1st Year Mathematics
Imperial College London

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Analysis I

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About these notes

These notes are not affiliated with Richard Thomas (or even *Professor* Richard Thomas!) The LaTeX source is available on `github` - it would be great if someone would update changes to the courses so they'll still be useful to later years. Notes for other courses are available on `dropbox`:

1st / 2nd Year	**3rd / 4th Year**
Foundations of Analysis	Galois Theory
Analysis I	Algebraic Number Theory
Analysis II	Analytical Number Theory
Complex Analysis	Elliptic Curves
Geometry & Linear Algebra	Modular Forms
Algebra I	Algebra III
Algebra II	Commutative Algebra
Methods I	Lie Algebras
Methods II	Measure & Integration
Multivariable Calculus	Functional Analysis
Differential Equations	Algebraic Topology
Metric Spaces & Topology	Differential Topology
Intro to Numerical Analysis	Complexity
Mechanics	General Relativity

- Karim Bacchus

Syllabus

A rigorous treatment of the concept of a limit, as applied to sequences, series and functions.

- Real and complex sequences. Convergence, divergence and divergence to infinity. Sums and products of convergent sequences. The Sandwich Test. Sub-sequences, monotonic sequences, Bolzano-Weierstrass Theorem. Cauchy sequences and the general principle of convergence.

- Real and complex series. Convergent and absolutely convergent series. The Comparison Test for non-negative series and for absolutely convergent series. The Alternating Series Test. Rearranging absolutely convergent series. Radius of convergence of power series. The exponential series.

- Limits and continuity of real and complex functions. Left and right limits and continuity. Maxima and minima of real valued continuous functions on a closed interval. Inverse Function Theorem for strictly monotonic real functions on an interval.

- An introduction to differentiability: definitions, examples, left and right derivative.

Appropriate books

K. G. Binmore, *Mathematical Analysis, A Straightforward Approach* (Cambridge University Press).

Contents

0 Preliminaries

Lecture 1

M1F stuff:

- $\forall -$ for any, **fix any**, for all, every...
- $\exists -$ there exists
- $\mathbb{N} = \{1, 2, 3, \dots\}$

Theorem 0.1: Triangle Inequality

(See Question Sheet 1)

$$|a + b| \leq |a| + |b|$$

Corollary 0.2.

$$\big||a| - |b|\big| \leq |a - b|$$

PROOF.

$$\begin{aligned} |a - b| < \epsilon &\iff b - \epsilon < a < b + \epsilon \\ &\iff a \in (b - \epsilon, b + \epsilon) \\ &\iff b \in (a - \epsilon, a + \epsilon) \\ &\implies \big||a| - |b|\big| < \epsilon \end{aligned}$$

■

Lots of other versions, see Question Sheet 1 - *don't try to memorise them!*

Clicker Question 0.3. Fix $a \in \mathbb{R}$. What does the statement

$$\forall \epsilon > 0,\ |x - a| < \epsilon\ (*)$$

mean for the number x?

Answer: $x = a$.

PROOF. Assume $x \neq a$. Take $\epsilon := \frac{1}{2}|x - a| > 0$. Then $(*)$ does not hold. ■

1 Sequences

Lecture 2

A sequence $(a_n)_{n\geq 1}$ of real (or complex, etc.) numbers is an infinite list of numbers $a_1,\ a_2,\ a_3, \dots$ all in $\mathbb{R}$ (or $\mathbb{C}$, etc.) Formally:

> **Definition.** A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$

Notation: We let $a_n \in \mathbb{R}$ denote $a(n)$ for $n \in \mathbb{N}$. The data $(a_n)_{n=1,2,\dots}$ is equivalent to the function $a : \mathbb{N} \to \mathbb{R}$ because a function a is determined by its values a_n over all $n \in \mathbb{N}$.

We will denote a by $a_1,\ a_2, \dots$ or $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n\geq 1}$ or even just (a_n).

Remark 1.1. a_i's could be repeated, so (a_n) is *not* equivalent to the set $\{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$. E.g. $(a_n) = 1,\ 0,\ 1,\ 0, \dots$ is different from $(b_n) = 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1, \dots$

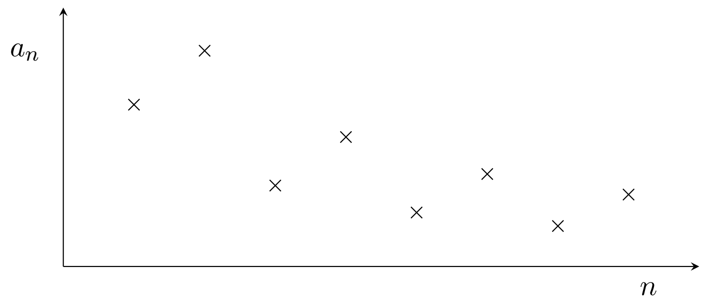
We can describe a sequence in may ways, e.g. formula for a_n as above $a_n = \frac{1-(-1)^n}{2}$, or a recursion e.g. $c_1 = 1 = c_2$, $c_n = c_{n-1} + c_{n-2}$ for $n \geq 3$, or a summation (see next section) e.g. $d_n = \sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Convergence of Sequences

We want to *rigorously* define $a_n \to a \in \mathbb{R}$, or "a_n converges to a as $n \to \infty$" or "a is the limit of (a_n)".

Idea: a_n should get closer and closer to a. Not necessarily monotonically, e.g.:

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{2n} & n \text{ even} \end{cases} \qquad \text{we want } a_n \to 0$$

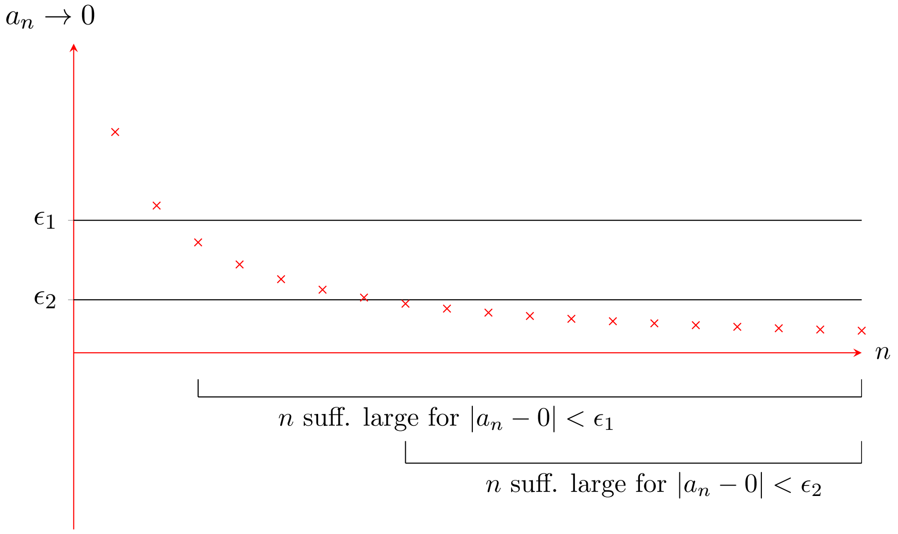


Also notice that $\frac{1}{n}$ gets closer and closer to -1! So we want to say instead that a_n gets *as close as we like to a*. We will measure this with $\epsilon > 0$. We phrase "a_n gets *arbitrarily* close to a" by "a_n gets to within ϵ of a for *any* $\epsilon > 0$".

> **Definition** (Mestel). $u_n \to u$ if $\forall n$ sufficiently large, $|u_n - u|$ is *arbitrarily small.*
>
> Define a real number $b \in \mathbb{R}$ to be arbitrarily small if it is smaller than any $\epsilon > 0$ i.e. $\forall \epsilon > 0,\ |b| < \epsilon$.

Definition Mestel says that once n is large enough, $|u_n - u|$ is less than every $\epsilon > 0$, i..e it's zero, i.e. $u_n = u$. We want to *reverse* the order of specifying n and ϵ.

i.e. we want to say that to get *arbitrarily close to the limit* a (i.e. $|a_n - a| < \epsilon$), we need to go sufficiently far down the sequence. Then if I change $\epsilon > 0$ to be smaller, I simply go further down the sequence to get within ϵ of a.



There will not be a "n sufficiently large" that works for all ϵ at once! (unless $a_n = a$ eventually.)

But for *any* (fixed) $\epsilon > 0$ we want there to be an n sufficiently large such that $|a_n - a| < \epsilon$. So we change "$\exists n$ such that $\forall \epsilon$" to "$\forall \epsilon,\ \exists n.$". *This allows n to depend on ϵ.*

Definition (Nestel)**.** $a_n \to a$ if $\forall \epsilon > 0,\ \exists n \in \mathbb{N}$ such that $|a_n - a| < \epsilon$.

e.g.

$$a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \quad \text{satisfies } a_n \to 0 \text{ according to Prof. Nestel.}$$

We want to modify this to say eventually $|a_n - a| < \epsilon$ *and it stays there!*

Ignore Mestel and Nestel's definition! Lecture 3

Definition (Convergence)**.** We say that $a_n \to a$ iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that "} n \geq N \implies |a_n - a| < \epsilon \text{"}$$

This says that *however close* ($\forall \epsilon > 0$) I want to get to the limit a, there's a point in the sequence ($\exists N \in \mathbb{N}$) beyond which ($n \geq N$) my a_n is indeed that close to the limit a ($|a_n - a| < \epsilon$).

Remark 1.2. N depends on ϵ! $N = N(\epsilon)$

Equivalently:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that"} \forall n \geq N,\ |a_n - a| < \epsilon \text{"}$$

or equivalently

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon,\ \forall n \geq N_\epsilon$$

Clicker Question 1.3. Given a sequence of real numbers $(a_n)_{n\geq 1}$. Conisder

$$\boxed{\forall n \geq 1,\ \exists \epsilon > 0 \text{ such that } |a_n| < \epsilon}$$

This means?

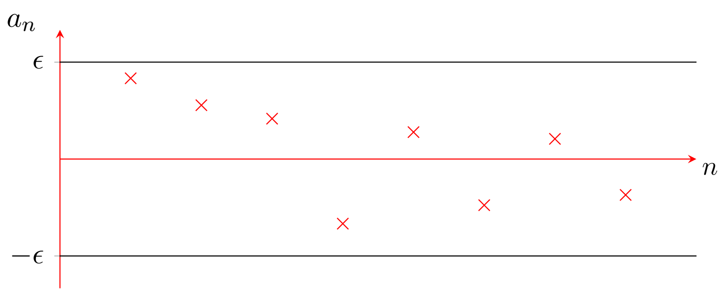
Answer: It always holds.

Proof. Fix any $n \in \mathbb{N}$. Take $\epsilon = |a_n| + 1$. ■

What about

$$\boxed{\exists \epsilon > 0 \text{ such that } \forall n \geq 1,\ |a_n| < \epsilon}$$

Answer: (a_n) is bounded.

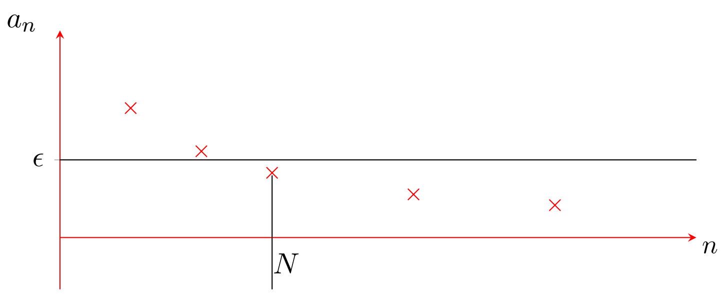


Proof. $\iff a_n \in (-\epsilon, \epsilon)\ \forall n \iff |a_n|$ is bounded by ϵ. ■

Definition. If a_n does not converge to a for any $a \in \mathbb{R}$, we say that a_n *diverges.*

Example 1.4. I claim that $\frac{1}{n} \to 0$ as $n \to \infty$

Rough working: Fix $\epsilon > 0$. I want to find $N \in \mathbb{N}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N$.



Since $a_n = \frac{1}{n}$ is monotonic, it is *sufficient* to ensure that $\frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$ (This *implies* $\frac{1}{n} \leq \frac{1}{N} < \epsilon,\ \forall n \geq N$).

Proof. Fix $\epsilon > 0$. Pick any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. (This is the Archimedean axiom of $\mathbb{R}$. Notice N depends on ϵ!!). Then $n \geq N \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. ■

Method to prove $a_n \to a$

(I) Fix $\epsilon > 0$

(II) Calculate $|a_n - a|$

(II$'$) Find a good estimate $|a_n - a| < b_n$

(III) Try to solve $a_n - a < b_n < \epsilon$ $(*)$

(IV) Find $N \in \mathbb{N}$ s.t. $(*)$ holds whenever $n \geq N$

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see examples below)

Example 1.5. $a_n = \dfrac{n+5}{n+1}$

Rough Working

$$|a_n - 1| = \left|\frac{n+5}{n+1} - 1\right| = \frac{4}{n+1}$$

This is $< \epsilon \iff n + 1 > 4/\epsilon \iff n > 4/\epsilon$, so take $N \geq 4/\epsilon$.

Proof. Fix $\epsilon > 0$. Pick N such that $N \geq 4/\epsilon$. Then $\forall n \geq N$,

$$|a_n - 1| = \frac{4}{n+1} \leq \frac{4}{N+1} < \frac{4}{N} \leq \epsilon$$ ∎

Example 1.6. $a_n = \dfrac{n+2}{n-2} \to 1$

Rough Working

$$|a_n - 1| = \left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2}$$

We want $\frac{4}{n-2} < \epsilon$. We want implications in the $\Longleftarrow$ direction (i.e. $\frac{4}{n-2} < \epsilon \Longleftarrow n \geq N$) *not* $\Longrightarrow$ direction. i.e. $\frac{4}{n-2} < \epsilon \Longrightarrow \frac{4}{n} < \epsilon$.

But if we take $N = \frac{4}{\epsilon}$, we need the *opposite* implication, we *need* $\frac{4}{n-2} < \epsilon$. We *need* to estimate $\frac{4}{n-2} < b_n$, and then solve $b_n < \epsilon$. So we make denominator smaller.

To make $n-2$ smaller, make 2 bigger! e.g. $\frac{n}{2} > 2$ for $n > 4$. Then $\frac{4}{n-2} < \frac{4}{n-n/2} = \frac{8}{n}$

Also want $b_n = \frac{8}{n} < \epsilon \iff n > 8/\epsilon$. So take $N > \max(8/\epsilon, 4)$.

Proof. Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \max(8/\epsilon, 4)$. Then $n \geq N \Longrightarrow n > 8/\epsilon$ (1) and $n > 4$ (2) $\Longrightarrow$

$$\left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2} \underbrace{<}_{(2)} \frac{4}{n - n/2} = \frac{8}{n} \underbrace{<}_{(1)} \epsilon$$ ∎

Lecture 4

We can also define limits for *complex sequences.*

Definition. $a_n \in \mathbb{C}$, $\forall n \geq 1$. We say $a_n \to a \in \mathbb{C}$ iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon$$

(i.e. $\sqrt{\Re(a_n - a)^2 + \Im(a_n - a)^2} < \epsilon$)

This is equivalent (see problem sheet!) to $(\Re a_n) \to \mathfrak{a}$ *and* $(\Im a_n) \to \Im a$

Example 1.7. Prove $a_n = \dfrac{e^{in}}{n^3 - n^2 - 6} \to 0$ as $n \to \infty$

Rough Working

$$|a_n - a| = \left|\frac{e^{in}}{n^3 - n^2 - 6}\right| = \left|\frac{1}{n^3 - n^2 - 6}\right|$$

Estimate $\left|\dfrac{1}{n^3 - n^2 - 6}\right| < \dfrac{1}{c_n}$ by making c_n smaller than $n^3 - n^2 - 6$ (But not too small! We want $c_n \to \infty$). So let $c_n = n^3 -$ *something bigger than* $n^2 + 6$.

Take off $\frac{n^3}{2}$ to make the expression simple. For $n \geq 4$, we have $\frac{n^3}{2} > n^2 + 6$.

So for $n \geq 4$

$$\left|\frac{1}{n^3 - n^2 - 6}\right| < \frac{1}{n^3 - n^3/2} = \frac{2}{n^3}$$

and this is $< \epsilon$ for $n > \sqrt[3]{\frac{2}{\epsilon}}$.

Proof. $\forall \epsilon > 0$, choose $N \geq \max(4, \sqrt[3]{2/\epsilon})$. Then $\forall n \geq N$

$$|a_n - 0| = \left|\frac{1}{n^3 - n^2 - 6}\right| < \frac{1}{n^3 - n^3/2} = \frac{2}{n^3} \leq \frac{2}{N^3} \leq \epsilon \qquad \blacksquare$$

Example 1.8. Set $\delta = 10^{-1000000}$, $a_n = (-1)^n \cdot \delta$. Prove that a_n does not converge.

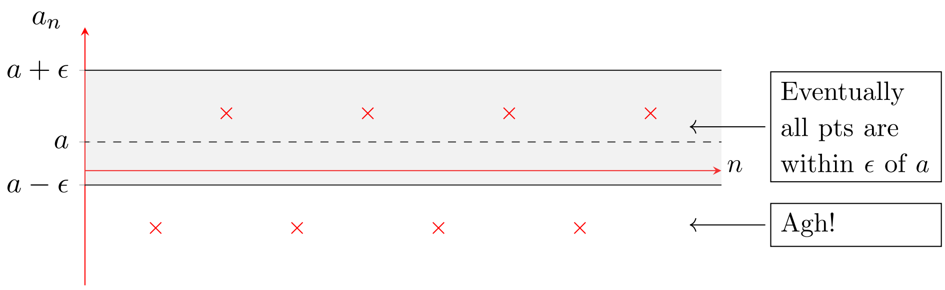
We want to show that the following is false:

$$\exists a \text{ s.t. } \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n - a| < \epsilon$$

i.e. we need to prove

$$\forall a,\ \exists \epsilon > 0 \text{ s.t. } \forall N \in \mathbb{N},\ \exists n \geq N \text{ s.t. } |a_n - a| \geq \epsilon$$

Rough: Assume for contadiction that $a_n \to a$, i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \implies |a_n - a| < \epsilon$



For small enough $\epsilon > 0$, the fact that a is within ϵ of δ (a_{2n}) and $-\delta$ (a_{2n+1}) will be a contradiction.

