Pricing via Quasi Monte Carlo Methods, Variance Reduction Techniques, and Hedging for Asian Options in the Black-Scholes World, with an Application on Historical Market Data

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Erdős Institute
Quantitative Finance Boot Camp Project

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Outline

- **Part I:** Convergence analysis of Quasi Monte Carlo (QMC) methods for multi-look option pricing.
- **Part II:** Variance reduction estimators.
- **Part III:** Hedging simulation for Asian call options.
- **Part IV:** Hedging strategy on historical market data with rolling volatilities.

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Monte Carlo vs quasi Monte Carlo methods*

- **MC**: Construct GBM path obtaining normal d-dimensional n samples $Z = \Phi^{-1}(U)$, for $U \sim \text{Unif}[0,1]^d$ (i.e. independent $\{Z_i\}_{i=1}^d$) via the inverse CDF method.
- **QMC**: Does not sample U but constructs algorithmically (Sobol low discrepancy) sequence of n points in $[0,1]^d$ that covers the space as evenly as possible.
- The QMC estimator can made unbiased and random by avaraging several QMC estimators obtained using R "scrambled" (i.e., randomly perturbed) replicates of the original Sobol sequence via Owen scrambling or digital shifts.

Caflisch R. E., Monte Carlo and quasi–Monte Carlo Methods, 1998. Figures from Wikipedia.

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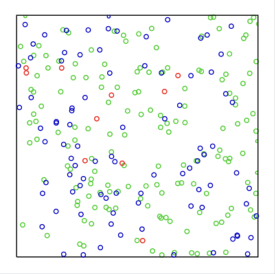

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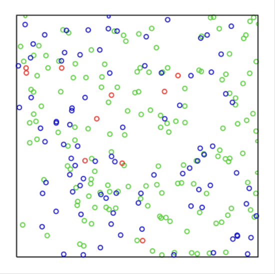

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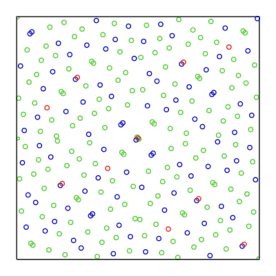

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Dimension reduction for Brownian path construction*

- **Standard incremental construction**: $W_{t_{i+1}} - W_{t_i} = \sqrt{t_{i+1} - t_i} Z_i$ and $S_{t_{i+1}} = S_{t_i} \exp((r - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma(W_{t_{i+1}} - W_{t_i}))$.
- In high dimensions, the coverage of Sobol sequences deteriorates. It is then crucial to utilize the first coordinates (covering more uniformly their ambient subspace) of each Sobol point to generate most of the variance of the final constructed path.
- **Spectral/PCA**: $W = AZ$, where $AA^\top = \Sigma = P \Lambda P^\top$, with $\Sigma_{i,j} = Cov(W_{t_i}, W_{T_j})$, and $A = P\Lambda^{1/2}$.
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Arithmetic vs Geometric Asian call options*

- **Arithmetic**: $\max\{A_T - K, 0\}$ where $A_T = \frac{1}{d}\sum_{i=1}^{d} S_{t_i}$.

- **Geometric**: $\max\{G_T - K, 0\}$ where $G_T = (\prod_{i=1}^{d} S_{t_i})^{1/d}$, which can be priced in closed form:

$$e^{-rT}\mathbb{E}[\max\{G_T - K, 0\}] = e^{-rT}\left[S_0 e^{\mu_d T}\Phi(d_1) - K\Phi(d_2)\right],$$

with

$$\sigma_d = \frac{\sigma^2(d+1)(2d+1)}{6d^2}, \qquad \mu_d = (r - \frac{1}{2}\sigma^2)\frac{d+1}{2d} + \frac{1}{2}\sigma_d^2,$$

$$d_1 = \frac{\ln(S_0/K) + (\mu_d + (1/2)\sigma_d^2)T}{\sigma_d\sqrt{T}}, \qquad d_2 = d_1 - \sigma_d\sqrt{T}.$$

