1st Year Mathematics
Imperial College London

Spring 2015

Analysis I

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Syllabus

A rigorous treatment of the concept of a limit, as applied to sequences, series and functions.

Sequences

Real and complex sequences. Convergence, divergence and divergence to infinity. Sums and products of convergent sequences. The Sandwich Test. Sub-sequences, monotonic sequences, Bolzano-Weierstrass Theorem. Cauchy sequences and the general principle of convergence.

Series

Real and complex series. Convergent and absolutely convergent series. The Comparison Test for non-negative series and for absolutely convergent series. The Alternating Series Test. Rearranging absolutely convergent series. Radius of convergence of power series. The exponential series.

Continuity

Limits and continuity of real and complex functions. Left and right limits and continuity. Maxima and minima of real valued continuous functions on a closed interval. Inverse Function Theorem for strictly monotonic real functions on an interval.

Differentiability

An introduction to differentiability: definitions, examples, left and right derivative.

Appropriate books

K. G. Binmore, *Mathematical Analysis, A Straightforward Approach* (Cambridge University Press).

W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill) (not recommended as an introduction)

Contents

0 Preliminaries

Lecture 1

M1F stuff:

- $\forall -$ for any, **fix any**, for all, every...
- $\exists -$ there exists
- $\mathbb{N} = \{1, 2, 3, \dots\}$

Theorem 0.1: Triangle Inequality

(See Question Sheet 1)

$$|a + b| \leq |a| + |b|$$

Corollary 0.2.

$$\big||a| - |b|\big| \leq |a - b|$$

Proof.

$$\begin{aligned} |a - b| < \epsilon &\iff b - \epsilon < a < b + \epsilon \\ &\iff a \in (b - \epsilon, b + \epsilon) \\ &\iff b \in (a - \epsilon, a + \epsilon) \\ &\implies \big||a| - |b|\big| < \epsilon \end{aligned}$$

∎

Lots of other versions, see Question Sheet 1 - *don't try to memorise them!*

Clicker Question 0.3. Fix $a \in \mathbb{R}$. What does the statement

$$\forall \epsilon > 0, \ |x - a| < \epsilon \ (*)$$

mean for the number x?

Answer: $x = a$.

Proof. Assume $x \neq a$. Take $\epsilon := \frac{1}{2}|x - a| > 0$. Then $(*)$ does not hold. ∎

1 Sequences

Lecture 2

A sequence $(a_n)_{n\geq 1}$ of real (or complex, etc.) numbers is an infinite list of numbers $a_1,\ a_2,\ a_3,\ldots$ all in $\mathbb{R}$ (or $\mathbb{C}$, etc.) Formally:

> **Definition.** A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$

Notation: We let $a_n \in \mathbb{R}$ denote $a(n)$ for $n \in \mathbb{N}$. The data $(a_n)_{n=1,2,\ldots}$ is equivalent to the function $a : \mathbb{N} \to \mathbb{R}$ because a function a is determined by its values a_n over all $n \in \mathbb{N}$.

We will denote a by $a_1,\ a_2,\ldots$ or $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n\geq 1}$ or even just (a_n).

Remark. a_i's could be repeated, so (a_n) is *not* equivalent to the set $\{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$. E.g. $(a_n) = 1,\ 0,\ 1,\ 0,\ldots$ is different from $(b_n) = 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1,\ldots$

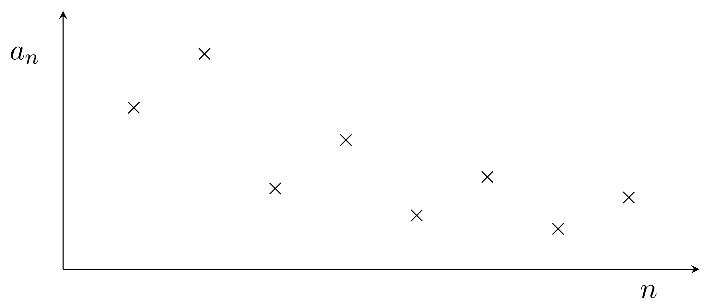
We can describe a sequence in may ways, e.g. formula for a_n as above $a_n = \frac{1-(-1)^n}{2}$, or a recursion e.g. $c_1 = 1 = c_2$, $c_n = c_{n-1} + c_{n-2}$ for $n \geq 3$, or a summation (see next section) e.g. $d_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Convergence of Sequences

We want to *rigorously* define $a_n \to a \in \mathbb{R}$, or "a_n converges to a as $n \to \infty$" or "a is the limit of (a_n)".

Idea: a_n should get closer and closer to a. Not necessarily monotonically, e.g.:

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{2n} & n \text{ even} \end{cases} \qquad \text{we want } a_n \to 0$$

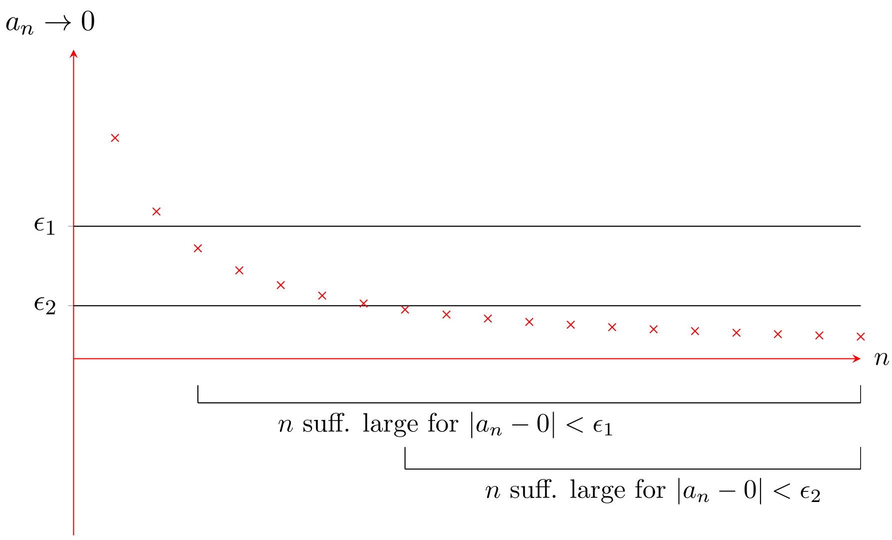


Also notice that $\frac{1}{n}$ gets closer and closer to -1! So we want to say instead that a_n gets *as close as we like to a*. We will measure this with $\epsilon > 0$. We phrase "a_n gets *arbitrarily* close to a" by "a_n gets to within ϵ of a for *any* $\epsilon > 0$".

