1st Year Mathematics
Imperial College London

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Mechanics

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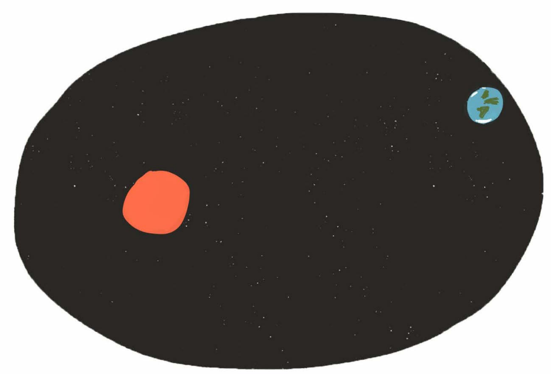


Comments and corrections should be sent to `kb514@ic.ac.uk`

Syllabus

This introductory course on Applied Mathematics is centred on Newtonian mechanics - the consequences of Newtons laws. Some of the course overlaps with A-level Applied Mathematics. It includes far-reaching ideas on energy, linear and angular momentum, simple oscillatory systems and motion under central forces such as planetary motion.

- **Kinematics of point particles:** Vectors and vector algebra; position, velocity, and acceleration in three dimensions; polar coordinates; intrinsic coordinates and path curvature.
- **Kinetics and Newton's laws:** Definition of mass, momentum, inertia, and force; Axioms, or Laws of Motion
- **Forces:** Gravitation; forces that constrain motion: normal force and tension; friction; forces that depend on velocity: drag forces; forces that depend on position: spring forces.
- **Oscillators:** Simple, damped, and forced oscillators; amplitude and phase difference; resonance.
- **Energy:** Kinetic and potential energies; conservative forces; stability of and motion about fixed points; potential wells and escape; energy diagrams.
- **Angular momentum:** Central forces; orbital equation; effective potential.
- **Systems of (interacting) particles:** Two body systems; centre of mass; moment of inertia; total momentum, angular momentum, and energy for systems; variable mass systems; torque;
- **Rigid body motion:** Rigid body kinematics; continuous mass distributions; rigid body dynamics with rotation about a single axis

Appropriate books

D. Kleppner and R. J. Kolenkow *An Introduction to Mechanics.*

G. R. Fowles and G. L. Cassiday *Analytical Mechanics.*

R. Feynman *The Feynman Lectures.*

T. W. B. Kibble and F. H. Berkshire *Classical Mechanics.*

Contents

1 Kinematics

Lecture 1

What we are after is an *equation of motion* to find the position of an object for all times. Ingredients to an equation of motion:

1) Kinematics - Description of motion
2) Kinetics - Newton's laws
3) Mathematical Description of Forces - Describe forces in terms of kinematic quantities

Cartesian Coordinates

For a point particle there are three key kinematic quantities.

1. Positon: $\vec{r}(t)$
2. Velocity: $\vec{v}(t)$
3. Acceleration: $\vec{a}(t)$

   In general, $\vec{r}(t),\ \vec{v}(t),\ \vec{a}(t) \in \mathbb{R}^3$.

We can use different coordinate systems to describe our quantities:

(i) Cartesian

(ii) Polar

(iii) Intrinsic

Consider the path of a particle through space:

Definition. We write the *position* at time t as

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

We can also write this as

$$[\vec{r}(t)] = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \text{ so, } \hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \hat{\jmath} = \begin{bmatrix} 0 \\ j \\ 0 \end{bmatrix},\ \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Magnitude of $\vec{r}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

r is the distance from the origin.

Direction of $\vec{r}$

$$\hat{r} = \vec{r}/r = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}$$

So, we can write

$$\vec{r} = r(t)\hat{r}(t)$$

This is the starting point for polar coordinates.

Lecture 2

Last Time:

Position:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

At Δt later

$$\vec{r}(t+\Delta t) = x(t+\Delta t)\hat{\imath} + y(t+\Delta t)\hat{\jmath} + z(t+\Delta t)\hat{k}$$

Definition. Define $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

Define the *velocity* of the particle at time t

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

Since $\hat{\imath}, \hat{\jmath}, \hat{k}$ are constant in time

$$\begin{aligned}\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k}\end{aligned}$$

Writing $\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \dot{f}$,

$$\vec{v}(t) = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

Definition.

$$v = |\vec{v}| = [v_x^2 + v_y^2 + v_z^2]^{1/2}$$

is the magnitude of the velocity or *speed* of the particle.

Thus, the *direction* of motion is

$$\hat{v} = \vec{v}/v, \ \ |\hat{v}| = 1$$

$\hat{v}$ is also the unit tangent to the path.

Define the *acceleration*

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$$

$\vec{a}(t)$ tells us how the velocity is changing at time t.

Recall that we can write $\vec{v} = v(t)\hat{v}(t)$, then

$$\vec{a} = \frac{d}{dt}(v\hat{v}) = \frac{dv}{dt}\hat{v} + v\frac{d\hat{v}}{dt}$$

Question from the audience

$$\vec{r}(t) = r(t)\vec{r}(t)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\vec{r}) = \frac{dr}{dt}\vec{r} + r\frac{d\hat{r}}{dt}$$

What we have done: started with $\vec{r}$ and we differentiated w.r.t t to find $\vec{v}$ and $\vec{a}$.

Starting with $\vec{a}$ we can integrate to find $\vec{v}$, then $\vec{r}$.

$$\int_{t_0}^{t} \vec{a}dt' = \int_{t_0}^{t} \frac{d\vec{v}}{dt'}\,dt' = \vec{v}(t) = \vec{v}(t_0)$$

$$\implies \vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}\,dt'$$

We can integrate this component. For example

$$v_x(t) = v_x(t_0)\int_{t_0}^{t} a_x\,dt'$$

To determine $\vec{v}(t)$ uniquely, we need to know $\vec{v}(t_0)$ (constant vector).

Similarly

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(t')\,dt'$$

Thus, starting with $\vec{a}(t)$, we need to know *both* $\vec{r}(t_0)$ and $\vec{v}(t_0)$ to find $\vec{r}(t)$ uniquely!

What allows us to connect the mathematics to the physical world is that these quantities are measurable.

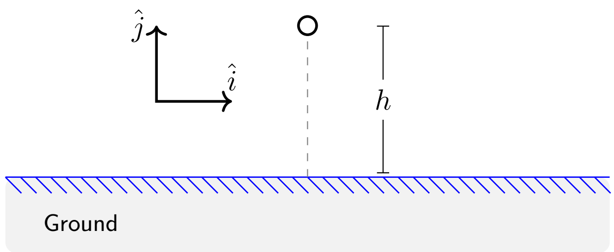
Definition (SI Units)**.**

- *Time:* Measured in seconds, s
- *Distance:* Measured in metres, m
- *Velocity:* "Metres per second", m/s or ms^{-1}
- *Accelerations*: "Metres per second squared"", m/s^2 or ms^{-2}

Example 1.1. Near the surface of the earth, the acceleration due to gravity is constant!

$$g = 9.8ms^{-2}$$

Suppose: an object is dropped *from rest* at height h above the ground. Find $\vec{r}(t)$:



Align the coordinates such that $\hat{j}$ points upwards. We know that $\vec{a} = -g\hat{j}$. No motion in other directions. Problem is 1D!.

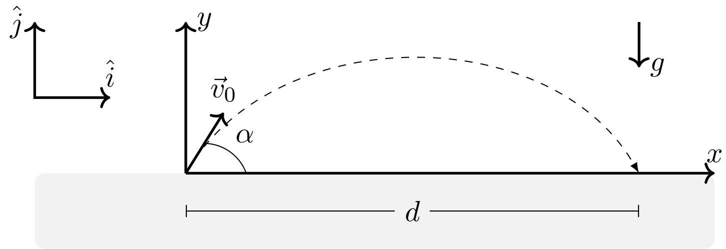
We know initially ($t = 0$) that $y(0) = h \implies v_y(0) = 0$. Integrate twice:

$$y(t) = h - \frac{1}{2}gt^2$$

Lecture 3

Recap:

Example 1.2 (Projectile).



