Physique des marchés
modèles d'agents I

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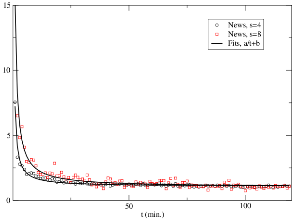
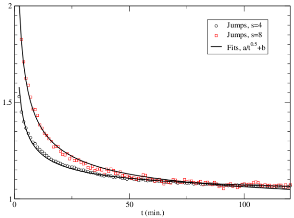
Motivation 1: endogenous fluctuations

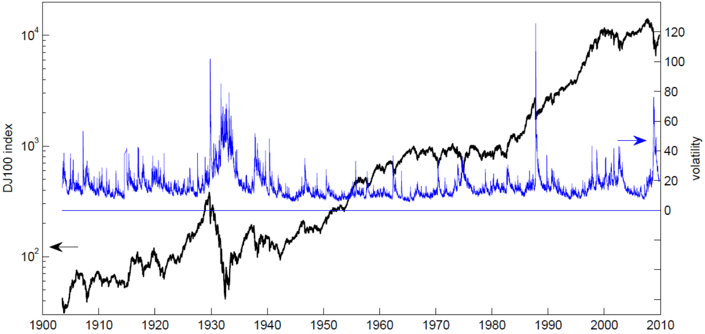
Volatility: endo- vs exogenous





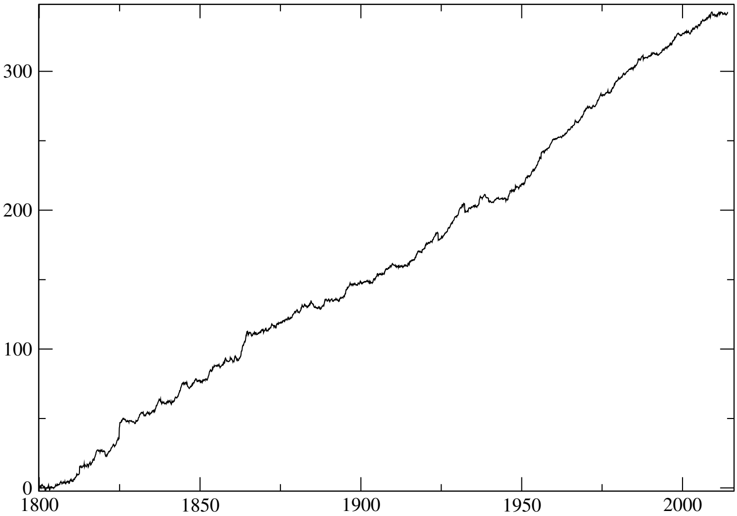
Motivation 2: eternal instabilities

100 years of volatility [article]



Motivation 3: weakly predictable prices

200 years of trend following [article]



From agents to markets

- Financial agent = strategies + learning
- Financial markets = interacting agents

Research program

1. Interacting strategies,
   implicit learning (weakest dies)
2. Behavioural finance:
   How do human beings learn?
3. Interacting learning agents

Market = interacting strategies

- You: a single trader, backtesting and using a strategy
- Market: N traders, each using a strategy

How do strategies modify price dynamics?

Generic (log-) price dynamics

1. Black-Scholes dynamics (log-price)

$$p_{t+1} = p_t + \epsilon_{t+1}$$

   where ϵ_t are i.i.d.

2. Exogenous influence E_t

$$p_{t+1} = p_t + \epsilon_{t+1} + E_{t+1}$$

3. Agents' influence A_t

$$p_{t+1} = p_t + \epsilon_{t+1} + E_{t+1} + A_{t+1}$$

Influence of the agents

- Agents wish to hold

$$x_{t+1} = \text{function}(past_t)$$

- They hold

$$x_t$$

- Transaction size

$$\omega_{t+1} = x_{t+1} - x_t$$

- Price impact

$$A_{t+1} = I(\omega_{t+1})$$

Price impact: assumptions

- Round trip: (e.g. buy, wait sell) impact

$$I(x) + I(-x) = 0: \text{ symmetric impact}$$

- Constant impact

- No price manipulation: split $x = x_1 + x_2$

$$I(x_1 + x_2) = I(x_1) + I(x_2)$$

$$\implies I(x) = \frac{x}{\lambda}$$

Influence of trend-followers

[Farmer and Joshi (2002)]