> **Definition** (Mestel). $u_n \to u$ if $\forall n$ sufficiently large, $|u_n - u|$ is *arbitrarily small.*
>
> Define a real number $b \in \mathbb{R}$ to be arbitrarily small if it is smaller than any $\epsilon > 0$ i.e. $\forall \epsilon > 0,\ |b| < \epsilon$.

Definition Mestel says that once n is large enough, $|u_n - u|$ is less than every $\epsilon > 0$, i..e it's zero, i.e. $u_n = u$. We want to *reverse* the order of specifying n and ϵ.

i.e. we want to say that to get *arbitrarily close to the limit* a (i.e. $|a_n - a| < \epsilon$), we need to go sufficiently far down the sequence. Then if I change $\epsilon > 0$ to be smaller, I simply go further down the sequence to get within ϵ of a.



There will not be a "n sufficiently large" that works for all ϵ at once! (unless $a_n = a$ eventually.)

But for *any* (fixed) $\epsilon > 0$ we want there to be an n sufficiently large such that $|a_n - a| < \epsilon$. So we change "$\exists n$ such that $\forall \epsilon$" to "$\forall \epsilon$, $\exists n$.". *This allows n to depend on ϵ.*

Definition (Nestel)**.** $a_n \to a$ if $\forall \epsilon > 0,\ \exists n \in \mathbb{N}$ such that $|a_n - a| < \epsilon$.

e.g.

$$a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \quad \text{satisfies } a_n \to 0 \text{ according to Prof. Nestel.}$$

We want to modify this to say eventually $|a_n - a| < \epsilon$ *and it stays there!*

Lecture 3

Definition (Actual Convergence)**.** We say that $a_n \to a$ iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that "} n \geq N \implies |a_n - a| < \epsilon \text{"}$$

This says that *however close* ($\forall \epsilon > 0$) I want to get to the limit a, there's a point in the sequence ($\exists N \in \mathbb{N}$) beyond which ($n \geq N$) my a_n is indeed that close to the limit a ($|a_n - a| < \epsilon$).

Remark. N depends on ϵ! $N = N(\epsilon)$

Equivalently:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that "} \forall n \geq N,\ |a_n - a| < \epsilon \text{"}$$

or equivalently

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon,\ \forall n \geq N_\epsilon$$

Clicker Question 1.1. Given a sequence of real numbers $(a_n)_{n\geq 1}$. Conisder

$$\boxed{\forall n \geq 1,\ \exists \epsilon > 0 \text{ such that } |a_n| < \epsilon}$$

This means?

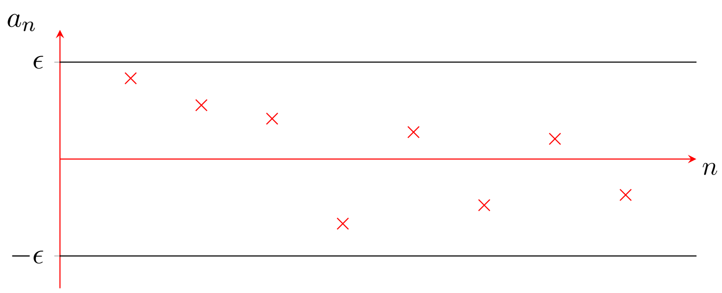
Answer: It always holds.

Proof. Fix any $n \in \mathbb{N}$. Take $\epsilon = |a_n| + 1$. ∎

What about

$$\boxed{\exists \epsilon > 0 \text{ such that } \forall n \geq 1,\ |a_n| < \epsilon}$$

Answer: (a_n) is bounded.

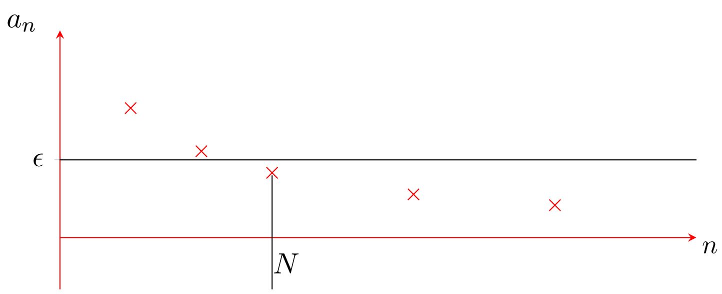


Proof. $\iff a_n \in (-\epsilon, \epsilon)\ \forall n \iff |a_n|$ is bounded by ϵ. ∎

Definition. If a_n does not converge to a for any $a \in \mathbb{R}$, we say that a_n *diverges.*

Example 1.2. I claim that $\frac{1}{n} \to 0$ as $n \to \infty$

Rough working: Fix $\epsilon > 0$. I want to find $N \in \mathbb{N}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N$.



Since $a_n = \frac{1}{n}$ is monotonic, it is *sufficient* to ensure that $\frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$ (This *implies* $\frac{1}{n} \leq \frac{1}{N} < \epsilon,\ \forall n \geq N$).

Proof. Fix $\epsilon > 0$. Pick any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. (This is the Archimedean axiom of $\mathbb{R}$. Notice N depends on ϵ!!). Then $n \geq N \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. ∎

Method to prove $a_n \to a$

(I) Fix $\epsilon > 0$

(II) Calculate $|a_n - a|$

(II′) Find a good estimate $|a_n - a| < b_n$

(III) Try to solve $a_n - a < b_n < \epsilon \quad (*)$

(IV) Find $N \in \mathbb{N}$ s.t. $(*)$ holds whenever $n \geq N$

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see examples below)

Example 1.3. I claim that $\frac{1}{n} \to 0$ as $n \to \infty$

Rough working: Fix $\epsilon > 0$. I want to find $N \in \mathbb{N}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N$.

Since $a_n = \frac{1}{n}$ is monotonic, it is *sufficient* to ensure that $\frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$ (This *implies* $\frac{1}{n} \leq \frac{1}{N} < \epsilon,\ \forall n \geq N$).