In this coordinate system $\vec{a} = -g\hat{j}$. Choose that $t = 0$ when the object is released. Based on this:

$$\vec{r}(0) = \vec{0},\ \vec{v}(0) = \vec{v}_0 = v_0 \cos\alpha\hat{i} + v_0 \sin\alpha\hat{j}$$

Integrate our acceleration to find $\vec{v}(t)$

$$\begin{aligned}\vec{v}(t) &= \vec{v}(0) + \int_0^t -g\hat{j}\,dt' = \vec{v}_0 - gt\hat{j} \\ &= v_0\cos\alpha\hat{i} + (v_0\sin\alpha - gt)\hat{j}\end{aligned}$$

Integrate our velocity to find the position

$$\begin{aligned}\vec{r}(t) &= \vec{r}(0) + \int_0^t \vec{v}(t)\,dt' = \vec{0} + \int_0^t \vec{v}_0 - gt'\hat{j}\,dt' \\ &= \vec{v}_0 t - \tfrac{1}{2}t^2 g\hat{j} = v_0 t\cos\alpha\hat{i} + [v_0 t\sin\alpha - \tfrac{1}{2}t^2 g]\hat{j}\end{aligned}$$

Maximise the *range* using α as the control parameter. Finding the time t_H when the object hits the ground. $y(t_H) = 0$ where $y = \vec{r}\cdot\hat{j}$.

$$y = \vec{r}\cdot\hat{j} = v_0 t_H \sin\alpha - \tfrac{1}{2}t_H^2 g = 0$$

Two values of t_H:

$$t_H = 0,\ t_H = \frac{2v_0\sin\alpha}{g}$$

To find the range:

$$x(t_H) = v_0\cos\alpha\left[\frac{2v_0\sin\alpha}{g}\right] = \frac{v_0^2}{g}\sin 2\alpha$$

For $0 \leq \alpha \leq \pi/2$, the range is maximum for $\alpha = \pi/4$.

Example 1.3 (Circular Motion).

$$\vec{r}(t) = R\sin(\Omega t)\hat{\imath} + R\cos(\Omega t)\hat{\jmath}$$

where R, Ω are positive constraints.

Distance from the origin

$$r = |\vec{r}| = [R^2\sin^2\Omega t + R^2\cos^2\Omega t]^{1/2} = R$$

Path is a circle of radius R, centred at the origin.

Differentiate $\vec{r}(t)$ to find $\vec{v}(t) = R\Omega\cos\Omega t\hat{\imath} - R\Omega\sin\Omega t\hat{\jmath}$

Find the speed: $v = |\vec{v}| = R\Omega$, the speed is constant.

Clockwise or anticlockwise?

Direction of motion

$$\hat{v} = \vec{v}/v = \cos\Omega t\hat{\imath} - \sin\Omega t\hat{\jmath}$$

At $t = 0$, $\hat{v}(0) = \hat{\imath}$, so it moves clockwise around the circle!

Interpretation of Ω:

Introduce $\theta = -\Omega t + \pi/2$, $\frac{d\theta}{dt} - \Omega$.

The parameter Ω is angular speed. Differentiate our $\vec{v}(t)$ we find

$$\begin{aligned}\vec{a}(t) &= -R\Omega^2\sin[\Omega t]\hat{\imath} - R\Omega^2\cos[\Omega t]\hat{\jmath} \\ &= -\Omega^2\vec{r}\end{aligned}$$

Acceleration is pointing in towards the circle.

Lecture 4

Vector Operations

Lecture 5

Already we have seen vector addition and subtraction are useful:

Addition: $\vec{r} = \vec{r}_C + \vec{r}_\Omega$

Subtraction: velociites relative to a moving observer $\vec{v}_{A,O} = \vec{v}_A - \vec{v}_O$

Vector Products: Vector products are also useful and do arise in describing mechanical phenomena:

(i) Scalar product (dot product)

(ii)

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

Polar Coordinates

Intrinsic Coordinates

Lecture 7

Coordinates that are intrinsic to the path of our particle. We know the path!

Distance travelled between t and $t + \Delta t$:

$$\Delta s = |\Delta \vec{r}| = \left| [x(t+\Delta t) - x(t)]\hat{\imath} + [y(t+\Delta t) - y(t)]\hat{\jmath} + [z(t+\Delta t) - z(t)]\hat{k} \right|$$

For $\Delta t << 1$:

$$x(t+\Delta t) = x(t) + \Delta t \frac{dx}{dt} + \mathcal{O}(\Delta t^2)$$

Doing the same for our other components:

$$\Delta s = \underbrace{\left| \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k} \right|}_{\vec{v}} \Delta t + \mathcal{O}(\Delta t^2)$$

Thus,

$$\frac{\Delta s}{\Delta t} = v + \mathcal{O}(\Delta t)$$

Taking $\lim \Delta t \to 0$

$$\boxed{\frac{ds}{dt} = v = \dot{s} \implies s(t) = \int_0^t v(t')\, dt'}$$

Both t and s are ways of parametrizing our curve (path). Instead of writing $\vec{r}(t)$, we can write $\vec{r}(s)$.

Definition. s is what we call the *arc length.*

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$. But $\frac{ds}{dt} = v$. So, $\frac{d\vec{r}}{ds} = \hat{v}$, the unit tangent at evert point s. Thus

$$\vec{v}(s) = \dot{s}\hat{v}$$

Also

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{s}\,\frac{d\vec{r}}{ds}\right) = \ddot{s}\frac{\mathrm{d}\vec{r}}{ds} + \dot{s}\frac{d}{dt}\left(\frac{d\vec{r}}{ds}\right) = \ddot{s}\hat{v} + \dot{s}\frac{d^2\vec{r}}{ds^2}\frac{ds}{dt}$$

So

$$\vec{a}(s) = \ddot{s}\hat{v} + \dot{s}^2\frac{d^2\vec{r}}{ds^2} = \ddot{s}\hat{v} + \kappa\dot{s}^2\hat{n}$$

Writing $\frac{d^2\vec{r}}{ds^2} = \kappa\hat{n}$, where $\kappa = \left|\frac{d^2\vec{r}}{ds^2}\right|$, $\hat{n} = \frac{1}{\kappa}\frac{d^2\vec{r}}{ds^2}$.

It turns out that κ is the curvature of the path. What about $\hat{n}$?

Recall: $|\hat{v}| = 1$. So $\frac{d}{ds}(\hat{v} \cdot \hat{v} = 1) \implies 2\hat{v} \cdot \frac{d\hat{v}}{ds} = 0 \implies 2\kappa(\hat{v} \cdot \hat{n}) = 0$

So, if $\kappa \neq 0$, then $\hat{v} \cdot \hat{n} = 0$. Thus $\hat{n}$ is the unit normal to the path.

Tangential component of the acceleration $a_t = \vec{a} \cdot \hat{v} = \ddot{s}$

Normal component of the acceleration $a_n = \vec{a} \cdot \hat{n} = \kappa \dot{s}^2$, where $\kappa \approx 1/R$

Key things to note: $\hat{v}, \hat{n}, \kappa$ depend only on the path. Knowing $\vec{r}(s)$, we can find these quantities.

$\dot{s}$ and $\ddot{s}$ depend on how the particle is moving along the path.

Example 1.4 (Circular Motion)**.**

Cartesian: $\vec{r}(t) = R\sin(\omega t)\hat{\imath} + R\cos(\omega t)\hat{\jmath}$. Differentiating finds $\vec{v}(t)$ and $\vec{a}(t)$.

Polars: $r = R$ and $\theta = \frac{\pi}{2}\omega t \implies \dot{r} = \ddot{r} = 0$ and $\dot{\theta} = -\omega,\ \ddot{\theta} = 0$. So

$$\begin{aligned}
\vec{r} &= R\hat{r} \\
\vec{v} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = -R\omega\hat{\theta} \\
\vec{a} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = -R\omega^2\hat{r}
\end{aligned}$$

Intrinsic: $(s(0) = 0)$ The speed is given by $v = R\omega = \dot{s} \implies \ddot{s} = 0$. Integrate to find s

$$s = R\omega t \implies t = \frac{s}{R\omega}$$

Substitute this into our expression for $\vec{r}(t)$

$$\vec{r}(s) = R\sin(s/R)\hat{\imath} + R\cos(s/R)\hat{\jmath}$$

$$\implies \hat{v} = \frac{d\vec{r}}{ds} = \cos(s/R)\hat{\imath} - \sin(s/R)\hat{\jmath}, \quad \frac{d^2\vec{r}}{ds^2} = -\frac{1}{R}[\sin(s/R)\hat{\imath} + \cos(s/R)\hat{\jmath}]$$

$$\implies \kappa = \left|\frac{d^2\vec{r}}{ds^2}\right| = \frac{1}{R},\ \hat{n} = -\sin(s/R)\hat{\imath} - \cos(s/R)\hat{\jmath}$$

$$\begin{aligned}
\vec{v}(s) &= R\omega\hat{v} \\
\vec{a}(s) &= \ddot{s}\hat{v} + \kappa\dot{s}^2\hat{n} = \frac{1}{R}(R\omega)^2\hat{n} = R\omega^2\hat{n}
\end{aligned}$$