- Agents wish to hold

$$x_{t+1} = K \times (p_t - p_{t-\theta})$$

- They hold

$$x_t = K \times (p_{t-1} - p_{t-\theta-1})$$

- Transaction size

$$\omega_{t+1} = x_{t+1} - x_t = K \times (r_t - r_{t-\theta})$$

- Price impact

$$A_{t+1} = I(\omega_{t+1}) = \frac{K}{\lambda} \times (r_t - r_{t-\theta})$$

Influence on price dynamics

- $$\begin{aligned} p_{t+1} &= p_t + \epsilon_{t+1} + A_{t+1} \\ &= \epsilon_{t+1} + \frac{K}{\lambda}(r_t - r_{t-\theta}) \end{aligned}$$

- Hyp: $K/\lambda = \alpha$

- For $\theta = 1$
$$C_r(1) = \frac{E(r_t r_{t-1})}{E(r^2)} = \frac{\alpha}{\alpha + 1}$$

- For $\theta > 1$: θ equations for $C_r(\theta)$

Influence of strategies on price dynamics

- Trend-following $\rightarrow$ $C_r(1) > 0$
- Mean-reversion $\rightarrow$ $C_r(1) < 0$

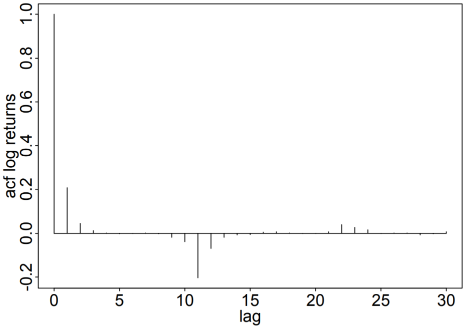


FIGURE 2. The autocorrelation function for Equation 10 with $\alpha = 0.2$ and $\theta = 10$. The positive coefficients for small τ indicate short term trends in prices, and the negative coefficients indicate longer term oscillations.

Influence of strategies on price dynamics

- Price returns are now predictable
- New traders add $+\beta(r_t - r_{t-\theta'})$

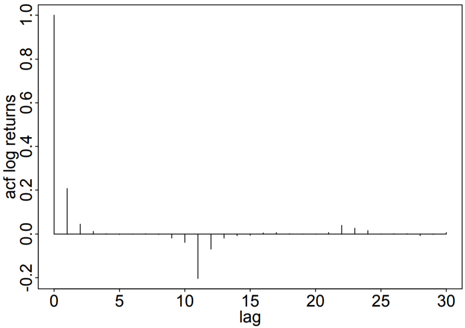


FIGURE 2. The autocorrelation function for Equation 10 with $\alpha = 0.2$ and $\theta = 10$. The positive coefficients for small τ indicate short term trends in prices, and the negative coefficients indicate longer term oscillations.

How to remove predictability

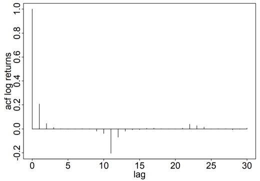


FIGURE 2. The autocorrelation function for Equation 10 with $\alpha = 0.2$ and $\theta = 10$. The positive coefficients for small τ indicate short term trends in prices, and the negative coefficients indicate longer term oscillations.