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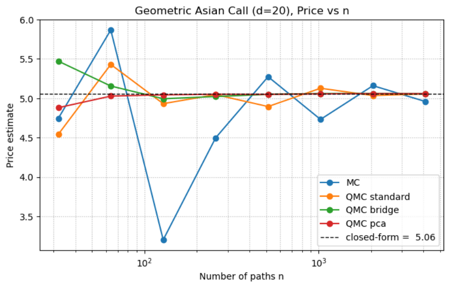
Convergence and computational times of the proposed methods



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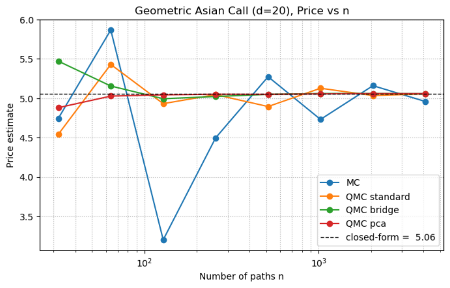

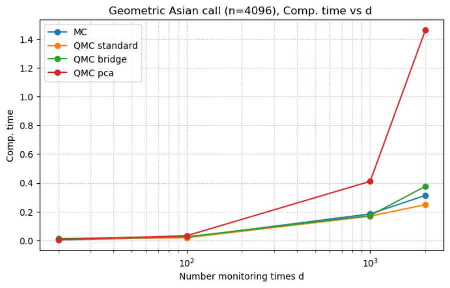

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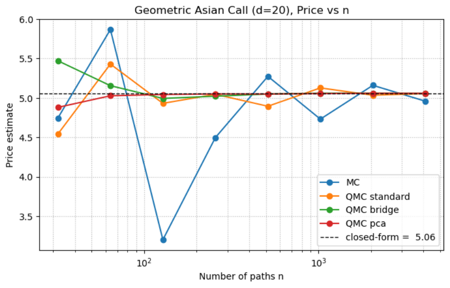

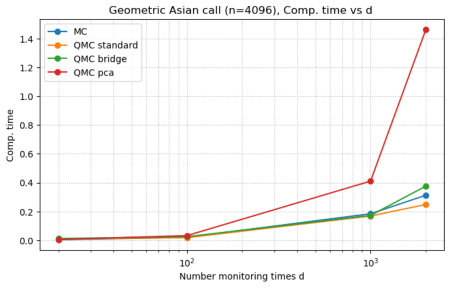

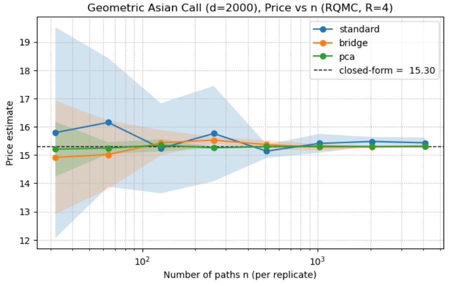

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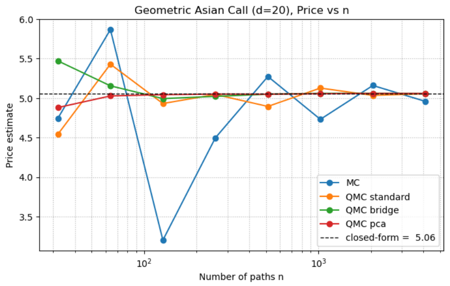

