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1st Year Mathematics
Imperial College London

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Mathematical Methods I

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Syllabus

The course supplies a firm grounding to A-level topics such as differentiation, integration, complex numbers and series expansions.

- Functions: Polynomial, rational, exponential, logarithmic, trigonometric and hyperbolic functions. Odd, even and inverse functions.
- Limits: basic properties and evaluation. Continuity & discontinuous functions.
- Differentiation: First principles, differentiability; logarithmic and implicit differentiation; higher derivatives; Leibniz's formula; stationary points and points of inflexion; curve sketching; parametric representation, polar co-ordinates.
- Power Series Expansions: The Mean Value Theorem. Taylor's Theorem with remainder. Infinite power series, radius of convergence. Ratio test; Taylor and Maclaurin Series. De l'Hopital's rule. : definition as Riemann limit; indefinite definite integrals; the fundamental theorem of calculus; integration by substitution and by parts; partial fractions; Existence of improper and infinite integrals. Integrals over areas and volumes.
- Complex Numbers: definition; the complex plane; standard and polar representation; de Moivre's Theorem; exp(z) and log(z)
- First order differential equations. Separable, homogeneous and linear equations. Special cases. Linear higher order equations with constant coefficients,

Course content at `http://www.ma.ic.ac.uk/~ajm8/M1M1`

Appropriate books

M. Liebeck *A Concise Introduction to Pure Mathematics.*

K. Houston *How to Think Like a Mathematician.*

E. Hurst and M. Gould *Bridging the Gap to University Mathematics.*

None are particularly recommended. The course is completely self-contained. But Stephenson's "Maths for Science Students", and Kreyszig's of a similar name are useful.

Contents

0 Introduction

Lecture 0

Example 1.1. Consider the definiton of the exponential function:

$$\begin{aligned} exp(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

So $e^{100} = 1 - 100 + \frac{1}{2}100^2 - \frac{1}{6}100^3 + \dots \approx 0$

Is this really true?
Yes, but this is not obvious. The series converges (i.e. tends to an answer) for all x. We will see this later...

Example 1.2. $f = \dfrac{4}{3 + \cos x}$

Can we write the power series $f = a_0 + a_1 x + a_2 x^2 + \dots = \sum a_n x^n$? where a_n are known constants?

Yes, we find (somehow) that $f = 1 + \frac{x^2}{8} + \frac{x^4}{192} + \dots$

Does this series converge? Using Maple to find the series and plotting $f - \sum_0^{200} a_n x^n$, we actually find that at ± 3.60ish, the difference is non-zero and it fails to converge. This is (apparently) amazing evidence of the existence of the complex plane...

Example 1.3. Limits

$\lim_{x \to \infty} (\sin x) =?$, it is undefined.

$\lim_{x \to \infty} \left(\dfrac{\sin x}{x} \right) = 0$, from the sandwich theorem as 0 is squeezed between $\frac{1}{x}$ and $-\frac{1}{x}$

$\lim_{x \to \infty} \left(\dfrac{1}{x \sin x} \right) =?$, undefined once more, since whenever $x = n\pi$, the denominator is 0.

$\lim_{n \to \infty} \left(\dfrac{\sin n}{n} \right)$, $n \in \mathbb{R}$ is not at all obvious (depends on how well you can approximate π)

Clearly we have work to do....

1 Functions

Lecture 1
14.10.2014

A function takes an "input" and gives a **unique** output: $f(x) : x \in \mathbb{R}$.
$f(x)$ is the output or function value at x.

$f : \mathbb{R} \to \mathbb{R}$ (alternative notation: f "maps" input $\in \mathbb{R}$ to output $\in \mathbb{R}$)

f may not be defined for all reals. A function *should* be defined along with the **domain** of values over which it applies, e.g.:

$f(x) = \sqrt{x^2 - 1}$ for $x \geq 1$

Notation: $[a, b]$ means $\forall x : \ a \leq x \leq b$ $\qquad (a, b)$ means $\forall x : \ a < x < b$
These are called **closed** and **open** intervals respectively.

- So $x \geq 1$ could be written as $x \in [1, \infty)$
  By convention ∞ is never a closed interval since it is not a real number.

We also define the **range** of a fucntion to be the set of possible values $f(x)$ as it takes values of the domain. So $f(x) = \sqrt{x^2 - 1}$ in $[1, \infty)$ has the **range** $[0, \infty)$.

Note: $\sqrt{\ }$ is always positive conventionally, otherwise it maps to more than one value $\implies f$ is not a function. Hence $\sqrt{x^2} = |x|$, *not necessarily* x.

How might we define functions?

(i) An explicit formula
e.g. $f(x) = x^2 \sin(x)$
(as the domain is not given, we assume it applies for all x or all sensible x.)

$$f(x) = \frac{x + 2}{x - 1} \qquad \text{"sensible" here means } x \neq 1$$

(ii) Split ranges, e.g.

$$f(x) = \begin{cases} x \text{ if } x > 1 \\ \sin(x^2) \text{ if } 0 < x \leq 1 \\ e^x \text{ if } x \leq 0 \end{cases}$$

(iii) As a solution to an equation
e.g. $f'' + x^2 f = 0, f(0) = 1, f'(0) = 0$ *may* define a function

Similarly we could define $f(x) = \displaystyle\int_0^x t^t \, \mathrm{d}t$

(Note: we use a different letter for the *dummy* variable, t)

(iv) In words
e.g. $f(x)$ = "the maximum amount by which x exceeds an integer for $n = 1, 2, ...100$"

(v) An implicit definition
e.g. $f(x)$ given by $f(x) + \frac{1}{2}\sin[f(x)] = x$

(or $y + \sin y = x$: we can't solve for y in terms of x easily) given x, not easy to calculate $f(x)$

(vi) As a limit
e.g. x^{x^x} or more formally: $f_1 = x^x$, $f_{n+1} = x^{f_n}$ for $n \geq 1$

If this process tends to a limit as $n \to \infty$ we may have defined a function.

