NUMERICAL ANALYSIS FOR PARTIAL DIFFERENTIAL EQUATIONS
A.Y. 2019/2020

Lecturer: Prof. A. Quarteroni T.A. and Tutor: Dr. F. Regazzoni

1. Consider the problem

$$\begin{cases} -\operatorname{div}(\mu\nabla u) + \operatorname{div}(\mathbf{b}u) + \sigma u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_{\mathrm{D}} \\ \mu\nabla u \cdot \mathbf{n} = g & \text{on } \Gamma_{\mathrm{N}} \end{cases} \tag{1}$$

where $\overline{\partial\Omega} = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}}$, $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$ (with Γ_{D} of positive measure) and μ, $\mathbf{b}$, σ, f and g are suitably given functions.

   (a) Introduce its Galerkin approximation (G).

   (b) Recall its stability and convergence properties.

   (c) Discuss existence, uniqueness, stability and convergence of (G) depending upon the assumptions of the problem data μ, $\mathbf{b}$, σ, f and g.

2. In the square $\Omega = (-1, 1) \times (-1, 1)$ we consider the following problem:

$$\begin{cases} -\Delta u + 2u = (20\pi^2 + 1)\sin(2\pi x)\sin(4\pi y) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_{\mathrm{D}} \\ \mu\nabla u \cdot \mathbf{n} = 2\pi\sin(4\pi y) & \text{on } \Gamma_{\mathrm{N}} \end{cases} \tag{2}$$

where $\mathbf{n}$ is the outward-pointing unit normal to $\partial\Omega$, $\Gamma_{\mathrm{N}} = \{(x, y) \in \mathbb{R}^2 \colon x = 1, -1 \le y \le 1\}$ and Γ_{D} is the remaining part of the boundary.

   (a) Write the $SEM - NI$ formulation for problem (2).

   (b) Knowing that the exact solution to problem (2) is given by $u_{ex}(x, y) = \sin(2\pi x)\sin(4\pi y)$, use the provided MATLAB® code (`CHQZ_lib_1.0`) to compute the $L^2(\Omega)$ and $H^1(\Omega)$ errors as a function of the mesh size h and the polynomial degree N. First, fix $N = 2$ and let $h = 0.5, 0.25, 0.125, 0.0625$. Then, fix $h = 0.5$ and let $N = 2, 4, 6, 8, 10$. Gather these results in a table and provide a comment on them, moving from the theoretical knowledge.

   (c) Let now N_x and N_y be the polynomial approximation degree with respect to the x and y coordinates. Fix $h = 0.25$ and compute the $L^2(\Omega)$ errors by varying N_x and N_y, respectively. In particular consider the cases:

      - $N_x = 2$ and $N_y = 2, 4, 6, 8, 10$;

- $N_x = 2, 4, 6, 8, 10$ and $N_y = 2$.

Gather these results in a table and provide a comment on them

3. Consider the advection-diffusion problem

$$\begin{cases} -\operatorname{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

   (a) Introduce its SUPG (Streamline upwind Petrov-Galerkin) approximation.

   (b) Recall its stability and convergence properties.

   (c) Compare the error estimates obtained for GLS method with that of the classical Galerkin method and comment on their differences.

4. Fix $T = 1$ and consider the parabolic equation in $\Omega = (0, 1)^2$ defined by:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{in } \Omega \times (0, \mathrm{T}), \\ u = u_0 & \text{in } \Omega \times \{0\}, \\ u = 0 & \text{on } \partial\Omega \times (0, \mathrm{T}). \end{cases} \tag{4}$$

   (a) Write the semi-discrete Galerkin finite element formulation of (4).

   (b) Introduce the θ-method for the time integration of the problem obtained at the previous point and discuss the numerical stability of the scheme. In particular, prove the stability properties of the backward Euler discretization (assume for simplicity $f = 0$).

   (c) Let $u_{ex}(x, y, t) = 10x(x-1)y(y-1)e^{-t}$ be the exact solution to (4). Based on the provided MATLAB® code (`CG_FEM_PAR`), use the Crank-Nicolson method ($\theta = 1/2$) coupled with linear finite elements to compute the numerical solution of (4). Fix $\Delta t = 0.01$ and, for four levels of grid refinements (1, 2, 3, 4), compute the L^2-norm $||u_{ex}(T) - u_h(T)||_{L^2(\Omega)}$ and the H^1-norm $||u_{ex}(T) - u_h(T)||_{H^1(\Omega)}$. Report the results in a table and comment on them.

   (d) Repeat the previous point by choosing $\theta = 1/4$. Compute numerically the CFL condition

$$\Delta t \leq \frac{2}{(1 - 2\theta)|\lambda_{\max}|}, \tag{5}$$

   where $|\lambda_{\max}| = \max_i |\lambda_i|$. Here, λ_i, $i = 1, 2, \ldots,$, are the eigenvalues of the generalized eigenvalue problem

$$A\mathbf{x} = \lambda M \mathbf{x},$$

   where A and M are the stiffness and mass matrices obtained through space discretization with finite elements, respectively. Report the results in a table and comment on them.
   Hint: use the MATLAB® function `lambda = eigs(A,M,1,'largestreal')`.

5. Consider the time dependent Stokes problem:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \mathbf{x} \in \Omega \subset \mathbb{R}^d, t > 0 \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0} & \mathbf{x} \in \partial\Omega, t > 0 \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega, t = 0 \end{cases} \tag{6}$$

with $d = 2, 3$, for $\nu \in \mathbb{R}$, $\nu > 0$ and $\mathbf{f}$, $\mathbf{u}_0$ being two given functions with convenient regularity.

(a) Derive the weak formulation of (6).

(b) Derive the divergence-free formulation of (6).

(c) Motivate why the divergence-free formulation is unsuitable for numerical approximation.

6. Consider the time dependent Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \mathbf{x} \in \Omega \subset \mathbb{R}^d, t > 0 \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0} & \mathbf{x} \in \partial\Omega, t > 0 \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \mathbf{x} \in \Omega, t = 0 \end{cases} \tag{7}$$

with $d = 2, 3$, for $\nu \in \mathbb{R}$, $\nu > 0$ and $\mathbf{f}$, $\mathbf{u}_0$ being two given functions with suitable regularity.

(a) Derive the weak formulation of (7).

(b) Introduce the Chorin-Temam operator splitting method.

(c) Discuss its algorithmic aspects.

(d) Illustrate its stability and convergence properties.