Example 1.5 (Helical Path). Lecture 8

$$\vec{r}(s) = b\cos(ks)\hat{\imath} + b\sin(ks)\hat{\jmath} + s\sqrt{1-b^2k^2}\hat{k}$$

Tangent:

$$\vec{v} = \frac{d\vec{r}}{ds} = -bk\sin(ks)\hat{\imath} + bk\cos(ks)\hat{\jmath} + \sqrt{1-b^2k^2}\hat{k}$$

Curvature and Normal:

$$\frac{d^2\vec{r}}{ds^2} = -bk^2\cos(ks)\hat{\imath} - bk^2\sin(ks)\hat{\jmath}$$

$$\kappa = \left|\frac{d^2\vec{r}}{ds^2}\right| = bk^2,\ \hat{n} = -\cos(ks)\hat{\imath} - \sin(ks)\hat{\jmath}$$

Take $s = ct\ (c > 0) \implies \dot{s} = c,\ \ddot{s} = 0$. Thus

$$\vec{v} = c\hat{v},\ \vec{a} = c^2bk^2\hat{n}$$

Take the case where our path lies in the $xy-$plane and we know $y(x)$

Then $ds^2 = dx^2 + dy^2$. Since $dy = \frac{dy}{dx}dx \implies ds^2 = (1 + (\frac{dy}{dx})^2)dx^2$

$$\implies \frac{ds}{dx} = \sqrt{1+y'^2}$$

$$s(x) = \int_{x_0}^{x}(1+y'^2)^{1/2}dx$$

Highlights that s just depends on the path.

Position $\vec{r}(x) = x\hat{\imath} + y(x)\hat{\jmath}$

Tangent to the path

$$\hat{v} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dx}\frac{dx}{ds} = \frac{d\vec{r}}{dx}\left(\frac{ds}{dx}\right)^{-1}$$

$$\frac{d\vec{r}}{dx} = \hat{\imath} + y'\hat{\jmath}, \quad \left(\frac{ds}{dx}\right)^{-1} = [1+y'^2]^{-1/2}$$

$$\implies \hat{v} = [1+y'2]^{-1/2}[\hat{\imath} + y'\hat{\jmath}]$$

Curvature and Normal

$$\frac{d^2\vec{r}}{ds^2} = \frac{d}{dx}\left(\frac{d\vec{r}}{ds}\right)\left(\frac{ds}{dx}\right)^{-1} = \left(\frac{d}{dx}\left(\frac{d\vec{r}}{ds}\right)\frac{dx}{ds}\right)$$

$$\frac{d}{dx}\frac{d\vec{r}}{ds} = \frac{d\hat{v}}{dx} = -\frac{1}{2}[1+y'^2]^{-3/2} \times (2y'y'') \times [\hat{\imath} + y'\hat{\jmath}] + [1+y'^2]^{-1/2}y''\hat{\jmath}$$

Example 1.6. $y = x^2,\ y' = 2x,\ y'' = 2$

$$\frac{ds}{dx} = [1+y'^2]^{1/2}$$

2 Kinetics and Newtons Laws

Newton's Laws

Lecture 9

Definition.

- *Mass, m* - "Quantity of Matter", measured in kg (scalar)
- *Momentum, $\vec{p} = m\vec{v}$* - "Quantity of Motion" (vector)
- *Inertia* - "Vis Insita" (innate force of matter). The resistance of an object to change its state of motion.
- *Force* - An action that changes an objects state of motion

Theorem 2.1: Newton's First Law

Every body has inertia.

Theorem 2.2: Newton's Second Law

The net force on an object is equal to the rate of change of momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

Theorem 2.3: Newton's Third Law

If $\vec{F}_{AB}$ is the force on object A due to object B. Then $\vec{F}_{BA} = -\vec{F}_{AB}$.

3 Forces

Gravity

Contraint Forces

Friction

Drag Force

Lecture 15

Example of a force that depends on the velocity of an object.

Motion of bodies through fluid.

Fluid has: ρ: density and η: viscosity

To move through the fluid, the body exerts a force on the fluid: $\vec{F}_{FB}$

By Newton's III Law

$$\vec{F}_{BF} = -\vec{F}_{FB}$$

The drag force

$$\vec{F}_D = (\vec{F}_{BF} \cdot \hat{v})\hat{v}$$

In general to find $\vec{F}_D$ is a challenging problem!

To find $\vec{u}$ we need to solve the Navier-Stokes Equations. From $\vec{u}$ we can obtain $\vec{F}_D$.

Fortunately this calculation can be done for two limiting cases; at low and at high speeds:

Low Speeds

At low speeds $|\vec{v}| << 1$, then

$$\vec{F}_D = -C_D\vec{v}$$

Where C_D is the *drag co-efficient*

- This depends linearly on $\vec{v}$.
- Always opposite the direction of motion
- For a sphere $C_D = 6\pi R\eta$
- C_D depends on (i) the size of the object, (ii) the viscosity of the fluid

If $\vec{u} \neq 0$ meaning there is a background flow:

$$\vec{F}_D = -C_D(\vec{v} - \vec{u})$$

only a drag force if there's relative motion to the fluid.

High Speeds

$$\vec{F}_D = -C_D|\vec{v}|\vec{v}$$

- Opposes the motion
- Depends quadratically on the speed
- Changes $C_D = \frac{1}{2}\rho R^2 K$
- Draf Force is not all of $\vec{F}_{BF}$

Example 3.1.

$\vec{F}_D = -C_D\vec{v}$

Force Diagram:

Newton's Second Law:

$$m\frac{d\vec{v}}{dt} = \vec{F}_D + \vec{F}_g = -C_D\vec{v} - mg\hat{j}$$

First, seek the solution, $\vec{v}_\infty$, where $\frac{d\vec{v}}{dt} = 0$, the *steady state solution*

$$\implies 0 = -C_D\vec{v} - mg$$
$$\implies \vec{v}_\infty = -\frac{mg}{C_D}\hat{j}$$

Using linearity of the equation

$$\vec{v} = \vec{v}_\infty + \vec{w}$$

Substitute this into Newton's Second Law:

$$m\frac{d}{dt}(\vec{v}_\infty + \vec{w}) = -C_D(\vec{v}_\infty + \vec{W}) - mg\hat{j}$$

$$m\frac{d\vec{w}}{dt} = mg\hat{j} - C_D\vec{w} - mg\hat{j}$$

$$\implies \frac{d\vec{w}}{dt} = -\frac{C_D}{m}\vec{w}$$
$$\implies \vec{w} = \vec{w}_0 e^{-C_D t/m}$$

Thus

$$\vec{v} = -\frac{mg}{C_D}\hat{j} + \vec{w}_0 e^{-C_D t/m}$$

Initial condition: $t = 0$, $\vec{v} = \vec{v}_0 \implies \vec{w}_0 = \vec{v}_0 + \frac{mg}{C_D}\hat{j}$. So

$$\vec{v} = \vec{v}_0 e^{-C_D t/m} - \frac{mg}{C_D}\hat{j}[1 - e^{-C_D t/m}]$$

As $t \to \infty$, $\vec{v} \to -\frac{mg}{C_D}\hat{j} = \vec{v}_\infty$ as expected.

The ratio C_D/m controls how quickly this limit is reached.

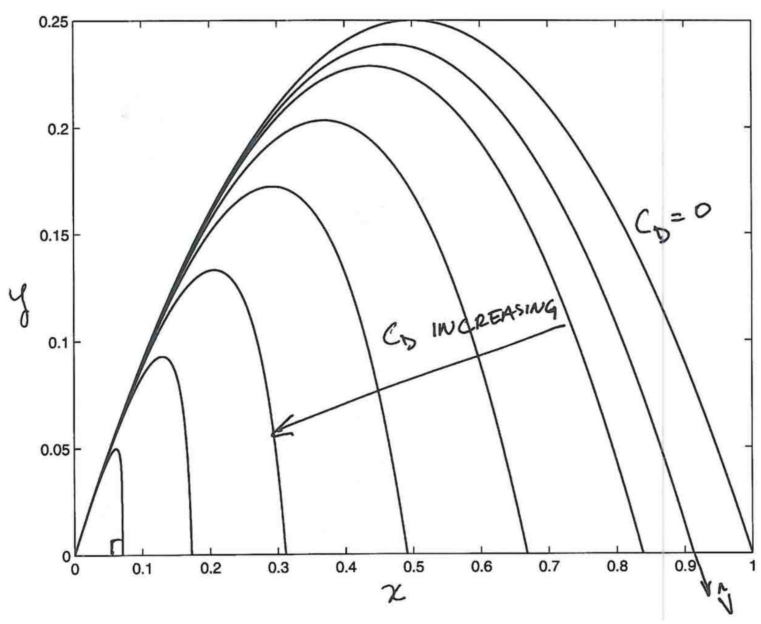
Taking $\vec{v}_0 = 0$

$$\vec{v} = -\frac{mg}{C_D}\hat{j}[1 - e^{-C_D t/m}]$$

Integrating our general expression to find the position:

$$\vec{r}(t) = \vec{r}_0 - \frac{mgt}{C_D}\hat{j} + \frac{m}{C_D}[\vec{v}_0 + \frac{mg}{C_D}\hat{j}] \times (1 - e^{-C_D t/m})$$

Projectiles: $\vec{r}_0 = 0$, $\vec{v}_0 = v_0 \cos\alpha\hat{i} + v_0 \sin\alpha\hat{j}$



4 Oscillators

5 Energy

Lecture 19

Energy gives us another viewpoint on mechanical systems.