... And so on. There are lots of ways of defining functions.

How many functions are there?

It turns out it's (a very large[1]...) infinity

Most functions are horrible horrendous. Even ones which look nice can be nasty...

> **Example 1.4.** $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

The graph crosses the x-axis an infinite number of times between $[0, n]$

[1] see M1F, there are different sizes of "infinities"

Example 1.5. $f(x) = e^{-\frac{1}{x^2}}$

[floatPos=htb,capWidth=0.5,capPos=r,capVPos=c,objectPos=c]figure [2][Caption beside object and vertically centered]$f(x)$ is so flat at zero that the Macluarin (Taylor) Series converges to 0. This is NOT the right answer. We then call this a non-analytical function.fig:1 $pattern = on4ptoff4pt]$

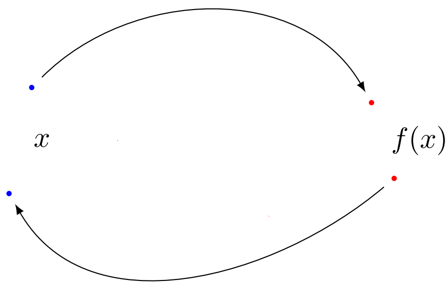
Example 1.6. $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^4 x)}{n^2} = \sin x + \frac{\sin 16x}{4} + \frac{\sin 81x}{9} + \dots$

This function is continous everywhere, differentiable nowhere.

Inverse functions

Lecture 2
16.10.2014

Suppose we have a function f



Domain f Range of f (also known as image of f)

If it is possible to find a function $g(f(x)) = x \quad \forall x$ in domain of f, then g is called the inverse of f. It's often denotes as f^{-1}.

The domain of f = the range of f^{-1}
& The range of f = the domain of f^{-1}

Do inverses always exist?
Clearly not if ($\geq$) two x values give the same value of $f(x) = y$ say. As we cannot determine a unique x value given y.

In practice, we try to solve $f(x) = y$ for x. This may find the inverse or tell us that there is a problem.

Example 1.7. Find the inverse of $f(x) = \frac{x-1}{x+3} \qquad (x \neq 3)$

$$y = \frac{x-1}{x+3} \implies y(x+3) = x - 1$$

$\implies x(y-1) = -3y - 1$

$\implies x = \dfrac{3y+1}{1-y} = f^{-1}(y)$

[floatPos=htb,capWidth=0.5,capPos=r,capVPos=c,objectPos=c]figure [Caption beside object and vertically centered]The graph $y = f(x)$ helps us understand what is going on. So we sketch it, noting that $y = \dfrac{x-1}{x+3} = 1 - \dfrac{4}{x+3}$. For inverse to exist, all lines $y =$ constant must intersect $y = f(x)$ oncee once and only once, which is clearly the case. fig:1

Example 1.8. Does f^{-1} exist? for $y = x + \frac{1}{x} = f(x)$

Note that if $x = 2$, then $f(2) = 2.5$, but $f(\frac{1}{2}) = 2.5$ also...
maybe if we restrict the domain of f, we can find a sensible inverse.

Lets try $y = x + \frac{1}{x} \implies xy = x^2 + 1$

$\implies x^2 - xy + 1 = 0$

$\implies x = \dfrac{y \pm \sqrt{y^2-4}}{2}$

Which root should we take? Sketching $f(x)$, we can note that $y = k$ intersects the graph: $\begin{cases} \text{Not at all} & -2 < k < 2 \\ \text{Once} & k = \pm 2 \\ \text{Twice} & k > 2 \text{ or } k < -2 \end{cases}$

[floatPos=htb,capWidth=0.5,capPos=r,capVPos=c,objectPos=c]figure [Caption beside object and vertically centered]The graph $y = x + \dfrac{1}{x}$ fig:1

Suppose we restrict the domain of f to be $|x| \leq 1$ (exlcluding x =0).

Then we can define the inverse function:

$$f^{-1}(y) = \begin{cases} \dfrac{y - \sqrt{y^2-4}}{2} & \text{if } y \geq 2 \\ \\ \dfrac{y + \sqrt{y^2-4}}{2} & \text{if } y \leq 2 \end{cases}$$

If instead we restrict the domain of f to be $|x| \geq 1$ then: $f^{-1}(y) = \begin{cases} \dfrac{y + \sqrt{y^2-4}}{2} & \text{if } y \geq 2 \\ \\ \dfrac{y - \sqrt{y^2-4}}{2} & \text{if } y \leq 2 \end{cases}$

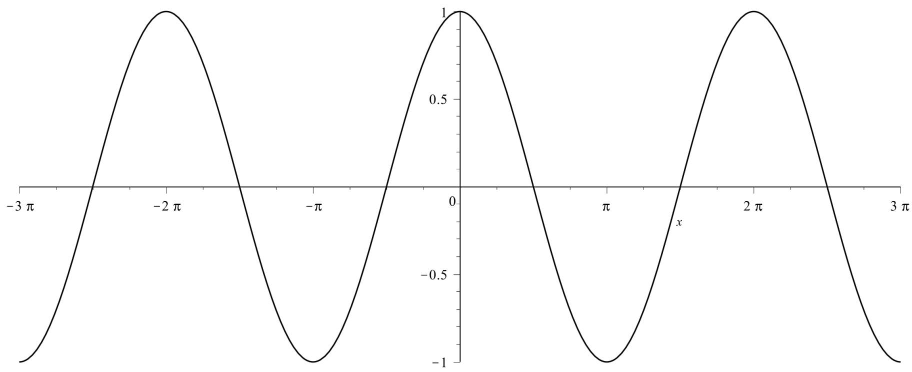
> So a little care is required.