- ACF induced by strategy use

$$C_r(\tau) \simeq \alpha^\tau - \alpha^{(\theta+2-\tau)}, \;\; \tau \leq \theta + 1$$

- Use anti-strategy $\alpha \to -\alpha$ modifies predictability

- Predictability depends on the fraction of agents using $+\alpha$ and $-\alpha$

Measures of price predictability

Strategy $\longleftrightarrow$ predictability measure $H \geq 0$

$$H = |E(g)| = |E(x_t r_{t+1})|$$

Examples:

- Value investors

$$H_{\text{value}} = E(|(p_t - p_{\text{fond}}) r_{t+1}|)$$

- Trend followers

$$H_{TF,\theta} = |E[r_{t+1} \kappa \cdot (p_t - p_{t-\theta})]|$$

- State-based strategies

$$H = \sum_{\mu} P(\mu) |E(r_{t+1} | \mu_t = \mu)|$$

Measures of price predictability

Strategy $\longleftrightarrow$ predictability measure $H \geq 0$

$$H = |E(g)| = |E(x_t r_{t+1})|$$

Consequences:

1. As many H as strategy families
2. Arbitrage-free: define which strategy
3. Markets: arbitrage-free only w.r.t. all possible strategies

Dynamics of price predictability

1. Some trader uses a strategy
   - Decreases predictability — average gain
   - Introduces predictability — average loss
2. On average, makes money if gains – losses > 0
3. If gains – losses > 0, global predictability decreases

Dynamics of conditional predictability

1. Predictability conditional on $\mu_t \iff$

$$E(r_{t+1}|\mu_t) \neq 0$$

2. If agents detect and exploit it a little, for $t' > t$,

$$\begin{gathered} |E(r_{t'+1}|\mu_{t'} = \mu_t)| < |E(r_{t+1}|\mu_t)| \\ \text{sign } E(r_{t'+1}|\mu_{t'} = \mu_t) = \text{sign } E(r_{t+1}|\mu_t) \end{gathered}$$

3. If conditional predictability is perfectly exploited

$$E(r_{t'+1}|\mu_{t'} = \mu_t) = 0$$

4. Overshooting ($t' > t$)

$$\text{sign } E(r_{t'+1}|\mu_{t'} = \mu_t) = -\text{sign } E(r_{t+1}|\mu_t)$$

How to think about predictability and market states?

A computational view of market efficiency

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there is simpler (and earlier)

Predictability dynamics model: wish list

Repeated game

- No a priori optimal predictor
- Optimal predictor depends on all other agents
- Optimal predictor gradually found by agents
- Optimal predictor changes with time
- Obvious predictability disappears with time

Red Queen effect (Lewis Carroll):

It takes all the running you can do,
to keep in the same place

El Farol Bar Problem

Arthur (1997) [link]

- El Farol Bar: Irish music on Thursday nights
- 100 customers
- 60 seats

To go or not to go?

El Farol Bar Problem: subtle problem

Repeated game

- No a priori optimal predictor
- Optimal predictor depends on all other agents
- Obvious predictability disappears with time
- Optimal predictor gradually found by agents
- Optimal predictor changes with time

Red Queen effect (Lewis Carroll):

It takes all the running you can do,
to keep in the same place

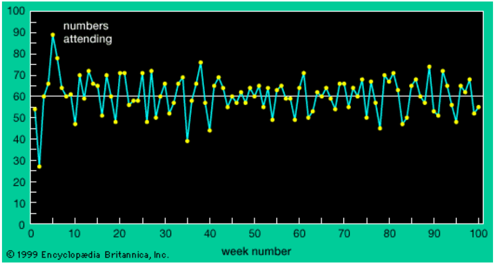
El Farol Bar Problem

Repeated game

- Deduction: 1 single best answer $\rightarrow$ all fail
- Induction: each agent has a set of predictors; trial and error
- Predictor examples:
  1. average over last 3 weeks
  2. 100-previous week
  3. 42
  4. ...
- Learning:
  1. update performance of each predictor each week
  2. discard bad predictors
  3. find new predictors

El Farol Bar problem: results

Arthur (1997)



Average attendance converges to resource level

Collective learning? [answer]

El Farol $\rightarrow$ Minority Game

- 100 customers, 60 seats $\rightarrow$ N customers, $N/2$ seats

Minority Game Challet&Zhang (1997)

- N agents
- 2 choices $a_i \in \{-1, +1\}$
- Aggregate choice

$$A = \sum_i a_i$$

- Payoff to agent i

$$-a_i A$$

- NB: El Farol's payoff, $L =$number of seats

$$-a_i(A - [2L - N])$$

Minority Game: examples

- Competition for limited resource
  - seats
  - space
  - speed
  - food
- Minority mechanism when population learns resource level
- Signature of Minority Game mechanism:

$$-a_i A$$

Are market minority games?