PROOF. Fix $a \in \mathbb{R}$. Take $\epsilon = \delta$ (or $\epsilon < \delta$ will do).

Then if $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \epsilon$ this implies

(i) $|a_{2N} - a| < \epsilon \iff a \in (\delta - \epsilon, \delta + \epsilon) \implies a > \delta - \epsilon = 0$

(ii) $|a_{2N+1} - a| < \epsilon \iff a \in (-\delta - \epsilon, -\delta + \epsilon) \implies a < -\delta + \epsilon = 0$, ※

(or use triangle inequality:

$$|\delta - (-\delta)| \leq |\delta - a| + |a - (-\delta)| < \epsilon + \epsilon \implies 2\delta < 2\epsilon = 2\delta \text{ ※})$$

So $a_n \not\to a$, but this holds $\forall a \in \mathbb{R}$, so a_n does not converge. ∎

Clicker Question 1.9. Fix $(a_n)_{n\geq 1}$, $a_n \in \mathbb{R}$. Then

$$\forall n, \ \exists \epsilon > 0 \text{ s.t. } |a_n| < \epsilon \text{ means?}$$

Answer: Nothing. This is always true. Take $\epsilon = |a_n| + 1$

Lecture 5

Theorem 1.10: Uniqueness of Limits

Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$

Idea: For n large, a_n should be close to a and to b. So a arbitrarily close to $b \implies a = b$.

PROOF 1.

(i) $\forall \epsilon$, $\exists N_a$ s.t. $\forall n \geq N_a$, $|a_n - a| < \epsilon$

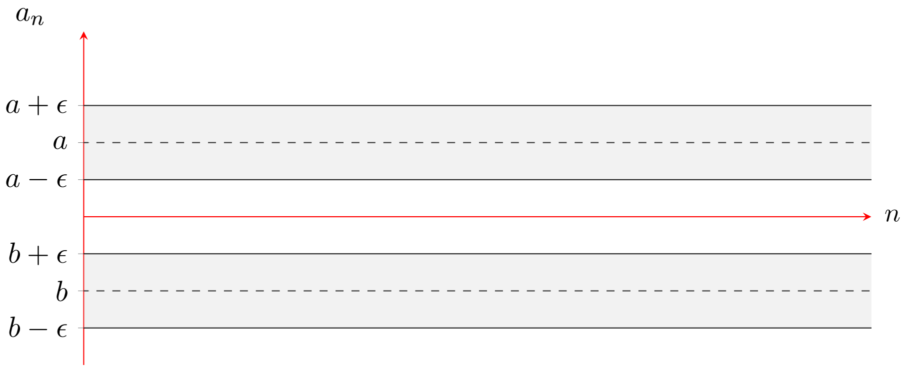
(ii) $\forall \epsilon$, $\exists N_b$ s.t. $\forall n \geq N_b$, $|a_n - b| < \epsilon$

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$, (i) and (ii) hold, so

$$|a - b| = |(a - a_n) + (a_n - b)| \leq |a - a_n| + |a_n - b| < 2\epsilon \implies |a - b| = 0! \quad ∎$$

(recall! if not, set $\epsilon = \frac{1}{2}|a - b| > 0$ to get a contradiction)

Proof 2. By contradiction. Assume $a \neq b$.



Eventually a_n is in *both* corridors. So if I choose ϵ sufficiently small so that corridors don't overlap to get a contradiction.

Set $\epsilon = \frac{|a-b|}{2} > 0$. Then $\exists N_a, N_b$ such that $\forall n \geq N_a, N_b$, we have

$$|a_n - a| < \epsilon \text{ and } |a_n - b| < \epsilon$$

w.l.o.g. $a > b$. Then $a_n > a - \epsilon$ and $a_n < b$

$$\begin{aligned} &\implies b + \epsilon > a - \epsilon \\ &\implies 2\epsilon > a - b = 2\epsilon \text{ ⨳} \end{aligned}$$

■

Clicker Question 1.11. Prove $\frac{1}{n-2} \to 0$. Student Answer: Fix $\epsilon > 0$.

(i) We want $|\frac{1}{n-2} - 0| = \frac{1}{n-2} < \epsilon$

(ii) $\implies n - 2 > 1/\epsilon$

(iii) $\implies n > 2 + 1/\epsilon$

(iv) $\implies n > 1/\epsilon$ $(*)$

(v) So take $N > 1/\epsilon$, then

(vi) $\forall n \geq N$, $n > 1/\epsilon$ which is $(*)$

(vii) So $\frac{1}{n-2} \to 0$

(viii) (This is correct)

Answer: (iv) is wrong.

Theorem 1.12: Algebra of Limits

$a_n \to a$ and $b_n \to b$ then:

(i) $a_n + b_n \to a + b$

(ii) $a_n b_n \to ab$

(iii) $\frac{a_n}{b_n} \to \frac{a}{b}$ $(b \neq 0)$

Proof of (i). Fix any $\epsilon > 0$. Then $\exists N_a \in \mathbb{N}$ such that $\forall n \geq N_a,\ |a_n - a| < \epsilon/2$ and $\exists N_b \in \mathbb{N}$ such that $\forall n \geq N_b,\ |b_n - b| < \epsilon/2$. Set $N = \max\{N_a, N_b\}$, so

$$\begin{aligned}|(a_n + b_n) - (a+b)| &\leq |a_n - a| + |b_n - b| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad \blacksquare\end{aligned}$$

Proof of (ii). *Rough working:*

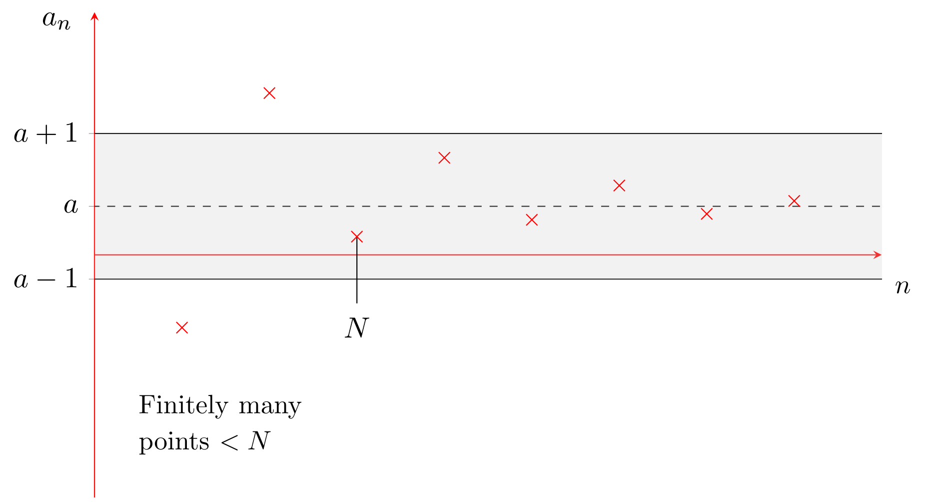
$$\begin{aligned}|a_n b_n - ab| &= |(a_n - a)b - a_n b + a_n b_n| \\ &\leq |a_n - a||b| + |a_n||b_n - b|\end{aligned}$$

We can easily make $|a_n - a| < \epsilon/2$ if I take $|a_n - a| < \frac{\epsilon}{2|b|}$.

We need to show that $|a_n| < A$, so that I can take $|b_n - b| < \frac{\epsilon}{2A}$.

Lemma 1.13. *If $a_n \to a$, then (a_n) is bounded: $\exists A \in \mathbb{R}$ s.t. $|a_n| < A,\ \forall n$.*

Proof of Lemma.



Fix $\epsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N,\ |a_n - a| < 1 \implies |a_n| < 1 + |a|$.

Then (a_n) is bounded by $\max\{a_1, a_2, \ldots, a_{N-1}, a+1\}$. $\blacksquare$

Fix $\epsilon > 0$. Then $\exists N_a$ such that $\forall n \geq N_a$, $|a_n - a| < \dfrac{\epsilon}{2(|b|+1)}$ (we add 1 in case $|b| = 0$)

and $\exists N_b$ such that $\forall n \geq N_b$, $|b_n - b| < \dfrac{\epsilon}{2A}$.

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$

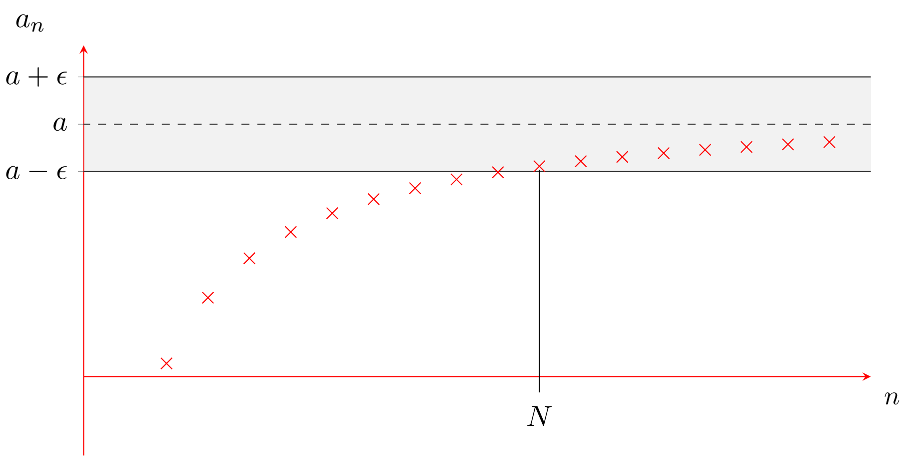
$$\begin{aligned}|a_n b_n - ab| &\leq |a_n - a||b_n| + |b_n - b||a| \\ &< \frac{\epsilon}{2}\frac{|b|}{|b|+1} + A\frac{\epsilon}{2A} \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad \blacksquare\end{aligned}$$

See exercise sheet for proof of 1.12iii.

Theorem 1.14

If (a_n) is bounded above *and* monotonically increasing then a_n is *convergent*.

Idea:



Eventually we get in the epsilon corridor (shaded area) because $a-\epsilon$ is *not* an upper bound. We stay in there because monotonic and bounded by a.

Proof. Fix $\epsilon > 0$. $a-\epsilon$ is *not* an upper bound for $\{a_n : n \in \mathbb{N}\}$ (because a is the *smallest* upper bound). So $\exists N \in \mathbb{N}$ such that $a_N > a-\epsilon$. Monotonic so $\forall n \geq N$ we have

$$a \geq a_n \geq a_N > a-\epsilon \implies |a_n - a| < \epsilon$$

■

Lecture 6 *Remark* 1.15. Now it's easier to handle things like $a_n = \dfrac{n^2+5}{n^3-n+6}$.

Dividing by n^3, we get $a_n = \dfrac{1/n+5/n^3}{1-1/n^2+6/n^3}$.

Use the fact that $1/n \to 0$ as $n \to \infty$ (Recall proof: $\forall\epsilon > 0$, let $N_\epsilon > 1/\epsilon$, then $n \geq N_\epsilon \implies n > 1/\epsilon \implies 1/n < \epsilon$), and the algebra of limits to deduce that

$$a_n \to \frac{0+5.0^3}{1-0^2+6.0^3} = 0.$$

Cauchy Sequences

Gives a way of proving convergence *without* knowing the limit.

Definition. A sequence is Cauchy iff

$$\forall\epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n,m \geq N,\ |a_n - a_m| < \epsilon$$

Remark 1.16. $m,n \geq N$ are arbitrary. It is not enough to say that $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a_{n+1}| < \epsilon$. See ex sheet.

Proposition 1.17. *If $a_n \to a$ then (a_n) is Cauchy.*

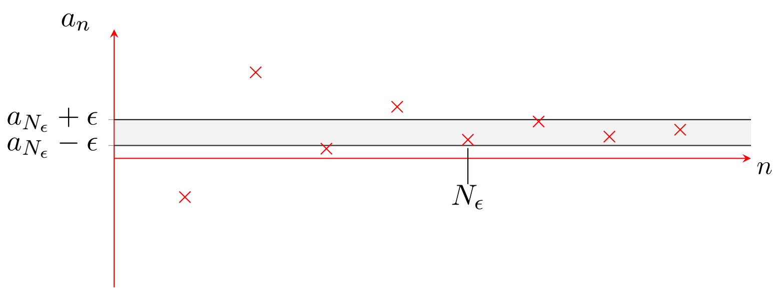
Proof. $a_n \to a \implies \forall \epsilon > 0,\ \exists N$ s.t. $n \geq N \implies |a_n - a| < \epsilon/2$ (1)

So $m \geq N \implies |a_m - a| < \epsilon/2$ (2). So

$$m, n \geq N \implies |a_n - a_m| \leq |a_n - a| + |a_m - a| < \underbrace{\epsilon/2}_{(1)} + \underbrace{\epsilon/2}_{(2)} = \epsilon \qquad \blacksquare$$

We want to prove converse: Cauchy $\implies$ Convergence.

We need a candidate for the limit a



We will produce an auxiliary sequence which is *monotonic* (+ bounded) $\implies$ convergence. $b_n := \sup\{a_i : i \geq n\}$. Then picture shows that $b_{N_\epsilon} \in (a_{N_\epsilon} - \epsilon, a_{N_\epsilon} + \epsilon]$ and b_n's are monotonically *decreasing* because $b_{n+1} = \sup\{a_i : i \geq n+1\}$, a subset of $\{a_i : i \geq n\}$.

So b_ns converge to $\inf\{b_n : n \in \mathbb{N}\}$. We will show that a_n's converge to same number, a, using Cauchy condition.

Lemma 1.18. *(a_n) is Cauchy $\implies$ (a_n) is bounded*

Proof. Pick $\epsilon = 1$, then $\exists N$ such that $\forall n, m \geq N$, $|a_n - a_m| < 1$. In particular $|a_n| < 1 + |a_N|$ $\forall n \geq N$ (take $m = N$), so

$$|a_n| \leq \max\{|a_1|, |a_2|, \ldots |a_{N-1}|, 1 + |a_N|\}\ \forall N \in \mathbb{N} \qquad \blacksquare$$

Theorem 1.19

(a_n) is a Cauchy sequence of real numbers $\implies$ a_n convergent.

Corollary 1.20. *(a_n) Cauchy $\iff$ (a_n) convergent. (Ex: Show not true in $\mathbb{Q}$!)*

Proof. (a_n) Cauchy $\implies$ bounded. So we can define $b_n = \sup\{a_i : i \geq n\}$. Then define $a = \inf\{b_n : n \in \mathbb{N}\}$ and we prove that $a_n \to a$.

Fix $\epsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n, m \geq N \implies |a_n - a_m| < \epsilon/2 \iff a_n - \epsilon/2 < a_m < a_n + \epsilon/2$. Take supremum over all $m \geq i \geq N$

$$\begin{aligned}
&\implies a_n - \epsilon/2 < \sup\{a_m : m \geq i\} \leq a_n + \epsilon/2 \\
&\text{i.e. } a_n - \epsilon/2 < b_i \leq a_n + \epsilon/2 \\
&\implies a_n - \epsilon/2 \leq \underset{\substack{\| \\ a}}{\inf\{b_i : i \geq N\}} \leq a_n + \epsilon/2 \\
&\iff |a - a_n| \leq \epsilon/2 < \epsilon \quad \forall n \geq N.
\end{aligned}$$

(We used: $S \subseteq \mathbb{R}$ is bounded satisfying $x < M\ \forall x \in S$. Then $\sup S \leq M$.) $\blacksquare$

Lecture 7

Example 1.21. Prove that if $\left|a_{n+1}/a_n\right| \to L$, $L < 1$, then $a_n \to 0$

Idea: $a_N \approx c.L^n$ for $n >> 0$, $L < 1 \implies a_n \to 0$.

To turn this in to a proof, we want $\left|a_{n+1}/a_n\right|$ to be less than $\alpha < 1$! We can't take $\alpha = L$! We can take $\alpha = L + \epsilon$ (because $\left|a_{n+1}/a_n\right|$ is *not* equal to L; it just tends to it). So we need $L + \epsilon < 1$, so take $\epsilon = \frac{1-L}{2}$.

Proof. Fix $\epsilon = \frac{1-L}{2} > 0$ (because $L < 1$). $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon \implies \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon = L + \frac{1-L}{2} = \frac{1+L}{2} < 1$$

So inductively we find that

$$|a_{N+k}| \leq \frac{1+L}{2}|a_{N+k-1}| \leq \left(\frac{1+L}{2}\right)^2 |a_{N+k-2}| \leq \cdots \leq \left(\frac{1+L}{2}\right)^k |a_N| \ (*)$$

[Ex sheet: $\alpha^k \to 0$ as $k \to \infty$ if $|\alpha| < 1$]

Applying this to $\alpha = \frac{1+L}{2} < 1$. $\exists M > 0$ s.t. $\forall m \geq M$

$$\left(\frac{1+L}{2}\right)^M < \frac{\epsilon}{1+|a_N|}$$

(as before we add 1 in denominator in case $|a_N| = 0$)

So by $(*)$ we have $|a_{N+m}| < \dfrac{\epsilon|a_N|}{1+|a_N|} < \epsilon \ \forall m \geq M$. Rewriting this:

$$\forall n \geq N + M, \ |a_n| < \epsilon$$ ∎

Subsequences

Definition. A *subsequence* of (a_n) is a new sequence $b_i = a_{n(i)}$ $\forall i \in \mathbb{N}$ where $n(1) < n(2) < \cdots < n(i) < \dots$ $\forall i \implies n(i) \geq i$ (Ex: prove this by induction)

[Formally $n(i)$ is a function $\mathbb{N} \to \mathbb{N}$ with $i \mapsto n(i)$ which is strictly monotonically increasing.] "Just go down the sequence faster, missing some terms out"

Example 1.22. $a_n = (-1)^n$ has subsequences:

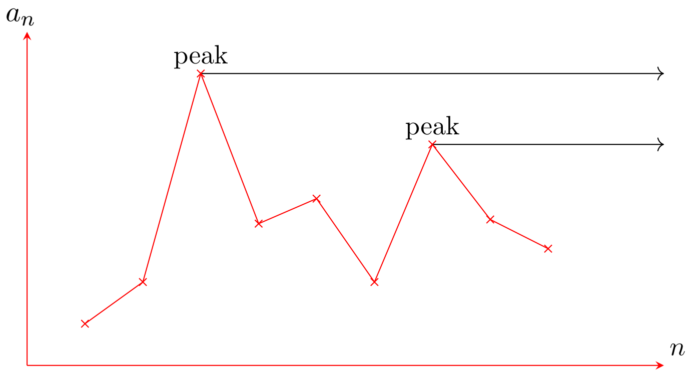
- $b_n = a_{2n}$, so $b_n = 1 \ \forall n \implies b_n \to 1$
- $c_n = a_{2n+1}$, so $c_n = -1 \ \forall n \implies c_n \to -1$
- $d_n = a_{3n}$, so $d_n = (-1)^n (= a_n!)$ doesn't converge.
- $e_n = a_{n+17}$, so $e_n = (-1)^{n+1} = -a_n$ doesn't converge.