Trigonometric Functions

Trigonometry: trignon metron - The Measuring of Triangles

Later we will define $\cos(x), \sin(x), \tan(x)$, but you already know them.

Inverse Cosine If $f(x) = \cos(x)$ then $f^{-1}(x) = \cos^{-1}(x)$ (or $\arccos(x)$) exists for some x and some agreed domain of $f(x)$



The natural domain to restrict is $0 \leq x \leq \pi$. Then $\cos^{-1}(y)$ exists uniquely, provided $|y| \leq 1$.

Now remove the restrictions on x. Solve the equation $\cos x = \alpha$

General solution: $\boxed{x = 2n\pi \pm \cos^{-1} \alpha \quad (n \in \mathbb{Z})}$

But remember $0 \leq \cos^{-1} \leq \pi$ always!

Inverse Sine

L0.5

We restrict $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ i.e. $|x| \leq \frac{\pi}{2}$

Then the inverse function $\sin^{-1} y$ exists, if $|y| \leq 1$

So $-\frac{\pi}{2} \leq \sin^{-1} \leq \frac{\pi}{2}$

Then the equation $\sin x = \beta$ has the general solution:

$x = 2n\pi + \sin^{-1}\beta$ if n is even
$x = 2n\pi - \sin^{-1}\beta$ if n is odd
$= \boxed{x = 2n\pi + (-1)^n \sin^{-1}\beta \quad (n \in \mathbb{Z})}$

.

Inverse Tan

[floatPos=htb,capWidth=0.5,capPos=r,capVPos=c,objectPos=c]figure [Caption beside object and vertically centered] We restrict $-\frac{\pi}{2} \leq \tan^{-1} y \leq \frac{\pi}{2}$ defined $\forall y$. So finally $tanx = \gamma$ has general solution: $\boxed{x = n\pi + \tan^{-1}\gamma}$ fig:1

Parity - Even & Odd Functions

Lecture 3
17.10.2014

A function $f(x)$ defined over a symmetric domain (i.e. $[-a, a]$) is called even $\iff f(-x) = f(x)$ and odd $\iff f(-x) = -f(x)$

e.g. x^2 is even, $\sin x$ is odd. Functions need not be even or odd.

But any function (over a symmetric domain) can be written as the sum of an even function & an odd function.

Example 1.9. $\dfrac{x}{x+1} = \dfrac{x}{x+1}\dfrac{(x-1)}{(x-1)} = \underbrace{\dfrac{x^2}{x^2-1}}_{\text{Even}} - \underbrace{\dfrac{x}{x^2-1}}_{\text{Odd}}$

Example 1.10. $\cos(x+3) = \underbrace{\cos x \cos 3}_{\text{Even}} - \underbrace{\sin x \sin 3}_{\text{Odd}}$

In general, how do we write $f(x) = f_e(x) + f_o(x)$?

$$f(x) = f_e(x) + f_o(x) \tag{1.11}$$

$$\implies f(-x) = f_e(-x) + f_o(-x)$$

$$\implies f(-x) = f_e(x) - f_o(x) \tag{1.12}$$

Solving (1) and (2) by adding:

$\implies f_e(x) = \frac{1}{2}[f(x) + f(-x)]$

and similarly $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$

To prove that we can always find these two functions, start again from other way:

Define f_e & f_o as above, note:

(i) f_e is even

(ii) f_o is odd

(iii) $f_e + f_o = f$

This proves that any f has an even part and an odd part.

Example 1.13. Redo example 6. $f(x) = \dfrac{x}{x+1} = f_e(x) + f_o(x)$

$\implies f_e(x) = \frac{1}{2}\left[\dfrac{x}{x+1} + \dfrac{-x}{1-x}\right]$

$= \frac{1}{2}\left[\dfrac{x - x^2 - x^2 - x}{(1+x)(1-x)}\right] = \dfrac{-x^2}{1-x^2} = \dfrac{x^2}{x^2-1}$ as before.

Evaluating Integrals Parity is a great help when evaluating integrals.

What is $\displaystyle\int_{-\pi}^{\pi} \frac{x+\sin(x^3)}{1+e^{x^2}}\,\mathrm{d}x$?

Replacing x by $-x$ we can see that the integrand is an odd function since $f(-x) = -f(x)$

Hence the Integral $= 0$.

In general $I = \boxed{\displaystyle\int_{-a}^{a} f_o(x)\,\mathrm{d}x = 0}$

Substituting $t = -x$

$$\begin{aligned}
\implies I &= \int_{a}^{-a} f_o(-t)\,(-\mathrm{d}x) \\
&= \int_{-a}^{a} f_o(-t)\,(\mathrm{d}x) \qquad \text{[Use - sign to swap limits]} \\
&= -I
\end{aligned}$$

Hence $I = -I \implies I = 0$.

We conclude that if $f(x)$ is odd, then $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$

Later we will deal with power series $f(x) = a_0 + a_1x + a_2x^2 + ...$, where a_i is a given constant for $i \in \mathbb{N}$

If $f(x)$ is even then $a_1 = 0, a_3 = 0$ etc., i.e. $a_{odd} = 0$
If $f(x)$ is odd then $a_0 = 0, a_2 = 0$ etc., i.e. $a_{even} = 0$

So even/odd functions only have even/odd powers of x.

Periodicity

We say a function $f(x)$ is T-periodic if and only if $f(x+T) = f(x) \forall x$, where $T > 0$ and T is the smallest value for which this holds.

So although $\sin(x + 4\pi) = \sin x \ \forall x$, we do not say that $\sin x$ is 4π periodic, as $\sin(x + 2\pi) = \sin(x)$ as well.

$f(x)$	period
$\cos^2 x$	π
$\cos\lvert x\rvert$	2π
$\lvert\cos x\rvert$	π
$\sin(\alpha x)$	$\frac{2\pi}{\alpha} \quad (\alpha \neq 0)$
3	depends on definition, say period $= 0$ & change definiton
$\sin\lvert x\rvert$	not periodic
$\lvert\sin x\rvert$	π

Are there any other periodic functions (other than the trigonometric ones)?