Speculative gain, round trip

- Agent i decides to buy $a_i(t)$ shares at time t
- Price paid $p(t+1) = p(t) + A(t)$, where

$$A(t) = \sum_{j=1}^{N} a_j(t)$$

  A: excess demand

- Agent i sells $a_i(t') = -a_i(t)$ shares at time t'
- Price obtained $p(t'+1) = p(t') + A(t')$

Speculative gain

Round trip gain:

$$
\begin{aligned}
g &= a_i(t)[p(t'+1) - p(t+1)] \\
&= a_i(t)[p(t') - p(t)] + a_i(t)A(t') - a_i(t)A(t) \\
&= \underbrace{a_i(t)[p(t') - p(t)]}_{\text{gain if no impact}} \underbrace{- a_i(t')A(t')}_{\text{impact}} \underbrace{- a_i(t)A(t)}_{\text{impact}}
\end{aligned}
$$

- impact terms: Minority Games
- gain if no impact: delayed majority game

$$
\begin{aligned}
\hat{g} &= a_i(t)[p(t') - p(t)] \\
&= \sum_{s=t+1}^{t'-1} a_i(t)A(s) = a_i(t) \sum_{s=t+1}^{t'-1} A(s)
\end{aligned}
$$

Minority Game: strategies

- strategy: action given μ: market state
- market states $\mu \in \{1, \cdots, P\}$:
  - Louvain
  - Past losing choices
  - ...
- generically,

μ	$--$	$-+$	$+-$	$++$
a_1^μ	$-$	$-$	$-$	$+$
a_2^μ	$+$	$-$	$+$	$+$

Minority Game: agents

agent i:

- 2 strategies $a_{i,1}$ and $a_{i,2}$ drawn at random

μ	$--$	$-+$	$+-$	$++$
$a^{\mu}_{i,1}$	$+$	$-$	$-$	$+$
$a^{\mu}_{i,2}$	$+$	$+$	$-$	$+$

- 2 scores $U_{i,1}$ and $U_{i,2}$: cumulative payoffs $-a^{\mu(t)}_{i,s}A(t)$

$$U_{i,1}(t+1) = U_{i,1}(t) - a^{\mu(t)}_{i,1}A(t)$$

Minority Game: dynamics

- Rule: use the best strategy

$$s_i(t) = \arg\max U_{i,s}(t)$$

- Update cumulated payoffs

$$U_{i,s}(t+1) = U_{i,s}(t) - a_{i,s}^{\mu(t)} A(t)$$

where

$$A(t) = \sum_i a_i(t) = \sum_i a_{i,s_i(t)}^{\mu(t)}$$

- Market state update: e.g.

$$\mu(t) \to \mu(t+1) = [2\mu(t) + \operatorname{sign} A(t)] \,\mathrm{MOD}\, 2^M$$

