1st Year Mathematics
Imperial College London

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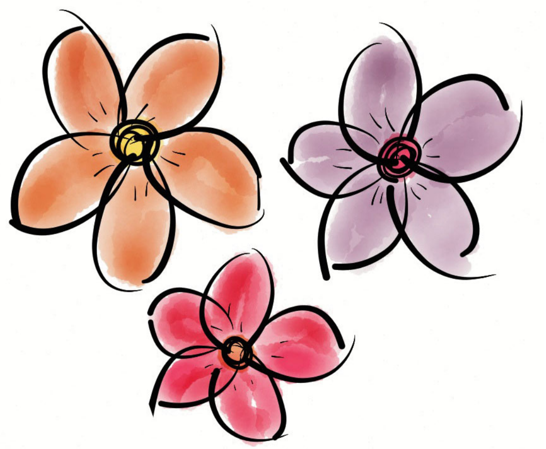
Algebra I

Lectured by:
Prof. J. Britnell

Humbly typed by:
Karim Bacchus



Comments and corrections should be sent to kb514@ic.ac.uk.

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Syllabus

Introductions to three topics in abstract algebra: The theory of vector spaces and linear transformations, the theory of groups and the theory of rings.

- Vector spaces: Linear maps, rank-nullity theorem, connections with linear equations and matrices.
- Groups: Axioms, examples. Cyclic groups, dihedral groups, symmetric groups. Lagranges theorem and applications.
- Rings: Polynomial rings, rings of the form $\mathrm{Z}[\sqrt{d}]$. Euclids algorithm for certain rings. Uniqueness of factorisation for these rings. Applications to Diophantine Equations.

Appropriate books

J. Fraleigh and R. Beauregard, *Linear Algebra*

S. Lipschutz and M. Lipson, *Linear Algebra*

J. B. Fraleigh, *A First Course in Abstract Algebra*

R. Allenby, *Rings, Fields and Groups*

I. N. Herstein, *Topics in Algebra*

Contents

1 Vector Spaces

Spanning Sets and Bases (M1GLA Review)

Lecture 1

Let V be a vector space, and let $S = \{v_1, \dots, v_k\}$ be a finite subset of V. Then the *span* of S is the set $\{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k, \lambda_1, \dots, \lambda_k \text{ scalar}\}$.

* Span S is a subspace of V.
* If Span $S = V$, then we say that S is a *spanning set* for V.
* If V has a finite spanning set, then we say that V is finite dimensional.

Assume from now on that V is finite dimensional. The set S is *linearly independent* if the only solution to the equation $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = \mathbf{0}$ is $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

A *basis* for V is a linearly independent spanning set.

* V has a basis
* Every basis of V has the same size. This is the *dimension* of V, written dim V.

Suppose that dim $V = n$.

* Any spanning set of size n is a basis.
* Any linearly independent set of size n is a basis.
* Every spanning set contains a basis as a subset.
* Every linearly independent set is contained as a subset in a basis.
* Any subset of V of size $< n$ is **not** a spanning set.
* Any subset of V of size $> n$ is **not** linearly independent.
* If W is a subspace of V then dim $W \leq$ dim V eith equality only when $W = V$.

Every vector space has associated with it a set of *scalars*. E.g. $\mathbb{R}^n$ has the scalar set $\mathbb{R}$. The scalars come as a structure called a *field* (To be defined in the Ring Theory Section). I'll write F for the field of scalars. It will usually be safe to assume that $F = \mathbb{R}$. Other fields include $\mathbb{C}, \mathbb{Q}$, integers and $(\mathbb{Z} \backslash p\mathbb{Z})^\times$.

More on Subspaces

> **Definition 1.** Let V be a vector space, and let U and W be subspaces of V. The *intersection* of U and W is $U \cap W = \{v : v \in U \text{ and } v \in W\}$. The *subspace sum* (or just *sum*) of U and W is $U + W = \{u + w : u \in U, w \in W\}$

Remark 2. Note that $\mathbf{0} \in U$ and $\mathbf{0} \in W$, so if $u \in U$, then $u = \mathbf{0} \in U + W$, and similarly if $w \in W$, then $w = \mathbf{0} \in U + W$. So $U \subseteq U + W$ and $W \subseteq U + W$. ($U + W$ usually contains many other vectors)

> **Example 3.** Let $V \in \mathbb{R}^2$. Let $U = \text{Span}\{(1,0)\}$, and $W = \text{Span}\{(0,1)\}$. So $U = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$, $W = \{(0, \lambda : \lambda \in \mathbb{R}\}$.

> We see that $U + W$ contains (λ, μ) for all $\lambda, \mu \in \mathbb{R}$ and so $U + W = V$.

Proposition 4. *$U \cap W$ and $U + W$ are both subspaces of V.*

Proof. Do $U + W$ first. Checking the subspace axioms:

(i) $\mathbf{0} \in U$ and $\mathbf{0} \in W \implies \mathbf{0} = \mathbf{0} + \mathbf{0} \in U + W$. So $U + W \neq \emptyset$.

(ii) Suppose $v_1, v_2 \in U + W$. Then $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$, where $u_1, u_2 \in U$ and $w_1, w_2 \in W$.

Now $v_1 + v_2 = u_1 + w_1 + u_2 + w_2 = (u_1 + u_2) + (w_1 + w_2) \in U + W \implies$ closed under addition.

(iii) Let $v \in U + W$. Then $v = u + w$ where $u \in U$ and $w \in W$. Let $\lambda \in F$. Then $\lambda v = \lambda(u + w) = \lambda u + \lambda w \in U + W$.

Now do $U \cap W$:

(i) $\mathbf{0} \in U$ and $\mathbf{0} \in W$, so $\mathbf{0} \in U \cap W$ by definition.

(ii) Suppose $v_1, v_2 \in U \cap W$. Then $v_1, v_2 \in U$ and so $v_1 + v_2 \in U$ since U is closed. Similarly for $v_1 + v_2 \in W$. Hence $v_1 + v_2 \in U \cap W$.

(iii) Suppose $v \in U \cap W$, and $\lambda \in F$. Then $v \in U$ and so $\lambda v \in U$. $v \in W \implies \lambda v \in W$. Hence $\lambda v \in U \cap W$. ∎

Proposition 5. *Let V be a vector space and let U & W be subspaces. Suppose that $U =$ Span$\{u_1, \dots, u_r\}$ and $W =$ Span$\{w_1, \dots, w_s\}$. Then $U + W =$ Span$\{u_1, \dots, u_r, w_1, \dots, w_s\}$.* *Lecture 2*

Proof. By inclusion both ways:

Notice that $u_i \in U \subseteq U + W$ (by Remark 2) and similarly $w_i \in W \subseteq U + W$, $\forall i$. Since $U + W$ is a subspace of V, so $U + W$ is closed under linear combinations, so $\mathrm{Span}\{u_1, \dots, u_r, w_1, \dots, w_s\} \subseteq U + W$.

For the reverse inclusion, let $v \in U + W$.
Then $v = u + w$ for some $u \in U, w \in W$. Since $U = \mathrm{Span}\{u_1, \dots, u_r\}$, we have $u = \lambda_1 u_1 + \cdots + \lambda_r u_r$, for some $\lambda_1, \dots, \lambda_r \in F$. Similarly $w = \mu_1 w_1 + \cdots + \mu_s w_s$, for some $\mu_1, \dots, \mu_s \in F$.

Now $v = \lambda_1 u_1 \dots \lambda_r u_r + \mu_1 w_1 \dots \mu_s w_s \in \mathrm{Span}\{u_1, \dots, u_r, w_1, \dots, w_s\}$. Hence $U + W \subseteq \mathrm{Span}\{u_1, \dots, u_r, w_1, \dots, w_s\}$.

∎

> **Examples 6.**
>
> **Question:** Let $V = \mathbb{R}^4, U = \mathrm{Span}\{(1, 1, 2, -3), (1, 2, 0, -3)\}$ and $W = \mathrm{Span}\{(1, 0, 5, -4), (-1 - 3, 0, 5)\}$. Find a basis for $U + W$.
>
> **Answer:** By Proposition 5, we have:

$U+W=\text{Span}\{(1,1,2,-3),(1,2,0,-3),(1,0,5,-4),(-1,-3,0,5)\}$.

We then just row reduce the matrix:

$$\begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & 2 & 0 & -3 \\ 1 & 0 & 5 & -4 \\ -1 & -3 & 0 & 5 \end{pmatrix} \to \textit{Echelon Stuff} \to \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The three non-zero rows are linearly independent, and have the same span as the original four vectors, so a basis for $U+W$ is:

$$\{(1,1,2,-3),(0,1,-2,0),(0,0,1,-1)\} \text{ (or just the first 3 vectors).}$$

Question: What about a basis for $U \cap W$?

Answer: If $v \in U \cap W$, then $v \in U = \text{Span}\{(1,1,2,-3),(1,2,0,-3)\}$. So $v = a(1,1,2,-3)+b(1,2,0,-3)$ for $a,b \in \mathbb{R}$.

And $v \in W = \text{Span}\{(1,0,5,-4),(-1,-3,0,5)\}$. So $v = c(1,0,5,-4)+d(-1,-3,0,5)$. So we have:

$$a(1,1,2,-3)+b(1,2,0,-3)-c(1,0,5,-4)-d(-1,3,0,5) = \mathbf{0} \ (*)$$

$(*)$ gives us 4 simultaneous equations, which we can encode as a matrix equation:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & 0 & -3 \\ 2 & 0 & -5 & 0 \\ -3 & -3 & 4 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0}$$

We find the solution space by row reducing:

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & 0 & -3 \\ 2 & 0 & -5 & 0 \\ -3 & -3 & 4 & -5 \end{pmatrix} \to \textit{Echelon Stuff} \to \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There is one line of solutions, given by $a = 5d, b = -4d, c = 2d$. So pick $d = 1$, then $a = 5, b = -4, c = 2, d = 1$.

So $v = a(1,1,2,-3)+b(1,2,0,-3) = (5,5,10,-15)-(4,8,0,-12) = (1,-3,10,3)$.

We can check our solutions with c,d: $v = c(1,0,5,-4)+d(-1,-3,0,5) = (1,-3,10,3)$ as expected. So $U \cap W$ is 1-dimensional, and a basis is $\{(1,-3,10,3)\}$.

Theorem 7

Let V be a vector space, and let U and W be subspaces. Then dim $U + W =$ dim $U+$ dim $W-$ dim $U \cap W$.

Proof. Let dim $U = r$, dim $W = s$, dim $U \cap W = m$. Let $\{v_1, \dots, v_m\}$ be a basis for $U \cap W$.

Then $\{v_1, \dots, v_m\}$ is a linearly independent subset of U. So it is contained in some basis for U. So there exists $u_1, \dots, u_{r-m}$ in U such that $\{v_1, \dots, v_m, u_1, \dots, u_{r-m}\}$ is a basis for U. Similarly, there exists $w_1, \dots, w_{s-m}$ such that $\{v_1, \dots, v_m, w_1, \dots, w_{s-m}\}$.

Now let $B = \{v_1, \dots, v_m, u_1, \dots, u_{r-m}, w_1, \dots, w_{s-m}\}$. Then Span $B = U + W$ by Proposition 5.

Claim. B is linearly independent.

Proof of claim:

Suppose that $\alpha_1 v_1 + \dots \alpha_m v_m + \beta_1 u_1 + \dots \beta_{r-m} u_{r-m} + \gamma_1 w_1 + \dots \gamma_{s-m} w_{s-m} = \mathbf{0}$. For $\alpha_i, \beta_i, \gamma_i \in F$. Then:

$$\sum_{i=1}^{s-m} \gamma_i w_i = -\sum_{i=1}^{m} \alpha_i v_i - \sum_{i=1}^{r-m} \beta_i u_i \in U$$

$$\implies \sum_{i=1}^{s-m} \in U \cap W \implies \sum_{i=1}^{s-m} = \sum_{i=1}^{m} \delta_i v_i, \text{ for some } \delta_i \in F$$

$$\text{Hence: } \sum_{i=1}^{m} \delta_i v_i - \sum_{i=1}^{s-m} \gamma_i w_i = \mathbf{0}.$$

But $\{v_1, \dots, v_m, w_1, \dots, w_{s-m}\}$ is linearly independent (being a basis for W). So $\gamma_i = \delta_i = 0, \forall i$. Since $\gamma_i = 0 \ \forall i$, we have:

$$\sum_{i=1}^{m} \alpha_i v_i + \sum_{i=1}^{r-m} \beta_i u_i = \mathbf{0}.$$

But $\{v_1, \dots, v_m, u_1, \dots, u_{r-m}\}$ is linearly independent (being a basis for U). So $\alpha_i = \beta_i = 0, \forall i$. Hence B is linearly independent. So B is a basis for $U + V$. So dim $U + W = |B| = m + (r - m) + (s - m) = r + s - m$. ∎

Example 8. Question: Let $V = \mathbb{R}^3$, and let $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Similarly let $W = \{(x, y, z) \in \mathbb{R}^3 : -x + 2y + z = 0\}$. Find bases for $U, W, U \cap W, U + W$. Lecture 3

Answer: A general element of U is $(x, y, -x - y) = x(1, 0, -1) + y(0, 1, -1) \implies$

$U = \text{Span}\{(1,0,-1),(0,1,-1)\}$. Since this set is clearly linearly independent, it's a basis for U.

A general element of W is $(x,y,x-2y) = x(1,0,1)+y(0,1,-2) \implies W = \text{Span}\{(1,0,1),(0,1,-2)\}$. Again, clearly linearly independent, so a basis for W.

Suppose that $v = (x,y,z)$ lies in $U \cap W$. Then $v \in U$, and so $z = -x-y$. But also $v \in W$, and so $z = x-2y$. Hence $-x-y = x-2y \implies y = 2x$.

So a general element of $U \cap W$ is $(x,2x,-3x) = x(1,2,-3) \implies U \cap W = \{(1,2,-3)\}$ is a basis for $U \cap W$.

As in the proof of theorem 7, we find a basis for U, and W, each of which contain a basis for $U \cap W$. So any linearly independent subset of U of size 2 is a basis for U, i.e. $U = \{(1,2,-3),(1,0,-1)\}$. Similarly $W = \{(1,2,-3),(1,0,1)\}$. So a spanning set for $U+W = \{(1,2,-3),(1,0,-1),(1,0,1)\}$.

By Theorem 7, we know $\dim U+W = \dim U + \dim W - \dim U \cap W = 2+2-1 = 3$. Hence our spanning set is a basis for $U+W$.

Rank of a Matrix

Definition 9. Let A be an $m \times n$ matrix with entries from F. Define the *row-span* of A by $\text{RSp}(A) = \text{Span}\{\text{rows of } A\}$. This is a subspace of F^n. The *column-span*, $\text{CSp}(A)$ of A is $\text{Span}\{\text{columns of } A\}$, again a subspace of F^n.

The *row-rank* of A is $\dim \text{RSp}(A)$. The *column-rank* of A is $\dim \text{CSp}(A)$ i.e. the number of linearly independent rows / columns.

Example 10. $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$, $F = \mathbb{R}$

$\text{RSp}(A) = \text{Span}\{(3,1,2),(0,-1,1)\}$. Since the two vectors are linearly independent, we have that the row-rank $= \dim \text{RSp}(A) = 2$.

$$\text{CSp}(A) = \text{Span}\left\{\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}.$$

This is linearly dependent, since $\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

So $\text{CSp}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \implies \text{CSp}(A) = 2$.

How do we calculate the row-rank of a matrix A?

Procedure 11.

Step 1: *Use row operations to reduce the matrix A, to row echelon form.*

$$A_{ech} = \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Step 2: *The row-rank of A is the number of non-zero rows in A_{ech}, and the non-zero rows of A_{ech} form a basis for $RSp(A)$. [i.e. you don't need to go back to original vectors get the basis.]*

Justification We need to show:

(i) $\mathrm{RSp}(A) = \mathrm{RSp}(A_{ech})$

(ii) The non-zero rows of A_{ech} are linearly independent.

To show (1), recall that A_{ech} is obtained from A by a series of row-operations:

$$\begin{cases} r_i := r_i + \lambda r_j & (i \neq j) \\ r_i := \lambda r_i & (\lambda \neq 0) \\ r_i \longleftrightarrow r_j & (i \neq j) \end{cases}$$

Suppose that A' is obtained from A by one row-operation. Then it is clear than every row of A' lies in $\mathrm{RSp}(A)$. So $\mathrm{RSp}(A') \subseteq \mathrm{RSp}(A)$. But every row operation is invertible by another row operation. i.e.

$$\begin{cases} r_i := r_i + \lambda r_j & \text{has inverse} \;\; r_i := r_i - \lambda r_j \\ r_i := \lambda r_i & \text{has inverse} \;\; r_i := r_i - 1/\lambda r_j \\ r_i \longleftrightarrow r_j & \text{has inverse} \;\; r_i \longleftrightarrow r_j \end{cases}$$

So we have $\mathrm{RSp}(A) \subseteq \mathrm{RSp}(A')$, and so the row spaces are equal. It follows that $\mathrm{RSp}(A_{ech}) = \mathrm{RSp}(A)$.

For (2), consider the form of A_{ech}, and denote $r_1, \dots r_k$ as the non-zero rows. Say r_i has it's leading entry in column c_i. Suppose that $\lambda_1 r_1 + \cdots + \lambda_k r_k = \mathbf{0}$ $(*)$.

Look at the co-ordinate corresponding to column c_1. The only contribution is $\lambda_1 r_1$ since all of the other rows have 0 in that co-ordinate. Since r_1 has 1 in this co-ordinate $\implies \lambda_1 = 0$.

Since $\lambda_1 = 0$, the only contribution to c_2 is $\lambda_2 r_2$. So $\lambda_2 = 0$. We can continue this argument for $c_2, \dots, c_i, \dots, c_k \implies$ the only solution to $(*)$ is $\lambda_1, \dots, \lambda_k = 0 \implies$ linearly independence of vectors.

Lecture 4

Example 12. Find the row-rank of $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ -1 & 4 & 15 \end{pmatrix}$

Answer: Reduce A to echelon form:

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ -1 & 4 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 10/3 \\ 0 & 0 & 0 \end{pmatrix} = A_{ech}$$

Since A_{ech} has two non-zero rows, the row-rank of A is 2. (Note: Scaling the second row to make the leading entry 1 was not necessary)

Example 13. Find the dimension of $W = \text{Span}\{(-1,1,0,1),(2,3,1,0),(0,1,2,3)\} \subseteq \mathbb{R}^4$

Answer: Notice that $W = \text{RSp}\begin{pmatrix} -1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} = A$

$$\implies A_{ech} = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 9 & 12 \end{pmatrix}$$

There are 3 non-zero rows in A_{ech}, so the row rank is 3.
Hence $\dim W = 3$.

Procedure 14. *The columns of A are the rows of A^T. So apply Procedure 11 to the matrix A^T. (Alternatively, use column operations to reduce A to "column echelon form", and then count the no. of non-zero columns).*

Example 15. Let $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ -1 & 4 & 15 \end{pmatrix}$

The column rank of A is the row-rank of A^T.

$$A^T = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 5 & 0 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

There are two non-zero rows, so row-rank$(A^T) = 2$, and so column-rank $(A) = 2$. A basis for $\text{RSp}(A^T)$ is $\{(1,2,-1),(0,-3,6)\}$, so a basis for $\text{CSp}(A)$ is the transponse of these vectors.

