb{R} \to \mathbb{R}$ is bijective and cts $\implies$ f is strictly monotonic*

Proof. We prove this on any closed bounded interval $[a, b]$ (Hence monotonic on $\mathbb{R}$! Ex!)

f is bijective, so $f(a) \neq f(b)$, w.l.o.g. $f(b) > f(a)$. Suppose for contradiction $\exists c \in (a, b)$ such that $f(c) \notin (f(a), f(b))$.

w.l.o.g. take $f(c) > f(b)$. Then fix $d \in (f(b), f(c))$. By IVT applied to:

- $f|_{[a,c]}$, we find $\exists x \in (a, c)$ such that $f(x) = d$.
- $f|_{[c,b]}$, we find $\exists y \in (c, b)$ such that $f(y) = d$.

But $y > x \implies x \neq y$, so f is not injective ⨳.

So $\forall c \le b$, we find that $f(c) \le f(b)$, and f injective $\implies f(c) < f(b)$. ∎

Theorem 3.29

$f : \mathbb{R} \to \mathbb{R}$ bijective and cts $\implies f^{-1} : \mathbb{R} \to \mathbb{R}$ cts.

Proof. By Lemma f is strictly monotonic, w.l.o.g. strictly increasing.

We want to show f^{-1} is continuous at $y \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be $f^{-1}(y_0)$, so $f(x_0) = y_0$.

Fix $\epsilon > 0$.

Let $\delta := \min\{f(x_0 + \epsilon) - y_0, y_0 - f(x_0 - \epsilon)\}$.

Then $|y - y_0| < \delta \implies y \in (y_0 - \delta, y_0 + \delta) \subseteq (f(x_0 - \epsilon), f(x_0 + \epsilon))$.

Applying f^{-1} preserves order

$$\implies f^{-1}(y) \in (x_0 - \epsilon, x_0 + \epsilon) \iff |f^{-1}(y) - f^{-1}(y_0)| < \epsilon. \qquad \blacksquare$$

Corollary 3.30. $E : \mathbb{R} \to \mathbb{R}$, $E(x) : \sum \frac{x^n}{n!}$ *is a continuous bijection* $\mathbb{R} \to (0, \infty)$ *with continuous inverse* $\log : (0, \infty) \to \mathbb{R}$.

We already showed that E is continuous, never takes the value 0 ($E(-x) = E(x)^{-1}$) is unboundedly poisitive for $x \geq 0$ ($E(x) \geq 1+x$) and positive for $x < 0$ ($E(-x) = E(x)^{-1}$). So by IVT it takes *every* value in $(0, \infty)$ (Ex!).

We also showed it is strictly monotonically increasing ($E(y) = E(y - x)E(x) > E(x)$ for $y > x$). So by previous result it's a bijection to $(0, \infty)$ with a continuous inverse.

Lecture 25

Theorem 3.31

$f : \mathbb{R}^n \to \mathbb{R}^m$ is cts at $\mathbf{a} = (a_1, \ldots, a_n)$ if and only if $f_i : \mathbb{R}^n \to \mathbb{R}$ is cts at a_i $\forall i$. (With $f = (f_1, \ldots, f_m)$).

(i.e. f_i is $\pi_i \circ f$ where $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is the projection to the ith coordinate $\pi_i(x_1, \ldots, x_m) = x_i$.)

Proof. Easy way is $\implies$:

Highbrow: $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is continuous, so $\pi_i \circ f - f_i$ is continuous.

First Principles: Fix $\epsilon > 0$. Then f is cts at $\vec{a} \implies \exists \delta > 0$ such that $|\vec{x} - \vec{a}| < \delta \implies |f(\vec{x}) - f(\vec{a})| < \epsilon$ $(*)$. But this implies $|f_i(\vec{x}) - f_i(\vec{a})| < \epsilon$ because

$$\begin{aligned} |f(\vec{x}) - f(\vec{a})| &= \sqrt{\sum_{j=1}^{m} (f_j(\vec{x}) - f_j(\vec{a}))^2} \\ &\geq \sqrt{(f_i(\vec{x}) - f_i(\vec{a}))^2} \\ &= |f_i(\vec{x}) - f_i(\vec{a})| \end{aligned}$$

Proof of $\impliedby$:

Suppose f_i cts at a_i $\forall i$. Fix $\epsilon > 0$. Then $\exists \delta_i > 0$ such that

$$|\vec{x} - \vec{a}| < \delta_i \implies |f_i(\vec{x}) - f_i(\vec{a})| < \epsilon$$

Set $\delta = \min\{(\delta_i\} > 0$, so that

$$
\begin{aligned}
|\vec{x}-\vec{a}| < \delta &\implies |f_i(\vec{x}) - f_i(\vec{a})| < \epsilon \ \forall i \\
\implies |f(\vec{x}) - f(\vec{a})| &= \sqrt{\sum_{i=1}^{m} (f_i(\vec{x}) - f_i(\vec{a}))^2} \\
&\leq \sqrt{\sum_{i=1}^{m} \epsilon^2} = \sqrt{m}.\epsilon
\end{aligned}
$$

■

So we can study the continuity of $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of their coordinates f_i in $\mathbb{R}^m$. But *not* in terms of the restoration of f to coordinate axises in $\mathbb{R}^n$.

Example 3.32. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

On any horizontal line $y = c$ it results to the function

$$
\begin{aligned}
f(x,c) = \frac{cx}{c^2+x^2} &\quad \text{if } c \neq 0 \\
\text{or } f(x,0) \equiv 0 \ \forall x &\quad \text{if } c = 0
\end{aligned}
$$

Both are continuous functions $\mathbb{R} \to \mathbb{R}$.

Similarly on any vertical line $x = c$, f restricts to a continuous function:

$$
\begin{aligned}
f(c,y) = \frac{cy}{c^2+y^2} &\quad \text{if } c \neq 0 \\
\text{or } f(0,y) \equiv 0 \ \forall y &\quad \text{if } c = 0
\end{aligned}
$$

But f is *not* continuous at $(0,0)$

Idea: on line $y = x$, f is $\begin{cases} \dfrac{x.x}{x^2+x^2} = \frac{1}{2} & \forall x \neq 0 \\ 0 & x = 0 \end{cases}$

Pick $\epsilon = \frac{1}{2}$. Then for any $\delta > 0$, take $x = \frac{\delta}{2}$ so that $(x,x) \in B_\epsilon(0,0)$. But $f(x,x) = \frac{1}{2} \notin B_\epsilon(f(0,0)) = B_\epsilon(0)$. So f is not continuous at $(0,0)$. ■

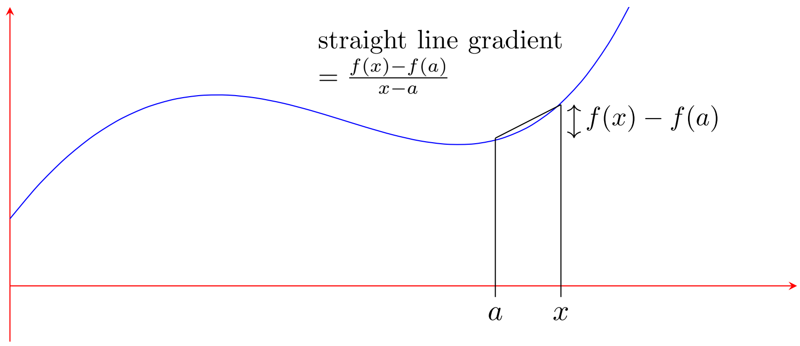
Exercise: Converse is true: if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous then f is continuous on restriction to any line in $\mathbb{R}^n$; more generally $f|_S : S \to \mathbb{R}^m$ is continuous $\forall S \subseteq \mathbb{R}^n$

4 Differentiation

Differentiability

Definition. f is *differentiable* at a iff $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$, i.e.

$$\forall\epsilon > 0,\ \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \implies \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon.$$



Example 4.1. $f(x) = x^2$ is differentiable at all $a \in \mathbb{R}$ with $f'(a) = 2a$

Proof. Fix $a \in \mathbb{R}$

$$\frac{f(x)-f(a)}{x-a} = \frac{x^2-a^2}{x-a} = x+a$$

$$\implies \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists and equals } 2a$$

∎

or from first principles:

$$\left|\frac{f(x)-f(a)}{x-a} - 2a\right| = |x+a-2a| = |x-a|$$

So fixing $\epsilon > 0$, take $\delta = \epsilon$ so that $|x-a| < \delta \implies \left|\frac{f(x)-f(a)}{x-a} - 2a\right| < \epsilon$ ∎