We can turn any function into a periodic one.
Any function defined on a finite interval can be extended into a periodic function over all $\mathbb{R}$ by copying.

Define $f(x + L) = f(x)$ to replicate the behaviour $\forall x$.

Lecture 4
21.10.2014

Polynomials

An nth order polynomial in x is a function of the form: $f(x) = \sum_{a=0}^{N} a_n x^n$ where $a_N \neq 0$

N is called the **degree** or **order** of the polynomial.. a_n for $n = 0, ..., N$ are called the coefficients. If a_n is real $\forall n$, we say the polynomial is real (even if x may be complex).

Theorem 1.14: The Fundamental Theorem of Algebra

Every polynomial has a root (possibly complex). In general, we call a value α a root of $f(x)$ if $f(\alpha) = 0$.

Proof. See next year's course M2PM3 on Complex Analysis...

Corollary 1.15. *If c is a root of an Nth order polynomial $P_N(x)$, then we can write $P_N(x) = (x - c)P_{N-1}(x)$.*

Corollary 1.16. *Every Nth order polynomial has precisely N roots, allowing for repeated roots. e.g. $(x - 1)^2$ has roots 1,1.*

Corollary 1.17. *If $P(x)$ is a real polynomial with a complex root $\alpha + i\beta$ (α, β real, $\beta \neq 0$), then it also has root $\alpha - i\beta$ (the complex conjugate).*

Corollary 1.18. *Every real polynomial can be written as:*
$P_N(x) = A(x - r_1)(x - r_2)...(x - r_M)((x - \alpha_1)^2 + \beta^2)((x - \alpha_2)^2 + \beta_2^2)...((x - \alpha_L)^2 + \beta_L^2)$

Where $r_1..r_M$ are the real roots, and $(\alpha \pm i\beta), ...(\alpha_L \pm i\beta_L)$ are the complex roots and $M + 2L = N$

i.e. Any real polynomial can be written as a product of real linear and quadratic factors.

N.B. If the polynomial is not real (i.e. if at least one coefficient is strictly complex), then the complex roots need not be in conjugate pairs.

Roots of polynomials

- Linear: $ax + b = 0$, one trivial root
- Quadratic: $ax^2 + bx + c = 0$

  Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ OR $x = \dfrac{2c}{-b \pm \sqrt{b^2 - 4ac}}$

  Exercise 1: Show these are the same.
  Exercise 2: Use your calculator to solve the equation $\epsilon x^2 + (1 + \epsilon)x + 1 = 0$, where ϵ is very small (i.e. 10^{-12}).

  Subtracting two numbers which are very close together leads to severe accuracy loss c.f. Patriot missiles. The "Best" formula to solving a quadratic depends on

a, b and c in practice.

- Cubics: $ax^3 + bx^2 + cx + d = 0$

  There is a formula[2], but it has very little practical use.

- Quartics: $ax^4 + bx^3 + cx^2 + dx + e = 0$

  There is also a general formula... once again not worth knowing[3]

- Quintics: $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

  There is no general formula to express the roots in terms of radicals. See Galois Theory in 3rd year. But one can easily find the roots in practice for any particular case.

- N ¿ 5: Similarly no formula. However considering general N:

  $$\begin{aligned} P_N(x) &= a_N x^N + a_{N-1} x^{N-1} + \dots \\ &= a_N x^N \left[1 + \frac{a_{N-1}}{a_N}\frac{1}{x} + \frac{a_{N-2}}{a_N}\frac{1}{x^2} + \dots\right] \end{aligned}$$

  As $|x| \to \infty, \dfrac{1}{x^N} \to 0$, So for large $|x|, P_N \approx a_N x^N$. Hence if $a_N > 0$ WLOG, then as $x \to \pm\infty$, $P_{2N}(x) \to +\infty$ and $P_{2N+1} \to \pm\infty$

Rational Functions

Lecture 5
23.10.2014

A function of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials is called a rational function.
We *could* require that the order of $P <$ order of Q. If this doesn't happen, we can use polynomial division to write:

$$\frac{P}{Q} = R(x) + \frac{S(x)}{Q(x)} \quad \text{where } R \text{ and } S \text{ are also polynomial}$$

$$\text{e.g. } \frac{x^2+x}{x-1} = \frac{x^2 - x + 2x}{x-1} = x + \frac{2x}{x-1} = x + \frac{2x - 2 + 2}{x-1} = (x+2) + \frac{2}{x-1}$$

We could also require that P and Q have no common factors i.e. $\nexists \alpha : P(\alpha) = 0 = Q(\alpha)$ Let's do this! (for simplicity). Any zero of $Q(x)$ is then a **singularity** or **pole** or infinity of $\frac{P}{Q}$ and is important. This behaviour is illustrated by...

---

[2] $x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 - \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 - \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$

[3] or even typesetting...

Partial Fraction Decomposition

Suppose Q has degree N, with no repeated root, i.e.

$$Q = \lambda(x-r_1)(x-r_2)\dots(x-r_N) \quad \text{where } \lambda \neq 0, r_i \neq r = j \text{ unless } i = j \text{ \& } r_i \in \mathbb{C}$$

Then we can write:

$$\frac{P}{Q} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \dots + \frac{A_N}{x-r_N} + \overbrace{R(x)}^{\text{if needed}}$$

We can easily find A_i by multiplying through Q:

$$\begin{aligned} P &= \frac{A_1 Q}{x-r_1} + \frac{A_2 Q}{x-r_2} + \dots + \frac{A_N Q}{x-r_N} \\ &= A_1\lambda(x-r_2)(x-r_3)\dots(x-r_N) + \frac{A_2 Q}{x-r_2} + \dots + \frac{A_N Q}{x-r_N} \qquad (*) \end{aligned}$$

Putting $x = r_1$ into $(*)$; $Q(r_1) = 0$, so:

$$P(r_1) = A_1\lambda(r_1-r_2)(r_1-r_3)\dots(r_1-r_N)$$

Using the product rule for differentiation, we also now have:

$$Q'(x) = \lambda[(x-r_2)\dots(x-r_N)] + \lambda(x-r_1)[\text{ a load of stuff }]$$

So $Q'(r_1) = \lambda(r_1-r_2)(r_1-r_3)\dots(r_1-r_N)$, hence $P(r_1) = A_1 Q'(r_1)$ or $A_1 = \dfrac{P(r_1)}{Q'(r_1)}$

So obviously $\boxed{A_i = \dfrac{P(r_i)}{Q'(r_i)} \quad i = 1, 2, ..., N}$

What could go wrong?

(a) *What if (some of) the roots are complex?*
Algebra still works. But for some purposes we may prefer to keep things real.

$$\text{e.g.} \quad \frac{3}{x^3+1} = \frac{3}{(x+1)(x^2-x+1)} = \frac{3}{(x+1)(x-\omega)(x+\omega*)}$$
$$= \frac{A_1}{x+1} + \frac{A_2}{x-\omega} + \frac{A_3}{x-\omega*}$$

Using the formula we obtained for A_i, we get $A_1 = 1, A_2 = -\omega, A_3 = -\omega*$

$$\begin{aligned} \frac{3}{x^3+1} &= \frac{1}{x+1} - \frac{\omega}{x-\omega} - \frac{\omega*}{x-\omega*} \\ &= \frac{1}{x+1} - \frac{x-2}{x^2-x+1} \end{aligned}$$

Alternative Partial Fraction Form for real polynomials:

$$\frac{P}{Q} = R + \frac{A_1}{(x-r_1)} + \dots + \underbrace{\frac{Cx+D}{Cx^2+\delta x+\delta}}_{\text{for complex roots}} + \dots$$

(b) *What if there are repeated roots?*

Lecture 6
24.10.2014

Limits and Power Series

A function of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$

We wil asume for now that the infinie sum converges (i.e. tends to al imit) for at least some values of x. We will also assume we can manipulate infinite series sensibly. The a_n are called cefficients (may be $\in \mathbb{C}$)

2 Infinite Series

The Exponential Function

$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$ In fact this series converges $\forall x$

Forget everything we know about e^x for now... What can we deduce about $f(x)$

If $x > 0, f(x) > 1$ by inspection, and also x increases as $f(x)$ increases.

Question 6 on the problem sheets proved that $\forall x, y, f(x)f(y) = f(x+y)$

Setting $y = -x$, yields $f(-x) = \frac{1}{f(x)}$, which tells us about when $0 < x < 1$

- $f(x) \to 0$, as $x \to -\infty$ etc.

It follows that $f(x)$ has an inverse functin $g(x)$ whose domain is $(0, \infty)$ and range $(-\infty, \infty)$, so we know that $x = g(f(x)) \forall x \quad \& \quad x = f(g(x))\ , x > 0$

Now consider $x^2 = x.x = f(g(x)).f(g(x)) = f(2g(x))$, obviously by induction $x^n = f(ng(x))$ for $n \in \mathbb{N}$

I) $x^\alpha = f(\alpha g(x)) = e^{\alpha \log x}$ for $x > 0$, any arbitrary α
II) $a^x = f(x(g(a))$ for $a > 0$, any abitrary x

From the definiton of f, $a^x = 1 + x(g(a) + \frac{1}{2}[xg(a)]^2 + \frac{1}{3!}[xg(a)]^3 + \dots$

Choose a such taht $g(a) = 1$, then we have:
$a^x = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \dots = 2.718281828459...$

Let;s call this value (...wait for it), e
Then $g(e) = 1$, so $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots !!$

From now we can use all the properties of the exponential function, e.g. $e^x e^y = e^{x+y}, \quad (e^a)^b = e^{ab}, \quad e^0 = 1$

Calling $g(x) = \log(x) = ln(x)$, we have the usual properties which follows from the first problem sheet:

- $\log(uv) = \log(u) + \log(v)$
- $\log(1) = 0$
- "$\log(0) = -\infty$"
- $\log(\frac{u}{v}) = \log(u) - \log(v)$
- $\log(a^b) = b \log a$
- $a^b = e^{b \log a}$

(We will not consider logarithms to different bases)

e^x is defined $x \in R$, **what if x is complex or purely imaginary?**

Write $x = i\theta, \theta \in R$, then define $e^{i\theta}$ to be:

$f(i\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{1}{2}(i\theta^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \dots$

$= (1 - \frac{1}{2}\theta^2 + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots)$

Now define $coz(\theta) = (1 - \frac{1}{2}\theta^2 + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots)$ and $zin(\theta) = (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots)$

We then have: $\boxed{e^{i\theta} = coz(\theta) + izin(\theta)}$

Lecture 7
28.10.2014

A better proof that $coz(x+y) = coz(x)coz(y) - zin(x)zin(y)$ (see appendix for series proof), uses complex numbers, and the fact that:

$$\boxed{\exp(x)\exp(y) \equiv \exp(x+y) \quad (*)}$$

Proof. Consider $\exp(i\theta)\exp(i\phi) \equiv \exp(i\theta + i\phi)$

$$\implies [\cos(\theta) + i\sin(\theta)][\cos(\phi) + i\sin(\phi)] \equiv \cos(\theta+\phi) + i\sin(\theta+\phi)$$

Expanding and equating the real parts gives required result.[4]

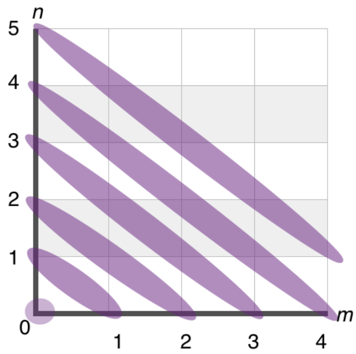
□

Now we prove identity $(*)$...

