1st Year Mathematics
Imperial College London

Autumn 2014

Geometry and Linear Algebra

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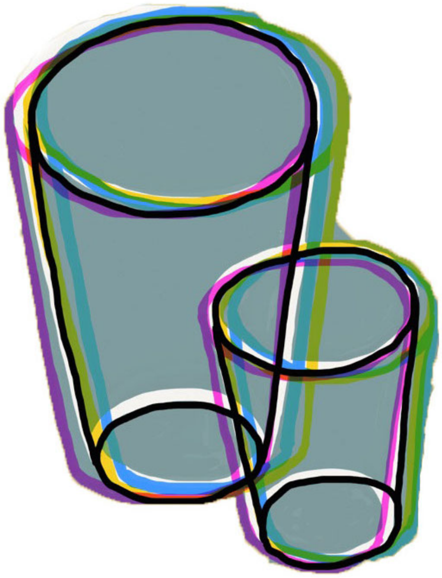

Syllabus

An introductory course involving basic material, which will be widely used later.

- Number systems. Integers, rational numbers, real numbers, decimal expansions for rationals and reals.
- Inequalities, complex numbers.
- Induction; examples and applications.
- Sets, functions, countability, logic.
- Permutations and combinations. The Binomial Theorem.
- Equivalence relations and arithmetic modulo n.
- Euclids algorithm.
- Introduction to limits.

Appropriate books

M. Liebeck *A Concise Introduction to Pure Mathematics.*
K. Houston *How to Think Like a Mathematician.*
E. Hurst and M. Gould *Bridging the Gap to University Mathematics.*

Contents

1 Vectors in $\mathbb{R}^2$

$\mathbb{R}$ = the set of real numbers. This has the properties of being: *Lecture 1 14/10*

- Commutative: $ab = ba$
- Associative: $a + (b + c) = (a + b) + c$
- Distributive: $c(a + b) = ca + cb$
- If $a < b,\ b > 0$ then $ca < cb$

$\mathbb{R}^2$ = the set of ordered pairs (a, b) where $a, b \in \mathbb{R}$ i.e. $(a, b) \neq (b, a)$ in general. Elements of $\mathbb{R}^2$ are *points*, or *vectors*. $\mathbf{0} = (0, 0)$ = origin of $\mathbb{R}^2$. A *scalar* is an element of $\mathbb{R}$.

Properties:

(i) Addition (sum): $(a, b) + (a', b') = (a + a', b + b')$

(ii) Scalar Multiplication: If $\lambda = \mathbb{R}$ $(a, b) \in \mathbb{R}^2$, then $\lambda \cdot (a, b) = (\lambda a, \lambda b)$. $v \in \mathbb{R}^2 \implies \lambda v \in \mathbb{R}^2$.

Exercise: Check $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$ where $\lambda \in \mathbb{R},\ v_1, v_2 \in \mathbb{R}^2$

Lines in $\mathbb{R}^2$. L is a *line in* $\mathbb{R}^2$ if $\exists u, v \in \mathbb{R}^2$ $v \neq 0$ with $L = \{u + \lambda v \mid \lambda \in \mathbb{R}\}$. If $u = (a_1, b_1),\ v = (a_2, b_2)$ then $L = \{(a_1 + \lambda b_1, a_2 + \lambda b_2) \mid \lambda \in \mathbb{R}\}$.

Examples: x line $\{(0, 0) + \lambda(1, 0) \mid \lambda \in \mathbb{R}\}$, y line $\{(0, 0) + \lambda(0, 1) \mid \lambda \in \mathbb{R}\}$. $L = \{x + y = 1\} = \{(1, 0) + \lambda(-1, 1) \mid \lambda \in \mathbb{R}\}$. Note that v is the vector for which L is parallel to. We check from the components $x = 1 + (-1)\lambda = 1 - \lambda$. $y = 0 + 1 \cdot \lambda = \lambda$. So $x + y = 1$.

