1st Year Mathematics
Imperial College London

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Mathematical Methods II

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Unofficial notes, *not* endorsed by Imperial College.

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† To make this readable, this isn't an exact copy of his notes.

Syllabus

This course continues and extends the techniques introduced in M1M1, with further differential equations and partial differentiation. There are brief introductions to difference equations, curve fitting and scaling, together with a variety of applications of integration.

- First and second order differential equations.
- Homogeneous and inhomogeneous linear differential equations.
- Systems of linear differential equations matrix solution.
- Phase plane analysis: Qualitative analysis of solutions of differential equations and stability.
- Bifurcation of first order non-linear differential equations.
- Partial differentiation: Definitions, implicit partial differentiation, total differential, change of variables.
- Functions of two variables: Taylors theorem, stationary pointe and their classification, contours.
- Vector calculus: Definitions and physical meaning of grad, div, curl.
- Applications of partial differentiation: Optimisation and Lagrange multipliers.
- Applications of integration area under curves, arc length, surface area and volume of revolution; double integrals geometry, mass, moments of inertia; simple triple integrals.

Appropriate books

For the first part of the course on Introduction to Ordinary Differential Equations:
W. E. Boyce. *Elementary differential equations and boundary value problems*, Wiley.
M. Braun, *Differential equations and their applications*, Springer Verlag.
F. Diacu, *An introduction to differential equations*, Freeman
R. Redheffer, *Differential equations: theory and applications*, Jones and Bartlett.

For the section on qualitative analysis of nonlinear ODEs, the first few chapters of this:
S.H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*, Westview Press

For the second part of the course on Introduction to multivariate calculus, another book:
H.M. Schey, *Div, grad, curl and all that*, Norton and Company

Contents

1 Basics

1.1 Sets

Lecture 1

A set S is a collection of objects (called the *elements* of the set)

Example 1.1. A way to specify a set is to list the objects (between curly brackets):

$$S = \{1, 3, 7\}$$

The order of elements is unimportant, as is repetition:

$$\{1, 2\} = \{2, 1\} = \{1, 1, 2\}$$

I say $S_1 \subset S_2$ (S_1 *contained* in S_2) if every element of S_1 is also an element of S_2. I can write this as a *statement*: $x \in S_1 \implies x \in S_2$

I say $S_1 = S_2$ if $S_1 \subset S_2$ & $S_2 \subset S_1$. Elements can be sets: $S = \{1, 2, \{1, 2\}\}$.
But there is one thing you are never allowed to do:

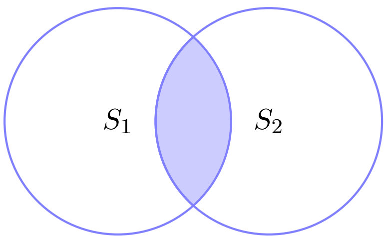
Axiom 1.2 (Foundation Axiom)**.** $S \notin S$

More things about sets:

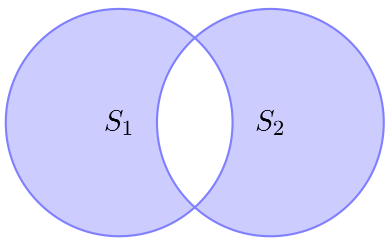
- $a \notin S$ "a is not an element of S"
- $S_1 \cup S_2$ "S_1 union S_2" $= \{x \mid x \in S_1 \text{ or } x \in S_2 \text{ (or both)}\} = \{x : x \in S_1 \text{ or } x \in S_2\}$
- $S_1 \cap S_2$ "S_1 intersection S_2" $= \{x \mid x \in S_1 \text{ and } x \in S_2\}$
- $S_1 \backslash S_2$ "S_1 take away S_2" $= \{x \mid x \in S_1 \& x \notin S_2\}$
- $S_1 \triangle S_2$ "symmetric difference" $= \{x \mid x \in S_1 \text{ or } x \in S_2 \text{ but not both }\}$
$= (S_1 \cup S_2) \backslash (S_1 \cap S_2)$

When you reason about sets & other mathematical objects, it is useful to draw pictures:

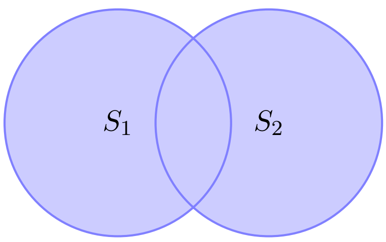
$S_1 \cap S_2$



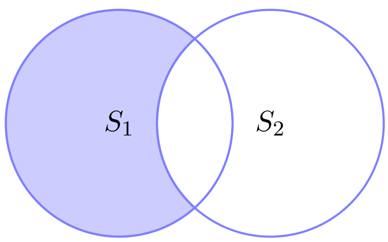
$S_1 \triangle S_2$



$S_1 \cup S_2$



$S_1 \backslash S_2$



Sometimes (often) it is not practical to list all the elements of a set:

Examples 1.3.

$\mathbb{Z}$ = set of integers $= \{0, +1, -1, 2, -2, 3, -3, \dots\}$

$\mathbb{N}$ = set of natural numbers $= \{0, 1, 2, 3, \dots\} = \{n \in \mathbb{Z} \mid n \geq 0\}$

$\mathbb{Q}$ = set of rational numbers $= \{x \mid x = \frac{p}{q},\ p \in \mathbb{Z}, q \in \mathbb{N}\backslash\{0\}\}$

$\mathbb{R}$ = set of real numbers

$\mathbb{C}$ = set of complex numbers

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Lecture 2

2 Multivariable Calculus

2.1 Stationary Points

Lecture 25

Definition. For $u(\vec{x})$ with $\vec{x} \in \mathbb{R}^2$, the *stationary points*, $\vec{x}^*$, occur when

$$\frac{\partial u}{\partial x}(\vec{x}^*) = \frac{\partial u}{\partial y}(\vec{x}^*) = 0$$

What about the character? Maximum / Minimum / Saddle Point.