Proof. Fix $\epsilon > 0$. Pick any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. (This is the Archimedean axiom of $\mathbb{R}$. Notice N depends on ϵ!!). Then $n \geq N \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. ∎

Lecture 4

We can also define limits for *complex sequences.*

Definition. $a_n \in \mathbb{C},\ \forall n \geq 1$. We say $a_n \to a \in \mathbb{C}$ iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon$$

(i.e. $\sqrt{\Re(a_n - a)^2 + \Im(a_n - a)^2} < \epsilon$)

This is equivalent (see problem sheet!) to $(\Re a_n) \to \mathfrak{a}$ *and* $(\Im a_n) \to \Im a$

Example 1.4. I claim that $\frac{1}{n} \to 0$ as $n \to \infty$

Rough working: Fix $\epsilon > 0$. I want to find $N \in \mathbb{N}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N$.

Since $a_n = \frac{1}{n}$ is monotonic, it is *sufficient* to ensure that $\frac{1}{N} < \epsilon \iff N > \frac{1}{\epsilon}$ (This *implies* $\frac{1}{n} \leq \frac{1}{N} < \epsilon,\ \forall n \geq N$).

Proof. Fix $\epsilon > 0$. Pick any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. (This is the Archimedean axiom of $\mathbb{R}$. Notice N depends on ϵ!!). Then $n \geq N \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. ∎

Theorem 1.5

Convergence $\implies$ Bounded

Proof. Fix $\epsilon = 1$. Then (a_n) is bounded by $\max\{a_1, a_2, \ldots, a_{N-1}, a+1\}$. ∎

Theorem 1.6

Monotonic and Bounded $\implies$ Convergent

Proof. Set $a = \sup a_n$. $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $a_N \in (a - \epsilon, a)$.
Monotonic so

$$\forall n \geq N,\ a_n \geq a_N \implies |a_n - a| < \epsilon$$

∎

Theorem 1.7

(Algebra of Limits) $a_n \to a$ and $b_n \to b$ then:

(i) $a_n + b_n \to a + b$

(ii) $a_n b_n \to ab$

(iii) $\frac{a_n}{b_n} \to \frac{a}{b}$ $(b \neq 0)$

Proof. $\forall \epsilon > 0,\ \exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1,\ |a_n - a| < \epsilon$ and $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2,\ |b_n - b| < \epsilon$. Set $N = \max\{N_1, N_2\}$ Then:

1. $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < 2\epsilon$
2. $|a_n b_n| \leq |a_n - a||b_n| + |b_n - b||a| < k\epsilon$
3. $|\frac{a_n}{b_n} - \frac{a}{b}| \leq \frac{|a_n - a||b|}{|b||b_n|} + \frac{|b_n - b||a|}{|b||b_n|} < k\epsilon$ ∎

Cauchy Sequences

Definition. A sequence is Cauchy iff

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N,\ |a_n - a_m| < \epsilon$$

Theorem 1.8

Convergence $\implies$ Cauchy

Proof. $\forall\epsilon > 0,\ \exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1,\ |a_n - a| < \epsilon$ and $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2,\ |a_m - a| < \epsilon$.
Set $N = \max\{N_1, N_2\}, \forall n, m \geq N : |a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon$. ∎

Theorem 1.9

Cauchy $\implies$ Convergence

Proof (1.) Fix $\epsilon > 0$. $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$:
$a_m \in (a_n - \frac{\epsilon}{2}, a_n + \frac{\epsilon}{2}) \implies a_n - \frac{\epsilon}{2} < a_m < a_n + \frac{\epsilon}{2}$.
Set $b_i := \sup\{a_m : m \geq i \geq N\} \implies a_n - \frac{\epsilon}{2} < b_i \leq a_n + \frac{\epsilon}{2}$.
Set $a := \inf\{b_i : i \geq N\} \implies a_n - \frac{\epsilon}{2} \leq a \leq a_n + \frac{\epsilon}{2}$
$\implies |a_n - a| \leq \frac{\epsilon}{2} < \epsilon$. ∎

Proof (2.) Bounded by $\{a_1, \ldots, a_{N-1}, a_{N+1}\}$ so by BW Theorem, exists a convergent subsequence $a_k \to a$:
$\forall\epsilon > 0,\ \exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1,\ |a_k - a| < \epsilon$ and $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2,\ |a_n - a_k| < \epsilon$.
Set $N = \max\{N_1, N_2\}, \forall n, k \geq N$: $|a_n - a| \leq |a_n - a_k| + |a_k - a| < 2\epsilon$. ∎

Subsequences

Theorem 1.10: Bolzano-Weierstrass

Bounded $\implies$ Convergent Subsequence

Proof (1.) If finitely many peak points, we can set $y_i := x_{N(i)}\ \forall i \in \mathbb{N}$ such that $\forall i > j,\ N(i) > N(j)$ and $x_{N(i)} > x_{N(j)}$. So y_i is monotonic increasing $\implies$ convergent.

If infinitely many peak points, we can set $y_i := x_{N(i)}\ \forall i \in \mathbb{N}$ such that $\forall i > j,\ N(i) > N(j)$ and $x_{N(i)} < x_{N(j)}$. So y_i is monotonic decreasing $\implies$ convergent. ∎

Proof (2.) Definte the subsequence of a_n, $a_n(i)$ to be the set of points $a_n \in [A_i, B_i] \subseteq [A_j, B_j]\ \forall i > j$, where $[A_i]$ is the interval defined:

- $[A_0, B_0] = [-R, R]$, with R being the bounds of a_n.
- $[A_i, B_i]$ is one of the sets $[A_{i-1}, \frac{A_{i-1}, Bi-1}{2}]$ or $[\frac{A_{i-1}, Bi-1}{2}, Bi - 1]$ which contains infinitely many points.

Then our convergent subsequence is $(b_i)_{i\in\mathbb{N}} = a_{n_{i-1}(i)}$, valid since $n_i(i+1) > n_{i-1}(i)\ \forall i \in \mathbb{N}$. **Claim:** (b_i) is convergent.

