1st Year Mathematics
Imperial College London

Spring 2015

Mechanics

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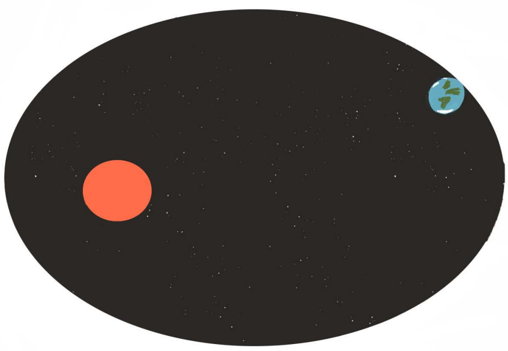


Syllabus

This introductory course on Applied Mathematics is centred on Newtonian mechanics - the consequences of Newtons laws.

Kinematics of point particles

Vectors and vector algebra; position, velocity, and acceleration in three dimensions; polar coordinates; intrinsic coordinates and path curvature.

Newton's Laws

Definition of mass, momentum, inertia, and force; Axioms, or Laws of Motion.

Forces

Gravitation; forces that constrain motion: normal force and tension; friction; forces that depend on velocity: drag forces; forces that depend on position: spring forces.

Oscillators

Simple, damped, and forced oscillators; amplitude and phase difference; resonance.

Energy

Kinetic and potential energies; conservative forces; stability of and motion about fixed points; potential wells and escape; energy diagrams.

Angular Momentum

Central forces; orbital equation; effective potential. Systems of (interacting) particles: Two body systems; centre of mass; moment of inertia; total momentum, angular momentum, and energy for systems; variable mass systems; torque.

Rigid Body Motion

Rigid body kinematics; continuous mass distributions; rigid body dynamics with rotation about a single axis

Appropriate books

D. Kleppner and R. J. Kolenkow *An Introduction to Mechanics.*

G. R. Fowles and G. L. Cassiday *Analytical Mechanics.*

R. Feynman *The Feynman Lectures.*

T. W. B. Kibble and F. H. Berkshire *Classical Mechanics.*

Contents

1 Kinematics

For a point particle there are three key kinematic quantities.

1. Positon: $\vec{r}(t)$
2. Velocity: $\vec{v}(t)$
3. Acceleration: $\vec{a}(t)$

   In general, $\vec{r}(t),\ \vec{v}(t),\ \vec{a}(t) \in \mathbb{R}^3$.

We can use different coordinate systems to describe our quantities:

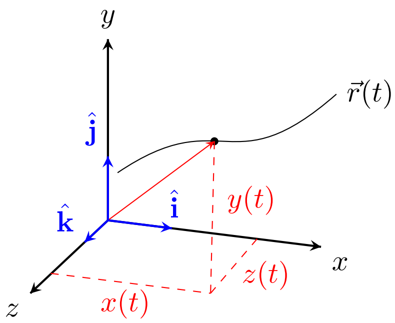
(i) Cartesian

(ii) Polar

(iii) Intrinsic

Cartesian Coordinates

Consider the path of a particle through space:



Definition. We write the *position* at time t as

$$\vec{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$$

We can also write this as

$$[\vec{r}(t)] = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \text{ so, } \hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Magnitude of $\vec{r}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

r is the distance from the origin.

Direction of $\vec{r}$

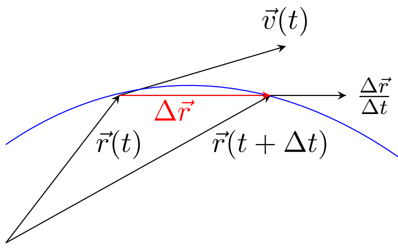
$$\hat{\mathbf{r}} = \vec{r}/r = \frac{x}{r}\,\hat{\mathbf{i}} + \frac{y}{r}\,\hat{\mathbf{j}} + \frac{z}{r}\,\hat{\mathbf{k}}$$

So, we can write

$$\vec{r} = r(t)\,\hat{\mathbf{r}}(t)$$

This is the starting point for polar coordinates.

Last Time: Lecture 2



Position:

$$\vec{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + zd(t)\,\hat{\mathbf{k}}$$

At Δt later

$$\vec{r}(t+\Delta t) = x(t+\Delta t)\,\hat{\mathbf{i}} + y(t+\Delta t)\,\hat{\mathbf{j}} + z(t+\Delta t)\,\hat{\mathbf{k}}$$

Definition. Define $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

Define the *velocity* of the particle at time t

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

Since $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are constant in time

$$\begin{aligned}\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\,\hat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\,\hat{\mathbf{j}} + \frac{\mathrm{d}z}{\mathrm{d}t}\,\hat{\mathbf{k}}\end{aligned}$$

Writing $\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \dot{f}$,

$$\vec{v}(t) = \dot{x}\,\hat{\mathbf{i}} + \dot{y}\,\hat{\mathbf{j}} + \dot{z}\,\hat{\mathbf{k}} = v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}$$

Definition.

$$v = |\vec{v}| = [v_x^2 + v_y^2 + v_z^2]^{1/2}$$

is the magnitude of the velocity or *speed* of the particle.

Thus, the *direction* of motion is

$$\hat{\mathbf{v}} = \vec{v}/v, \ \ |\,\hat{\mathbf{v}}| = 1$$

$\hat{\mathbf{v}}$ is also the unit tangent to the path.

Define the *acceleration*

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\,\hat{\mathbf{i}} + \ddot{y}\,\hat{\mathbf{j}} + \ddot{z}\,\hat{\mathbf{k}}$$

$\vec{a}(t)$ tells us how the velocity is changing at time t.

Recall that we can write $\vec{v} = v(t)\,\hat{\mathbf{v}}(t)$, then

$$\vec{a} = \frac{d}{dt}(v\,\hat{\mathbf{v}}) = \underbrace{\frac{dv}{dt}\,\hat{\mathbf{v}}}_{\substack{\text{Due to change}\\ \text{in speed}}} + \underbrace{v\frac{d\,\hat{\mathbf{v}}}{dt}}_{\substack{\text{Due to change}\\ \text{in direction}}}$$

Starting with $\vec{a}$ we can integrate to find $\vec{v}$, then $\vec{r}$.

$$\int_{t_0}^{t} \vec{a}dt' = \int_{t_0}^{t} \frac{d\vec{v}}{dt'}\, dt' = \vec{v}(t) = \vec{v}(t_0)$$

$$\implies \vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}\, dt'$$

We can integrate this component. For example

$$v_x(t) = v_x(t_0) \int_{t_0}^{t} a_x\, dt'$$

To determine $\vec{v}(t)$ uniquely, we need to know $\vec{v}(t_0)$ (constant vector).

Similarly

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(t')\, dt'$$

Thus, starting with $\vec{a}(t)$, we need to know *both* $\vec{r}(t_0)$ and $\vec{v}(t_0)$ to find $\vec{r}(t)$ uniquely!

What allows us to connect the mathematics to the physical world is that these quantities are measurable.

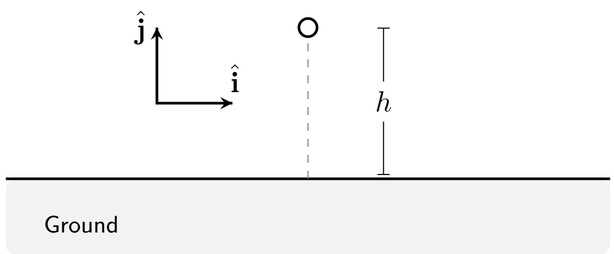
Definition (SI Units)**.**

- *Time:* Measured in seconds, s
- *Distance:* Measured in metres, m
- *Velocity:* "Metres per second", m/s or ms^{-1}
- *Accelerations*: "Metres per second squared"", $\mathrm{m/s}^2$ or ms^{-2}

Example 1.1. Near the surface of the earth, the acceleration due to gravity is constant!

$$g = 9.8ms^{-2}$$

Suppose: an object is dropped *from rest* at height h above the ground. Find $\vec{r}(t)$:

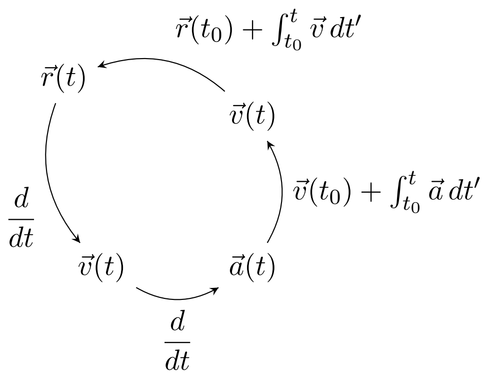


Align the coordinates such that $\hat{\mathbf{j}}$ points upwards. We know that $\vec{a} = -g\hat{\mathbf{j}}$. No motion in other directions. Problem is 1D!

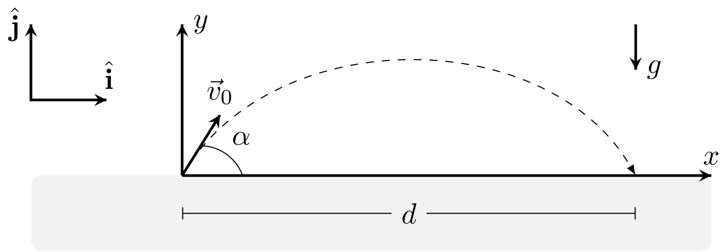
We know initially $(t = 0)$ that $y(0) = h \implies v_y(0) = 0$. Integrate twice:

$$y(t) = h - \frac{1}{2}gt^2$$

Recap: Lecture 3



Example 1.2 (Projectile)**.**



In this coordinate system $\vec{a} = -g\,\hat{\mathbf{j}}$. Choose that $t = 0$ when the object is released. Based on this:

$$\vec{r}(0) = \vec{0},\ \vec{v}(0) = \vec{v}_0 = v_0 \cos\alpha\,\hat{\mathbf{i}} + v_0 \sin\alpha\,\hat{\mathbf{j}}$$

Integrate our acceleration to find $\vec{v}(t)$

$$\begin{aligned}\vec{v}(t) &= \vec{v}(0) + \int_0^t -g\,\hat{\mathbf{j}}\, dt' = \vec{v}_0 - gt\,\hat{\mathbf{j}} \\ &= v_0 \cos\alpha\,\hat{\mathbf{i}} + (v_0 \sin\alpha - gt)\,\hat{\mathbf{j}}\end{aligned}$$

Integrate our velocity to find the position

$$\begin{aligned}\vec{r}(t) &= \vec{r}(0) + \int_0^t \vec{v}(t)\, dt' = \vec{0} + \int_0^t \vec{v}_0 - gt'\,\hat{\mathbf{j}}\, dt' \\ &= \vec{v}_0 t - \tfrac{1}{2}t^2 g\,\hat{\mathbf{j}} = v_0 t\cos\alpha\,\hat{\mathbf{i}} + [v_0 t \sin\alpha - \tfrac{1}{2}t^2 g]\,\hat{\mathbf{j}}\end{aligned}$$

Maximise the *range* using α as the control parameter. Finding the time t_H when the object hits the ground. $y(t_H) = 0$ where $y = \vec{r} \cdot \hat{\mathbf{j}}$.

$$y = \vec{r} \cdot \hat{\mathbf{j}} = v_0 t_H \sin\alpha - \tfrac{1}{2}t_H^2 g = 0$$

Two values of t_H:

$$t_H = 0,\ t_H = \frac{2v_0 \sin\alpha}{g}$$

To find the range:

$$x(t_H) = v_0 \cos\alpha \left[\frac{2v_0 \sin\alpha}{g}\right] = \frac{v_0^2}{g} \sin 2\alpha$$

For $0 \leq \alpha \leq \pi/2$, the range is maximum for $\alpha = \pi/4$.

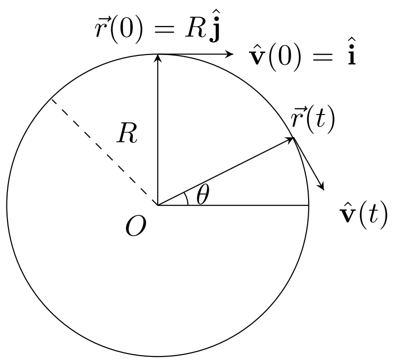
Example 1.3 (Circular Motion)**.**

$$\vec{r}(t) = R\sin(\Omega t)\,\hat{\mathbf{i}} + R\cos(\Omega t)\,\hat{\mathbf{j}} \quad (R, \Omega > 0)$$

Distance from the origin

$$r = |\vec{r}| = [R^2 \sin^2 \Omega t + R^2 \cos^2 \Omega t]^{1/2} = R$$

Path is a circle of radius R, centred at the origin.



Differentiate $\vec{r}(t)$ to find

$$\vec{v}(t) = R\Omega \cos\Omega t\,\hat{\mathbf{i}} - R\Omega \sin\Omega t\,\hat{\mathbf{j}}$$

Find the speed: $v = |\vec{v}| = R\Omega$, the speed is constant.

Clockwise or anticlockwise?

Direction of motion

$$\hat{\mathbf{v}} = \vec{v}/v = \cos\Omega t\,\hat{\mathbf{i}} - \sin\Omega t\,\hat{\mathbf{j}}$$

At $t = 0$, $\hat{\mathbf{v}}(0) = \hat{\mathbf{i}}$, so it moves clockwise around the circle!

Interpretation of Ω*:* Introduce $\theta = -\Omega t + \pi/2$, $\frac{d\theta}{dt} - \Omega$.

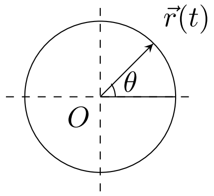
The parameter Ω is angular speed. Differentiate our $\vec{v}(t)$ we find

$$\begin{aligned}\vec{a}(t) &= -R\Omega^2 \sin[\Omega t]\,\hat{\mathbf{i}} - R\Omega^2 \cos[\Omega t]\,\hat{\mathbf{j}} \\ &= -\Omega^2 \vec{r}\end{aligned}$$

Acceleration is pointing in towards the circle.

Last time: Lecture 4

$$\vec{r}(t) = R\sin\Omega t\,\hat{\mathbf{i}} + R\cos\Omega t\,\hat{\mathbf{j}}$$



$$\theta = -\Omega t + \tfrac{\pi}{2}$$

$$\vec{r}(t) = R\cos[\theta(t)]\,\hat{\mathbf{i}} + R\sin[\theta(t)]\,\hat{\mathbf{j}}$$

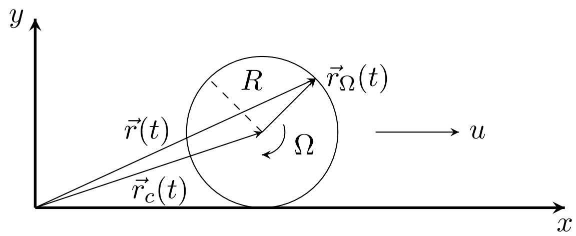
Insert expression for θ

$$\begin{aligned}
\vec{r}(t) &= R\cos[\pi/2 - \Omega t]\,\hat{\mathbf{i}} + R\sin[\pi/2 - \Omega t]\,\hat{\mathbf{j}} \\
&= R(\cancelto{0}{\cos\pi/2}\cos\Omega t + \cancelto{1}{\sin\pi/2}\sin\Omega t)\,\hat{\mathbf{i}} \\
&+ R(\cancelto{1}{\sin\pi/2}\cos\Omega t - \cancelto{0}{\cos\pi/2}\sin\Omega t)\,\hat{\mathbf{j}} \\
&= R\sin\Omega t\,\hat{\mathbf{i}} + R\cos\Omega t\,\hat{\mathbf{j}}
\end{aligned}$$

Example 1.4 (Wheel rolling without slipping)**.**

Describe the position of a point on the surface of the wheel

$$\vec{r}(t) = \vec{r}_\Omega(t) + \vec{r}_c(t)$$



$$\begin{aligned}
\vec{r}_\Omega(t) &= R\cos\theta(t)\,\hat{\mathbf{i}} + R\sin\theta(t)\,\hat{\mathbf{j}} \\
\vec{r}_c(t) &= x_c(t)\,\hat{\mathbf{i}} + R\hat{\mathbf{j}}
\end{aligned}$$

Rolling at a constant angular speed: Ω.
We are translating to the right with constant velocity: u.

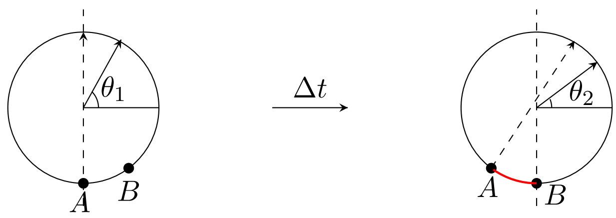
Suppose initially $x_c(0) = 0,\ \theta(0) = \pi/2$.

Then $x_c(y) = ut$ and $\theta = \pi/2 - \Omega t$.

$$\begin{aligned}
\vec{r}_\Omega(t)f &= R\sin\Omega t\,\hat{\mathbf{i}} + R\cos\Omega t\,\hat{\mathbf{j}} \\
\vec{r}_c(t) &= ut\vec{i} + R\vec{j} \\
\vec{r}(t) &= [ut + R\sin(\Omega t)]\,\hat{\mathbf{i}} + R[1 + \cos\Omega t]\,\hat{\mathbf{j}}
\end{aligned}$$

Here, u, and Ω are still independent of one another. We use "rolling without slipping" to connect them:

Rolling without slipping:



Rolling without slipping implies that the distance travelled equals the arc length between A and B.

$$\frac{\Delta x_c}{\Delta t} = -R\frac{(\theta_2 - \theta_1)}{\Delta t}$$

Taking the limit as $\Delta t \to 0$

$$\dot{x}_c = -R\dot{\theta}$$

For our problem

$$\dot{x}_c = u, \quad \dot{\theta} = -\Omega, \quad \boxed{u = R\Omega}$$

with this expression

$$\vec{r}(t) = R[\Omega t + R\sin(\Omega t)]\,\hat{\mathbf{i}} + R[1 + \cos\Omega t]\,\hat{\mathbf{j}}$$

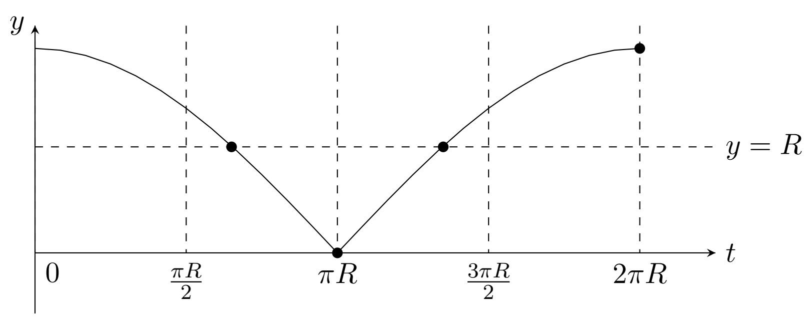
Differentiate w.r.t t

$$\vec{v}(t) = R\Omega[1 + cos(\Omega t)]\,\hat{\mathbf{i}} - R\Omega\sin(\Omega t)\,\hat{\mathbf{j}}$$

Differentiating again

$$\begin{aligned}\vec{a}(t) &= -R\Omega^2[\sin(\Omega t)\,\hat{\mathbf{i}} + \cos(\Omega t)\,\hat{\mathbf{j}}] \\ &= \vec{a}_\Omega(t)\end{aligned}$$

Sketch the path:



At $x_c = \pi R \implies ut = \pi R$

$$y = \frac{\pi R}{u} = \frac{\pi}{\Omega}$$

$$\begin{aligned}\vec{v}(\pi/\Omega) &= R\Omega[1 + \cos[\Omega\pi/\Omega]]\,\hat{\mathbf{i}} - R\Omega\sin[\Omega\pi/\Omega]\,\hat{\mathbf{j}} \\ &= \vec{0}\end{aligned}$$

Another way of expressing the condition is that the point on the wheel touching the ground has zero velocity *relative* to the ground!