Assume L, M are two lines in $\mathbb{R}^2$, with $L = \{u = \lambda v \mid \lambda \in \mathbb{R}\}$ and $v \neq 0$, $M = \{a + \mu b \mid \mu \in \mathbb{R}\}$ and $b \neq 0$.

Proposition. The two lines are the same ($L = M$) if and only if the following holds:

(i) $v = \alpha b$ for some $\alpha \in \mathbb{R}$

(ii) $L \cap M \neq \emptyset$ (L and M have a point in common)

Proof. ($\implies$) Assume that $L = M$. We know that $u \in L \implies u \in M \implies u = a + \mu b$ for some $\mu \in \mathbb{R}$. Also $u + v = L$ ($\lambda = v$) $\implies u + v \in M \implies u + v = a + \mu_1 b$ for some $\mu_1 \in \mathbb{R} \implies v = (a + \mu_1 b) - (a - \mu b) = (\mu_1 - \mu)b = \alpha b$. Since $L = m$, surely $L \cap M = \emptyset$.

($\impliedby$) Assume $v = \alpha b \implies L = \{u + \lambda \alpha b \mid \lambda \in \mathbb{R}\}$. We also know that $L \cap M \neq \emptyset \implies \exists c \in L \cap M \implies c \in L \implies c = u + \lambda_0 \alpha b$, for some $\lambda_0 \in \mathbb{R}$. Then also $c \in M \implies c = a + \mu b$, for some $\mu \in \mathbb{R} \implies u + \lambda_0 \alpha b = a + \mu b \implies u = a + \mu b - \lambda_0 \alpha b = a + (\mu - \lambda_0 \alpha) b$ $(*)$.

By $(*)$, a point inside $L = u + \lambda \alpha b = a + (\mu - \lambda_0 \alpha) b + \lambda \alpha b = a + (\mu - \lambda_0 \alpha + \lambda \alpha) b \in M$. Hence any point in L is inside M, so $L \subseteq M$. Similarly by symmetry $M \subseteq L \implies L = M$. $\square$

2 Matrices

Inverses

Theorem 5.1. Let A be a square matrix. If there exists a square matrix B such that $AB = I$ then this B is unique and satisfies $BA = I$.

Proof. (Also gives a method for finding this B!)
Let X be the square matrix with unknown entires. We want to solve the equation $AX = I$. The entires of X are x_{ij}. We have n^2 unknowns and n^2 equations. We record this as follows: $(A \mid I)$, a $n \times 2n$ matrix.

This is n systems of linear equations in n variables, e.g. for each column of I we have the following system:

$$A \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = \text{ the } j\text{th column of } I = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

All these n systems have the same co-efficient matrix A, so we solve them using the same process. Apply reduction to echelon form. Perform elementary row operations on the matrix $(A \mid I)$

$$(A_{ech} \mid I) = \left(\begin{array}{cccc|c} 1 & & * & & \\ & 1 & & & \\ & & 1 & & D \\ 0 & & & 1 & \end{array}\right)$$

Claim: The matrix on the LHS cannot have any rows made entirely of zeros.

Proof of Claim. Remember that D is obtained by row operations from I. We know that two matrices that are obtained from each other by row operations define equivalent linear systems. This means that the linear system $I(y_1, \ldots, y_n) = 0$ has the same solutions as $D(y_1, \ldots, y_n) = 0$. But $(0, \ldots, 0)$ is the only solution to this. Now if D has an all zero row, the system $D(y_1, \ldots, y_n) = 0$ has free variables, hence infinitely many solutions. This contradiction proves that D does not have an all zero row. □

Therefore there is a non-zero entry in the bottom row of D. Say this entry is in the jth column. Then the system (1.1) has no solutions (follows from the echelon form method). Therefore, if the matrix on the left has a bottom row made of zeros, then $AX = I$ has no solutions. So A has on right inverse. It remains to consider the case when the matrix on the left has no all-zero rows:

$$(A_{ech} \mid I) = \left(\begin{array}{cccc|c} 1 & & & * & \\ & 1 & & & \\ & & \ddots & & D \\ 0 & & & 1 & \end{array}\right)$$

Perform more elementary row operations to clear the entries above the main diagonal (This is possible because all diagonal entries equal 1). After this step, we obtain:

$$(I \mid E) = \left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & E \\ 0 & & & 1 & \end{array}\right)$$

Since row operations don't change the solution of our linear system, we have $IX = E$. Hence E is a unique solution of the system $AX = I$, i.e. $AE = I$. We've proven that if the right inverse exists it can be obtained by the procedure, and it is unique.