Next we work up to

Theorem 1.23: Bolzano-Weierstrass

If (a_n) is a *bounded* sequence of real numbers then it has a *convergent subsequence*.

CHEAP PROOF. Use "peak points" of (a_n)



We say that a_j is a *peak point* iff $a_k < a_j$ $\forall k > j$. Either

(i) (a_n) has a finite no. of peak points

(ii) (a_n) has an infinite no. of peak points

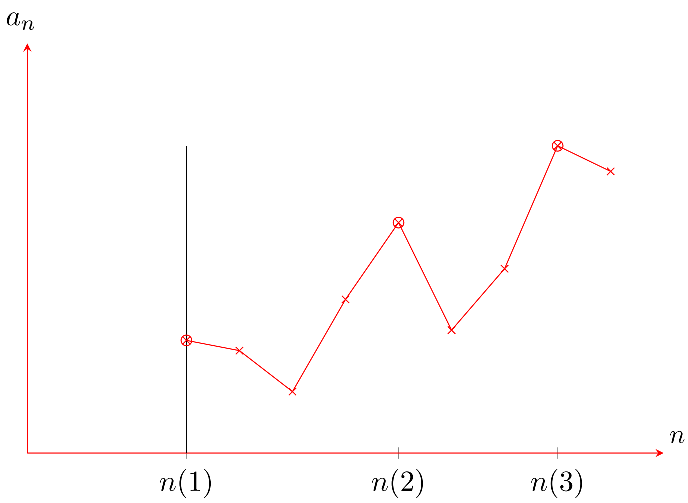
Case (i): Pick $n(1) \geq \max(j_1, \ldots, j_k)$ where $a_{j1}, \ldots, a_{jk}$ are the finite no. of peak points.

"Go beyond the (finitely many) peak points".

$a_{n(1)}$ is not a peak point $\implies \exists n(2) > n(1)$ s.t. $a_{n(2)} \geq a_{n(1)}$.

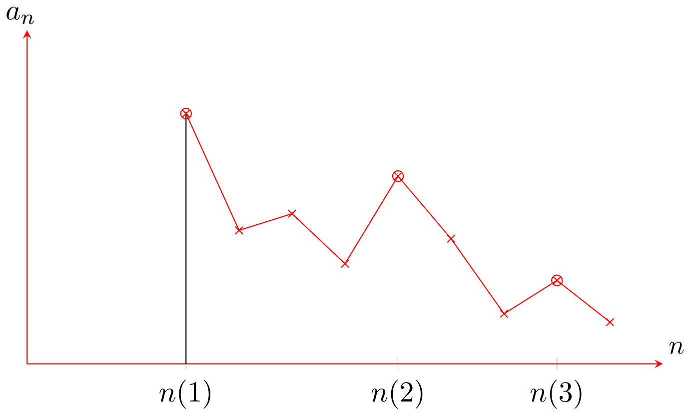
Similarly $a_{n(2)}$ not a peak point $\implies \exists n(3) > n(2)$ s.t. $a_{n(3)} \geq a_{n(2)}$.

Recursively no peak pints beyond $n(1) \implies$ we get $n(i) > n(i-1) > \cdots > n(1)$ s.t. $a_{n(i)} \geq a_{n(i-1)}$ $\forall i$.



i.e. $a_{n(i)}$ is a monotonically increasing subsequence of a_n. $(a_n)_{n\geq 1}$ bounded $\implies$ $(a_{n(i)})_{i\geq 1}$ is bounded $\implies a_{n(i)}$ is convergent (to $\sup\{a_{n(i)} : i \in \mathbb{N}\}$.

Case (ii): $\exists$ infinitely many peak points. Call these peak points $a_{n(1)}, a_{n(2)}, \dots$ where $n(1) > n(2) > \dots$



$a_{n(i+1)} \leq a_{n(i)}$ because $n(i+1) > n(i)$ and $a_{n(i)}$ is a peak point $\implies$ $(a_{n(i)})_{i \geq 1}$ is monotonically decreasing and bounded $\implies$ convergent (to $\inf\{a_{n(i)} : i \in \mathbb{N}\}$. ■

Lecture 8

Proposition 1.24. *If $a_n \to a$ as $n \to \infty$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$*

PROOF.

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |a_n - a| < \epsilon \ (*)$$

But $\forall i \geq N$, then $n(i) \geq i \geq N \implies$ by $(*)$, $|a_{n(i)} - a| < \epsilon$. ■

This gives us another proof that $(-1)^n$ is not convergent, because if $(-1)^n \to a$, then by Prop 1.24, $(-1)^{2n} \to a$ and $(-1)^{2n+1} \to a \implies a = 1$ and $a = 1$, ⋇

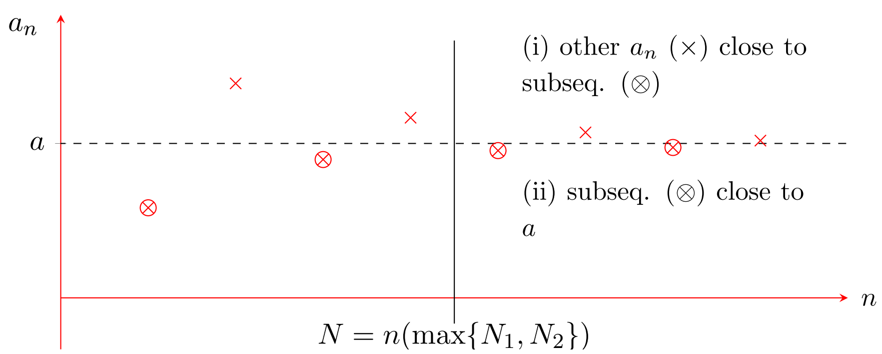
We also get another proof of "Cauchy $\implies$ convergence" using BW (Bolzano-Weierstrass). If a_n is Cauchy ($\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N\ |a_n - a_m| < \epsilon$), then a_n is convergent ($\exists a$ s.t. $a_n \to a$)

PROOF. We know that a_n is bounded (by $\max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a_N| + 1)\}$. So by BW, $\exists$ a convergent subsequence $a_{n(i)},\ i \geq 1$ s.t. $a_{n(i)}) \to a$ as $i \to \infty$ for some $a \in \mathbb{R}$.

So fix $\epsilon > 0$. We have:

(i) $\exists N_1$ s.t. $\forall n, m \geq N_1,\ |a_n - a_m| < \epsilon$

(ii) $\exists N_2$ s.t. $\forall i \geq N_2,\ |a_{n(i)} - a| < \epsilon$



Set $N = n(\max\{N_1, N_2\}) \geq \max\{N_1, N_2\} \geq N_1$. Then $\forall n \geq N$ we have

$$\begin{aligned} |a_n - a| &= |(a_n - a_N) + (a_N - a)| \\ &\leq |a_n - a_N| + |a_N - a| \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

■

Aside: Fix $c > 0$. Then $a_n \to a$ iff

$$\boxed{\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ s.t. } n \geq N_\epsilon \implies |a_n - a| < c\epsilon (*)}$$

Ex: Show $\implies$

PROOF $\impliedby$. Fix $\epsilon > 0$. Set $e' = \epsilon/c > 0$. Then $(*) \implies$

$$\exists N_\epsilon \in \mathbb{N} \text{ s.t. } n \geq N_\epsilon \implies |a_n - a| < c\epsilon' = \epsilon$$ ■

Beware! Do not let c depend on ϵ (Nor N!), e.g. if we let $c = \frac{1}{\epsilon}$ then $(*)$ becomes $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N,\ |a_n - a| < 1$ and $a_n = \frac{1}{2} \forall n,\ a = 0$ satisfies this!

We can also go the other way round: Cauchy theorem $\implies$ BW.

PROOF 2 OF BW. Take a bounded sequence (a_n). We want to find a convergent subsequence.

Given $a_n \in [-R, R]\ \forall n$, repeatedly subdivide to make this interval smaller. So either

(i) $\exists$ infinite number of a_n's in $[-R, 0]$

(ii) $\exists$ infinite number of a_n's in $[0, R]$

Pick one of these intervals with inifnite number of a_n's; call it $[A_1, B_1]$, length $2R/2$.

Now subdivide again; call $[A_2, B_2]$ one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ with infinitely many a_n's in it with length $2R/2^2$ etc.

We get a sequence of intervals $[A_n, B_n]$ of length $2R/2^n$ each containing an infinite number of a_ns which are nested: $[A_{k+1}, B_{k+1}] \subseteq [A_k, B_k]$

Now we use a *diagonal argument.* Let $b_i = a_{n(i)}$ be an elements of the sequence in $[A_i, B_i]$ s.t. $n(i) > n(i-1)$. (This is possible because $\exists$ infinite no. of elements of sequence in $[A_i, B_i]$.

Claim: $b_i = a_{n(i)}$ is convergent.

Fix $\epsilon > 0$. Take $N_\epsilon > \frac{2R}{\epsilon}$, so that $\frac{2R}{2^{N_\epsilon}} < \frac{2R}{N_\epsilon} < \epsilon$. Then $\forall i, j \geq N_\epsilon$ we have

$$|b_i - b_j| < \frac{2R}{2^{N_\epsilon}} < \epsilon$$

beacause $b_i, b_j \in [A_{N_\epsilon}, B_{N_\epsilon}] \implies (b_i)$ Cauchy $\implies$ convergent. ■

2 Series

Lecture 9

Definition. An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \text{ or } a_1 + a_2 + \dots$$

where $(a_i)_{i \geq 1}$ is a sequence.

Convergence of Series

Definition. We say that the series $\sum a_n = A \in \mathbb{R}$ (or "converges to $A \in \mathbb{R}$") iff the sequence of partial sums $S_n := \sum_{i=1}^{n} a_i \in \mathbb{R}$ converges to $A \in \mathbb{R}$; $S_n \to A$ as $n \to \infty$.

$$\begin{tikzcd}
& \text{Sequence of partial sums } (s_n) \arrow[dl, leftrightarrow, "a_n=s_n-s_{n-1}"'] \arrow[dr, leftrightarrow, "s_n=\sum_{i=1}^n a_i"] & \\
\text{Sequence } (a_n) \arrow[rr, leftrightarrow, "\text{Equivalent Information}"] & & \text{Series } \sum a_n
\end{tikzcd}$$

Example 2.1. $a_n = x^n,\ n \geq 0$. Consider $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$.

Define $s_n = \sum_{i=0}^{n} x^i = 1 + x + \dots + x^n$ then $xS_n = x + \dots + x^n + x^{n+1} \implies S_n - xS_n = 1 - x^{n+1}$

$$\implies S_n = \begin{cases} \frac{1-x^{n+1}}{1-x} & x \neq 1 \\ n+1 & x = 1 \end{cases}$$

So for $|x| < 1$, we see that

$$S_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \to \frac{1}{1-x} \text{ as } n \to \infty$$

(Question Sheet 3: proves that $r^n \to 0$ if $|n| < 1$)

So we have proved that (s_n) is convergent and $\sum x^n = \frac{1}{1-x} \in \mathbb{R}$ for $|x| < 1$.

For $|x| \geq 1$, $a_n = x^n$ does not $\to 0$ as $n \to \infty$. So $\sum a_n = \sum x^n$ is *not* a real number (does not converge) by the following result:

Theorem 2.2

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$

Proof. $S_n - S_{n-1} = a_n$. If $S_n \to S$ then $S_{n-1} \to S$ (Ex). So by the algebra of limits a_n is convergent and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$. ■

Proof from first principles. Fix $\epsilon > 0$. $s_n \to s$, so

$$\exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N,\ |s_n - s| < \epsilon$$

$$\begin{aligned} \implies |a_n| &= |s_n - s_{n-1}| \\ &\leq |s_n - s| + |s_{n-1} - s| \\ &< \epsilon + \epsilon, \text{ for } n-1 \geq N. \end{aligned}$$

So $\forall n \geq N+1$, $|a_n| < 2\epsilon$. ■

Remark 2.3. Converse is *not* true. E.g. $a_n = \frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ is *not* convergent.

Example 2.4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent†

Proof. (Trick) First do $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ and use $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) \\ &= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty \end{aligned}$$

$\implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent to 1.

Ao now compare the partial sums σ_n of $\sum \frac{1}{n^2}$ to those of $\sum \frac{1}{n(n+1)} = 1$

$$\begin{aligned} \sigma_n = \sum_{i=1}^{n} \frac{1}{i^2} &= 1 + \sum_{j=1}^{n-1} \frac{1}{(j+1)^2} \\ &\leq 1 + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} \\ &= 1 + s_{n-1} \end{aligned}$$

s_{n-1} is a bounded (by 1) monotonically increasing sequence (because $\frac{1}{n(n+1)} > 0$), convergent to 1. So $s_{n-1} < 1\ \forall n \implies \sigma_n < 2 \implies$ bounded above monotonic increasing sequence $\implies \sigma_n$ is convergent $\implies \sum \frac{1}{n^2}$ is convergent. ■

†Famously to $\pi^2/6$ - see Basel Problem.

Similarly $\sum \frac{1}{n^k}$ is convergent for $k \geq 2$ because $\frac{1}{n^k} \leq \frac{1}{n^2}$. In fact $\zeta(k) = \sum \frac{1}{n^k}$ is convergent for $k \in (1, \infty)$... See later!

Theorem 2.5: Algebra of Limits for Sequences

If $\sum a_n = A \in \mathbb{R}$ and $\sum b_n = B \in \mathbb{R}$, then $\sum(\lambda a_n + \mu b_n) = \lambda A + \mu B \in \mathbb{R}$.

Put differently, if $\sum a_n$, $\sum b_n$ converge, then so does $\sum(\lambda a_n + \mu b_n)$ and it equals $\lambda \sum a_n + \mu \sum b_n$.

PROOF. Partial sums (to n terms) of $\sum(\lambda a_n + \mu b_n)$ is

$$\sum_{i=1}^{n}(\lambda a_i + \mu b_i) = \lambda \sum_{i=1}^{n} \lambda a_i + \sum_{i=1}^{n} \mu b_i \to \lambda \sum_{i=1}^{\infty} a_n + \mu \sum_{i=1}^{\infty} \mu b_n$$

as $n \to \infty$ by the algebra of limits for sequences. So the partial sums converge. ∎

Lecture 10

Theorem 2.6: Comparison Test

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (and $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$)

PROOF. Call the partial sums A_n, B_n respectively. Then

$$0 \leq A_n \leq B_n \leq \sum_{i=1}^{\infty} b_i = \lim_{n\to\infty} B_n$$

So A_n is bounded and monotonically increasing $\implies$ convergent.

(Question Sheet 3 shows that if $A_n \leq B_n$ and $A_n \to A$, $B_n \to B$, then $A \leq B$) ∎

Proposition 2.7. *Suppose $a_n \geq 0$ $\forall n$. Then $\sum_{n=1}^{\infty} a_n$ converges iff $S_N = \sum_{n=1}^{N} a_N$ is bounded above and $\sum_{n=1}^{\infty} a_n$ diverges to ∞ (i.e. $S_n \to +\infty$ as $N \to \infty$) iff $S_N = \sum_{n=1}^{N} a_n$ is an unbounded sequence.*

PROOF. $a_n \geq 0 \iff (S_n)$ is monotonic increasing. So (S_n) bounded $\iff$ convergent.

S_N unbounded $\iff \forall R > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ S_n > R \iff S_n \to +\infty$. ∎

Ex: (Converse of Comparison Test) If $0 \leq a_n \leq b_n$ then $\sum a_n$ diverges to $\infty \implies \sum b_n$ diverges to ∞

Example 2.8. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha > 1$ is convergent.

PROOF. (Trick!) Arrange the partial sum as follows:

$$\begin{aligned}1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\cdots=1&+\left(\frac{1}{2^\alpha}+\frac{1}{3^\alpha}\right)+\left(\frac{1}{4^\alpha}+\cdots+\frac{1}{7^\alpha}\right)\\&+\left(\frac{1}{8^\alpha}+\cdots+\frac{1}{15^\alpha}\right)\\&+\left(\frac{1}{16^\alpha}+\cdots+\frac{1}{31^\alpha}\right)\\&+\dots\end{aligned}$$

Note that the kth bracketed term:

$$\left(\frac{1}{(2^k)^\alpha}+\cdots+\frac{1}{(2^{k+1}-1)^\alpha}\right)\leq\frac{1}{2^{k\alpha}}+\cdots+\frac{1}{2^{k\alpha}}=\frac{2^k}{2^{k\alpha}}=\frac{1}{2^{k(\alpha-1)}}$$

So any partial sum is less than some finite sum of these bracketed terms, i.e. for some sufficiently large N:

$$S_N<\sum_{k=0}^{N}\frac{1}{2^{k(\alpha-1)}}=\frac{1-\frac{1}{2^{(N+1)(\alpha-1)}}}{1-\frac{1}{2^{(\alpha-1)}}}\leq\frac{1}{1-\frac{1}{2^{\alpha-1}}}$$

because $\alpha>1$, so $\left|\frac{1}{2^{\alpha-1}}\right|<1$, so denominator >0.

So partial sums are bounded above $\implies$ convergent. ∎

Definition. Say that the series $\sum_{n=1}^\infty a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^\infty |a_n|$ is convergent

Example 2.9. $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent, but it is convergent.

Rough Working. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=(1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\dots$, the kth bracket $\frac{1}{2k-1}-\frac{1}{2k}=\frac{1}{2k(2k-1)}$. This is positive and $\leq\frac{1}{2k(2k-2)}=\frac{1/4}{k(k-1)}$, seen earlier sum of these is convergent.

So cancellation between consecutive terms is enough to make series converge by comparison with $\sum\frac{1}{k(k-1)}$.

PROOF. Fix $\epsilon>0$. Then use 2 things

(1) $\sum\frac{1}{2k(2k-1)}$ is convergent

(2) $\frac{(-1)^{n+1}}{n}\to 0$

By (1) $\exists N_1$ such that $\forall n\geq N_1$, $\sum_{n=1}^\infty\frac{1}{k(k-1)}<\epsilon$

By (2) $\exists N_2$ such that $\forall n\geq N_2$, $\left|\frac{(-1)^{n+1}}{n}\right|<\epsilon$

Set $N = \max(N_1, N_2)$. Then $\forall n \geq N$, we have:

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \left(\frac{1}{2j-1} - \frac{1}{2j}\right) + \delta = \sum_{k=1}^{j} \frac{1}{2k(2k-1)} + \delta$$

where $\delta = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd.} \\ 0 & \text{if } n \text{ is even.} \end{cases}$ $\quad (j = \lfloor \frac{n}{2} \rfloor)$ $j = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd.} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$

$$\implies S_n = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} - \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} \frac{1}{2k(2k-1)} + \delta$$

$$\text{So } \left| S_n - \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \right| \leq \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} \frac{1}{2k(2k-1)} + \frac{1}{n} < \epsilon + \epsilon$$

for all $n \geq 2N$ (so that $\lfloor \frac{n}{2} \rfloor + 1 > N$) ∎

Lecture 11

Theorem 2.10

If (a_n) is absolutely convergent, then it is convergent.

Proof. Let $S_n = \sum_{i=1}^{n} |a_i|$, $\sigma + n = \sum_{i=1}^{n} a_i$ be the partial sums.

We're assuming that S_n converges. Therefore S_n is Cauchy:

$$\forall \epsilon > 0 \ \exists N_\epsilon \text{ such that } n > m \geq N_\epsilon \implies |S_n - S_m| < \epsilon \iff |a_{m+1} + \cdots + |a_n| < \epsilon$$

i.e. the terms in the tail of the series contribute little to the sum

$\implies |a_{m+1} + \cdots + a_n| < \epsilon$ by the triangle inequality $\implies |\sigma_n - \sigma_m| < \epsilon \implies (\sigma_n)$ is Cauchy $\implies \sum a_i$ is convergent. ∎

Example 2.11. $\sum_{n=1}^{\infty} z_n$ is convergent for $|z| < 1$, divergent for $|z| \geq 1$

Proof. $\sum_{n=1}^{\infty} z_n$ is absolutely convergent because we showed that $\sum_{n=1}^{\infty} |z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$

For $|z| \geq 1$, the individual terms z^n have $|z^n| \geq 1$, so $z^n \not\to 0$, so $\sum z^n$ divergent. ∎

∗Re-arrangement of Series∗

This section was non-examinable in 2015

Beware. Do not rearrange series and sum them in a different order unless you can prove the result is the same.

> **Example 2.12.** $\sum(-1)^{n+1} = 1 - 1 + 1 - 1 + \dots$
>
> either this "$=$" $(1-1) + (1-1) + \dots = 0$
>
> or this "$=$" $1 - (1-1) + (1-1) + \dots = 1$

A better (convergent) example

> **Example 2.13.** $a_n := 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$
>
> (See later for proof of result, it's the series for $\log(1+x) = x - \frac{x^2}{2} + \dots$ putting $n = 1$, which is on our radius of convergence!)
>
> Reorder the sum as follows:
>
> $$\begin{array}{ccccccccc} 1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \dots \\ & -\frac{1}{2} & & -\frac{1}{4} & & -\frac{1}{6} & & \dots \end{array}$$
>
> $$\begin{array}{ccccccccc} =1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \dots \\ -\frac{1}{2}[& 1 & & +\frac{1}{2} & & +\frac{1}{3} & & \dots] \end{array}$$
>
> Terms with even denominator appear only in bottom row $(\times - \frac{1}{2})$
>
> Terms with odd denominator appear in the top row $(\times 1)$ + bottom row $\times - \frac{1}{2} \implies (\times \frac{1}{2})$ in total.
>
> So $a = \frac{1}{2}[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots] \implies a = a/2$, ⋇ (But clearly $a \geq \frac{1}{2} > 0$)

This happened because when I reordered I went along the bottom row twice as fast as I went along the top row. Since the top and bottom row diverges to ∞, I'm computing $\infty - \infty$, and originally I did this like $(a + n) - n$ as $n \to \infty$. Now I'm doing it like $(a + n) - (n + \frac{a}{2})$ as $n \to \infty$.

In fact I can rearrange the sum to converge to anything I like.

> **Example 2.14.** Rearrange $a_n = \dfrac{(-1)^{n+1}}{n} \to 42$.
>
> We reorder the sum as follows
>
> (i) Take only off terms $a_{2n+1} > 0$ until their sum is > 42. We can do this as $1 + \frac{1}{3} + \dots$ diverges to ∞!
>
> (ii) Now take only even terms $a_{2n} < 0$ until sum gets < 42
>
> (iii) Repeat (i) and (ii) to fade.
>
> We can do each step because $\sum a_{2n+1}$ diverges to ∞ and $\sum a_{2n} \to -\infty$. We use all the terms eventually (so this is really a reordering of the whole sum)
>
> Why? If not then we must eventually only take terms of one type (w.l.o.g. the even -ve terms) but these sum to $-\infty$, ⋇. At point they reach < 42 we switch back to odd +ve terms.
>
> Finally proof that the reordered sum converges to 42
>
> $$a_n \to 0 \text{ so } \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n| < \epsilon\ (*)$$

> So now we go to a point in the reordering where we have used all a_i up to N and then further to the point where the partial sum crosses 42. At this point, $(*)$ holds, so I'm within ϵ of 42. from this point on the sum is always within ϵ of 42 by design and by $(*)$.
>
> $$\implies |s_n - 42| < \epsilon \text{ from this point on} \quad \blacksquare$$

Lecture 12 More generally if (a_n) is a sequence whose terms tend to zero, $a_n \to 0$ and such that:

- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n$ diverges $(\to \infty)$
- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n$ diverges $(\to -\infty)$

then I can rearrange the series $\sum a_n$ (1) to make it converge to *any* number I like $\in \mathbb{R}$ or (2) to make it diverge to ∞ or (3) to $-\infty$.

For (1), the Algorithm is same as for $\sum \frac{(-1)^n}{n}$

(i) Pick +ve terms until partial sums are $>$ my fixed real number, a

(ii) Now pick -ve terms until partial sum is $< a$

(iii) Go back to (i) and repeat.

If however $a_n \to 0$ and

- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n \to \infty$
- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n$ converges

Then however I rearrange $\sum a_n$ it will always diverge to $+\infty$

Similarly if $a_n \to 0$ and

- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n$ converges
- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n \to -\infty \implies \sum a_n$ diverges to $-\infty$ (however rearranged)

Final case: $a_n \to 0$ and

- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n \geq 0}} a_n$ converges
- $\displaystyle\sum_{\substack{n \text{ s.t.} \\ a_n < 0}} a_n$ converges

This is the *good case* where *however* you rearrange, $\sum a_n$ is *absolutely convergent* to the same limit, $\sum_{a_n \geq 0} a_n + \sum_{a_n < 0} a_n$. We will prove this next time.

Remark 2.15. Rearrange partial sums only. $a+b = b+a$ is fine. Infinite sums are tricky!

Definition (Rearrangement of a Sequence)**.** If $M : \mathbb{N} \to \mathbb{N}$ is a bijection (i.e. a reordering!) then define $b_m := a_{M(m)}$. Then $(b_m)_{m \geq 1}$ is a rearrangement of (a_n).

e.g. if $M(1), M(2), M(3), M(4), \ldots$ is $5, 1, 6, 2, \ldots$ then $b_1, b_2, b_3, b_4, \ldots$ is $a_5, a_1, a_6, a_2, \ldots$.

Theorem 2.16

Suppose that $\sum a_n$ is absolutely convergent. Then

(1) $\sum_{a_n \geq 0} a_n$ is convergent to A (say)

(2) $\sum_{a_n < 0} a_n$ is convergent to B (say)

(3) $\sum a_n = A + B$

(4) $\sum b_m = A + B$ where (b_m) is any rearrangement of (a_n)

Proof. Key Idea: $\sum |a_n|$ is convergent so has a small "tail", so by the triangle inequality $\sum a_n$ has an even smaller tail so should converge. Lecture 13

But what to? No idea, so we use the Cauchy criterion!

(1) $s_n = \sum_{i=1}^n a_i$, $\sigma_n = \sum_{i=1}^n |a_i|$. σ_n convergent $\implies$ σ_n is Cauchy.

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, \ |\sigma_n - \sigma_m| < \epsilon$$

w.l.o.g. $n \geq m$, this says

$$\sum_{i=m+1}^{n} |a_i| < \epsilon \implies \left| \sum_{i=m+1}^{m} a_i \right| < \epsilon \iff |s_n - s_m| < \epsilon$$

So (s_n) is Cauchy $\implies$ s_n is convergent.

(2) $\sum_{a_n \geq 0} a_n$ is also convergent because the partial sums are monotonic increasing, bounded above by $\sum |a_n|$. Similarly $\sum_{a_n < 0} a_n$ is decreasing, $\geq -\sum |a_n|$, so also cvgt.

(3) Let $A = \sum_{a_n \geq 0} a_n$ and $B = \sum_{a_n < 0} a_n$. Then $\forall \epsilon > 0$

$$\exists N_1 \text{ s.t. } n \geq N_1 \implies \left| \sum_{a_n \geq 0}^{\text{first } n \text{ terms}} -A \right| < \epsilon$$

$$\exists N_2 \text{ s.t. } n \geq N_2 \implies \left| \sum_{a_n < 0}^{\text{first } n \text{ terms}} -B \right| < \epsilon$$

Let N be $\max(I, J)$ where I is the N_ith $a_i \geq 0$ (the N_ith positive term) and a_J the N_Jth -ve term. Then $\forall n \geq N$

$$\left| \sum_{i=1}^{n} -(A+B) \right| \leq \left| \sum_{a_i \geq 0}^{n} a_i - A \right| + \left| \sum_{a_i < 0}^{n} a_i - B \right| < \epsilon + \epsilon = 2\epsilon$$

So $\sum_{i=1}^{n} \to A + B$ as $n \to \infty$.

(4) Finally (b_m) is a rearrangement of (a_n). We want to show that $\sum b_m$ converges to $A+B$ as well.

Pick $M \in \mathbb{N}$ such that $b_1, b_2, \dots, b_M$ contains all of $P_1, P_2, \dots, P_I$ and $N_1, N_2, \dots, N_J$ where P_i is the ith $a_i \geq 0$ and N_J is the jth $a_j < 0$.

[i.e. we're far enough down the rearranged series to have included all significant $a_i \geq 0$ and $a_i < 0$ which sum to $< \epsilon$ by (1) and (2)]

Then $\forall m \geq M$ we have

$$\begin{aligned}\left|\sum_{i=1}^{m} b_i - (A+B)\right| &\leq \left|\sum_{b_i \geq 0}^{m} b_i - A\right| + \left|\sum_{b_i < 0}^{m} b_i - B\right| \\ &\leq \left|\sum_{a_k \geq 0}^{I} a_k + \delta - A\right| + \left|\sum_{a_k < 0}^{J} a_k + \delta' - B\right| \\ &< \epsilon + \epsilon = 2\epsilon\end{aligned}$$

(where $\delta =$ sum of $a_k \geq 0$ with $k > I$ and $\delta' =$ sum of $a_k < 0$ with $k > J$) ∎

Tests for convergence

We already met the first test:

Theorem 2.5: Comparison I

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (and $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$)

Recall proof from earlier: $s_n = \sum a_i$ is monotonic increasing and bounded above by $\sum b_i \in \mathbb{R}$.

Theorem 2.18: Comparison II - Sandwich Test

Suppose $c_m \leq a_n \leq b_n$ and $\sum c_n$, $\sum b_n$ are both convergent. Then $\sum a_n$ is convergent.

Proof. Use Cauchy. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$

$$\left|\sum_{i=m+1}^{n} b_i\right| < \epsilon, \quad \left|\sum_{i=m+1}^{n} c_i\right| < \epsilon$$

since the partial sums of b_i, c_i are Cauchy. Therefore

$$-\epsilon < \sum_{i=m+1}^{n} c_i \leq \sum_{i=m+1}^{n} a_i \leq \sum_{i=m+1}^{n} b_i < \epsilon$$

$$\implies \left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i\right| < \epsilon \implies \left(\sum_{i=1}^{n} a_i\right) \text{ is Cauchy.}$$ ∎

Theorem 2.19: Comparison III

If $\frac{a_n}{b_n} \to l \in \mathbb{R}$ then $\sum b_n$ absolutely convergent $\implies \sum a_n$ is absolutely convergent. Lecture 14

PROOF. Pick $\epsilon = 1$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$\left|\frac{a_n}{b_n} - l\right| < 1 \implies \left|\frac{a_n}{b_n}\right| < |l| + 1 \implies |a_n| < (|l| + 1)|b_n|$$

So now by the comparison test $\sum_{n\geq N} |b_n|$ convergent $\implies$ $\sum_{n\geq N} |a_n|$ convergent $\implies$ $\sum_{n\geq 1} |a_n|$ convergent. ∎

We have used the obvious fact that if $\sum_{n\geq N} c_n$ is convergent then $\sum_{n\geq 1} c_n$ is also convergent (and vice-versa). Ex: proof this!

Theorem 2.20: Alternating Series Test.

Given an alternating sequence a_n where $a_{2n} \geq 0$, $a_{2n+1} \leq 0\ \forall n$. Then $|a_n|$ monotonic decreasing to 0 $\implies \sum a_n$ convergent

PROOF. Write $a_n = (-1)^n b_n,\ b_n \geq 0\ \forall n$. Consider the partial sums $S_n = \sum_{i=1}^{n}(-1)^n b_n$.

Observe that:

(1) $S_i \leq S_{2n}\ \forall i \geq 2n$

(2) $S_i \geq S_{2n+1}\ \forall i \geq 2n+1$

Since if $i = 2j$ is even, then

$$\begin{aligned} S_{2j} &= S_{2n} + a_{2n+1} + \cdots + a_{2j} \\ &= S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} \leq S_2 n \end{aligned}$$

If $i = 2j+1$ is odd, then similarly:

$$S_{2j} = S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} - b_{2j+1} \leq S_2 n$$

So now $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ |b_n| < \epsilon$. So $\forall n, m \geq 2n$, we have:

$$S_{2N+1} \leq S_n,\ S_m \leq S_{2N}$$

$$\begin{aligned} \text{So } |S_n - S_m| &\leq |S_{2N+1} - S_{2N}| \\ &= b_{2n+1} < \epsilon \end{aligned}$$

∎

Theorem 2.21: Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

PROOF. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon \implies |a_{n+1}| < (r+\epsilon)|a_n|$$

Set $\alpha := r + \epsilon = \frac{1+r}{2} < 1$.

Inductively

$$|a_{N+m}| < \alpha|a_{N+m-1}| < \cdots < \alpha^m|a_N|$$

So $\forall k \geq N$

$$|a_k| < \alpha^{k-N}|a_N| = C\alpha^k$$

Then

$$C\sum_{k=N}^{n} \alpha^k = \frac{C(\alpha^N - \alpha^n)}{1-\alpha} \to \frac{C'}{1-\alpha} \text{ as } n \to \infty, \text{ since } \alpha < 1$$

So by the comparison test $\sum_{k\geq N} |a_k|$ is convergent $\implies \sum_{k\geq 1} |a_k|$ is convergent ∎

The point is that the ratio test, when it applies, says that $a_n \approx r^n$ i.e. decays exponentially. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.

Example 2.22. $a_n = \dfrac{100^n(\cos n\theta + i\sin n\theta)}{n!} = \dfrac{(100e^{i\theta})^n}{n!}$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(100e^{i\theta})^{n+1}/(n+1)!}{(100e^{i\theta})^n/n!} = \frac{100}{n+1} \to 0$$

So by the ratio test, $\sum a_n$ is absolutely convergent $\implies \sum a_n$ is convergent.

Theorem 2.23: Root Test

Lecture 15

If $\lim_{n\to\infty} |a_n|^{1/n} = r < 1$, then $\sum a_n$ is absolutely convergent.

PROOF. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left||a_n|^{1/n} - r\right| < \epsilon \implies |a_n|^{1/n} < r + \epsilon$$

Set $\alpha := r + \epsilon = \frac{1+r}{2} < 1$, so that $|a_n| < \alpha^n$. Then

$$\sum_{k=1}^{n} \alpha^k = \frac{\alpha(1-\alpha^n)}{1-\alpha} \to \frac{\alpha}{1-\alpha} \text{ as } n \to \infty \text{ since } \alpha < 1$$

So by the comparison test $\sum_{k\geq 1} |a_k|$ is convergent. ∎

Power Series

Theorem 2.24: Radius of Convergence

Consider the series $\sum a_n z^n$ $(*)$, $z, a_n \in \mathbb{C}$.

Then $\exists R \in [0, \infty]$ such that $|z| < R \implies (*)$ is aboslutely convergent, $|z| > R \implies$ $(*)$ divergent

Proof. Define $R = \sup S = \{|z| : a_n z^n \to 0\}$ or $R = \infty$ if the set is unbounded. (1) Suppose $|z| < R$. $|z|$ not an upperbound for $S \implies \exists w$ such that $|w| > |z|$ and $a_n w^n \to 0$. Then

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq A \left|\frac{z}{w}\right|^n$$

Since $\left|\frac{z}{w}\right| < 1 \implies \sum |a_n z^n|$ cvgt. Similarly $|z| > R \implies \sum |a_n z^n|$ divergent.