Vector Operations

Already we have seen vector addition and subtraction are useful: Lecture 5

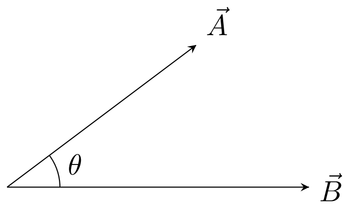
Addition: $\vec{r} = \vec{r}_c + \vec{r}_\Omega$

Subtraction: Velocities relative to a moving observer $\vec{v}_{A,O} = \vec{v}_A - \vec{v}_O$

Vector Products

Vector products are also useful and do arise in describing mechanical phenomena:

(i) Scalar product (dot product)



$$\begin{aligned}\vec{A}\cdot\vec{B} &= |\vec{A}||\vec{B}|\cos\theta \\ \vec{A} &= A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}} \\ \vec{B} &= B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}} \\ \vec{A}\cdot\vec{B} &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

We can write $|\vec{A}| = (\vec{A}\cdot\vec{A})^{1/2}$

The dot product can be used to pick out the component of a vector in a particular direction:

$$\hat{\mathbf{n}},\ |\,\hat{\mathbf{n}}| = 1 \text{ (Direction)}$$

The component of $\vec{A}$ in $\hat{\mathbf{n}}$-direction is

$$A_n = \vec{A}\cdot\,\hat{\mathbf{n}}$$

E.g. $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, then

$$\begin{aligned}\vec{A}\cdot\,\hat{\mathbf{i}} &= A_x(\,\hat{\mathbf{i}}\cdot\,\hat{\mathbf{i}}) + A_y\cancel{(\,\hat{\mathbf{j}}\cdot\hat{\imath})} + A_z\cancel{(\,\hat{\mathbf{i}}\cdot\,\hat{\mathbf{k}})} \\ &= A_x\end{aligned}$$

Example 1.5. Acceleration tangent to the path of point on the surface of a wheel:

Recall:

$$\vec{v}(t) = R\Omega[1+\cos(\Omega t)]\,\hat{\mathbf{i}} - R\Omega\sin(\Omega t)\,\hat{\mathbf{j}}$$

$$\vec{a}(t) = -R\Omega^2[\sin(\Omega t)\,\hat{\mathbf{i}} + \cos(\Omega t)\,\hat{\mathbf{j}}]$$

Direction tangent to the path:

$$\frac{\vec{v}}{v} = \hat{\mathbf{v}},\ |\hat{\mathbf{v}}| = 1$$

Find: speed

$$v^2 = |\vec{v}|^2 = R^2\Omega^2(1 + 2\cos\Omega t + \cos^2\Omega t + \sin^2\Omega t)$$

$$v = R\Omega\sqrt{2(1+\cos\Omega t)}$$

The component of the acceleration tangent to the path

$$a_t = \frac{\vec{a}\cdot\vec{v}}{v}$$

$$\vec{a}\cdot\vec{v} = -R^2\Omega^3(\sin\Omega t + \sin\Omega t\cos\Omega t - \sin\Omega t\cos\Omega t]$$

$$= -R^2\Omega^3\sin\Omega t$$

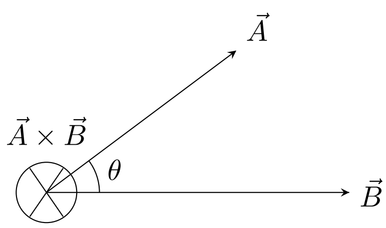
$$a_t = -\frac{R\Omega^2\sin\Omega t}{\sqrt{2(1+\cos\Omega t)}}$$

Check that $a_t = \dfrac{dv}{dt}$.

(ii) Vector product (cross product)

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{\mathbf{i}}(A_yB_z - B_yA_z) + \hat{\mathbf{j}}(A_zB_x - A_xB_z) + \hat{\mathbf{k}}(A_xB_y - A_yB_x)$$



$$|\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta$$

The direction is given by the "right hand rule"

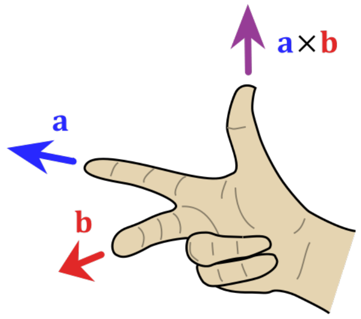


Figure 1: Right Hand Rule

Example 1.6. Introduce $\vec{\omega} = -\Omega\,\hat{\mathbf{k}}$, $(\Omega > 0)$

$$\vec{r}_\Omega = R\sin\Omega t\,\hat{\mathbf{i}} + R\cos\Omega t\,\hat{\mathbf{j}}$$

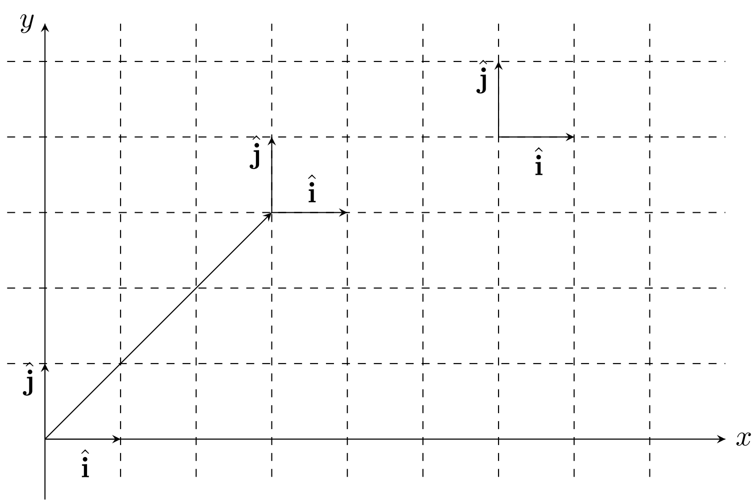
$$\begin{aligned}\vec{\omega}\times\vec{r}_\Omega &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -\Omega \\ R\sin\Omega t & R\cos\Omega t & 0\end{vmatrix} \\ &= R\Omega\cos\Omega t\,\hat{\mathbf{i}} - R\Omega\sin\Omega t\,\hat{\mathbf{j}} \\ &= \frac{d\vec{r}_\Omega}{dt} = \vec{v}_\Omega\end{aligned}$$

Ω: Angular speed

$\vec{\omega}$: Angular velocity

$-\hat{\mathbf{k}}$ is the axis of rotation

What we have been doing really is thinking inside the box:



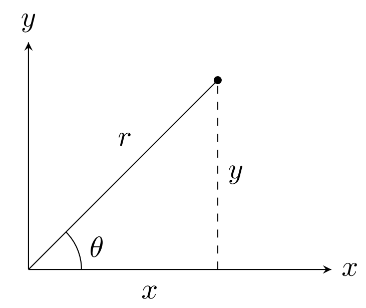
$\hat{\mathbf{i}}$: Points along lines of constant y in the direction of increasing x

$\hat{\mathbf{j}}$: Points along lines of constant x in the direction of increasing y

Polar Coordinates

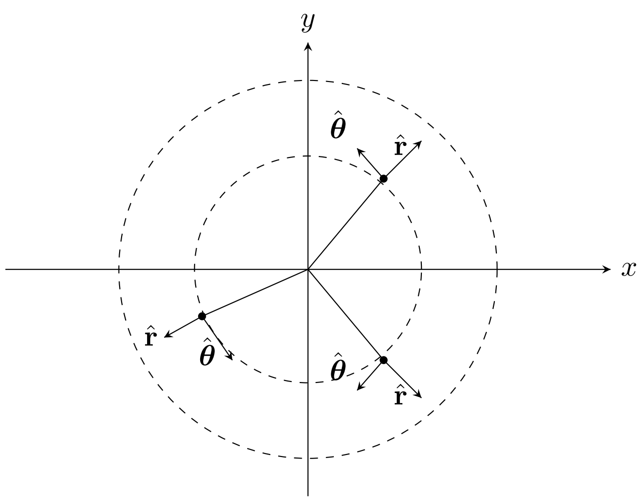
Lecture 6

- 2D Motion
- Certain systems simplify in these coordinates



We can relate x and y to r and θ:

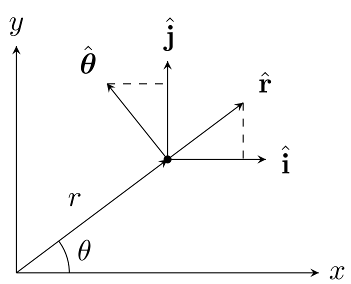
$$x = r\cos\theta, \quad r = [x^2 + y^2]^{1/2}$$
$$y = r\sin\theta, \quad \theta = \arctan(y/x)$$



$\hat{\mathbf{r}}$: Unit vector pointing in the direction of increasing r along lines of constant θ.

$\hat{\boldsymbol{\theta}}$: Points in the direction of increasing θ tangent to the curves of constant r.

Consider:



$$|\,\hat{\mathbf{r}}| = |\,\hat{\boldsymbol{\theta}}| = 1$$
$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

- They depend on θ
- $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$: *orthonormal*

Definition (Kinematic Quantities in Polar Coordinates)**.**

Position:

$$\begin{aligned}\vec{r} &= x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\\ &= r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}\\ &= r[\underbrace{\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}}_{\hat{\mathbf{r}}}]\end{aligned}$$

$$\boxed{\vec{r} = r(t)\,\hat{\mathbf{r}}(t)}$$

Velocity:

$$\begin{aligned}\vec{v} &= \frac{d}{dt}(\vec{r})\\ &= \frac{d}{dt}(r\,\hat{\mathbf{r}})\\ &= \dot{r}\,\hat{\mathbf{r}} + r\frac{d\,\hat{\mathbf{r}}}{dt}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\,\hat{\mathbf{r}} &= \frac{d}{dt}[\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}]\\ &= -\sin\theta\dot{\theta}\,\hat{\mathbf{i}} + \cos\theta\dot{\theta}\,\hat{\mathbf{j}}\\ &= \dot{\theta}[-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}]\\ &= \dot{\theta}\,\hat{\boldsymbol{\theta}}\end{aligned}$$

$$\boxed{\vec{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}}$$

Acceleration:

$$\begin{aligned}\vec{a} &= \frac{d}{dt}(\vec{v})\\ &= \frac{d}{dt}[\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}]\\ &= \ddot{r}\,\hat{\mathbf{r}} + \dot{r}\frac{d\,\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\,\hat{\boldsymbol{\theta}}}{dt}\end{aligned}$$

$$\begin{aligned}\frac{d\,\hat{\boldsymbol{\theta}}}{dt} &= \frac{d}{dt}[-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}]\\ &= -\cos\theta\dot{\theta}\,\hat{\mathbf{i}} - \sin\theta\dot{\theta}\,\hat{\mathbf{j}}\\ &= -\dot{\theta}\,\hat{\mathbf{r}}\end{aligned}$$

Thus

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}}$$

Examples 1.7.

(i) Uniform Circular Motion:

$$\vec{r}(t) = R\sin\Omega t\,\hat{\mathbf{i}} + R\cos\Omega t\,\hat{\mathbf{j}}$$

The corresponding expression in polars:

$$r = R, \quad \theta(t) = \frac{\pi}{2} - \Omega t$$

$$\boxed{\vec{r} = R\,\hat{\mathbf{r}}}$$

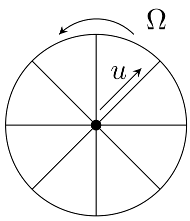
$$\vec{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

$$\boxed{\vec{v} = -R\Omega\,\hat{\boldsymbol{\theta}}}$$

$$\vec{a} : \quad \ddot{r} = \ddot{\theta} = 0$$

$$\boxed{\vec{a} = -R\Omega^2\,\hat{\mathbf{r}}}$$

(ii) Bead moves outwards with speed u as the wheel turns with angular speed Ω:



$(u, \Omega > 0)$

If at $t = 0$:

$$r(0) = 0 \;\;\&\;\; \theta(0) = 0$$

$$\implies r(t) = ut \;\;\&\;\; \theta(t) = \Omega t$$

Then $\dot{r} = u, \ddot{r} = 0, \dot{\theta} = \Omega, \ddot{\theta} = 0$

$$\vec{r} = ut\,\hat{\mathbf{r}}$$

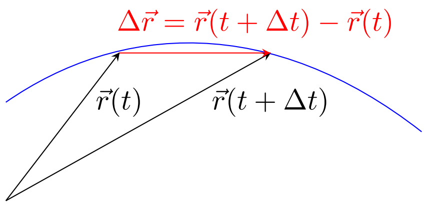
$$\vec{v} = u\,\hat{\mathbf{r}} + u\Omega t\,\hat{\boldsymbol{\theta}}$$

$$\vec{a} = -u\Omega^2 t\,\hat{\mathbf{r}} + 2u\Omega\,\hat{\boldsymbol{\theta}}$$

Polar coordinates really come in handy when the forces have certain symmetries, e.g. Central forces: $\vec{F} = F(r)\,\hat{\mathbf{r}}$

Intrinsic Coordinates

Coordinates that are intrinsic to the path of our particle. We know the path! Lecture 7



Distance travelled between t and $t + \Delta t$:

$$\begin{aligned}\Delta s &= |\Delta\vec{r}| \\ &= \left|[x(t+\Delta t) - x(t)]\,\hat{\mathbf{i}} + [y(t+\Delta t) - y(t)]\,\hat{\mathbf{j}} + [z(t+\Delta t) - z(t)]\,\hat{\mathbf{k}}\right|\end{aligned}$$

For $\Delta t << 1$:

$$x(t+\Delta t) = x(t) + \Delta t\frac{dx}{dt} + \mathcal{O}(\Delta t^2)$$

Doing the same for our other components:

$$\Delta s = \underbrace{\left|\frac{dx}{dt}\,\hat{\mathbf{i}} + \frac{dy}{dt}\,\hat{\mathbf{j}} + \frac{dz}{dt}\,\hat{\mathbf{k}}\right|}_{\vec{v}}\Delta t + \mathcal{O}(\Delta t^2)$$

Thus,

$$\frac{\Delta s}{\Delta t} = v + \mathcal{O}(\Delta t)$$

Taking $\lim \Delta t \to 0$

$$\boxed{\frac{ds}{dt} = v = \dot{s}}$$

$$\implies s(t) = \int_0^t v(t')\,dt'$$

Both t and s are ways of parametrizing our curve (path). Instead of writing $\vec{r}(t)$, we can write $\vec{r}(s)$.

> **Definition.** s is what we call the *arc length.*
>
> *Velocity*:
>
> $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$
>
> But $\dfrac{ds}{dt} = v$. So, $\dfrac{d\vec{r}}{ds} = \hat{\mathbf{v}}$, the unit tangent at evert point s. Thus
>
> $$\boxed{\vec{v}(s) = \dot{s}\,\hat{\mathbf{v}}}$$

Acceleration:

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{d}{dt}\left(\dot{s}\,\frac{d\vec{r}}{ds}\right) \\ &= \ddot{s}\frac{\mathrm{d}\vec{r}}{ds} + \dot{s}\frac{d}{dt}\left(\frac{d\vec{r}}{ds}\right) \\ &= \ddot{s}\,\hat{\mathbf{v}} + \dot{s}\frac{d^2\vec{r}}{ds^2}\frac{ds}{dt}\end{aligned}$$

Writing $\dfrac{d^2\vec{r}}{ds^2} = \kappa\,\hat{\mathbf{n}}$, where $\kappa = \left|\dfrac{d^2\vec{r}}{ds^2}\right|$, $\hat{\mathbf{n}} = \dfrac{1}{\kappa}\dfrac{d^2\vec{r}}{ds^2}$, we have:

$$\boxed{\vec{a}(s) = \ddot{s}\,\hat{\mathbf{v}} + \kappa\dot{s}^2\,\hat{\mathbf{n}}}$$

It turns out that κ is the *curvature* of the path. What about $\hat{\mathbf{n}}$?

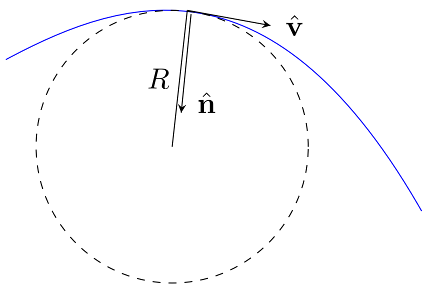
Recall: $|\,\hat{\mathbf{v}}| = 1$. So

$$\begin{aligned}&\frac{d}{ds}(\,\hat{\mathbf{v}}\cdot\,\hat{\mathbf{v}} = 1) \\ \implies\;& 2\,\hat{\mathbf{v}}\cdot\frac{d\,\hat{\mathbf{v}}}{ds} = 0 \\ \implies\;& 2\kappa(\,\hat{\mathbf{v}}\cdot\,\hat{\mathbf{n}}) = 0\end{aligned}$$

So, if $\kappa \neq 0$, then $\hat{\mathbf{v}}\cdot\,\hat{\mathbf{n}} = 0$. Thus $\hat{\mathbf{n}}$ is the *unit normal* to the path.

Tangential component of the acceleration $a_t = \vec{a}\cdot\,\hat{\mathbf{v}} = \ddot{s}$

Normal component of the acceleration $a_n = \vec{a}\cdot\,\hat{\mathbf{n}} = \kappa\dot{s}^2$, where $\kappa \approx 1/R$



Key things to note: $\hat{\mathbf{v}}$, $\hat{\mathbf{n}}, \kappa$ depend only on the path. Knowing $\vec{r}(s)$, we can find these quantities.

$\dot{s}$ and $\ddot{s}$ depend on how the particle is moving along the path.

Example 1.8 (Circular Motion).