Proof. Fix $\epsilon > 0$. Take $N_\epsilon > \frac{2R}{\epsilon}$, so that $\frac{2R}{2^{N_\epsilon}} < \frac{2R}{N_\epsilon} < \epsilon$. Then $\forall i, j \geq N_\epsilon\ |b_i - b_j| < \frac{2R}{2^{N_\epsilon}} < \epsilon$ since $b_i, b_j \in [A_{N_\epsilon}, B_{N_\epsilon}] \implies (b_i)$ Cauchy $\implies$ convergent. ∎

2 Series

Convergence of Series

Definition. $\sum a_n = A \in \mathbb{R}$ if and only if the partial sums converge, so

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ \left|\sum_{k=1}^{n} a_k - A\right| < \epsilon$$

Theorem 2.1

$\sum a_n \to a \implies a_n \to 0$

Proof. $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ s.t. $\left|\sum_{i=1}^{n} a_i - a\right| < \epsilon$ and $\left|\sum_{i=1}^{n+1} a_i - a\right| < \epsilon$

$$\implies |a_{n+1} - a| \leq \left|\sum_{i=1}^{n} a_i - a\right| + \left|\sum_{i=1}^{n+1} a_i - a\right| < 2\epsilon$$

∎

Theorem 2.2

(S_n) bounded and $a_n \geq 0 \implies \sum a_n$ convergent.

Proof. Let $a = \sup S_n$. $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $S_n \in (a - \epsilon, a)$

$$a_n \geq 0 \implies \forall n \geq N\ S_n \geq S_N \implies |S_n - a| < \epsilon$$ ∎

Tests for convergence

Lecture 5

Theorem 2.3: Comparison I

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. (and $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$)

Proof. Call the partial sums A_n, B_n respectively. Then

$$0 \leq A_n \leq B_n \leq \sum_{i=1}^{\infty} b_i = \lim_{n\to\infty} B_n$$

So A_n is bounded and monotonically increasing $\implies$ convergent. ∎

Proposition 2.4. *Suppose $a_n \geq 0\ \forall n$. Then $\sum_{n=1}^{\infty} a_n$ converges iff $S_N = \sum_{n=1}^{N} a_N$ is bounded above and $\sum_{n=1}^{\infty} a_n$ diverges to ∞ (i.e. $S_n \to +\infty$ as $N \to \infty$) iff $S_N = \sum_{n=1}^{N} a_n$ is an unbounded sequence.*

Proof. $a_n \geq 0 \iff (S_n)$ is monotonic increasing. So (S_n) bounded $\iff$ convergent.

S_N unbounded $\iff \forall R > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ S_n > R \iff S_n \to +\infty$. ∎

Example 2.5. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha},\ \alpha > 1$ is convergent.

Proof. (Trick!) Arrange the partial sum as follows:

$$\begin{aligned} 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots = 1 &+ \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \cdots + \frac{1}{7^\alpha}\right) \\ &+ \left(\frac{1}{8^\alpha} + \cdots + \frac{1}{15^\alpha}\right) \\ &+ \left(\frac{1}{16^\alpha} + \cdots + \frac{1}{31^\alpha}\right) \\ &+ \dots \end{aligned}$$

Note that the kth bracketed term:

$$\left(\frac{1}{(2^k)^\alpha} + \cdots + \frac{1}{(2^{k+1}-1)^\alpha}\right) \leq \frac{1}{2^{k\alpha}} + \cdots + \frac{1}{2^{k\alpha}} = \frac{2^k}{2^{k\alpha}} = \frac{1}{2^{k(\alpha-1)}}$$

So any partial sum is less than some finite sum of these bracketed terms, i.e. for some sufficiently large N:

$$S_N < \sum_{k=0}^{N} \frac{1}{2^{k(\alpha-1)}} = \frac{1 - \frac{1}{2^{(N+1)(\alpha-1)}}}{1 - \frac{1}{2^{(\alpha-1)}}} \leq \frac{1}{1 - \frac{1}{2^{\alpha-1}}}$$

because $\alpha > 1$, so $\left|\frac{1}{2^{\alpha-1}}\right| < 1$, so denominator > 0.

So partial sums are bounded above $\implies$ convergent. ∎

Definition. Say that the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent

Example 2.6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent, but it is convergent.

Rough Working. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \dots$, the kth bracket $\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)}$. This is positive and $\leq \frac{1}{2k(2k-2)} = \frac{1/4}{k(k-1)}$, seen earlier sum of these is convergent.

So cancellation between consecutive terms is enough to make series converge by comparison with $\sum \frac{1}{k(k-1)}$.

Proof. Fix $\epsilon > 0$. Then use 2 things

(1) $\sum \frac{1}{2k(2k-1)}$ is convergent

(2) $\frac{(-1)^{n+1}}{n} \to 0$

By (1) $\exists N_1$ such that $\forall n \geq N_1$, $\sum_{n=1}^{\infty} \frac{1}{k(k-1)} < \epsilon$

(2) $\exists N_2$ such that $\forall n \geq N_2$, $\left|\frac{(-1)^{n+1}}{n}\right| < \epsilon$

Set $N = \max(N_1, N_2)$. Then $\forall n \geq N$, we have:

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \left(\frac{1}{2j-1} - \frac{1}{2j}\right) + \delta = \sum_{k=1}^{j} \frac{1}{2k(2k-1)} + \delta$$

where $\delta = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd.} \\ 0 & \text{if } n \text{ is even.} \end{cases}$ $\quad (j = \lfloor \frac{n}{2} \rfloor)\ j = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd.} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$

$$\implies S_n = \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} - \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} \frac{1}{2k(2k-1)} + \delta$$

$$\text{So } \left|S_n - \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)}\right| \leq \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} \frac{1}{2k(2k-1)} + \frac{1}{n} < \epsilon + \epsilon$$

for all $n \geq 2N$ (so that $\lfloor \frac{n}{2} \rfloor + 1 > N$) ∎

Theorem 2.7

If (a_n) is absolutely convergent, then it is convergent.

Proof. Let $S_n = \sum_{i=1}^{n} |a_i|$, $\sigma + n = \sum_{i=1}^{n} a_i$ be the partial sums.