(2) Suppose $|z| > R$. Then $a_n z^n \not\to 0$ as $n \to \infty \implies \sum a_n z^n$ does not converge. ∎

Clicker Question 2.25. What is the radius of convergence for $\sum \frac{z^n}{n}$?

Answer: $R = 1$, in fact the series

(i) $\sum z^n$

(ii) $\sum \frac{z^n}{n}$

(iii) $\sum \frac{z^n}{n^2}$

all have this R.

Proof. The ratio test gives $\left|\frac{z^{n+1}}{z^n} \cdot f(n)\right|$ where f is a rational function of n of degree 0. $= |zf(n)| \to |z|$ as $n \to \infty$. So convergent for $|z| < 1$ and divergent for $|z| > 1$. ∎

But notice different behaviours on $|z| = 1$.

(i) Never converges on $|z| = 1$ as $z^n \not\to 0$

(ii) Convergent for some $|z| = 1$ (in fact $z \neq 1$), divergent for others

(iii) Also convergent $\forall z$ with $|z| = 1$ (comparison with $\sum \frac{1}{n^2}$)

Products of Series

Consider

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\ &\text{“}=\text{''}\ a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \\ &= \sum_{n=0}^{\infty} c_n z^n\end{aligned}$$

where $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b + 0, \dots c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$.

So we set $c_n = \sum_{i=0}^n a_i b_{n-i}$ and ask when is the product $\sum a_n z^n \sum b_n z^n$ equal to $\sum c_n z^n$? We can also do this without the z^n's:

Definition. Given series $\sum a_n$, $\sum b_n$, their *Cauchy Product* is the series $\sum c_n$ where $c_n := \sum_{i=0}^n a_i b_{n-i}$.

Theorem 2.26: Cauchy Product

Lecture 16

If $\sum a_n, \sum b_n$ are absolutely convergent, then $\sum c_n$ is absolutely convergent to $(\sum a_n) \cdot (\sum b_n)$

Proof. See handout on blackboard. Non-examinable.

Corollary 2.27. *If $\sum A_n z^n$ and $\sum B_n z^n$ have radius of convergence R_A and R_B respectively, then $\sum c_n z^n$ has radius of convergence $R_C \geq \min\{R_A, R_B\}$.*

PROOF. By the previous theorem, for $|z| < \min\{R_A, R_B\}$ $(*)$ we have $\sum A_n z^n$ and $\sum B_n z^n$ absolutely convergent $\implies \sum c_n z^n$ absolutely convergent to their product.

In fact $|c_n z^n| \to 0$ so $|z| < R_c$. So by $(*)$, $R_c \geq \min\{R_A, R_B\}$. ∎

Example 2.28. $\sum z^n$ has $R_A = 1$, $1 - z$ has $R_B = \infty$ So their cauchy product $\sum c_n z^n$ has $R_c \geq 1$.

Ex: Check $c_0 = 1, c_n = 0 \ \forall n \geq 1$, so in fact $R_c = \infty$.

But we only know that $\sum c_n z^n = 1 = (\sum z^n)(1 - z)$ when $|z| < 1 = \min\{R_A, R_B\}$.

Exponential Power Series

Definition (Exponential Series)**.**

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

Ratio test: $|a_{n+1}/a_n| = \frac{z}{n+1} \to 0$ as $n \to \infty \ \forall z \in \mathbb{C} \implies E(z)$ is absolutely convergent $\forall z \in \mathbb{C}$.

Proposition 2.29. $E(z)E(w) = E(z + w)$

PROOF. By Cauchy product theorem

$$E(z)E(w) = \sum_{n=0}^{\infty} c_n$$

where $c_n = \sum_{i=0}^n \dfrac{z^i}{i!} \dfrac{w^{n-i}}{(n-i)!} \implies c_n = \dfrac{(z+w)^n}{n!}$. ∎

Corollary 2.30. *$E(z) \neq 0$ and $\dfrac{1}{E(z)} = E(-z)$*

PROOF. $E(z)E(-z) = E(0) = 1$. ∎

Definition. $e := E(1) = \sum \frac{1}{n!} \in (-0, \infty)$

Corollary 2.31. *$E(n) = e^n$ for $n \in \mathbb{N}$*

PROOF. $E(n) = E(1 + (n-1)) = E(1)E(n-1) = \cdots = (E(1))^n$. ∎

Proposition 2.32. *$E(q) = e^q$ for $q \in \mathbb{Q}$ (recall rational powers of $a \in \mathbb{R}$ were defined in M1F)*

PROOF. Suppose $q > 0$; write $q = \frac{m}{n}$, $m, n \in \mathbb{N}$. Then

$$E(q) = E(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}}) = E(\tfrac{1}{n})^m$$

But

$$E(\frac{1}{n})^n = E(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}) = E(1) = e$$

$$\implies E(\frac{1}{n}) = e^{1/n} \text{ and } E(q) = E(\frac{1}{n})^m = e^{m/n} = e^q$$

If $q = \frac{-m}{n}$ then $E(q) = 1/E(m/n) = \frac{1}{e^{m/n}} = e^{-m/n} = e^q$. ∎

So we know that $E(x) = e^x\ \forall x \in \mathbb{Q}$. Later we define $e^x\ \forall x \in \mathbb{R}$ by *continuity* and we will show $E(x)$ is also continuous and so $E(x) = e^x\ \forall x \in \mathbb{R}$.

Some useful properties of $E(x)$: Lecture 17

(i) $x \geq 0 \implies E(x) \geq 1$ and $x > 0 \implies E(x) > 1$ (obvious from series)

(ii) $E(x) > 0\ \forall x \in \mathbb{R}$

(iii) $E(x)$ is strictly increasing for $x \in \mathbb{R}$: $x < y \implies E(y) = E(x)E(y-x) > E(x).1$

(iv) $|x| < 1$ then $|E(x) - 1| < \frac{|x|}{1-|x|}$

(v) $\mathbb{R} \ni x \mapsto E(x)$ is a continuous bijection onto $(0, \infty)$. (proven later)

(vi) So we can define $\log : (0, \infty) \to \mathbb{R}$ as inverse of E, i.e. $y = \log x$ defined by $\iff x = e^y$ with the usual log properties

We can also define a^x for $a \in (0, \infty), x \in \mathbb{R}$ by $a^x = E(x \log a)$

Ex: If $x \in \mathbb{Q}$ this agrees with Corti's definition.

And trig functions $\cos\theta = \Re E(i\theta)$, $\sin\theta = \Im E(i\theta)$ etc.

Ex: $E(i\theta + i\phi) = E(i\theta)E(i\phi)$ implies what?

3 Continuity

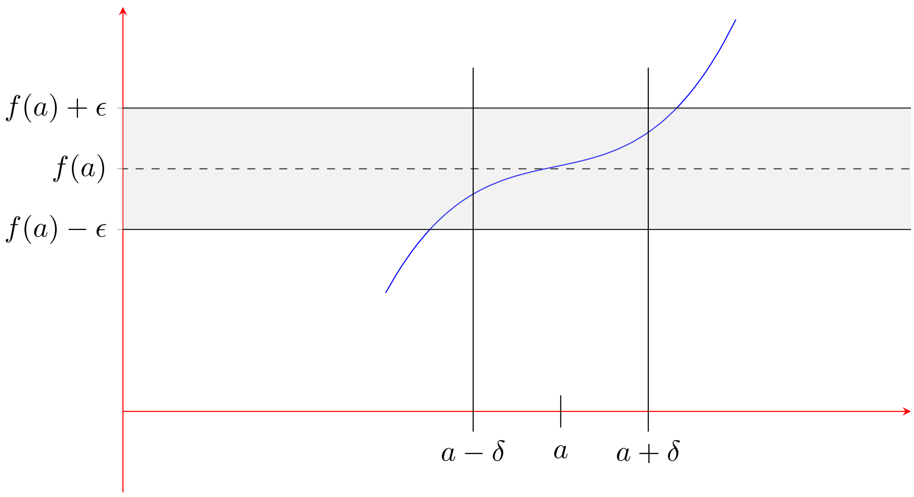
Continuity and Limits

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *continuous at* $a \in \mathbb{R}$ if and only if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

So δ depends on a, ϵ. "Once x is close to a, then $f(x)$ is close to $f(a)$".

More precisely: "However close (i.e. within ϵ) I want $f(x)$ to be to $f(a)$, I can arrange it by taking x close (i.e. within δ) to a".



Equivalently: $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon\ \forall x$ with $|x - a| < \delta$

Or: $\forall \epsilon > 0,\ \exists \delta > 0$ such that $f(a - \delta, a + \delta) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$

Where $S \subseteq R$ then $f(S)$ is the set $\{f(x) : x \in S\}$

Or: $\forall \epsilon,\ \exists \delta > 0$ such that $f^{-1}(f(a) - \epsilon, f(a) + \epsilon) \supseteq (a - \delta, a + \delta)$

Where $f : A \to B \subset T$ then $f^{-1}(T) = \{a \in A : f(a) \in T\}$ [Don't need f^{-1} to exist !!]

Example 3.1.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then f is not continuous at $x = 0$

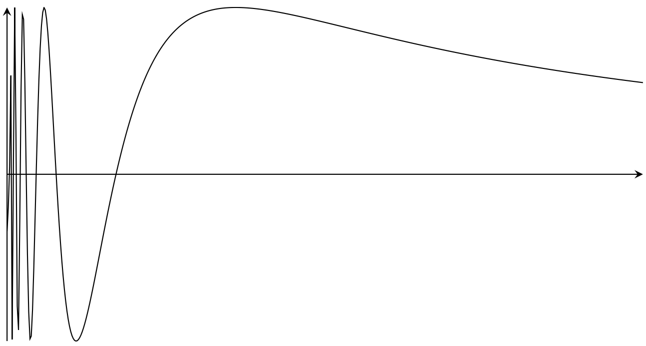
Proof. Take $\epsilon = 1$ (or $0 < \epsilon < 1$). Then if f is continuous at $x = 0$ we know that $\exists \delta > 0$ such that $|f(x) - f(0)| < 1\ \forall x \in (0 - \delta, 0 + \delta)$ $(*)$. In particular, take $x = \delta/2$ to find that $|1 - 0| < 1$ by $(*)$. ∎

"Jump discontinuity" is another type of discontinuity

Example 3.2.

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ r & x = 0 \end{cases}$$

Then f is discontinuous at $x = 0$ (for any r).



Idea of proof: If f is continuous at $x = 0$, then $f(x) \in (r - \epsilon, r + \epsilon)$ is close to $f(0) = r$ for $x \in (-\delta, \delta)$. In particular, $f(x)$ and $f(y)$ are close to each other (within 2ϵ). But $f(x)$ could be $+1$ and $f(y)$ could be -1, ⨳.

Proof. Fix $\epsilon \in (0, 1]$. If f is continuous at 0, then $\exists \delta > 0$ such that $|f(x) - f(0)| < \epsilon \ \forall x \in (\delta, \delta)$. In particular, $\forall x, y \in (-\delta, \delta), |f(x) - f(y)| < 2\epsilon \leq 2$, by the triangle inequality.

Now choose $n \in \mathbb{N}$, $n > \frac{1}{\delta}$. Then take $x = \frac{1}{(4n+1)\pi/2} \in (0, \delta)$, $y = \frac{1}{(4n+3)\pi/2} \in (0, \delta)$. Then

$$|\sin(1/x) - \sin(1/y)| = |1 - (-1)| = 2 \text{ ⨳}$$

■

Example 3.3. $f : \mathbb{R} \to \mathbb{R}$, $f = mx + c$ is continuous at a, $\forall a \in \mathbb{R}$. Lecture 18

Rough working: We want

$$\begin{aligned} |f(x) - f(a)| < \epsilon &\iff |(mx + c) - (ma + c)| < \epsilon \\ &\iff |mx(-a)| < \epsilon \\ &\iff |x - a| < \frac{\epsilon}{|m|} \text{ if } m \neq 0 \\ &\impliedby |x - a| < \frac{\epsilon}{|m| + 1} \end{aligned}$$

So set $\delta := \epsilon/(1 + |m|)$. Then $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$

Proof. Set $\delta := \frac{\epsilon}{1+|m|} > 0$. Then when $|x - a| < \delta$ we have

$$\begin{aligned} |(mx + c) - (ma + c)| &= |f(x) - f(a)| \\ &= |m||x - a| \\ &< |m|\delta = \epsilon \frac{|m|}{|m| + 1} < \epsilon \end{aligned}$$

■

Example 3.4. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ Proposition: f continuous on $\mathbb{R}$ (i.e. at a, $\forall a \in \mathbb{R}$)

Rough working:

$$|f(x) - f(a)| = |x^2 - a^2| = |x + a||x - a|$$

we want this to be $< \epsilon$, i.e. $|x - a| < \frac{\epsilon}{|x+a|} * (*)$

But we can't let δ depend on x!!

Problem: If $|x - a| < \frac{\epsilon}{R} \forall R > 0$, then $|x - a| = 0$.

Solution: I only care about x close to a; within 1 say.

So, so long as I choose $\delta \leq 1$, then I know that

$$|x - a| < \delta \implies |x + a| \leq |x - a| + 2|a| \leq 1 + 2|a|$$

So now $|x - a| < \dfrac{\epsilon}{1 + 2|a|} \implies (*)$

So to ensure both conditions we set $\delta = \min\{1, \epsilon/(1 + 2|a|)\}$

Proof. Fix $\epsilon > 0$, $a \in \mathbb{R}$. Set $\delta = \min\{1, \frac{\epsilon}{1+2|a|}\}$. Then $|x - a| < \delta \implies$

(i) $|x - a| < 1 \implies |x + a| < 1 + 2|a|$

(ii) $|x - a| < \dfrac{\epsilon}{1 + 2|a|}$

$$\implies |x^2 - a^2| = |x - a||x + a| < \frac{\epsilon}{1 + 2|a|} \cdot (1 + 2|a|) = \epsilon$$

∎

Clicker Question 3.5. Fix $a, b \in \mathbb{R}$. Then $x < a \implies x < b$ tells us?

Answer: $a \geq b$.

Prove that $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is discontinuous at $x = 0$

Student answer:

(i) Suppose f is cts at 0

(ii) Then $\forall \epsilon > 0,\ \exists \delta > 0$ s.t.

(iii) $|x| < \delta \implies |f(x) - f(0)| = |1/x| < \epsilon$

(iv) $\implies |1/(x/2)| = |2/x| < 2\epsilon$

(v) But $|x| < \delta \implies |x/2| < \delta$ so

(vi) should get that $|f(x/2) - f(0)| = |1/(x/2)| < \epsilon$

(vii) This contradicts $(*)$

(viii) So f is not continuous at 0

Answer: (vii) is the problem. (vi) $\implies$ (iv) doesn't contradict (iv).

Notice the definition of continuity makes sense whenever I have a notion of distance. e.g. in $\mathbb{R}^n$ use $|\vec{x}-\vec{y}| := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. Lecture 19

Definition. $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } |\vec{x}-\vec{a}| < \delta \implies |f(\vec{x}) - f(\vec{a})| < \epsilon$$

Notation: The ϵ-ball around $\vec{a} \in \mathbb{R}^n$ is $B_\epsilon(\vec{a}) := \{\vec{x} \in \mathbb{R}^n | |\vec{x}-\vec{a}| < \epsilon\}$

So if $n = 1$, $B_\epsilon(a) = (a-\epsilon, a+\epsilon) \subseteq \mathbb{R}$.

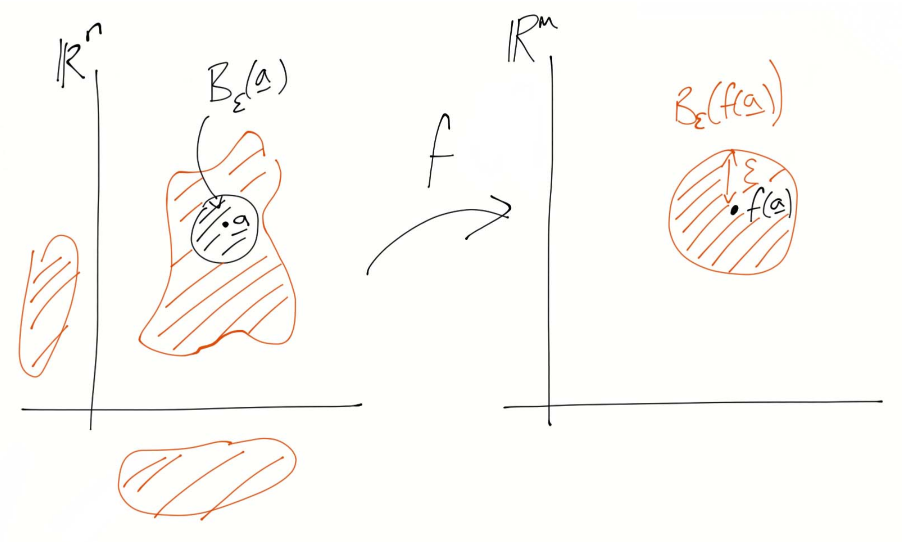
Using this we can rewrite our definition of continuity:

Definition. $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{a} \in \mathbb{R}^n$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } f(B_\delta(\vec{a})) \subseteq B_\epsilon(f(\vec{a}))$$

So every point within δ of $\vec{a}$ gets mapped by f to within ϵ of $f(\vec{a})$, equivalently

$$\boxed{\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } B_\epsilon(\vec{a}) \subseteq f^{-1}(B_\epsilon(f(\vec{a})))}$$

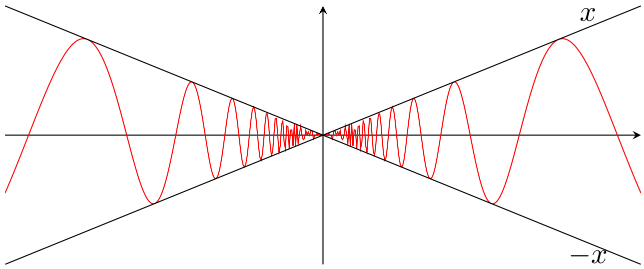


$f^{-1}(B_\epsilon(f(\vec{a}))) := \{\vec{x} \in \mathbb{R}^n : f(\vec{x}) \in B_\epsilon(f(\vec{a}))\}$

Continuity at $\vec{a}$ says that $\vec{a}$ is in the "interior" of $f^{-1}(B_\epsilon(f(\vec{a})))$, i.e. $\exists$ a small ball $B_\delta(\vec{a})$ around it which is also in $f^{-1}(B_\epsilon(f(\vec{a})))$.

So continuity at $\vec{a} \iff$ If $\vec{x}$ moves a tiny bit around $\vec{a}$ then $f(\vec{x})$ moves a tiny bit around $f(\vec{a})$.

Example 3.6. $f(x) = \begin{cases} x \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$



Proposition 3.7. *f is continuous at 0*

PROOF. Fix $\epsilon > 0$. Then

$$|f(x) - f(0) = |x \sin \frac{1}{x}| \leq |x|$$

Take $\delta = \epsilon$. Then $|x| < \delta \implies |x| < \epsilon \implies |f(x) - f(0)| < \epsilon$ ∎

Proposition 3.8. *$E : \mathbb{C} \to \mathbb{C}$ defined by $E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is continuous (i.e. continuous at a, $\forall a \in \mathbb{C}$)*

[Ex: from this show that $x \mapsto \sin x$ is continuous on $\mathbb{R}$]

Rough working:

$$\begin{aligned} |E(z) - E(a)| &= |E(a)(E(z-a) - E(0))| \\ &= |E(a)||E(z-a) - 1| \\ &\leq |E(a)| \cdot \frac{|z-a|}{1 - |z-a|} \end{aligned}$$

for $|z - a| < 1$ (see earlier lecture)

$$\begin{aligned} \text{We want this to be } < \epsilon &\iff |z-a| < \frac{\epsilon}{|E(a)|}(1 - |z-a|) \\ &\iff (1 + \epsilon/|E(a)|)|z-a| < \frac{\epsilon}{|E(a)|} \\ &\iff |z-a| < \epsilon/|E(a)|/(1 + \epsilon/|E(a)|) \end{aligned}$$

PROOF. Fix $\epsilon > 0$. Set $\delta = \dfrac{\epsilon}{|E(a)| + \epsilon}$ $(*)$

Then we calculate that

$$\begin{aligned} |E(z) - E(a)| &\leq |E(a)| \frac{|z-a|}{1 - |z-a|} \\ &< |E(a)| \cdot \frac{\delta}{1 - \delta} \end{aligned}$$

for all z with $|z - a| < \delta$. But by $(*)$, $\dfrac{\delta}{1-\delta} = \dfrac{\epsilon}{|E(a)|}$.

So $|z - a| < \delta \implies |E(z) - E(a)| < \epsilon$. ∎

Theorem 3.9

$f, g : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R} \implies (f+g), f \cdot g$ are cts at a.

Proof. Fix $\epsilon > 0$.

$$\exists \delta_1 > 0 \text{ such that } |x - a| < \delta_1 \implies |f(x) - f(a)| < \epsilon$$

and

$$\exists \delta_2 > 0 \text{ such that } |x - a| < \delta_1 \implies |g(x) - g(a)| < \epsilon$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x$ such that $|x - a| < \delta$:

$$|(f+g)(x) - (f+g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < 2\epsilon$$

For (2): Similarly

$$|f(x)g(x) - f(a)g(a)| \leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)| \ (*)$$

We need a bound on $|g(x)|$. We cannot bound $g(x)$ $\forall x$! But near a, $g(x)$ is close to $g(a)$, so we can bound $g(x)$ near a

Take $\epsilon = 1$

$$\exists \delta_1 > 0 \text{ s.t. } |x - a| < \delta_1 \implies |g(x) - g(a)| < 1 \implies |g(x)| < 1 + |g(a)| \ (A)$$

Now fix any $\epsilon > 0$. Then

$$\exists \delta_2 > 0 \text{ s.t. } |x - a| < \delta_2 \implies |f(x) - f(a)| < \epsilon/1 + |g(a)| \ (B)$$

(to cancel $|g(x)| < 1 + |g(a)|$ in $(*)$)

$$\exists \delta_3 > 0 \text{ s.t. } |x - a| < \delta_3 \implies |g(x) - g(a)| < \frac{\epsilon}{1 + |f(a)|} \ (C)$$

(to cancel $|f(a)|$ in $(*)$)

Set $\delta := \min\{\delta_1, \delta_2, \delta_3\}$. Then $|x - a| < \delta \implies (A), (B), (C)$ all hold.

Substitute into $(*)$ to find

$$\begin{aligned}|f(x)g(x) - f(a)g(a)| &< 1 + |g(a)| \frac{\epsilon}{1 + |g(a)|} + |f(a)| \frac{\epsilon}{1 + |f(a)|} \\ &\leq \epsilon + \epsilon = 2\epsilon\end{aligned}$$

∎

Theorem 3.10

$f : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ cts at $f(a) \in \mathbb{R}$, then $g \circ f$ cts at a

Idea of Proof: We want $g(f(x))$ to be close (within ϵ) to $g(f(a))$.

But g is continuous at $f(a)$! So sufficient for $f(x)$ to be close (within δ_g to $f(a)$. But f is continuous at a! so we can arrange this (by taking $\epsilon = \delta_g$ by taking x to be close (within δ_g) to a.

Lecture 20 PROOF. Fix $\epsilon > 0$. g is continuous at $f(a)$, so

$$\exists \delta > 0 \text{ s.t. } |g - f(a)| < \delta \implies |g(y) - g(f(a))| < \epsilon$$

Also f is continuous at a, so

$$\exists \eta > 0 \text{ s.t. } |x - a| < \eta \implies |f(x) - f(a)| < \delta$$

Hence $|x - a| < \eta \implies |f(x) - f(a)| < \delta \implies |g(f(x)) - g(f(a))| < \epsilon$. ■

Corollary 3.11. $a^x := E(x \log a),\ a > 0$ *is continuous* $\forall x \in \mathbb{R}$

PROOF. It is a composition $\mathbb{R} \xrightarrow[x \mapsto x\log a]{} \mathbb{R} \xrightarrow[y \mapsto E(y)]{} \mathbb{R}$ of two functions. ■

Ex: Show $\sin 1/x$ is continuous for $\mathbb{R}\backslash\{0\} \to \mathbb{R}$. (i.e. show $1/x$ is continuous from first principles, $\sin x$ is continuous using continuity of $E(x)$ and compose!)

Example 3.12. Suppose $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is continuous. Then $1/f$ is continuous.

PROOF. Pick $a \in \mathbb{R}$. Show $1/f(x)$ is continuous at a:

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| = \frac{1}{|f(x)f(a)|}|f(x) - f(a)|\ (*)$$

We need to bound $f(x)$ below! Need $|f(x)| >$ some $\eta > 0 \iff \frac{1}{|f(x)|} < \frac{1}{\eta}$

We can't, but we can near a! $f(a) \neq 0$, so take $\epsilon' = |f(a)|/2 > 0$. Then

$$\exists \delta' \text{ s.t. } |x - a| < \delta' \implies |f(x) - f(a)| < \epsilon' = \frac{|f(a)|}{2}$$

$$\implies |f(x)| > |f(a)| - \epsilon = \frac{|f(a)|}{2}$$

$$\text{So by } (*)\text{, we have } \left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| < \frac{1}{|f(a)|/2 \cdot |f(a)|}|f(x) - f(a)|$$
$$= \frac{2}{|f(a)|^2}|f(x) - f(a)|$$

Fix $\epsilon > 0$. Set $\epsilon'' = \min\left(\frac{|f(a)|}{2}, \frac{\epsilon}{2}|f(a)|^2\right) > 0$

Then $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon''$ (by continuity of f at a) $\implies (1)|f(x)| > |f(a)| - \epsilon'' \geq |f(a)| - |f(a)|/2$ and $(2)|f(x) - f(a)| < \frac{\epsilon}{2}|f(a)|^2$. So

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| = \frac{1}{|f(x)||f(a)|}|f(x) - f(a)|$$
$$< \frac{1}{|f(a)|/2 \cdot |f(a)|} \cdot \frac{\epsilon}{2}|f(a)|^2 = \epsilon \quad ■$$

Theorem 3.13

$f : \mathbb{R} \to \mathbb{R}$ is cts at $a \in \mathbb{R}$ iff $\forall$ sequences $x_n \to a$, $f(x_n) \to f(a)$

In one direction this is somewhat easy: if $x_n \to a$ and f is continuous at a, then $f(x_n)$ gets close to $f(a0$ as x_n gets close to $a \implies f(x_n) \to f(a)$.

The converse is *much harder*. If I want to see if f is continuous, I can test with a sequence $x_n \to a$ to see if $f(x_n)$ if close to $f(a)$ when n is large. But x_n's doesn't cover all x's! But if I use *all* sequences $x_n \to a$ then I do cover all x and get a theorem.

Proof. If f is cts at a, fix $\epsilon > 0$. $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. Now $x_n \to a$, so $\exists N \in \mathbb{N}$ such that $n \geq N \implies |x_n - a| < \delta \implies |f(x_n) - f(a)| < \epsilon$.

Suppose f is not cts at $a \in \mathbb{R}$ for contradiction.

Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in (a - \delta, a + \delta)$ such that $|f(x) - f(a)| \geq \epsilon$.

Choose $\delta = \frac{1}{n}$. $\exists x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ such that $|f(x_n) - f(a)| \geq \epsilon$.

So $|x_n - a| < \frac{1}{n}$ $\forall n \implies x_n \to a$. But $f(x_n) \not\to f(a)$, ※. ■

Example 3.14. $f(x) = \begin{cases} \sin 1/x & x \neq 0 \\ 0 = 0 \end{cases}$

This is *not* continuous at 0. But if we take $x_n \to 0$, then $f(x_n) = \sin(n\pi) = 0$ $\forall n$, so $f(x_n) \to f(0)$. so this sequence does not defect.

Have to choose a different sequence e.g. $x_n = \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \dots$, gives $\sin \frac{1}{x_n} = (-1)^{n+1} \not\to f(0) \implies f$ discontinuous at 0.

To get this problem of sequences not covering the whole of an interval $(a - \delta, a + \delta)$ (so having to consider all sequences at once - nasty), we can let x run through all of $\mathbb{R}$ with the following definition:

Definition. $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$. Lecture 21

We say that $f(x) \to b$ as $x \to a$ (or "$\lim_{x \to a} f(x) = b$") iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - b| < \epsilon$$

"x close to a (but not equal!!) $\implies f(x)$ close to b"

Example 3.15. $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$

Then $\lim_{x \to 0} f(x) = 0$

e.g. We can talk about $\lim_{x \to 0} f(x)$ for $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$.

Theorem 3.16

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ iff $f(x) \to f(a)$ as $x \to a$

Proof. f is continuous at $a \in \mathbb{R}$ says (1):

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

Whereas $f(x) \to f(a)$ as $x \to a$ says (2):

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

So (1) $\implies$ (2).

Suppose (2). Then I get (1) except for when $|x - a| = 0$. But when $|x - a| = 0$, then $x = a$, so $f(x) = f(a)$, so $|f(x) - f(a)| < \epsilon$, so I still get (1). ∎

Can extend the definition of continuity to functions defined on subsets of $\mathbb{R}$ or $\mathbb{R}^n$ e.g.

Definition. $f : S \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^n$, is continuous at $\vec{a} \in S$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } (0 < |\vec{x} - \vec{a}| < \delta \text{ and } x \in S) \implies |f(\vec{x}) - f(\vec{a})| < \epsilon$$

Example 3.17. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$

This is discontinuous. But $f|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ is continuous.

Related to this is one-sided continuity:

Definition. $f : \mathbb{R} \to \mathbb{R}$ is *right continuous* at $a \in \mathbb{R}$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } x \in [a, a + \delta) \implies |f(x) - f(a)| < \epsilon$$

Ex: f is right continuous at $a \in \mathbb{R} \iff f|_{[a,\infty)} : [a, \infty) \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$.

Ex: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f$ is both right and left continuous at $a \in \mathbb{R}$

Definition. $f(x) \to b$ as $x \to a_+$ "as x tends to a from above" means

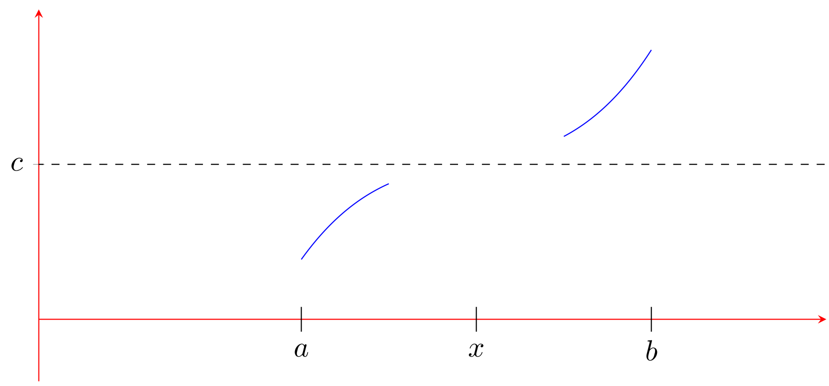
$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } x \in (a, a + \delta) \implies |f(x) - f(a)| < \epsilon$$

Ex: Just as before find that f is right continuous at $a \iff f(x) \to f(a)$ as $x \to a_+$

Intermediate Value Theorem

Theorem 3.18: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ cts, $c \in (f(a), f(b))$, then $\exists x \in [a, b]$ such that $f(x) = c$



If f is continuous it must cross the line $y = c$ at some point $x \in [a, b]$.

Corollary 3.19. *Any odd degree polynomial over* $\mathbb{R}$ *has a root* $\in \mathbb{R}$

Proof. w.l.o.g. $p(x) = x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$

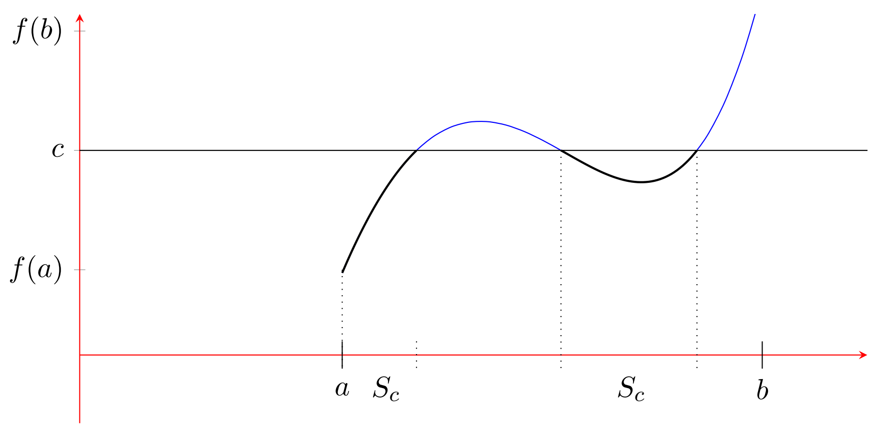
If we write this as $p(x) = x^{2n+1}(1 + \frac{a_{2n}}{x} + \cdots + \frac{a_0}{x^{2n+1}})$ then we see that $p(x) < 0$ for $x << 0$, and $p(x) > 0$ for $x >> 0$.

So we can find $a, b \in \mathbb{R}$ such that $p(a) < 0$, $p(b) > 0$.

So we apply IVT to $p|_{[a,b]} : [a, b] \to \mathbb{R}$ with $c = 0$ to find an $x \in [a, b]$ with $p(x) = c = 0$. ∎

We used the facts (proved in earlier lectures) that $mx + c$ is continuous and the product/sum of continuous functions are also continuous $\implies p(x)$ is continuous.

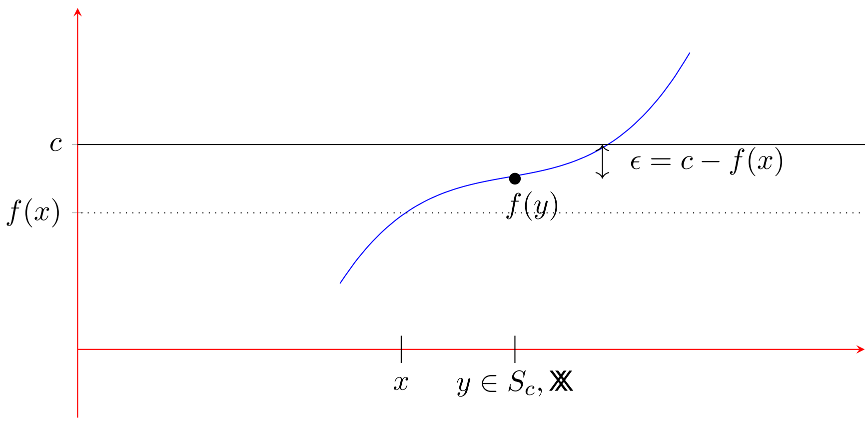
Proof of IVT. Lecture 22



Consider $S_c = \{y \in [a, b] : f(y) \leq c\}$. Define $x := \sup S_c$ ($S_c \neq \emptyset$ since $a \in S_c$ and bounded above by b so sup exists)

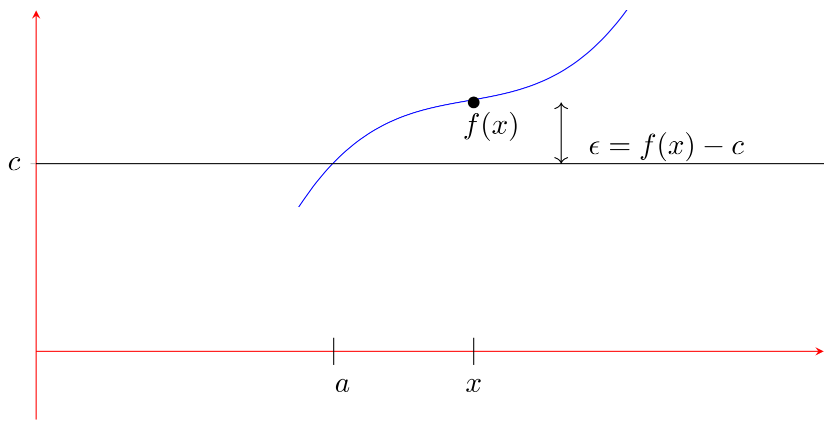
Claim: $f(x) = c$. *Proof:*

(i) Suppose $f(x) < c$.