Finally we now prove that $EA = I$:

Consider the equation $EY = I$, where $Y = (y_{ij})$ is a square matrix with unknown entires y_{ij}. Reverse row operations from the first part of the proof to so $(E \mid I) \mapsto (I \mid A)$. $EY = I$ is equivalent to $IY = A$, that is $Y = A$. Therefore $EA = I$. $\square$

Finding inverses of 2×2 matrices is easy:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ consider } B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I$$

Case 1. $(ad - bc)$ is non-zero. Then

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ is the inverse of } A$$

Case 2. $ad - bc = 0$. In this case $AB = 0$. Then the inverse does not exist (If $CA = I$, then $C(AB) = (CA)B = IB = B$. If A is non-zero then B is non-zero, so we get a contradiction for $0 = B$.) Hence A is not invertible.

Determinants

Corollary 6.4. If A' is obtained from A by row operations, then $\det(A') \neq 0$ if and only if $\det(A) \neq 0$.

Proof. A direct consequence of Proposition 6.3. □

Theorem 6.5. Let A be a 3×3 matrix. Then A^{-1} exists if and only if $\det(A) \neq 0$.

Proof. Recall that A can be reduced to echelon form by row operations. Let A' be the matrix in echelon form to which A reduces. Then $\det(A') \neq 0 \iff \det(A) \neq 0$. Hence we are in Case 2 (If A' has an all zero-row then we expand in this row and $\det(A') = 0$.) In Case 2, A' can be reduced to I by further row operations. By Theorem 5.1, A is invertible.

We need to prove that if A^{-1} exists, then $\det(A) \neq 0$. Indeed, if A^{-1} exists, then the echelon form of A has no all-zero rows. Then A can be reduced to I by row operations. Row operations can only multiply det by a non-zero number, and they can be reversed. Therefore, det $(A) \neq 0$. □ *Lecture 13 11/11*

Remark. For any square matrices A and B of the same size $\det(AB) = \det(A)\det(B)$. If A^{-1} exists, then $AA^{-1} = I$, so $\det(A)\det(A^{-1}) = 1$. Hence $\det(A) \neq 0$ if A^{-1} exists.

Final Comment. If A is a square matrix, then $Ax = 0$ has non-zero solutions if and only if $\det(A) = 0$. (Indeed if $Ax = 0$ has a non-zero solution, then it has at least two distinct solutions, so it has infinitely many solutions. Then A^{-1} does't exist, and $\det(A) = 0$).

Eigenvalues and Eigenvectors

Definition. Let A be a $n \times n$ matrix. Then a non-zero vector, v, is called an *eigenvector* of A if $Av = \lambda v$ for some $\lambda \in \mathbb{R}$. In this case λ is called an *eigenvalue* of A corresponding to the eigenvector v.

Remarks. A scalar multiple of an eigenvector is also an eigenvector with the same eigenvalue.

Example.

Definition. The determinant of $tI - A$ is called the *characteristic polynomial* of A (For us $n = 3$, or $n = 2$). For example

Proposition 7.1. Let A be a 2×2 or 3×3 matrix. Then the eigenvalues of A are the roots of the characteristic polynomial of A, i.e. every eigenvalue λ satisfies $\det(\lambda I - A) = 0$. The eigenvectors of A with eigenvalue λ are non-zero solutions of the system of linear equations $(\lambda I - A)v = 0$.

Proof. The real numbers λ for which $\det(\lambda I - A) = 0$ are by definition the roots of the characteristic polynomial of A. Hence v is a non-zero solution of $(\lambda I - A)v = 0$. $\square$