Cartesian:

$$\vec{r}(t) = R\sin(\omega t)\,\hat{\mathbf{i}} + R\cos(\omega t)\,\hat{\mathbf{j}}$$
$$\vec{v}(t) = \dots$$
$$\vec{a}(t) = \dots$$

Polars: $r = R$ and $\theta = \frac{\pi}{2}\omega t \implies \dot{r} = \ddot{r} = 0$ and $\dot{\theta} = -\omega,\ \ddot{\theta} = 0$. So

$$\vec{r} = R\,\hat{\mathbf{r}}$$
$$\vec{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = -R\omega\,\hat{\boldsymbol{\theta}}$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}} = -R\omega^2\,\hat{\mathbf{r}}$$

Intrinsic: $(s(0) = 0)$

The speed is given by $v = R\omega = \dot{s} \implies \ddot{s} = 0$. Integrate to find s

$$s = R\omega t \implies t = \frac{s}{R\omega}$$

Substitute this into our expression for $\vec{r}(t)$

$$\vec{r}(s) = R\sin(s/R)\,\hat{\mathbf{i}} + R\cos(s/R)\,\hat{\mathbf{j}}$$

Tangent:

$$\hat{\mathbf{v}} = \frac{d\vec{r}}{ds} = \cos(s/R)\,\hat{\mathbf{i}} - \sin(s/R)\,\hat{\mathbf{j}}$$

Curvature and Normal:

$$\frac{d^2\vec{r}}{ds^2} = -\frac{1}{R}[\sin(s/R)\,\hat{\mathbf{i}} + \cos(s/R)\,\hat{\mathbf{j}}$$

$$\kappa = \left|\frac{d^2\vec{r}}{ds^2}\right| = \frac{1}{R},\quad \hat{\mathbf{n}} = -\sin(s/R)\,\hat{\mathbf{i}} - \cos(s/R)\,\hat{\mathbf{j}}$$

Thus

$$\vec{v}(s) = R\omega\,\hat{\mathbf{v}}$$
$$\vec{a}(s) = \ddot{s}\,\hat{\mathbf{v}} + \kappa\dot{s}^2\,\hat{\mathbf{n}}$$
$$= \frac{1}{R}(R\omega)^2\,\hat{\mathbf{n}}$$
$$= R\omega^2\,\hat{\mathbf{n}}$$

Lecture 8

Example 1.9 (Helical Path)**.**

$$\vec{r}(s) = b\cos(ks)\,\hat{\mathbf{i}} + b\sin(ks)\,\hat{\mathbf{j}} + s\sqrt{1-b^2k^2}\,\hat{\mathbf{k}}$$

Tangent:

$$\hat{\mathbf{v}} = \frac{d\vec{r}}{ds} = -bk\sin(ks)\,\hat{\mathbf{i}} + bk\cos(ks)\,\hat{\mathbf{j}} + \sqrt{1-b^2k^2}\,\hat{\mathbf{k}}$$

Curvature and Normal:

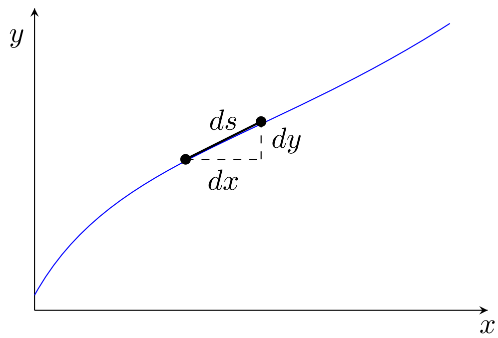
$$\frac{d^2\vec{r}}{ds^2} = -bk^2\cos(ks)\,\hat{\mathbf{i}} - bk^2\sin(ks)\,\hat{\mathbf{j}}$$

$$\kappa = \left|\frac{d^2\vec{r}}{ds^2}\right| = bk^2,\quad \hat{\mathbf{n}} = -\cos(ks)\,\hat{\mathbf{i}} - \sin(ks)\,\hat{\mathbf{j}}$$

Take $s = ct$ $(c > 0)$ $\implies$ $\dot{s} = c,\ \ddot{s} = 0$. Thus

$$\vec{v} = c\,\hat{\mathbf{v}},\ \vec{a} = c^2bk^2\,\hat{\mathbf{n}}$$

Take the case where our path lies in the $xy-$plane and we know $y(x)$



Then $ds^2 = dx^2 + dy^2$. Since $dy = \dfrac{dy}{dx}dx$

$$ds^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)dx^2$$

$$\implies \frac{ds}{dx} = \sqrt{1+y'^2} \quad y' = \frac{dy}{dx}$$

$$s(x) = \int_{x_0}^{x}(1+y'^2)^{1/2}dx$$

Highlights that s just depends on the path.

Position:

$$\vec{r}(x) = x\,\hat{\mathbf{i}} + y(x)\,\hat{\mathbf{j}}$$

Tangent to the path

$$\hat{\mathbf{v}} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dx}\frac{dx}{ds} = \frac{d\vec{r}}{dx}\left(\frac{ds}{dx}\right)^{-1}$$

$$\frac{d\vec{r}}{dx} = \hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}$$
$$\left(\frac{ds}{dx}\right)^{-1} = [1+y'^2]^{-1/2}$$
$$\implies \boxed{\hat{\mathbf{v}} = [1+y'^2]^{-1/2}[\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}]}$$

Curvature and Normal:

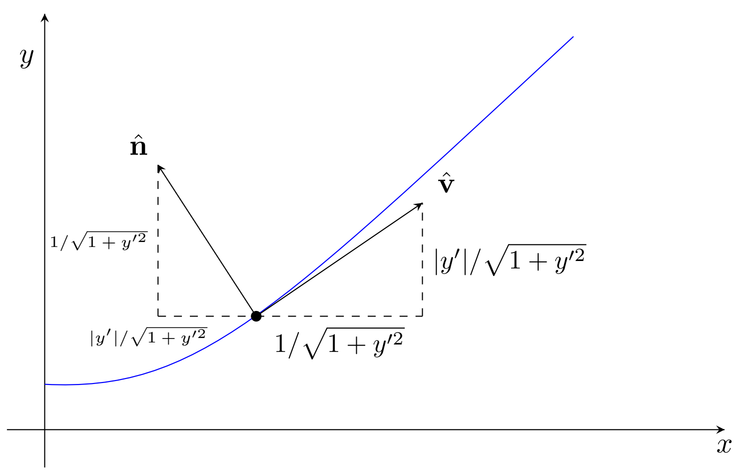
$$\begin{aligned}\frac{d^2\vec{r}}{ds^2} &= \frac{d}{dx}\left(\frac{d\vec{r}}{ds}\right)\left(\frac{ds}{dx}\right)^{-1}\\ &= \left(\frac{d}{dx}\left(\frac{d\vec{r}}{ds}\right)\frac{dx}{ds}\right)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{d\vec{r}}{ds}\right] &= \frac{d\,\hat{\mathbf{v}}}{dx}\\ &= -\frac{1}{2}[1+y'^2]^{-3/2} \times (2y'y'') \times [\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}}]\\ &+ [1+y'^2]^{-1/2}y''\,\hat{\mathbf{j}}\\ &= [1+y'^2]^{-3/2}y'' \times (-y'\,\hat{\mathbf{i}} + (1+y'^2-y'^2)\,\hat{\mathbf{j}})\\ &= \frac{y''}{[1+y'^2]^{3/2}}(-y'\,\hat{\mathbf{i}} + \hat{\mathbf{j}})\end{aligned}$$

$$\implies \frac{d^2\vec{r}}{ds^2} = \frac{y''}{[1+y'^2]^{3/2}}\left(-\frac{y'}{[1+y'^2]^{1/2}}\,\hat{\mathbf{i}} + \frac{1}{[1+y'^2]^{1/2}}\,\hat{\mathbf{j}}\right)$$

$$\boxed{\kappa = \left|\frac{d^2\vec{r}}{ds^2}\right| = \frac{|y''|}{[1+y^2]^{3/2}}}$$

$$\boxed{\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d^2\vec{r}}{ds^2} = \frac{y''}{|y''|}\frac{1}{[1+y'^2]^{1/2}}(-y'\,\hat{\mathbf{i}} + \hat{\mathbf{j}})}$$



Example 1.10. $y = x^2,\ y' = 2x,\ y'' = 2$

$$\begin{aligned}\frac{ds}{dx} &= [1 + y'^2]^{1/2} \\ &= [1 + 4x^2]^{1/2}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{v}} &= \frac{d\vec{r}}{dx}\left(\frac{ds}{dx}\right)^{-1} \\ &= (\,\hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}})(1 + 4x^2)^{-1/2}\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{n}} &= (1 + 4x^2)^{-1/2}(-2x\,\hat{\mathbf{i}} + \,\hat{\mathbf{j}}) \\ \kappa &= \frac{2}{[1 + 4x^2]^{3/2}}\end{aligned}$$

Now all we need is $\dot{s}$ and $\ddot{s}$. If $\dot{s} = c,\ \ddot{s} = 0$, then

$$\vec{v} = c\,\hat{\mathbf{v}},\ \vec{a} = c^2\kappa\,\hat{\mathbf{n}}.$$

2 Kinetics and Newtons Laws

Newton's Laws

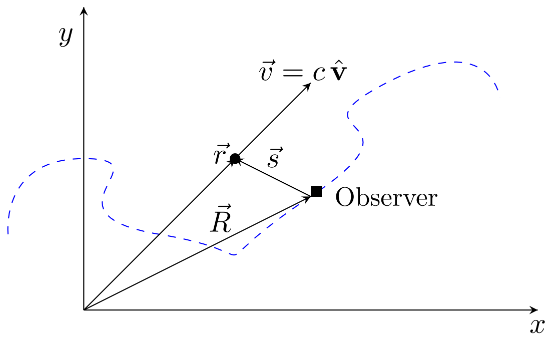
Lecture 9

Definition.

- *Mass, m* - "Quantity of Matter", measured in kg (scalar)
- *Momentum, $\vec{p} = m\vec{v}$* - "Quantity of Motion" (vector)
- *Inertia* - "Vis Insita" (innate force of matter). The resistance of an object to change its state of motion.
- *Force* - An action that changes an objects state of motion

Theorem 2.1: Newton's First Law

Every body has inertia!



$$\vec{r} = \vec{R} + \vec{s}, \ \ \vec{s} = \vec{r} - \vec{R}$$

$$\frac{d\vec{s}}{dt} = \frac{d\vec{r}}{dt} - \frac{d\vec{R}}{dt}$$

$$\frac{d^2\vec{s}}{dt^2} = \cancel{\frac{d^2\vec{r}}{dt^2}} - \frac{d^2\vec{R}}{dt^2}$$

If $\dfrac{d^2\vec{R}}{dt^2} \neq 0$, then the object will not be maintaining its state of motion.

Inertial frame $\dfrac{d^2\vec{R}}{dt^2} = 0$.

Theorem 2.2: Newton's Second Law

The net force on an object is equal to the rate of change of momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

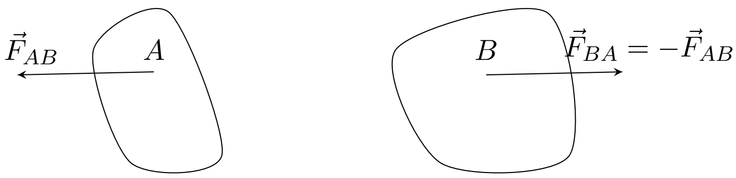
Alternatively

$$\vec{p}(t) - \vec{p}(0) = \int_0^t \vec{F}(t')\,dt'$$

If the mass is constant we get the familiar $\vec{F} = m\vec{a}$.

Theorem 2.3: Newton's Third Law

If $\vec{F}_{AB}$ is the force on object A due to object B, then $\vec{F}_{BA} = -\vec{F}_{AB}$.



Theorem 2.4: Conservation of Linear Momentum

Momentum is conserved in a closed system with no external forces.

Proof. Total momentum:

$$\vec{p}_T = \vec{p}_A + \vec{p}_B$$

$$\frac{d\vec{p}_T}{dt} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt}$$

Assume $\vec{F}_{AB}$ is the only force on $A \implies$ only force on B is $-\vec{F}_{AB}$.

$$\frac{d\vec{p}_T}{dt} = \vec{F}_{AB} - \vec{F}_{AB} = \vec{0}$$

∎

3 Forces

Kinds of forces: Lecture 10

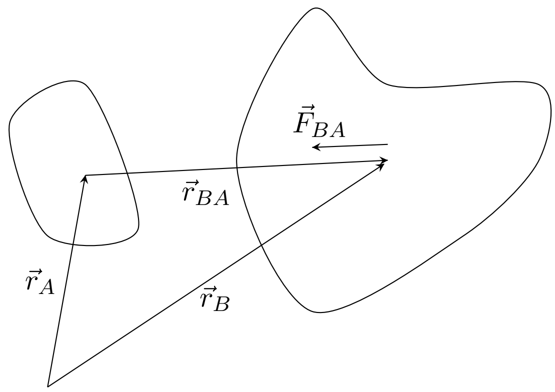
(i) Constraint Forces

(ii) Forces can also depend on our kinematic quantities

(iii) Forces can depend on velocity. “Drag Force”

(iv) Forces can also depend on position

Gravity



The force on m_b due to m_A is

$$\vec{F}_{BA} = -\frac{Gm_A m_b}{r_{BA}^2}\,\hat{\mathbf{r}}_{BA}$$

$$r_{BA} = |\vec{r}_{BA}|$$

$$\hat{\mathbf{r}}_{BA} = \frac{\vec{r}_{BA}}{r_{BA}}$$

- G is the gravitational constant

$$G = 6.67 \times 10^{-11} m^3/kgs^2$$

- $\vec{F}_{BA}$ is attractive
- The magnitude of force decays like $1/r_{BA}^2$
- This is a central force - $\vec{F} = F(r)\,\hat{\mathbf{r}}$

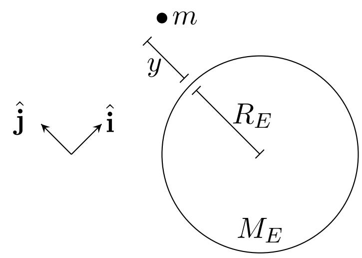
Recall that we said that the acceleration due to gravity is constant!

$\implies |\vec{F}_g| = mg$

We can say this because the force due to gravity acts from the centre of the earth, and the change in height of our object is small compared to the radius of the earth.

And the change in height of our object be small compared to the radius of the earth.

How small does the height need to be? How do we show this?



Using Newton's Formula:

$$\vec{F}_g = -\frac{GmM_E}{(y+R_E)^2}\,\hat{\mathbf{j}}$$

If $y = 0$

$$\vec{F}_g = -\frac{GmM_E}{R_E^2}\,\hat{\mathbf{j}}$$

If we write

$$\vec{F}_g = -mg\,\hat{\mathbf{j}}$$

The $g = \frac{GM_E}{R_E^2} = 9.8m/s^2$. Rewrite $\vec{F}_g$:

$$\vec{F}_g = -mg[1 + y/R_E]^{-2}\,\hat{\mathbf{j}}$$

We know that $y/R_E << 1$. This allows us to use a taylor series about $y/R_E = 0$ to approximate $[1 + y/R_E]^{-2}$. Taylor series about $x = 0$:

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + ...$$

In our case $x = y/R_E$

$$f(x) = [1+x]^{-2} \quad f(0) = 1$$
$$f'(x) = -2[1+x]^{-3} \quad f'(0) = -2$$
$$f''(x) = 6[1+x]^{-4} \quad f''(0) = 6$$
$$\implies \vec{F}_g = mg(1 - 2\frac{y}{R_E})\,\hat{\mathbf{j}} + \mathcal{O}(y^2/R_E^2)$$

So taking $\vec{F}_g = -mg\,\hat{\mathbf{j}}$ is equivalent to using the first term in the taylor series.

> **Example 3.1** (For Felix). $y = 39 \times 10^3 m \quad R_E = 6371 \times 10^3 m.$
>
> Suppose we use the constant force
>
> $$\vec{F}_g = -mg\,\hat{\mathbf{j}}$$
>
> Using our linear approximation
>
> $$\vec{F}_g = -0.9878mg\,\hat{\mathbf{j}}$$
>
> Actual $\vec{F}_g = -0.9879mg\,\hat{\mathbf{j}}$.

We are interested in using the force to predict where the object will be.

Using Newton's II Law and the Newtonian Gravity

$$m\frac{d^2y}{dt^2} = -\frac{GmM_E}{(R_E+y)^2}$$

Constant approximation:

$$m\frac{d^2y}{dt^2} = -mg$$

$$\implies y(t) = y_0 + v_0 t + \frac{1}{2}gt^2$$

Linear approximation:

$$m\frac{d^2y}{dt^2} = -mg(1-2y/R_E)$$

Contraint Forces

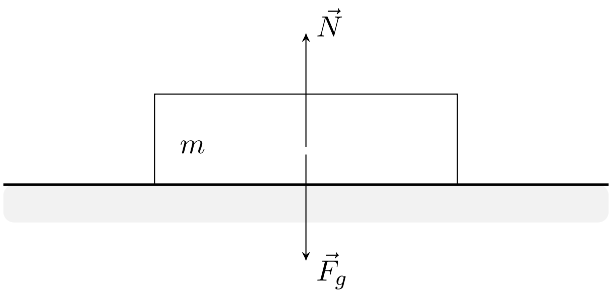
Force that arise only to satisfy or enforce a particular constraint on the motion of a body. These forces arise when we have Lecture 11

(i) Surfaces

(ii) Wires

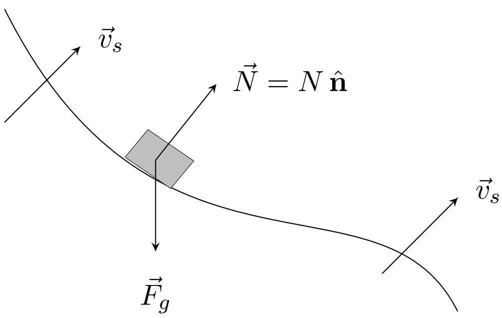
(iii) Strings and Bars

Example 3.2.



$\vec{N}$ exists to keep the object from going through the surface.

Definition. $\vec{N}$ is called the *Normal* or *Reaction* Force.



$\vec{N}$ acts in a direction normal (or perpendicular) to the surface.

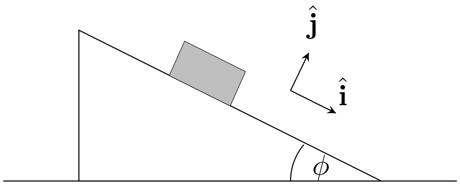
If our surface has a velocity $\vec{v}_s$, then

$$\vec{v} \cdot \hat{\mathbf{n}} = \vec{v}_s \cdot \hat{\mathbf{n}}$$

$\vec{N}$ only exists when the object is in contact with the surface $\implies N \geq 0$. For wires, everything is more or less the same, except that $N \in \mathbb{R}$.

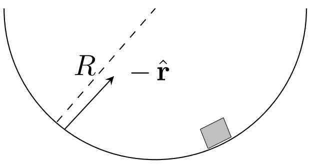
We can use different coordinate systems to find the normal force:

(i)



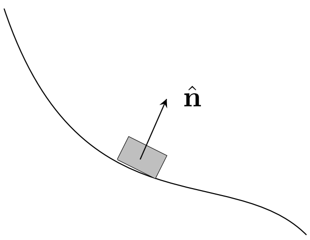
$\vec{N} = N\,\hat{\mathbf{j}}$.
Constraint: $\dot{y} = 0$ ($\vec{v}_s = 0$)

(ii)



$\vec{N} = -N\,\hat{\mathbf{r}}$.
Constraint: $\dot{r} = 0$.

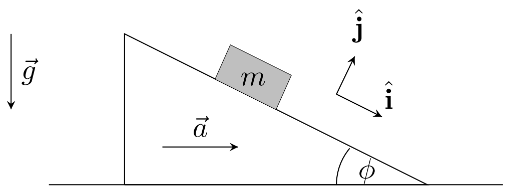
(iii)



(In 2D) $\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d^2\vec{r}}{ds^2}$.

$\vec{N} = N\,\hat{\mathbf{n}}$.
Constraint: $\vec{v} \cdot \hat{\mathbf{n}} = 0$.

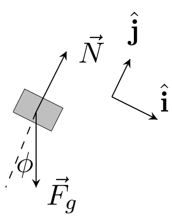
Example 3.3.



$\vec{a}$ is constant. Find: $\vec{a}_m$ (acceleration of m) and $\vec{N}$.

Setting up the equations is important! [See: Kleppner & Kolenkow §2.4]

Force diagram:



Constraints:

$$\vec{v}_s \cdot \hat{\mathbf{n}} = \vec{v} \cdot \hat{\mathbf{n}}$$

$$\hat{\mathbf{n}} = \hat{\mathbf{j}} \implies \dot{y}_s = \dot{y}_m$$

$$\implies \ddot{y}_s = \ddot{y}_m$$

Express forces in the coordinate system:

$$\vec{N} = N\,\hat{\mathbf{j}}$$

$$\vec{F}_g = mg[\sin\phi\,\hat{\mathbf{i}} - \cos\phi\,\hat{\mathbf{j}}]$$

$$\vec{a} = a[\cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}]$$

Use Newton's Laws:

$$m\vec{a}_m = \vec{F}_g + \vec{N}$$

$$ma_{mx} = mg\sin\phi$$

$$ma_{my} = -mg\cos\phi + N$$

From our constraint:

$$\implies a\sin\phi = a_{my}$$

Substitute this into Newton's Second Law:

$$ma\sin\phi = -mg\cos\phi + N$$

$$\implies \boxed{N = m[a\sin\phi + g\cos\phi]}$$

Our final unknown is given directly by Newton's II: $a_{mx} = g\sin\phi$, so

$$\boxed{\vec{a}_m = g\sin\phi\,\hat{\mathbf{i}} + a\sin\phi\,\hat{\mathbf{j}}}$$

The mass will not move relative to the surface if:

$$\vec{a}_m = \vec{a}$$

$$\implies \vec{a}_m \cdot \hat{\mathbf{i}} = \vec{a} \cdot \hat{\mathbf{i}}$$

$$\implies g\sin\phi = a\cos\phi$$

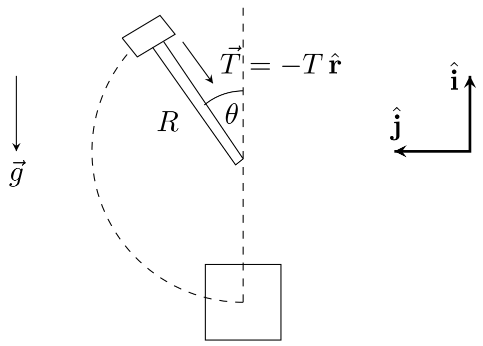
$$a/g = \tan\phi$$

If $g\sin\phi < a\cos\phi$ the block slides off the top.

If $g\sin\phi > a\cos\phi$ the block slides down the ramp.

Lecture 12

Example 3.4 (OK Go Video)**.**

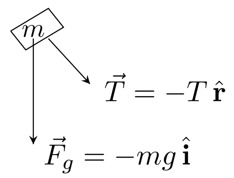


We know that the distance between m and a point in space remains constant. We can use polar coordinates to solve the problem because we have information about $r, (\dot{r}, \ddot{r})$ or $\theta, \dot{\theta}, \ddot{\theta}$. In this case we know $r = R$ (constant) $\implies \dot{r} = \ddot{r} = 0$. This will be enforced by $\vec{T}$, the *tension*. In our coordinate system:

$$\vec{T} = -T\,\hat{\mathbf{r}}$$

Find: T and also $\vec{v}$ at $\theta = \pi$.

Force diagram:



$$(\,\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) \times \cos\theta$$

$$+(\,\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\,\hat{\mathbf{j}}) \times -\sin\theta$$

$$\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} = \,\hat{\mathbf{i}}$$

$$\implies \vec{F}_g = -mg[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}]$$

Newton's Laws:

$$m\vec{a} = \vec{T} + \vec{F}_g$$

$$\implies m[(\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}] = -T\,\hat{\mathbf{r}} - mg\cos\theta\,\hat{\mathbf{r}} + mg\sin\theta\,\hat{\boldsymbol{\theta}}$$

$\hat{\mathbf{r}}$ component:

$$m(\ddot{r} - r\dot{\theta}^2) = -T - mg\cos\theta \tag{i}$$

$\hat{\boldsymbol{\theta}}$ component:

$$m(r\ddot{r} + 2\dot{r}\dot{\theta}) = mg\sin\theta \tag{ii}$$

Use the constraint in Newton's II Law: $r = R$, $\dot{r} = \ddot{r} = 0$.

$$\text{(i)} \implies -mR\dot{\theta}^2 = -T - mg\cos\theta$$

$$\text{(ii)} \implies mR\ddot{\theta} = mg\sin\theta$$

Take (ii) $\times\dot{\theta} \implies mR\ddot{\theta}\dot{\theta} = mg\sin\theta\dot{\theta}$.

Notice

$$\ddot{\theta}\dot{\theta} = \frac{1}{2}\frac{d}{dt}(\dot{\theta}^2)$$

$$\sin\theta\dot{\theta} = \frac{d}{dt}(-\cos\theta)$$

$$\frac{d}{dt}(\frac{1}{2}\dot{\theta}^2 + \frac{g}{R}\cos\theta) = 0$$

$$\implies \frac{1}{2}\dot{\theta}^2 + \frac{g}{R}\cos\theta = K \text{ (constant!)}$$

Take $t = 0, \theta = \theta_0, \dot{\theta} = 0$

$$\implies K = \frac{g}{R}\cos\theta_0$$

$$\implies \dot{\theta}^2 = \frac{2g}{R}[\cos\theta_0 - \cos\theta]$$

In polar coordinates:

$$\vec{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

$$\vec{v} = R\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

At the bottom when the hammer hits the TV, $\theta = \pi$

$$\implies \dot{\theta}^2 = \frac{2g}{R}[\cos\theta_0 + 1]$$

$$\vec{v} = R\sqrt{\frac{2g}{R}[\cos\theta_0 + 1]}\,\hat{\boldsymbol{\theta}}$$

At $\theta = \pi$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{j}}$. We can find the tension from (i):

$$-mR\dot{\theta}^2 = -T - mg\cos\theta$$

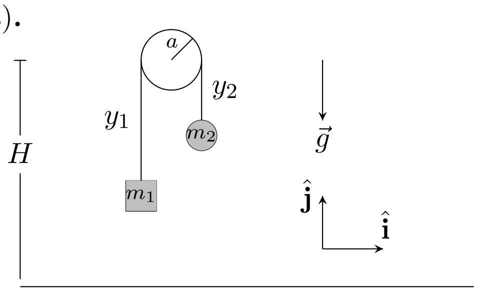
Using our expression for $\dot{\theta}^2$

$$+mR[\frac{2g}{R}[\cos\theta_0 - \cos\theta]] = +T + mg\cos\theta$$

$$\implies \boxed{T = 2mg\cos\theta_0 - 3mg\cos\theta}$$

Lecture 13

Example 3.5 (Strings)**.**



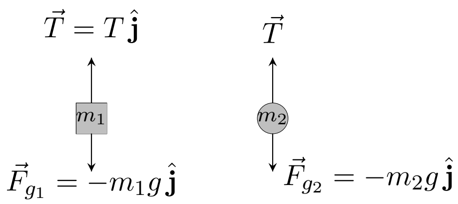
l is the length of the string. This is fixed $\implies \dot{l} = 0 \implies \ddot{l} = 0$.

$$l = (H - y_1) + (H - y_2) + \pi a$$

$\dot{l} = 0$, so if $\dot{H} = 0$ (fixed pulley height),

$$\begin{aligned} 0 = \dot{l} &= -\dot{y}_1 - \dot{y}_2 \\ \implies \dot{y}_1 &= -\dot{y}_2 \\ \implies \ddot{y}_1 &= \ddot{y}_2 \end{aligned}$$

Force diagram:



Newton's Second Law:

1D Problem in the $\hat{\mathbf{j}}$ direction:

$$m_1\ddot{y}_1 = -m_1 g + T \tag{i}$$
$$m_s\ddot{y}_2 = -m_2 g + T \tag{ii}$$

Using the constraint

$$\text{(i)} \implies -m_1\ddot{y}_2 = -m_1 g + T$$

Subtract (ii)

$$-(m_1 + m_2)\ddot{y}_2 = (m_2 - m_1)g$$
$$\implies \boxed{\ddot{y}_2 = \frac{m_1 - m_2}{m_1 + m_2} g}$$

We can also find T: (i) $\times m_2 +$ (ii) $\times m_1 \implies$

$$0 = -2m_1 m_2 g + T(m_1 + m_2)$$

$$\boxed{T = \frac{2m_1 m_2}{(m_1 + m_2)} g}$$

Use of Intrinsic Coordinates

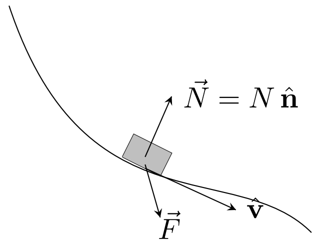
We know $\vec{r}(s)$ or $\vec{r} = x\,\hat{\mathbf{i}} + y(x)\,\hat{\mathbf{j}}$

(i) $\hat{\mathbf{v}} = \frac{d\vec{r}}{ds}$

(ii) $\frac{d^2\vec{r}}{ds^2} = \kappa\,\hat{\mathbf{n}}, \kappa = \left|\frac{d^2\vec{r}}{ds^2}\right|$, $\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d^2\vec{r}}{ds^2}$

Describe acceleration:

$$\vec{a} = \ddot{s}\,\hat{\mathbf{v}} + \kappa\dot{s}^2\,\hat{\mathbf{n}}$$

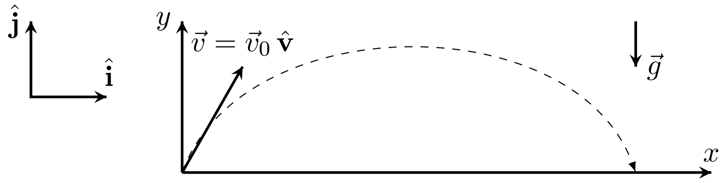


From Newton's Second Law:

$$m\ddot{s} = \vec{F}\cdot\hat{\mathbf{v}}$$

$$m\kappa\dot{s}^2 = \vec{F}\cdot\hat{\mathbf{n}} + N$$

Example 3.6.



At $t = 0$, $v = \dot{s} = v_0 > 0$ and $x = 0$.

For what values of v_0 does the object leave the surface before reaching $x = \pi/2$?

$\Longrightarrow$ we need to see if the normal force goes to zero.

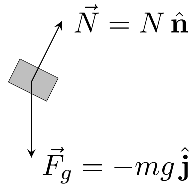
$$y'(x) = \cos x, \quad y''(x) = -\sin x$$

$$\begin{aligned}\hat{\mathbf{v}} &= [1 + y'^2]^{-1/2}(\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}})\\ &= [1 + \cos^2 x]^{-1/2}(\hat{\mathbf{i}} + \cos x\,\hat{\mathbf{j}})\end{aligned}$$

$$\kappa = \frac{|y''|}{[1 + y'^2]^{3/2}} = \frac{\sin x}{[1 + \cos^2 x]^{3/2}}$$

$$\begin{aligned}\hat{\mathbf{n}} &= \frac{y''}{|y''|}[1 + y'^2]^{-1/2}[-y'\,\hat{\mathbf{i}} + \hat{\mathbf{j}}]\\ &= -[1 + \cos^2 x]^{-1/2}[-\cos x\,\hat{\mathbf{i}} + \hat{\mathbf{j}}]\end{aligned}$$

Force diagram:



Newton's Laws:

$$\begin{aligned} m\ddot{s} &= \vec{F}_g \cdot \hat{\mathbf{v}} \\ &= -mg\cos x[1+\cos^2 x]^{-1/2} \qquad \text{(i)} \\ m\kappa\dot{s}^2 &= \vec{F}_g \cdot \hat{\mathbf{n}} + N \\ &= mg[1+\cos^2 x]^{-1/2} + N \qquad \text{(ii)} \end{aligned}$$

Multiply (i) $\times\dot{s}$:

$$\begin{aligned} m\ddot{s}\dot{s} &= -mg[1+\cos^2 x]^{-1/2}\cos x\dot{s} \\ \ddot{s}\dot{s} &= \frac{d}{dt}\left(\frac{\dot{s}^2}{2}\right) \\ \dot{s} &= \frac{ds}{dx}\dot{x} = [1+\cos^2 x]^{1/2}\dot{x} \end{aligned}$$

Substitute these in:

$$\begin{aligned} m\frac{d}{dt}\left(\frac{\dot{s}^2}{2}\right) &= -mg\cos x\dot{x} \\ &= -\frac{d}{dt}(g\sin x) \\ \implies \frac{d}{dt}\left(\frac{\dot{s}^2}{2} + g\sin x\right) &= 0 \\ \implies \frac{\dot{s}^2}{2} + g\sin x &= K \end{aligned}$$

Initial conditions: $t = 0, \dot{s} = v_0, x = 0$

$\implies K = v_0^2/2, \implies \dot{s}^2 = v_0^2 - 2g\sin x$

Substitute this into (ii)

$$m \times \frac{\sin x}{[1+\cos^2 x]^{3/2}}(v_0^2 - 2g\sin x) = mg[1+\cos^2 x]^{-1/2} + N$$

$$N = \frac{m}{[1+\cos^2 x]^{3/2}}(v_0^2 \sin x - 2g\sin^2 x - g(1+\cos^2 x))$$

$$N = \frac{m}{[1+\cos^2 x]^{3/2}}[v_0^2 \sin x - g(2+\sin^2 x)]$$

Set $N = 0$

$$\implies v_0^2 \sin x - g(2+\sin^2 x) = 0$$

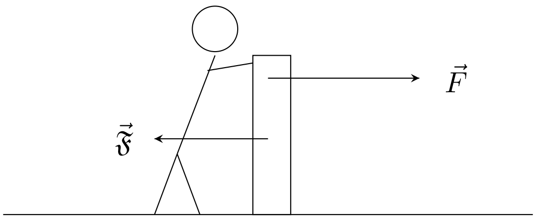
$$v_0^2 = \frac{g(2+\sin^2 x)}{\sin x}$$

We can show that this has a minimum at $x = \pi/2 \implies v_0^2 = 3g \implies v_0 = \sqrt{3g}$.

If $v_0 > \sqrt{3g}$ it will leave the surface before reaching $x = \pi/2$.

Friction

Friction arises when one object is in contact with another: Lecture 14



$\vec{\mathfrak{F}}$ is the force due to friction. $|\vec{\mathfrak{F}}| < \mathfrak{F}_{max}$, friction acts like a constraint force.

If $|\vec{F}| < \mathfrak{F}_{max} \implies \vec{\mathfrak{F}} = -\vec{F}$.

$\mathfrak{F}_{max}$ depends on:

(a) The materials of the objects.

(b) The normal force.

It is independent of the area of contact and velocity.

$$\mathfrak{F}_{max} = \mu|\vec{N}| \ (\mu > 0)$$

Definition. μ is the *coefficient of friction.*

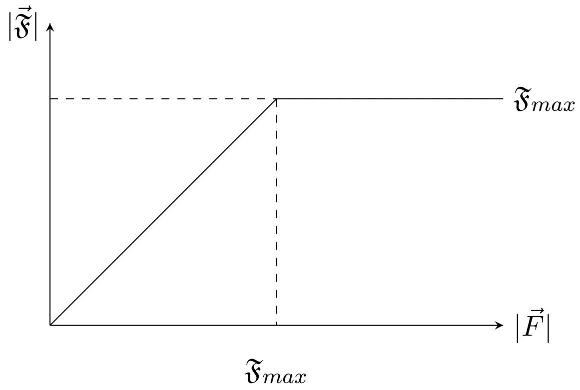
Typically $0 < \mu \leq 1$.

Once $|\vec{F}| > \mathfrak{F}_{max}$:

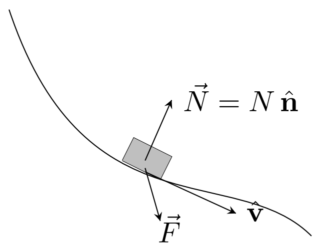
- The object moves.
- $|\vec{\mathfrak{F}} = \mu|\vec{N}| = \mathfrak{F}_{max}$
- Friction opposes the motion of the object.

Before reaching $\mathfrak{F}_{max}$ the direction opposes the "would be" motion.

Graphically:



More generally, $0 \leq |\vec{\mathfrak{F}}| \leq \mu|\vec{N}|$.



There is no relative motion then

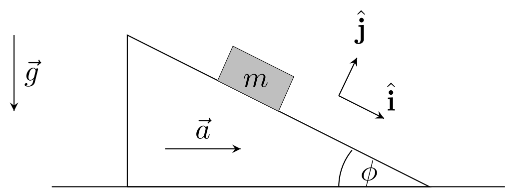
$$\vec{v} = \vec{v}_s \tag{i}$$

Friction works to satisfy (i).

If there is relative motion, $\vec{v} \neq \vec{v}_s$ and $|\vec{\mathfrak{F}}| = \mu|\vec{N}|$ acts to restore (i).

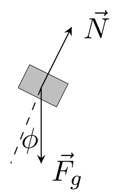
This is an example of a "mathematical model" - we can describe the phenomena using mathematics, then use this description to predict other phenomena.

Example 3.7.



At what value of ϕ does the block begin to slide? $\implies$ Friction acting like a constraint.

Force diagram:



Since there is no relative motion $\vec{v} = \vec{v}_s = 0$, $\dot{y} = \ddot{y} = \dot{x} = \ddot{x} = 0$

No acceleration, we have static equilibrium. Newton's Second Law;

$$m\ddot{x} = 0 = mg\sin\phi - \mathfrak{F} \tag{i}$$

$$m\ddot{y} = 0 = -mg\cos\phi + N \tag{ii}$$

$$\implies N = mg\cos\phi$$

$$\mathfrak{F} = mg\sin\phi$$

We also know

$$\mathfrak{F} \leq \mu N = \mu mg\cos\phi$$

Also have

$$\mathfrak{F} = mg\sin\phi \leq \mu mg\cos\phi$$

or

$$\mu \geq \tan\phi$$

Thanks Euler!

So when $\mathfrak{F} = \mathfrak{F}_{max}$

$$\implies \mu = \tan\phi$$

For $\mu < \tan\phi$ the object is moving in the $\hat{\mathbf{i}}$ direction and $\ddot{x} \neq 0$.

Newton's Second Law

$$m\ddot{x} = mg\sin\phi - \mathfrak{F}_{max}$$

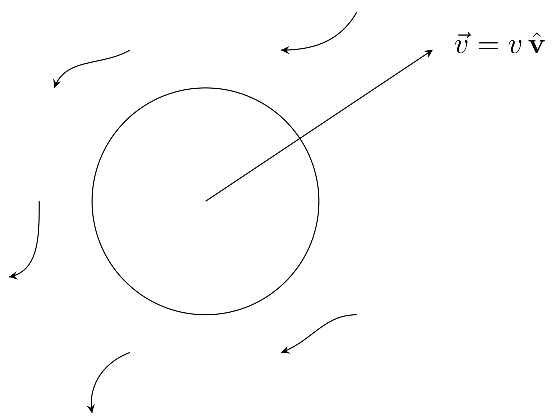
$$= mg\sin\phi - \mu mg\cos\phi$$

$$\implies \ddot{x} = g[\sin\phi - \mu\cos\phi]$$

Drag Force

Example of a force that depends on the velocity of an object. Lecture 15

Motion of bodies through fluid.



Fluid has: ρ: density and η: viscosity

To move through the fluid, the body exerts a force on the fluid: $\vec{F}_{FB}$

By Newton's III Law

$$\vec{F}_{BF} = -\vec{F}_{FB}$$

The drag force

$$\vec{F}_D = (\vec{F}_{BF} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}}$$

In general to find $\vec{F}_D$ is a challenging problem! To find $\vec{u}$ we need to solve the Navier-Stokes Equations. From $\vec{u}$ we can obtain $\vec{F}_D$. Fortunately this calculation can be done for two limiting cases; at low and at high speeds:

Low Speeds

At low speeds $|\vec{v}| << 1$, then

$$\vec{F}_D = -C_D\vec{v}$$

Where C_D is the *drag co-efficient*

- This depends linearly on $\vec{v}$.
- Always opposite the direction of motion
- For a sphere $C_D = 6\pi R\eta$
- C_D depends on (i) the size of the object, (ii) the viscosity of the fluid

If $\vec{u} \neq 0$ meaning there is a background flow:

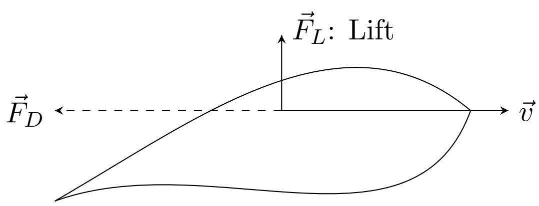
$$\vec{F}_D = -C_D(\vec{v} - \vec{u})$$

only a drag force if there's relative motion to the fluid.

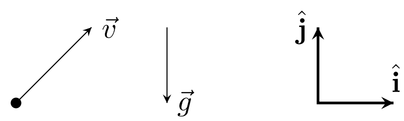
High Speeds

$$\vec{F}_D = -C_D|\vec{v}|\vec{v}$$

- Opposes the motion
- Depends quadratically on the speed
- Changes $C_D = \frac{1}{2}\rho R^2 K$
- Drag Force is not all of $\vec{F}_{BF}$

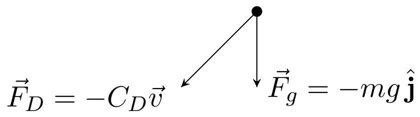


Example 3.8.



$\vec{F}_D = -C_D\vec{v}$

Force Diagram:



Newton's Second Law:

$$m\frac{d\vec{v}}{dt} = \vec{F}_D + \vec{F}_g = -C_D\vec{v} - mg\,\hat{\mathbf{j}}$$

First, seek the solution, $\vec{v}_\infty$, where $\dfrac{d\vec{v}}{dt} = 0$, the *steady state solution*

$$\implies 0 = -C_D\vec{v} - mg$$
$$\implies \vec{v}_\infty = -\frac{mg}{C_D}\,\hat{\mathbf{j}}$$

Using linearity of the equation

$$\vec{v} = \vec{v}_\infty + \vec{w}$$

Substitute this into Newton's Second Law:

$$m\frac{d}{dt}(\vec{v}_\infty + \vec{w}) = -C_D(\vec{v}_\infty + \vec{W}) - mg\,\hat{\mathbf{j}}$$
$$m\frac{d\vec{w}}{dt} = mg\,\hat{\mathbf{j}} - C_D\vec{w} - mg\,\hat{\mathbf{j}}$$
$$\implies \frac{d\vec{w}}{dt} = -\frac{C_D}{m}\vec{w}$$
$$\implies \vec{w} = \vec{w}_0 e^{-C_D t/m}$$

Thus

$$\vec{v} = -\frac{mg}{C_D}\,\hat{\mathbf{j}} + \vec{w}_0 e^{-C_D t/m}$$

Initial condition: $t = 0$, $\vec{v} = \vec{v}_0 \implies \vec{w}_0 = \vec{v}_0 + \frac{mg}{C_D}\,\hat{\mathbf{j}}$. So

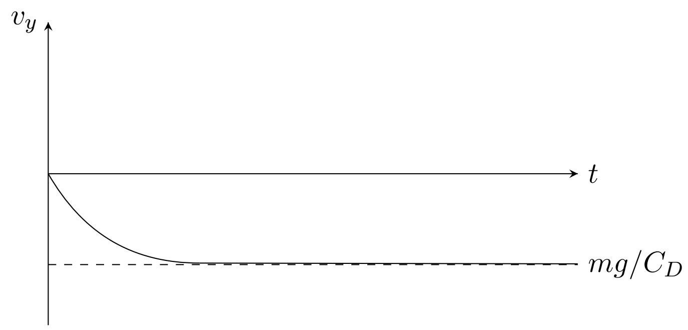
$$\vec{v} = \vec{v}_0 e^{-C_D t/m} - \frac{mg}{C_D}\,\hat{\mathbf{j}}[1 - e^{-C_D t/m}]$$

As $t \to \infty$, $\vec{v} \to -\frac{mg}{C_D}\,\hat{\mathbf{j}} = \vec{v}_\infty$ as expected.

The ratio C_D/m controls how quickly this limit is reached.

Taking $\vec{v}_0 = 0$

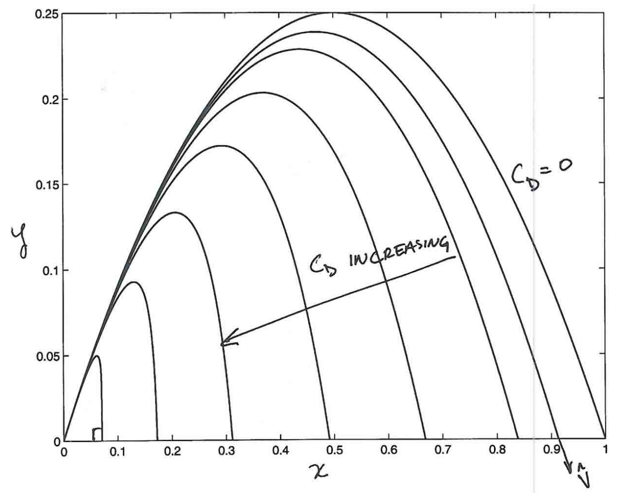
$$\vec{v} = -\frac{mg}{C_D}\,\hat{\mathbf{j}}[1 - e^{-C_D t/m}]$$



Integrating our general expression to find the position:

$$\vec{r}(t) = \vec{r}_0 - \frac{mgt}{C_D}\hat{\mathbf{j}} + \frac{m}{C_D}[\vec{v}_0 + \frac{mg}{C_D}\hat{\mathbf{j}}] \times (1 - e^{-C_D t/m})$$

Projectiles: $\vec{r}_0 = 0$, $\vec{v}_0 = v_0 \cos\alpha\,\hat{\mathbf{i}} + v_0 \sin\alpha\,\hat{\mathbf{j}}$



4 Oscillators

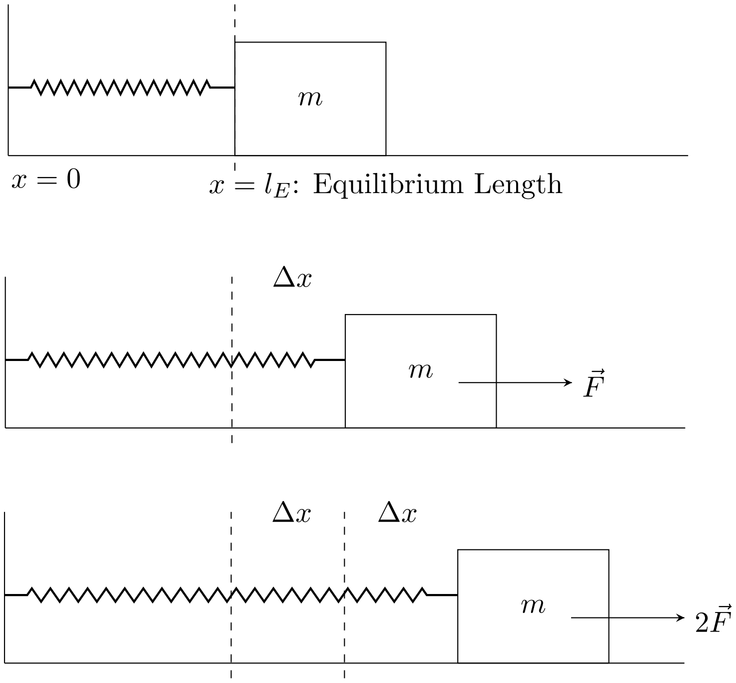
1660: Ceiiinosssttuv

Lecture 16

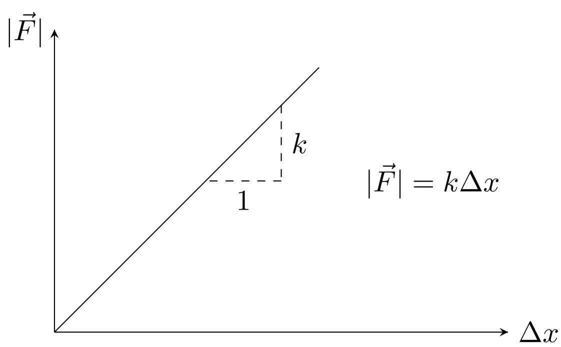
1678: "Ut tensio sic vis", "As is the extension, so the force" - Robert Hooke

Spring Force

- Example of a force that depends on position
- Spring forces are related to the deformations of solids



There is a linear relationship between $|\vec{F}|$ and Δx:



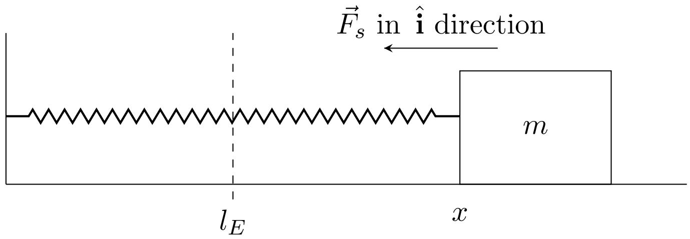
Definition. k is the *spring constant.*

k depends on:

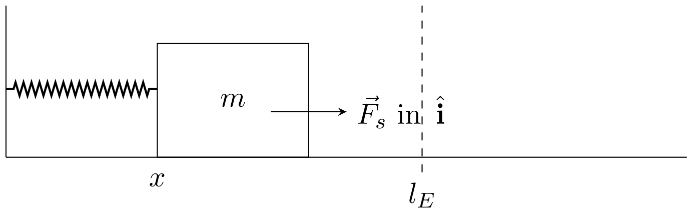
(i) Material

(ii) Geometry of the spring

The spring acts in a way to restore its equilibrium length.



$|\vec{F}_s| = k(x - l_E)$



$|\vec{F}_s| = k(l_E - x)$

The spring force

$$\vec{F}_s = -k(x - l_E)\,\hat{\mathbf{i}}$$

If $l_E = 0$

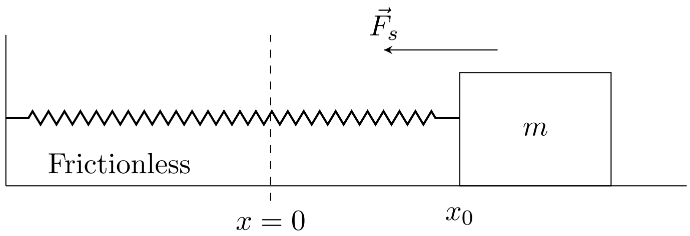
$$\vec{F}_s = -kx\,\hat{\mathbf{i}}$$

In 3D:

$$\vec{F}_s = -k(\vec{r} - \vec{r}_E)$$

If $\vec{r}_E = 0$

$$\vec{F}_s = -k\vec{r}$$



We only need to worry about the spring force (other forces balance). Newton's Second Law

$$m\ddot{x} = -kx \;(*)$$

We seek a solution of the form $x = Ce^{\alpha t}$, $\dot{x} = C\alpha e^{\alpha t}$, $\ddot{x} = C\alpha^2 e^{\alpha t}$. Substituting this into $(*)$

$$m[C\alpha^2 e^{\alpha t}] = -kCe^{\alpha t}$$

$$Ce^{\alpha t}[m\alpha^2 + k] = 0$$

$$\implies \alpha^2 = -\frac{k}{m}$$

$$\alpha = \pm i\sqrt{\frac{k}{m}}$$

The general solution is

$$x(t) = C_1 e^{i\sqrt{k/m}t} + C_2 e^{-i\sqrt{k/m}t}$$

x is real but C_1 and C_2 are complex. The equivalent general solution is

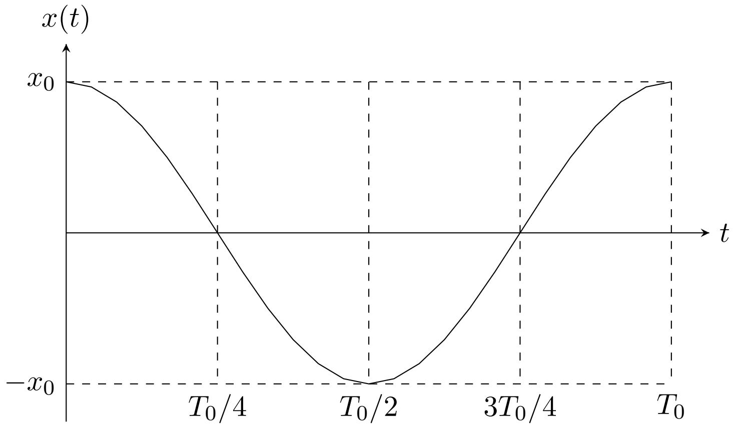
$$x(t) = C_3 \cos\sqrt{k/m}t + C_4 \sin\sqrt{k/m}t$$
$$\dot{x}(t) = -C_3\sqrt{k/m}\sin\sqrt{k/m}t + C_4\sqrt{k/m}\cos\sqrt{k/m}t$$

Initial conditions:

$$t = 0,\ \dot{x} = 0 \implies C_4 = 0$$
$$t = 0,\ x - x_0 \implies C_3 = x_0$$

So the solution is

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}t$$



$x(t)$ has:

- Amplitude of oscillation $A = x_0$
- Period $T_0 = 2\pi\sqrt{\frac{m}{k}}$
- Frequency: $\omega_0 = \sqrt{\frac{k}{m}} \implies \omega_0 = \frac{2\pi}{T_0}$

Amplitude just depends on the initial conditions. The frequency depends solely on k & m. This system is an example of a *simple harmonic oscillator*.

Similar equations arises with a pendulum:

Newton's Second Law:

$$m(\ddot{r} - r\dot{\theta}) = mg\cos\theta - T$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\sin\theta$$

Since $r = l$, $\dot{r} = \ddot{r} = 0$:

$$\implies -ml\dot{\theta}^2 = mg\cos\theta - T$$

$$ml\ddot{\theta} = -mg\sin\theta$$

If the angle θ remains small, $\theta << 1$, $\sin\theta \approx \theta$. When we do this

$$ml\ddot{\theta} = -mg\theta$$

$$\implies \ddot{\theta} + \frac{g}{l}\theta = 0$$

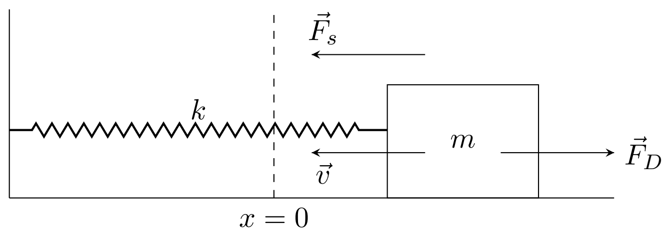
This is the same as our equation for the spring with

$$x \longrightarrow \theta$$

$$\frac{k}{m} \longrightarrow \frac{g}{l}$$

For a pendulum $\omega_0 = \sqrt{\frac{g}{l}}$.

Damped Harmonic Oscillator



Lecture 17

Take: $\vec{F}_D = -C_D\vec{v}$. Motion is still 1D!

Newton's Second Law:

$$m\ddot{x} = -kx - C_D\dot{x}$$

$$\ddot{x} + \frac{C_D}{m}\dot{x} + \frac{k}{m}x = 0$$

Recall: $\omega_0^2 = \frac{k}{m}$, $\mu = \frac{C_D}{2m}$

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = 0 \qquad (*)$$

Look for a solution of the form $x = Ce^{\alpha t}$. Plug this into $(*)$:

$$Ce^{\alpha t}[\alpha^2 + 2\mu\alpha + \omega_0^2] = 0$$

Solving our quadratic equation for α

$$\alpha = \frac{-2\mu \pm \sqrt{4\mu^2 - 4\omega_0^2}}{2}$$

Two solutions:

$$\alpha_1 = -\mu + \sqrt{\mu^2 - \omega_0^2}$$

$$\alpha_2 = -\mu - \sqrt{\mu^2 - \omega_0^2}$$

The general solution is

$$x(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}$$

We know that both $\mu, \omega_0 > 0$, but there are three cases to consider:

(i) $\mu > \omega_0$: Over damped

(ii) $\mu < \omega_0$: Under damped

(iii) $\mu = \omega_0$: Critically damped

Case (i):

$\mu^2 - \omega_0^2 > 0$

$\implies$ α_1 and α_2 are real. We also know $\mu > \sqrt{\mu^2 - \omega_0^2} \implies$ both $\alpha_1, \alpha_2 < 0$.

Our solutions decay exponentially to zero.

Case (ii):

$$\begin{aligned} \mu < \omega_0 &\implies \mu^2 - \omega_0^2 < 0 \\ &\implies \alpha_1 = -\mu + i\sqrt{\omega_0^2 - \mu^2} \\ &\qquad \alpha_2 = -\mu - i\sqrt{\omega_0^2 - \mu^2} \end{aligned}$$

$\implies$ α_1 and α_2 are complex and in fact complex conjugates of one another.

General solution:

$$\begin{aligned} x(t) &= C_1 e^{-\mu + i\omega_D)t} + C_2 e^{-\mu - i\omega_D)t} \\ \omega_D &= \sqrt{\omega_0^2 - \mu^2} \\ x(t) &= e^{-\mu t}[C_1 e^{i\omega_D t} + C_2 e^{-\omega_D t}] \end{aligned}$$

- Drag force modifies the frequency: $\omega_D < \omega_0$
- Amplitude decays with time.
- $x(t)$ still goes to zero at $t \to \infty$.

Case (iii):

$$\mu = \omega_0 \implies \alpha_1 = \alpha_2 = -\mu$$

Rethink the general solution:

$$x(t) = C_1 e^{-\mu t} + C_2 t e^{-\mu t}$$

- No oscillations
- $x \to 0$ as $t \to \infty$. This happens more rapidly than any solution in case (i).

Example 4.1.

$\dot{x} = 0, x(0) = L$. $\mu < \omega_0 \implies$ under damped.

The general solution:

$$x(t) = e^{-\mu t}[K_1 \cos \omega_D t + K_2 \sin \omega_D t]$$
$$\dot{x}(t) = \mu e^{-\mu t}[K_1 \cos \omega_D t + K_2 \sin \omega_D t] + \omega_D e^{-\mu t}[-K_1 \sin \omega_D t + K_2 \cos \omega_D t]$$

Apply the initial conditions:

$$x(0) = L = K_1$$
$$\dot{x}(0) = 0$$
$$\implies -\mu L + \omega_D K_2 = 0$$
$$\implies K_2 = \frac{\mu L}{\omega_D}$$

Solution:

$$x(t) = Le^{-\mu t}\left[\cos \omega_D t + \frac{\mu}{\omega_D}[\sin \omega_D t\right]$$

We can also express the general solution using an *amplitude* and a *phase*:

$$x(t) = \underbrace{A(t)}_{\text{Amplitude}} \cos(\omega_D t - \underbrace{\phi)}_{\text{Phase}}$$
$$x(t) = A(t)[\cos \omega_D t \cos \phi + \sin \omega_D t \sin \phi]$$
$$= [A(t) \cos \phi] \cos \omega_D t + [A(t) \sin \phi] \sin \omega_D t$$

By comparing with our previous expression for the general solution:

$$A(t) \cos \phi = K_1 e^{-\mu t}$$
$$A(t) \sin \phi = K_2 e^{-\mu t}$$
$$\implies \tan \phi = K_2/K_1$$
$$A = e^{-\mu t}[K_1^2 + K_2^2]^{1/2}$$

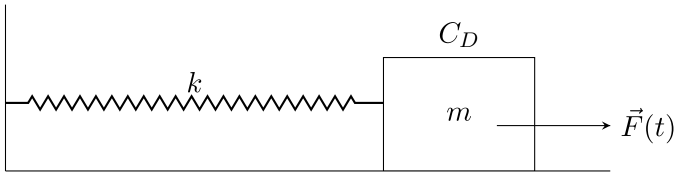
For our example $K_1 = L, K_2 = \frac{\mu L}{\omega_D}$

$$\tan \phi = \mu/\omega_D$$
$$A = e^{-\mu t}\left[L^2 + \frac{\mu^2 L^2}{\omega_D^2}\right]^{1/2}$$

As $t \to \infty, A \to 0$.

We see in this case *without* any external forcing that the amplitude and phase depend on the initial conditions.

Forced, Damped Oscillator



Lecture 18

$$F(t) = F_0 \cos \omega t$$

In general $\omega \neq \omega_0 \neq \omega_D$.