Take $\epsilon = c - f(x) > 0$. f is cts at x, so $\exists \delta > 0$ such that $\forall y \in (x, x+\delta) \cap [a,b]$, $|f(y) - f(x)| < \epsilon$. Hence $f(y) < f(x) + \epsilon = c$. So $y \in S_c \implies x \neq \sup S_c$.

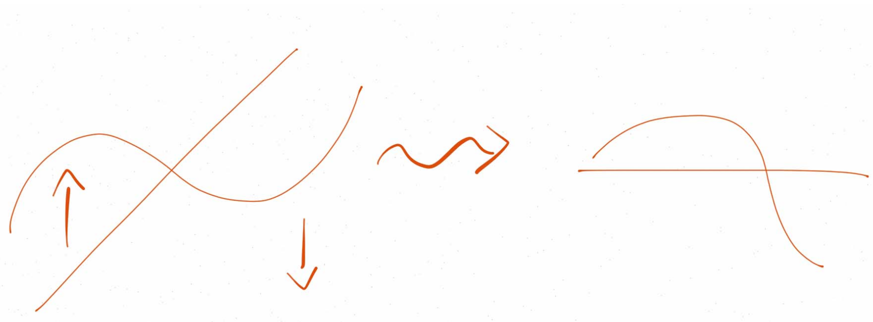
(ii) Suppose $f(x) > c$.



Take $\epsilon = f(x) - c > 0$. f is cts at x, so $\exists \delta > 0$ such that $\forall y \in (x-\delta, x) \cap [a,b]$, $|f(y) - f(x)| < \epsilon$. Hence $f(y) > f(x) - \epsilon = c \implies x - \delta$ is an upperbound for S_c, so $x \neq \sup S_c$. ∎

Proposition 3.20. *Suppose $f : [0,1] \to [0,1]$ is continuous. Then it has a fixed point (i.e. $\exists x \in [0,1]$ s.t. $f(x) = x$)*

Idea of proof: Rotate picture to make it look like IVT.



Proof. Set $g(x) = f(x) - x$, $g : [0,1] \to [0,1]$ is continuous.

So $g(0) = f(0) - 0 \geq 0$, $g(1) = f(1) - 1 \leq 0$

So by IVT $\exists x \in [0,1]$ s.t. $g(x) = 0 \iff f(x) = x$ ■

So if during the lecture you watch last weeks lecture on Panopto, using pause, fast-forward, rewind, play (but no jumping!) then at some point you will be watching a time in the lecture which equals the time now. (No matter where you start or end.)

Definition. $S \subseteq \mathbb{R}^n$,$f : S \to \mathbb{R}$. Then we say that f is bounded above if $\exists M \in \mathbb{R}$ s.t. $f(\vec{x}) \leq M \ \forall \vec{x} \in S$.

Similar for bounded below, bounded is both.

Example 3.21. $f(x) = \frac{1}{x} : (0,1] \to \mathbb{R}$ is not bounded above

Proof. Suppose $\frac{1}{x} \leq M \ \forall x \in (0,1]$ (Then $M > 0$!).

Then take $x = \min\{\frac{1}{2m}, 1) \implies x \leq 1/2m \implies 1/x \geq 2M > M$, ※. ■

Also $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f : [0,1] \to \mathbb{R}$ is also unbounded. Note that f is not continuous at 0!

So $\begin{cases} \text{discontinuous functions can be unbounded} \\ \text{continuous functions can be unbounded on non-closed intervals} \end{cases}$

But..

Theorem 3.22

$f : [a,b] \to \mathbb{R}$ cts $\implies f$ is bounded.

Ex: Give a function $f : [a,b] \cap \mathbb{Q} \to \mathbb{R}$ which is continuous and unbounded.

Proof. Suppose not. Then $\forall N \in \mathbb{N}$, N is not an unpperbound, so $\exists x_N \in [a,b]$ such that $|f(x_n)| > N$. Lecture 23

By BW Theorem, exists convergent subsequence, $y_i := x_{N(i)}, y_i \to y \in [a,b]$. With $|f(y_i)| = |f(x_{N(i)})| > N(i) \geq i$ $(*)$.

Fix $\epsilon = 1$, then

$$\exists \delta > 0 \text{ such that } \forall x \in (y - \delta, y + \delta) : |f(x) - f(y)| < 1 \implies |f(x)| < |f(y)| + 1.$$

Since $y_i \to y$,

$$\exists N \text{ such that } \forall n \geq N \ |y_n - y| < \delta \implies y_n \in (y - \delta, y + \delta) \implies |f(y_n)| < |f(y)| + 1.$$

By $(*)$, $n \leq |f(y_n)| < |f(y)| + 1 \ \forall n \geq N$, not true by the Archimedean Axiom ※. ■

SLICKER PROOF. Suppose not. Then $\forall N \in \mathbb{N}$, N is not an unpperbound, so $\exists x_N \in [a, b]$ such that $|f(x_n)| > N$.

By BW Theorem, exists cvgt subsequence, $y_i := x_{N(i)}, y_i \to y \in [a, b]$. With $|f(y_i)| = |f(x_{N(i)})| > N(i) \geq i$ $(*)$. f is cts at $y \implies f(y_i) \to f(y)$, contradicting $(*)$. ∎

Extreme Value Theorem

Theorem 3.23: Extreme Value Theorem

$f : [a, b] \to \mathbb{R}$ cts $\implies f$ bounded and attains its bounds.

So max $f(x)$ exists (not just sup)

PROOF. By boundedness theorem, $\exists \sup_{x\in[a,b]} f(x) = s$. Suppose for contradiction $\nexists c \in [a, b]$ such that $f(x) = s$.

2 proofs:

(1) Then $s - f(x) > 0$ $\forall x \in [a, b]$, so $g(x) = \frac{1}{s-f(x)} : [a, b] \to \mathbb{R}$ is well defined and cts. So $g(x)$ is bounded by $M > 0 \implies \frac{1}{s-f(x)} \leq M \implies f(x) \leq s - \frac{1}{M}$, so $s \neq \sup f(x)$, ⨳.

(2) From M1F $\exists$ a sequence $x_n \in [a, b]$ such that $f(x) \to \sup_{x\in[a,b]} f(x) = s$. BW Theorem $\implies$ exists subsequence $y_i := x_{N(i)}$ such that $y_i \to c \in [a, b]$. f is cts $\implies f(y_i) \to f(x)$. Since $f(y_i) \to s$, by uniqueness of limits, $f(c) = s$. ∎

Combining IVT + EVT we get

Theorem 3.24

$f : [a, b] \to \mathbb{R}$ is continuous then $\exists c, d \in [a, b]$ s.t. $\mathrm{im} f = f[a, b]$ is the interval $[f(c), f(d)]$.

PROOF. EVT $\implies \exists c, d$ s.t. $f[a, b] \subseteq [f(c), f(d)]$ $(*)$

Given any $y \in [f(c), f(d)]$ the IVT $\implies \exists x$ between c and d s.t. $f(x) = y$, so $(*)$ is onto. ∎

Inverse Function Theorem

Proposition 3.25. *If $f : [a, b] \to \mathbb{R}$ is continuous and strictly increasing ($x > y \implies f(x) > f(y)$), then f is a bijection $[a, b] \to [f(a), f(b)]$*

PROOF. $f(a)$ is a minimum of $f[a, b]$ because $x > a \implies f(x) > f(a)$. $f(b)$ is maximum. So by previous result $f[a, b] = [f(a), f(b)]$. We just need too show that f is injective:

If $x \neq y$, w.l.o.g. $x \neq y$ then $x < y \implies f(x) < f(y) \implies f(x) \neq f(y)$. So f is injective. ∎

So $\exists$ inverse $g := f^{-1} : [f(a), f(b)] \to [a, b]$

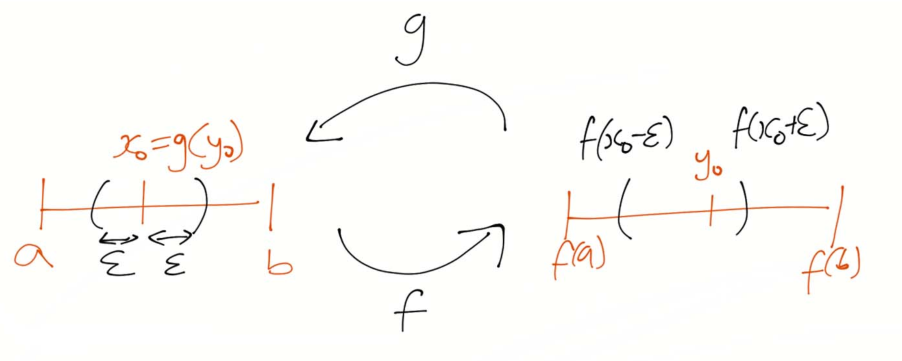
Proposition 3.26. *g is continuous (and also strictly increasing - Ex!)*

Proof. Fix $\epsilon > 0$ and $y_0 \in [f(a), f(b)]$.

Set $\delta := \min(f(g(y_0) + \epsilon) - f(g(y_0)), f(g(y_0)) - f(g(y_0) - \epsilon))$

$= \min(f(x_0 + \epsilon) - y_0, y_0 - f(x - \epsilon))$ where $x_0 = g(y_0)$

Picture:



In this definition we use the convention that if $x_0 - \epsilon < a$ then by $f(x_) - \epsilon$ I mean $f(a)$ if $x_0 + \epsilon > b$ then $f(x_0 + \epsilon)$ means $f(b)$.

(Equivalently I've extended f to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(x) = \begin{cases} f(a) & x \leq a \\ f(x) & x \in [a, b] \\ f(b) & x \geq b \end{cases}$)

So δ was chosen s.t. $(y_0 - \delta, y_0 + \delta) \subseteq (f(x_0 - \epsilon), f(x + \epsilon))$, so $y \in (y_0 - \delta, y_0 + \delta) \cap [a, b]$ then $f(x_0 - \epsilon) < y < f(x_0 + \epsilon)$

Apply $g \implies x_0 - \epsilon < g(y) < x_0 + \epsilon$. Recall $x_0 = g(y_0) \implies |g(y) - g(y_0)| < \epsilon$. ∎

Corollary 3.27. *$\sqrt{x} : [0, \infty) \to [0, \infty)$, $x^{1/n} : [0, \infty) \to [0, \infty)$, $n \in \mathbb{N}$ are continuous.*

Simpler exposition: Fix $f : \mathbb{R} \to \mathbb{R}$ bijective and continuous. Before we prove f^{-1} is continuous we prove

Lemma 3.28. *$f : \mathbb{R} \to \mathbb{R}$ is bijective and cts $\implies$ f is strictly monotonic* *Lecture 24*

Proof. We prove this on any closed bounded interval $[a, b]$ (Hence monotonic on $\mathbb{R}$! Ex!)

f is bijective, so $f(a) \neq f(b)$, w.l.o.g. $f(b) > f(a)$. Suppose for contradiction $\exists c \in (a, b)$ such that $f(c) \notin (f(a), f(b))$.

w.l.o.g. take $f(c) > f(b)$. Then fix $d \in (f(b), f(c))$. By IVT applied to:

- $f|_{[a,c]}$, we find $\exists x \in (a, c)$ such that $f(x) = d$.
- $f|_{[c,b]}$, we find $\exists y \in (c, b)$ such that $f(y) = d$.

But $y > x \implies x \neq y$, so f is not injective ※.

So $\forall c \leq b$, we find that $f(c) \leq f(b)$, and f injective $\implies f(c) < f(b)$. ∎

Theorem 3.29

$f : \mathbb{R} \to \mathbb{R}$ bijective and cts $\implies f^{-1} : \mathbb{R} \to \mathbb{R}$ cts.

PROOF. By Lemma f is strictly monotonic, w.l.o.g. strictly increasing.

We want to show f^{-1} is continuous at $y \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be $f^{-1}(y_0)$, so $f(x_0) = y_0$.

Fix $\epsilon > 0$.

Let $\delta := \min\{f(x_0 + \epsilon) - y_0, y_0 - f(x_0 - \epsilon)\}$.

Then $|y - y_0| < \delta \implies y \in (y_0 - \delta, y_0 + \delta) \subseteq (f(x_0 - \epsilon), f(x_0 + \epsilon))$.

Applying f^{-1} preserves order

$$\implies f^{-1}(y) \in (x_0 - \epsilon, x_0 + \epsilon) \iff |f^{-1}(y) - f^{-1}(y_0)| < \epsilon. \qquad \blacksquare$$

Corollary 3.30. *$E : \mathbb{R} \to \mathbb{R}$, $E(x) : \sum \frac{x^n}{n!}$ is a continuous bijection $\mathbb{R} \to (0, \infty)$ with continuous inverse* $\log : (0, \infty) \to \mathbb{R}$.

We already showed that E is continuous, never takes the value 0 ($E(-x) = E(x)^{-1}$) is unboundedly poisitive for $x \geq 0$ ($E(x) \geq 1+x$) and positive for $x < 0$ ($E(-x) = E(x)^{-1}$). So by IVT it takes *every* value in $(0, \infty)$ (Ex!).

We also showed it is strictly monotonically increasing ($E(y) = E(y - x)E(x) > E(x)$ for $y > x$). So by previous result it's a bijection to $(0, \infty)$ with a continuous inverse.

Theorem 3.31

Lecture 25 $f : \mathbb{R}^n \to \mathbb{R}^m$ is cts at $\mathbf{a} = (a_1, \dots, a_n)$ if and only if $f_i : \mathbb{R}^n \to \mathbb{R}$ is cts at a_i $\forall i$. (With $f = (f_1, \dots, f_m)$).

(i.e. f_i is $\pi_i \circ f$ where $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is the projection to the ith coordinate $\pi_i(x_1, \dots, x_m) = x_i$.)

PROOF. Easy way is $\implies$:

HIGHBROW: $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is continuous, so $\pi_i \circ f - f_i$ is continuous.

FIRST PRINCIPLES: Fix $\epsilon > 0$. Then f is cts at $\vec{a} \implies \exists \delta > 0$ such that $|\vec{x} - \vec{a}| < \delta \implies |f(\vec{x}) - f(\vec{a})| < \epsilon$ $(*)$. But this implies $|f_i(\vec{x}) - f_i(\vec{a})| < \epsilon$ because

$$\begin{aligned} |f(\vec{x}) - f(\vec{a})| &= \sqrt{\sum_{j=1}^{m} (f_j(\vec{x}) - f_j(\vec{a}))^2} \\ &\geq \sqrt{(f_i(\vec{x}) - f_i(\vec{a}))^2} \\ &= |f_i(\vec{x}) - f_i(\vec{a})| \end{aligned}$$

PROOF OF $\impliedby$:

Suppose f_i cts at a_i $\forall i$. Fix $\epsilon > 0$. Then $\exists \delta_i > 0$ such that

$$|\vec{x} - \vec{a}| < \delta_i \implies |f_i(\vec{x}) - f_i(\vec{a})| < \epsilon$$

Set $\delta = \min\{(\delta_i\} > 0$, so that

$$\begin{aligned}|\vec{x} - \vec{a}| < \delta &\implies |f_i(\vec{x}) - f_i(\vec{a})| < \epsilon \ \forall i \\ \implies |f(\vec{x}) - f(\vec{a})| &= \sqrt{\sum_{i=1}^{m} (f_i(\vec{x}) - f_i(\vec{a}))^2} \\ &\leq \sqrt{\sum_{i=1}^{m} \epsilon^2} = \sqrt{m}.\epsilon\end{aligned}$$

∎

So we can study the continuity of $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of their coordinates f_i in $\mathbb{R}^m$. But *not* in terms of the restoration of f to coordinate axises in $\mathbb{R}^n$.

Example 3.32. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

On any horizontal line $y = c$ it results to the function

$$f(x, c) = \frac{cx}{c^2 + x^2} \quad \text{if } c \neq 0$$

$$\text{or } f(x, 0) \equiv 0 \ \forall x \quad \text{if } c = 0$$

Both are continuous functions $\mathbb{R} \to \mathbb{R}$.

Similarly on any vertical line $x = c$, f restricts to a continuous function:

$$f(c, y) = \frac{cy}{c^2 + y^2} \quad \text{if } c \neq 0$$

$$\text{or } f(0, y) \equiv 0 \ \forall y \quad \text{if } c = 0$$

But f is *not* continuous at $(0, 0)$

Idea: on line $y = x$, f is $\begin{cases} \dfrac{x.x}{x^2 + x^2} = \frac{1}{2} & \forall x \neq 0 \\ 0 & x = 0 \end{cases}$

Pick $\epsilon = \frac{1}{2}$. Then for any $\delta > 0$, take $x = \frac{\delta}{2}$ so that $(x, x) \in B_\epsilon(0, 0)$. But $f(x, x) = \frac{1}{2} \notin B_\epsilon(f(0, 0)) = B_\epsilon(0)$. So f is not continuous at $(0, 0)$. ∎

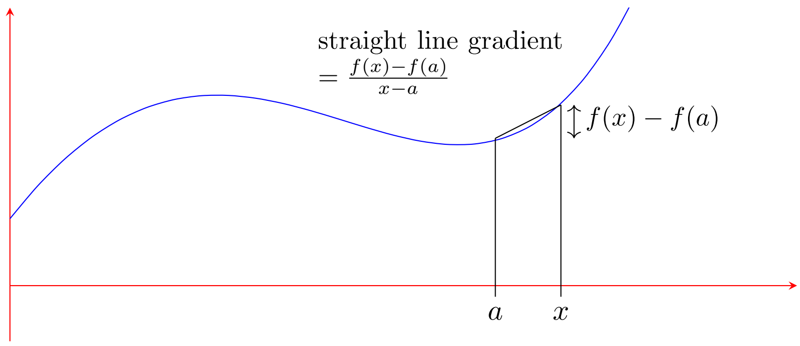
Ex: Converse is true: if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous then f is continuous on restriction to any line in $\mathbb{R}^n$; more generally $f|_S : S \to \mathbb{R}^m$ is continuous $\forall S \subseteq \mathbb{R}^n$

4 Differentiation

Differentiability

Definition. f is *differentiable* at a iff $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$, i.e.