Theorem 16

For any matrix A, the row-rank of A is equal to the column-rank of A.

Proof. Let the rows of A be $r_1, \dots, r_m$, so $r_i = (a_{i1}, a_{i2}, \dots, a_{in})$.

Let the columns of A be $c_1, \dots, c_n$, so $c_j = (a_{1j}, a_{2j}, \dots, a_{mj})^T$.

Let k be row-rank of A. Then $\mathrm{RSp}(A)$ has basis $\{v_1, \dots, v_k\}, v_i \in F^n$. Every row r_i is a linear combination of $v_1, \dots, v_k$. Say that:

$$r_i = \lambda_{i1}v_1 + \lambda_{i2}v_2 + \cdots + \lambda_{il}v_k \ (*)$$

Let $v_i = (b_{i1}, \dots, b_{in})$. Looking at the jth co-ordinate in $(*)$, we have $a_{ij} = \lambda_{i1}b_{1j} + \lambda_{i2}b2j + \cdots + \lambda_{ik}b_{kj}$

$$c_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix} = \begin{pmatrix} \lambda_{11}b_{1j} + \lambda_{12}b_{2j} + \cdots + \lambda_{1k}b_{kj} \\ \lambda_{21}b_{2j} + \lambda_{22}b_{2j} + \cdots + \lambda_{2k}b_{kj} \\ \dots \\ \lambda_{m1}b_{1j} + \lambda_{m2}b_{2j} + \cdots + \lambda_{mk}b_{kj} \end{pmatrix}$$

So every column of A is a linear combination of $(\lambda_{1i}, \lambda_{2i}, \dots, \lambda_{mi})^T$ for $1 \leq i \leq k$. So the column space of A is spanned by k vectors and so column-rank$(A) \leq k =$ row-rank(A).

But row-rank$(A) =$ column-rank(A^T), column-rank$(A) =$ row-rank(A^T).

By the argument above, column-rank$(A^T) \leq$ row-rank(A). So row-rank$(A) \leq$ column-rank(A). Hence row-rank$(A) =$ column-rank(A). ∎

Definition 17. Let A be matrix. The *rank* of A, written $rk(A)$, is the row-rank of A. (which is also the column-rank of (A).)

Example 18. Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 \end{pmatrix}$

Notice that $r_3 = r_1 + r_2$. So a basis for $RSp(A)$ is $\{(1, 2, -1, 0), (-1, 1, 0, 1)\}$

Write the rows of A as linear combinations of $\{v_1, v_2\}$:

$$r_1 = 1v_1 + 0v_2 \quad r_2 = 0v_1 + 1v_2 \quad r_3 = 1v_1 + 1v_2$$

(The scalars here are the λ_{ij} from the proof of Theorem 16)

According to the proof, a spanning set of for $CSp(A)$ is given by:

$$\{(1, 0, 1)^T, (0, 1, 1)^T\}$$

We verify this - We have $c_1 = (1, -1, 0)^T = w_1 - w_2$. $c_2 = (2, 1, 3)^T = 2w_1 + w_2$, $c_3 = (-1, 0, -1)^T = -w_1$, and $c_4 = (0, 1, 1)^T = w_2$.

Lecture 5 **Proposition 19.** *Let A be an $n \times n$ (square) matrix. Then the following statements are equivalent:*

(i) $rk(A) = n$ (A has "full rank")

(ii) The rows of A form a basis for F^n

(iii) The columns of A form a basis of F^n

(iv) A is invertible (so $detA \neq 0$, row reduced to I etc.)

Proof. (1) $\iff$ (2):

$$\begin{aligned}\text{rk}(A) = n &\iff \dim \text{RSp}(A) = n \\ &\iff \text{RSp}(A) = F^n \\ &\iff \text{Rows of } A \text{ are spanning a set for } F^n \text{ of size } n \\ &\iff \text{The rows of } A \text{ form a basis for } F^n\end{aligned}$$

(1) $\iff$ (3): The same, using columns instead.

(1) $\iff$ (4):

$$\begin{aligned}\text{rk}(A) = n &\iff A_{ech} = I \\ &\iff A \text{ can be row-reduced to } I \\ &\iff A \text{ is invertible.}\end{aligned}$$

∎

Linear Transformations

Suppose that V and W are vector spaces over a field F. Let $T : V \to W$ be a function.

* Say that T "preserves addition", if whenever $T : V \mapsto W$ and $T : v_2 \mapsto w_2$, we also have $T : v_1 + v_2 \mapsto w_1 + w_2$. (Briefly: $T(v_1 + v_2) = Tv_1 + Tv_2$.)
* Say that T "preserves scalar multiplcation", if whenever $T : v \mapsto w$ and $\lambda \in F \implies T : \lambda v \mapsto \lambda w$. (Briefly: $T(\lambda v) = \lambda T(v)$)

Definition 20. The function $T : V \to W$ is a *linear transformation* (or *linear map*), if it preserves addition and scalar multiplication. So:

$$T(v_1 + v_2) = Tv_1 + Tv_2 \text{ and } T(\lambda v) = \lambda(T(v)), \forall v_1, v_2, v \in V \;\&\; \lambda \in F$$

Examples 21.

(a) $T : \mathbb{R}^2 \to \mathbb{R}, T(x, y) = x + y$. I claim this is a linear transformation.
Check it preserves addition:

$$\begin{aligned}T((x_1, y_1) + (x_2, y_2)) &= T((x_1 + x_2, y_1 + y_2)) \\ &= x_1 + x_2 + y_1 + y_2 \\ &= x_1 + y_1 + x_2 + y_2 \\ &= T((x_1, y_1)) + T((x_2, y_2))\end{aligned}$$

And T also preserves scalar multiplicaton, since if $\lambda \in \mathbb{R}$, then:

$$T(\lambda(x,y)) = T((\lambda x, \lambda y)) = \lambda x + \lambda y = \lambda(x+y) = \lambda T((x,y))$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}, T(x,y) = x + y + 1$. This is not linear.
For example, $2T((1{,}0)) = 4$, but $T((2,0)) = 3$, so it doesn't preserve scalar multiplication $\implies$ not a linear map.

(c) $T : \mathbb{R} \to \mathbb{R}\ T(x) = \sin(x)$, is not linear.
$2T(\pi/2) = 2$, but $T(2 \times \pi/2) = T(\pi) = 0$. Again it doesn't preserve scalar multiplication, so not a linear map.

(d) Let V be the space of all polynomials in a single variable x with co-efficients from $\mathbb{R}$. Define $T : v \to V$ by $T(f(x)) = \frac{d}{dx} f(x)$. Then T is a linear transformation.

(I) $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \implies$ preserves addition

(II) $\frac{d}{dx}(\lambda f(x)) = \lambda \frac{d}{dx} f(x) \implies$ preserves scalar multiplication

Proposition 22. *Let A be an $m \times n$ matrix over F. Define $T : F^n \to F^m$ by $T(b) = Av$. This is a linear transformation. (We say that T is a* matrix transformation.*)*

Proof.

1. $$\begin{aligned} T(v_1 + v_2) &= A(v_1 + v_2) \\ &= Av_1 + Av_2 \\ &= Tv_1 + Tv_2 \quad \forall v_1, v_2 \in F^n \implies T \text{ preserves addition.} \end{aligned}$$

2. $$\begin{aligned} T(\lambda v) &= A(\lambda v) = \lambda Av \\ &= \lambda Tv \quad \forall v \in F^n, \lambda \in F \implies T \text{ preserves scalar multiplication.} \end{aligned}$$ ∎

Examples 23.

(a) Define a map $T : \mathbb{R}^3 \to \mathbb{R}^2$ by:

$$T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 - 2a_2 + a_3 \\ a_1 + a_2 - 2a_3 \end{pmatrix}$$

Then T is linear because:

$$T \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and so T is a matrix transformation. So Proposition 22 applies.

(b) Define $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ to be a rotation through an angle of θ (aniticlockwise). Then ρ_θ is linear since it is given by the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Proposition 24. (Basic properties of linear transformations)

Let $T : V \to W$ be a linear transformation.

(i) If $\mathbf{0}_v$ is the zero vector in V and $\mathbf{0}_w$ is the zero vector in w, then $T(\mathbf{0}_v) = \mathbf{0}_w$

(ii) Suppose that $v_1, \ldots, v_k \in V$ and that $v = \lambda_1 v_1 + \ldots \lambda_k v_k$ ($\lambda_i \in F$). Then $Tv = \lambda_1 Tv_1 + \lambda_2 Tv_2 + \cdots + \lambda_k Tv_k$.

Proof. (i). Since T preserves scalar multiplication, for any $v \in V$, we have $T(0v) = 0Tv$, so $T(\mathbf{0}_v) = \mathbf{0}_w$. ∎

Lecture 6 *Proof.* (ii) We observe:

$$\begin{aligned} T(v) &= T(\lambda_1 v_1 + \ldots \lambda_k v_k) \\ &= T(\lambda_1 v_1 + \ldots \lambda_{k-1} v_{k-1}) + T(\lambda_k v_k) \\ &= T(\lambda_1 v_1 + \ldots \lambda_{k-1} v_{k-1}) + \lambda_k v_k \end{aligned}$$

Now a straightforward argument by induction tells us that:

$$T(v) = \lambda_1 T(v_1) + \lambda_2 T(v_2) + \ldots \lambda_k T(v_k)$$ ∎

Example 25. Question: Find a linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^3$, which sends $(1,0) \to (1,-1,2)$ and sends $(0,1) \to (0,1,3)$.

Answer: Notice that $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$. So a general element of $\mathbb{R}^2$ is $a(1,0) + b(0,1)$, for $a, b \in \mathbb{R}$.

So if T is a solution to the question, then we must have $T(a(1,0) + b(0,1)) = a(1,-1,2) + b(0,1,3)$, by Proposition 24(ii).

So we are *forced* to take $T(a,b) = (a, b-a, 2a+3b)$. This is indeed a linear transformation, since it is a matrix transformation. And we do have $T(0,1) = (1,-1,2)$ and $T(0,1) = (0,1,3)$ as required. So this is a solution and it is the unique solution.

Proposition 26. *Let V and W be vector spaces over F. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let $\{w_1, \ldots, w_n\}$ be any n vectors in W. Then there is a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i \ \forall i$.*

Remark. The vectors $w_1, \ldots w_n$ don't have to be linearly independent, or even distinct.

Proof. Suppose that $v \in V$. Then there exists unique scalars $\lambda_1, \ldots, \lambda_n$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

Define $T : V \to W$ by $T(v) = \lambda_1 w_1 + \dots \lambda_n w_n$. (This makes sense, since the scalars λ_i are uniquely determined by v.)

Show that T is linear:
Take $u, v \in V$. Write $u = \mu_1 v_1 + \cdots + \mu_n v_n$, $v = \lambda_1 v_1 + \dots \lambda_n v_n$. Then $u + v = (\mu_1 + \lambda_1)v_1 + \cdots + (\mu_n + \lambda_n)v_n$. Now by the definition of T, we have:

$$T(u) = \mu_1 w_1 + \cdots + \mu_n w_n \text{ and } T(v) = \lambda_1 w_1 + \cdots + \lambda_n w_n$$
$$\text{Also } T(u + v) = (\mu_1 + \lambda_1)w_1 + \cdots + (\mu_n + \lambda_n)w_n$$

So $T(u + v) = T(u) + T(v)$, so T preserves addition.

Now let $\pi \in F$. We have $\pi = \pi\lambda_1 v_1 + \cdots + \pi\lambda_n v_n$ ∎

Remark 27. Once we know what a linear transformation does on a basis, we know all about it. This gives a convenient shorter way of defining a linear transformation.

Example 28. Let V be the vector space of polynomials in a variable x over $\mathbb{R}$ of degree ≤ 2. A basis for V is $\{1, x, x^2\}$.

Pick three "vectors" in V; $w_1 = 1 + x$, $w_2 = x - x^2$, $w_3 = 1 + x^2$. By Proposition 26, there should be a unique linear transformation, $T : V \to V$ such that $T(1) = w_1, T(x) = w_2, T(x^2) = w_3$.

Let's work out what T does to an arbitrary polynomial, $c + bx + ax^2$ from V. We must have $c+bx+ax^2 \longmapsto c(1+x)+b(x-x^2)+a(1+x^2) = (a+c)+(b+c)x+(a-b)x^2$.

Kernels and Images

Definition 29. Let $T : V \to W$ be a linear transformation. The *image* of T is the set $\{Tv : v \in V\} \subseteq W$. The *kernel* of T is the set of $\{v \in V : Tv = \mathbf{0}\}$. We write $\mathrm{Im}(T)$ for the image, and $\mathrm{Ker}(T)$ for the kernel. Lecture 7

Example 30. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by:

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

So:

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix}$$

So $\mathrm{Im}(T)$ is the set:

$$\begin{aligned}
&\left\{\begin{pmatrix} 3x_1 + x_2 + 2x_3 \\ -x_1 + x_3 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}\right\} \\
&= \left\{x_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R}\right\} \\
&= CSp(A)
\end{aligned}$$

The kernel of T is the set:

$$\left\{\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) \in \mathbb{R}^3 : 3x_1 + x_2 + 2x_3 = 0, \ -x_1 + x_3 = 0\right\}$$

This is $\{v \in \mathbb{R}^3 : Av = \mathbf{0}\}$, the solution space of $Av = \mathbf{0}$. (Solved this in M1GLA). (In this case the kernel is the span $\{(1, -5, 1)^T\}$)

Proposition 31. *Let $T : V \to W$ be a linear transformation. Then:*

(i) Im(T) is a subspace of W

(ii) Ker(T) is a subspace of V

(In general we write $U \leq V$ to mean that U is a subspace of V. So Proposition 31 says $\mathrm{Im}(T) \leq W$, $\mathrm{Ker}(T) \leq V$).

Proof. (i) $\mathrm{Im}(T) \leq W$:

Certainly $T(\mathbf{0}) \in \mathrm{Im}(T)$, so $\mathrm{Im}(T) \neq \emptyset$.

Suppose that $w_1, w_2 \in \mathrm{Im}(T)$. Then there exists $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. Now $T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$. (Since T preserves addition). So $w_1 + w_2 \in Im\ T$, so $\mathrm{Im}(T)$ is closed under addition.

Now suppose that $w \in \mathrm{Im}T$ and $\lambda \in F$. Then there exists $v \in V$ such that $Tv = w$. Now $T(\lambda v) = \lambda T(v) = \lambda w$. (Since T preserves scalar multiplication). So $\lambda w \in \mathrm{Im}(T)$, so $\mathrm{Im}(T)$ is closed under scalar multiplication.

(ii) $\mathrm{Ker}(T) \leq V$:

We know that $T(\mathbf{0}) = \mathbf{0}w$. So $\mathbf{0}v \in Ker\ T \implies Ker\ T \neq \emptyset$.

Suppose that $v_1, v_2 \in \mathrm{Ker}(T)$. So: $Tv_1 = Tv_2 = \mathbf{0}$. Now $T(v_1 + v_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ (Since T preserves addition). So $v_1 + v_2 \in \mathrm{Ker}(T)$, and so $\mathrm{Ker}(T)$ is closed under addition.

Now suppose we have $v \in \mathrm{Ker}(T), \lambda \in F$. Then $Tv = \mathbf{0}$. Now $T(\lambda v) = \lambda Tv = \lambda \mathbf{0} = \mathbf{0}$. So $\lambda v \in \mathrm{Ker}(T)$. So $\mathrm{Ker}(T)$ is closed under scalar muliplication. ∎

Example 32. Let V_n be the vector space of polynomials in a variable x over $\mathbb{R}$ of degree $\leq n$.

We have $V_0 \leq V_1 \leq V_2 \leq \dots$

Define $T : V_n \to V_{n-1}$ by $T(f(x)) = \dfrac{d}{dx} f(x)$.

We have $\mathrm{Ker}(T) = \{\text{constant polynomials}\} = V_0$.

If $g(x) = V_{n-1}$, Let $f(x)$ be the antiderivative (integral) of $g(x)$. Since $\deg g(x) \leq n-1$, we have $\deg f(x) \leq n$. And $T(f(x)) = \frac{d}{dx} f(x) = g(x)$.

So $g(x) \in \mathrm{Im}(T)$, and so $\mathrm{Im}(T) = V_{n-1}$.

Proposition 33. *Let $T : V \to W$ be a linear transformation. Let $v_1, v_2 \in V$. Then $Tv_1 = Tv_2 \iff T(v_1 - v_2) = \mathbf{0}$. $v_1 - v_2 \in$ Ker(T).*

Proof. $Tv_1 = Tv_2 \iff Tv_1 - Tv_2 = 0 \iff T(v_1 - v_2) = 0$
(Since T preserves addition and multiplication by -1). ∎

Proposition 34. *Let $T : V \to W$ be a linear transformation. Suppose that $\{v_1, \dots, v_n\}$ is a basis for V. Then Im$(T) =$ Span$\{Tv_1, \dots, Tv_n\}$.*

Proof. Let $w \in \mathrm{Im}(T)$. Then there exists $v \in V$ such that $T(v) = w$. We can write v as a linear combination of basis vectors:

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots \lambda_n v_n.$$

Now $Tv_1 = \lambda_1 Tv_1 + \dots \lambda_n Tv_n$ by Proposition 24(ii). So $w = \lambda_1 Tv_1 + \dots + \lambda_n Tv_n \in \mathrm{Span}\{Tv_1, \dots, Tv_n\} \geq \mathrm{Im}(T)$

Since $Tv_i \in \mathrm{Im}(T)$ for all i, $\mathrm{Span}\{Tv_1, \dots Tv_n\} \leq \mathrm{Im}(T)$. So $\mathrm{Im}(T) = \mathrm{Span}\{Tv_1, \dots, Tv_n\}$. ∎

Proposition 35. *Let A be an $m \times n$ matrix. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be given by $Tx = Ax$.* Lecture 8

(i) Ker(T) is the solution space to the equation $Ax = \mathbf{0}$.

(ii) Im(T) is the column space CSp(A).

(iii) dim Im(T) is the rank rk(A).

(Compare with Example 30).

Proof. (i): This is immediate from the definitions.

(ii): Take the standard basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, that is $e_i = (0, 0, \ldots, 1, 0, 0)^T$, where the 1 is in the ith position. $T(e_i) = A(0, 0, \ldots, 1, 0, 0)^T = c_i$, the ith column of A. By Proposition 34, $\mathrm{Im}(T) = \mathrm{Span}\{T_{e_1}, \ldots, T_{e_n}\} = \mathrm{Span}\{c_1, \ldots, c_n\} = \mathrm{CSp}(A)$.

(iii): By (ii), $\dim \mathrm{Im}(T) = \dim \mathrm{CSp}(A) = \text{column-rank} = \mathrm{rk}(A)$ ∎

Example 36. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by:

$$Tx = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 1 & 4 & 7 \end{pmatrix} x \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Question: Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Answer: To find $\mathrm{Ker}(T)$, we solve $Ax = \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 7 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\implies x_1 - x_3 = 0; x_2 + 2x_3 = 0$. So we must have $x_1 = x_3, x_2 = -2x_3$

So a basis for $\mathrm{Ker}(T)$ is $\{(1, -2, 1)^T\}$

For $\mathrm{Im}(T)$, we notice from the row-reduced matrix above, that $rk(A) = 2$. So dim $\mathrm{Im}(T)$ is 2. Since $\mathrm{Im}(T)$ is $CSp(A)$, a basis can be obtained by taking any two lin. indep. columns. So $\{(1, -1, 1)^T, (2, 0, 4)^T\}$ is a basis.