Proof. Consider $\exp(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $\exp(y) = \displaystyle\sum_{m=0}^{\infty} \frac{y^m}{m!}$

Then $\exp(x)\exp(y) = \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^n}{n!}\frac{y^m}{m!}$

[4]Much easier than the series manipulation on handout 1!



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[Caption beside object and vertically centered] Assume it does not matter in which order we add up all the terms i.e. we can add the diagonals $m + n = p$:fig:1

Indeed counting the terms diagonally, writing $m + n = p$:

$$\begin{aligned}
\exp(x)\exp(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^n}{n!}\frac{y^m}{m!} &= \sum_{p=0}^{\infty}\sum_{n=0}^{p}\left(\frac{x^n y^{p-n}}{p!}\right) \\
&= \sum_{p=0}^{\infty}\sum_{n=0}^{p}\left(\frac{pCnx^n y^{p-n}}{p!}\right) \\
&= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{n=0}^{p} pCnx^n y^{p-n} \\
&= \sum_{p=0}^{\infty}\frac{(x+y)^p}{p!} = \exp(x+y)
\end{aligned}$$

□

We now have $\cos and \sin$ You will prove on sheet 2 question 1 that they are 2π-periodic and all the trigonmetric formulae that follow, i.e. $cos(A + B)$ etc. *Remember them, or be able to derive them in 15 seconds!*

Other infinite series which you should know

$\log(x) \neq \sum_{n=0}^{\infty} a_n x^n$ for any a_n (try putting $x = 0$, and you would get "$-\infty = a_0$"...bullshit)

$\log(1+x) = 0 + x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}x^n}{n}$

$\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n(-1)^n, \ |x| < 1$
This is an example of the geometric series $a + ar + ar^2 + \cdots = \frac{a}{1-r}, \ |r| < 1$

N.B. We could integrate the series for $\frac{1}{1+x}$ to obtain $\log(1+x) + c = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$

assuming integrating term by term is allowed. Letting $x = 0 \implies c = 0$

The Binomial Series The Geometric series is a special case of the Binomial Series:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-p)}{(p+1)!}x^{p+1} + \ldots \quad (\alpha \in \mathbb{R})$$

This series converges provided $|x| < 1$. Note, if $\alpha \in \mathbb{N}$, then eventually the coefficients become zero and the power series **terminates** as a polynomial of degree N. The series then becomes the binomial theorem: $(1+x)^n = \sum_{k=0}^{n} nCkx^k$

Proof by:

(i) Leave it to M1F

(ii) Induction

[Lecture 8, more stuff on series... Panopto broke :S]

Maclaurin Series

Lecture 9
31.10.2014

What Kind of functions have a power series?

i.e. When can we write $f(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n = \sum_{n=0}^{\infty} a_nx^n$?

Note that if this is true then $f(0) = a_0$. So if $f(x)$ is **differentiable** (see later) and it is legitimate to differentiate an infinite series term by term, then $f'(x) = a_1 + 2a_2 + \dots na_n = \sum_{n=0}^{\infty} na_nx^{n-1}$. Now we put $x = 0$, to get $f'(0) = a_1$.

In general, if the function $f(x)$ can be differentiated r times (and so can the series), then $f^{(r)}(0) = a_rr!x^0 + 0 + 0 + 0 + \dots \implies \boxed{a_r = \frac{f^{(r)}}{r!}}$

(if $n < r$ in sum, one of the prefactors of x^{n-r} is 0. If $n > r, x^{n-r}$ is 0, when $x = 0$. So only one term, $n = r$, remains.)

So formally, if the function has a power series then we expect the **Maclaurin Series** (or the Taylor series about $x = 0$) to be:

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \dots + \frac{x^r f^{(r)}(0)}{r!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

We suspect therefore, the only those functions with an abitrary number of derivatives of $x = 0$ have the power series expansion.

So $\log(x), x^{3.1}, \sin(x^{\frac{1}{2}})$, or $|x^3|$ do not have series expansions. There are some functions for which the power series exists but converges to a different function. Such functions are called **non-analytical** functions. e.g.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We shall not worry about such functions anymore.

So a Maclaurin series exists $\iff f^{(n)}(0)$ exists $\forall n$

e.g. $f(x) = e^{-x}\sin(2x) = 2x - 2x^2 + \mathcal{O}(x^3)$

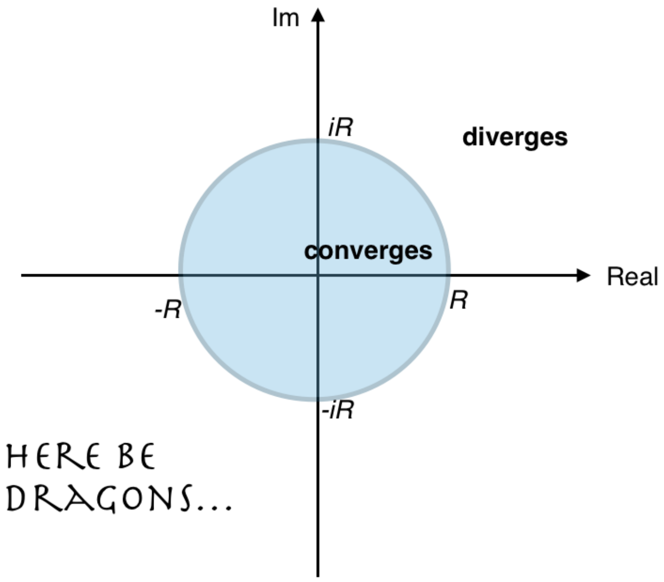
Next lecture we will look at the Analysis very briefly. Next term you will do it "properly".