Newton's Laws

$$m\ddot{x} = -kx - C_D\dot{x} + F_0 \cos \omega t$$

Using $\mu = C_D/2m$, $\omega^2 = k/m$

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \qquad (*)$$

Inhomogeneous equation

$$x(t) = x_{CF}(t) + x_{PI}(t)$$

We know x_{CF} from looking at the damped harmonic oscillator. To find $x_{PI}(t)$ let's look for solutions of the form

$$\begin{aligned} x_{PI}(t) &= K_3 \cos \omega t + K_4 \sin \omega t \\ \dot{x}_{PI} &= -K_3\omega \sin \omega t + K_4\omega \cos \omega t \\ \ddot{x}_{PI} &= -K_3\omega^2 \cos \omega t - K_4\omega^2 \sin \omega t \end{aligned}$$

Plus this all into $(*)$

$$\begin{aligned} &[-K_3\omega^2 + 2\mu\omega K_4 + K_3\omega_0^2 - \frac{F_0}{m}] \cos \omega t \\ +&[-K_4\omega^2 - 2\mu\omega K_4 + K_4\omega_0^2] \sin \omega t = 0 \end{aligned}$$

Only possible if everything in the brackets is zero!

$$\begin{aligned} \implies 2\mu\omega K_4 &= K_3(\omega^2 - \omega_0^2) + \frac{F_0}{m} \\ 2\mu\omega K_3 &= -K_4(\omega^2 - \omega_0^2) \end{aligned}$$

Solving for K_3 & K_4

$$\begin{aligned} K_3 &= -\frac{F_0}{m} \frac{\omega^2 - \omega_0^2}{4\mu^2\omega^2 + (\omega^2 - \omega_0^2)^2} \\ K_4 &= \frac{F_0}{m} \frac{2\mu\omega}{4\mu^2\omega^2 + (\omega^2 - \omega_0^2)^2} \end{aligned}$$

The complete general solution:

$$x(t) = K_1 e^{\alpha_1 t} + K_2 e^{\alpha_2 t} + K_3 \cos \omega t + K_4 \sin \omega t$$

As $t \to \infty$, $x_{CF}(t) \to 0$, but $x_{PI}(t)$ does not.

$$x(t) \to K_3 \cos \omega t + K_4 \sin \omega t$$

Write the steady state solution as

$$x(t) \to A \cos[\omega t - \phi]$$

We know from yesterday

$$A = [K_3^2 + K_4^2]^{1/2}, \ \ \tan \phi = K_4 / K_3$$

$$\begin{aligned} A &= [K_3^2 + K_4^2]^{1/2} \\ \tan \phi &= K_4 / K_3 \\ A &= \frac{F_0}{m} \frac{1}{[4\mu^2\omega^2 + (\omega^2 - \omega_0^2)^2]^{1/2}} \\ \tan \phi &= -\frac{2\mu\omega}{\omega^2 - \omega_0^2} \end{aligned}$$

Since this is the steady state, for this case A & ϕ are independent of the initial conditions.

Amplitude:

- $\omega = 0$

  $\implies A(\omega = 0) = \frac{F_0}{m} \frac{1}{\omega_0^2} = A_0$
- $\omega \to \infty$.

  A decays like $\omega^{-2} \implies A \to 0$

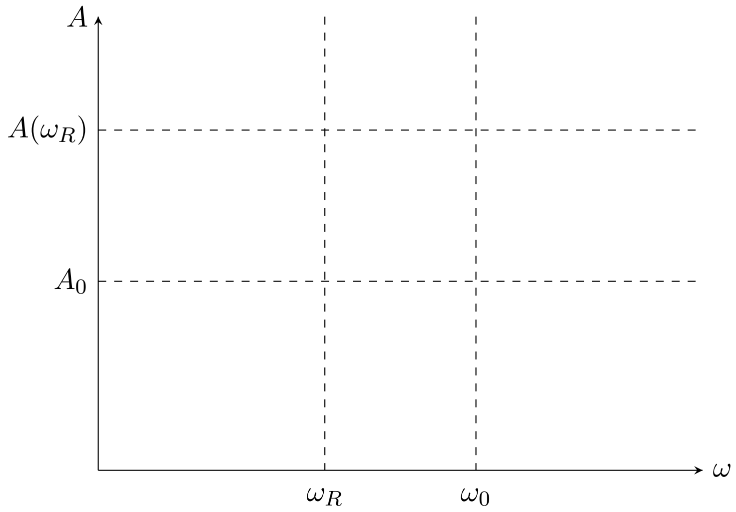
What happens in between? Look at $\frac{dA}{d\omega}$

$$\frac{dA}{d\omega} = -\frac{F_0}{m} \frac{8\mu^2\omega + 4(\omega^2 - \omega_0^2)\omega}{[4\mu^2\omega^2 + (\omega^2 - \omega_0^2)^2]^{3/2}}$$

Find where $\frac{dA}{d\omega} = 0$

- $\frac{dA}{d\omega} \to 0$ as $\omega \to \infty$
- Consider the numerator:

$$\begin{aligned} 4\omega[2\mu^2 + (\omega^2 - \omega_0^2)] &= 0 \\ \implies \omega = 0 \ \& \ \omega^2 = \omega_0^2 - 2\mu^2 &\equiv \omega_R^2 \end{aligned}$$



> **Definition.** *Resonance*: When the system responds dramatically when forced at a particular frequency.

Phase: $(0 \leq \phi < \pi)$

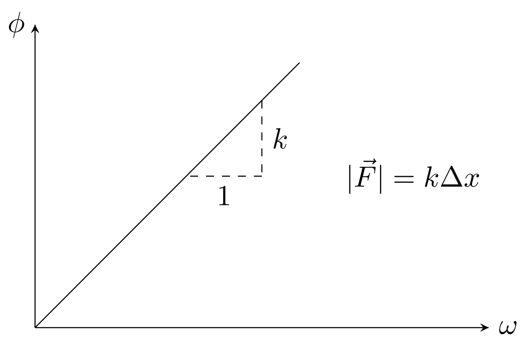
$$\tan\phi = -\frac{2\mu\omega}{\omega^2 - \omega_0^2}$$

For $\omega = 0 \implies \tan\phi = 0 \implies \phi = 0$.

$$\begin{aligned}
\frac{d\tan\phi}{d\omega} &= -\frac{2\mu}{\omega^2 - \omega_0^2} + 2\mu\omega(\omega^2 - \omega_0^2)^{-2}2\omega \\
&= \frac{2\mu(\omega^2 + \omega_0^2)}{(\omega^2 - \omega_0^2)^2} \\
&> 0 \text{ For } 0 \leq \omega < \omega_0 \\
&\implies \tan\phi \text{ is increasing.}
\end{aligned}$$

As $\omega \to \omega_0$, $\tan\phi \to \infty \implies \phi \to \pi/2$.

As $\omega \to \infty, \tan\phi \to 0 \implies \phi \to \pi$.



5 Energy

Lecture 19

Energy gives us another viewpoint on mechanical systems.

1D: From Newton's 2nd Law

$$
\begin{aligned}
m\ddot{x} &= F(x, \dot{x}, t) \\
\implies m\ddot{x}\dot{x} &= F\dot{x}
\end{aligned}
$$

Since $\ddot{x}\dot{x} = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right)$

$$
\implies \boxed{\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = F\dot{x}} \tag{5.1}
$$

Call $T = \frac{1}{2}m\dot{x}^2$ and integrate (5.1) with respect to time

$$
\begin{aligned}
\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} T \,\mathrm{d}t &= \int_{t_1}^{t_2} F\dot{x}\,\mathrm{d}t \\
\implies T(t_2) - T(t_1) &= \int_{x(t_1)}^{x(t_2)} F\,\mathrm{d}x
\end{aligned}
$$

Definition.

- $T = \frac{1}{2}m\dot{x}^2$ is the *kinetic energy.*
- $F\dot{x}$ is the *rate of work.*
- $W_{12} = \displaystyle\int_{x(t_1)}^{x(t_2)} F\,\mathrm{d}x$ is the *work done* on m by F.
- $V(x) = -\displaystyle\int F\,dx + C$ is the *potential energy.*
- $T + V = E$ is the *total energy.*
- A force that can be written in terms of a potential ($\vec{F} = -\vec{\nabla}V$) is *conservative.*

Theorem 5.2: Conservation of Energy

Under conservative forces, the total energy of a system is constant.

Proof. Suppose that $F = F(x)$ is a conservative force.

Then $V(x) = -\int F\,dx + C$ or $F = -\dfrac{dV}{dx}$. Integrating:

$$
\begin{aligned}
\int_{x(t_1)}^{x(t_2)} F\,\mathrm{d}x &= \int_{x(t_1)}^{x(t_2)} -\frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x \\
\implies T(t_2) + V(t_2) &= T(t_1) + V(t_1) = E
\end{aligned}
$$

∎

Proof 2. We also have from (5.1)

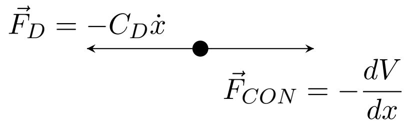
$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) - \dot{F}(x) = 0$$

Since $\dot{F}(x) = \dfrac{dV}{dx}\dfrac{dx}{dt} = \dfrac{dV}{dt}$

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 - V\right) &= 0 \\ \implies T + V &= E, \text{ a constant} \end{aligned}$$

∎

Not all forces are conservative!

> **Example 5.3.** $F_D = -C_D\dot{x}$ is not conservative.
>
> Suppose that
>
> 
>
>
> Newton's Second Law:
>
> $$\begin{aligned} m\ddot{x} &= F_{CON} + F_D \\ \implies m\ddot{x} + \frac{dV}{dx} &= -C_D\dot{x} \end{aligned}$$
>
> Multiplying by $\dot{x}$ and rearranging the terms:
>
> $$\begin{aligned} \frac{d}{dt}(\underbrace{T+V}_{E}) &= -C_D\dot{x}^2 \le 0 \\ \implies \frac{dE}{dt} &\le 0 \\ \implies &\text{ Energy decreases with time} \end{aligned}$$

Examples of Conservative Forces:

- Gravity: $F = -mg \implies V = mgx + C$
- Spring Force: $F = -kx \implies V = \frac{1}{2}kx^2 + C$

We can choose C for our convenience.

Recall that forces that are related to a potential are called *conservative forces*. Another Lecture 20
way to think about conservative forces is through the *work done*:

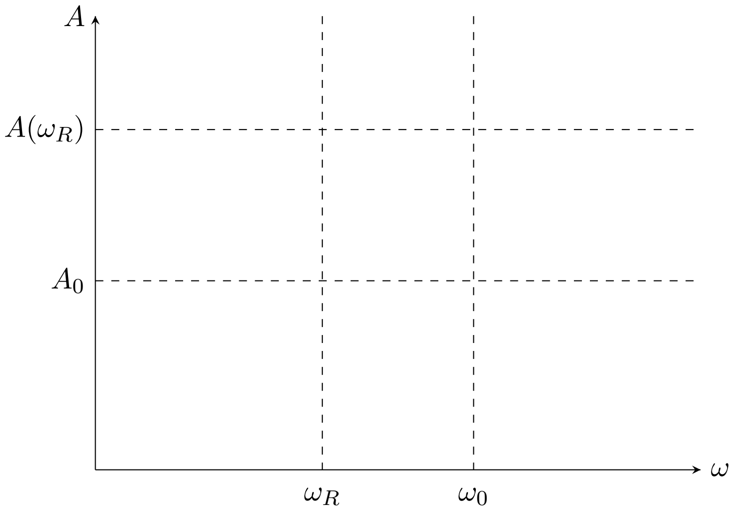
$$W_{12} = \int_{x(t_1)}^{x(t_2)} F \,\mathrm{d}x$$

If the forces is conservative $F = -\dfrac{dV}{dx} \implies W_{12} = -V(x_2) + V(x_1)$.

Hence the work done just depends on the initial and final position. It is path independent! We also saw that as a result:

$$T(t_1) + V(t_1) = T(t_2) + V(t_2) = E, \text{ the total energy}$$

Potential Wells



Suppose we know $\dot{x}$ and x at $t = 0$. With this, we can find

$$E = \frac{1}{2}m\dot{x}^2(0) + V(x(0))$$

And we know this for all times.

> **Definition.** The points x_0, x_1 and x_2 are where $V = E$. These points are called *turning points.*

Oscillations between Turning Points

At the turning points, for example $V(x_1) = E$, we know that $T(x_1) = 0 \implies \dot{x_1} = 0$.

We know that if the particle is between x_0 and x_1, it will oscillate between these points forever! We say that this particle is *trapped*!

Period of oscillation between x_0 and x_1:

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

Solve for $\dot{x}$

$$\frac{dx}{dt} = \dot{x} = \pm\left[\frac{2}{m}(E - V(x))\right]^{1/2} \tag{5.4}$$

We need to choose the correct root based on $\dot{x}$ at a particular point in time. Suppose we know going from x_0 to x_1, $\dot{x} > 0$.

We need to integrate (5.5) to find the time it takes to go from x_0 to x_1

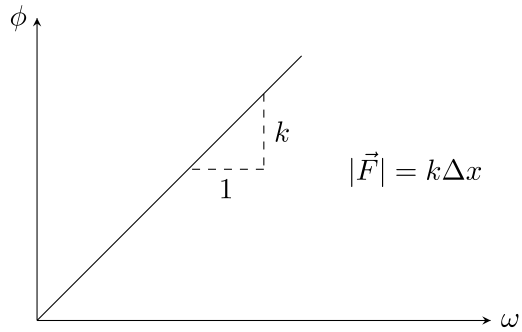
$$\begin{aligned}\int_{x_0}^{x_1} \frac{dx}{\left[\frac{2}{m}(E - V(x))\right]^{1/2}} &= \int_{t_0}^{t_1} dt \\ &= T_{osc}/2\end{aligned}$$

Thus

$$T_{osc} = 2\int_{x_0}^{x_1} \frac{dx}{\left[\frac{2}{m}(E - V(x))\right]^{1/2}} \tag{5.5}$$

Example 5.6 (Spring).

Spring: $V = \frac{1}{2}kx^2$



Initially $x(0) = L$, $\dot{x}(0) = 0$

$$E = \frac{1}{2}m\dot{x}(0) + V(L) = \frac{1}{2}kL^2$$

Then

$$\begin{aligned} T_{osc} &= 2\int_{-L}^{L} \frac{dx}{[\frac{2}{m}(\frac{1}{2}kL^2 - \frac{1}{2}kx^2)]^{1/2}} \\ &= 2\sqrt{\frac{m}{k}}\int_{-L}^{L} \frac{dx}{[L^2 - x^2]^{1/2}} \\ &= 2\sqrt{\frac{m}{k}}\int_{-L}^{L} \frac{dx}{L[1 - (x/L)^2]^{1/2}} \\ u &= x/L \\ &= 2\sqrt{\frac{m}{k}}\int_{-1}^{1} \frac{du}{[1 - u^2]^{1/2}} \\ &= 2\sqrt{\frac{m}{k}}\arcsin u\Big|_{-1}^{1} = 2\pi\sqrt{\frac{m}{k}} \end{aligned}$$

So $T_{osc} = 2\pi\sqrt{\frac{m}{k}}$, $\omega_0 = \sqrt{\frac{k}{m}}$

Escape

Suppose the particle is at x_A. What speed does is need to not be trapped, i.e. $x \to \infty$ as $t \to \infty$? Lecture 21

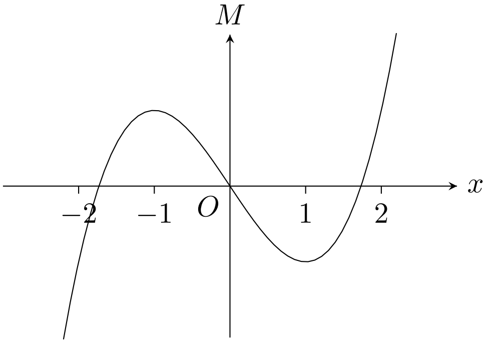
Initial speed: u

$$E = \frac{1}{2}mu^2 + V(x_A)$$

We want $E > E^*$ to allow our particle to escape. $E* = V(X_1)$. We require then

$$V(X_1) < \frac{1}{2}mu^2 + V(x_A)$$

$$\implies u > \sqrt{\frac{2}{m}(V(X_1) - V(x_A))}$$



Stability

Definition. *Equilibrium Points* are where $\frac{dV}{dx} = 0 \implies F = 0 \implies m\ddot{x} = 0$

We say that an equilibrium point is

- *stable* if $\frac{d^2V}{dx^2} > 0$ (Minimum) e.g. X_0
- *unstable* if $\frac{d^2V}{dx^2} < 0$ (Maximum) e.g. X_1

Oscillations near Equilibrium Point

Suppose we are near and very close to a stable equilibrium point, X_0, so $|x - X_0| << 1$.

Taylor expansion of $V(x)$ about X_0:

$$V(x) = V(X_0) + V'(X_0)(X - X_0) + \frac{1}{2}V''(X_0)(X - X_0)^2 + \dots \tag{5.7}$$

Since X_0 is an equilibrium point, we know $V'(X_0) = 0$

$$V(x) = V(X_0) + \frac{1}{2}V''(X_0)(X - X_0)^2$$

Since X_0 is a stable equilibrium point $V''(X_0) > 0$

$$F = \frac{-dV}{dx} = -(x - X_0)V''(X_0)$$

From Newton's 2nd Law

$$m\ddot{x} = -(x - X_0)V''(X_0)$$

Taking $X = x - X_0$

$$m\ddot{X} + V''(X_0)X = 0$$

This looks like the simple harmonic oscillator with $k = V''(X_0)$.

Since $\omega_0 = \sqrt{\frac{k}{m}}$, the frequency of small oscillation is $\omega_0 = \sqrt{\frac{V''(X_0)}{m}}$

$$\implies T_{osc} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{V''(X_0)}}$$

Example 5.8 (Lennard-Jones Potential)**.**

Used to model interactions between neutral atoms or molecules and Molecular dynamics simulations.

$$V(x) = \epsilon\left[\left(\frac{r_0}{x}\right)^{12} - 2\left(\frac{r_0}{x}\right)^{6}\right]$$

$$\underbrace{\epsilon > 0,\ r_0 > 0}_{\text{constants}} \quad (x > 0)$$

As $x \to 0, V \to \infty$. As $x \to \infty, V \to 0$.

$$\frac{dV}{dx} = \epsilon[-12r_0^{12}x^{-13} + 12r_0^{6}x^{-7}$$

Set $\dfrac{dV}{dx} = 0$:

$$\implies 0 = 1 - r_0^6 x^{-6}$$

Equilibrium point as $x = r_0$.

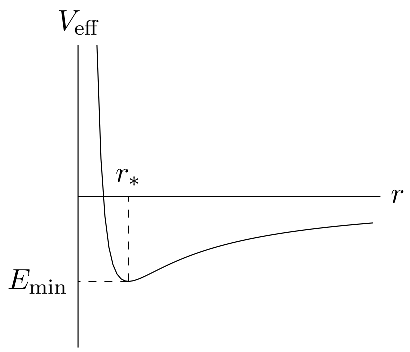
Stability:

$$\frac{d^2V}{dx^2} = \epsilon[156r_0^{12}x^{-14} - 84r_0^6x^{-8}]$$

$$V''(r_0) = 72\epsilon r_0^{-2} > 0$$

$$\implies \text{stable.}$$

In fact $V(r_0) = -\epsilon$:



What occurs for different values of $E = T + V$?

If $E > 0 \implies$ single turning point at $x = x_0 \implies x \geq x_0$. Our particle won't be trapped by the potential.

If $-\epsilon < E < 0$ the particle is trapped.

Suppose that

$$|x - r_0| << 1$$

We can find the period of small oscillations:

$$\begin{aligned} T_{OSC} &= 2\pi\sqrt{\frac{m}{V''(r_0)}} \\ &= \frac{\pi r_0}{3}\sqrt{\frac{m}{2\epsilon}} \end{aligned}$$

Lecture 22

What about energy when the motion is not restricted to a line?

$$m\frac{d\vec{v}}{dt} = \vec{F}$$

Take the dot product with $\vec{v}$:

$$m\frac{d\vec{v}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v}$$

Since $v^2 = \vec{v} \cdot \vec{v}$

$$2v\frac{dv}{dt} = 2\vec{v} \cdot \frac{d\vec{v}}{dt}$$

6 Angular Momentum

Central Forces

We will consider forces of the form

Lecture 23

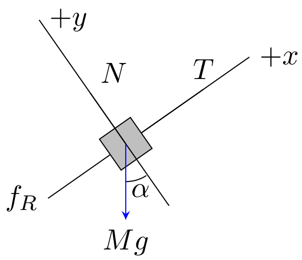
$$\vec{F} = F(r)\,\hat{\mathbf{r}}$$

Magnitude depends on the distance from the origin.