Example 2.1. Looking at $f(x)$, $x \in \mathbb{R}$:

$$f(x) = f(x^*) + \cancel{\frac{df}{dx}(x^*)\delta x} + \frac{1}{2}\frac{d^2 f}{dx^2}(x^*)(\delta x)^2 + \mathcal{O}(\delta x^3)$$

$$\implies f(x) - f(x^*) = \frac{1}{2}\frac{d^2 f}{dx^2}(x^*)(\delta x)^2 + \mathcal{O}(\delta x^3)$$

So $\frac{d^2 f}{dx^2}(x^*) > 0 \iff f(x) - f(x^*) > 0$. Hence local minimum when $f(x) - f(x^*) > 0$.

For $u = u(x, y)$, consider the taylor expansion about a stationary point $(x^*, y^*) = \vec{x}^*$ (so $\frac{\partial u}{\partial x}(\vec{x}^*) = \frac{\partial u}{\partial y}(\vec{x}^*) = 0$):

$$u(\vec{x}^* + \delta\vec{x}) = u(\vec{x}^*) + \cancel{\left[\frac{\partial u}{\partial x}(\vec{x}^*)\delta x + \frac{\partial u}{\partial y}(\vec{x}^*)\delta y\right]} + \frac{1}{2}\delta\vec{x}^* \underbrace{\begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix}_{\vec{x}^*}}_{\equiv\, H(\vec{x}^*)} \delta\vec{x} + \mathcal{O}(||\delta\vec{x}||^3)$$

Thus:

$$u(\vec{x}^* + \delta\vec{x}) - u(\vec{x}^*) = \frac{1}{2}\delta\vec{x}^* H(\vec{x}^*)\delta\vec{x} + \mathcal{O}(||\delta\vec{x}||^3)$$

So we get:

$$\text{Local} \begin{cases} \text{Minimum:} & \forall \delta\vec{x}^*,\ \delta\vec{x}^* H(\vec{x}^*)\delta\vec{x} > 0 \\ & ||\delta\vec{x}|| \text{ is small} \\ \text{Maximum:} & \forall \delta\vec{x}^*,\ \delta\vec{x}^* H(\vec{x}^*)\delta\vec{x} < 0 \\ & ||\delta\vec{x}|| \text{ is small} \end{cases}$$

Note that $\forall\vec{x}$, $\vec{x}^* A\vec{x} > 0 \iff A$ is positive definite $\iff$ All eigenvalues λ_i are positive.

So for $H(\vec{x}^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ (continuity $\implies \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \implies H(\vec{x}^*) = H^T(\vec{x}^*)$), we have that:

(1) $H(\vec{x}^*)$ is always diagonalisable:

$$v^{-1}Hv = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(2) v is orthogonal (i.e. $v^{-1} = v^T$), so:

$$v^T H v = \Lambda \implies H = v \Lambda v^T$$

(3) All eigenvalues are real:

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}, \ \lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \ \tau^2 - 4\Delta > 0$$

$$\begin{aligned} (A+C)^2 - 4(AC - B^2) &= A^2 + C^2 + 2AC - 4AC + 4B^2 \\ &= A^2 + C^2 - 2AC + 4B^2 \\ &= (A-C)^2 + 4B^2 > 0 \end{aligned}$$

Theorem 2.2

$H = H^T$ is positive definite $\iff$ λ_1, λ_2 are positive.

Proof. $\lambda_1, \lambda_2 > 0 \implies$ positive definite:

$$\begin{aligned} \vec{x}^T H \vec{x} = \vec{x}^T V \Lambda V^T \vec{x} &= \vec{x}^T (\vec{v_1} \ \vec{v_2}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \vec{v_1}^T \\ \vec{v_2}^T \end{pmatrix} \vec{x} \\ &= (\vec{x}^T \cdot \vec{v_1} \ \vec{x}^T \cdot \vec{v_2}) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \vec{v_1}^T \cdot \vec{x} \\ \vec{v_2}^T \cdot \vec{x} \end{pmatrix} \\ &= \lambda_1 (\vec{x}^T \vec{v_1})^2 + \lambda_2 (\vec{x}^T \vec{v_2})^2 > 0 \end{aligned}$$

Positive definite $\implies \lambda_1, \lambda_2 > 0$:

$$\vec{x}^T H \vec{x} = \lambda_1 (\vec{x}^T \vec{v_1})^2 + \lambda_2 (\vec{x}^T \vec{v_3})^2$$

Assume $\vec{x} = \vec{v_1}$, then $\begin{cases} \vec{v_1}^T \cdot \vec{v_2} = 0 \\ \vec{v_1}^T \cdot \vec{v_1} = 1 \end{cases}$ from $VV^T = V^T V = I$

$$\vec{v_1}^T H \vec{v_1} = \lambda_1 \cdot 1 + \lambda_2 \cdot 0 \text{ and } \vec{v_2}^T H \vec{v_2} = \lambda_2$$

So $\lambda_1, \lambda_2 > 0$ is necessary so that $\vec{x}^T H \vec{x} > 0$ ∎

Going back to the character of stationary points:

(i) Minimum:

(ii) Maximum:

(iii) Saddle-Point:

(iv) If λ_1 or λ_2 or both are zero, we need to go to higher derivatives

Example 2.3. $u(x, y) = (x - y)(x^2 + y^2 - 1)$

(i) Contour lines for $u = 0$, $(x - y)(x^2 + y^2 - 1) = 0 \implies$

(ii) Stationary points: