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High-frequency data and limit order books
A few exercises on Poisson processes

Exercise 1 (Poisson characterization with stationary and independent increments).
Let $(N_t)_{t\geq 0}$ be a counting process such that the following conditions are satisfied :

(C1) N has independent increments ;

(C2) N has stationary increments ;

(C3) for any interval I with finite length, $N(I)$ is integrable ;

(C4) there exists an interval I such that $\mathbf{P}(N(I) > 0) > 0$.

(C5) the process is simple : $\lim_{\epsilon \to 0} \frac{\mathbf{P}(N(\epsilon) \geq 2)}{\epsilon} = 0$.

The aim of the exercise is to show that the number of points in any interval is Poisson distributed. Some standard results are recalled along the way.

1. We start by showing that the expectation of N_t is linear with respect to t.
   (a) Let $m(t) = \mathbf{E}[N_t]$, $t \geq 0$. Prove that for all $s, t \geq 0$, $m(t+s) = m(t) + m(s)$.
   (b) Prove that there exists $\lambda > 0$ such that $m(t) = \lambda t$.
2. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence with values in $[0,1]$ such that $\lim_{n\to+\infty} np_n = \mu > 0$. Let Z_n be a r.v. with binomial distribution with parameters $n, p_n)$. Prove that for all $k \in \mathbb{N}$, $\lim_{n\to+\infty} \mathbf{P}(Z_n = k) = e^{-\mu}\frac{\mu^k}{k!}$.
3. Let $t > 0$. For all $n \in \mathbb{N}^*$, let $t_j = \frac{jt}{n}$ and $X_j^{(n)} = N_{t_j} - N_{t_{j-1}}$, $j = 1, \ldots, n$. Let also $Y_j^{(n)} = \min(X_j^{(n)}, 1)$, $j = 1, \ldots, n$, and $N_t^{(n)} = \sum_{j=1}^{n} Y_j^{(n)}$.
   (a) Show that the random variables $Y_j^{(n)}$, $j = 1, \ldots, n$ are i.i.d. with Bernoulli distribution. Let $p_n = \mathbf{P}(Y_1^{(n)} = 1)$.
   (b) Prove that $\lim_{n\to+\infty} \mathbf{P}(N_t^{(n)} \neq N_t) = 0$.
   (c) Prove that for all $k \in \mathbb{N}$, $\left|\mathbf{P}(N_t = k) - \mathbf{P}(N_t^{(n)} = k)\right| \leq \mathbf{P}(N_t^{(n)} \neq N_t)$.
4. Let Z be an integer-valued random variable. Prove that $\mathbf{E}[Z] = \sum_{l=1}^{\infty} \mathbf{P}(Z \geq l)$.
5. Prove that $\lim_{n\to\infty} \mathbf{E}[N_t^{(n)}] = \mathbf{E}[N_t]$.
6. Prove that N_t has a Poisson distribution with parameter λt.

Exercise 2 (Superposition of independent Poisson processes).
Let $(N_t^{(1)})_{t\geq 0}$ and $(N_t^{(2)})_{t\geq 0}$ be two independent Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$. Let $N_t = N_t^{(1)} + N_t^{(2)}$. Let $\lambda = \lambda_1 + \lambda_2$.

1. Prove that for all $0 \leq s < t$, and all $k \in \mathbb{N}$, $\mathbf{P}(N_t - N_s = k) = e^{-\lambda(t-s)}\frac{\lambda^k (t-s)^k}{k!}$.
2. Prove that for all $0 \leq s < t \leq u < v$, $N_t - N_s$ and $N_v - N_u$ are independent.

3. Conclude.
4. Answer the first question again using the characteristic function this time.

Exercise 3 (Some conditional distributions).
Let N be a time-homogeneous Poisson process with parameter $\lambda > 0$. Let $(T_n)_{n\geq 1}$ be the sequence of associated events.

1. Determine the conditional distribution of N_s given $N_t = l$, for all $0 \leq s \leq t$, $l \in \mathbb{N}$.
2. Determine the conditional distribution of T_1 given $N_t = 1$.
3. Determine the conditional distribution of $(T_1, \dots, T_n)$ given $N_t = n$.

Exercise 4 (A family of martingales).
Let N be a time-homogeneous Poisson process with parameter $\lambda > 0$. Let $\alpha \in \mathbb{R}$ and $Y_t^{(\alpha)} = \exp\left(\alpha N_t - \lambda t(e^{\alpha} - 1)\right)$. Prove that $(Y_t^{(\alpha)})_{t\geq 0}$ is a martingale (w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the Poisson process).