Key Results of Analysis

For any power series $\sum_{n=0}^{\infty} a_n x^n$ $(a_n, x \in \mathbb{C})$, $\exists \mathbb{R}$ such that:

- if $|x| < R$ series converges
- if $|x| > R$ series does not converge
- if $|x| = R$ anything can happen
- if $R = 0$ series converges only for $x = 0$
- if "$R = \infty$" series converges $\forall x$

R is called the **radius of convergence**:



Example 1.19. $(x+\alpha)^{\alpha}$ $\quad a \neq 0 \in \mathbb{R}, \alpha \in \mathbb{R}$

$\implies$ $a^{\alpha}(1+\frac{x}{a})^{\alpha}$ assuming $a > 0$ (otherwise if $a < 0$, write $a = -b, (x-b)^{\alpha} = b^{\alpha}(\frac{x}{b}-1)^{\alpha}$)

$= (1+t)^{\alpha}$ we "know" this requires $|t| < 1 \implies |\frac{x}{a}| < 1 \implies |x| < |a|$, so $\boxed{R = |a|}$.

What limits the circle of convergence?

In practice, the series converges in as big a circle as it can i.e. until it reaches a singular point.

e.g. $\frac{1}{x^2+1}$ is singular (infinite) when $x = i$, so $R \not> 1$ (else series would have to converge at $x = i$, but it can't as function is infinite there), so $R = 1$ for this function.

[pages = 1,2]series.pdf

[*Lectures 10, 11, 12 on basic analysis and differentiation, see handout and panopto*]

3 Differentiation

Lecture 13
10.11.2014

The product rule and chain rule extend to more than two functions e.g.

$(fgh)' = f'(gh) + f(gh)' = f'(gh) + fg'h + (fg)h'$

Similarly (by defining $g(h(x)) \equiv k(x)$):
$f[g(h[x])]' = f[k(x)]' = f'(k(x)).k'(x) = f'(g(h(x)))g'(h(x))h'(x)$

It's easier to just remember that:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dt}\frac{dt}{d\Omega}\frac{d\Omega}{d\,\mathrm{ch}}\frac{d\,\mathrm{ch}}{d\xi}\frac{d\xi}{dx} \text{ etc.}$$

"First Principles" differentiation

What is $\frac{\mathrm{d}}{\mathrm{d}x}(e^x)$?

$$= \lim_{\epsilon\to 0}\left(\frac{e^{x+\epsilon} - e^x}{\epsilon}\right) = e^x \lim_{\epsilon\to 0}\left(\frac{e^x - 1}{\epsilon}\right) = e^x \lim_{\epsilon\to 0}\left(\frac{1 + \epsilon + \mathcal{O}(\epsilon^2) - 1}{\epsilon}\right)$$

$$= e^x \lim_{\epsilon\to 0}[1 + \mathcal{O}(\epsilon)] = e^x$$

Excerise: Show that $\frac{dx}{dx} = 1$ from first principles

So $1 = \frac{dx}{dx} = \frac{dx}{dy}\frac{dy}{dx}$ (using chain rule). i.e. $\boxed{\frac{dy}{dx} = \frac{1}{dx/dy}}$

N.B. Next term you will meet partial derivatives, $\frac{\partial u}{\partial x}$; The chain rule for partial differentiation is more complicated, and $\frac{\partial u}{\partial x} \neq \frac{1}{\partial x/\partial u}$ necessarily.

Inverse Functions

If $y = \log x$, *What is* $\frac{dy}{dx}$*?*

Write $x = e^y \implies \frac{dx}{dy} = e^y \implies \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

What is $\frac{\mathrm{d}}{\mathrm{d}x}(\sin^- 1x)$*?*

$$\frac{dx}{dy} = \frac{d}{dy}(\sin y) = \frac{d}{dy}\left[y - \frac{y^3}{6} + \frac{y^5}{120} + \dots \frac{y^n}{n!}\right] = \left[1 - \frac{y^2}{2} + \frac{y^4}{24} + \dots\right] = \cos y$$

Or we could use first principles:

$$\begin{aligned}
\frac{d}{dy}[\sin y] = \lim_{\epsilon \to 0}\left[\frac{\sin(y+\epsilon) - \sin(y)}{\epsilon}\right] &= \lim_{\epsilon \to 0}\left[\frac{\sin y \cos \epsilon + \sin \epsilon \cos y - \sin y}{\epsilon}\right] \\
&= \lim_{\epsilon \to 0}\left[\frac{\sin \epsilon}{\epsilon}\cos y\right] + \lim_{\epsilon \to 0}\left[\sin y\left(\frac{\cos \epsilon - 1}{\epsilon}\right)\right] \\
&= \lim_{\epsilon \to 0}\left[\frac{\epsilon + \mathcal{O}(\epsilon^3}{\epsilon}\right]\cos y + \sin y \lim_{\epsilon \to 0}\left[\frac{1 - \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^4) - 1}{\epsilon}\right] \\
&= \cos y
\end{aligned}$$

Similarly $\frac{d}{dx}(\cos x) = -\sin x$.

$$\frac{d}{dx}(\tan x) = \left(\frac{\sin(x)}{\cos(x)}\right) = 1 + \frac{sin^2x}{\cot x} = \frac{cos^2x + sin^2x}{\cos^2 x} = \frac{1}{cos^2x} = \sec^2 x$$

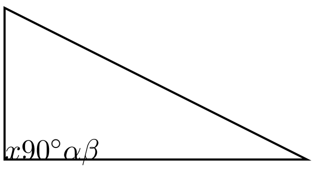
Return to $\frac{d}{dx}(\sin^{-1} x)$; $\quad x = \sin y$

$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

Hence $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}[\sin^{-1} x]$

Exercise Show $\frac{d}{dx}(\cos x) = -\frac{1}{\sqrt{1-x^2}}$

Hence $\frac{d}{dx}[\sin^{-1} x + \cos^{-1} x] = 0$, which is clear from the right angled triangle.
Since $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} x$, $\alpha + \beta = \frac{\pi}{2}$, obviously $\frac{d}{dx}\frac{\pi}{2} = 0$