We're assuming that S_n converges. Therefore S_n is Cauchy:

$$\forall \epsilon > 0\ \exists N_\epsilon \text{ such that } n > m \geq N_\epsilon \implies |S_n - S_m| < \epsilon \iff |a_{m+1} + \dots + |a_n| < \epsilon$$

i.e. the terms in the tail of the series contribute little to the sum

$\implies |a_{m+1} + \dots + a_n| < \epsilon$ by the triangle inequality $\implies |\sigma_n - \sigma_m| < \epsilon \implies (\sigma_n)$ is Cauchy $\implies \sum a_i$ is convergent. ∎

Example 2.8. $\sum_{n=1}^{\infty} z_n$ is convergent for $|z| < 1$, divergent for $|z| \geq 1$

Proof. $\sum_{n=1}^{\infty} z_n$ is absolutely convergent because we showed that $\sum_{n=1}^{\infty} |z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$

For $|z| \geq 1$, the individual terms z^n have $|z^n| \geq 1$, so $z^n \not\to 0$, so $\sum z^n$ divergent. ∎

Beware. Do not rearrange series and sum them in a different order unless you can prove the result is the same.

Theorem 2.9: Comparison II - Sandwich Test

Suppose $c_m \leq a_n \leq b_n$ and $\sum c_n,\ \sum b_n$ are both convergent. Then $\sum a_n$ is convergent.

Proof. Use Cauchy. $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n, m > N$

$$\left|\sum_{i=m+1}^{n} b_i\right| < \epsilon, \ \left|\sum_{i=m+1}^{n} c_i\right| < \epsilon$$

since the partial sums of $b_i,\ c_i$ are Cauchy. Therefore

$$-\epsilon < \sum_{i=m+1}^{n} c_i \leq \sum_{i=m+1}^{n} a_i \leq \sum_{i=m+1}^{n} b_i < \epsilon$$

$$\implies \left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i\right| < \epsilon \implies \left(\sum_{i=1}^{n} a_i\right) \text{ is Cauchy.}$$

∎

Theorem 2.10: Comparison III

If $\frac{a_n}{b_n} \to l \in \mathbb{R}$ then $\sum b_n$ absolutely convergent $\implies \sum a_n$ is absolutely convergent.

Proof. Pick $\epsilon = 1$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$:

$$\left|\frac{a_n}{b_n} - l\right| < 1 \implies \left|\frac{a_n}{b_n}\right| < |l| + 1 \implies |a_n| < (|l|+1)|b_n|$$

So now by the comparison test $\sum_{n \geq N} |b_n|$ convergent $\implies \sum_{n \geq N} |a_n|$ convergent $\implies \sum_{n \geq 1} |a_n|$ convergent. ∎

Theorem 2.11: Alternating Series Test.

Given an alternating sequence a_n where $a_{2n} \geq 0,\ a_{2n+1} \leq 0\ \forall n$. Then $|a_n|$ monotonic decreasing to $0 \implies \sum a_n$ convergent

Proof. Write $a_n = (-1)^n b_n,\ b_n \geq 0\ \forall n$. Consider the partial sums $S_n = \sum_{i=1}^n (-1)^n b_n$.

Observe that:

(1) $S_i \leq S_{2n}\ \forall i \geq 2n$

(2) $S_i \geq S_{2n+1}\ \forall i \geq 2n+1$

Since if $i = 2j$ is even, then

$$\begin{aligned} S_{2j} &= S_{2n} + a_{2n+1} + \cdots + a_{2j} \\ &= S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} \leq S_2 n \end{aligned}$$

If $i = 2j+1$ is odd, then similarly:

$$S_{2j} = S_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \cdots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} - b_{2j+1} \leq S_2 n$$

So now $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n \geq N,\ |b_n| < \epsilon$. So $\forall n, m \geq 2n$, we have:

$$S_{2N+1} \leq S_n,\ S_m \leq S_{2N}$$

$$\text{So } |S_n - S_m| \leq |S_{2N+1} - S_{2N}| = b_{2n+1} < \epsilon$$

■

Theorem 2.12: Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Proof. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon \implies |a_{n+1}| < (r+\epsilon)|a_n|$$

Set $\alpha := r + \epsilon = \frac{1+r}{2} < 1$.

Inductively

$$|a_{N+m}| < \alpha |a_{N+m-1}| < \cdots < \alpha^m |a_N|$$

So $\forall k \geq N$

$$|a_k| < \alpha^{k-N} |a_N| = C\alpha^k$$

Then

$$C \sum_{k=N}^{n} \alpha^k = \frac{C(\alpha^N - \alpha^n)}{1-\alpha} \to \frac{C'}{1-\alpha} \text{ as } n \to \infty, \text{ since } \alpha < 1$$

So by the comparison test $\sum_{k \geq N} |a_k|$ is convergent $\implies \sum_{k \geq 1} |a_k|$ is convergent ■

Theorem 2.13: Root Test

If $\lim_{n\to\infty} |a_n|^{1/n} = r < 1$, then $\sum a_n$ is absolutely convergent.

Proof. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\left| |a_n|^{1/n} - r \right| < \epsilon \implies |a_n|^{1/n} < r + \epsilon$$

Set $\alpha := r + \epsilon = \frac{1+r}{2} < 1$, so that $|a_n| < \alpha^n$. Then

$$\sum_{k=1}^{n} \alpha^k = \frac{\alpha(1-\alpha^n)}{1-\alpha} \to \frac{\alpha}{1-\alpha} \text{ as } n \to \infty \text{ since } \alpha < 1$$

So by the comparison test $\sum_{k\geq 1} |a_k|$ is convergent. ∎

∗Re-arrangement of Series∗

Cauchy Product

Theorem 2.14

(Cauchy Product) $\sum |a_n| \to a$ and $\sum |b_n| \to b$, then: $\sum |c_n| \to ab$ with $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.

Proof. See handout on blackboard. No need to reproduce in exam.

Radius of Convergence

Theorem 2.15: Radius of Convergence

For $\sum a_n z^n$, $\exists R \in [0, \infty]$ such that $|z| < R \implies$ absolute convergence

Proof. Define $R = \sup S = \{|z| : a_n z^n \to 0\}$ or $R = \infty$ if the set is empty. Suppose $|z| < R$. $|z|$ not an upperbound for $S \implies \exists w$ such that $|w| > |z|$ and $a_n w^n \to 0$. Then

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq A \left|\frac{z}{w}\right|^n$$

Since $\left|\frac{z}{w}\right| < 1 \implies \sum |a_n z^n|$ cvgt. Similarly $|z| > R \implies \sum |a_n z^n|$ divergent. ∎

3 Continuity

Continuity and Limits

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *continuous at* $a \in \mathbb{R}$ if and only if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

So δ depends on a, ϵ. "Once x is close to a, then $f(x)$ is close to $f(a)$".

More precisely: "However close (i.e. within ϵ) I want $f(x)$ to be to $f(a)$, I can arrange it by taking x close (i.e. within δ) to a".

Equivalently: $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f(x) - f(a)| < \epsilon\ \forall x$ with $|x - a| < \delta$

Or: $\forall \epsilon > 0,\ \exists \delta > 0$ such that $f(a - \delta, a + \delta) \subseteq (f(a) - \epsilon, f(a) + \epsilon)$

Where $S \subseteq R$ then $f(S)$ is the set $\{f(x) : x \in S\}$

Or: $\forall \epsilon,\ \exists \delta > 0$ such that $f^{-1}(f(a) - \epsilon, f(a) + \epsilon) \supseteq (a - \delta, a + \delta)$

Where $f : A \to B \subset T$ then $f^{-1}(T) = \{a \in A : f(a) \in T\}$ [Don't need f^{-1} to exist !!]

Example 3.1.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then f is not continuous at $x = 0$

Proof. Take $\epsilon = 1$ (or $0 < \epsilon < 1$). Then if f is continuous at $x = 0$ we know that $\exists \delta > 0$ such that $|f(x) - f(0)| < 1\ \forall x \in (0 - \delta, 0 + \delta)$ $(*)$. In particular, take $x = \delta/2$ to find that $|1 - 0| < 1$ by $(*)$. ■

"Jump discontinuity" is another type of discontinuity,

Example 3.2.

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ r & x = 0 \end{cases}$$

Then f is discontinuous at $x = 0$ (for any r).

Idea of proof: If f is continuous at $x = 0$, then $f(x) \in (r - \epsilon, r + \epsilon)$ is close to $f(0) = r$ for $x \in (-\delta, \delta)$. In particular, $f(x)$ and $f(y)$ are close to each other (within

2ϵ). But $f(x)$ could be $+1$ and $f(y)$ could be -1, ⋇.

Proof. Fix $\epsilon \in (0,1]$. If f is continuous at 0, then $\exists \delta > 0$ such that $|f(x) - f(0)| < \epsilon \ \forall x \in (\delta, \delta)$. In particular, $\forall x, y \in (-\delta, \delta), |f(x) - f(y)| < 2\epsilon \leq 2$, by the triangle inequality.

Now choose $n \in \mathbb{N}$, $n > \frac{1}{\delta}$. Then take $x = \frac{1}{(4n+1)\pi/2} \in (0, \delta)$, $y = \frac{1}{(4n+3)\pi/2} \in (0, \delta)$. Then

$$|\sin(1/x) - \sin(1/y)| = |1 - (-1)| = 2 \text{ ⋇}$$

■

Theorem 3.3

$f, g : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R} \implies (f+g)$ cts at a.

Proof. Fix $\epsilon > 0$.

$$\exists \delta_1 > 0 \text{ such that } |x - a| < \delta_1 \implies |f(x) - f(a)| < \epsilon$$

and

$$\exists \delta_2 > 0 \text{ such that } |x - a| < \delta_1 \implies |g(x) - g(a)| < \epsilon$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x$ such that $|x - a| < \delta :$

$$|(f+g)(x) - (f+g)(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < 2\epsilon$$

■

Theorem 3.4

$f, g : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R} \implies (fg)$ cts at a.

Proof. Take $\epsilon = 1$. $\exists \delta_1 > 0$ such that $|x - a| < \delta_1 \implies g(x) < 1 + g(a)$.
Fix $\epsilon > 0, \exists \delta_2 > 0$ such that $|x - a| < \delta_1 \implies |f(x) - f(a)| < \epsilon$ and $\exists \delta_3 > 0$ such that $|x - a| < \delta_2 \implies |g(x) - g(a)| < \epsilon$.

Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\forall x$ such that $|x_a| < \delta$:
$|f(x)g(x) - f(a)g(a)| \leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)| < k\epsilon$. ■

Theorem 3.5

$f : \mathbb{R} \to \mathbb{R}$ cts at $a \in \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ cts at $f(a) \in \mathbb{R}$, then $g \circ f$ cts at a

Proof. Fix $\epsilon > 0$. $\exists \delta > 0$ s.t. $|g - f(a)| < \delta \implies |g(y) - g(f(a))| < \epsilon$.
Also $\exists \eta > 0$ such that $|x - a| < \eta \implies |f(x) - f(a)| < \delta$.

Hence $|x - a| < \eta \implies |f(x) - f(a)| < \delta \implies |g(f(x)) - g(f(a))| < \epsilon$. ■

Theorem 3.6

$f : \mathbb{R} \to \mathbb{R}$ is cts at $a \in \mathbb{R}$ iff $\forall$ sequences $x_n \to a$, $f(x_n) \to f(a)$

Proof. If f is cts at a, fix $\epsilon > 0$. $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. Now $x_n \to a$, so $\exists N \in \mathbb{N}$ such that $n \geq N \implies |x_n - a| < \delta \implies |f(x_n) - f(a)| < \epsilon$.

Suppose f is not cts at $a \in \mathbb{R}$ for contradiction. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x \in (a - \delta, a + \delta)$ such that $|f(x) - f(a)| \geq \epsilon$. Set $\delta = \frac{1}{n}$. $\exists x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$ such that $|f(x_n) - f(a)| \geq \epsilon$. So $|x_n - a| < \frac{1}{n}$ $\forall n \implies x_n \to a$. But $f(x_n) \not\Longrightarrow f(a)$, a contradiction. ∎

Theorem 3.7

$f : [a, b] \to \mathbb{R}$ cts $\implies f$ is bounded.