1D: From Newton's 2nd Law

$$m\ddot{x} = F(x, \dot{x}, t) \implies m\ddot{x}\dot{x} = F\dot{x}$$

Since $\ddot{x}\dot{x} = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right)$

$$\boxed{\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = F\dot{x}} \tag{5.1}$$

Call $T = \frac{1}{2}m\dot{x}^2$ and integrate (5.1) with respect to time

$$\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} T \,\mathrm{d}t = \int_{t_1}^{t_2} F\dot{x}\,\mathrm{d}t$$
$$\implies T(t_2) - T(t_1) = \int_{x(t_1)}^{x(t_2)} F\,\mathrm{d}x$$

Definition. We call $T = \frac{1}{2}m\dot{x}^2$ the *kinetic energy*, $F\dot{x}$ the *rate of work*.

$W_{12} = \displaystyle\int_{x(t_1)}^{x(t_2)} F\,\mathrm{d}x$ is the *work done* on m by F.

Define $V(x) = -\displaystyle\int F\,dx + C$ is *potential energy*. $T + V = E$, the *total energy*.

A force that can be written in terms of a potential ($\vec{F} = -\vec{\nabla}V$) is *conservative*.

Theorem 5.2: Conservation of Energy

Under conservative forces, the total energy of a system is constant.

Proof. Suppose that $F = F(x)$, $V(x) = -\int F\,dx + C$ or $F = -\frac{dV}{dx}$

$$\int_{x(t_1)}^{x(t_2)} F\,\mathrm{d}x = \int_{x(t_1)}^{x(t_2)} -\frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x$$

$$\implies T(t_2) + V(t_2) = T(t_1) + V(t_1) = E$$

More generally, from (5.1)

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) - F(\dot{x}) = 0$$

Since $F(\dot{x}) = \dfrac{dV}{dx}\dfrac{dx}{dt} = \dfrac{dV}{dt}$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 - V\right) = 0$$

$$\implies T + V = E, \text{ a constant}$$

∎

Not all forces are conservative!

Example 5.3. $F_D = -C_D\dot{x}$ is not conservative.

Suppose that

Newton's Second Law:

$$m\ddot{x} = F_{CON} + F_D$$
$$\implies m\ddot{x} + \frac{dV}{dx} = -C_D\dot{x}$$

Multiplying by $\dot{x}$ and rearranging the terms:

$$\frac{d}{dt}(\underbrace{T+V}_{E}) = -C_D\dot{x}^2 \leq 0$$

$$\implies \frac{dE}{dt} \leq 0 \implies \text{ Energy decreases with time}$$

Examples of Conservative Forces

Examples 5.4.

- Gravity: $F = -mg \implies V = mgx + C$
- Spring Force: $F = -kx \implies V = \frac{1}{2}kx^2 + C$

We can choose C for our convenience.

Lecture 20

Recall that forces that are related to a potential are called *conservative forces.*

Another way to think about conservative forces is through the *work done*:

$$W_{12} = \int_{x(t_1)}^{x(t_2)} F \, \mathrm{d}x$$

If the forces is conservative $F = -\dfrac{dV}{dx} \implies W_{12} = -V(x_2) + V(x_1)$.

Hence the work done just depends on the initial and final position. It is path independent! We also saw that as a result:

$$T(t_1) + V(t_1) = T(t_2) + V(t_2) = E, \text{ the total energy}$$

Potential Wells

Suppose we know $\dot{x}$ and x at $t = 0$. With this, we can find

$$E = \frac{1}{2}m\dot{x}^2(0) + V(x(0))$$

And we know this for all times.

> **Definition.** The points x_0, x_1 and x_2 are where $V = E$. These points are called *turning points*.

Oscillations between Turning Points

At the turning points, for example $V(x_1) = E$, we know that $T(x_1) = 0 \implies \dot{x_1} = 0$.

We know that if the particle is between x_0 and x_1, it will oscillate between these points forever! We say that this particle is *trapped*!

Period of oscillation between x_0 and x_1:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Solve for $\dot{x}$

$$\frac{dx}{dt} = \dot{x} = \pm\left[\frac{2}{m}(E - V(x))\right]^{1/2} \tag{5.5}$$

We need to choose the correct root based on $\dot{x}$ at a particular point in time. Suppose we know going from x_0 to x_1, $\dot{x} > 0$.

We need to integrate (5.5) to find the time it takes to go from x_0 to x_1

$$\begin{aligned} \int_{x_0}^{x_1} \frac{dx}{\left[\frac{2}{m}(E - V(x))\right]^{1/2}} &= \int_{t_0}^{t_1} dt \\ &= T_{osc}/2 \end{aligned}$$

Thus

$$T_{osc} = 2\int_{x_0}^{x_1} \frac{dx}{\left[\frac{2}{m}(E - V(x))\right]^{1/2}} \tag{5.6}$$

> **Example 5.7** (Spring).
>
> Spring: $V = \frac{1}{2}kx^2$
>
> Initially $x(0) = L$, $\dot{x}(0) = 0$
>
> $$E = \frac{1}{2}m\dot{x}(0) + V(L) = \frac{1}{2}kL^2$$

Then

$$
\begin{aligned}
T_{osc} &= 2\int_{-L}^{L} \frac{dx}{[\frac{2}{m}(\frac{1}{2}kL^2 - \frac{1}{2}kx^2)]^{1/2}} \\
&= 2\sqrt{\frac{m}{k}} \int_{-L}^{L} \frac{dx}{[L^2 - x^2]^{1/2}} \\
&= 2\sqrt{\frac{m}{k}} \int_{-L}^{L} \frac{dx}{L[1 - (x/L)^2]^{1/2}} \\
u &= x/L \\
&= 2\sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{du}{[1 - u^2]^{1/2}} \\
&= 2\sqrt{\frac{m}{k}} \arcsin u \Big|_{-1}^{1} = 2\pi\sqrt{\frac{m}{k}}
\end{aligned}
$$

So $T_{osc} = 2\pi\sqrt{\frac{m}{k}},\ \omega_0 = \sqrt{\frac{k}{m}}$

Escape

Suppose the particle is at x_A. What speed does is need to not be trapped, i.e. $x \to \infty$ as $t \to \infty$? Lecture 21

Initial speed: u

$$E = \frac{1}{2}mu^2 + V(x_A)$$

We want $E > E^*$ to allow our particle to escape. $E* = V(X_1)$. We require then

$$V(X_1) < \frac{1}{2}mu^2 + V(x_A)$$

$$\implies u > \sqrt{\frac{2}{m}(V(X_1) - V(x_A))}$$

Stability

Definition. *Equilibrium Points* are where $\dfrac{dV}{dx} = 0 \implies F = 0 \implies m\ddot{x} = 0$

We say that an equilibrium point is

- *stable* if $\dfrac{d^2V}{dx^2} > 0$ (Minimum) e.g. X_0
- *unstable* if $\dfrac{d^2V}{dx^2} < 0$ (Maximum) e.g. X_1

Oscillations near Equilibrium Point

Suppose we are near and very close to a stable equilibrium point, X_0, so $|x - X_0| << 1$.

Taylor expansion of $V(x)$ about X_0:

$$V(x) = V(X_0) + V'(X_0)(X - X_0) + \frac{1}{2}V''(X_0)(X - X_0)^2 + \dots \tag{5.8}$$

Since X_0 is an equilibrium point, we know $V'(X_0) = 0$

$$V(x) = V(X_0) + \frac{1}{2}V''(X_0)(X - X_0)^2$$

Since X_0 is a stable equilibrium point $V''(X_0) > 0$

$$F = \frac{-dV}{dx} = -(x - X_0)V''(X_0)$$

From Newton's 2nd Law

$$m\ddot{x} = -(x - X_0)V''(X_0)$$

Taking $X = x - X_0$

$$m\ddot{X} + V''(X_0)X = 0$$

This looks like the simple harmonic oscillator with $k = V''(X_0)$.