Exercise: $f(x) = x^3$, $f(x) = |x|$

Proposition 4.2. *If f is differentiable at $a \in \mathbb{R}$ then $f$is continuous at a* *Lecture 26*

Proof. If f is differentiable at a then

$$\forall\epsilon > 0\ \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \implies \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon$$
$$\implies |f(x)-f(a)| < |x-a|(|f'(a)|+\epsilon).$$

Fix $\epsilon > 0$, set $\delta = \epsilon$. Then

$$0 < |x-a| < \delta \implies |f(x)-f(a)| < \epsilon(|f'(a)|+\epsilon) = k\epsilon$$

(also true for $x = a \implies |f(x)-f(a)| = 0$.) ∎

Highbrow Proof. Note that $f(x) = f(a) + (x-a)\frac{f(x)-f(a)}{x-a}$, $x \neq a$. Taking $\lim_{x\to a}$

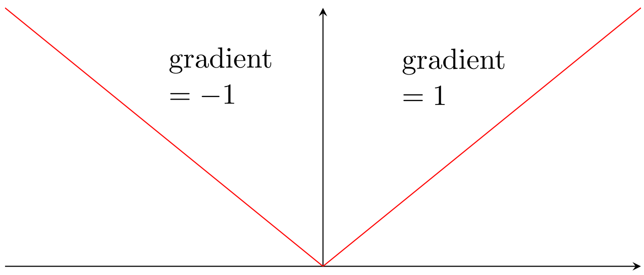
$$\lim_{x\to a} f(x) = f(a) + 0.f'(a) \implies f \text{ cts at } a$$ ∎

The converse is *not* true.

Example 4.3. $f(x) = |x|$ is continuous at $x = 0$ but not diff'ble at $x = 0$ since

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

So $\lim_{x\to o} \frac{f(x)-f(0)}{x-0}$ does not exist (Ex)



So left and right derivates do exist, they're just not equal.

Definition. Left derivative of f at a is $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ iff it exists. Right derivative is $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$.

$\lim_{x\to a^-} g(x)$ exists and equals $\lim_{x\to a^+} g(x) \iff \lim_{x\to a} g(x)$ exists.

So f is differentiable at a iff the left and right derivatives of f exist at a and are equal.

Anything else you might guess is also false: e.g. "if f is differentiable everywhere then is f' continuous?" No!

Theorem 4.4: Product Rule

$f, g : \mathbb{R} \to \mathbb{R}$ differentiable at $a \in \mathbb{R}$. Then fg is differentiable at a with $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

Proof.

$$\begin{aligned} \frac{f(x)g(x) - f(a)g(a)}{x-a} &= \frac{(f(x)-f(a))g(x) + (g(x)-g(a))f(a)}{x-a} \\ &= g(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{g(x)-g(a)}{x-a} \end{aligned}$$

Taking $\lim_{x\to a} \implies (fg)'(a) = g(a)f'(a) + f(a)g'(a)$ by cty of g and algebra of limits. ∎

Corollary 4.5. *$f(x) = x^k$ has $f'(x) = kx^{k-1}$*

Proof. Induction! ∎

Then $g(x) := 1/f(x)$ is defined in a neighbourhood of a, and it is differentiable with $g'(a) = \dfrac{f'(a)}{f^2(a)}$

Proof. See old question sheet.

f is continuous at $a \implies \exists \delta > 0$ s.t. $\forall x \in (a-\delta, a+\delta),\ |f(x)| > \frac{|f(a)|}{2}$. So g is defined on $(a-\delta, a+\delta)$.

Working on this and $(a-\delta, a+\delta) \ni x$ we calculate

$$\begin{aligned}\frac{g(x)-g(a)}{x-a} &= \frac{1/f(x) - 1/f(a)}{x-a} \\ &= \frac{f(a)-f(x)}{(x-a)f(a)f(x)} \\ &\to -f'(a) \cdot \frac{1}{f(a)f(a)} \quad \text{as } x \to a\end{aligned}$$

∎

Example 4.6. $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x \in \mathbb{R}$

If we could differentiate term by term we would conclude that

$$E'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (k = n-1)$$

So Mestel *guesses* that $E' = E$

Claim: $E'(0) = 1$

Proof.

$$\begin{aligned}\frac{E(x)-E(0)}{x-0} &= \frac{\sum \frac{x^n}{n!}}{x} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} \quad (k = n+1)\end{aligned}$$