Logarithmic Derivative

$$\frac{d}{dx}(\log(u(x))) = \frac{d}{du}\log u . \frac{du}{dx} = \frac{1}{u}u'$$

This is called a logarithmic derivative. Useful when performing integration e.g.

$$I = \int \frac{e^x + \cos x}{1 + e^x + \sin x}\,\mathrm{d}x$$

Observe that $u = 1 + e^x + \sin x$ and $u' = e^x + \cos x \implies I = \log(1 + e^x + \sin x) + c$

Implicit Differentiation

If y is given implicitly in terms of x e.g. $y + \tan y = x$, we can differentiate the entire equation term by term with respect to x:

$$\begin{aligned}\frac{d}{dx}(y+\tan y) &= \frac{d}{dx}(x) = 1\\ \implies \frac{dy}{dx}\frac{d}{dy}(y+\tan y) &= 1\\ \implies \frac{dy}{dx}(1+\sec^2 y) &= 1\\ \implies \frac{dy}{dx} &= \frac{1}{\sec^2 y + 1} = \frac{1}{2+\tan^2 y} = \frac{1}{2+(x-y)^2}\end{aligned}$$

Higher Derivatives

If $f(x)$ is differentiable in $(a, b) \iff f'(x)$ is defined on (a, b).

Maybe f' is also differentiable. If so, we write it as f'' or $\dfrac{d^2 f}{dx^2}$ or $\left(\dfrac{d}{dx}\right)^2 f$

BUT NOT EVER EVER $\left(\dfrac{df}{dx}\right)^2$

Continuing, we can write the n'th derivative (if it exists) as $f^{(n)}(x)$ or $\dfrac{d^n f}{dx^n}$ or $\left(\dfrac{d}{dx}\right)^n f$.
These forms are useful for Taylor / Maclaurin series.

Leibniz' Rule

How can we (easily) differentiate a product many times?

Suppose f and g are differentiable an arbitrarily number of times. What is $(fg)'$? Use the product rule:

$(fg)' = f'g + fg'$
$(fg)'' = (f'g + fg')' = (f'g)' + (fg')' = f''g + 2f'g' + fg''$
$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$

We spot a pattern, and make an inspired (but intelligent) guess:

$$(fg)^n = \sum_{r=0}^{n} \binom{n}{k} f^{(r)} g^{(n-r)} \quad \} \text{ Leibniz' Formula} \quad (*)$$

This is very similar to the binomial theorem:
$(f+g)^n = \sum_{r=0}^{n} \binom{n}{k} f^r g^{n-r}$ (which I had hoped would have been proved in M1F)

Proof. Use induction to prove $(*)$

(A) Take $n = 1 : (fg)' = f'g + fg'$ by product rule, so true for $n = 1$

(B) Assume $(*)$ holds when $n = k$, and try to prove it then holds for $n = k + 1$. Hence:

$(fg)^{(k)} = \sum_{r=0}^{k} \binom{k}{r} f^{(r)} g^{(k-r)}$, and differentiate again:

$$\implies (fg)^{(k+1)} = \sum_{r=0}^{k} \binom{k}{r} \left[f^{(r+1)} g^{(k-r)} + f^{(r)} g^{(k-r+1)} \right]$$

$$= f^{(s)} g^{(k+1-s)} \left[\binom{k}{s-1} + \binom{k}{s} \right]$$

Lemma[5]: $\binom{k}{s-1} + \binom{k}{s} = \binom{k+1}{s}$

Assuming lemma, we obtain: $(fg)^{(k+1)} = \sum_{s=0}^{k+1} \binom{k+1}{s} f^{(s)} g^{(k+1-s)}$, as required.

(C) Hence by induction $(*)$ holds for all natural N

□

Note: Leibniz is very useful is one of the functions $\omega \log f$ is a polynomial, as then the high derivatives vanish (are zero)

Example 1.20. What is $\frac{d}{dx}(x^2 \sin x)$?

Using Leibniz, $f = \sin x, g = x^2$:

$$\begin{aligned} \frac{dy}{dx} &= (\sin x)^{(100)} x^2 + \binom{100}{1} \sin x^{(99)} .2x + \binom{100}{2} \sin x^{(98)} .2 + 0 + 0 + \dots \\ &= \sin x[x^2 - 9900] - 200x \cos x \end{aligned}$$

Example 1.21. : What is the Maclaurin series for $y = \sin^{-1} x$?

$y' = \frac{1}{\sqrt{1-x^2}}, y'' = x(1-x^2)^{-3/2}$

$\implies (1 - x^2)y'' = xy'$

[5] *proof*: A) Pester Alessio, B) Pester Emma, C) Use Pascal's Triangle, D) Use Factorials

Differentiate entire equation n times using Leibniz rule:

$$[(1-x^2)y'']^{(n)} = [xy']^{(n)} = xy^{(n+1)} + ny^{(n)}$$

Note that: $[(1-x^2)y'']^{(n)} = (1-x^2)y^{(n+2)} + n(2x)y^{(n+1)} + \dfrac{n(n-1)}{2}(-2)y^{(n)}$

Hence $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0$

So at $x = 0$, we have:
$y^{(n+1)}(0) = n^2y^{(n)}(0) \quad \forall n$

Now $y = \sin^{-1} x \implies y(0) = 0$, and $y' = \dfrac{1}{\sqrt{1-x^2}} \implies y'(0) = 1$
Means all even derivatives are zero.

$y^{(3)}(0) = 1, y^{(5)}(0) = 3^2, y^{(7)}(0) = 5^2.3^2$

So $\sin^{-1} x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(n-2)^2(n-4)^2 \ldots 5^2 \times 3^2 \times 1$