NUMERICAL ANALYSIS FOR PARTIAL DIFFERENTIAL EQUATIONS
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Lab 3
Galerkin-finite element method for elliptic problems

Exercise 1

In the unit square $\Omega \equiv (0,1) \times (0,1)$ we consider the following problem:

$$\begin{cases} -\Delta u = 8\pi^2 \sin(2\pi x)\sin(2\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

1. Give the weak formulation and prove the well posedness of the problem.
2. Implement in Matlab a linear finite element solver for problem (1), focusing in particular on the assembly of the linear system and of the load vector stemming after the Galerkin finite element approximation.
3. Compute the $H^1(\Omega)$ and $L^2(\Omega)$ errors as function of the mesh grid h, knowing that the exact solution to problem (1) is $u_{ex} = \sin(2\pi x)\sin(2\pi y)$. Gather these results in a table and provide a comment on them, moving from the theoretical knowledge.

Exercise 2

In the unit square $\Omega \equiv (0,1) \times (0,1)$ we consider the following problem:

$$\begin{cases} -\Delta u + 2u = 0, & \text{in } \Omega, \\ u = g \equiv e^{x+y}, & \text{on } \partial\Omega. \end{cases}$$

1. Give the weak formulation and prove the well posedness of the problem. Find the analytical solution.
2. Give the linear finite element approximation using Matlab on uniform grids.
3. Compute the $H^1(\Omega)$ and $L^2(\Omega)$ errors as function of the mesh grid h. Gather these results in a table and provide a comment on them, moving from the theoretical knowledge.

Exercise 3

A simple model used in oceanography is due to Stommel[1]. In this model ocean is assumed to be flat and with uniform depth H, and no vertical movement of the water free boundary it is considered (since it is small compared with the horizontal one). Only Coriolis force, wind action

[1] Stommel H. (1948) The westward intensification of wind-driven ocean currents. *Trans. Amer. Geophys. Union* 29(202).

on the surface and friction at the bottom are included. Incompressibility implies that there exists a function ψ, called *stream function*, related to the velocity components by the equations

$$u = -\frac{\partial \psi}{\partial y}, \; v = \frac{\partial \psi}{\partial x}.$$

In Stommel model for a rectangular ocean $\Omega = (0, L_x) \times (0, L_y)$, ψ is solution to the following elliptic problem

$$\begin{cases} -\Delta\psi - \alpha\dfrac{\partial \psi}{\partial x} = \gamma \sin(\pi y / L_y), & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

with $\alpha = \dfrac{H\beta}{R}$, $\gamma = \dfrac{W\pi}{RL_y}$, being R the friction coefficient on the bottom, W a coefficient associated with the surface wind, $\beta = df/dy$ where f is the Coriolis parameter, which is in general function only of y (latitude).

1. Give the weak formulation of (2).
2. Compute the analytical solution of the model (2) for a constant β.
3. Solve the problem numerically with linear finite elements, using the Stommel's parameters: $L_x = 10^5\ m$, $L_y = 2\pi 10^4\ m$, $H = 200\ m$, $W = 0.3\ 10^{-7} m^2 s^{-2}$, $R = 0.6\ 10^{-3} m s^{-1}$. Assume at first $\beta = 0$ and then $\beta = 5\ 10^{-10} m^{-1} s^1$. Comment the results.