Theorem 37: Rank Nullity Theorem.

Let $T : V \to W$ be a linear transformation. Then $\dim \mathrm{Im}(T) + \dim \mathrm{Ker}(T) = \dim(V)$. (rank $T = \dim \mathrm{Im}(T)$, nullity $= \dim \mathrm{Ker}(T)$)

Proof. Let $\{u_1, \ldots, u_s\}$ be a basis for $\mathrm{Ker}(T)$, and $\{w_1, \ldots, w_r\}$ be a basis for $\mathrm{Im}(T)$. For each $i \in \{1, \ldots, r\}$, there exists some $v_i \in V$ such that $T(v_i) = w_i$ (since $w_i \in \mathrm{Im}(T)$).

I claim that $B = \{u_1, \ldots, u_s\} \cup \{v_1, \ldots, v_r\}$ is a basis for V.
This needs to be proved:

(i) Show that $V = \mathrm{Span}(B)$

Take $v \in V$. Then $Tv \in \mathrm{Im}(T)$, and so $Tv = \mu_1 w_1 + \ldots \mu_r w_r$, for some $\mu_i \in F$. Define

$\bar{v} \in V$ by $\mu_1 v_1 + \dots \mu_r v_r$. Then $T\bar{v} = \mu_1 w_1 + \dots \mu_r w_r = Tv$.
So $v - \bar{v} \in \text{Ker}(T)$ by Proposition 33. So $v - \bar{v} = \lambda_1 u_1 + \cdots + \lambda_s u_s$ for some $\lambda_i \in F$.
Now $v = \bar{v} + \lambda_1 u_1 + \dots \lambda_s u_s = \mu_1 v_1 + \cdots + \mu_r v_r + \lambda_1 u_1 + \cdots + \lambda_s u_s \in \text{Span}(B)$.

(ii) Show that B is linearly independent. Suppose that:

$$\lambda_1 u_1 + \dots \lambda_1 u_s + \mu_1 v_1 + \cdots + \mu_r v_r = \mathbf{0} \ (*)$$

We want to show that $\lambda_i = 0$ and $\mu_i = 0$ for all i. Apply T to both sides of $(*)$. Since $Tu_i = \mathbf{0}$ for all i, we get $\mu_1 w_1 + \cdots + \mu_r w_r = \mathbf{0}$. But $\{w_1, \dots, w_r\}$ is linearly independent (a basis for $\text{Im}(T)$). So $\mu_i = 0 \ \ \forall i$. Now from $(*)$, we have:

$$\lambda_1 u_1 + \dots \lambda_s u_s = \mathbf{0}$$

But $\{u_1, \dots, u_s\}$ is linearly independent (a basis for $\text{Ker}(T)$). So $\lambda_i = 0$ for all i. Hence B is a basis.

So dim $V = |B| = r + s =$ dim Im(T)+dim Ker(T). ■

Corollary 38. *Let A be an $m \times n$ matrix. Let U be the solution space to $Ax = \mathbf{0}$. Then:*

$$\textit{dim } U = n - \textit{rk}(A)$$

Proof. Let T be the matrix transformation $\mathbb{R}^n \to \mathbb{R}^m$ given by $Tx = Ax$. Then $rk(A)$ =dim Im(T), and $\text{Ker}(T) = U$. So the result follows by Theorem 37. ■

Matrix of Linear Transformation

Lecture 9

Definition 39. Let V be a vector space over a field F. Let $B = \{v_1, \dots, v_n\}$ be a basis for V. (Actually we should write $(v_1, \dots, v_n)$, because we care about the order of the basis vectors. But that notation could be confusing, so we'll continue to use set brackets).

Any $v \in V$ can be written uniquely as a linear combination $v = \lambda_1 v_1 + \dots \lambda_n v_n$. Define the *vector of V with respect to B* to be $[V]_B = (\lambda_1 \dots \lambda_n)^T$.

Examples 40.

(a) Let $V = \mathbb{R}^n$, and let E be the standard basis $\{e_1, \dots, e_n\}$, where $e_i = (0 \dots, 1, \dots, 0)^T$ (1 in the ith position). Let $v = (a_1, \dots, a_n)^T$. Then $[V]_E = V$, since $v = a_1 e_1 + \cdots + a_n e_n$.

(b) Let $V = \mathbb{R}^2$, and let $B = \{\underbrace{(1,1)^T}_{v_1}, \underbrace{(0,1)^T}_{v_2}\}$. Let $v = (1,3)^T$. What is $[v]_B$? We have to solve $V = \lambda_1 v_1 + \lambda_2 v_2$. We have the matrix equation:

$$\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = v. \quad \text{So: } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solve by row reducing:

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

from which we find $\lambda_1 = 1, \lambda_2 = 2$. So $[v]_B = (1,2)^T$

Definition 41. Let V be a vector space of dimension n over F. Let $B = \{v_1, \dots, v_n\}$ be a basis. Let $T : V \to V$ be a linear transformation. For all $i \in \{1, \dots, n\}$, we have $Tv_i = a_{1i} v_1 + a_{2i} v_2 + \dots a_{ni} v_n$.

The matrix of T with respect to the basis B, is:

$$[T]_B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Notice that the ith column of $[T]_B$ is $[Tv_i]_B$.

Examples 42.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by:

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Take E to be the standard basis $\{\underset{e_1}{(1,0)^T}, \underset{e_2}{(0,1)^T}\}$

Notice that $Te_1 = (2,1)^T = 2e_1 + e_2$, $Te_2 = (-1,2)^T = -e_1 + 2e_2$

So $[T]_E = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, the matrix we started with.

(b) Let $B = \{\underset{v_1}{(1,1)^T}, \underset{v_2}{(0,1)^T}\}$. Let T be as above. We have $T(v_1) = (1,3)^T$, which we know from Example 40, is $v_1 + 2v_2$. And $T(v_2) = (-1,2)^T$, which we can calculate is $-v_1 + 3v_2$.
So $[T]_B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

(Observe that the matrices $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ have many features in common - same determinants and same trace. So the same characteristic polynomial. This is not an accident).

(c) Let V be the space of polynomials of degree ≤ 2 in a variable x over $\mathbb{R}$. Let $T : V \to V$ be defined by $T(f(x)) = \frac{d}{dx} f(x)$.

We have the basis $B = \{1, x, x^2\}$ for V. Then:

$$\begin{aligned} T(1) &= 0 = 0.1 + 0.x + 0.x^2 \\ T(x) &= 1 = 1.1 + 0.x + 0.x^2 \\ T(x^2) &= 2x = 0.1 + 2.x + 0.x^2 \end{aligned}$$

So $[T]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Take a polynomial $f(x) \in V$. Write $f(x)$ in terms of B as $f(x) = a.1 + b.x + c.x^2$.

So $[xf(x)]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Now $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 0 \end{pmatrix}$

And $\frac{d}{dx} f(x) = b + 2cx$, so $[Tf(x)]_B = (b, 2c, 0)^T = [T]_B [f(x)]_B$
This *always* happens.

Proposition 43. *Let $T : V \to V$ be a linear transformation. Let B be a basis for V. Let $v \in V$. Then $[Tv]_B = [T]_B [v]_B$.*

Proof. Let $B = \{v_1, \ldots, v_n\}$. Let $v = \lambda_1 v_1 + \ldots \lambda_n v_n$. Let $[T]_B = (a_{ij})$

$$\begin{aligned} \text{We have: } Tv &= T(\lambda_1 v_1 + \ldots \lambda_n v_n) \\ &= \lambda_1 T v_1 + \ldots \lambda_n T v_n \end{aligned}$$

But $Tv_i = a_{1i} v_1 a_{2i} v_2 + \cdots + a_{ni} v_n$

$$\begin{aligned}\text{So } Tv &= \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{n} a_{ji} v_j) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \lambda_i a_{ji} \right) v_j\end{aligned}$$

(interchanging the order of summation). So

$$[Tv]_B = \begin{pmatrix} \sum_{i=1}^{n} \lambda_i a_{1i} \\ \sum_{i=1}^{n} \lambda_i a_{2i} \\ \vdots \\ \sum_{i=1}^{n} \lambda_i a_{ni} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$= [T]_B [v]_B \text{ as required.}$$

∎

Eigenvalues and Eigenvectors

Definition 44. Let $T : V \to V$ be a linear transformation. We say that v, is an *eigenvector* of T if $v \neq \mathbf{0}$, and $Tv = \lambda v$ for some $\lambda \in F$. We say that λ is an *eigenvalue* of T. Lecture 10

Example 45. Let $V = \mathbb{R}^2$ and let $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ -x_1 + x_2 \end{pmatrix}$.

We see that $T\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right) = 2\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$, and so $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ is an eigenvector of T with eigenvalue 2.

Note that $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

We have $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \iff A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

and so the eigenvectors and eigenvalues of T are the same as those of A. (We know how to find those from M1GLA). This will always work if T is a matrix transformation.

How do we find eigenvalues and eigenvectors in general?
Use the fact that we can represent any $T : V \to V$ as a matrix transformation.

Proposition 46. *Let $T : V \to V$ and let $B = \{v_1, \dots, v_n\}$ be a basis for V. Then:*

(i) Eigenvalues of T are the same as eigenvalues of the matrix $[T]_B$

(ii) The eigenvectors of T are those vectors v, such that $[v]_B$ is an eigenvector of $[T]_B$. (So if $[v]_B = (\lambda_1, \dots, \lambda_n)^T$, then $v = \lambda_1 v_1 + \dots \lambda_n v_n$.)

Proof.

$$\begin{aligned} Tv = \lambda v &\iff [Tv]_B = [\lambda v]_B \\ &\iff [T]_B[v]_B = \lambda[v]_B \text{ by Proposition 43} \\ &\iff [v]_B \text{ is an eigenvector for } [T]_B \text{ with eigenvalue } \lambda \end{aligned}$$

∎

Example 47. $V =$ Space of polynomials in x of degree ≤ 2 over $\mathbb{R}$. Let $T : V \to V$ be given by $T(f(x)) = f(x+1) - f(x)$ [Ex: Check T is linear]

Question: Calculate the eigenvalues and eigenvetors of T.

Answer: Let $B = \{1, x, x^2\}$, a basis for V. We have $T(1) = 0, T(x) = x + 1 - x =$

$1, T(x^2) = (x+1)^2 - x^2 = 2x + 1$.

$$[T]_B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial is then:

$$\begin{vmatrix} \lambda & -1 & -1 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3$$

So the only eigenvalue is 0. Find eigenvectors $[T]_B$ by solving:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solutions are vectors of the form $(a, 0, 0)^T$, so the eigenvectors of $[T]_B$ are $\{(a, 0, 0)^T \quad a \in F \backslash \{0\}\}$. So the eigenvectors of T are polynomials $f(x)$ such that $[f(x)]_B = (a, 0, 0)^T \quad (a \neq 0)$

So these are polynomials $a.1 + 0.x + 0.x^2$, which are the non-zero constant polynomials.

Diagonalisation of Linear Transformations

Proposition 48. *Let $T : V \to V$ be a linear transformation. Let B be a basis for V. Then $[T]_B$ is a* diagonal *if and only if every basis vector in B is an eigenvector for T.*

Proof. Let $e_1, \ldots, e_n$ be the standard basis vectors, $(1, 0, \ldots, 0)^T, (0, 1, \ldots, 0)^T$ etc. in F^n, where $n = \dim V$. Let A be an $n \times n$ matrix. Then A is diagonal if and only if e_1 is an eigenvector of A for all i.

So $[T]_B$ is diagonal if and only if all the e_i are eigenvectors.

But $e_i = [v_i]_B$, where $B = \{v_1, \ldots, v_n\}$, since $v_i = 0v_1 + 0v_2 + \cdots + 1v_i + \ldots 0v_n$. So e_i is an eigenvector for $[T]_B$ if and only if v_i is an eigenvector for T, by Proposition 46. ∎

Definition 49. A linear transformation $T : V \to V$ is *diagonalisable* if there is a basis of V such that every element of V is an eigenvector of T.

Examples 50.

(a) V, T as in Example 47. $T(f(x)) = f(x+1) - f(x)$.
Is T diagonalisable?.

We calculated earlier that the eigenvectors of T all lie in $Span\{1 + 0.x + 0.x^2\}$

which is a one-dimensional subspace of V. So the eigenvectors do not span V, and so there is no basis of V consisting of the eigenvector of T. So T is not diagonalisable.

(b) V as before (polynomial space of degree ≤ 2)
Define $T : V \to V$ by $T(f(x)) = f(1-x)$, a basis for V.
(Exercise: check T is linear).
We have $T(1) = 1,\ T(x) = 1-x,\ T(x^2) = (1-x)^2 = 1 - 2x + x^2$. So:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial is $(\lambda-1)^2(\lambda+1)$ [the diagonals are the roots]. So the eigenvalues are 1,-1.

We need to know whether there exists two linearly independent eigenvectors with eigenvalue 1. Using M1GLA techniques...

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

are both eigenvectors with eigenvalue 1. They are linearly independent. We have $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ as an eigenvector with eigenvalue -1.

$$\text{So } C = \{(1,0,0)^T, (0,1,-1)^T, (1,-2,0)^T\}$$

is a basis for V, whose elements are eigenvectors of T.
(Or $C = \{1, x - x^2, 1 - 2x\}$).

$$\text{So } T \text{ is diagonalisable and } [T]_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Change of Basis

Let V be a vector space. Let $B = \{v_1, \dots, v_n\}$ and $C = \{w_1, \dots, w_n\}$. Lecture 11
For j in $\{1, \dots, n\}$, we can write $w_j = \lambda_{1j}v_1 + \lambda_{2j}v_2 + \dots \lambda_{nj}v_n$.

Write:

$$P = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} \\ \vdots & & & \vdots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nn} \end{pmatrix} = (\lambda_{ij})$$

So the jth column of this matrix of P is $[w_j]_B$.

Proposition 51.

(i) $P = [T]_B$, *where* T *is the unique linear transformation* $V \to V$, *such that* $Tv_i = w_i$, *for all* i.

(ii) For all vectors $v \in V$, *we have* $P[v]_C = [v]_B$.

Proof. (i) We know that $[T]_B[v_i]_B = [Tv_i]_B = [w_i]_B$. And $[T]_B$ is the only matrix with this property. So it is enough to show that $P[v_i]_B = [w_i]_B$.

But $[v_i]_B = e_i = (0, \dots, 1, \dots, 0)^T$ (1 in ith row), and so $P[v_i]_B = Pe_i = i$th column of P is $[w_i]_B$, as above. ∎

Proof. (ii) First note that $P[w_i]_C = Pe_i = [w_i]_B$, as we saw in (i).

Now if $v \in V$ then we can write $v = a_1w_1 + \dots + a_nw_n,\ a_i \in F$. Now $[v]_C = (a_1, \vdots, a_n)^T = a_1e_1 + \dots a_ne_n$. So:

$$\begin{aligned} P[v]_C &= a_1Pe_1 + \dots + a_nPe_n \\ &= a_1[w_1]_B + \dots a_n[w_n]_B \\ &= [a_1w_1 + \dots a_nw_n]_B \quad ∎ \end{aligned}$$

Definition 52. The Matrix P as defined above is the *change of basis matrix* from B to C.

Proposition 53. *Let V, B, C, P be all as above. Let $T : V \to V$ be a linear transformation.*

(i) P is invertible, and its inverse is the change of basis matrix from C to B.

(ii) $[T]_C = P^{-1}[T]_BP$

Proof. (i) Let Q be the change of basis matrix from C to B. Then $Q[v]_B = [v]_C$, for all $v \in V$. We calculate:

$$PQe_i = PQ[v_i]_B = P[v_i]_C = [v_i]_B = e_i$$

It follows that $PQ = I$, and so $Q = P^{-1}$. ∎

Proof. (ii)

$$\begin{aligned} P^{-1}[T]_BPe_i &= P^{-1}[T]_BP[w_i]_C \\ &= P^{-1}[T]_B[w_i]_B \\ &= P^{-1}[Tw_i]_B \\ &= [Tw_i]_C \\ &= [T]_C[w_i]_C \\ &= [T]_C\ e_i \end{aligned}$$

So the ith column of $P^{-1}[T]_BP$ is the same as the ith column of $[T]_C$ for all i, and so $P^{-1}[T]_BP = [T]_C$, as required. ∎

Example 54. Let $V = \mathbb{R}^2$, and let $T : V \to V$ be given by:

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let $B = E = \{e_1, e_2\}$. Let $C = \{(1,1)^T, (1,2)^T\}$. The basis elements of C are eigenvectors of the matrix $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

The change of basis matrix from B to C is $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

We have $[T]_B = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$

$$\begin{aligned} [T]_C &= P^{-1}[T]_B P \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

so we have diagonalised T. (M1GLA - $D = P^{-1}AP$)

- End of Linear Algebra -

2 Group Theory

Groups

Lecture 12

Let S be a set. A *binary operation* on S is function $S \times S \to S$. $S_\times$ takes pairs of elements, (S_1, S_2), to some element $S_1 * S_2$.

Examples 55.

(i) $S = \mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}, \mathbb{C}$):

$a * b = a + b$, or $a * b = ab$, or $a * b = a - b$.
But *not* $a * b = a/b$. (Since, for example, b could be 0.)

(ii) $S = \backslash\{0\}$:

Now $a * b = a/b$ is a binary operation.

(iii) $S = \mathbb{Z}$ (or $\mathbb{Q}, \mathbb{R}$)

$a * b = min(a, b)$

(iv) S any set at all:

$a * b = a$

(v) $S = \{1, 2, 3\}$, $a * b$ defined by a table:

$a\backslash b$	1	2	3
1	1	2	1
2	2	1	2
3	1	2	3

Suppose that S is a set with binary operation $*$. Then the expression "$a * b * c$" is ambiguous: it could mean $(a * b) * c$ or $a * (b * c)$.

Example: $S =, \; a * b = a - b$. In general $(a - b) - c \neq a - (b - c)$. (except when $c = 0$).

Definition 56. If $(a * b) * c = a * (b * c)$ for all $a, b, c, \in S$, then we say that $*$ is *associative*.

If an operation $*$ is associative, then $a * b * c$ is now unambiguous. So we can omit brackets in expressions of this sort.

Example 57 (For motivation).
Question: Solve $x + 1 = 2, \quad x \in \mathbb{Z}$

Answer: (carefully explaining our reasoning)

(i) Use the fact that -1 in $\mathbb{Z}$. Add it to both sides.

$$(x + 1) + (-1) = 2 + (-1)$$

(ii) Use associativity of $+$ to rewrite the left-hand side:

$$x + (1 + (-1)) = 2 + (-1) \implies x + 0 = 1$$

(iii) Use the fact that $x + 0 = x$ for all x, so $x = 1$.

A group is a set with a binary operation $*$ in which equations $x * b = c$ can be solved for x in these three steps.