Since $\omega_0 = \sqrt{\dfrac{k}{m}}$, the frequency of small oscillation is $\omega_0 = \sqrt{\dfrac{V''(X_0)}{m}}$

$$\implies T_{osc} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{V''(X_0)}}$$

Example 5.9 (Lennard-Jones Potential)**.**

Used to model interactions between neutral atoms or molecules and Molecular dynamics simulations.

Lecture 22

6 Angular Momentum

Central Forces

Lecture 23

We will consider forces of the form

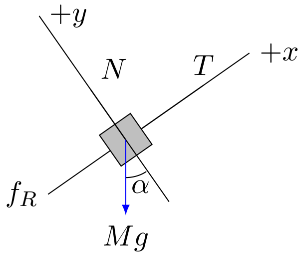
$$\vec{F} = F(r)\hat{r}$$

Magnitude depends on the distance from the origin.

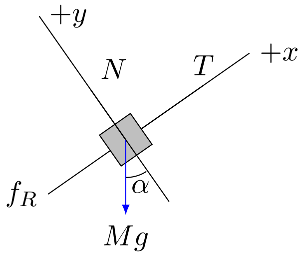
Direction $\hat{r}$ is repulsive; away from the origin. $-\hat{r}$: attractive; towards the origin.

Example 6.1 (Gravity).

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$



Suppose that



Polar coordinates are perfect for these problems

Newton's Second Law:

$$m(\ddot{r} - r\dot{\theta}^2) = F \tag{6.2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{6.3}$$

Multiply (6.3) by r

$$m(r^2\ddot{\theta} + 2\dot{r}r\dot{\theta}) = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \implies mr^2\dot{\theta} = mh = \text{constant}$$

Definition. $h = r^2\dot{\theta}$ - *angular momentum per unit mass*

Angular momentum, $\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

Theorem 6.4: Conservation of Angular Momentum

Under a central force (no torque), the total angular momentum is conserved.

Proof. In polars, $\vec{r} = r\hat{r}$, $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$\vec{J} = \vec{r} \times m\vec{v} = (r\hat{r}) \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = mr\dot{r}(\hat{r} \times \hat{r}) + mr^2\dot{\theta}(\hat{r} \times \hat{\theta})$$
$$\implies \vec{J} = mr^2\dot{\theta}\hat{k} = mh\hat{k} = \text{ constant}$$ ∎

Energy

For a force to be conservative $\vec{F} = -\vec{\nabla}V$. In 2D

$$\vec{F} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} \tag{6.5}$$

Since $\vec{F} = \vec{F}(r)\hat{r}$ we need $V = V(r)$

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\frac{\partial r}{\partial x}$$

Since $r = (x^2 + y^2)^{1/2}$, $\dfrac{\partial r}{\partial x} = \frac{1}{2}[x^2 + y^2]^{1/2} \times (2x) = x/r = \cos(\theta)$. Thus

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\cos\theta$$

Similarly

$$\frac{\partial V}{\partial y} = \frac{dV}{dr}\frac{\partial r}{\partial y} = \frac{dV}{dr}\sin\theta$$

Thus the force, by (6.5), is

$$\begin{aligned}\vec{F} &= -\frac{dV}{dx}\cos\theta\hat{i} - \frac{dV}{dy}\sin\theta\hat{j} \\ &= -\frac{dV}{dr}\hat{r}\end{aligned}$$

So for a central force to be conservative

$$\vec{F}(r) = -\frac{dV}{dr}$$

From the Conservation of Energy

$$\frac{1}{2}mv^2 + V(r) = E$$

Since $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$\boxed{E = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] + V(r)} \tag{6.6}$$

Orbital Equation

Find the trajectories or shapes or orbits as a function of θ. It's solution is $u(\theta) = 1/r(\theta)$. We know $h = r^2\dot{\theta} = \dot{\theta}u^{-2} \implies \dot{\theta} = hu^2$. Thus

$$\dot{r} = \frac{d}{dt}(u^{-1}) = -u^{-2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\dot{\theta} = h^2u^2\frac{d^2u}{d\theta^2}$$

Also

$$r\dot{\theta}^2 = u^{-1}(hu^2)^2 = h^2u^3$$

Write $F(r) = F(u^{-1})$ and substitute into (6.2) from Newton's 2nd Law:

$$m(h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3) = F(u^{-1})$$

Giving our orbital equation:

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}F(u^{-1})} \tag{6.7}$$

Example 6.8. $r(\theta) = c\theta^2$ $(c > 0)$. Find $F(r)$:

$$u = c^{-1}\theta^{-2}, \quad \frac{du}{d\theta} = -2c^{-1}\theta^{-3}, \quad \frac{d^2u}{d\theta^2} = 6c^{-1}\theta^{-4} = 6u^2$$

From the Orbital Equation (6.6)

$$F(u^{-1}) = -mh^2u^2(u + 6cu^2) = -mh^2(u^3 + 6cu^4)$$

$$\implies F(r) = -mh^2(r^{-3} + 6cr^{-4})$$

Kepler's Laws

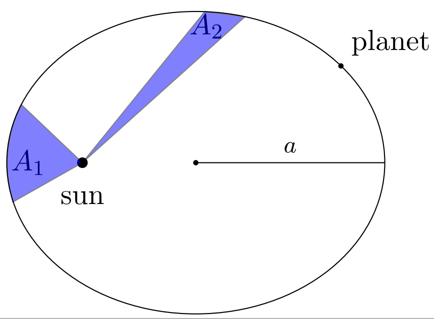
Lecture 24

Theorem 6.9: Kepler's Laws

I Orbits of Planets are Ellipses

II Law of Equal Areas: If $\Delta t_1 = \Delta t_2$ then $A_1 = A_2$

III The time period of orbit, $T \propto a^3$



Proof of Kepler's First Law. Inverse square law:

$$F(r) = -k/r^2 \implies F(u^{-1}) = -ku^2$$

Substituting into our orbital equation (6.6)

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2u^2} \quad (*)$$

This resembles

$$m\frac{d^2x}{dt^2} + kx = F_0$$

The general solution to $(*)$ is $u = A\cos(\theta - \theta_0) + \frac{k}{mh^2}$; wlog take $\theta_0 = 0$ so

$$u(\theta) = A\cos(\theta) + \frac{k}{mh^2}$$

$$\implies r(\theta) = \frac{(mh^2/k)}{1 + \frac{Amh^2}{k}\cos\theta} \tag{6.10}$$

This is the form of an ellipse in polar coordinates (see Problem 10, P.S. 1)

$$r(\theta) = \frac{l}{1 + e\cos\theta}$$

Where $l = \frac{mh^2}{k}$, $e = \frac{Amh^2}{k}$.

$e = [1 - b^2/a^2]^{1/2}$, $l = a(1 - e^2)$ ∎

We see that E is related to A.

We can get the family of orbits by considering the energy; equation (6.5) gives

$$E = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] + V(r)$$

$F(r) = -kr^{-2} = -\frac{dV}{dr}$, so $V(r) = -kr^{-1} \implies V(u^{-1}) = -ku$.

Also $\dot{r} = -h\frac{du}{d\theta}$, and $r^2\dot{\theta}^2 = h^2r^{-2} = h^2u^2$. So the energy is

$$E = \frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku$$

Using the fact $u(\theta) = A\cos(\theta) + \dfrac{k}{mh^2}$, $\dfrac{du}{d\theta} = -A\sin\theta$ and simplifying the trig we get

$$E = \frac{1}{2}mh^2A^2 - \frac{1}{2}\frac{k^2}{mh^2}$$

$$\implies A = \sqrt{\frac{2E}{mh^2} + \frac{k^2}{(mh^2)^2}}$$

From (6.9), the eccentricity of the orbit, $e = (1 - b^2/a^2)^{1/2}$, is

$$e = \frac{Amh^2}{k} = \frac{mh^2}{k}\sqrt{\frac{2E}{mh^2} + \frac{k^2}{(mh^2)^2}} = \sqrt{1 + \frac{2Emh^2}{k^2}}$$

This parameter e actually allows our solution $r(\theta)$ to describe a whole family of orbits.

Examples 6.11.