Proof (1.) Suppose for contradiction f is unbounded. Then $\forall N \in \mathbb{N}$, N is not an unpperbound, so $\exists x_N \in [a, b]$ such that $|f(x_n)| > N$.
By BW Theorem, exists cvgt subsequence, $y_i := x_{N(i)}, y_i \to y \in [a, b]$. With $|f(y_i)| = |f(x_{N(i)} > N(i) \geq i$ $(*)$. Fix $\epsilon = 1$, then $\exists \delta > 0$ such that $\forall x \in (y - \delta, y + \delta) : |f(x) - f(y)| < 1 \implies |f(x)| < |f(y)| + 1$. Since $y_i \to y$, $\exists N$ such that $\forall n \geq N$ $|y_n - y| < \delta \implies y_n \in (y - \delta, y + \delta) \implies |f(y_n)| < |f(y)| + 1$. By $(*)$, $n \leq |f(y_n)| < |f(y)| + 1$ $\forall n \geq N$, contradicting the Archimedean Axiom.

∎

Proof (2.) Suppose for contradiction f is unbounded. Then $\forall N \in \mathbb{N}$, N is not an unpperbound, so $\exists x_N \in [a, b]$ such that $|f(x_n)| > N$.
By BW Theorem, exists cvgt subsequence, $y_i := x_{N(i)}, y_i \to y \in [a, b]$. With $|f(y_i)| = |f(x_{N(i)} > N(i) \geq i$ $(*)$.
f is cts at $y \implies f(y_i) \to f(y)$, contradicting $(*)$. ∎

Intermediate Value Theorem

Theorem 3.8: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ cts, $c \in (f(a), f(b))$, then $\exists x \in [a, b]$ such that $f(x) = c$

Proof. Consider $S_c = \{y \in [a, b] : f(y) \leq c\}$. Define $x := \sup S_c$ ($S_c \neq \emptyset$ since $a \in S_c$ and bounded above by b so sup exists)
Claim: $f(x) = c$. *Proof:*

(i) Suppose $f(x) < c$. Take $\epsilon = c - f(x) > 0$. f is cts at x, so $\exists \delta > 0$ such that $\forall y \in (x, x + \delta) \cap [a, b]$, $|f(y) - f(x)| < \epsilon$. Hence $f(y) < f(x) + \epsilon = c$. So $y \in S_c \implies x \neq \sup S_c$.

(ii) Suppose $f(x) > c$. Take $\epsilon = f(x) - c > 0$. f is cts at x, so $\exists\delta > 0$ such that $\forall y \in (x-\delta, x) \cap [a,b]$, $|f(y) - f(x)| < \epsilon$. Hence $f(y) > f(x) - \epsilon = c \implies x - \delta$ is an upperbound for S_c, so $x \neq \sup S_c$. ∎

Extreme Value Theorem

Theorem 3.9: Extreme Value Theorem

$f : [a,b] \to \mathbb{R}$ cts $\implies f$ bounded and attains its bounds.

Proof (1.) By boundedness theorem, $\exists \sup_{x\in[a,b]} f(x) = s$. Suppose for contradiction $\nexists c \in [a,b]$ such that $f(x) = s$. Then $s - f(x) > 0\ \forall x \in [a,b]$, so $g(x) = \frac{1}{s-f(x)} : [a,b] \to \mathbb{R}$ is well defined and cts. So $g(x)$ is bounded by $M > 0 \implies \frac{1}{s-f(x)} \leq M \implies f(x) \leq s - \frac{1}{M}$, so $s \neq \sup f(x)$, a contradiction. ∎

Proof (2.) $\exists$ a sequence $x_n \in [a,b]$ such that $f(x) \to \sup_{x\in[a,b]} f(x) = s$. BW Theorem $\implies$ exists subsequence $y_i := x_{N(i)}$ such that $y_i \to c \in [a,b]$. f is cts $\implies f(y_i) \to f(x)$. Since $f(y_i) \to s$, by uniqueness of limits, $f(c) = s$. ∎

Theorem 3.10

$f : \mathbb{R} \to \mathbb{R}$ is bijective and cts $\implies f$ is strictly monotoinc

Proof. Fix any interval $[a,b] \subseteq \mathbb{R}$. f is bijective, so $f(a) \neq f(b)$, w.l.o.g. $f(b) > f(a)$. Suppose for contradiction $\exists c \in (a,b)$ such that $f(c) \notin (f(a), f(b))$. w.l.o.g. take $f(c) > f(b)$. Then fix $d \in (f(b), f(c))$. By IVT: $f|_{[a,c]}$, $\exists x \in (a,c)$ such that $f(x) = d$. Also $f|_{[c,b]}$, $\exists y \in (c,b)$ such that $f(y) = d$.
But $x \neq y$, a contradiction since f is injective. Hence $\forall c \leq b, f(x) \leq f(b)$. f injective $\implies f(c) < f(b)$. ∎

Inverse Function Theorem

Theorem 3.11

$f : \mathbb{R} \to \mathbb{R}$ bijective and cts $\implies f^{-1} : \mathbb{R} \to \mathbb{R}$ cts.

Proof. By Theorem 3.8, f is strictly monotonic, w.l.o.g. strictly increasing. Fix $y_0 \in \mathbb{R}$. Let $f^{-1}(y_0) = x_0 \in \mathbb{R}$. Fix $\epsilon > 0$.
Set $\delta = \min\{f(x_0+\epsilon) - y_0, y_0 - f(x_0 - \epsilon)\}$. Then $|y - y_0| < \delta \implies y \in (y_0 - \delta, y_0 + \delta) \subseteq (f(x_0 - \epsilon), f(x_0 + \epsilon))$. Applying f^{-1} preserves order $\implies f^{-1}(y) \in (x_0 - \epsilon, x_0 + \epsilon \implies |f^{-1}(y) - f^{-1}(y_0)| < \epsilon$. ∎

Theorem 3.12

$f : \mathbb{R}^n \to \mathbb{R}^m$ is cts at $\mathbf{a} = (a_1, \dots, a_n)$ if and only if $f_i : \mathbb{R}^n \to \mathbb{R}$ is cts at a_i $\forall i$. (With $f = (f_1, \dots, f_m)$).

Proof. Fix $\epsilon > 0$. Then f is cts at $\mathbf{a} \implies \exists \delta > 0$ such that $|\mathbf{x} - \mathbf{a}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{a}) < \epsilon$ $(*)$. Since $|f(\mathbf{x}) - f(\mathbf{a})| = \sqrt{\sum_{j=1}^m (f_j(\mathbf{x}) - f_j(\mathbf{a}))^2} \geq \sqrt{(f)i(\mathbf{x}) - f_i(\mathbf{a}))}^2 = |f_i(\mathbf{x} - f_i(\mathbf{a})|$, $(*) \implies |f_i(\mathbf{x}) - f(\mathbf{a})| < \epsilon$.

