High-frequency data and limit order books

Lecture 6 - April 1st, 2025

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Table of contents

Table of contents

Market impact I

Market impact can be described as the response of the market to a given order flow.

- First aspect of market impact is purely mechanical.
- Market impact of a given order is affected by existing correlations in order flows: liquidity taking/momentum trading, liquidity providing/mean-reversion, etc.
- Market participants split large investment decisions into multiple child orders sent successively to the exchange. The total (parent) order is called a metaorder.
- Market impact can be described in terms of quantities (function of the size of the order/metaorder) and in terms of time (instantaneous, temporary, permanent, transient, etc.).

Market impact II

- Market impact can be analyzed via response functions [5]. If p_n is the mid-price just before the n-th trade et $\epsilon_n \in \{-1, 1\}$ the sign of the trade, the response function w.r.t time is

$$\mathcal{R}(\ell) = \langle (p_{n+\ell} - p_n)\epsilon_n \rangle.$$

  This can be measured on the market but should not be interpreted as the impact of a single order as it is affected by all correlated subsequent orders.
- Propagator models define the dynamics of the (mid-)price as a sum of all the impact of past trades:

$$p_n = \sum_{m<n} g(n-m)\epsilon_m + \sum_{m<n} \eta_m.$$

  η_m's are exogenous noise terms and g is the propagator (isolated impact of a single trade). Correlations of order flows are thus crucial to response functions (see also Lab below).

Market impact III

- If V_n is the size of the n-th trade, a volume-dependent response function for the trades is:

$$\mathcal{R}(\ell, V) = \langle (p_{n+\ell} - p_n)\epsilon_n | V_n = V \rangle.$$

- Empirical studies on trades suggest that a variable separation is valid:

$$\mathcal{R}(\ell, V) = \mathcal{R}(\ell)\, f(V),$$

with $f(V)$ being strongly concave (suggested forms for $f(V)$: $V^{-\alpha}$ with $\alpha \approx 0.2 - 0.4$, $\log V$, etc.)

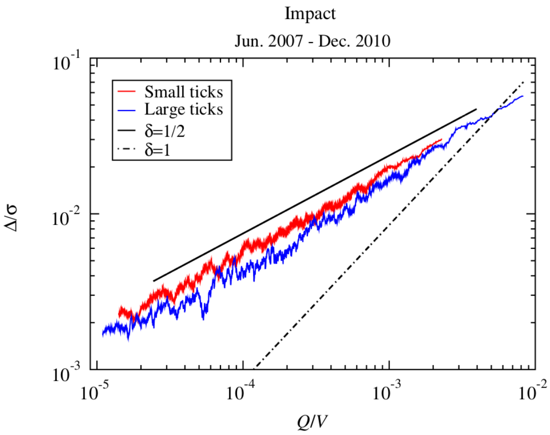
The square-root law I

- Let $I(Q)$ be the average price variation between the first and last child orders of a metaorder of size Q. A widely-used transaction cost model assumes a square-root law for the price impact:

$$I(Q) = \alpha\sigma\sqrt{\frac{Q}{V}},$$

  where α is a constant, σ is the daily volatility of the asset and V is the (average) daily traded volume.
- Many variants of the square-root law are implemented in trading firms/software.
- Empirical verification on proprietary datasets only.
- Rate of trading and optimal execution ?

The square-root law II



Source : [10]

- *(Left)* Empirical market impact of metaorders in CFM (2007-2010, proprietary data).
- Observations valid on equities, futures, cryptocurrencies. Universal ? [9]
- Modeling challenges: trading models compatible with a square-root (or power-law with an exponent smaller than one) impact ? arbitrage in these trading models ?
- Further reading: [4, 7, 3]

Table of contents

Optimal execution as a mean-variance problem I

- Framework proposed in [2] (known as the Almgren-Chriss model).
- At time 0 a trader holds an initial inventory of Q_0 shares and needs to liquidate it by a (finite) time horizon T.
- N trading periods of length $\tau = T/N$. Set $t_n = n\tau$, $n = 0, \ldots, N$.
- At the start of the trading period $[t_n, t_{n+1})$, the trader decides the number of shares q_{n+1} to be bought (> 0) or sold (< 0) in the coming period.
- The trading rate is

$$\nu_{n+1} = \frac{q_{n+1}}{\tau}.$$

- The set of admissible (deterministic) execution strategies is therefore

$$\Theta = \left\{ (\nu_1, ..., \nu_N) \in \mathbb{R}^N : \sum_{n=0}^{N-1} \nu_{n+1}\tau = -Q_0 \right\}.$$

Optimal execution as a mean-variance problem II

- Let Q_n be the trader's inventory at time t_n. We have

$$Q_{n+1} = Q_n + \nu_{n+1}\tau.$$

- In the absence of action by the trader, the stock (mid-)price would be a random walk (fundamental value). The model assumes a permanent linear impact of the trader's actions. The (mid-)price dynamics is thus

$$S_{n+1} = S_n + \sigma\sqrt{\tau}Z_{n+1} + \gamma\nu_{n+1}\tau,$$

  for $\sigma > 0$, $\gamma > 0$ and $(Z_n)_n$ a sequence of i.i.d. random variables with standard Gaussian distribution.

Optimal execution as a mean-variance problem III

- There is no model of the limit order book. At each trading period, the trader will face execution costs (spread, walking the LOB and/or temporary impact). The average execution price per share on the trading period $[t_n, t_{n+1})$:

$$\tilde{S}_{n+1} = S_n + \eta \nu_{n+1}$$

for some $\eta > 0$.

- The dynamics of the cash position of the trader is therefore

$$X_{n+1} = X_n - \nu_{n+1}\tau \tilde{S}_{n+1} = X_n - \nu_{n+1} S_n \tau - \eta \nu_{n+1}^2 \tau.$$

- We assume that the trader seeks to maximize the expected final wealth while minimizing risk, i.e. has the objective function

$$\mathbb{E}[X_N] - \lambda \mathbb{V}[X_N].$$

for some risk-aversion parameter $\lambda > 0$.

Optimal execution as a mean-variance problem IV

- Direct computations for the terminal wealth of the trader yields:

$$X_N = X_0 + Q_0 S_0 - \frac{\gamma}{2} Q_0 + \sigma\sqrt{\tau} \sum_{i=0}^{N-1} Q_{n+1} Z_{n+1} - \sum_{i=0}^{N-1} \nu_{n+1}^2 \overbrace{\left(\eta - \frac{\gamma\tau}{2}\right)}^{\eta'} \tau.$$

- The trader thus faces the optimization problem

$$\min_{\nu\in\Theta} \left\{ \sum_{i=0}^{N-1} \nu_{n+1}^2 \eta' \tau + \lambda\sigma^2\tau \sum_{i=0}^{N-1} Q_{n+1}^2 \right\},$$

which is rewritten w.r.t. the variables Q_n as:

$$\min_{Q\in\Theta_Q} \left\{ \frac{\eta'}{\tau} \sum_{i=0}^{N-1} (Q_{n+1} - Q_n)^2 + \lambda\sigma^2\tau \sum_{i=0}^{N-1} Q_{n+1}^2 \right\}$$

Optimal execution as a mean-variance problem V

over the set of admissible inventory curves

$$\Theta_Q = \left\{ (Q_1, ..., Q_{N-1}) \in \mathbb{R}^{N-1} \right\}.$$

(Q_0 given and $Q_N = 0$).

- If $\eta' = \eta - \frac{\gamma\tau}{2} > 0$, the objective is convex and the optimal solution can be explicitly computed

$$Q_n^* = Q_0 \frac{\sinh(\kappa(T - t_n))}{\sinh(\kappa T)}$$

where κ satisfies:

$$\frac{2}{\tau^2}(\cosh(\kappa\tau) - 1) = \frac{\lambda\sigma^2}{\eta'} = \frac{\lambda\sigma^2}{\eta - \frac{\gamma\tau}{2}}.$$

Optimal execution as a mean-variance problem VI

- Typical optimal trading curve in the Almgren-Chriss framework

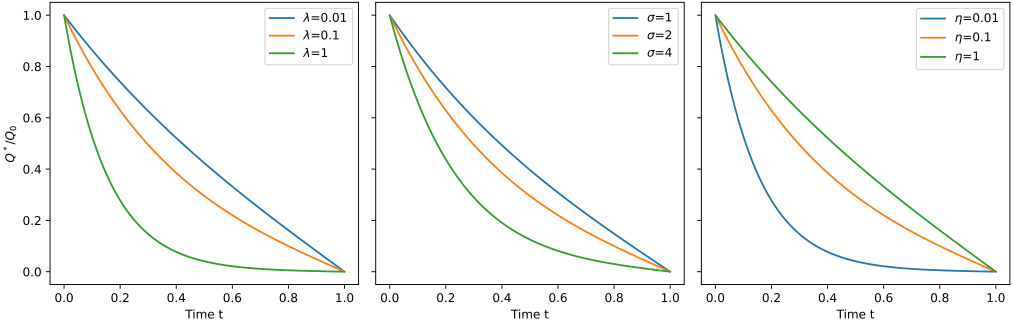


Table of contents

Optimal execution in a block-shaped LOB I

- Proposed by Obizhaeva and Wang in [8] (originally released as a preprint in 2005), extended in [1] to more general shapes.
- A trader needs to buy a total quantity X_0 before a (finite) time horizon T and may trade at discrete times $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$.
- Set of possible strategies: $\Theta = \left\{ x = (x_{t_0}, \ldots, x_{t_N}) \in [0, +\infty)^{N+1} : \sum_{i=0}^{N} x_{t_i} = X_0 \right\}$.
- The model assumes continuous prices (a_t ask price at time t) and a block-shaped LOB with available size q at all prices. On the ask side, the LOB is represented at time t by

$$q_t(p) = q\mathbf{1}_{[a_t, +\infty)}(p)$$

- There exists a "fundamental" mid-price f_t (a Brownian motion). *In the absence of trades*, the LOB moves with f_t at all times with a constant spread s.

Optimal execution in a block-shaped LOB II

- At time $t_0 = 0$, LOB is defined with ask price $a_{t_0} = f_{t_0} + \frac{s}{2}$ and ask LOB density $q_{t_0}(p) = q\mathbf{1}_{[a_{t_0},+\infty)}(p)$.
- If the trader trades the quantity x_{t_0} at $t_0 = 0$, then the new ask price satisfies

$$x_{t_0} = \int_{a_{t_0}}^{a_{t_0+}} q_{t_0}(p)\, dp, \quad \text{i.e.} \quad a_{t_0+} = f_0 + \frac{s}{2} + \frac{x_{t_0}}{q},$$

  and the average execution price $\overline{p_{t_0}}$ of this trade satisfies

$$\overline{p_{t_0}} x_{t_0} = \int_{a_{t_0}}^{a_{t_0+}} p q_{t_0}(p)\, dp, \quad \text{i.e.} \quad \overline{p_{t_0}} = f_{t_0} + \frac{s}{2} + \frac{x_{t_0}}{2q}.$$

Optimal execution in a block-shaped LOB III

- The model assumes a permanent linear impact on the mid-price : *in the absence of other trades*, as $t \to +\infty$, the LOB would be defined with

$$a_t - f_t \xrightarrow[t\to+\infty]{} \frac{s}{2} + \lambda x_{t_0} \quad \text{and} \quad q_t(p) = q\mathbf{1}_{[a_t,+\infty)},$$

for a given $\lambda \in \left[0, \frac{1}{q}\right]$.

- The LOB is assumed to be resilient and liquidity returns exponentially fast: *in the absence of other trades*,

$$a_t = f_t + \lambda x_{t_0} + \frac{s}{2} + x_{t_0} \overbrace{\left(\frac{1}{q} - \lambda\right)}^{\kappa:=} e^{-\rho t},$$

for some $\rho > 0$.

Optimal execution in a block-shaped LOB IV

- Now, considering all trades on $[0, t)$, the ask price at time t becomes

$$a_t = f_t + \sum_{t_i < t} \lambda x_{t_i} + \frac{s}{2} + \sum_{t_i < t} x_{t_i} \kappa e^{-\rho(t - t_i)}.$$

- The ask price after trading x_{t_n} at time t_n is $a_{t_n+} = a_{t_n} + \frac{x_{t_0}}{q}$ and the average cost of this trade is $\overline{p_{t_n}} = a_{t_n} + \frac{x_{t_n}}{2q}$. Hence, the optimal trading problem is written

$$\min_{x \in \Theta} \mathbf{E} \left[\sum_{i=0}^{N} x_{t_n} \left(f_{t_n} + \sum_{i=0}^{n-1} \lambda x_{t_i} + \frac{s}{2} + \sum_{i=0}^{n-1} x_{t_i} \kappa e^{-\rho(t_n - t_i)} + \frac{x_{t_n}}{2q} \right) \right].$$

Optimal execution in a block-shaped LOB V

- This problem can be explicitly solved.
  Sketch of the proof: using the new variables $X_{t_n} = X_0 - \sum_{i=0}^{n-1} \lambda x_{t_i}$, (remaining quantity to be traded remaining before time t_n) and $D_{t_n} = \sum_{i=0}^{n-1} x_{t_i} \kappa e^{-\rho(t-t_i)}$, ("deviation" from the impacted steady state), the problem takes a simpler form:

$$\min_{x\in\Theta} \mathbf{E}\left[\sum_{i=0}^{N} x_{t_n}\left(f_{t_n} + \lambda(X_0 - X_{t_n}) + \frac{s}{2} + D_{t_n} + \frac{x_{t_n}}{2q}\right)\right].$$

  This can be solved using a backward induction (dynamic programming principle). One finds that the optimal traded quantity x_{t_n} at time t_n is linear in X_{t_n} and D_{t_n}. For explicit formulas, see [8, Proposition 1].

Optimal execution in a block-shaped LOB VI

- Typical shape of the optimal trading strategy in the Obizhaeva & Wang framework:

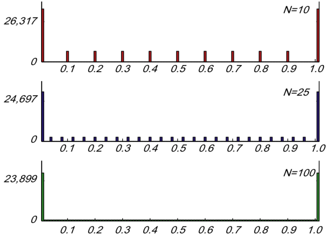


Fig. 2. Optimal strategy with N fixed discrete trading intervals. This figure plots the optimal trades for N fixed intervals, where N is 10, 25, and 100 for respectively the top, middle and bottom panels. The initial order to trade is set at $X_0 = 100{,}000$ units, the time horizon is set at $T = 1$ day, the market depth is set at $q = 5{,}000$ units, the price-impact coefficient is set at $\lambda = 1/(2q) = 10^{-4}$, and the resilience coefficient is set at $\rho = 2.231$.

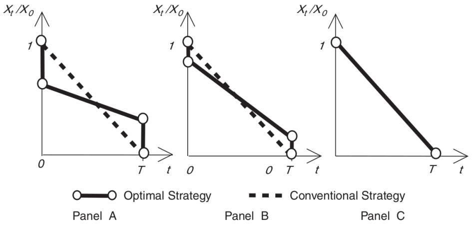


Fig. 4. Optimal strategy versus simple strategy from the conventional models. The figure plots the time paths of remaining order to be executed for the optimal strategy (solid line) and the simple strategy obtained from the conventional models (dashed line), respectively. The order size is set at $X_0 = 100{,}000$, the initial ask price is set at \$100, the market depth is set at $q = 5{,}000$ units, the (permanent) price-impact coefficient is set at $\lambda = 1/(2q) = 10^{-4}$, and the trading horizon is set at $T = 1$ day, which is assumed to be 6.5 hours (390 minutes). Panels A, B, and C plot the strategies for $\rho = 0.001$, 2 and 1,000, respectively.

Source: [8]

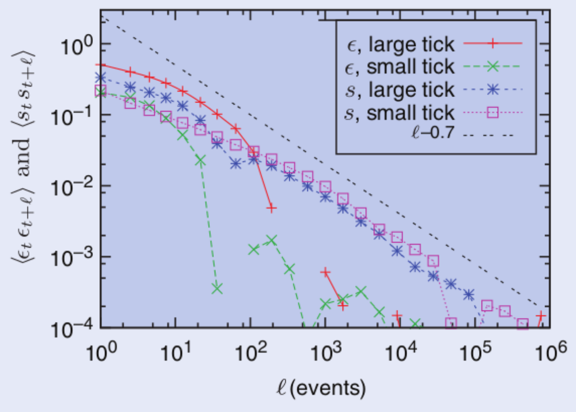
Table of contents

Order typology

- [6] proposes several propagator models to analyze impact of LOB events.
- Orders are classified by type (limit, market, cancel) and aggressiveness (the order moves the price, or does not move the price).
- Any order occurring at time t (*here in event time*) has a type π_t, a sign ϵ_t, a side s_t and a gap Δ_t:

π	Event definition	Event sign definition	Gap definition ($\Delta_{\pi,\epsilon}$)
$\pi = \mathrm{MO}^0$	Market order, volume < outstanding volume at the best	$\epsilon = \pm 1$ for buy/sell market orders	0
$\pi = \mathrm{MO}'$	Market order, volume ≥ outstanding volume at the best	$\epsilon = \pm 1$ for buy/sell market orders	Half of the first gap behind the ask ($\epsilon = 1$) or bid ($\epsilon = -1$)
$\pi = \mathrm{CA}^0$	Partial cancellation of the bid/ask queue	$\epsilon = \mp 1$ for buy/sell side cancellation	0
$\pi = \mathrm{LO}^0$	Limit order at the current best bid/ask	$\epsilon = \pm 1$ for buy/sell limit orders	0
$\pi = \mathrm{CA}'$	Complete cancellation of the best bid/ask	$\epsilon = \mp 1$ for buy/sell side cancellation	Half of the first gap behind the ask ($\epsilon = 1$) or bid ($\epsilon = -1$)
$\pi = \mathrm{LO}'$	Limit order inside the spread	$\epsilon = \pm 1$ for buy/sell limit orders	Half distance of the limit order from the earlier best quote on the same side

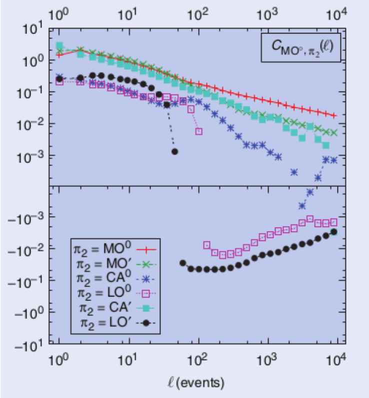
Sign and side autocorrelations



Source: [6]

- Side autocorrelations exhibit long range memory: market orders persistently hit one side, and attract liquidity on the same side.

Signed event-event autocorrelations



Source: [6]

- A signed event-event correlation function is defined as:

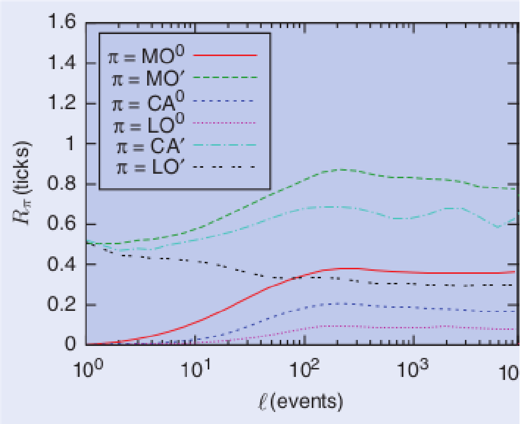
$$C_{\pi_1,\pi_2}(\ell) = \frac{\langle \mathbf{1}_{\pi_t=\pi_1}\epsilon_t\,\mathbf{1}_{\pi_{t+l}=\pi_2}\epsilon_{t+\ell}\rangle}{\mathbf{P}(\pi_1)\mathbf{P}(\pi_2)}$$

Response functions

- The response function $R_\pi(\ell)$ describes the average price movement ℓ events after the occurrence of an order of a given type π:

$$R_\pi(\ell) = \langle (p_{t+\ell} - p_t)\epsilon_t \mid \pi_t = \pi \rangle$$

- This response function is a global market response : it is the combination of a "bare" impact of the order and the correlation functions.



Source: [6]

A constant impact model I

- Exact mid-price movement:

$$p_{t+\ell} = p_t + \sum_{t \le t' < t+\ell} \epsilon_{t'} \Delta_{\pi_{t'}, \epsilon_{t'}, t'}.$$

- Approximate mid-price movement:

$$p_{t+\ell} \approx p_t + \sum_{t \le t' < t+\ell} \epsilon_{t'} \Delta^R_{\pi},$$

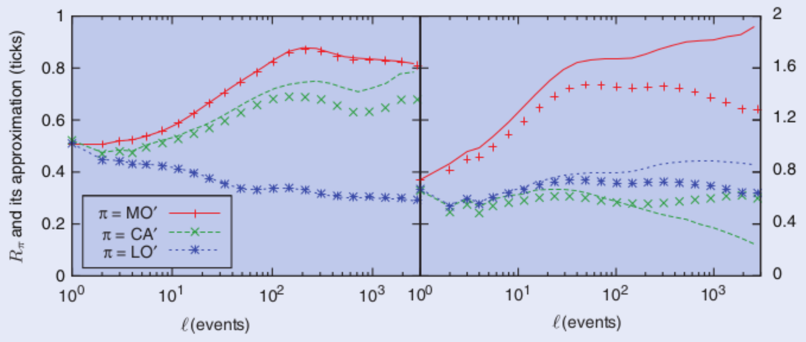
  where $\Delta^R_\pi = \langle \Delta_{\pi_{t'}, \epsilon_{t'}, t'} \mid \pi_{t'} = \pi \rangle$.

- The response function becomes:

$$R_\pi(\ell) = \langle (p_{t+\ell} - p_t)\epsilon_t \mid \pi_t = \pi \rangle \approx \sum_{0 \le t' < t+\ell} \sum_{\pi_1} \Delta^R_{\pi_1} \mathbf{P}(\pi_1) C_{\pi, \pi_1}(t').$$

A constant impact model II

- The model works very well for large tick stocks (dense LOB), but less for small tick stocks (rough gap approximation).



Source: [6]

Table of contents

Lab 5 - Questions

The goal of this lab is to reproduce some of the results of [6]. Observe that in order to plot a function of the lag ℓ, you may not need to compute the function for all ℓ.

1. Plot the sign $\langle \epsilon_t, \epsilon_{t+\ell} \rangle$ and side autocorrelation $\langle s_t, s_{t+\ell} \rangle$ functions.
2. Compute and plot the empirical response functions $R_\pi(\ell)$ for each of the 6 types of events π.
3. Compute the signed event-event correlations $C_{\pi_1,\pi_2}(\ell)$. (Here, you will need these functions of all $\ell = 1, \ldots, \bar{\ell}$).
4. Plot the function $C_{MO',\pi_2}(\ell)$. Note that a signed log-scale similar to the one presented in [6] can be obtained in matplotlib with `ax.set_yscale('symlog')`.
5. Compute the theoretical responses in the constant impact model, and compare them to the empirical ones. Comment.

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