Now by the comparison test

$$\sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} \le \sum_{k=1}^{\infty} |x^k| = \frac{|x|}{1-|x|} \to 0$$

So $\lim_{x \to 0} \dfrac{E(x)-E(0)}{x-0}$ exists and equals 1. ∎

So now we have

Proposition 4.7. *E is differentiable everywhere with $E' = E$*

Proof.

$$\begin{aligned}\frac{E(x)-E(a)}{x-a} &= E(a)\cdot\frac{E(x-a)-E(a)}{x-a}\\ &\to E(a)E'(0)\\ &= E(a)\end{aligned}$$

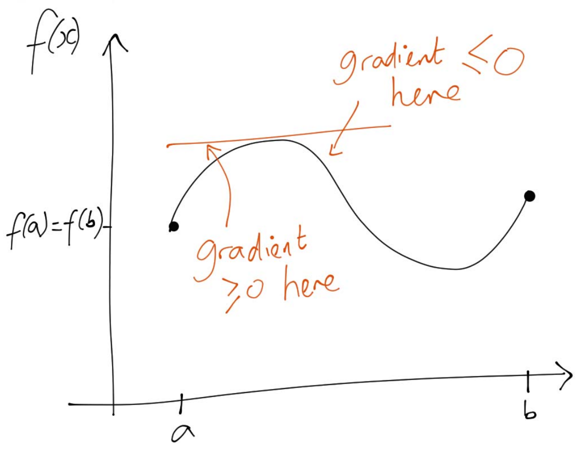
■

Rolle's Theorem

Theorem 4.8: Rolle's Theorem

Lecture 27

$f : [a,b] \to \mathbb{R}$ cts on $[a,b]$, differentiable on (a,b) such that $f(a) = f(b)$. Then $\exists c \in (a,b)$ such that $f'(c) = 0$.



Proof.

Case 1. f is constant on $[a,b]$. Then set $c = \frac{a+b}{2}$, so $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = 0$.

Case 2. f takes values $< f(a)$. Then replace f by $-f$ and consider Case 3.

Case 3. f takes values $> f(a)$. Therefore $\sup\{f(x) : x \in [a,b]\} > f(a)$ by EVT is realised by some $c \in (a,b)$. Now $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. Consider

$$x > c,\ f(x) \le f(c) \implies \frac{f(x)-f(c)}{x-c} \le 0 \implies \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} \le 0$$

$$x < c,\ f(x) \le f(c) \implies \frac{f(x)-f(c)}{x-c} \ge 0 \implies \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} \ge 0$$

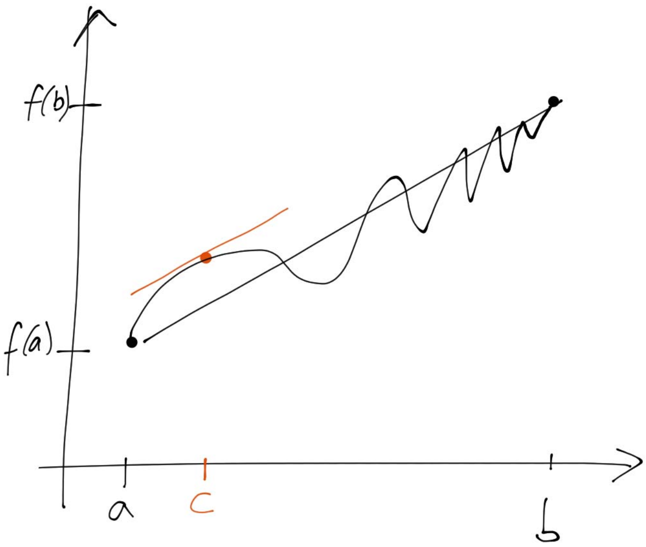
Hence $\frac{f(x)-f(c)}{x-c} = 0$.