Suppose f_i cts at a_i $\forall i$. Fix $\epsilon > 0$. Then $\exists \delta_i > 0$ such that $|\mathbf{x} - \mathbf{a}| < \delta_i \implies |f_i(\mathbf{x}) - f_i(\mathbf{a})| < \epsilon$. Set $\delta = \min\{(\delta_i\} > 0$, so that $|\mathbf{x} - \mathbf{a}| < \delta \implies |f_i(\mathbf{x}) - f_i(\mathbf{a})| < \epsilon$ $\forall i \implies |f(\mathbf{x}) - f(\mathbf{a})| = \sqrt{\sum_{i=1}^m (f_i(\mathbf{x}) - f_i(\mathbf{a}))^2} \leq \sqrt{\sum_{i=1}^m \epsilon^2} = \sqrt{m}.\epsilon$ ∎

4 Differentiation

Differentiability

Definition. f is *differentiable* at a iff $\lim_{x \to a} \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right|$ exists, i.e.

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \epsilon.$$

Theorem 4.1

f differentiable at $a \in \mathbb{R} \implies$ cts at a.

Proof (1.) If f is differentiable at a then

$$\begin{aligned} \forall \epsilon > 0\ \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta &\implies \left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \epsilon \\ &\implies |f(x)-f(a)| < |x-a|(|f'(a)| + \epsilon). \end{aligned}$$

Fix $\epsilon > 0$, set $\delta = \epsilon$. Then

$$0 < |x-a| < \delta \implies |f(x)-f(a)| < \epsilon(|f'(a)| + \epsilon) = k\epsilon$$

(also true for $x = a \implies |f(x)-f(a)| = 0$.) ∎

Proof (2.) Note that $f(x) = f(a) + (x-a)\frac{f(x)-f(a)}{x-a}$, $x \neq a$. Taking $\lim_{x \to a}$

$$\lim_{x \to a} f(x) = f(a) + 0.f'(a) \implies f \text{ cts at } a$$ ∎

Rolle's Theorem

Theorem 4.2: Rolle's Theorem

$f : [a,b] \to \mathbb{R}$ cts on $[a,b]$, differentiable on (a,b) such that $f(a) = f(b)$. Then $\exists c \in (a,b)$ such that $f'(c) = 0$.

Proof.

Case 1. f is constant on $[a,b]$. Then set $c = \frac{a+b}{2}$, so $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = 0$.

Case 2. f takes values $< f(a)$. Then replace f by $-f$ and consider Case 3.

Case 3. f takes values $> f(a)$. Therefore $\sup \{f(x) : x \in [a,b]\} > f(a)$ by EVT is realised by some $c \in (a,b)$. Now $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. Consider

$$x > c,\ f(x) \le f(c) \implies \frac{f(x)-f(c)}{x-c} \le 0 \implies \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0$$

$$x < c,\ f(x) \leq f(c) \implies \frac{f(x)-f(c)}{x-c} \geq 0 \implies \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$$

Hence $\dfrac{f(x)-f(c)}{x-c} = 0.$ ∎

Mean Value Theorem

Theorem 4.3: Mean Value Theorem

If $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof. Let $g(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a)$, which is cts on $[a,b]$ and diff'ble on (a,b). $g(a) = f(a) = g(b)$. By Rolle's Theorem

$$\exists c \in (a,b) \text{ such that } g'(c) = 0 \implies g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}.$$ ∎

Rules for Differentiation

Theorem 4.4: Product Rule

$f, g : \mathbb{R} \to \mathbb{R}$ differentiable at $a \in \mathbb{R}$. Then fg is differentiable at a with $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

Proof.

$$\begin{aligned} \frac{f(x)g(x)-f(a)g(a)}{x-a} &= \frac{(f(x)-f(a))g(x)+(g(x)-g(a))f(a)}{x-a} \\ &= g(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{g(x)-g(a)}{x-a} \end{aligned}$$

Taking $\lim_{x\to a} \implies (fg)'(a) = g(a)f'(a) + f(a)g'(a)$ by cty of g and algebra of limits. ∎

Theorem 4.5: Chain Rule

$g : \mathbb{R} \to \mathbb{R}$ diff'ble at $a \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ diff'ble at $g(a) \in \mathbb{R}$, then $f \circ g$ diff'ble at a with $(f \circ g)'(a) = f'(g(a))g'(a)$

Proof. Define $F(g) = \begin{cases} \frac{f(y)-f(b)}{g-b} & y \neq b \\ f'(g) & y = b \end{cases}$ with $b = g(a)$.

f is diff'ble at $b \implies \lim_{y\to b} F(y) \to f'(b) = F(b)$. Hence F is cts at b. g is diff'ble at $a \implies$ cts at $a \implies F \circ g$ is cts at $a \implies F(g(x)) \to F(g(a)) = f'(b)$ as $x \to a$. Then: $(f \circ g)'(a) = \lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a} = F(g(x))\frac{g(x)-g(a)}{x-a} = f'(b)g'(a) = f'(g(a))g'(a)$. ∎

Theorem 4.6

If $f : \mathbb{R} \to \mathbb{R}$ is diff'ble at $a \in \mathbb{R}$ with $f'(a) \neq 0$ and f is bijective with inverse $g = f^{-1}$, then g is diff'ble at $b = f(a)$ with $g'(b) = \frac{1}{f'(g(b))} = \frac{1}{f'(a)}$.

Proof. Lemma: $f'(a) \neq 0 \implies \exists \delta > 0$ such that $f(x) \neq f(a)$ for $x \in (a-\delta, a+\delta) \backslash \{0\}$.

So $\frac{g(y)-g(b)}{y-b} = \frac{x-a}{f(x)-f(a)} = 1/\frac{f(x)-f(a)}{x-a}$ with $x = g(y),\ y \neq b$. As $y \to b$, $g(y) \to g(b) = a$ since f diff'ble at $a \implies f$ cts at $a \implies g$ cts at $b \implies x \to a \implies$ RHS $\to \frac{1}{f'(a)}$. ∎

- End of Analysis I -