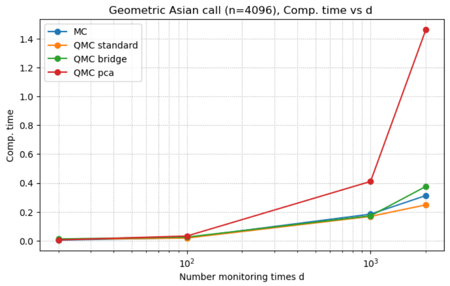

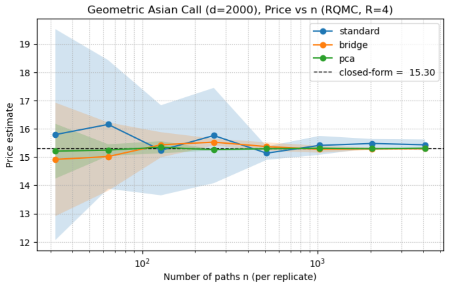

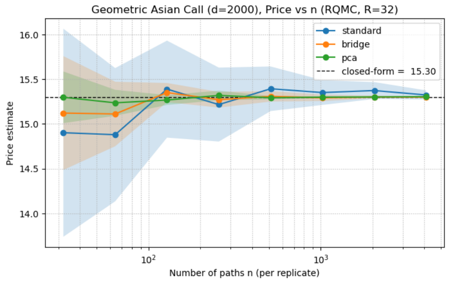

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Antithetic and geometric control variates*

- Let the arithmetic Asian option price estimate be

$$\widehat{f}^{\text{Ari}}(S) = \frac{1}{n}\sum_{i=1}^{n} f^{\text{Ari}}(S^i), \qquad S = [S^1, ..., S^n], \qquad S^i = [S^i_{t_1}, ..., S^i_{t_d}]\,,$$

computed evaluating the payoffs $f^{\text{Ari}}(S^i)$ for each simulated GBM path S^i.

- **Antithetic**: Given corresponding GBM paths $S^{ant} = [S^{\text{ant},1}, ..., S^{\text{ant},n}]$ generated by flipping the signs of the normal samples used to generate the original GBM paths, the antithetic estimator is

$$\frac{1}{2}(\widehat{f}^{\text{Ari}}(S) + \widehat{f}^{\text{Ari}}(S^{\text{ant}}))\,.$$

- **Geometric control**: Introducing the corresponding geometric Asian call option payoffs and empirical estimators $f^{\text{Geom.}}$ and $\widehat{f}^{\text{Geom}}$, the gemetric control variate estimator is

$$\widehat{f}^{\text{Arit}}(S) + b^*(\mathbb{E}[\widehat{f}^{Geom}(S)] - \widehat{f}^{Geom}(S))\,,$$

where $b^* = \dfrac{\sum_{i=1}^{n}(f^{\text{Ari}}(S^i) - \widehat{f}^{\text{Ari}}(S))(f^{\text{Geo}}(S^i) - \widehat{f}^{\text{Geo}})}{\sum_{i=1}^{n}(f^{\text{Geo}}(S^i) - \widehat{f}^{\text{Geo}}(S))^2}$.

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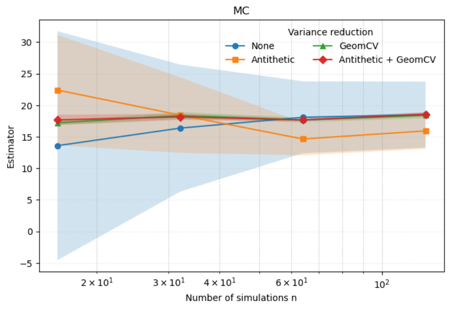
Variance reduction results



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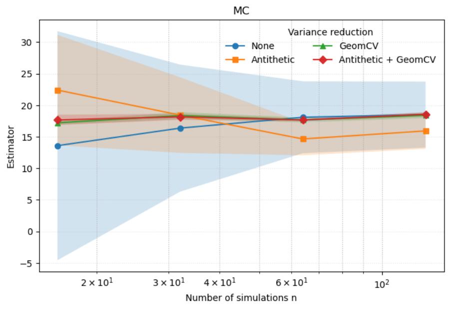