■

Mean Value Theorem

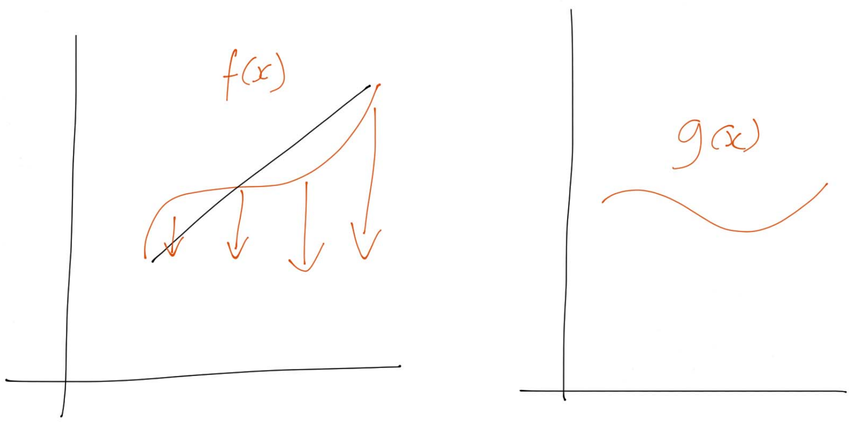
Theorem 4.9: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is cts on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Note: we can write this as $f(b) = f(a) + (b - a)f'(c)$, $c \in (a, b)$. Compare this to Taylor's Theorem - we're taking just the first 2 terms of.

Idea of Proof: Turn MVT into Rolle.



Proof. Let $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$, which is cts on $[a, b]$ and diff'ble on (a, b). $g(a) = f(a) = g(b)$. By Rolle's Theorem applied to g

$$\exists c \in (a, b) \text{ such that } g'(c) = 0 \implies g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

∎

Corollary 4.10. *If $f'(x) = 0\ \forall x \in (a,b)$. Then f is a constant: $f(x) = f(a)\ \forall x \in [a,b]$*

Proof. Suppose for a contradiction that $\exists d \in [a,b]$ s.t. $f(d) \neq f(a)$. Then by MVT applied to $f|_{[a,d]} : [a,d] \to \mathbb{R}$, $\exists c \in (a,d)$ s.t. $f'(c) = \dfrac{f(d)-f(a)}{d-a} \neq 0$, ※ ∎

Theorem 4.11: Chain Rule

Lecture 28

$g : \mathbb{R} \to \mathbb{R}$ diff'ble at $a \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ diff'ble at $g(a) \in \mathbb{R}$, then $f \circ g$ diff'ble at a with $(f \circ g)'(a) = f'(g(a))g'(a)$

i.e.

$$\frac{d}{dx} f(g(x))\bigg|_{x=a} = \frac{df}{dx}(g(a))\frac{dg}{dx}(a) = \frac{df}{dy}\bigg|_{y=g(a)} \frac{dg}{dx}(a) \text{“} = \text{”} \frac{df}{dg}\frac{dg}{dx}$$

Idea of proof:

$$\frac{f(g(x)) - f(g(a))}{x-a} = \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a} \to f'(g(a)) \cdot g'(a)$$

problem with this is that $g(x) - g(a)$ might be zero

$\left(\dfrac{h(x)-h(a)}{x-a}\right.$ is not defined at $x = a$, so define it to be $h'(a)$ at $x = a\left.\right)$

Proof. Define $F(g) = \begin{cases} \dfrac{f(y)-f(b)}{g-b} & y \neq b \\ f'(g) & y = b \end{cases}$ (‡) where $b = g(a)$.

f is diff'ble at $b \implies \lim_{y\to b} F(y) \to f'(b) = F(b)$ as $y \to b$. So F is cts at $b = g(a)$ $(*)$.

g is diff'ble at $a \implies$ cts at a.

By $(*) \implies F \circ g$ is cts at $a \implies F(g(x)) \to F(g(a)) = f'(b)$ as $x \to a$ $(**)$.

So now we can follow the rough proof to write

$$\frac{f(g(x)) - f(g(a))}{x} = F(g(x))\frac{g(x)-g(a)}{x-a}$$

Now take $\lim_{x\to a}$ to get $(f \circ g)'(a)$ exists and equals $f'(b)g'(a)$ by $(**)$ ∎

Ex: “Sum Rule” f, g are differentiable at $a \implies f + g$ are differentiable at a with $(f+g)'(a) = f'(a) + g'(a)$. Pre-ex: Algebra of limit for $\lim_{x\to a}$ is on Question Sheet.

Rough: $f : \mathbb{R} \to \mathbb{R}$ is differentiable and bijective, $g = f^{-1} : \mathbb{R} \to \mathbb{R}$.

Suppose g is differentiable. Then by the chain rule $f \circ g(y) = y \implies f'(g(y_0))g'(y_0) = 1\ \forall y_0 \implies g'(y) = \frac{1}{f'(g(y))}$.