(i) Bounded Trajectories

- $E = -k^2/2mh^2 \implies e = 0$ [Circle]
- $E < 0 \implies 0 < e < 1$ [Ellipse]

(ii) Unbounded Trajectories

- $E = 0 \implies e = 1$ [Parabola]
- $E > 0 \implies e > 1$ [Hyperbola]

Effective Potential

Lecture 25

Consider the energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

Since $h = r^2\dot{\theta}$, $h^2 = r^4\dot{\theta}^2 \implies r^2\dot{\theta}^2 = h^2/r^2$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{mh^2}{r^2} + V(r)$$

Definition. The *Effective Potential*, $V_{EFF} = \dfrac{1}{2}\dfrac{mh^2}{r^2} + V(r)$

$$\implies E = \frac{1}{2}m\dot{r}^2 + V_{EFF}$$

What we've done is written our energy in such a way that it looks like what we had with 1D motion!

$$x \longrightarrow r$$
$$V(x) \longrightarrow V_{EFF}(r)$$

Definition. *Turning points* occur when $V_{EFF} = E$. This tells us where $\frac{1}{2}m\dot{r}^2 = 0 \implies \dot{r} = 0$. This tells us about the boundedness of our orbit.

Equilibria

In 1D: $V'(x_0) = 0 \implies F(x_0) = 0$, where x_0 is the equilibrium point

If $\dot{x} = 0$ and $x = x_0$ at $t = 0$, then $m\ddot{x} = 0$ and $x = x_0$ $\forall t$

$$V_{EFF} = \frac{1}{2}\frac{mh^2}{r^2} + V(r)$$
$$\implies \frac{dV_{EFF}}{dr} = -mh^2r^{-3} + \underbrace{V'(r)}_{-F(r)}$$

Newton's 2nd Law's $\hat{r}$ component (equation (6.2))

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$
$$\implies m\ddot{r} = F(r) + \frac{mh^2}{r^3} = \frac{dV_{EFF}}{dr}$$

Suppose that $V'_{EFF}(r_0) = 0$. If $r = r_0$ and $\dot{r} = 0$ at $t = 0$, then $m\ddot{r} = 0 \implies r = r_0$ $\forall t$. So we have a constant $r \implies$ Circular Trajectory

Stability

$R = r - r_0$, $|R| << 1$, then the Taylor expansion about r_0:

$$V_{EFF}(r) = V_{EFF}(r_0) + RV'_{EFF}(r_0) + \frac{1}{2}R^2V''_{EFF}(r_0) + \dots \tag{6.12}$$

Since at r_0, $V'_{EFF}(r_0) = 0$

$$V_{EFF}(r) = V_{EFF}(r_0) + \frac{1}{2}R^2V''_{EFF}(r_0)$$

Differentiating

$$V'_{EFF}(r) = RV''_{EFF}(r)$$

Using this in Newton's Second Law:

$$m\ddot{r} = -RV''_{EFF}(r_0)$$

or

$$m\ddot{R} + RV''_{EFF}(r_0) = 0$$

- If $V''_{EFF}(r_0) > 0 \implies$ a minimum, so the circular orbit is stable.
- If $V''_{EFF}(r_0) < 0 \implies$ a maximum, so the circular orbit is unstable.

Example 6.13. $F(r) = -kr^{-2}$ $(k > 0) \implies V(r) = -kr^{-1}$

$$\implies V_{EFF}(r) = -kr^{-1} + \frac{1}{2}mh^2r^{-2}$$
$$\implies V'_{EFF}(r) = kr^{-2} - mh^2r^{-3}$$

Setting this equal to zero

$$r^{-3}(kr - mh^2) = 0$$

This is satisfied as $r \to \infty$ or at $r_0 = mh^2/k$

$$V''_{EFF}(r) = -2kr^{-3} + 3mh^2r^{-4}$$

So at the equilibria point

$$V''_{EFF}(mh^2/k) = \left(\frac{k}{mh^2}\right)^4 (3mh^2 - 2k(mh^2/k)) = \left(\frac{k}{mh^2}\right)^4 (mh^2) > 0$$

This is a stable circular trajectory.

$$V'_{EFF}(\frac{mh^2}{k}) = -k\left(\frac{k}{mh^2}\right) + \frac{1}{2}mh^2\left(\frac{k^2}{(mh^2)^2}\right) = -\frac{k^2}{2mh^2}$$

Thus

$$E_{MIN} = -\frac{k^2}{2mh^2}.$$

We reach the same family of orbits as Example 6.10 by differing values of E:

(i) Bounded Trajectories

- $E = E_{MIN} = -k^2/2mh^2 \implies r = \frac{mh^2}{k} \implies$ Circular Orbit
- $E_{MIN} < E < 0 \implies$ two turning points $\implies$ Bounded Orbit [Ellipse]

(ii) Unbounded Trajectories when $E \geq 0$ since we have only a single turning point. In particular

- $E = 0 \implies$ Parabola
- $E > 0 \implies$ Hyperbola

7 Systems of Particles

Lecture 26

Definition.

- N: Total number of particles
- $\vec{r}_i$: Position of particle i
- $\vec{v}_i$: Velocity of particle i
- $\vec{F}_i$: Force on particle i
- m_i: Mass of particle i

Consider the average motion of the system:

Definition. *Centre of Mass*, $\vec{r}_{cm}$:

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$$

Where $M = \sum_{i=1}^{N} m_i$ is the *total mass.*

Momentum

The total momentum $\vec{p}$ is

$$\begin{aligned}\vec{p} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i &= \sum_i m_i \frac{d\vec{r}_i}{dt} \\ &= \frac{d}{dt}(\sum_i m_i \vec{r}_i) \\ &= \frac{d}{dt}(M\vec{r}_{cm}) \\ &= M\frac{d\vec{r}_{cm}}{dt} = M\vec{v}_{cm}\end{aligned}$$

Where $\vec{v}_{cm}$ is the velocity of the centre of mass.

$$\vec{F}_i = \vec{F}_i^{EXT} + \sum_{j=1}^{N} \vec{F}_{ij}$$

where $\vec{F}_i^{EXT}$ is the external forces on particle i, $\vec{F}_{ij}$ is the force on i due to j

Example 7.1.

Here $\vec{F}_{gi}$(Force due to gravity on i) is the only external force on $i \implies \vec{F}_i^{EXT} = \vec{F}_{gi}$

Note that

(i) $\vec{F}_{ii} = \vec{0}$

(ii) $\vec{F}_{ij} = -\vec{F}_{ji}$ By Newton's Third Law

Theorem 7.2: Newton's Second Law for a System

The external force is equal to the rate of change of momentum of the centre of mass

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT}$$

Where the total external force on the system $\vec{F}^{EXT} = \sum_i \vec{F}_i^{EXT}$.

Proof. For particle i,

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i = \vec{F}_i^{EXT} + \sum_{j=1}^{N} \vec{F}_{ij}$$

$$\implies \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i = \sum_i \vec{F}_i^{EXT} + \sum_i \sum_j \vec{F}_{ij}$$

Due to Newton's Third Law $\sum_i \sum_j \vec{F}_{ij} = \vec{0}$. We are then left with

$$\sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i^{EXT}$$

$$\implies \frac{d}{dt}(\sum_i \vec{p}_i) = \vec{F}^{EXT}$$

$$\implies M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT}$$ ∎

(i) If there is no external forces then

$$M\frac{d\vec{v}_{cm}}{dt} = 0 = \frac{d\vec{p}}{dt}$$

(The conservation of momentum)

(ii) If there are external forces then the centre of mass moves as though it were a point particle of mass m subject to force $\vec{F}^{EXT}$

Two Body Problems

$\vec{F}_1 = m_1 g\hat{i} + \vec{F}_{12}$

$\vec{F}_2 = m_2 g\hat{i} + \vec{F}_{21}$

The total external force:

$$\vec{F}^{EXT} = m_1 g\hat{i} + m_2 g\hat{i} = Mg\hat{i} \ (M = m_1 + m + 2)$$

Thus

$$M\frac{d\vec{v}_{cm}}{dt} = Mg\hat{i} \implies \frac{d\vec{v}_{cm}}{dt} = g\hat{i}$$

For two body problems this is half of the information.

$$m_1\frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1^{EXT} + \vec{F}_{12} \tag{7.3}$$

$$m_2\frac{d^2\vec{r}_2}{dt^2} = \vec{F}_2^{EXT} + \vec{F}_{21} \tag{7.4}$$

Calling $\dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$, and adding the equations

$$m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} = \vec{F}_1^{EXT} + \vec{F}_2^{EXT}$$

$$M\frac{d}{dt}\left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{M}\right) = \vec{F}_1^{EXT} + \vec{F}_2^{EXT}$$