Definition 58. A *group* is a set G, with a binary operation $*$, satisfying the following axioms:

(i) **Associativity:** $(a * b) * c = a * (b * c)$, for all $a, b, c \in G$

(ii) **Identity Axiom:** There is an element $e \in G$ such that $x * e = e * x = x$ for all $x \in G$.

(iii) **Inverses Axiom:** For every $x \in G$, there exists $y \in G$ such that $x * y = e$, where e is the element from the identity axiom.

Notes.

(i) Most books also list another axiom:
0. **Closure:** If $x, y \in G$ then $x * y \in G$.

For us this is implied by the fact that $*$ is a binary operation on G.

(ii) The element e from (2) is called the *identity element* of G.

(iii) For $x \in G$, the element y from (3) is the *inverse* of x.

Examples 59.

$(G, *)$	Associativity	Identity	Inverse	Group
$(\mathbb{Z}, +)$	Yes	0	$-x$	Yes
$(\mathbb{Z}, -)$	No	No $(0 - n \neq n)$	No	No
$(\mathbb{Z}, \times)$	Yes	1	No	No
$(\mathbb{Q}, +)$	Yes	0	$-x$	Yes
$(\mathbb{Q}, \times)$	Yes	1	No, 0 has no inverse	No
$(\mathbb{Q}\backslash\{0\}, \times)$	Yes	1	$1/x$	Yes
$(\mathbb{R}\backslash\{0\}, \times)$	Yes	1	$1/x$	Yes
$(\mathbb{C}\backslash\{0\}, \times)$	Yes	1	$1/x$	Yes
$(\{1, -1, i, -i\} \subset \mathbb{C}, \times)$	Yes	1	$1/x$	Yes

Proposition 60. *Let $(G, *)$ be a group.*

(i) G has exactly one identity element.

(ii) Every element of G has exactly one inverse.

(iii) (Left cancellation) If $x, y, z \in G$, and $x * y = x * z$, then $y = z$

(iv) (Right cancellation) If $x, y, z \in G$ and $x * z = y * z$, then $x = y$

Lecture 13 *Proof.* (1) Let e, f be identity elements for G. Then $e*x = x$ for all $x \in G$. So $e*f = f$. But $x*f = x$ for all $x \in G$, so $e*f = e$. Hence $e = f$. ∎

Proof. (2) Suppose y, z be inverses for x. Look at $y*x*z$. Since $y*x = e$, we have $(y*x)*z = e*z = z$. But also $x*z = e$, and so $y*(x*z) = y*e = y$. So $y = z$. ∎

Proof. (3) [(4) similar] Let w be the inverse of x. Since $x*y = x*z$, we have $w*(x*y) = w*(x*z)$. By associativity $(w*x)*y = (w*x)*z$. But $w*x = e$, so $e*y = e*z$, and so $y = z$. ∎

The two most common notations for groups are:

(i) **Additive notation:** We write $+$ for $*$ and 0 for e. The inverse of x is $-x$. We write (for instance) $2x$ for $x + x$ etc. (This is normally used when the group operation really "is" addition in some set).

(ii) **Multiplicative notation:** We write xy for $x*y$. We write e or 1 for the identity. Write x^{-1} for the inverse. Write (for instance) x^2 for $x*x$.

We will usually use multiplicative notation.

We say that a group G is *finite* if it has finitely many elements. In this case we say that $|G|$ is the *order* of G.

Examples 61.

(i) Let F be a field of scalars (say $F = \mathbb{R}, \mathbb{C}$). Let S be the set of $n \times n$ matrices, with entries from F. Let $*$ be matrix multiplication. *Is* $(S, *)$ *a group?*

Certainly $*$ is a binary operation on S, since if $A, B \in S$, then $AB \in S$.

- Associativity Yes. $(AB)C = A(BC)$
- Identity: Yes. I_n.
- Inverses: No. Non-invertible matrices exist.

So S is not a group.

(ii) Let G be the set of invertible $n \times n$ matrices over F. Let $*$ be matrix multiplication. *Is* $(G, *)$ *a group?*

Check that $*$ is a binary operation on G. If A has inverse A^{-1} and if B has inverse B^{-1}. Then AB has inverse $B^{-1}A^{-1}$. So if $A, B \in G$, then $AB \in G$.

- Associativity: Yes, as above.
- Identity: Yes, $I_n \in G$.
- Inverse: Yes, by definition of G.

So G is a group - the general linear group of dimension n over F. We write $GL_n(F)$.

Notice $GL_1(F) = \{(x) : x \in F \backslash \{0\}\}$. So this is really just $(F \backslash \{0\}, \times)$ in disguise. $GL_2(F) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d, \in F, ad - bc \neq 0\}$

Definition 62.

(i) Let G be a group (we write G multiplicatively.) Let $a, b \in G$. If $ab = ba$, then we say that a and b *commute*.

(ii) If $xy = yx$ for all $x, y \in G$, then G is *abelian*.
[Neils Henrik Abel 1802-1829
- like many great mathematicians in his time,
he never met his 30th birthday]

Most groups are not abelian. Example: in $GL_2()$, we have:

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

But

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

So $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 2 \end{smallmatrix}\right)$ do not commute, so $GL_2(\mathbb{R})$ is not abelian.

But many of the groups we have seen so far are abelian.

Examples: $(\mathbb{Z}, +), (F, +), (F \backslash \{0\}, \times), GL_1(F), (\{1, -1, i, -i\}, \times)$ are all abelian.

Definition 63. Let X be a set. A function $f : X \to X$ is a *permutation* of X if it is a bijection. (Injective + surjective).

Examples 64.

(i) $X = \{1, 2, 3, 4\}$. $f : 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 1$ is a permutation.

(ii) $X = \mathbb{Z}, f : n \mapsto n + 3$ is a permutation.

(iii) $X = \mathbb{Z},\ f : n \mapsto 3n$ is *not* a permutation, (not surjective).

Notation for permutations (First attempt.) Lecture 14

Assume that $X = \{1, \ldots, n\}$. Let $f : X \to X$ be a permutation. We can write f as a matrix with two rows:

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ f(1) & f(2) & \ldots & f(n) \end{pmatrix}$$

is called *two-row rotation.*

Examples: if $f : 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 1$, we can write f as $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$. If g is $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{smallmatrix}\right)$, then $g : 1 \mapsto 3,\ 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 4$.

Let f and g be permutations of a set X. We define the *composition*, $f \circ g$ by $f \circ g(x) = f(g(x))$ for all $x \in X$. For example, if f and g are as in the example above, we have $f \circ g(1) = 4, f \circ g(2) = 2, f \circ g(3) = 3, f \circ g(4) = 1$. So $f \circ g = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{smallmatrix}\right)$.

Proposition 65. *Let X be any set. Let S be the set of all permutations of X. Let $\circ$ be the composition operation as above. Then $(S, \circ)$ is a group, called the* symmetric group *on X, written Sym(X).*

Proof. We first check that $\circ$ is a binary operation on S. Certainly if $f : X \to X$ and $g : X \to X$, then $f \circ g : X \to X$. The composition of two bijections is a bijection. So if $f, g \in S$, then $f \circ g \in S$.

Now the group axioms:

(i) Associativity: If $x \in X$, and $f, g, h \in S$, then $(f \circ g) \circ h(x) = (f \circ g)(h(x)) = f(g(h(x))) = f(g \circ h(x)) = f \circ (g \circ h(x))$. Since they agree on all $x \in X$, we have $(f \circ g) \circ h = f \circ (g \circ h))$.

(ii) Identity: Let e be the permutation defined by $e(x) = x$ for all $x \in X$. Then we have $e \circ f(x) = f(x) = f \circ e(x)$ for all $f \in S$. So $e \circ f = f \circ e = f$.

(iii) Inverses: Bijections have inverses.

∎

Further notation: We almost always write symmetric groups multiplicatively. So write fg for $f \circ g$, and so on.
When $X = \{1, \ldots, n\}$, we write S_n for the Sym(X).

> **Examples 66.** $S_1 = \mathrm{Sym}(\{1\}) = \{e\} = (1, 1)^T$.
>
> $S_2 = \mathrm{Sym}\{1, 2\} = \{e, \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)\}$
>
> $S_3 = \mathrm{Sym}\{1, 2, 3\} = \{e, \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)\}$.
> We have $|S_1| = 1, |S_2| = 2, |S_3| = 6$

Proposition 67. *The group S_n has order n!*

Proof. By induction. Inductive hypothesis. if X and Y are the two set of size n, then the number of bijections form X to Y is $n!$. (This gives us the result by taking $Y = X$.)

Base case $n = 1$ is obvious.
Inductive step: Suppose the result is true for n:

Let $|X| = |Y| = n+1$. Take $x \in X$, $y \in Y$. The number of bijections $f : X \to Y$ such that $f(x) = y$ is equal to the number of bijections $X \backslash \{x\} \to Y \backslash \{y\}$, which is $n!$ by the inductive hypothesis. So the total number of bijections $X \to Y$ is $(n+1)n! = (n+1)!$. ■

The Group Table. Let G be a finite group. We can record the multiplication (binary operation) in G in a table called the *Group Table* or *Cayley Table* of G. If $G = \{a, b, c, \dots\}$, write:

	a	b	c	$\dots$
a	aa	ab	ac	
b	ba	bb	bc	$\dots$
c	ca	cb	cc	
$\vdots$		$\vdots$		

Example 68. Let $G = S_3$

Write $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Then $a^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, and $a^3 = e$.

Write $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Then $b^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $a^2b\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

So $S_3 = \{e, a, a^2, b, ab, a^2b\}$. To work out the group table, it is helpful to check: $b^2 = e, ba = a^2b, ba^2 = ab$. Now it is easy to write down:

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

Notice that every element of the group appears exactly once in each row and each column. (This follows from left and right cancellation laws.)

Subgroups

Definition 69. Let $(G, *)$ be a group. A *subgroup* of G is a subset of G which is itself a group under the operation $*$.

Examples 70.

(i) $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$. Both are subgroups of $(\mathbb{R}, +)$. All are subgroups of $(\mathbb{C}, +)$. But $(\mathbb{N}, +)$ is not a subgroup of any - it has no inverses.

(ii) $(\mathbb{R}\backslash\{0\}, \times)$ is not a subgroup of $(\mathbb{R}, +)$. The group operation is different.

(iii) $\{e\}$ is a subgroup of any group (where e is the identity element). The *trivial* subgroup.

(iv) Every group is a subgroup of itself.

Lecture 15 *Recall:* a subgroup of a group G is a subset of G which is a group under the same operation as G.

Proposition 71. *(Subgroup Criteria) Let G be a group. We write G multiplicatively. Let $H \subseteq G$ be a subset. Then H is a subgroup if and only if the following conditions hold:*

(i) $e \in H$, where e is the identity of G.

(ii) If $a, b \in H$, then $ab \in H$, for all $a, b \in G$.

(iii) If $a \in H$, then $a^{-1} \in H$, where a^{-1} is the inverse of a in G.

Proof. **"if"**. Condition (2) says that the binary operation on G restricts to a binary operation on H. Since the operation is associative on G, it is also associative on H. Condition (1) gives us an identity, and Condition (3) gives inverses. So if (1),(2),(3) hold then H is a subgroup.

"only if". Certainly (2) must hold if H is a subgroup, since we need the binary operation on G to restrict to a binary operation on H. If H is a subgroup, then H has an identity, e_H. Write e_G for the identity of G. Then $e_G e_H = e_H$, and also $e_H e_H = e_H$. Now $e_G = e_H$, by right cancellation. So $e_G \in H$, and so (1) holds.

Let $a \in H$. Let b be the inverse of a in G, and let c be the inverse of a in H. $ab = e_G = e_H = ac$. So $b = c$ by left cancellation. So $b \in H$, and so (3) holds. ∎

Example 72. Let $G = GL_2()$. For $n \in \mathbb{Z}$, define $u_n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, so $u_n \in G$, for all $n \in \mathbb{Z}$. Define $U = \{u_n \ : \ n \in \mathbb{Z}\}$. Then U is a subgroup of G. Check the subgroup criteria:

(i) $e_G = I = u_0 \in U$

(ii) $u_m u_n = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & m+n \\ 0 & 1 \end{smallmatrix}\right) = u_{m+n} \in U$

(iii) From (1) and (2), we see that $u_m^{-1} = u_{-m}$, so $u_m^{-1} \in U$. So U is a subgroup.

Notation: If H is a subgroup of G, we write $H \leq G$. (We can write $H < G$ if $H \neq G$).

Powers in groups: Let G be a a group written multiplicatively. Let $g \in G$. We can write $g^1 = g$, $g^2 = gg$, $g^3 = ggg$, and so on. We also write $g^0 = e$, and $g^{-n} = (g^{-1})^n$. So now g^n is defined for all $n \in \mathbb{Z}$.

Proposition 73. *(a) $(g^n)^{-1} = g^{-n}$.*

(b) $g^m g^n = g^{m+n}$.

(c) $(g^m)^n = g^{mn}$.

Proof omitted.

Caution: It is not generally true that $a^n b^n = (ab)^n$. (Though true for Abelian Group)

Cyclic Subgroups

Proposition 74. *Let G be a group, and let $g \in G$. Define $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$. Then $\langle g \rangle$ is a subgroup of G.*

Proof. Check subgroup criteria:

(i) $g^0 = e$

(ii) $g^m g^n = g^{m+n}$

(iii) $(g^n)^{-1} = g^{-n}$ ∎

Definition 75. The subgroup $\langle g \rangle$ is the *cyclic subgroup* generated by g.

Examples 76.

(i) $U = \{u_n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \ : \ n \in \mathbb{Z}\}$ is $\langle u_1 \rangle$. (Easy induction to show that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for all $n \in Z$)

(ii) In $(\mathbb{Z}, +)$, what is $\langle 3 \rangle$?
Note that we write this group additively, write ng instead of g^n. So $\langle 3 \rangle = \{n.3 \ : \ n \in \mathbb{Z}\}$. So $\langle 3 \rangle$ contains precisely all multiples of 3.

(iii) $G = S_3$. What are the cyclic subgroups? From the group table we can easily calculate:
$\langle e \rangle = \{e\}$
$\langle a \rangle = \{e, a, a^2\}$, since $a^3 = e$.
$\langle a^2 \rangle = \{e, a, a^2\}$
$\langle b \rangle = \{e, b\}$ since $b^2 = e$
$\langle ab \rangle = \{e, ab\}$, since $(ab)^2 = e$
$\langle a^2 b \rangle = \{e, a^2 b\}$, since $(a^2 b)^2 = e$

Recall: $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ Lecture 16

Definition 77. A group is *cyclic* if $G = \langle g \rangle$ for some $g \in G$. In this case g is a *generator* for G.

Examples 78.

(i) $(\mathbb{Z}, +)$ is cyclic, since $\mathbb{Z} = \langle 1 \rangle$. Another generator is -1. There are no other generators.

(ii) $\{1, -1, i, -i\}$ is cyclic, since it is equal to $\langle i\rangle$, and $\langle -i\rangle$. So i and $-i$ are generators. 1 and -1 are not.

(iii) Let $\Omega_n = \{$ complex nth roots of unity $\}$, under complex multiplication. Let $w = e^{2\pi i/n}$. Then $\langle w\rangle = \{e^{2\pi ik/n} : k \in \mathbb{Z}\} = \Omega_n$. So Ω_n is a a group - a cyclic subgroup of $\mathbb{C}$.

Since $|\Omega_n| = n$, it follows that there exists a group of order n for $n \in \mathbb{N}$.

(iv) S_3 is *not* cyclic. We have calculated all of its cyclic subgroups (Example 76, 3), and none of them were equal to S_3.

The Order of a Group Element

Definition 79. Let G be a group, and $g \in G$. The *order* of g is the least positive integer k, such that $g^k = e$, if such an integer exists. Otherwise g has infinite order. Write $\mathrm{ord}(g)$ or $\mathrm{o}(g)$ for the order of g. Write $\mathrm{ord}(g) = \infty$ if g has infinite order.

Examples 80.

(i) In any group G, the identity e has order 1. No other element has order 1. (If $g^1 = e$ then $g = e$.)

(ii) Let $G = S_3$ Let $a = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$. Then $a^1 \neq e$, $a^2 = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right) \neq e$. But $a^3 = e$. So ord $(a) = 3$.

Let $b = \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$. Then $b \neq e$, but $b^2 = e$. So $\mathrm{ord}(b) = 2$. Can also check that $\mathrm{ord}(a^2) = 3$, $\mathrm{ord}(ab) = 2$, $\mathrm{ord}(a^2b) = 2$.

(iii) Let $G = (\mathbb{Z}, +)$. We know that $\mathrm{ord}(0) = 1$. Suppose $n \neq 0$. Then since $\underbrace{n + n + \dots n}_{k} = kn \neq 0$ for any positive integer k. We must have ord $(n) = \infty$.

(iv) $G = GL_2(\mathbb{C})$. Let $A = \begin{pmatrix} i & 0 \\ 0 & e^{2\pi i/3}\end{pmatrix}$. *What is the order of A?*

We see that for $k \in \mathbb{Z}$ we have $A^k = \begin{pmatrix} i^k & 0 \\ 0 & e^{2\pi ik/n}\end{pmatrix}$, which is equal to $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ if and only if $i^k = 1$ and $e^{2\pi ik/n} = 1$.

Now $i^k = 1$ if and only if $4 \mid k$, and $e^{2\pi ik/n} = 1$ if and only if $3 \mid k$. So $A^k = I$ if and only if $12 \mid k$. Since the smallest positive integer divisible by 12 is 12, we have $\mathrm{ord}(A) = 12$.

Proposition 81. *If G is a group and $g \in G$, then $|\langle g\rangle| =$ ord (g). ($\langle g\rangle$ is infinite $\iff$ ord $(g) = \infty$).*

Proof. Suppose first that $\mathrm{ord}(g) = \infty$. So $g^k \neq e$ for any $k \in \mathbb{N}$. We claim that if $m \neq n \in \mathbb{Z}$, then $g^m \neq g^n$. Suppose w.l.o.g. that $n > m$. Let $k = n - m$. Then $k \in \mathbb{N}$. Now $g^n = g^{m+k} = g^m g^k$. If $g^m = g^n$, then $g^m = g^m g^k$, and so $g^k = e$ by left cancellation. But this is a contradiction, and this proves the claim.

Now we have $g^0, g^1, g^2, g^3, \ldots$ are all distinct elements of $\langle g \rangle$, and so $\langle g \rangle$ is infinite.

Now suppose that $\mathrm{ord}(g) = k \in \mathbb{N}$. I claim that $\langle g \rangle = \{g^0, g^1, \ldots, g^{k-1}\}$, and that the elements $g^0, \ldots, g^{k-1}$, are distinct. (So $|\langle g \rangle| = k$). We use the fact that $n \in \mathbb{Z}$ can be written as $pk + q$, where $p, q \in \mathbb{Z}$ and $0 \leq q < k$. So $g^n = g^{pk+q} = (g^k)^p g^q = e^p g^q = g^q$. But g^q is one of the elements $g^0, \ldots, g^{k-1}$, and so $\langle g \rangle = \{g^0, \ldots, g^{k-1}\}$.