Suggests that if $f' \neq 0$, then g is differentiable with derivative $\frac{1}{f' \circ g}$

Theorem 4.12

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ with $f'(a) \neq 0$ and f is bijective with inverse $g = f^{-1}$, then g is differentiable at $b = f(a)$ with $g'(b) = \dfrac{1}{f'(g(b))} = \dfrac{1}{f'(a)}$.

Proof. Lemma: $f'(a) \neq 0 \implies \exists \delta > 0$ such that $f(x) \neq f(a)$ for $x \in (a-\delta, a+\delta)\backslash\{0\}$. (Proof is left as exercise - use $\lim_{x\to a}$ definition of f' and MVT)

So $\dfrac{g(y)-g(b)}{y-b} = \dfrac{x-a}{f(x)-f(a)} = 1/\dfrac{f(x)-f(a)}{x-a}$ where $x = g(y),\ y \neq b$.

As $y \to b$, $g(y) \to g(b) = a$ since f differentiable at $a \implies f$ cts at $a \implies g$ cts at $b \implies x \to a \implies$ RHS $\to \dfrac{1}{f'(a)}$. ∎

Felina. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

- $f(x) + f(y) = f(x+y)\ \forall x, y \in \mathbb{R}$
- f is continuous everywhere

What if f?

Observe $y = 0 : f(x) + f(0) = f(x),\ \forall x$, so $f(0) = 0$.

For $y = 1 : f(x) + f(1) = f(x+1)$

$$\begin{aligned} \text{Induction } f(x+2) &= f(x) + f(1) + f(1) \\ f(x+3) &= f(x) + 3f(1) \\ &\vdots \\ f(x+n) &= f(x) + nf(1) \\ \implies f(n) &= nf(1)\ (*) \end{aligned}$$

Similar mucking about should convince you that $f(x) = xf(1)$. We've proved that for $x \in \mathbb{N}$ by $(*)$. $f(1)$ is an unknown constant c. [Notice $f(x) = cx$ indeed satisfies the given assumptions]

Notice that $(*)$ holds for $n \in \mathbb{Z}$ too Lecture 29

$$f(-n) + f(n) = f(n-n) = f(0) = 0$$

$$\implies f(-n) = -f(n) = -nf(1) = -nc, \quad n \in \mathbb{N}$$

$(*)$ also holds for $\mathbb{Q}$

$$\underbrace{f(\tfrac{n}{m}) + \cdots + f(\tfrac{n}{m})}_{m \text{ copies}} = f(\tfrac{n}{m} + \cdots + \tfrac{n}{m}) = f(n) = cn$$

$$\implies f(\tfrac{n}{m}) = c\tfrac{n}{m} \quad \forall \tfrac{n}{m} \in \mathbb{Q},\ n, n \in \mathbb{Z}$$

Claim: $f(x) = cx$ $[c = f(1)]$ $\forall x \in \mathbb{Q}$

Idea: now is if $x \in \mathbb{R}$ then x is close to $y \in \mathbb{Q}$. f is continuous $\implies f(x)$ is close to $f(y) = cy$, close to cx. So $f(x)$ is arbitrarily ($\forall\epsilon$!) close to $cx \implies f(x) = cx$

(or we could use some machinery to say $\forall x \in \mathbb{R}$, $\exists (y_n) \to x$, $y_n \in \mathbb{Q}$. Then f is continuous $\implies f(y_n) = cy_n \to f(x)$ and $cy_n \to cx$. So uniqueness of limits $\implies f(x) = cx$.)

Proof. Fix $x \in \mathbb{R}$. Fix $\epsilon > 0$. M1F: $\exists y \in \mathbb{Q}$ s.t. $|y - x| < \epsilon \implies |cy - cx| < \epsilon/2$

$$\exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon/2$$

and by M1F again $\exists y \in \mathbb{Q}$ s.t. $|y - x| < \min\{\delta, \epsilon/2\}$.

So $|cy - cx| < \epsilon/2$ and $|f(x) - \underset{\substack{\| \\ cy}}{f(y)}| < \epsilon/2 \implies |f(x) - cx| < 2\epsilon/2 = \epsilon$.

This is true $\forall \epsilon > 0 \implies |f(x) - cx| = 0$ ∎

- End of Analysis I -