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}_1^{EXT} + \vec{F}_2^{EXT}$$

Lecture 27

Consider: $m_2 \times (7.4) - m_1 \times (7.3)$

$$m_1m_2\frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT} + m_2\vec{F}_{12} - m_1\vec{F}_{21}$$

Call $\vec{r}_{12} = (\vec{r}_1 - \vec{r}_2)$. Since $\vec{F}_{12} = -\vec{F}_{21}$

$$m_1m_2\frac{d^2\vec{r}_{12}}{dt^2} = m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT} + (m_1+m_2)\vec{F}_{12}$$

Divide through by M

$$\frac{m_1m_2}{M}\frac{d^2\vec{r}_{12}}{dt^2} = \frac{m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT}}{M} + \vec{F}_{12}$$

> **Definition.** Introduce $\mu = \dfrac{m_1m_2}{M}$, the *reduced mass.*

Then for our two body system we have:

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}_1^{EXT} + \vec{F}_2^{EXT} \tag{7.5}$$

$$\mu\frac{d^2\vec{r}_{12}}{dt^2} = \frac{m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT}}{M} + \vec{F}_{12} \tag{7.6}$$

If $\vec{F}_1^{EXT} = \vec{F}_2^{EXT} = 0$, then $M\dfrac{d\vec{v}_{cm}}{dt} = 0$, and $\mu\dfrac{d^2\vec{r}_{12}}{dt^2} = \vec{F}_{12}$.

If $\vec{F}_1^{EXT} = -m_1g\hat{j}$ and $\vec{F}_2^{EXT} = -m_2g\hat{j}0$, then $M\dfrac{d\vec{v}_{cm}}{dt} = -Mg\hat{j}$, and $\mu\dfrac{d^2\vec{r}_{12}}{dt^2} = \vec{F}_{12}$.

Example 7.7 (Spring).

Speing has a spring constant k and equilibrium lnegth l.

$$\vec{F}_{12} = -k(x_1 - x_2 - l)\hat{\imath}$$

Initially $x_1(0) = k,\ \dot{x}_1 = v_0$. $x_2(0) = \dot{x}_2(0) = 0$.

$\vec{F}_{12}$ is the only force in the $\hat{\imath}$ direction. No external forces in the $\hat{\imath}$ direction.

$$\implies M\ddot{x}_{cm} = 0 \implies \dot{x}_{cm} = C$$

We can find C using the conservation of momentum

$$\vec{p} = m\dot{x}_1 + m\dot{x}_2 = M\dot{x}_{cm}$$

At $t = 0, \dot{x}_1 = v_0$ and $\dot{x}_2 = 0$. Then $p = mv_0$. Since $M = 2m$:

$$\dot{x}_{cm} = v_0/2$$

For $x_{12} = x_1 - x_2$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}$$
$$\vec{F}_{12} = -k(x_1 - x_2 - l) = -k(x_{12} - l)$$

Using the equation for $\vec{r}_{12}$

$$\mu\ddot{x}_{12} = \vec{F}_{12}$$
$$\frac{m}{2}\ddot{x}_{12} = -k(x_{12} - l)$$
$$\ddot{x}_{12} + \frac{2k}{m}x_{12} = \frac{2kl}{m}$$

The general solution is

$$x_{12} = A\cos\omega t + B\sin\omega t + l$$

where $\omega^2 = \frac{2k}{m}$.

From our initial conditions $x_{12}(0) = x_1(0) - x_2(0) = l$ and $\dot{x}_{12} = v_0$.

$$\implies A = 0,\ B = v_0/\omega$$

Thus

$$x_{12} = \frac{v_0}{\omega}\sin\omega t + l$$
$$\dot{x}_{12} = v_0\cos\omega t$$

We can show that (in general)

$$\vec{r}_1 = \vec{r}_{cm} + \vec{m_2} M \vec{r}_{12}$$

$$\vec{r}_2 = \vec{r}_{cm} + \vec{m_1} M \vec{r}_{12}$$

Thus

$$x_1 = x_{cm} + \frac{1}{2} x_{12}$$

$$\dot{x}_1 = \dot{x}_{cm} + \frac{1}{2}\dot{x}_{12} = \frac{v_0}{2} + \frac{1}{2} v_0 \cos\omega t = \frac{v_0}{2}(1 + \cos\omega t)$$

Similarly

$$\dot{x}_2 = \frac{v_0}{2}(1 - \cos\omega t)$$

This is a push-me-pull-you system.

What about more than two particles?

Definition (Centre of Mass Coordinates)**.**

$$\vec{R}_i = \vec{r}_i - \vec{r}_{cm}$$

This is the position of particle i relative to the position of the centre of mass

$$\sum_i m_i \vec{R}_i = \underbrace{\sum_i m_i \vec{r}_i}_{M\vec{r}_{cm}} - \vec{r}_{cm} \underbrace{\sum_i m_i}_{M} = 0$$

Kinetic Energy

Lecture 28

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

We can write $\vec{v}_i = \vec{v}_{cm} + \dfrac{d\vec{R}_i}{dt}$, $\vec{u}_i = \dfrac{d\vec{R}_i}{dt}$, so $\vec{v}_i = \vec{v}_{cm} + \vec{u}_i$

$$\begin{aligned}
T &= \sum_i \frac{1}{2} m_i (\vec{v}_{cm} + \vec{u}_i) \cdot (\vec{v}_{cm} + \vec{u}_i) \\
&= \sum_i \frac{1}{2} [v_{cm}^2 + 2\vec{u}_i \cdot \vec{v}_{cm} + u_i^2 \\
&= \frac{1}{2} v_{cm}^2 \sum_i m_i + \vec{v}_{cm} \cdot \sum_i m_i \vec{u}_i + \frac{1}{2} \sum_i m_i u_i^2 \\
&= \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \sum_i m_i u_i^2 + \vec{v}_{cm} \cdot \sum_i m_i \vec{u}_i
\end{aligned}$$

Consider $\sum_i m_i \vec{u}_i = \sum_i m_i \dfrac{d\vec{R}_i}{dt} = \dfrac{d}{dt}(\sum_i m_i \vec{R}_i) = 0$. Then

$$\boxed{T = \frac{1}{2} M v_{cm}^2 + \sum_i \frac{1}{2} m_i u_i^2} \tag{7.8}$$

Angular Momentum

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

For central forces where the motion was restricted to a plane $\vec{J} = mh\hat{k}$ = constant vector.

What causes $\vec{J}$ to change?

$$\frac{d\vec{J}}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

$$= m\cancelto{0}{[\vec{v} \times \vec{v}]} + \vec{r} \times \vec{F} = \vec{\tau}$$

Definition. $\vec{\tau} = \vec{r} \times \vec{F}$ is the *Torque* or the *Moment*.

- $\vec{\tau}$ is in the direction out of the screen
- $|\vec{\tau}| = |\vec{F}||\vec{r}| \sin\phi$

For central forces

Since $\phi = 0 \implies \vec{\tau} = 0$.

For a system, the total angular momentum

$$\vec{J} = \sum_i \vec{J}_i = \sum_i \vec{r}_i \times m_i\vec{v}_i$$

$$\implies \vec{\tau} = \frac{d\vec{J}}{dt} = \sum_i \frac{d\vec{J}_i}{dt} = \sum_i \vec{r}_i \times \vec{F}_i$$

Write $\vec{F}_i = \vec{F}_i^{EXT} + \sum_j \vec{F}_{ij}$. Then we have

$$\vec{\tau}\frac{d\vec{J}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT} + \sum_i \sum_j \vec{r}_i \times \vec{F}_{ij} \tag{7.9}$$

Theorem 7.10: Conservation of Angular Momentum for a System

If there is no net torque, the angular momentum is conserved.

Proof (for two body system). Suppose we have two particles. Then the double sum is

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

By Newton's Third Law $\vec{F}_{12} = -\vec{F}_{21}$. Thus

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12}$$

If $\vec{F}_{12}$ is parallel to $\vec{r}_1 - \vec{r}_2$, then $(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = 0$.

This is the case if $\vec{F}_{12}$ is a central force, i.e. no torque.

Thus if $\vec{F}_{ij}$ is a central force for all i and j. Then

$$\sum_i \sum_j \vec{r}_i \times \vec{F}_{ij} = \vec{0}$$

Then

$$\frac{d\vec{J}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT} = \vec{\tau}^{EXT}$$

where $\vec{\tau}^{EXT}$ is the total external torque on the system.

So if $\vec{\tau}^{EXT} = \vec{0}$ then $\dfrac{d\vec{J}}{dt} = \vec{0}$, hence the angular momentum is conserved. ∎

Example 7.11.