Suppose $g^i = g^j$, where $0 \leq i < j \leq k-1$. Let $l = j - i$. Then $g^i = g^j = g^{i+l}$, and so $g^l = e$ by left cancellation. But l is a positive integer less than k, so this contradicts the assumption that $\mathrm{ord}(g) = k$. This proves the claim. ■

We have seen Proposition 81 illustrated in several examples already.

Examples 82.

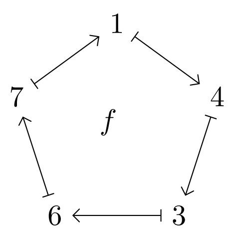
(i) Comparing the list of cyclic subgroups of S_3 with the list of the orders of elements, we saw that $\mathrm{ord}(g) = |\langle g \rangle|$ in each case.

(ii) In the case $G = (\mathbb{Z}, +)$, we saw that $\langle 3 \rangle = 3\mathbb{Z} = \{3k : k \in \mathbb{Z}\}$. So $|\langle 3 \rangle| = \infty$, and we have also seen that $\mathrm{ord}(3) = \infty$.

(iii) If $w = e^{2\pi i/n}$, then clearly $\mathrm{ord}(w) = n$. And $\langle w \rangle = \Omega_n$ has order n too.

Cycles

Let $f = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 3 & 2 & 7 & 1 & 8 \end{smallmatrix}\right)$. Looking at the successive images of 1 under the permutation Lecture 17
f, we get back to 1 again after 5 steps via 4,3,6,7.



We say that 1,4,3,6,7 forms a 5-cycle of f. Write (1 4 3 6 7) for this cycle.

Similarly: $2 \underset{f}{\rightleftarrows} 5$ is a 2-cycle, written (2 5).

Finally 8 is a fixed point or 1-cycle of f. We can write (8) if we like – but usually we do not write out 1-cycles.

We can think of cycles as permutations in their own right. Take everything outside the cycle to be fixed.

So (1 4 3 6 7) is the permutation: $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 6 & 3 & 5 & 7 & 1 & 8 \end{smallmatrix}\right)$.

And (2 5) is the permutation: $\left(\begin{smallmatrix}1&2&3&4&5&6&7&8\\1&5&3&4&2&6&7&8\end{smallmatrix}\right)$.

And (8) represents the identity permutation.

Notice that $f = (1\ 4\ 3\ 6\ 7)(2\ 5)$ (where the cycles are multiplied by composition, as elements of S_8). So we have factorised f into cycles. These cycles have no common points – they are disjoint. This is the *disjoint cycle notation* for f.

Method 83. *To calculate the disjoint cycle notation for a permutation $f \in S_n$.*

Step 1. Pick the least element $i \in \{1, \ldots, n\}$ which we haven't used yet. (Initially, we choose $i = 1$.) Open a new cycle with i.

Step 2. Continue the cycle with successive images of i under f until we get back to i again. Then close the cycle.

Step 3. If all $i \in \{1, \ldots, n\}$ have appeared, then stop. Otherwise go back to Step 1.

Examples 84. $f = \begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\3&6&1&7&5&8&2&4&10&9\end{pmatrix}$.

Open a cycle with 1. Continue the cycle with $f(1) = 3$. Since $f(3)$ is 1 again, close the cycle. (1 3).

Now open a new cycle with 2. Continue it with 6,8,4,7. Since $f(7) = 2$, we then close the cycle. (2 6 8 4 7).

Start a cycle with 5. But 5 is a fixed-point of f, so close the cycle immediately. (5).

Finally, start a cycle with 9. Continue it with $f(9) = 10$. But $f(10) = 9$, so close the cycle. (9 10).

Now all points have appeared, so we stop. So the disjoint cycle notation for f is $f = (1\,3)(2\,6\,8\,4\,7)(5)(9\,10)$. We usually omit 1-cycles. So $f = (1\,3)(2\,6\,8\,4\,7)(9\,10)$.

Proposition 85. *Method 83 always works. Every permutation in S_n can be written as a product of disjoint cycles.*

Proof. First we show that whenever we open a new cycle at Step 1, we are able to close it at Step 2. (We always get back to the starting point eventually.)

Suppose that x is the starting point. Since n is finite, the points $x, f(x), f^2(x), \ldots$ cannot be all distinct. So there must exist some least k such that $f^k(x) = f^j(x)$, for some $j < k$. Suppose that $j \neq 0$. Then $f^{-1}f^k(x) = f^{-1}f^j(x)$, and so $f^{k-1}(x) = f^{j-1}(x)$.

But this contradicts the assumption that k was the *least* to give a repeat. So by contradiction, we must have $j = 0$. So $f^k(x) = f^0(x) = x$. So the first repeated term in the cycle is x itself, and we close the cycle at that point.

Next we check that the cycles arising from Method 83. are disjoint. Let c and d be two cycles. Suppose that c starts with x and d starts with y. Suppose that c was constructed first. Then y cannot be in the cycle c (since otherwise we would not have used it to start a new cycle.)

Suppose z is in both c and d. So $z = f^i(x) = f^j(y)$ for some i, j. But now $f^{i-j}(x) = y$ This implies that y is in the cycle c, which is a contradiction. Hence c and d are disjoint. ∎

Multiplication. To multiply permutations given in disjoint cycle notation, recall that $fg = f \circ g$, so $fg(x) = f(g(x))$. Now use Method 83 to get the disjoint cycle notation of fg.

Example: $f = (1\ 3\ 5)(2\ 4\ 6)$, $g = (1\ 2\ 3\ 4)(5\ 6) \in S_6$. Calculate $fg : 1 \mapsto 4 \mapsto 3 \mapsto 6 \mapsto 1 \quad 2 \mapsto 5 \mapsto 2$. Hence $fg = (1\ 4\ 3\ 6)(2\ 5)$.

Inverses. These are very easy Just write the cycles backwards.

If $f = (1\ 3\ 5)(2\ 4\ 6)$, then $f^{-1} = (5\ 3\ 1)(6\ 4\ 2) = (1\ 5\ 3)(2\ 6\ 4)$.

Non-uniqueness.

i. Order of the cycles doesn't matter.

ii. The choice of starting point in each cycle doesn't matter.

iii. We can include or exclude 1-cycles.

Example 86. In disjoint cycle notation, the group table for S_3 from Example 68 becomes: Lecture 18

	e	(123)	(132)	(23)	(12)	(13)
e	e	(123)	(132)	(23)	(12)	(13)
(123)	(123)	(132)	e	(12)	(13)	(23)
(132)	(132)	e	(123)	(13)	(23)	(12)
(23)	(23)	(13)	(12)	e	(132)	(123)
(12)	(12)	(23)	(13)	(123)	e	(132)
(13)	(132)	(12)	(23)	(132)	(123)	e

Remark 87. Disjoint cycles commmute. (If cycles c_1 and c_2 have no points in common, then $c_1c_2 = c_2c_1$). Example: $(12)(345) = (345)(12)$.

Cycles which are not disjoint don't usually commute. Example: $(12)(13) = (132)$. $(13)(12) = (123)$.

Definition 88. The *cycle shape* of a permutation is the sequence of cycle lengths in descending order of size when the permutation is written in disjoint cycle notation. We include $1-$cycles.

Example: $f = (12)(3)(456)(7)(89) \in S_9$ then f has cycle shape $(3, 2, 2, 1, 1)$. We abbreviate this to $(3, 2^2, 1^2)$. The identity of S_9 has cycle shape $(1, 1, 1, 1, 1, 1, 1, 1, 1)$, or (1^9).

Examples 89. What are the cycle shapes in S_4, and how many elements are there of each shape?

Cycle shapes are given by weakly decreasing sequences of positive integers adding up to 4. There are (4), $(3, 1)$, (2^2), $(2, 1^2)$ and (1^4). (These are the *partitions* of the integer 4). How many of each type?

(4): May start the cycle at 1. There are 3 choices for $f(1)$. Then there are 2 choices for $f^2(1)$. This determines the cycle. So 6 possible 4-cycles. (These are (1234), (1243), (1324), (1342), $(1423), (1432)$.)

$(3, 1)$: There are 4 choices for the fixed-point (or 1-cycle). Once the fixed point is chosen, there are 2 choices for the 3-cycle. So there are 8 possible permutations with this shape.

(2^2): 1 is in a 2-cycle, and there are 3 choices for the other point in that cycle. This determines the permutation. So there are only 3 permutations with this shape. These are $(12)(34)$, $(13)(24)$, $(14)(23)$.

$(2, 1^2)$: There are $\binom{4}{2}$ ways of choosing the two fixed points. This determines the permutations. So there are 6 permutations.

(1^4): Only the identity has this shape.

Check: we have $6 + 8 + 3 + 6 + 1 = 24 = |S_4|$.

Order of a Permutation

Proposition 90. *Let G be a group, and let $g \in G$. Suppose ord$(g) = d$. Then for all $k \in \mathbb{Z}$, we have $g^k = e \iff d \mid k$.*

Proof. By Euclid's Lemma, we can write $k = xd + y$, where $x, y \in \mathbb{Z}$ and $0 \le y < d$. Now $g^k = g^{xd+y} = (g^d)^x g^y = e^x g^y = g^y$. But $y < d$, and so we have $g^y = e \iff y = 0 \iff d \mid k$. ∎

Proposition 91. *Let G be a group, and let $a, b \in G$ be elements such that $ab = ba$. Then:*

(i) $a^{-1}b = ba^{-1}$

(ii) $a^i b^j = b^j a^i$, *for* $i, j \in \mathbb{Z}$.

(iii) $(ab)^k = a^k b^k$, *for* $k \in \mathbb{Z}$.

Proof.

1. We have $ab = ba$. So $a^{-1}(ab)a^{-1} = a^{-1}(ba)a^{-1}$. Hence $ba^{-1} = a^{-1}b$.

2. If $j < 0$, then replace b with b^{-1}. We know $ab^{-1} = b^{-1}a$ by 1. In this way, we may assume $j \geq 0$. Now by induction on j. [left as exercise.]

3. If $k < 0$, then write $d = a^{-1}$, $c = b^{-1}$. Then $(ab)^{-1} = cd$, and so $(ab)^k = (cd)^{-k}$. In this way we may assume $k \geq 0$. Now work by induction on k. [left as exercise.] ∎

Proposition 92. *Let f be a permutation with cycle shape $(r_1, r_2, \ldots, r_k)$. Then $ord(f) = lcm(r_1, r_2, \ldots, r_k)$.*

Proof. Write $f = c_1c_2 \ldots c_k$, where c_i has length r_i for all i, and the cycles c_i are disjoint from one another. Lecture 19

Recall *Remark 87*: disjoint cycles commute. So $c_ic_j = c_jc_i$, for all i, j. Let $t \in \mathbb{Z}$. **Claim:** $f^t = c_1^tc_2^t \ldots c_k^t$.

Proof of Claim. First observe that $f = c_1, \ldots, c_{k-1}c_k = c_kc_1, \ldots, c_{k-1}$, since c_k commutes with all of the other cycles. So c_k commutes with $c_1 \ldots c_{k-1}$. So by Prop 91.3, we have $f^t = ((c_1 \ldots c_{k-1})c_k)^t = (c_1 \ldots c_{k-1})^tc_k^t$. Now an easy induction on k gives that $f^t = c_1^tc_2^t \ldots c_k^t$, as required. ∎

Continuing the proof of Prop 92, we see that $f^t = e \iff c_i^t = e$, for all i. But $c_i^t = e$ if and only if $r_i \mid t$. So $f^t = e \iff r_i \mid t$, for all $i \iff t$ is divisible by $\mathrm{lcm}(r_1, r_2, \ldots, r_k)$.

So $\mathrm{ord}(f)$ is the last positive integer divisible by $\mathrm{lcm}(r_1, \ldots, r_k)$, which is $\mathrm{lcm}(r_1, \ldots, r_k)$ itself. ∎

Examples 93.

(i) $\mathrm{ord}((12)(3456)) = \mathrm{lcm}(2, 4) = 4$.

(ii) $\mathrm{ord}(13)(3456)) \neq 4-$ the cycles are not disjoint. We calculate $(13)(3456) = (13456)$, which has order 5.

(iii) Clearly 1-cycles never affect the order of a permutation.

(iv) Suppose we have a pack of 8 cards. We shuffle them by cutting the pack into 2 equal parts, and then interleaving the parts.
We get the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 & 8 \end{pmatrix}$. In disjoint cycle form, this is $(253)(467)$, which has order 3. So if we repeat our shuffle three times, the pack is back in its original order.

Exercise: How many shuffles do you need for a full pack of 52 cards to be shuffled back to its original order?

Lagrange's Theorem

Theorem 94: Lagrange's Theorem

Let G be a finite group. Let H be a subgroup of G. Then $|H|$ divides $|G|$.

Examples 95.

(i) $|S_3| = 6$. We have seen that the cyclic subgroups of S_3 have orders $1, 2, 3$. There are no other subgroups, except S_3 itself.

(ii) $G = \{1, -1, i, -i\}$, order 4. The subgroups of G are $\{1\}, \{1, -1\}$, and G, with orders $1, 2, 4$.

(iii) Let $C_n = \langle w \rangle \leq \mathbb{C}\backslash\{0\}$, where $w = e^{2\pi i/n}$. If d divides n, then $w^{n/d} = e^{2\pi i/d}$, which has order d. So $\langle w^{n/d} \rangle$ is a subgroup of C_n with order d. In fact all of the subgroups of C_n have this form, for some d.

Remark. Although it is true for the elementary groups above, it is not true in general that a group G has a subgroup of order d for every divisor.
Example: S_5 has order 120, but has no subgroup of order $15, 30, 40$.

Corollary 96. *(to Lagrange's Theorem) If G is a finite group, and $g \in G$, then ord(g) divides G.*

Proof. $\mathrm{ord}(g) = |\langle g \rangle|$, and this divides G by Lagrange's Theorem. ■

Corollary 97. *Suppose $|G| = n$. Then $g^n = e$, for all $g \in G$.*

Proof. We know $\mathrm{ord}(g) \mid n$ by Corollary 96. So $g^n = e$ by Prop 90. ■

Corollary 98. *If $|G|$ is a prime number, then G is cyclic.*

Proof. Let $g \in G$ be such that $g \neq e$. Then $\mathrm{ord}(g) \neq 1$, but $\mathrm{ord}(g) \mid |G|$. So $\mathrm{ord}(g) = |G|$ (Since $|G|$ is prime.) Hence $\langle g \rangle = G$, and so G is cyclic. ■

Some ideas for the proof of Lagrange's Theorem.

G is a finite group. $H \leq G$ is a subgroup. We shall divide the elements of G into disjoint subsets, so that every element is in exactly one subset, and each subset has the same size as H. Now if there are k subsets, then $|G| = k|H|$, and we are finished.

What are these subsets? One of these will be H itself. Now if $x \in G\backslash H$, we need a subset including x. We take $Hx = \{hx : h \in H\}$. Now if $y \in G\backslash(H \cup Hx)$, then we take Hy, and keep on going until we have used up all the elements of G.

Definition 99. Let G be a group, and let $H \leq G$. Let $x \in G$. The *right coset* Hx is the set $\{hx : h \in H\}$. [The *left coset* xH is the set $\{xh : h \in H\}$. Lecture 20

Claim 100. $|Hx| = |H|$, *for all* $x \in G$.

Proof. It is enough to show that $h_1x \neq h_2x$, if $h_1 \neq h_2$. But this is true by right cancellation, since $h_1x = h_2x \implies h_1 = h_2$. ∎

Claim 101. *Let* $x, y \in G$. *Then either* $Hx = Hy$, *or else* $Hx \cap Hy = \emptyset$. *(Two right cosets of* H *are equal or disjoint.)*

Proof. Suppose that Hx and Hy are not disjoint. So there exists $z \in Hx \cap Hy$. So there exists $h_1, h_2 \in H$ such that $z = h_1x = h_2y$. Now $y = h_2^{-1}h_1x$, and so $y \in Hx$. Now any hy in Hy is equal to $hh_2^{-1}h_1x$, which is in Hx. So $Hy \subseteq Hx$. By a similar argument, $Hx \subseteq Hy$ and so $Hx = Hy$. ∎

Claim 102. $x \in Hx$ *for all* $x \in G$.

Proof. $e \in H$, and so $x = ex \in Hx$. ∎

Proof of Lagrange's Theorem.

Consider the set $S = \{Hx : x \in G\}$, a set of cosets.

(i) Every $x \in G$ is in one member of S. (By Claim 102.)

(ii) Every $x \in G$ is in no more than one member of S. (By Claim 101.)

It follows that $|G| = \sum_{X \in S} |X|$

(iii) Every member of S has size $|H|$. (By Claim 100.)

Hence $|G| = k|H|$, where $|S| = k$. So $|H|$ divides $|G|$ as required. ∎

Remarks 103. (Further Properties of cosets)

(i) $H = Hx \iff x \in H$.

Proof. Notice that $H = He$, so is a coset. If $x \in H$, then certainly $H \cap Hx \neq \emptyset$, so $H = Hx$, by Claim 101. If $x \neq H$, then $H \neq Hx$, since $x \in Hx$. ∎

(ii) If $y \in Hx$, then $Hx = Hy$.

Proof. We have $y \in Hx \cap Hy$, and so $Hx \cap Hy \neq \emptyset$. So $Hx = Hy$, by Claim 101. ∎

(iii) If $x \notin H$, then Hx is *not* a subgroup of G.

Proof. $H \cap Hx = \emptyset$, by (1). Hence $e \notin Hx$. ∎

(iv) $Hx = Hy$ if and only if $xy^{-1} \in H$.

Proof. $Hx = Hy$ if and only if $x \in Hy$, and this occurs if and only if $x = hy$ for some $h \in H$. But this is equivalent to $xy^{-1} = h$. ∎

Definition 104. Let G be a finite group, and $H \leq G$. The integer $\frac{|G|}{|H|}$ is the *index* of H in G, written $|G : H|$. So $|G : H|$ is the number of right cosets of H in G. (If is also the number of left cosets.)

Examples 105. $G = S_3$.

(i) $H = \langle (123) \rangle = \{e, (123), (132)\}$.

One coset is $He = H$. Since $\frac{|G|}{|H|} = \frac{6}{3} = 2$, there will be only one other coset. So this must be $\{(12), (13), (23)\}$.

(ii) $H = \langle (12) \rangle = \{e, (12)\}$. This time $\frac{|G|}{|H|} = \frac{6}{2} = 3$. So we are looking for two more cosets. We have:

$H(123) = \{e(123), (12)(123)\} = \{(123), (23)\}$.
$H(132) = \{(132), (13)\}$.

Note that $H(23) = H(123)$, and $H(13) = H(132)$.

Remarks 106. (i) If $H = \{e\}$ then the right cosets of H in G are just singleton (1-element) sets $\{g\}$ for $g \in G$. (These are also the left cosets.)

(ii) If $H = G$ then the only right (or left) coset is H itself.

Modular Arithmetic

Lecture 21 **Recall:** Let $m \in \mathbb{N}$, and $a, b \in \mathbb{Z}$. We say that "a is congruent to b modulo m" if $a - b$ is divisible by m. (Equivalently: a and b give the same remainder when you divide by b.)

Write $a \equiv b \mod m$.

Definition 107. Fix $m \in \mathbb{N}$. For $a \in \mathbb{Z}$, define the *residue class* of a modulo m, $[a]_m$, by $[a]_m = \{s \in \mathbb{Z} : s \equiv a \pmod{m}\} = \{km + a : k \in \mathbb{Z}\}$.

Examples 108. $[0]_5 = \{5k : k \in \mathbb{Z}\} = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$.
$[1]_5 = \{\ldots, -14, -9, -4, 1, 6, 11, 16, \ldots\}$

Recall that every integer $n \in \mathbb{Z}$ can be written as $n = qm + r$, where $q, r \in \mathbb{Z}$, with $0 \leq r < m$. Now $n \equiv r \mod m$, so $[n]_m = [r]_m$. So $\mathbb{Z} = [0]_m \cup [1]_m \cup \cdots \cup [m-1]_m$. This is a *disjoint* union - every integer lies in exactly one of these residue classes.

Proposition 109. *For any $m \in \mathbb{N}$, the residue class $[0]_m$ is a subgroup of $(\mathbb{Z}, +)$. The other residue classes modulo m are cosets of $[0]_m$.*

Proof. To check that $[0]_m$ is a subgroup, check the subgroup criteria:

(i) We have $0 \in [0]_m$. So the identity is in $[0]_m$.

(ii) Let $a, b \in [0]_m$. Then $a = km$ and $b = lm$ for some $k, l \in \mathbb{Z}$. Now $a+b = (k+l)m \in [0]_m$. So $[0]_m$ is closed under $+$.

(iii) Let $a \in [0]_m$. Then $a = km$ for $k \in \mathbb{Z}$. Now the inverse of a is $-a = (-k)m$. So $-a \in [0]_m$. Hence $[0]_m$ is a subgroup.

Now $[r]_m = \{km + r : k \in \mathbb{Z}\} = \{x + r : x \in [0]_m\} = [0]_m + r$, which is the right coset of $[0]_m$ containing r. ∎

Notation If m is fixed and understood, we then just write $[r]$ for $[r]_m$. We write $\mathbb{Z}_m$ for the set of residue classes modulo m. So:
$\mathbb{Z}_m = \{[0]_m, [1]_m, \ldots, [m-1]_m\}$, a set of size m.

Definition 110. (Binary operation on $\mathbb{Z}_m$.)

$$[a]_m + [b]_m = [a+b]_m.$$

$$[a]_m \times [b]_m = [ab]_m.$$

We need to make sure that these operations are well defined. Suppose $[a]_m = [a']_m$ and $[b]_m = [b']_m$. Then $a' = a + km$, and similarly $b' = b + lm$, for some $k, l \in \mathbb{Z}$. Now $a' + b' = a + km + b + lm = a + b + (k+l)m$. So $[a' + b']_m = [a+b]_m$.

And $a'b' = (a + km)(b + lm) = ab + (al + bk + klm)m$, so $[a'b']_m = [ab]_m$. So both of these operations are well defined.

Proposition 111. $(\mathbb{Z}_m, +)$ *is a group.*

Proof. $([a]_m + [b]_m) + [c]_m = [(a+b)+c]_m = [a+(b+c)]_m = [a]_m + ([b]_m + [c]_m)$. (since $+$ is associative on $\mathbb{Z}$.)
$[0]_m$ is an identity. The inverse of $[a]_m$ is $[-a]_m$. ∎

Note that $(\mathbb{Z}_m, \times)$ is *not* a group, if $m > 1$. It is associative, and $[1]_m$ is an identity. But $[0]_m$ has no inverse, unless $[0]_m = [1]_m$. And $[0]_m \neq [1]_m$, unless $m = 1$.

How about $\mathbb{Z}_m^* = \mathbb{Z}_m \backslash \{[0]_m\}$?

Proposition 112. $(\mathbb{Z}_m^*, \times)$ *is a group if and only if m is a prime number.*

Proof. If $m = 1$, then $\mathbb{Z}_m^* = \emptyset$, which is not a group. So suppose $m > 1$.
If m is *not* prime, we can write $m = ab$, where $1 < a \leq b < m$. Now m does not divide a or b, so $[a]_m \neq [0]_m$ and $[b]_m \neq [0]_m$. So $[a]_m, [b]_m \in \mathbb{Z}_m^*$. But $[a]_m \times [b]_m = [m]_m = [0]_m \notin \mathbb{Z}_m^*$. So $\times$ not a binary operation on $\mathbb{Z}_m^*$.

Now suppose m *is* prime. Property of prime numbers: If p is prime, and $p \mid ab$, then p divides at least one of a or b. Suppose $[a]_m, [b]_m \in \mathbb{Z}_m^*$. Then m does not divide a or b. So m does not divide ab, by the Property above. Hence $[ab]_m \in \mathbb{Z}_m^*$. So $\times$ is a binary operation on $\mathbb{Z}_m^*$. Then the axioms:

Associativity: same argument as for $+$ on $\mathbb{Z}_m$

Identity: $[1]_m$.

Inverses: If $[a]_m \in \mathbb{Z}_m^*$, then m does not divide a. So $\text{hcf}(m, a) = 1$. By Euclid's Algorithm, there exists $x, y \in \mathbb{Z}$, with $mx + ay = 1$. Now $ay \equiv 1 \mod m$, and so $[a]_m \times [y]_m = [1]_m$. So $[a]_m^{-1} = [y]_m$. ∎

Examples 113.

Lecture 22

(i) $\mathbb{Z}_5^* = \{[1], [2], [3], [4]\}$. Is $\mathbb{Z}_5^*$ cyclic? Check the powers of $[2]$:
$[2]^2 = [4]$, $[2]^3 = [8] = [3]$, $[2]^4 = [16] = [1]$.

So $\text{ord}[2] = 4$, and so $\langle [2] \rangle = \mathbb{Z}_5^*$. So yes, $\mathbb{Z}_5^*$ is cyclic.

Fact: $\mathbb{Z}_p^*$ is cyclic for any prime p.

(ii) In $\mathbb{Z}_{31}^*$, what is $[7]^{-1}$?
We need to find $x, y \in \mathbb{Z}$ such that $31x + 7y = 1$. (Then $[y] = [7]^{-1}$)
We use Euclid's algorithm:
$31 = 4.7 + 3$
$7 = 2.3 + 1$

So $1 = 7 - 2.3 = 7 - 2.(31 - 4.7) = 9.7 - 2.31$

So $x = -2, y = 9$. Hence $[7]^{-1} = [9]$

Theorem 114: Fermat's Little Theorem

Let p be a prime, and let $n \in \mathbb{Z}$. If p does not divide n, then $n^{p-1} \equiv 1 \pmod{p}$.

Proof. If p does not divide n, then $[n]_p \in \mathbb{Z}_p^*$. So $\text{ord}[n]_p$ divides $|\mathbb{Z}_p^*| = p - 1$ by the Corollary to Lagrange's Theorem. Hence $[n]_p^{p-1} = [1]_p$. ∎

Corollary 115. *(Alternative statement of FLT). Let p be a prime, and let $n \in \mathbb{Z}$. Then $n^p \equiv n \pmod{p}$.*

Proof. If p does not divide n, then by FLT, $n^{p-1} \equiv n \pmod{p}$. So $n^p \equiv n \mod p$. If p does divide n, then $[n]_p = [0]_p$. So $[n]_p^p = [0]_p^p = [0]_p = [n]_p$. ∎

Example 116. Find the remainder when 6^{82} is divided by 17.
Answer: Note that $6^{82} = 6^{5.16+2} = (6^{16})^5(6^2) \equiv 1^5 6^2 \equiv 36 \equiv 2 \pmod{17}$, using FLT.

Definition 117. A *perfect number* is a natural number n which is the sum of its proper divisors. (i.e. divisors other than n itself.)

Examples: $6 = 1+2+3$. Also $28,\ 496,\ 8128,\ 35,550,336$, etc.

Theorem 118

If $2^m - 1$ is a prime number, then $n = 2^{m-1}(2^m - 1)$ is perfect. [Euclid] Every even perfect number is of this form. [Euler]

It is still unknown whether any odd perfect numbers exist. If they do, they are $> 10^{1500}$

Our interest is in the primes $2^m - 1$. These are *Mersenne primes*.
Examples: $3 = 2^2-1$, $7 = 2^3-1$, $31 = 2^5-1$, $127 = 2^7-1$. The largest known Mersenne prime is $2^{57,885,161} - 1$.

Proposition 119. *If* $2^m - 1$ *is prime, then* m *is prime. [The converse is* not *true]*

Proof. Recall the polynomial identity:

$$(x^c - 1) = (x-1)(x^{c-1} + x^{c-2} + \cdots + x + 1).$$

Suppose that m is not prime. So $m = ab$, where $a, b > 1$. Now

$$2^m - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)(2^{a(b-1)} + \cdots + 2^a + 1)$$

(using the polynomial identity with $x = 2^a$). So $2^a - 1$ divides $2^m - 1$. But since $1 < a < m$, we see that $2^a - 1 \neq 1, 2^m - 1$, so it is a proper divisor greater than 1. So $2^m - 1$ is not prime. ∎

When is $2^p - 1$ *prime?* Use the group $\mathbb{Z}_p^*$:

Proposition 120. *Let* $n = 2^p - 1$, *where* p *is prime. Suppose that* q *is a prime divisor of* n. *Then* $q \equiv 1 \pmod{p}$.

Proof. Since q divides n, we have q divides $2^p - 1$. So $2^p \equiv 1 \mod q$.
So $[2]_q^p = [1]_q$. So $\mathrm{ord}([2]_q)$ divides p. Since p is prime, $\mathrm{ord}([2]_q)$ is 1 or p.
Now $2 \not\equiv 1 \mod q$, so $[2]_q \neq [1]_q$. So $\mathrm{ord}([2]_q) \neq 1$. Hence $\mathrm{ord}([2]_q) = p$. So p divides

$|\mathbb{Z}_q^*| = q - 1$, by the Corollary to Lagrange's Theorem. Hence $q \equiv 1 \pmod{p}$. ∎

Remark 121. What this tells us is that when we look for prime divisors of $n = 2^p - 1$ (to test primality), we need only test primes which are $\equiv 1 \mod p$. (If $p = 57,885,161$, this is a serious saving!)

Exercise: Check that in fact, we only need check primes which are $1 \pmod{2p}$.

Dihedral Groups

Lecture 23

Definition 122. Let S be a subset of $\mathbb{R}^n$. Let $A \in GL_n(\mathbb{R})$. Write $AS = \{Av : v \in S\}$. If $AS = S$, we say A *preserves* S.

Proposition 123. *For any $S \subseteq \mathbb{R}^n$, the set $H = \{A \in GL_n(\mathbb{R}) : A \text{ preserves } S\}$ is a subgroup of $GL_n(\mathbb{R})$. [The symmetry group of S.]*

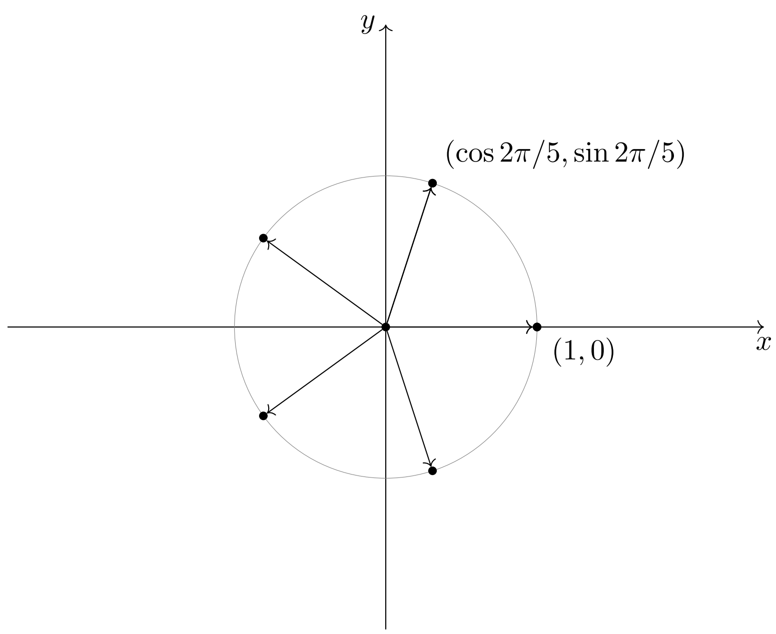
Proof. Check the subgroup criteria:

Certainly $IS = S$. So $I \in H$.

Suppose that $A, B \in H$. So $AS = S,\ BS = S$.
So $(AB)S = \{(AB)v : v \in S\} = \{(A(Bv) : v \in S\}$. But $BS = S$, and so $Bv \in S \iff v \in S$. So $(AB)S = \{Av : v \in S\} = AS = S$. Hence $AB \in H$.

For inverses suppose $A \in H$. So $AS = S$. Now $A^{-1}S = A^{-1}(AS) = (A^{-1}A)S = IS = S$. Hence $H \leq GL_n(\mathbb{R})$. ∎

Definition 124. Let $n > 2$. Let P_n be the regular n-gon in $\mathbb{R}^2$ with vertices $\begin{pmatrix} \cos 2\pi k/n \\ \sin 2\pi k/n \end{pmatrix}$, $k = 0, \dots n-1$. The *dihedral group* D_{2n} is the symmetry group of P_n. So $D_{2n} = \{A \in GL_2(\mathbb{R}) : AP_n = P_n\}$.

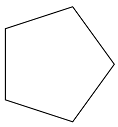


Remark We could instead take P_n to be the boundary of the n-gon, or just the set of vertices. We get the same symmetry group in each case. (Exercise: Check this)

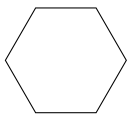
Proposition 125. *The group D_{2n} has order 2.*

Proof. (1) List $2n$ elements:

Clearly the rotation of $\mathbb{R}^2$ through $2\pi/n$, certainly preserves P_n. So any power of $\begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix}$ preserves P_n. Now $|\langle A \rangle| = n$, so we have n rotations in D_{2n}. We also have n reflections. If n is odd:



Here is an axis of symmetry passing through any vertex, and the midpoint of the opposite edge. There are n vertices, so n reflections.



If n is even, there is an axis of symmetry passing through any pair of opposite vertices, or the midpoints of opposite edges. This gives n reflections in this case as well.

(2) Show D_{2n} has at most $2n$ elements:

If $A \in D_{2n}$ and if v is a vertex of P_n, then Av is a vertex of P_n (by the Remark above.) Now if w is a vertex of P_n which is adjacent to v, then $A\{v, w\}$ is also a pair of adjacent vertices. But $\{v, w\}$ is a basis for $\mathbb{R}^2$. So once we know Av and Aw, we know what A is. Now the number of choices for Av and Aw is the number of paris of adjacent vertices of P_n. There are $2n$ such paris, so $|D_{2n}| \leq 2n$. Hence $|D_{2n}| = 2n$, as required. ∎

Corollary 126. *Every element of D_{2n} is either a rotation or a reflection.*

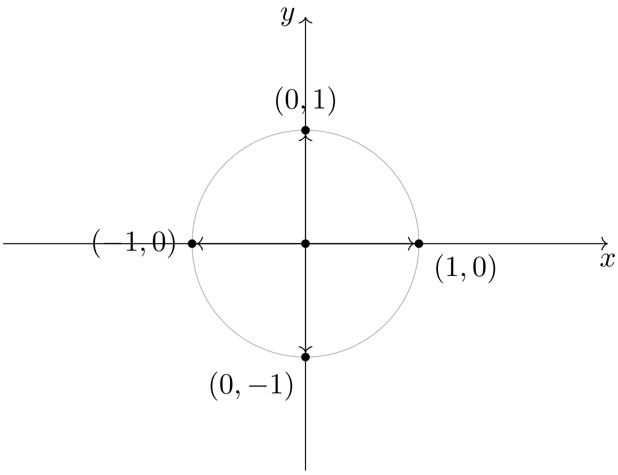
Proof. The $2n$ elements, rotation and reflection, constructed in (1) is a complete list. ∎

Remark. The rotations have determinant 1, and the reflections have determinant -1.

Proposition 127. *The set of rotations in D_{2n} is a subgroup, of index 2. The reflections form a coset of this subgroup.*

Proof. $\left\langle \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix} \right\rangle$ is a cyclic subgroup of order n, containing all of the rotations in D_{2n}. There is only one other coset of this subgroup containing everything else - namely the n reflections. ∎

Example 128. D_8 is the group of symmetries of a square.



Rotations: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Angles: 0 $\pi/2$ π $3\pi/2$

Reflections: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

3 Ring Theory

Many of the sets we've looked at so far have *two* binary operations, $+$ and $\times$. We may be interested in studying both at the same time. Lecture 24

For instance we might want to look at structures where equations like $x^2 + 1 = 2$, or $y^3 = x^2 + 2$ make sense, which involve both $+$ and $\times$.

Definition 129. A *ring* is a set R with two binary operations $+$ and $\times$ such that the following axioms hold:

(i) $(R, +)$ is an abelian group.

(ii) $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in R$. [ASSOCIATIVITY]

(iii) $x \times (y + z) = (x \times y) + (x \times z)$, and $(x + y) \times z = (x \times z) + (y \times z)$ for all $x, y, z \in R$ [DISTRIBUTIVITY] "multiplication distributes over addition"

R is a *ring with unity* if it also satisfies:

(iv) There exists $1 \in R$ such that $1 \times x = x \times 1 = x$.

R is a *commutative ring* if it also satisfies:

(v) $x \times y = y \times x$ for all $x, y \in R$.

In this course, all rings are assumed to be commutative rings with unity.

Examples 130.

(i) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ with their usual $+$ and $\times$.

(ii) $\mathbb{Z}_n$ for any $n > 0$, with $+$ and $\times$ defined on residue classes as before.

(iii) A non-commutative ring (so Axiom 5 fails): Let $n > 1$ and let $R = \mathrm{Mat}_n(\mathbb{R})$, the set of all $n \times n$ real matrices with the usual $+$ and $\times$.

Remarks 131.

(i) We write 0 for the identity of $(R, +)$.

(ii) We can write $x.y$ or xy for $x \times y$.

(iii) Fact: $0x = 0$ for any $x \in R$.

Proof. $0x = 0x + 0$. But $0x = (0 + 0)x = 0x + 0x$ by Distributivity. So $0x + 0 = 0x + 0x$. Hence $0x = 0$ by left cancellation in $(R, +)$. ∎

(iv) It is possible to have a ring R in which $0 = 1$. But in this case R only has one element. This ring is the *trivial ring*. We have seen it before - it is $\mathbb{Z}_1 = \{[0]_1\}$.

(v) The axioms do not say that multiplicative inverses exist. In fact if R is non-trivial, then it has at least one element with no inverse, since 0 is such an element.

Proof. $0x = 0$ for any x. So $0x \neq 1$ for any x unless $0 = 1$. But in this case R is trivial. ∎

Definition 132.

Let F be a field (e.g. $\mathbb{R}$). The *polynomial ring* $F[x]$ is the set of all polynomials in x with coefficients from F, with the usual $+$ and $\times$ for polynomials. Example: $f_1(x) = x^2 + 1$, $f_2(x) = x - 3$.

$$(f_1 + f_2)(x) = x^2 + x - 2 \quad (f_1 f_2)(x) = x^3 - 3x^2 + x - 3$$

Definition 133. A non-zero polynomial $f(x)$ has a *degree*, which is the largest power of x occurring in $f(x)$. Write deg $f(x)$ for the degree. Then we have deg $f(x) \geq 0$ for all $f(x)$. deg $f(x) = 0$ if and only if $f(x)$ is a non-zero constant polynomial.