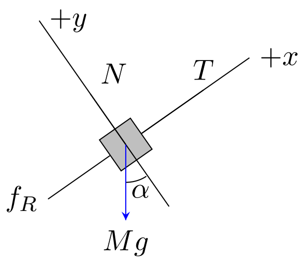
Direction $\hat{\mathbf{r}}$ is repulsive; away from the origin. $-\hat{\mathbf{r}}$: attractive; towards the origin.

Example 6.1 (Gravity)**.**

$$\vec{F} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}}$$



Suppose that



Polar coordinates are perfect for these problems

Newton's Second Law:

$$m(\ddot{r} - r\dot{\theta}^2) = F \tag{6.2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{6.3}$$

Multiply (6.3) by r

$$m(r^2\ddot{\theta} + 2\dot{r}r\dot{\theta}) = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \implies mr^2\dot{\theta} = mh = \text{constant}$$

Definition. $h = r^2\dot{\theta}$ - *angular momentum per unit mass*

Angular momentum, $\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

Theorem 6.4: Conservation of Angular Momentum

Under a central force (no torque), the total angular momentum is conserved.

Proof. In polars, $\vec{r} = r\,\hat{\mathbf{r}}$, $\vec{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$

$$\vec{J} = \vec{r} \times m\vec{v} = (r\,\hat{\mathbf{r}}) \times m(\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}) = mr\dot{r}(\,\hat{\mathbf{r}} \times \,\hat{\mathbf{r}}) + mr^2\dot{\theta}(\,\hat{\mathbf{r}} \times \,\hat{\boldsymbol{\theta}})$$

$$\implies \vec{J} = mr^2\dot{\theta}\,\hat{\mathbf{k}} = mh\,\hat{\mathbf{k}} = \text{ constant}$$ ∎

Energy

For a force to be conservative $\vec{F} = -\vec{\nabla}V$. In 2D

$$\vec{F} = -\frac{\partial V}{\partial x}\,\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\,\hat{\mathbf{j}} \tag{6.5}$$

Since $\vec{F} = \vec{F}(r)\,\hat{\mathbf{r}}$ we need $V = V(r)$

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\frac{\partial r}{\partial x}$$

Since $r = (x^2+y^2)^{1/2}$, $\dfrac{\partial r}{\partial x} = \frac{1}{2}[x^2+y^2]^{1/2} \times (2x) = x/r = \cos(\theta)$. Thus

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\cos\theta$$

Similarly

$$\frac{\partial V}{\partial y} = \frac{dV}{dr}\frac{\partial r}{\partial y} = \frac{dV}{dr}\sin\theta$$

Thus the force, by (6.5), is

$$\begin{aligned}\vec{F} &= -\frac{dV}{dx}\cos\theta\,\hat{\mathbf{i}} - \frac{dV}{dy}\sin\theta\,\hat{\mathbf{j}} \\ &= -\frac{dV}{dr}\,\hat{\mathbf{r}}\end{aligned}$$

So for a central force to be conservative

$$\vec{F}(r) = -\frac{dV}{dr}$$

From the Conservation of Energy

$$\frac{1}{2}mv^2 + V(r) = E$$

Since $\vec{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$

$$\boxed{E = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] + V(r)} \tag{6.6}$$

Orbital Equation

Find the trajectories or shapes or orbits as a function of θ. It's solution is $u(\theta) = 1/r(\theta)$. We know $h = r^2\dot{\theta} = \dot{\theta}u^{-2} \implies \dot{\theta} = hu^2$. Thus

$$\begin{aligned}\dot{r} &= \frac{d}{dt}(u^{-1}) \\ &= -u^{-2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}\end{aligned}$$

$$\begin{aligned}\ddot{r} &= -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) \\ &= -h\frac{d^2u}{d\theta^2}\dot{\theta} = h^2u^2\frac{d^2u}{d\theta^2}\end{aligned}$$

Also

$$r\dot{\theta}^2 = u^{-1}(hu^2)^2 = h^2u^3$$

Write $F(r) = F(u^{-1})$ and substitute into (6.2) from Newton's 2nd Law:

$$m\left(h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) = F(u^{-1})$$

Giving our orbital equation:

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}F(u^{-1})} \tag{6.7}$$

Example 6.8. $r(\theta) = c\theta^2$ $(c > 0)$. Find $F(r)$:

$$\begin{aligned}u &= c^{-1}\theta^{-2} \\ \frac{du}{d\theta} &= -2c^{-1}\theta^{-3} \\ \frac{d^2u}{d\theta^2} &= 6c^{-1}\theta^{-4} = 6u^2\end{aligned}$$

From the Orbital Equation (6.7)

$$\begin{aligned}F(u^{-1}) &= -mh^2u^2(u + 6cu^2) \\ &= -mh^2(u^3 + 6cu^4) \\ \implies F(r) &= -mh^2(r^{-3} + 6cr^{-4})\end{aligned}$$

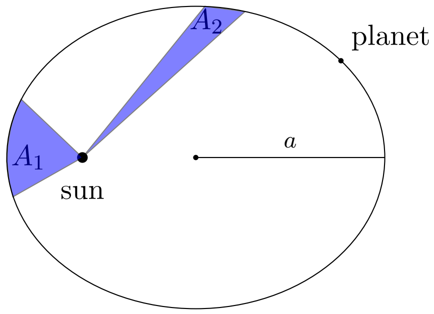
Kepler's Laws

Lecture 24

Theorem 6.9: Kepler's Laws

I Orbits of Planets are Ellipses

II Law of Equal Areas: If $\Delta t_1 = \Delta t_2$ then $A_1 = A_2$



III The time period of orbit, $T \propto a^3$

Proof of Kepler's First Law. Inverse square law:

$$F(r) = -k/r^2 \implies F(u^{-1}) = -ku^2$$

Substituting into our orbital equation (6.6)

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2u^2} \ (*)$$

This resembles

$$m\frac{d^2x}{dt^2} + kx = F_0$$

The general solution to $(*)$ is $u = A\cos(\theta - \theta_0) + \dfrac{k}{mh^2}$; wlog take $\theta_0 = 0$ so

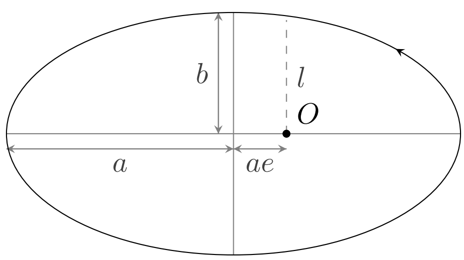
$$u(\theta) = A\cos(\theta) + \frac{k}{mh^2}$$

$$\implies r(\theta) = \frac{(mh^2/k)}{1 + \frac{Amh^2}{k}\cos\theta} \tag{6.10}$$

This is the form of an ellipse in polar coordinates (see Problem 10, P.S. 1)

$$r(\theta) = \frac{l}{1 + e\cos\theta}$$

Where $l = \dfrac{mh^2}{k}$, $e = \dfrac{Amh^2}{k}$. $e = [1 - b^2/a^2]^{1/2}$, $l = a(1 - e^2)$ ∎



We see that E is related to A.

We can get the family of orbits by considering the energy; equation (6.5) gives

$$E = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] + V(r)$$

$F(r) = -kr^{-2} = -\dfrac{dV}{dr}$, so $V(r) = -kr^{-1} \implies V(u^{-1}) = -ku$.

Also $\dot{r} = -h\dfrac{du}{d\theta}$, and $r^2\dot{\theta}^2 = h^2r^{-2} = h^2u^2$. So the energy is

$$E = \frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku$$

Using the fact $u(\theta) = A\cos(\theta) + \dfrac{k}{mh^2}$, $\dfrac{du}{d\theta} = -A\sin\theta$ and simplifying the trig we get

$$\begin{aligned} E &= \frac{1}{2}mh^2A^2 - \frac{1}{2}\frac{k^2}{mh^2} \\ \implies A &= \sqrt{\frac{2E}{mh^2} + \frac{k^2}{(mh^2)^2}} \end{aligned}$$

From (6.9), the eccentricity of the orbit, $e = (1 - b^2/a^2)^{1/2}$, is

$$\begin{aligned} e &= \frac{Amh^2}{k} \\ &= \frac{mh^2}{k}\sqrt{\frac{2E}{mh^2} + \frac{k^2}{(mh^2)^2}} \\ &= \sqrt{1 + \frac{2Emh^2}{k^2}} \end{aligned}$$

This parameter e actually allows our solution $r(\theta)$ to describe a whole family of orbits.

Examples 6.11.

(i) Bounded Trajectories

- $E = -k^2/2mh^2 \implies e = 0$ [Circle]
- $E < 0 \implies 0 < e < 1$ [Ellipse]

(ii) Unbounded Trajectories

- $E = 0 \implies e = 1$ [Parabola]
- $E > 0 \implies e > 1$ [Hyperbola]

Effective Potential

Lecture 25 Consider the energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

Since $h = r^2\dot{\theta},\ h^2 = r^4\dot{\theta}^2 \implies r^2\dot{\theta}^2 = h^2/r^2$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{mh^2}{r^2} + V(r)$$

> **Definition.** The *Effective Potential*, $V_{EFF} = \frac{1}{2}\frac{mh^2}{r^2} + V(r)$

$$\implies E = \frac{1}{2}m\dot{r}^2 + V_{EFF}$$

What we've done is written our energy in such a way that it looks like what we had with 1D motion!

$$x \longrightarrow r$$

$$V(x) \longrightarrow V_{EFF}(r)$$

> **Definition.** *Turning points* occur when $V_{EFF} = E$. This tells us where $\frac{1}{2}m\dot{r}^2 = 0 \implies \dot{r} = 0$. This tells us about the boundedness of our orbit.

Equilibria

In 1D: $V'(x_0) = 0 \implies F(x_0) = 0$, where x_0 is the equilibrium point

If $\dot{x} = 0$ and $x = x_0$ at $t = 0$, then $m\ddot{x} = 0$ and $x = x_0\ \forall t$

$$V_{EFF} = \frac{1}{2}\frac{mh^2}{r^2} + V(r)$$

$$\implies \frac{dV_{EFF}}{dr} = -mh^2r^{-3} + \underbrace{V'(r)}_{-F(r)}$$

Newton's 2nd Law's $\hat{\mathbf{r}}$ component (equation (6.2))

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$

$$\implies m\ddot{r} = F(r) + \frac{mh^2}{r^3} = \frac{dV_{EFF}}{dr}$$

Suppose that $V'_{EFF}(r_0) = 0$. If $r = r_0$ and $\dot{r} = 0$ at $t = 0$, then $m\ddot{r} = 0 \implies r = r_0\ \forall t$. So we have a constant $r \implies$ Circular Trajectory

Stability

$R = r - r_0$, $|R| << 1$, then the Taylor expansion about r_0:

$$V_{EFF}(r) = V_{EFF}(r_0) + RV'_{EFF}(r_0) + \frac{1}{2}R^2V''_{EFF}(r_0) + \dots \tag{6.12}$$

Since at r_0, $V'_{EFF}(r_0) = 0$

$$V_{EFF}(r) = V_{EFF}(r_0) + \frac{1}{2}R^2 V''_{EFF}(r_0)$$

Differentiating

$$V'_{EFF}(r) = R V''_{EFF}(r)$$

Using this in Newton's Second Law:

$$m\ddot{r} = -R V''_{EFF}(r_0)$$

or

$$m\ddot{R} + R V''_{EFF}(r_0) = 0$$

- If $V''_{EFF}(r_0) > 0 \implies$ a minimum, so the circular orbit is stable.
- If $V''_{EFF}(r_0) < 0 \implies$ a maximum, so the circular orbit is unstable.

Example 6.13. $F(r) = -kr^{-2}$ $(k > 0) \implies V(r) = -kr^{-1}$

$$\implies V_{EFF}(r) = -kr^{-1} + \frac{1}{2}mh^2 r^{-2}$$
$$\implies V'_{EFF}(r) = kr^{-2} - mh^2 r^{-3}$$

Setting this equal to zero

$$r^{-3}(kr - mh^2) = 0$$

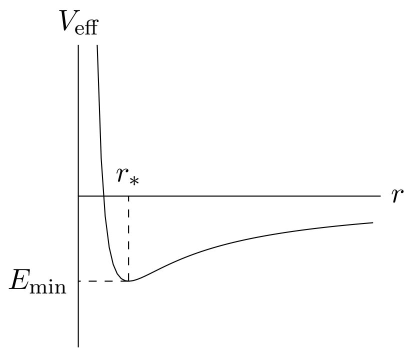
This is satisfied as $r \to \infty$ or at $r_0 = mh^2/k$

$$V''_{EFF}(r) = -2kr^{-3} + 3mh^2 r^{-4}$$

So at the equilibria point

$$V''_{EFF}(mh^2/k) = \left(\frac{k}{mh^2}\right)^4 (3mh^2 - 2k(mh^2/k)) = \left(\frac{k}{mh^2}\right)^4 (mh^2) > 0$$

This is a stable circular trajectory.



$$V'_{EFF}(\frac{mh^2}{k}) = -k\left(\frac{k}{mh^2}\right) + \frac{1}{2}mh^2\left(\frac{k^2}{(mh^2)^2}\right) = -\frac{k^2}{2mh^2}$$

Thus

$$E_{MIN} = -\frac{k^2}{2mh^2}.$$

We reach the same family of orbits as Example 6.10 by differing values of E:

(i) Bounded Trajectories

- $E = E_{MIN} = -k^2/2mh^2 \implies r = \frac{mh^2}{k} \implies$ Circular Orbit
- $E_{MIN} < E < 0 \implies$ two turning points $\implies$ Bounded Orbit [Ellipse]

(ii) Unbounded Trajectories when $E \geq 0$ since we have only a single turning point. In particular

- $E = 0 \implies$ Parabola
- $E > 0 \implies$ Hyperbola

7 Systems of Particles

Lecture 26

Definition.

- N: Total number of particles
- $\vec{r}_i$: Position of particle i
- $\vec{v}_i$: Velocity of particle i
- $\vec{F}_i$: Force on particle i
- m_i: Mass of particle i

Consider the average motion of the system:

Definition. *Centre of Mass*, $\vec{r}_{cm}$:

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$$

Where $M = \sum_{i=1}^{N} m_i$ is the *total mass.*

Momentum

The total momentum $\vec{p}$ is

$$\begin{aligned} \vec{p} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i &= \sum_i m_i \frac{d\vec{r}_i}{dt} \\ &= \frac{d}{dt}(\sum_i m_i \vec{r}_i) \\ &= \frac{d}{dt}(M\vec{r}_{cm}) \\ &= M\frac{d\vec{r}_{cm}}{dt} = M\vec{v}_{cm} \end{aligned}$$

Where $\vec{v}_{cm}$ is the velocity of the centre of mass.

$$\vec{F}_i = \vec{F}_i^{EXT} + \sum_{j=1}^{N} \vec{F}_{ij}$$

where $\vec{F}_i^{EXT}$ is the external forces on particle i, $\vec{F}_{ij}$ is the force on i due to j

Example 7.1.

Here $\vec{F}_{gi}$(Force due to gravity on i) is the only external force on $i \implies \vec{F}_i^{EXT} = \vec{F}_{gi}$

Note that

(i) $\vec{F}_{ii} = \vec{0}$

(ii) $\vec{F}_{ij} = -\vec{F}_{ji}$ By Newton's Third Law

Theorem 7.2: Newton's Second Law for a System

The external force is equal to the rate of change of momentum of the centre of mass

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT}$$

Where the total external force on the system $\vec{F}^{EXT} = \sum_i \vec{F}_i^{EXT}$.

Proof. For particle i,

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i = \vec{F}_i^{EXT} + \sum_{j=1}^{N} \vec{F}_{ij}$$

$$\implies \sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i = \sum_i \vec{F}_i^{EXT} + \sum_i \sum_j \vec{F}_{ij}$$

Due to Newton's Third Law $\sum_i \sum_j \vec{F}_{ij} = \vec{0}$. We are then left with

$$\sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_i^{EXT}$$

$$\implies \frac{d}{dt}(\sum_i \vec{p}_i) = \vec{F}^{EXT}$$

$$\implies M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT}$$ ∎

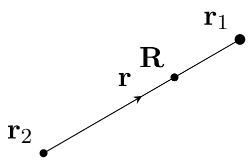
(i) If there is no external forces then

$$M\frac{d\vec{v}_{cm}}{dt} = 0 = \frac{d\vec{p}}{dt}$$

(The conservation of momentum)

(ii) If there are external forces then the centre of mass moves as though it were a point particle of mass m subject to force $\vec{F}^{EXT}$

Two Body Problems



$\vec{F}_1 = m_1 g\,\hat{\mathbf{i}} + \vec{F}_{12}$

$\vec{F}_2 = m_2 g\,\hat{\mathbf{i}} + \vec{F}_{21}$

The total external force:

$$\vec{F}^{EXT} = m_1 g\,\hat{\mathbf{i}} + m_2 g\,\hat{\mathbf{i}} = Mg\,\hat{\mathbf{i}}\ (M = m_1 + m + 2)$$

Thus

$$M\frac{d\vec{v}_{cm}}{dt} = Mg\,\hat{\mathbf{i}} \implies \frac{d\vec{v}_{cm}}{dt} = g\,\hat{\mathbf{i}}$$

For two body problems this is half of the information.

$$m_1\frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1^{EXT} + \vec{F}_{12} \tag{7.3}$$

$$m_2\frac{d^2\vec{r}_2}{dt^2} = \vec{F}_2^{EXT} + \vec{F}_{21} \tag{7.4}$$

Calling $\dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$, and adding the equations

$$m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} = \vec{F}_1^{EXT} + \vec{F}_2^{EXT}$$

$$M\frac{d}{dt}\left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{M}\right) = \vec{F}_1^{EXT} + \vec{F}_2^{EXT}$$

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}_1^{EXT} + \vec{F}_2^{EXT}$$

Lecture 27

Consider: $m_2 \times (7.4) - m_1 \times (7.3)$

$$m_1 m_2\frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT} + m_2\vec{F}_{12} - m_1\vec{F}_{21}$$

Call $\vec{r}_{12} = (\vec{r}_1 - \vec{r}_2)$. Since $\vec{F}_{12} = -\vec{F}_{21}$

$$m_1 m_2\frac{d^2\vec{r}_{12}}{dt^2} = m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT} + (m_1 + m_2)\vec{F}_{12}$$

Divide through by M

$$\frac{m_1 m_2}{M}\frac{d^2\vec{r}_{12}}{dt^2} = \frac{m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT}}{M} + \vec{F}_{12}$$

Definition. Introduce $\mu = \dfrac{m_1 m_2}{M}$, the *reduced mass.*

Then for our two body system we have:

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}_1^{EXT} + \vec{F}_2^{EXT} \tag{7.5}$$

$$\mu\frac{d^2\vec{r}_{12}}{dt^2} = \frac{m_2\vec{F}_1^{EXT} + m_1\vec{F}_2^{EXT}}{M} + \vec{F}_{12} \tag{7.6}$$

If $\vec{F}_1^{EXT} = \vec{F}_2^{EXT} = 0$, then:

$$M\frac{d\vec{v}_{cm}}{dt} = 0, \text{ and } \mu\frac{d^2\vec{r}_{12}}{dt^2} = \vec{F}_{12}$$

If $\vec{F}_1^{EXT} = -m_1 g\,\hat{\mathbf{j}}$ and $\vec{F}_2^{EXT} = -m_2 g\,\hat{\mathbf{j}}$, then:

$$M\frac{d\vec{v}_{cm}}{dt} = -Mg\,\hat{\mathbf{j}}, \text{ and } \mu\frac{d^2\vec{r}_{12}}{dt^2} = \vec{F}_{12}$$

Example 7.7 (Spring)**.**

Speing has a spring constant k and equilibrium lnegth l.

$$\vec{F}_{12} = -k(x_1 - x_2 - l)\,\hat{\mathbf{i}}$$

Initially $x_1(0) = k,\ \dot{x}_1 = v_0$. $x_2(0) = \dot{x}_2(0) = 0$.
$\vec{F}_{12}$ is the only force in the $\hat{\mathbf{i}}$ direction. No external forces in the $\hat{\mathbf{i}}$ direction.

$$\implies M\ddot{x}_{cm} = 0 \implies \dot{x}_{cm} = C$$

We can find C using the conservation of momentum

$$\vec{p} = m\dot{x}_1 + m\dot{x}_2 = M\dot{x}_{cm}$$

At $t = 0, \dot{x}_1 = v_0$ and $\dot{x}_2 = 0$. Then $p = mv_0$. Since $M = 2m$:

$$\dot{x}_{cm} = v_0/2$$

For $x_{12} = x_1 - x_2$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}$$

$$\vec{F}_{12} = -k(x_1 - x_2 - l) = -k(x_{12} - l)$$

Using the equation for $\vec{r}_{12}$

$$\mu\ddot{x}_{12} = \vec{F}_{12}$$
$$\frac{m}{2}\ddot{x}_{12} = -k(x_{12} - l)$$
$$\ddot{x}_{12} + \frac{2k}{m}x_{12} = \frac{2kl}{m}$$

The general solution is

$$x_{12} = A\cos\omega t + B\sin\omega t + l$$

where $\omega^2 = \frac{2k}{m}$.

From our initial conditions $x_{12}(0) = x_1(0) - x_2(0) = l$ and $\dot{x}_{12} = v_0$.

$$\implies A = 0,\ B = v_0/\omega$$

Thus

$$x_{12} = \frac{v_0}{\omega}\sin\omega t + l$$
$$\dot{x}_{12} = v_0\cos\omega t$$

We can show that (in general)

$$\vec{r}_1 = \vec{r}_{cm} + \vec{m_2}M\vec{r}_{12}$$
$$\vec{r}_2 = \vec{r}_{cm} + \vec{m_1}M\vec{r}_{12}$$

Thus

$$x_1 = x_{cm} + \frac{1}{2}x_{12}$$
$$\dot{x}_1 = \dot{x}_{cm} + \frac{1}{2}\dot{x}_{12} = \frac{v_0}{2} + \frac{1}{2}v_0\cos\omega t = \frac{v_0}{2}(1 + \cos\omega t)$$

Similarly

$$\dot{x}_2 = \frac{v_0}{2}(1 - \cos\omega t)$$

This is a push-me-pull-you system.

What about more than two particles?

Definition (Centre of Mass Coordinates)**.**

$$\vec{R}_i = \vec{r}_i - \vec{r}_{cm}$$

This is the position of particle i relative to the position of the centre of mass

$$\sum_i m_i\vec{R}_i = \underbrace{\sum_i m_i\vec{r}_i}_{M\vec{r}_{cm}} - \vec{r}_{cm}\underbrace{\sum_i m_i}_{M} = 0$$

Lecture 28

Kinetic Energy

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

We can write $\vec{v}_i = \vec{v}_{cm} + \dfrac{d\vec{R}_i}{dt}$, $\vec{u}_i = \dfrac{d\vec{R}_i}{dt}$, so $\vec{v}_i = \vec{v}_{cm} + \vec{u}_i$

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i (\vec{v}_{cm} + \vec{u}_i) \cdot (\vec{v}_{cm} + \vec{u}_i) \\ &= \sum_i \frac{1}{2} [v_{cm}^2 + 2\vec{u}_i \cdot \vec{v}_{cm} + u_i^2 \\ &= \frac{1}{2} v_{cm}^2 \sum_i m_i + \vec{v}_{cm} \cdot \sum_i m_i \vec{u}_i + \frac{1}{2} \sum_i m_i u_i^2 \\ &= \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \sum_i m_i u_i^2 + \vec{v}_{cm} \cdot \sum_i m_i \vec{u}_i \end{aligned}$$

Consider $\sum_i m_i \vec{u}_i = \sum_i m_i \dfrac{d\vec{R}_i}{dt} = \dfrac{d}{dt} (\sum_i m_i \vec{R}_i) = 0$. Then

$$\boxed{T = \frac{1}{2} M v_{cm}^2 + \sum_i \frac{1}{2} m_i u_i^2} \tag{7.8}$$

Angular Momentum

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

For central forces where the motion was restricted to a plane $\vec{J} = mh\,\hat{\mathbf{k}}$ = constant vector.

What causes $\vec{J}$ to change?

$$\begin{aligned} \frac{d\vec{J}}{dt} &= \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} \\ &= m\cancelto{0}{[\vec{v} \times \vec{v}]} + \vec{r} \times \vec{F} = \vec{\tau} \end{aligned}$$

Definition. $\vec{\tau} = \vec{r} \times \vec{F}$ is the *Torque* or the *Moment.*

- $\vec{\tau}$ is in the direction out of the screen
- $|\vec{\tau}| = |\vec{F}||\vec{r}| \sin\phi$

For central forces

Since $\phi = 0 \implies \vec{\tau} = 0$.

For a system, the total angular momentum

$$\vec{J} = \sum_i \vec{J}_i = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

$$\implies \vec{\tau} = \frac{d\vec{J}}{dt} = \sum_i \frac{d\vec{J}_i}{dt} = \sum_i \vec{r}_i \times \vec{F}_i$$

Write $\vec{F}_i = \vec{F}_i^{EXT} + \sum_j \vec{F}_{ij}$. Then we have

$$\vec{\tau}\frac{d\vec{J}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT} + \sum_i \sum_j \vec{r}_i \times \vec{F}_{ij} \tag{7.9}$$

Theorem 7.10: Conservation of Angular Momentum for a System

If there is no net torque, the angular momentum is conserved.

Proof (for two body system). Suppose we have two particles. Then the double sum is

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

By Newton's Third Law $\vec{F}_{12} = -\vec{F}_{21}$. Thus

$$\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12}$$

If $\vec{F}_{12}$ is parallel to $\vec{r}_1 - \vec{r}_2$, then $(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = 0$.

This is the case if $\vec{F}_{12}$ is a central force, i.e. no torque.

Thus if $\vec{F}_{ij}$ is a central force for all i and j. Then

$$\sum_i \sum_j \vec{r}_i \times \vec{F}_{ij} = \vec{0}$$

Then

$$\frac{d\vec{J}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT} = \vec{\tau}^{EXT}$$

where $\vec{\tau}^{EXT}$ is the total external torque on the system.

So if $\vec{\tau}^{EXT} = \vec{0}$ then $\dfrac{d\vec{J}}{dt} = \vec{0}$, hence the angular momentum is conserved. ∎

Example 7.11.

Each particle has mass m. Each mass has velocity $\vec{v}_i = \vec{\omega} \times \vec{r}_i$, with $\vec{\omega} = \omega\,\hat{\mathbf{k}}$

The angular momentum of particle i is:

$$\vec{J}_i = \vec{r}_i \times m_i \vec{v}_i = m[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)]$$

Recall that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$$\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = (\vec{r}_i \cdot \vec{r}_i)\vec{\omega} - \cancelto{0}{(\vec{r}_i \cdot \vec{\omega})}\vec{r}_i = r^2\omega\,\hat{\mathbf{k}}$$

Thus

$$\vec{J}_i = mr^2\omega\,\hat{\mathbf{k}}$$
$$\implies \vec{J} = \sum_i \vec{J}_i = 4mr^2\omega\,\hat{\mathbf{k}} = 2ml^2\omega\,\hat{\mathbf{k}}$$

Suppose that

$\vec{v}_i = \vec{\omega} \times \vec{r}_i \longrightarrow \vec{v}_i = \vec{\Omega} \times \vec{r}_i$. What's $\vec{\Omega}$?

Single the configuration changed to to internal, central forces, $\dfrac{d\vec{J}}{dt} = 0$

For our new configuration

$$\vec{J}_i = 2m[\vec{r}_i \times (\vec{\Omega} \times \vec{r}_i)] = 2mr_i^2\Omega\,\hat{\mathbf{k}} = \frac{ml^2\Omega}{2}\,\hat{\mathbf{k}}$$

The total angular momentum

$$\vec{J} = 2\vec{J}_i = ml^2\Omega\,\hat{\mathbf{k}}$$

Since $\dfrac{d\vec{J}}{dt} = 0 \implies \vec{J}_{before} = \vec{J}_{after}$

$$\implies 2ml^2\omega\,\hat{\mathbf{k}} = ml^2\Omega\,\hat{\mathbf{k}}$$
$$\implies \Omega = 2\omega$$

The angular speed doubles as a result of the change.

Centre of Mass Coordinates

Lecture 29

$$\vec{r}_i = \vec{r}_{cm} + \vec{R}_i$$
$$\vec{v}_i = \vec{v}_{cm} + \vec{u}_i, \quad \left(\vec{u}_i = \frac{d\vec{R}_i}{dt}\right)$$

Thus

$$\vec{J} = \sum_i (\vec{r}_{cm} + \vec{R}_i) \times m_i(\vec{v}_{cm} + \vec{u}_i)$$
$$= \sum_i \vec{r}_{cm} \times m_i\vec{v}_{cm} + \sum_i \vec{r}_{cm} \times m_i\vec{u}_i + \sum_i \vec{R}_i \times m_i\vec{v}_{cm} + \sum_i \vec{R}_i \times m_i\vec{u}_i$$
$$= \vec{r}_{cm} \times \vec{v}_{cm}(\sum_i m_i) + \vec{r}_{cm} \times (\sum_i m_i\vec{u}_i) + (\sum_i m_i R_i) \times \vec{v}_{cm} + \sum_i R_i \times m_i\vec{u}_i$$

We know that $\sum_i m_i = M$, $\sum_i m_i\vec{R}_i = \sum_i m_i\vec{u}_i = 0$. Thus

$$\vec{J} = \vec{r}_{cm} \times M\vec{v}_{cm} + \sum_i \vec{R}_i \times m_i\vec{u}_i$$

Call $\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i\vec{u}_i$

Recall that

$$\frac{d\vec{J}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{EXT} (= \vec{\tau}^{EXT})$$

Since $\vec{r}_i = \vec{r}_{cm} + \vec{R}_i$

$$\frac{d\vec{J}}{dt} == \sum_i \vec{r}_{cm} \times \vec{F}_i^{EXT}$$
$$= \vec{r}_{cm} \times \vec{F}^{EXT} + \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

We can show (P.S. 4 Problem 9)

$$\frac{d\vec{J}_{cm}}{dt} = \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

Call

$$\vec{\tau}_{cm}^{EXT} = \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

Complete Picture

(i) Momentum:

$$\vec{p} = M\vec{v}_{cm}$$
$$\frac{d\vec{p}}{dt} = M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT}$$

(ii) Angular Momentum:

$$\vec{J} = \vec{r}_{cm} \times M\vec{v}_{cm} + \vec{J}_{cm}$$
$$\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i\vec{u}_i$$
$$\frac{d\vec{J}}{dt} = \vec{r}_{cm} \times \vec{F}^{EXT} + \vec{\tau}_{cm}^{EXT}$$

8 Rigid Body Motion

Definition. *Rigid Body Motion* occurs when

$$\frac{d|\vec{r}\_\vec{r}_j|}{dt} = 0, \ \forall i, j$$

For such a system

$$\vec{v}_i = \vec{v}_{cm} + \underbrace{\vec{\omega} \times \vec{R}_i}_{\vec{u}_i}$$

Where $\vec{\omega}$ is the *angular velocity of the rigid body.*

We can also write

$$\vec{v}_i = \vec{V} + \vec{\omega} \times \vec{r}_i$$

where $\vec{V} = \vec{v}_{cm} - \vec{\omega} \times \vec{r}_{cm}$

To determine the motion of the system we'll need to find $\vec{v}_{cm}$ and $\vec{\omega}$. For $\vec{v}_{cm}$ we already have this!

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT} \tag{8.1}$$

What about $\vec{\omega}$?

$$\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i \vec{u}_i$$

For a rigid body $\vec{u}_i = \vec{\omega} \times \vec{R}_i$

$$\vec{J}_{cm} = \sum_i \vec{R}_i \times m_i(\vec{\omega} \times \vec{R})i) = \sum_i m_i(\vec{R}_i \times (\vec{\omega} \times \vec{R}_i))$$

From the identity for the triple vector product, we have

$$\vec{J}_{cm} = \sum_i m_i[R_i^2\vec{\omega} - (\vec{\omega} \cdot \vec{R}_i)\vec{R}_i]$$

Consider only planar motion: we have $\vec{\omega} = \omega\,\hat{\mathbf{k}}$, and $\vec{R}_i = X_i\,\hat{\mathbf{i}} + Y_i\,\hat{\mathbf{j}}$. Thus

$$\vec{\omega} \cdot \vec{R}_i = 0, \ \forall i$$

As a result:

$$\vec{J}_{cm} = \underbrace{(\sum_i m_i R_i^2)}_{I_{cm}}\,\vec{\omega} \tag{8.2}$$

Definition. I_{cm} is the *moment of inertia* about the centre of mass.

For this Rigid Body Motion $\frac{d|R_i|}{dt} = 0$. This means that I_{cm} is constant.

Consider

$$\frac{d\vec{J}_{cm}}{dt} = I_{cm}\frac{d\vec{\omega}}{dt} = \vec{\tau}_{cm}^{EXT} = \sum_i \vec{R}_i \times \vec{F}_i^{EXT}$$

For a rigid body undergoing planar motion:

$$M\frac{d\vec{v}_{cm}}{dt} = \vec{F}^{EXT} \tag{8.3}$$

$$I_{cm}\frac{d\omega}{dt} = \tau_{cm}^{EXT} \tag{8.4}$$

(Scalar Equation since all in $\hat{\mathbf{k}}$)

Kinetic Energy

$$T = \frac{1}{2}M\vec{v}_{cm}^2 + \frac{1}{2}\sum_i m_i\vec{u}_i^2$$

$\vec{u}_i = \vec{\omega}\times\vec{R}_i$, $u_i^2 = (\vec{\omega}\times\vec{R}_i)\cdot(\vec{\omega}\times\vec{R}_i)$

For planar motion $|\vec{\omega}\times\vec{R}_i| = |\vec{\omega}||\vec{R}_i| \implies u_i^2 = \omega^2 R_i^2$

$$T = \frac{1}{2}M\vec{v}_{cm}^2 + \frac{1}{2}\left(\sum_i m_i R_i^2\right)\omega^2$$

$$\implies T = \frac{1}{2}M\vec{v}_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

Lecture 30

Definition. The continuous case:

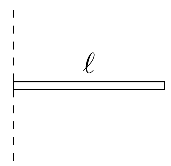
$$M = \sum_i m_i = \int_B dm$$

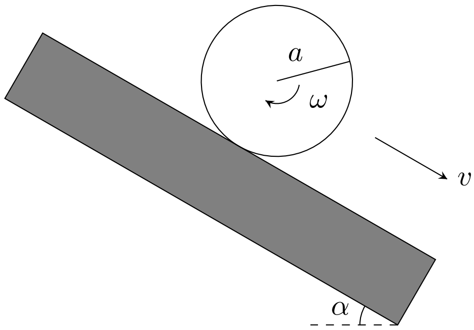
$$\vec{r}_{cm} = \frac{\sum_i m_i\vec{r}_i}{M} = \frac{\int_B \vec{r}\,dm}{M}$$

$$I_{cm} = \sum_i m_i R_i^2 = \int_B R^2\,dm$$

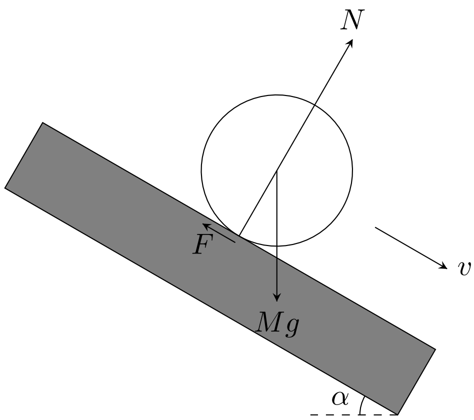
Equations of motion remain the same.

Example 8.5 (Uniform Rod)**.**





Alternatively, we may do it in terms of forces and torques,



* Parallel Axis Theorem *

Lecture 31

(Non-examinable in 2015)

Theorem 8.6: Parallel Axis Theorem

For an axis, P, parallel to the centre of mass

$$I_P = I_{CM} + Mr_{CM}^2$$

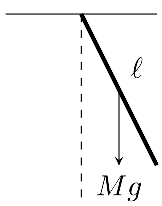
Proof.

$$I_P = \sum_i m_i r_i^2 = \sum_i m_i(\vec{r}_{CM} + \vec{R}_i)^2$$

$$= \sum_i m_i r_{CM}^2 + 2$$

∎

Felina. Consider a Physical Pendulum



Taking the pivot to be the origin because we know that it does not move.

Compute I, $\vec{J}$ and $\vec{\tau}$ relative (or about) the pivot rather than the centre of mass.

$$I_{cm}\dot{\omega} = \tau_{cm}^{EXT} \implies I_p\dot{\omega} = \tau_p^{EXT}$$

We can use the *parallel axis theorem* to find I_p. For point particles

$$\begin{aligned}
I &= \sum_i m_i r_i^2 \\
&= \sum_i m_i(\vec{r}_{cm} + \vec{R}_i)^2 \\
&= \sum_i m_i r_{cm}^2 + 2\vec{r}_{cm}\cdot(\sum_i m_i\vec{R}_i) + \sum_i m_i R_i^2 \\
&= Mr_{cm}^2 + I_{cm} \\
\implies I &= I_{cm} + Mr_{cm}^2
\end{aligned}$$

We can consider

$$I_p\dot{\omega} = \tau_p^{EXT}$$

We just need to consider the torque due to gravity.

$$\vec{\tau}_g = \vec{r}_{cm} \times \vec{F}_g$$

Since

$$\begin{aligned}
\vec{r}_{cm} &= r_{cm}[\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}] \\
\vec{F}_g &= Mg\,\hat{\mathbf{i}} \\
\vec{\tau}_g &= \vec{r}_{cm} \times \vec{F}_g \\
&= Mgr_{cm}\sin(\hat{\mathbf{j}} \times \hat{\mathbf{i}}) \\
&= -Mgr_{cm}\sin\theta\,\hat{\mathbf{k}}
\end{aligned}$$

We can write $\dot{\omega} = \ddot{\theta}$

$$\implies I_p\ddot{\theta} = -Mgr_{cm}\sin\theta$$

$$\ddot{\theta} + \frac{Mgr_{cm}}{I_p}\sin\theta = 0$$

Recall: For a point mass

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Take $\theta << 1$, $\sin\theta \equiv \theta$

$$\ddot{\theta} + \frac{Mgr_{cm}}{I_p}\theta = 0$$

Frequency:

$$\omega = \sqrt{\frac{Mgr_{cm}}{I_p}}$$

For a disc:

$$I_{cm} = MR^2/2$$

From parallel axis theorem

$$I_p = \frac{MR^2}{2} + Mr_{cm}^2$$

$$\omega = \sqrt{\frac{Mgr_{cm}}{\frac{MR^2}{2} + Mr_{cm}^2}}$$

$$= \sqrt{\frac{g}{l}}$$

Where $l = r_{cm}\left(1 + \frac{R^2}{2r_{cm}^2}\right)$

- End of Mechanics -