NUMERICAL ANALYSIS FOR PARTIAL DIFFERENTIAL EQUATIONS
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LAB 2
WEAK FORMULATION AND FINITE ELEMENT APPROXIMATION OF ELLIPTIC PROBLEMS

EXERCISE 1

Let $\Omega \subset \mathbb{R}^2$ be an open set having a regular enough boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\overset{\circ}{\Gamma}_D \cap \overset{\circ}{\Gamma}_N = \emptyset$. Let us consider the following advection-diffusion problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot (\mu \nabla u) + \mathbf{b} \cdot \nabla u = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma_D, \\ \mu \nabla u \cdot \mathbf{n} = g_N & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $\mathbf{b} : \Omega \to \mathbb{R}^2$ and $\mu : \Omega \to \mathbb{R}$ are two continuous functions, $f : \Omega \to \mathbb{R}$ belongs to $L^2(\Omega)$ and $g_D : \Gamma_D \to \mathbb{R}$, $g_N : \Gamma_N \to \mathbb{R}$ are regular enough functions. Moreover, let us suppose

$$0 < \mu_0 \leq \mu(\mathbf{x}) \leq \mu_1, \quad |\mathbf{b}(\mathbf{x})| \leq b_1, \quad \nabla \cdot \mathbf{b} = 0, \quad \text{a.e. in } \Omega.$$

1. Write the *weak formulation* of problem (1) in the general form

$$\text{find} \quad u \in V \quad \text{such that} \quad a(u, v) = F(v), \quad \forall v \in V. \tag{2}$$

   In particular, define properly the functional space V and the forms $a(\cdot, \cdot)$ and $F(\cdot)$.

2. Define *conditions* on the coefficients under which the solution of problem (2) is unique.

3. Let V_h be a suitable finite dimensional subspace of V. Write the *Galerkin formulation* of problem (2).

4. Analyse the *existence and uniqueness*, *stability* and *convergence* of the solution.

5. Let now V_h, defined at point 3, be the space of *linear finite elements*. Show that the Galerkin formulation is equivalent to the solution of the linear system $A\mathbf{u} = \mathbf{f}$ with dimension n. Define precisely the value of n and give an explicit representation of the matrix A and the vectors $\mathbf{u}$ and $\mathbf{f}$.