We do not define the degree of the zero polynomial $f(x) = 0$. (People who do define it to be $-\infty$)

Exercise: Show that $F[x]$ is indeed a ring.

Definition 134. Let R be a ring with unity. A *unit* in R is an element u such that there exists $w \in R$ with $uw = wu = 1$. So a unit in R is an element with a multiplicative inverse.

Examples 135.

(i) If R is $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, then all of its non-zero elements are units.

(ii) If $R = \mathbb{Z}$, then the units are $\{\pm 1\}$.

(iii) If p is prime and $R = \mathbb{Z}_p$, then all non-zero elements are units, since $\mathbb{Z}_p^*$ is a group under $\times$.

Lecture 25

(iv) Let $R = \mathbb{Z}_m$, with $n \neq 1$ and n is not prime. The units in $\mathbb{Z}_m$ are the residue classes $[x]_m$ for $x \in \mathbb{Z}$ such that $\text{hcf}(x, m) = 1$ (if $\text{hcf}(a, b) = 1$, we say that a, b are *coprime*)

Proof. Suppose that $[x]_m$ is a unit in $\mathbb{Z}_m$. Then there exists $[y]_m \in \mathbb{Z}_m$ such that $[x]_m[y]_m = [y]_m[x]_m = [1]_m$. But $[x]_m[y]_m = [xy]_m$, and so $xy \equiv 1 \mod m$. Hence $xy - 1 = km$ for some $k \in \mathbb{Z}$. Now $xy - km = 1$. But $xy - km$ is divisible by $\text{hcf}(x, m)$. Hence $\text{hcf}(x, m) = 1$.

Conversely, suppose $\text{hcf}(x, m) = 1$. Then Euclid's Algorithm tells us that there exists $y, z \in \mathbb{Z}$ such that $xy + mz = 1$. Now $xy \equiv 1 \mod m$ so $[x]_m[y]_m = [xy]_m = [1]_m$. Hence $[x]_m$ is a unit in $\mathbb{Z}_m$. ∎

(v) What are the units in $F[x]$, where F is a field?
Observation: Let $f_1(x)$, $f_2(x)$ be non-zero polynomials. Then deg $(f_1(x) \times$

$f_2(x)) = \deg f_1(x) + \deg f_2(x)$. In particular, it follows that $\deg (f_1(x) \times f_2(x)) \geq \deg f_1(x)$.

Suppose that $f_1(x)$ is a unit in $F[x]$. Then there exists $f_2(x)$ such that $f_1(x) \times f_2(x) = 1$. So $\deg f_1(x) \leq \deg 1 = 0$. Hence $\deg f_1(x) = 0$. Hence $f_1(x)$ is a constant polynomial.

Conversely if $f_1(x) = c$, a non-zero constant polynomial, then $f_2(x) = 1/c$ is an inverse for $f_1(x)$. So $f_1(x)$ is a unit. We have shown that units in $F[x]$ are precisely the non-zero constant polynomials.

(vi) A non-commutative example - the units in the ring $\mathrm{Mat}_n(F)$. ($n \times n$ matrices over F) are the invertible matrices. There are the elements of $GL_n(F)$.

Definition 136. Given a ring R with unity, we write $R^\times$ for the set of units in R.

Theorem 137

For any ring R with unity, $(R^\times, \times)$ is a group. [Called the unit group of R.]

Proof. First check that $\times$ gives a binary operation on $R^\times$. Suppose that $u_1, u_2 \in R^\times$. Then there exists $w_1, w_2 \in R^\times$ such that $u_1w_1 = w_1u_1 = 1$ and $u_2w_2 = w_2u_2 = 1$. $(u_1u_2)(w_1w_2) = u_1(u_2w_2)w_1 = u_11w_1 = u_1w_1 = 1$. Also $(w_2w_1)(u_1u_2) = w_2(w_1u_1)u_2 = w_21u_2 = w_2u_2 = 1$. So w_2w_1 is an inverse for u_1u_2, and so $u_1u_2 \in R^\times$.

Now check the group axioms:
Associativity is given by the ring axioms.
Identity: 1 is a unit since $1 \times 1 = 1$. So we have an identity in $R^\times$
Inverses: Let $u \in R^\times$. Then u has an inverse w in R. So $uw = wu = 1$. Now u is an inverse for w, and so $w \in R^\times$. Hence u has an inverse in $R^\times$. ∎

Remarks:

(i) The proof of Theorem 136 did not assume that R is commutative.

(ii) Since all units lie in $R^\times$, and since any inverse to a unit is itself a unit, it follows that every unit in R has a *unique* inverse. (since this is true in the unit group.) So we can write u^{-1} for the inverse of the unit u.

Examples 138.

(i) If R is $\mathrm{Mat}_n(F)$, then the unit group is $GL_n(F)$.

(ii) If R is $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ where p is prime, then $R^\times = R^* = R\backslash\{0\}$.

Definition 139. A *field* is a commutative ring, F, such that $F^* = F\backslash\{0\}$ (Every non-zero element is a unit.)

Arithmetic of Rings

Definition 140. Let R be a ring. Let $a, b \in R$. Say that a *divides* b if there exists $c \in R$ such that $ac = b$. We write $a \mid b$ to mean that a divides b.

Examples 141.

(i) If $R = \mathbb{Z}$ then "a divides b" means what we expect it to.

(ii) If $R = \mathbb{Q}$, $\mathbb{R}, \mathbb{C}$ or any other field, then $a \mid b$ for any $a, b \in R$, except when $a = 0$ and $b \neq 0$.

(iii) In a polynomial ring $F[x]$, "divides" means what it ought to. For example $x - 1$ divides $x^2 - 2x + 1$ in $\mathbb{Q}[x]$.

Lecture 26 **Proposition 142.**

(i) If $a \mid b$ then $a \mid bc$ for all $c \in R$.

(ii) If $a \mid b$ and $a \mid c$ then $a \mid b + c$

(iii) If $a \mid b$ and $b \mid c$ then $a \mid c$.

(iv) If $u \in R$ is a unit, then $u \mid b$ for all $b \in R$.

(v) If $a \mid b$ and if u is a unit, then $au \mid b$.

(vi) If u is a unit and $w \mid u$ then w is also a unit.

(vii) $a \mid 0$ for any $a \in R$.

(viii) If $0 \mid b$ then $b = 0$.

(*Proof.* Left as exercise)

Definition 143. Let R be a ring, and $r \in R$. We say that r is a *zero-divisor* if $\exists s \in R$ such that $s \neq 0$, and $rs = 0$. A ring R with no zero-divisors, apart from 0, is called an *integral domain*.

Examples 144.

(i) $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ (p prime), $F[x]$ (where $F = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) are all integral domains.

(ii) If $m > 1$, and m is not prime, then $\mathbb{Z}_m$ is *not* an integral domain, since if we have $m = ab$, $1 < a < m$, then $[a]_m$ and $[b]_m$ are non-zero, but their product is $[m]_m = [0]_m$, and so $[a]_m$ has a zero-divisor.

Proposition 145. *A unit in a ring R is not a zero-divisor (unless R is trivial.)*

Proof. Let u be a unit. Suppose that $ua = 0$ for some $a \in R$. Then $u^{-1}(ua) = u^{-1}0 = 0$. So (since $u^{-1}(ua) = 1a = a$) we have $a = 0$. Hence u is not a zero-divisor ∎

Corollary 145. Any field is an integral domain.

Proof. All non-zero elements in a field are units, and so they are not zero-divisors. ∎

Proposition 146. *Let R be an integral domain. Let $a, b \in R$ be such that $a \mid b$ and $b \mid a$. Then there exists a unit $u \in R$ such that $b = au$.*

Proof. Since $a \mid b$, there exists $c \in R$ with $b = ac$. And since $b \mid a$, there exists $d \in R$ such that $a = bd$. $(ac)d = bd = a$, and by associativity it follows that $a(cd) = a$. Hence $a(1 - cd) = 0$. So one of a or $1 - cd = 0$. If $a = 0$, then $b = 0$ since $a \mid b$, so $a.1 = b$. If $1 - cd = 0$, then $cd = 1$, and so d is an inverse for c, so c is a unit, with $b = ac$. ∎

Definition 147. Let R be an integral domain. Let r be a non-zero, non-unit in R. We say that r is *irreducible* if it can't be written as $r = st$, where $s, t \in R$, and neither s nor t is a unit. Otherwise r is *reducible.*

Examples 148.

(i) In $\mathbb{Z}$ the irreducible elements are the primes (positive and negative). Note that if p is prime then p has two factorisations, $p = 1 \times p = (-1) \times (-p)$. Here 1 and -1 are the units in $\mathbb{Z}$.

(ii) In a field there are no irreducible (or reducible) elements, since every element is either 0 or a unit.

(iii) $f(x) = x - 3$ is irreducible in $\mathbb{Q}[x]$ or $\mathbb{R}[x]$ or $\mathbb{C}[x]$.

(iv) $f(x) = x^2 - 2$ is irreducible in $\mathbb{Q}[x]$. But in $\mathbb{R}[x]$ or $\mathbb{C}[x]$, $f(x)$ is reducible since $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, and neither of these factors are units.

(v) $f(x) = x^2 + 1$ is irreducible in $\mathbb{Q}[x]$ and $\mathbb{R}[x]$, but is reducible in $\mathbb{C}[x]$ as $x^2 + 1 = (x + i)(x - i)$.

Remark. In an integral domain, every element is exactly one of: zero, a unit, irreducible or reducible.

Proposition 149. *Let F be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and let $f(x) \in F[x]$. Let $a \in F$. Then $f(a) = 0$ if and only if $x - a \mid f(x)$.*

Proof. Lecture 27

Suppose that $x - a \mid f(x)$. Then $f(x) = (x - a)g(x)$, for some $g(x) \in F[x]$. Now $f(a) = (a - a)g(a) = 0$.

For the converse, suppose that $f(a) = 0$. Let $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$, where $\alpha_0, \ldots, \alpha_n \in F$. Then $f(x) - f(a) = (\alpha_0 - \alpha_0) + (\alpha_1 x - \alpha_1 a) + \cdots + (\alpha_n x^n - \alpha_n a^n) = \alpha_1(x - a) + \alpha_2(x^2 - a^2) + \cdots + \alpha_n(x^n - \alpha^n)$.

Note that $x^k = a^k = (x - a)(x^{k-1} + x^{k-2}a + \ldots a^{k-1})$. So $x - a$ divides $x^k - a^k$ for all $k \in \mathbb{N}$. Hence $x - a$ divides $f(x) - f(a)$. But $f(a) = 0$ by assumption, and so $x - a$ divides $f(x)$. ∎

Theorem 150: Fundamental Theorem of Algebra

Let $f(x) \in \mathbb{C}[x]$ be a polynomial of degree greater than 0. Then $f(x)$ has a root in $\mathbb{C}$. (So there exists $a \in \mathbb{C}$ such that $f(a) = 0$.)

Proof. Next year's Complex Analysis course.

Corollary 151. *The irreducible elements in $\mathbb{C}[x]$ are the linear polynomials, $\alpha_1 x + \alpha_0$, where $\alpha_1 \neq 0$.*

Proof. First show that $\alpha_1 x + \alpha_0$ is irreducible. Suppose that $\alpha_1 x + \alpha_0 = f(x)g(x)$. Then deg $f(x)$+ deg $g(x) = 1$. So one of $f(x)$ or $g(x)$ has degree 0, and so is a unit in $\mathbb{C}[x]$. Hence $\alpha_1 x + \alpha_0$ is irreducible in $\mathbb{C}[x]$.

Conversely, suppose that $r(x)$ has degree d greater than 1. Then $r(x)$ has a root a by the Fundamental Theorem. So $r(a) = 0$, and so $x - a$ divides $r(x)$ by Proposition 149. So $r(x) = (x - a)s(x)$ for some $s(x) \in \mathbb{C}[x]$. Now deg $s(x) =$ deg $r(x) - 1 = d - 1 > 0$, so neither $s(x)$ nor $r(x)$ are units in $\mathbb{C}[x]$. So $r(x)$ is reducible. ∎

Exercise: What are the irreducible elements in the ring $\mathbb{R}[x]$?

The Rings $\mathbb{Z}[\sqrt{d}]$

Definition 152. Let $d \in \mathbb{Z}$ be a non-square. The ring $\mathbb{Z}[\sqrt{d}]$ is the set $\{x + y\sqrt{d} : x, y \in \mathbb{Z}\} \subseteq \mathbb{C}$, with the usual complex $+$ and $\times$.

Check that these give binary operations on our set:
We have $(x_1 + y_1\sqrt{d}) + (x_2 + y_2\sqrt{d}) = (x_1 + x_2) + (y_1 + y_2)\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$,
and $(x_1 + y_1\sqrt{d})(x_2 + y_2\sqrt{d}) = (x_1x_2 + dy_1y_2) + (x_1y_2 + x_2y_1)\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$.

The ring axioms are left as an exercise.

Example: $\mathbb{Z}[\sqrt{-1}] = \mathbb{Z}[i]$ is the set $\{x + yi : x, y \in \mathbb{Z}\}$, the ring of *Gaussian integers.*

Proposition 153. *Let d be a non-square in $\mathbb{Z}$. If $x_1 + y_1\sqrt{d} = x_2 + y_2\sqrt{d}$ for $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, then $x_1 = x_2$ and $y_1 = y_2$.*

Proof. Since d is a non-square in $\mathbb{Z}$, we know that $\sqrt{d} \notin \mathbb{Q}$. Suppose that $x_1 + y_1\sqrt{d} = x_2 + y_2\sqrt{d}$. Then $x_1 - x_2 = (y_2 - y_1)\sqrt{d}$. If $y_2 \neq y_1$, then $\sqrt{d} = \frac{x_1 - x_2}{y_2 - y_1} \in \mathbb{Q}$, which is a contradiction. So $y_2 = y_1$. Now $x_1 - x_2 = 0$, and so $x_2 = x_1$, as well. ∎

Definition 154. Define a function $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}$, by $N(x+y\sqrt{d}) = x^2 - dy^2$. N is the *norm map* on $\mathbb{Z}[\sqrt{d}]$.

Examples 155.

(i) In $\mathbb{Z}[i]$, the norm map is $x + iy \mapsto x^2 + y^2 = |x+iy|^2$.

(ii) More generally, if $d < 0$, then the norm map is $x + y\sqrt{d} \mapsto x^2 - dy^2 = x^2 + |d|y^2 = |x+y\sqrt{d}|^2$ (since $x + y\sqrt{d} = x + y\sqrt{|d|}i$).

(iii) In $\mathbb{Z}[\sqrt{2}]$, the norm map is $x+y\sqrt{2} \mapsto x^2 - 2y^2$. So for example $N(1+\sqrt{2}) = -1$. When $d > 0$, the norm map can take negative values.

Proposition 156. *(i) If $r \in \mathbb{Z}[\sqrt{d}]$, and if $N(r) = 0$, then $r = 0 = 0 + 0\sqrt{d}$*

(ii) If $r, s \in \mathbb{Z}[\sqrt{d}]$, then $N(rs) = N(r)N(s)$.

Proof (1). Let $r = x + y\sqrt{d}$, and suppose that $N(r) = 0$. Then $x^2 - dy^2 = 0$, and so $x^2 = dy^2$. Now if $y \neq 0$, then $d = \frac{x^2}{y^2} = \left(\frac{x}{y}\right)^2$, and so $\sqrt{d} \in \mathbb{Q}$. But this is impossible. Hence $y = 0$. Now $N(r) = x^2$, so $x^2 = 0$ and so $x = 0$. Proof of (2) in Homework Sheet. ∎

Proposition 157. *An element r of $\mathbb{Z}[\sqrt{d}]$ is a unit if and only if $N(r) = \pm 1$.* *Lecture 28*

Proof. (only if) Suppose that r is a unit. Then r has an inverse r^{-1}. Now $rr^{-1} = 1$, and so $N(r)N(r^{-1}) = N(1)$. But $1 = 1+0\sqrt{d}$, so $N(1) = 1$. Hence $N(r)$ has the inverse $N(r^{-1})$ in $\mathbb{Z}$, so $N(r)$ is a unit in $\mathbb{Z}$. So $N(r) = \pm 1$.

(if) Let $r = x + y\sqrt{d}$, and suppose that $N(r) = \pm 1$. Let $s = x - y\sqrt{d}$. Now $rs = x^2 - dy^2 = N(r) = \pm 1$. If $N(r) = 1$ then s an inverse for r. If $N(r) = -1$ then $-s$ is an inverse for r. In either case, r is a unit in $\mathbb{Z}[\sqrt{d}]$. ∎

Remark. If $d = -1$, the $\mathbb{Z}[\sqrt{d}] = \mathbb{Z}[i]$ has 4 roots, $\pm 1, \pm i$. If $d < -1$, then $\mathbb{Z}[\sqrt{d}]$ has only two units, ± 1. If $d > 0$ then $\mathbb{Z}[\sqrt{d}]$ has infinitely many units.

Remark. Elements that are irreducible in $\mathbb{Z}$ [primes] need not be irreducible in $\mathbb{Z}[\sqrt{d}]$.

Examples 158.

(i) 3 is not irreducible in $\mathbb{Z}[\sqrt{3}]$ since $3 = \sqrt{3} \times \sqrt{3}$, and $\sqrt{3}$ is not a unit.

(ii) 2 is not irreducible in $\mathbb{Z}[i]$, since $2 = (1+i)(1-i)$.

(iii) *Question:* Is 11 irreducible in $\mathbb{Z}[\sqrt{3}]$?

Answer: We have $N(11) = 11^2 = 121$. Suppose that $11 = ab$ in $\mathbb{Z}[\sqrt{-3}]$, where a and b are non-units. $N(a), N(b) \neq \pm 1$. We have $N(a)N(b) = N(11) = 121$. So we must have $N(a) = N(b) = \pm 11$. Since N takes non-negative values on $\mathbb{Z}[\sqrt{-3}]$, we have $N(a) = 11$. Let $a = x + y\sqrt{d}$. Then

> $x^2 + 3y^2 = 11$. But it is easy to check that no such x, y exist in $\mathbb{Z}$, which is a contradiction. So no such a, b exist. So 11 is irreducible in $\mathbb{Z}[\sqrt{-3}]$.

Highest Common Factor / Greatest Common Divisor

> **Definition 159.** Let R be a ring. Let $a, b, c \in R$. Then c is a *highest common factor* (hcf) or *greatest common divisor* (gcd) for a and b if
>
> (i) $c \mid a$ and $c \mid b$ (c is a common factor for a and b.)
>
> (ii) If $d \mid a$ and $d \mid b$ then $d \mid c$ for $d \in R$.

Remarks. We do not claim that a highest common factor necessarily exists for all $a, b \in R$. There exists rings R and elements a, b where this doesn't happen. Where hcfs do exist, they are not usually unique. For instance if $R = \mathbb{Z}$, and $a = 4,\ b = 6$, then the hcfs of a and b are $2, -2$.