Each particle has mass m. Each mass has velocity $\vec{v}_i = \vec{\omega} \times \vec{r}_i$, with $\vec{\omega} = \omega\hat{k}$

The angular momentum of particle i is:

$$\vec{J}_i = \vec{r}_i \times m_i\vec{v}_i = m[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)]$$

Recall that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$$\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = (\vec{r}_i \cdot \vec{r}_i)\vec{\omega} - \cancelto{0}{(\vec{r}_i \cdot \vec{\omega})}\,\vec{r}_i = r^2\omega\hat{k}$$

Thus

$$\vec{J}_i = mr^2\omega\hat{k}$$
$$\implies \vec{J} = \sum_i \vec{J}_i = 4mr^2\omega\hat{k} = 2ml^2\omega\hat{k}$$

Suppose that

$\vec{v}_i = \vec{\omega} \times \vec{r}_i \longrightarrow \vec{v}_i = \vec{\Omega} \times \vec{r}_i$. What's $\vec{\Omega}$?

Single the configuration changed to to internal, central forces, $\dfrac{d\vec{J}}{dt} = 0$

For our new configuration

$$\vec{J}_i = 2m[\vec{r}_i \times (\vec{\Omega} \times \vec{r}_i)] = 2mr_i^2 \Omega \hat{k} = \frac{ml^2\Omega}{2}\hat{k}$$

The total angular momentum

$$\vec{J} = 2\vec{J}_i = ml^2\Omega\hat{k}$$

Since $\frac{d\vec{J}}{dt} = 0 \implies \vec{J}_{before} = \vec{J}_{after}$

$$\implies 2ml^2\omega\hat{k} = ml^2\Omega\hat{k}$$
$$\implies \Omega = 2\omega$$

The angular speed doubles as a result of the change.

Centre of Mass Coordinates

Lecture 29

$$\vec{r}_i = \vec{r}_{cm} + \vec{R}_i$$
$$\vec{v}_i = \vec{v}_{cm} + \vec{u}_i, \quad \left(\vec{u}_i = \frac{d\vec{R}_i}{dt}\right)$$

Thus

$$\begin{aligned}\vec{J} &= \sum_i (\vec{r}_{cm} + \vec{R}_i) \times m_i(\vec{v}_{cm} + \vec{u}_i) \\ &= \sum_i \vec{r}_{cm} \times m_i\vec{v}_{cm} + \sum_i \vec{r}_{cm} \times m_i\vec{u}_i + \sum_i \vec{R}_i \times m_i\vec{v}_{cm} + \sum_i \vec{R}_i \times m_i\vec{u}_i \\ &= \vec{r}_{cm} \times \vec{v}_{cm}(\sum_i m_i) + \vec{r}_{cm} \times (\sum_i m_i\vec{u}_i) + (\sum_i m_iR_i) \times \vec{v}_{cm} + \sum_i R_i \times m_i\vec{u}_i\end{aligned}$$

We know that $\sum_i m_i = M$, $\sum_i m_i\vec{R}_i = \sum_i m_i\vec{u}_i = 0$. Thus

$$\vec{J} = \vec{r}_{cm} \times M\vec{v}_{cm} + \sum_i \vec{R}_i \times m_i\vec{u}_i$$

Call $\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i\vec{u}_i$

Recall that

$$\frac{d\vec{J}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT} (= \vec{\tau}^{EXT})$$

Since $\vec{r}_i = \vec{r}_{cm} + \vec{R}_i$

$$\begin{aligned}\frac{d\vec{J}}{dt} &== \sum_i \vec{r}_{cm} \times \vec{F}_i^{EXT} \\ &= \vec{r}_{cm} \times \vec{F}^{EXT} + \sum_i \vec{R}_i \times \vec{F}_i^{EXT}\end{aligned}$$

We can show (P.S. 4 Problem 9)

$$\frac{d\vec{J}_{cm}}{dt} = \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

Call

$$\vec{\tau}_{cm}^{EXT} = \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

Complete Picture

(i) Momentum:

$$\vec{p} = M\vec{v}_{cm}$$

$$\frac{d\vec{p}}{dt} = M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT}$$

(ii) Angular Momentum:

$$\vec{J} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{J}_{cm}$$

$$\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i\vec{u}_i$$

$$\frac{d\vec{J}}{dt} = \vec{r}_{cm} \times \vec{F}^{EXT} + \vec{\tau}_{cm}^{EXT}$$

8 Rigid Body Motion

Definition. *Rigid Body Motion* occurs when

$$\frac{d|\vec{r}\_\vec{r}_j|}{dt} = 0,\ \forall i, j$$

For such a system

$$\vec{v}_i = \vec{v}_{cm} + \underbrace{\vec{\omega} \times \vec{R}_i}_{\vec{u}_i}$$

Where $\vec{\omega}$ is the *angular velocity of the rigid body.*

We can also write

$$\vec{v}_i = \vec{V} + \vec{\omega} \times \vec{r}_i$$

where $\vec{V} = \vec{v}_{cm} - \vec{\omega} \times \vec{r}_{cm}$

To determine the motion of the system we'll need to find $\vec{v}_{cm}$ and $\vec{\omega}$. For $\vec{v}_{cm}$ we already have this!

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT} \tag{8.1}$$

What about $\vec{\omega}$?

$$\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i \vec{u}_i$$

For a rigid body $\vec{u}_i = \vec{\omega} \times \vec{R}_i$

$$\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i(\vec{\omega} \times \vec{R})i) = \sum_i m_i(\vec{R}_i \times (\vec{\omega} \times \vec{R}_i))$$

From the identity for the triple vector product, we have

$$\vec{J}_{cm} = \sum_i m_i[R_i^2\vec{\omega} - (\vec{\omega} \cdot \vec{R}_i)\vec{R}_i]$$

Consider only planar motion: we have $\vec{\omega} = \omega\hat{k}$, and $\vec{R}_i = X_i\hat{\imath} + Y_i\hat{\jmath}$. Thus

$$\vec{\omega} \cdot \vec{R}_i = 0,\ \forall i$$

As a result:

$$\vec{J}_{cm} = \underbrace{(\sum_i m_i R_i^2)}_{I_{cm}}\,\vec{\omega} \tag{8.2}$$

Definition. I_{cm} is the *moment of inertia* about the centre of mass.

For this Rigid Body Motion $\frac{d|R_i|}{dt} = 0$. This means that I_{cm} is constant.

Consider

$$\frac{d\vec{J}_{cm}}{dt} = I_{cm}\frac{d\vec{\omega}}{dt} = \vec{\tau}_{cm}^{EXT} = \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

For a rigid body undergoing planar motion:

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT} \tag{8.3}$$

$$I_{cm}\frac{d\omega}{dt} = \tau_{cm}^{EXT} \tag{8.4}$$

(Scalar Equation since all in $\hat{k}$)

Kinetic Energy

$$T = \frac{1}{2}M\vec{v}_{cm}^2 + \frac{1}{2}\sum_i m_i\vec{u}_i^2$$

$\vec{u}_i = \vec{\omega}\times\vec{R}_i$, $u_i^2 = (\vec{\omega}\times\vec{R}_i)\cdot(\vec{\omega}\times\vec{R}_i)$

For planar motion $|\vec{\omega}\times\vec{R}_i| = |\vec{\omega}||\vec{R}_i| \implies u_i^2 = \omega^2 R_i^2$

$$T = \frac{1}{2}M\vec{v}_{cm}^2 + \frac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2$$

$$\implies T = \frac{1}{2}M\vec{v}_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

Lecture 30

Definition. The continuous case:

$$M = \sum_i m_i = \int_B dm$$

$$\vec{r}_{cm} = \frac{\sum_i m_i\vec{r}_i}{M} = \frac{\int_B \vec{r}\,dm}{M}$$

$$I_{cm} = \sum_i m_i R_i^2 = \int_B R^2\,dm$$

Equations of motion remain the same.

Example 8.5 (Uniform Rod)**.**

* Parallel Axis Theorem *

Lecture 31

(Non-examinable in 2015)

Theorem 8.6: Parallel Axis Theorem

For an axis, P, parallel to the centre of mass

$$I_P = I_{CM} + Mr_{CM}^2$$

Proof.

$$\begin{aligned} I_P = \sum_i m_i r_i^2 &= \sum_i m_i(\vec{r}_{CM} + \vec{R}_i)^2 \\ &= \sum_i m_i r_{CM}^2 + 2 \end{aligned}$$

∎

Example 8.7 (Physical Pendulum)**.** Blah

- End of Mechanics -