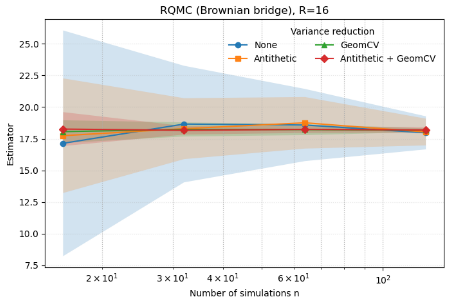

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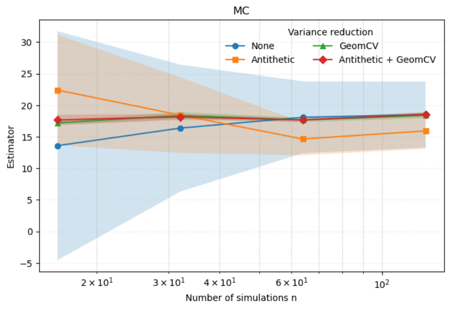

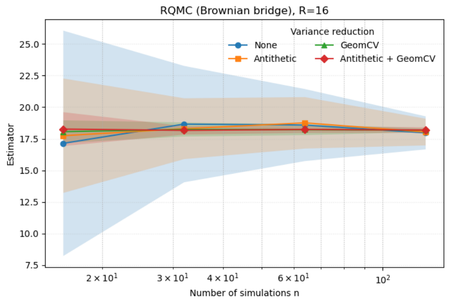

Method	Variance-Reduction	Replicates R	Mean	Std	b* (if GeomCV and R=1)	mean b* (if GeomCV and R>1)
MC	Antithetic	1	19.347618	1.459678	NaN	NaN
MC	Antithetic + GeomCV	1	18.390132	0.198007	1.116291	NaN
MC	GeomCV	1	18.211661	0.220735	1.107807	NaN
MC	None	1	14.714135	2.158316	NaN	NaN
RQMC (bridge)	Antithetic	16	17.997223	0.635903	NaN	NaN
RQMC (bridge)	Antithetic + GeomCV	16	18.211292	0.127033	NaN	1.131762
RQMC (bridge)	GeomCV	16	18.203494	0.160889	NaN	1.130927
RQMC (bridge)	None	16	18.313814	0.928195	NaN	NaN

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Hedging strategy for Asian options

- We simulate and compare the profits (from the seller's perspective) of an unhedged and Δ-hedged strategy for both geometric and arithmetic Asian options:

$$P_{\text{Unhedged}} = e^{-rT}\left(V_{t_0} - \max\{A_T - K, 0\}\right),$$

$$P_{\text{Hedged}} = e^{-rT}\left(V_{t_0} - \max\{A_T - K, 0\}\right) + \sum_{i=1}^{d}(e^{-rt_i}S_{t_i} - e^{-rt_{i-1}}S_{t_{i-1}})\Delta_{t_{i-1}}.$$

- V_{t_0} is the fair value (not discounted) price of the call option. In geometric case, evaluated in closed form. In arithmetic case, estimated with RQMC.
- $V_{t_i} = \mathbb{E}[\max\{G_T - K, 0\} \mid S_{t_0}, \ldots, S_{t_{i-1}}] = \widetilde{S}_{t_i} e^{\mu_i \tau}\Phi(d_{1,i}) - K\Phi(d_{2,i})$, where, replacing $\tau = T - t, \quad j = d - i, \quad \widetilde{S}_t = G_{t_i}^{i/d} S_{t_i}^{j/d}$, we have:

$$\sigma_i = \left(\frac{j}{d}\right)\frac{\sigma^2(j+1)(2j+1)}{6j^2}, \qquad \mu_i = \left(\frac{j}{d}\right)\left((r - \frac{1}{2}\sigma^2)\frac{j+1}{2j} + \frac{1}{2}\sigma_i^2\right),$$

$$d_{1,i} = \frac{\ln(\widetilde{S}_t/K) + (\mu_i + (1/2)\widetilde{\sigma}_i^2)\tau}{\sigma_i\sqrt{\tau}}, \qquad d_{2,i} = d_{1,i} - \sigma_i\sqrt{\tau}.$$

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Hedging strategy for Asian options

- We simulate and compare the profits (from the seller's perspective) of an unhedged and Δ-hedged strategy for both geometric and arithmetic Asian options:

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- V_{t_0} is the fair value (not discounted) price of the call option. In geometric case, evaluated in closed form. In arithmetic case, estimated with RQMC.
- $V_{t_i} = \mathbb{E}[\max\{G_T - K, 0\} \mid S_{t_0}, \dots, S_{t_{i-1}}] = \widetilde{S}_{t_i} e^{\mu_i \tau}\Phi(d_{1,i}) - K\Phi(d_{2,i})$, where, replacing $\tau = T - t, \quad j = d - i, \quad \widetilde{S}_t = G_{t_i}^{i/d} S_{t_i}^{j/d}$, we have:

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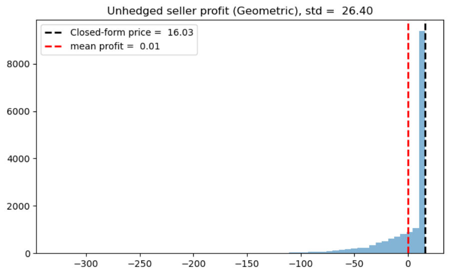
Hedging simulation under Black-Scholes model



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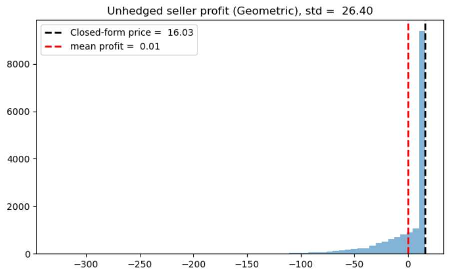

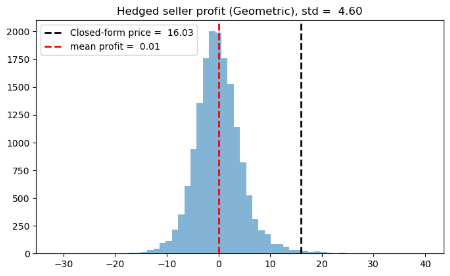

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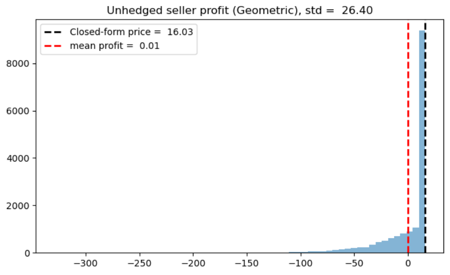

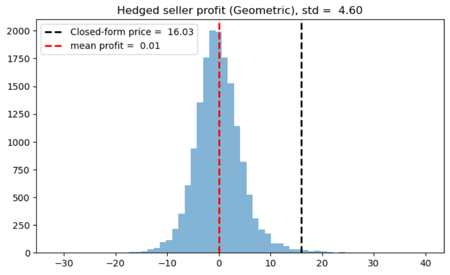

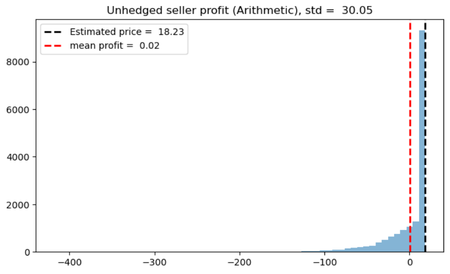

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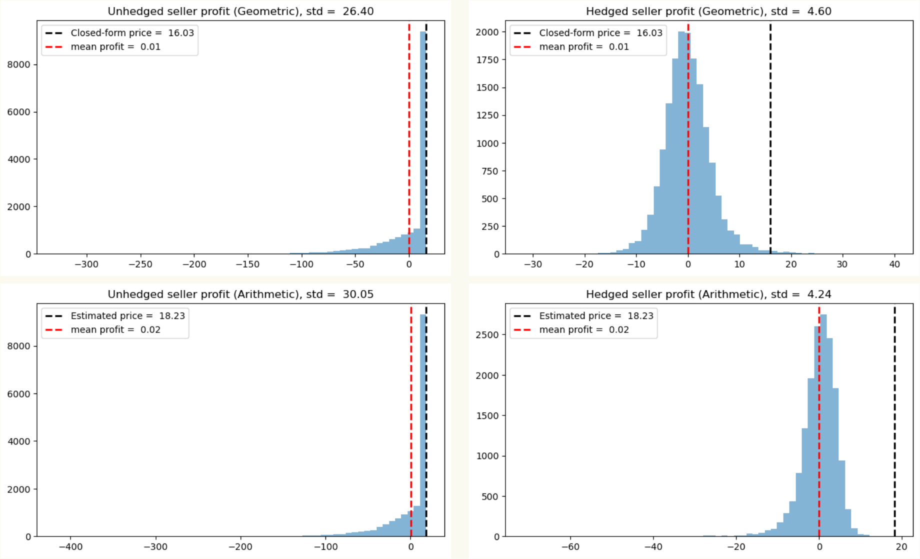

Outline

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Naive implementation of hedging strategy on real world data

- As done in class for European call options, investigated a naive application of the proposed hedging strategy for Asian options on historical market data using `yfinance` library.
- Considered period from 10/01/2023 to 10/01/2025.
- Used rolling volatilities to price options and compute Δ_{t_i}.
- For simplicity, kept interest rate $r = 0$.
- Real world data does not satisfy the assumptions of the Black-Scholes model but nevertheless the hedging strategy had an impact, avoiding big losses encountered with unhedged strategy.

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Results on historical data

Ticker	Estimated price (Arithmetic)	Unhedged profit (Arithmetic)	Hedged profit (Arithmetic)	Closed form price (Geometric)	Unhedged profit (Geometric)	Hedged profit (Geometric)
AAPL	8.676931	-32.588871	-4.810201	8.319129	-31.564316	-3.785646
TSLA	32.633448	-37.936391	-17.242443	28.112894	-31.353564	-10.659616
F	1.322963	-0.535866	0.521312	1.149855	-0.664168	0.393010
HD	16.249109	-74.715740	-2.464222	15.462736	-74.003440	-1.751921
GM	2.871196	-15.740514	-0.196709	2.624381	-15.328092	0.215714
SPY	18.754957	-125.648244	1.779597	18.063357	-123.087216	4.340625
^GSPC	194.576958	-1213.184831	23.110127	187.339490	-1191.153822	45.141135
VTI	9.629569	-61.713608	0.857163	9.259993	-60.521956	2.048815
QQQ	21.247848	-113.805108	2.347129	20.195554	-111.448860	4.703377
BA	14.975396	11.977883	-1.591923	13.995976	12.994070	-0.575736
UPS	11.989067	11.989067	-1.775236	11.056994	11.056994	-2.707309
UNP	12.673020	-18.135457	3.190448	12.018249	-18.561934	2.763971
WMT	2.455050	-22.825367	-2.738424	2.364278	-20.787103	-0.700160
COST	28.775709	-296.702087	-13.342789	27.526068	-286.888271	-3.528973
PG	7.344254	-7.884267	0.005513	7.040270	-7.969990	-0.080209
DIS	6.027461	-17.237755	-0.798699	5.659073	-16.984909	-0.545853
NKE	7.800542	7.800542	-2.688523	7.299502	7.299502	-3.189563
KO	4.388389	-5.968108	2.345506	4.092650	-6.029104	2.284511
IWM	9.848038	-38.195085	-1.628224	9.360633	-37.946959	-1.380098
DIA	12.350532	-73.286780	1.818982	11.969517	-72.357240	2.748522
XLK	9.041166	-50.877238	-0.597258	8.634125	-49.830748	0.449232
XLV	4.724687	-14.226620	0.763160	4.576647	-14.203918	0.785862
EFA	2.825587	-13.163118	0.183566	2.724056	-12.962369	0.384315
EEM	1.661689	-6.375586	-0.128753	1.598110	-6.246484	0.000348
HYG	1.407422	-8.268700	-0.067357	1.382972	-8.194377	0.006967