Proposition 160. *Let R be an integral domain. Let c be an hcf for a and b in R. Then d is a highest common factor for a and b if and only if $d = cu$, for some unit $u \in R$.*

Proof. (only if) Suppose that c and d are both hcfs for a and b. So by the 2nd condition in the definition of hcf, we have $c \mid d$ and $d \mid c$. Now by Proposition 146, we have $d = cu$ for a unit u.

(if) Let $d = cu$, where u is a unit. Since $c \mid a$ and $c \mid b$, we have $cu \mid a$ and $cu \mid b$, by Proposition 141.5. So $d \mid a$ and $d \mid b$, so d is a common factor for a and b. To show the 2nd hcf condition holds, let e be any common factor of a and b. So $e \mid a$ and $e \mid b$. Then $e \mid c$, since c is a hcf. So $e \mid cu$ by Proposition 141.3. Hence $e \mid d$. Hence d is an hcf for a and b. ∎

Lecture 29 *Reminder.* Euclids Lemma: If we have any two $a, b \in \mathbb{Z}$, with $b \neq 0$, then there exists q and $r \in \mathbb{Z}$, with $0 \leq r < |b|$, such that $a = qb + r$.

Lemma 161. *Let $f(x)$ and $g(x)$ be polynomials with coefficients from F, $(\mathbb{Q}, , \mathbb{C})$, with $g(x) \neq 0$. There exist polynomials $q(x)$ and $r(x) \in F[x]$ such that $f(x) = q(x)g(x)+r(x)$, and either $r(x) = 0$ or else deg $r(x) <$ deg $g(x)$.*

Example. Take $F = \mathbb{Q}$, $f(x) = x^4 + x^3 + 2x + 3$ and $g(x) = x^2 - 1$. Dividing $f(x)$ by $g(x)$, we get $f(x) = g(x)(x^2 + x + 1) + (3x + 4)$. So here $q(x) = x^2 + x + 1$, and $r(x) = 3x + 4$. Here $\deg r(x) = 1 < 2 = \deg g(x)$.

Proof of Lemma 161. Take $\deg f(x) = m$ and $\deg g(x) = n$. If $m < n$, take $q(x) = 0$ and $r(x) = f(x)$, and this satisfies the lemma. So we will assume that $m \geq n$. Now we argue by induction on $m = \deg f$. The base case is $m = 0$ (and so $n = 0$ too). $f(x) = a$ and $g(x) = b$ for $a, b \in F$, with $b \neq 0$. Take $q(x) = \frac{a}{b}$, $r(x) = 0$, and this satisfies the lemma.
Inductive step: Assume the lemma holds for all polynomials $f(x)$ of degree $< m$. Suppose $\deg f(x) = m$. Put

$$f(x) = a_m x^m + \cdots + a_1 x + a_0 \text{ and } g(x) = b_n x^n + \cdots + b_1 x + b_0$$

Define $f'(x) = f(x) - \frac{a_m}{b_n}x^{m-n}g(x)$. The coefficient of x^m in $f'(x)$ is 0, so deg $f'(x) < m$. So the inductive assumption applies to $f'(x)$, and so there exists $q'(x)$ and $r(x)$ such that $f'(x) = q'(x)g(x) + r(x)$, and with $r(x) = 0$ or deg $r(x) <$ deg $g(x)$.

But now $f(x) = f'(x) + \frac{a_m}{b_n}x^{m-n}g(x) = (q'(x) + \frac{a_m}{b_n}x^{m-n})g(x) + r(x)$. So take $q(x) = q'(x) + \frac{a_m}{b_n}x^{m-n}$, and this satisfies the lemma. ∎

Example: $f(x) = 3x^2 + 5x + 1$, $g(x) = 2x - 1$. Calculate $q(x)$ and $r(x)$ using polynomial long division.

Polynomial division working...

So $q(x) = \frac{3}{2}x + \frac{13}{14}, \quad r(x) = \frac{17}{4}$.

Proposition 162. *Let R be a ring. Let $a, b, q, r \in R$ such that $a = bq + r$. Then d is a highest common factor for a and b if and only if d is a highest common factor for b and r.*

Proof. We actually show that the common factors of a and b are the same as those of b and r. Suppose d divides b and r. Then d divides bq and r. So d divides $bq + r = a$. So d divides a and b.
Conversely, suppose d divides a and b. Note that $r = a - bq$. Now d divides a and bq, so d divides $a - bq = r$. ∎

Propositions 161 and 162 give us a Euclidean Algorithm for polynomials.

Example 163 (Euclidean Algorithm for polynomials.)**.**
$f(x) = x^3 - 2x^2 - 5x + 6$, $g(x) = x^2 - 2x - 3$.
First find $q(x)$ and $r(x)$:
Just Division Again...

We have $q(x) = x$, $r(x) = -2x + 6$. Now we look for a hcf of $g(x)$ and $r(x)$. (A "smaller" problem than the original.) So applying Euclid's Algorithm again, so now quotient is $-\frac{1}{2}x - \frac{1}{2}$, now remainder is 0. We've shown that $r(x) = -2x + 6$ divides $g(x)$. So $r(x)$ is a hcf for $r(x)$ and $g(x)$. So $r(x)$ is also a hcf for $f(x)$ and $g(x)$ by Proposition 162.

Note that -2 is a unit in $\mathbb{Q}[x]$, with inverse $-\frac{1}{2}$. So $x - 3$ is another highest common factor for $f(x)$ and $g(x)$.

Definition 164. Let R be an integral domain. A *Euclidean function* on R is a function $f : R\backslash\{0\} \to \mathbb{N} \cup \{0\}$, which satisfies the following two conditions: Lecture 30

(i) $f(ab) \geq f(a)$ for $a, b \in R\backslash\{0\}$

(ii) For all $a, b \in R$, $b \neq 0$, there exists $q, r \in R$ such that $a = qb + r$, either $r = 0$, or else $f(r) < f(b)$.

Examples 165.

(i) If $R = \mathbb{Z}$ then $f(n) = |n|$ is a Euclidean function.

(ii) If $R = F[x]$ then $\deg f(x)$ is a Euclidean function.

Euclidean functions are what we need for Euclidean algorithms:

Algorithm 166. *To find a hcf for a and b in a ring R with a Euclidean function f.*

Step 1. *Find q and r such that $a = qb + r$ and r is 0 or $f(r) < f(b)$*

Step 2. *If $r = 0$ then b is a hcf for a and b. Otherwise:*

Step 3. *Start the algorithm again, replacing a with b and b with r.*

Since $f(b) \in \mathbb{N} \cup \{0\}$, this algorithm must eventually terminate.

Definition 167. An integral domain with at least one Euclidean function is called a *Euclidean Domain.*

It can be very difficult to decide whether an Integral Domain is Euclidean.

Can we find a Euclidean function on $\mathbb{Z}[\sqrt{d}]$? Sometimes!

Proposition 168. *Let $d \in \{-2, -1, 2, 3\}$. Then the function $f(a) = |N(a)|$ is a Euclidean function on $\mathbb{Z}[\sqrt{d}]$.*

Proof. The first condition from Definition 164 is easy. Since $|N(b)| \geq 1$ for $b \neq 0$, we have $|N(ab)| = |N(a)||N(b)| \geq |N(a)|$.

For the second condition, let $a = x + y\sqrt{d}$ and $b = v + w\sqrt{d}$, with $b \neq 0$. In $\mathbb{C}$, calculate:

$$\frac{a}{b} = \frac{x + y\sqrt{d}}{v + w\sqrt{d}} = \frac{(x + y\sqrt{d})(v - w\sqrt{d})}{v^2 - dw^2} = \frac{1}{N(b)}((xv - ywd) + (yv - xw)\sqrt{d}).$$

$$\text{Put} \quad \alpha = \frac{xv - ywd}{N(b)}, \ \beta = \frac{yv - xw}{N(b)}, \in \mathbb{Q}$$

Set m, n to be the integers closest to α, β respectively. So

$$|\alpha - m| \leq \frac{1}{2} \quad \text{and} \quad |\beta - n| \leq \frac{1}{2} \quad (*)$$

$$\text{Put} \quad q = m + n\sqrt{d} \in \mathbb{Z}[\sqrt{d}], \text{ and } r = a - bq$$

We show that $|N(r)| < |N(b)|$:

Define $c = N(b)(\frac{a}{b} - q) = N(b)\frac{r}{b} \in \mathbb{C}$. So $bc = N(b)r$. We have:

$$\begin{aligned} c &= N(b)(\alpha + \beta\sqrt{d} - m - n\sqrt{d}) \\ &= N(b)(\alpha - m) + N(b)(\beta - n)\sqrt{d} \\ &\in \mathbb{Z}[\sqrt{d}], \text{ since } N(b)\alpha \text{ and } N(b)\beta \in \mathbb{Z} \end{aligned}$$

We have $N(bc) = N(b)N(c) = N(N(b)r) = N(b)^2N(r)$.

So $N(c) = N(b)N(r)$. But $N(c) = N(b)^2(\alpha - m)^2 - N(b)^2(\beta - n)^2 d$. $N(r) = N(b)(\alpha - m)^2 - d(\beta - n)^2)$.

Now from $(*)$, $|\alpha - m|$ and $|\beta - n| \leq \frac{1}{2}$. So if $-2 \leq d \leq 3$, it is easy to see that $|(\alpha - m)^2 - d(\beta - n)^2| < 1$. So $|N(r)| < |N(b)|$ as required. ■

Example 169. Find a hcf of $a = 4 + \sqrt{2}$ and $b = 2 - 2\sqrt{2}$ in $\mathbb{Z}[\sqrt{2}]$. Lecture 31

In $\mathbb{C}$ we calculate:

$$\frac{a}{b} = \frac{4+\sqrt{2}}{2-2\sqrt{2}} = \frac{(4+\sqrt{2})(2+2\sqrt{2})}{(2-2\sqrt{2})(2+2\sqrt{2})} = -3 - \frac{5}{2}\sqrt{2}$$

So set $q = -3 - 2\sqrt{2}$. ($q = -3 - 3\sqrt{2}$ would also work.)

Now $r = a - bq = 2 - \sqrt{2}$. (Notice $|N(r)| = 4 - 2 = 2$, $|N(b)| = |4 - 8| = 4$). Continue, replacing a with b and b with r. In $\mathbb{C}$:

$$\frac{b}{r} = \frac{2-2\sqrt{2}}{2-\sqrt{2}} = \sqrt{2} \in \mathbb{Z}[\sqrt{2}]$$

. So $q' = -\sqrt{2}$, $r' = 0$. So $r = 2 - \sqrt{2}$ divices $b = 2 - 2\sqrt{2}$, and so r is a hcf for b and r, hence r is a hcf for a and b.

Lemma 170. [Bézout's Lemma] Let R be a Euclidean domain. Let a and b be elements of R and let d be a hcf for a and b. Then there exists $s, t \in R$ such that $as + bt = d$.

Proof. Define $X = X_{a,b} = \{ax + by : x, y \in R\}$. So $X \subseteq R$. The ring R has a Euclidean function f. So if a and b are not both 0, then X has non-zero elements. So there exists some least $n \in \mathbb{N} \cup \backslash \{0\}$ such that $f(x) = n$ for some $x \in X \backslash \{0\}$. Let c be some elements of $X \backslash \{0\}$ such that $f(c) = n$. We show that c is a hcf for a and b:

Since $c = ax + by$ for some x, y, it is clear that any common factor of x and y must divide c.

We know that there exists q and $r \in R$ such that $a = qc + r$, and either $r = 0$ or $f(r) < f(c)$. Now $c = ax + by$, so $r = a - qc = a - q(ax + by) = a(1 - qx) - b(qy) \in X$. We can't have $f(r) < f(c)$, since $f(c)$ is the smallest possible f for elements of X. Hence $r = 0$, and so $c \mid a$. A similar argument shows that $c \mid b$, and so c is a common factor - and hence a hcf - for a and b.

Now let d be any hcf of a and b. Then $d = cu$ for some unit $u \in R$. Now $d = (ax+by)u = axu + byu$. So put $r = xu$ and $s = yu$, and we're done. ■

Unique Factorisation

Definition 170. Let R be an integral domain, and let $a, b \in R$. We say a and b are *coprime* if 1 is a hcf for a and b. (Equivalently, any unit is a hcf.)

Proposition 171. *Let R be a Euclidean Domain. Suppose a and b are coprime in R, and suppose a divides bc. Then a divides c.*

Proof. Since $a \mid bc$, we can write $ad = bc$, for some $d \in R$. Since R is Euclidean, and a and b are coprime, we use Bézout's Lemma: there exist $s, t \in R$ such that $as + bt = 1$. Now $c = 1c = (as + bt)c = asc + btc = asc + (bc)t = asc + (ad)t = a(sc + dt)$, which is divisible by a. ∎

Proposition 172 can fail if R is not Euclidean.

Example: $R = \mathbb{Z}[\sqrt{-3}]$. $a = 2,\ b = 1 + \sqrt{-3},\ c = 1 - \sqrt{-3}$. Then a and b are coprime, and 2 divides $bc = 4$. But a does not divide c. (This shows $\mathbb{Z}[\sqrt{-3}]$ is not Euclidean.)

Definition 172. A *unique factorisation domain* is an integral domain R with the following properties:

(i) For every $a \in R$, not zero and not a unit, there exists irreducible elements $p_1, \dots, p_s$ in R such that $a = p_1 \dots p_s$.

(ii) Let $p_1, \dots, p_s$ and $q_1, \dots, q_t$ be irreducible in R, such that $p_1 \dots p_s = q_1 \dots q_t$. Then $t = s$, and reordering the q_i if necessary, we have $p_i = q_i u_i$ for some unit u_i, for all $i \in \{1, \dots, s\}$. (Factorisation is unique "up to units".)

Example: $R = \mathbb{Z}$. Then every element except $0, 1, -1$ can be written as a product of irreducibles (primes), unique up to sign. $30 = 2 \times 3 \times 5 = (-2) \times 3 \times (-5)$.

Lecture 32 Not every integral domain is a UFD. We've seen that $\mathbb{Z}[\sqrt{-z}]$ is not a UDF.

Example 174. In $\mathbb{Z}[\sqrt{-5}]$ we can factorise 6 as 2×3, and also as $(1 + \sqrt{-5})(1 - \sqrt{-5})$. Are these factors irreducible?

We have $N(2) = 4$, $N(3) = 9$. $N(1 \pm \sqrt{-5}) = 6$. If $ab = 2$ or 3 or $(1 \pm \sqrt{-5})$, and if neither a nor b is a unit, then $N(a)$ must be ± 2 or ± 3. But it is easy to check that the equations $x^2 + 5y^2 = \pm 2$ or ± 3 has no solutions for $x, y \in \mathbb{Z}$. So all of $2, 3, 1 \pm \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

Are the factorisations of 6 the same "up to units"? No, since $N(2) \neq N(1 \pm \sqrt{-5})$. So 6 is not uniquely factorisable in $\mathbb{Z}[\sqrt{-5}]$

Theorem 173

Every Euclidean Domain is a Unique Factorisation Domain.

Proof. Omitted. (In Algebra II (?)) 

It follows that $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean Domain, by Example 174.

Proposition 174. *Let R be a UFD. Let p be an irreducible element of R, and let $a, b \in R$ be such that $p \mid ab$. Then either $p \mid a$ or $p \mid b$.*

Proof. If a or b is 0 or a unit, then the result is clear. So assume otherwise. So by the first UFD property, there exist irreducible elements $q_1, \dots, q_s$, $r_1, \dots, r_t$, such that $a = q_1 \dots q_s$, $b = r_1 \dots r_t$. So $ab = q_1 \dots q_s r_1 \dots r_t$.

Now $p \mid ab$, so $ab = pc$ for some $c \in R$. Suppose that $c = p_1 \dots p_u$ as a product of irreducibles. Then $ab = pp_1 \dots p_u$. Now by the second (uniqueness) property of UFDs, we have $p = q_i w$ or $r_i w$ for some i, and some unit, w. If $p = q_i w$ then $p \mid a$, and if $p = r_1 w$ then $p \mid b$, as required. ∎

Definition 175. A non-unit element, r, of a ring R is *prime* if it has the property that whenever $r \mid ab$ we have either $r \mid a$ or $r \mid b$.

In any integral domain, any prime element is irreducible. Why?

If r is prime, and $r = ab$, then $r \mid ab$, and then $r \mid a$ or $r \mid b$ by the prime property. Suppose $r \mid a$. Then since $a \mid r$, we have $r = au$ for a unit u. So $ab = au$, and so $a(b-u) = 0$. But R is an integral domain, so $b - u = 0$. Hence $b = u$, a unit.

In general the converse is not true - we've seen that 2 is irreducible but not prime in $\mathbb{Z}[\sqrt{-5}]$. (Example 174).

Felina. At the start of the Ring Theory section, we mentioned two equations:

(i) $x^2 - 2 = -1$ we could have solved (in $\mathbb{Z}$) then.

(ii) The other was $y^3 = x^2 + 2$. *Can we find all solutions $x, y \in \mathbb{Z}$?*

First notice that if $x^2+2 = y^3$ then both x and y are odd. (It is clear that if one of them is even, then so is the other. Suppose both are even. Then mod 4, we have $x^2 + 2 \equiv y^3$, and $x^2, y^3 \equiv 0$, so $2 \equiv 0 \mod 4$, a contradiction.)

Move into $\mathbb{Z}[\sqrt{-2}]$. We have $x^2 + 2 = (x + \sqrt{-2})(x - \sqrt{-2})$. Put $a = (x + \sqrt{-2})$ and $b = (x - \sqrt{-2})$. So $y^3 = ab$. Hence $N(y^3) = N(a)N(b)$.

Let d the hcf(a, b) in $\mathbb{Z}[\sqrt{-2}]$. Then d divides $a - b = 2\sqrt{-2}$. So $N(d)$ divides $N(2\sqrt{-2})$, so $N(d)$ divides 8. But $N(d)$ divides $N(a)$, so divides $N(y)^3 = y^6$, which is odd. So $N(d) = 1$.

So hcf(a, b) is a unit, and so a and b are coprime. So we have $y^3 = ab$, where a, b are co-prime. Since $\mathbb{Z}[\sqrt{-2}]$ is a UFD, we can factorise $y^3 = r_1^3 \dots r_t^3$, where $r_1, \dots, r_t$ are irreducible. Now for all i, we must have $r_i^3 \mid a$ or $r_1^3 \mid b$. So it's easy to see that $a = c^3$ and $b = d^3$ for $b, d \in \mathbb{Z}[\sqrt{-2}]$.

Let $c = m + n\sqrt{-2}$. So $(m + n\sqrt{-2})^3 = x + \sqrt{-2}$. So

$$(m + n\sqrt{-2})^3 = (m^3 - 6mn^2) + (3m^2n - 2n^3)\sqrt{-2} = x + \sqrt{-2}$$

So $m^3 - 6mn^2 = x$ and $3m^2n - 2n^3 = 1$. We have $(3m^2 - 2n^3)n = 1$, so $n \mid 1$, hence $n = \pm 1$, and $3m^2 - 2n^2 = n$, so $3m^2 - 2 = \pm 1$. So $n = \pm 1, m = \pm 1$. Now we have $x = \pm 5$. So $x^2 + 2 = 27, y = 3$. So the only solutions are $x = \pm 5, y = 3$.

- End of Algebra I -