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon.$$



Example 4.1. $f(x) = x^2$ is differentiable at all $a \in \mathbb{R}$ with $f'(a) = 2a$

Proof. Fix $a \in \mathbb{R}$

$$\frac{f(x)-f(a)}{x-a} = \frac{x^2-a^2}{x-a} = x + a$$

$$\implies \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists and equals } 2a$$

∎

or from first principles:

$$\left|\frac{f(x)-f(a)}{x-a} - 2a\right| = |x+a-2a| = |x-a|$$

So fixing $\epsilon > 0$, take $\delta = \epsilon$ so that $|x-a| < \delta \implies \left|\frac{f(x)-f(a)}{x-a} - 2a\right| < \epsilon$ ∎

Exercise: $f(x) = x^3$, $f(x) = |x|$

Lecture 26 **Proposition 4.2.** *If f is differentiable at $a \in \mathbb{R}$ then $f$is continuous at a*

Proof. If f is differentiable at a then

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < \epsilon$$
$$\implies |f(x)-f(a)| < |x-a|(|f'(a)| + \epsilon).$$

Fix $\epsilon > 0$, set $\delta = \epsilon$. Then

$$0 < |x-a| < \delta \implies |f(x)-f(a)| < \epsilon(|f'(a)| + \epsilon) = k\epsilon$$

(also true for $x = a \implies |f(x)-f(a)| = 0$.) ∎

Highbrow Proof. Note that $f(x) = f(a) + (x-a)\frac{f(x)-f(a)}{x-a}$, $x \neq a$. Taking $\lim_{x\to a}$

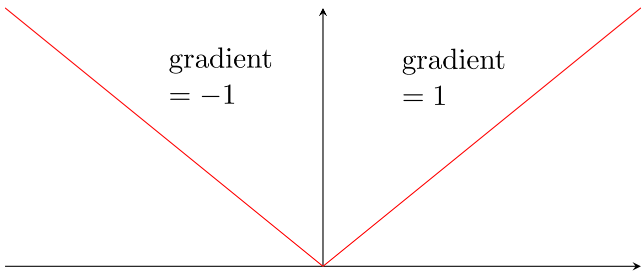
$$\lim_{x\to a} f(x) = f(a) + 0.f'(a) \implies f \text{ cts at } a \qquad \blacksquare$$

The converse is *not* true.

Example 4.3. $f(x) = |x|$ is continuous at $x = 0$ but not diff'ble at $x = 0$ since

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

So $\lim_{x\to o} \frac{f(x)-f(0)}{x-0}$ does not exist (Ex)



So left and right derivates do exist, they're just not equal.

Definition. Left derivative of f at a is $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ iff it exists. Right derivative is $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$.

$\lim_{x\to a^-} g(x)$ exists and equals $\lim_{x\to a^+} g(x) \iff \lim_{x\to a} g(x)$ exists.

So f is differentiable at a iff the left and right derivatives of f exist at a and are equal.

Anything else you might guess is also false: e.g. "if f is differentiable everywhere then is f' continuous?" No!

Theorem 4.4: Product Rule

$f, g : \mathbb{R} \to \mathbb{R}$ differentiable at $a \in \mathbb{R}$. Then fg is differentiable at a with $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

Proof.

$$\begin{aligned} \frac{f(x)g(x) - f(a)g(a)}{x-a} &= \frac{(f(x)-f(a))g(x) + (g(x)-g(a))f(a)}{x-a} \\ &= g(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{g(x)-g(a)}{x-a} \end{aligned}$$

Taking $\lim_{x\to a} \implies (fg)'(a) = g(a)f'(a) + f(a)g'(a)$ by cty of g and algebra of limits. $\blacksquare$

Corollary 4.5. *$f(x) = x^k$ has $f'(x) = kx^{k-1}$*

PROOF. Induction! ∎

Then $g(x) := 1/f(x)$ is defined in a neighbourhood of a, and it is differentiable with $g'(a) = \dfrac{f'(a)}{f^2(a)}$

PROOF. See old question sheet.

f is continuous at $a \implies \exists\delta > 0$ s.t. $\forall x \in (a-\delta, a+\delta),\ |f(x)| > \frac{|f(a)|}{2}$. So g is defined on $(a-\delta, a+\delta)$.

Working on this and $(a-\delta, a+\delta) \ni x$ we calculate

$$\begin{aligned}\frac{g(x)-g(a)}{x-a} &= \frac{1/f(x) - 1/f(a)}{x-a} \\ &= \frac{f(a)-f(x)}{(x-a)f(a)f(x)} \\ &\to -f'(a)\cdot\frac{1}{f(a)f(a)} \text{ as } x \to a\end{aligned}$$

∎

Example 4.6. $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x \in \mathbb{R}$

If we could differentiate term by term we would conclude that

$$E'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (k = n-1)$$

So Mestel *guesses* that $E' = E$

Claim: $E'(0) = 1$

PROOF.

$$\begin{aligned}\frac{E(x)-E(0)}{x-0} &= \frac{\sum \frac{x^n}{n!}}{x} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} \quad (k = n+1)\end{aligned}$$

Now by the comparison test

$$\sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} \leq \sum_{k=1}^{\infty} |x^k| = \frac{|x|}{1-|x|} \to 0$$

So $\lim_{x\to 0} \dfrac{E(x)-E(0)}{x-0}$ exists and equals 1. ∎

So now we have

Proposition 4.7. *E is differentiable everywhere with $E' = E$*

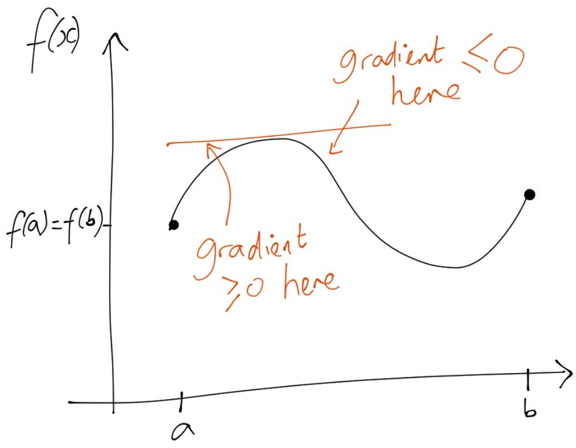
PROOF.
$$\begin{aligned}\frac{E(x)-E(a)}{x-a} &= E(a)\cdot\frac{E(x-a)-E(a)}{x-a} \\ &\to E(a)E'(0) \\ &= E(a)\end{aligned}$$

■

Rolle's Theorem

Theorem 4.8: Rolle's Theorem

$f : [a,b] \to \mathbb{R}$ cts on $[a,b]$, differentiable on (a,b) such that $f(a) = f(b)$. Then $\exists c \in (a,b)$ such that $f'(c) = 0$. Lecture 27



PROOF.

Case 1. f is constant on $[a,b]$. Then set $c = \frac{a+b}{2}$, so $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = 0$.

Case 2. f takes values $< f(a)$. Then replace f by $-f$ and consider Case 3.

Case 3. f takes values $> f(a)$. Therefore $\sup\{f(x) : x \in [a,b]\} > f(a)$ by EVT is realised by some $c \in (a,b)$. Now $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. Consider

$$x > c,\ f(x) \le f(c) \implies \frac{f(x)-f(c)}{x-c} \le 0 \implies \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} \le 0$$

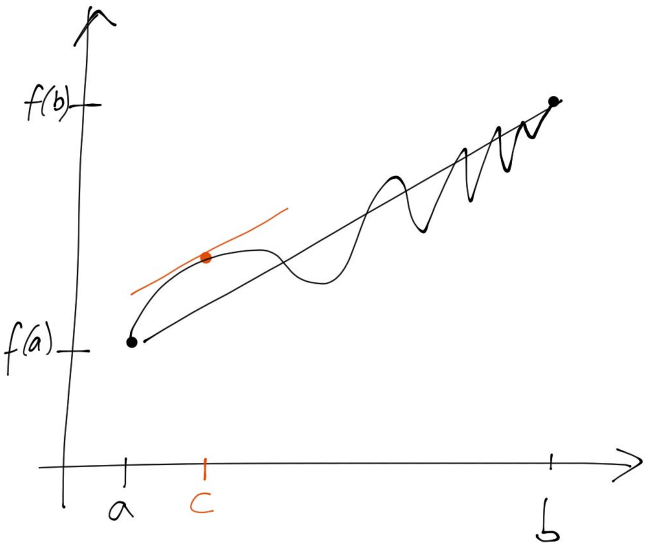
$$x < c,\ f(x) \le f(c) \implies \frac{f(x)-f(c)}{x-c} \ge 0 \implies \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} \ge 0$$

Hence $\frac{f(x)-f(c)}{x-c} = 0$. ■

Mean Value Theorem

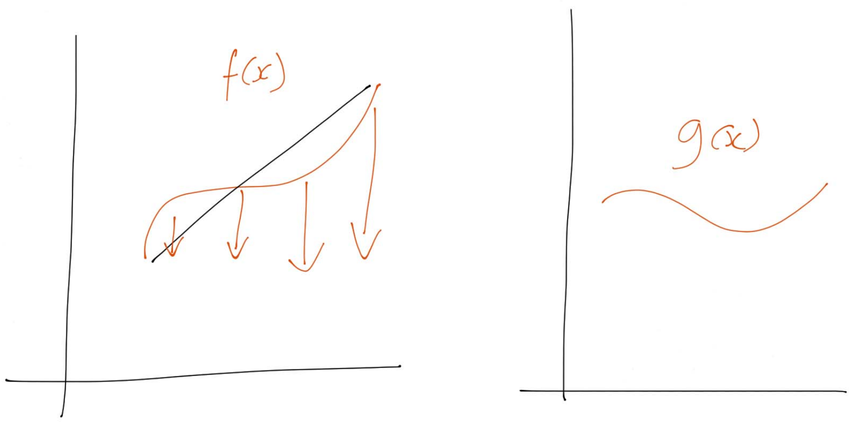
Theorem 4.9: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is cts on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Note: we can write this as $f(b) = f(a)+(b-a)f'(c)$, $c \in (a, b)$. Compare this to Taylor's Theorem - we're taking just the first 2 terms of.

Idea of Proof: Turn MVT into Rolle.



Proof. Let $g(x) = f(x)-\dfrac{f(b) - f(a)}{b - a}(x-a)$, which is cts on $[a, b]$ and diff'ble on (a, b). $g(a) = f(a) = g(b)$. By Rolle's Theorem applied to g

$$\exists c \in (a, b) \text{ such that } g'(c) = 0 \implies g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}. \qquad \blacksquare$$

Corollary 4.10. *If $f'(x) = 0\ \forall x \in (a,b)$. Then f is a constant: $f(x) = f(a)\ \forall x \in [a,b]$*

PROOF. Suppose for a contradiction that $\exists d \in [a,b]$ s.t. $f(d) \neq f(a)$. Then by MVT applied to $f|_{[a,d]} : [a,d] \to \mathbb{R}$, $\exists c \in (a,d)$ s.t. $f'(c) = \dfrac{f(d)-f(a)}{d-a} \neq 0$, ※ ∎

Theorem 4.11: Chain Rule

$g : \mathbb{R} \to \mathbb{R}$ diff'ble at $a \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ diff'ble at $g(a) \in \mathbb{R}$, then $f \circ g$ diff'ble at a with $(f \circ g)'(a) = f'(g(a))g'(a)$

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i.e.

$$\frac{d}{dx} f(g(x))\bigg|_{x=a} = \frac{df}{dx}(g(a))\frac{dg}{dx}(a) = \frac{df}{dy}\bigg|_{y=g(a)} \frac{dg}{dx}(a) \text{``}=\text{''} \frac{df}{dg}\frac{dg}{dx}$$

Idea of proof:

$$\frac{f(g(x)) - f(g(a))}{x-a} = \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a} \to f'(g(a)) \cdot g'(a)$$

problem with this is that $g(x) - g(a)$ might be zero

$\left(\dfrac{h(x)-h(a)}{x-a} \text{ is not defined at } x = a, \text{ so define it to be } h'(a) \text{ at } x = a\right)$

PROOF. Define $F(g) = \begin{cases} \dfrac{f(y)-f(b)}{g-b} & y \neq b \\ f'(g) & y = b \end{cases}$ (‡) where $b = g(a)$.

f is diff'ble at $b \implies \lim_{y \to b} F(y) \to f'(b) = F(b)$ as $y \to b$. So F is cts at $b = g(a)$ $(*)$.

g is diff'ble at $a \implies$ cts at a.

By $(*) \implies F \circ g$ is cts at $a \implies F(g(x)) \to F(g(a)) = f'(b)$ as $x \to a$ $(**)$.

So now we can follow the rough proof to write

$$\frac{f(g(x)) - f(g(a))}{x} = F(g(x))\frac{g(x)-g(a)}{x-a}$$

Now take $\lim_{x \to a}$ to get $(f \circ g)'(a)$ exists and equals $f'(b)g'(a)$ by $(**)$ ∎

Ex: "Sum Rule" f, g are differentiable at $a \implies f + g$ are differentiable at a with $(f+g)'(a) = f'(a) + g'(a)$. Pre-ex: Algebra of limit for $\lim_{x \to a}$ is on Question Sheet.

Rough: $f : \mathbb{R} \to \mathbb{R}$ is differentiable and bijective, $g = f^{-1} : \mathbb{R} \to \mathbb{R}$.

Suppose g is differentiable. Then by the chain rule $f \circ g(y) = y \implies f'(g(y_0))g'(y_0) = 1\ \forall y_0 \implies g'(y) = \frac{1}{f'(g(y))}$.

Suggests that if $f' \neq 0$, then g is differentiable with derivative $\frac{1}{f' \circ g}$

Theorem 4.12

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ with $f'(a) \neq 0$ and f is bijective with inverse $g = f^{-1}$, then g is differentiable at $b = f(a)$ with $g'(b) = \dfrac{1}{f'(g(b))} = \dfrac{1}{f'(a)}$.

PROOF. *Lemma:* $f'(a) \neq 0 \implies \exists \delta > 0$ such that $f(x) \neq f(a)$ for $x \in (a-\delta, a+\delta) \setminus \{0\}$. (Proof is left as exercise - use $\lim_{x \to a}$ definition of f' and MVT)

So $\dfrac{g(y) - g(b)}{y - b} = \dfrac{x - a}{f(x) - f(a)} = 1/\dfrac{f(x) - f(a)}{x - a}$ where $x = g(y),\ y \neq b$.

As $y \to b$, $g(y) \to g(b) = a$ since f differentiable at $a \implies f$ cts at $a \implies g$ cts at $b \implies x \to a \implies$ RHS $\to \dfrac{1}{f'(a)}$. ∎

Felina. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

- $f(x) + f(y) = f(x + y)\ \forall x, y \in \mathbb{R}$
- f is continuous everywhere

What if f?

Observe $y = 0 : f(x) + f(0) = f(x),\ \forall x$, so $f(0) = 0$.

For $y = 1 : f(x) + f(1) = f(x + 1)$

$$\begin{aligned} \text{Induction } f(x+2) &= f(x) + f(1) + f(1) \\ f(x+3) &= f(x) + 3f(1) \\ &\vdots \\ f(x+n) &= f(x) + nf(1) \\ \implies f(n) &= nf(1)\ (*) \end{aligned}$$

Similar mucking about should convince you that $f(x) = xf(1)$. We've proved that for $x \in \mathbb{N}$ by $(*)$. $f(1)$ is an unknown constant c. [Notice $f(x) = cx$ indeed satisfies the given assumptions]

Lecture 29 Notice that $(*)$ holds for $n \in \mathbb{Z}$ too

$$f(-n) + f(n) = f(n - n) = f(0) = 0$$

$$\implies f(-n) = -f(n) = -nf(1) = -nc, \quad n \in \mathbb{N}$$

$(*)$ also holds for $\mathbb{Q}$

$$\underbrace{f(\tfrac{n}{m}) + \cdots + f(\tfrac{n}{m})}_{m \text{ copies}} = f(\tfrac{n}{m} + \cdots + \tfrac{n}{m}) = f(n) = cn$$

$$\implies f(\tfrac{n}{m}) = c\tfrac{n}{m} \quad \forall \tfrac{n}{m} \in \mathbb{Q},\ n, n \in \mathbb{Z}$$

Claim: $f(x) = cx$ $[c = f(1)]$ $\forall x \in \mathbb{Q}$

Idea: now is if $x \in \mathbb{R}$ then x is close to $y \in \mathbb{Q}$. f is continuous $\implies f(x)$ is close to $f(y) = cy$, close to cx. So $f(x)$ is arbitrarily ($\forall\epsilon$!) close to $cx \implies f(x) = cx$

(or we could use some machinery to say $\forall x \in \mathbb{R}, \exists (y_n) \to x$, $y_n \in \mathbb{Q}$. Then f is continuous $\implies f(y_n) = cy_n \to f(x)$ and $cy_n \to cx$. So uniqueness of limits $\implies f(x) = cx$.)

Proof. Fix $x \in \mathbb{R}$. Fix $\epsilon > 0$. M1F: $\exists y \in \mathbb{Q}$ s.t. $|y - x| < \epsilon \implies |cy - cx| < \epsilon/2$

$$\exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon/2$$

and by M1F again $\exists y \in \mathbb{Q}$ s.t. $|y - x| < \min\{\delta, \epsilon/2\}$.

So $|cy - cx| < \epsilon/2$ and $|f(x) - \underset{\underset{cy}{\|}}{f(y)}| < \epsilon/2 \implies |f(x) - cx| < 2\epsilon/2 = \epsilon$.

This is true $\forall \epsilon > 0 \implies |f(x) - cx| = 0$ ∎

- End of Analysis I -