Minority Game: measurables

- Fluctuations

$$\sigma^2 = E_t(A^2)$$

$$\sigma^2 = N: \text{ coin tossing}$$

$$\sigma^2 \propto N^2: \text{ herding}$$

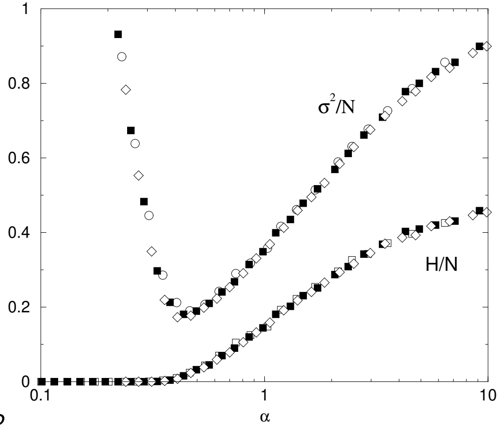
- Predictability

$$H = \frac{1}{P} \sum_{\mu} E_t(A|\mu)^2$$

$$H = 0 \iff E_t(A|\mu) = 0 \text{ for all } \mu$$

Minority Game: results

$\alpha = \frac{P}{N}, \quad P = 2^M$



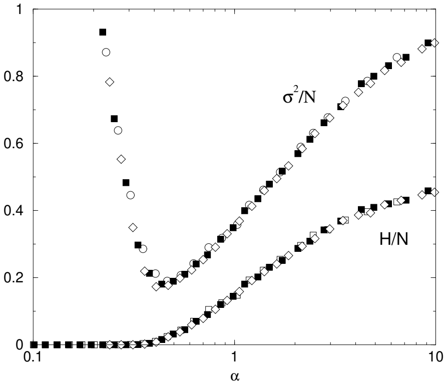
$N \gg P$ $\qquad$ $N \ll P$

Minority Game: results

$\alpha = \frac{P}{N}$

- $\sigma^2/N \to 1_-$, $N \ll P$
- σ^2/N: min at $P \simeq \alpha_c N$
- $\sigma^2 \propto N^2$: $N \gg P$

- $H > 0$, $P > \alpha_c N$
- $H = 0$, $P < \alpha_c N$



What happens at α_c?

- $H = 0$

- The agents solve P linear equations

$$E_t(A|\mu) = 0, \text{ for all } \mu$$

- Adding agents $\equiv$ adding variables to P linear equations

- Enough agents $\rightarrow$ enough variables

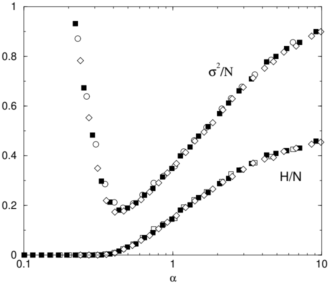
Why do the agents minimize H?

Scenario for large fluctuations and predictability

- Too few agents: too much information, random choices
- Predictability attracts more agents
- Predictability decreases
- When predictability vanishes, explosion of fluctuations
- No more predictability, too much fluctuations → less agents

Thus real markets should hover around a critical point

2 phases: predictable prices, unpredictable prices



Variable N games

Rationale:

- Bad player should not play
- Play only if score good enough: compare with fixed reward ϵ.
- Add N_p predictable players with a single strategy ("producers")

$$\Omega^\mu = \sum_{j=1}^{N_p} b_j^\mu \sim \mathcal{N}(0, N_p) \propto O(\sqrt{N_p})$$

Simplest model

- agent i: 1 strategy a_i^μ, 1 score $U_{i,t}$, agent $n_i(t)$

$$U_{i,t+1} = U_{i,t} - a_i^{\mu_t} A(t) - \epsilon$$

$$n_{i,t} = \theta(U_{i,t}) \in \{0,1\}\text{: play or not}$$

$$A(t) = \underbrace{\sum_i a_i^{\mu_t} n_{i,t}}_{} + \underbrace{\Omega^{\mu_t}}_{\text{producers}}$$

Average dynamics

Idea: coarse time by factor P

$$E(U_{i,t+1}|U_{i,t}) = U_{i,t} - E(a_i^{\mu_t} A(t)) - \epsilon E(n_{i,t})$$

Set

$$\phi_i = E(n_{i,t})$$
$$\tau = t/P$$

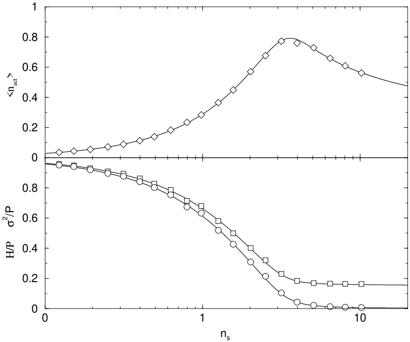
$$\frac{dU_{i,\tau}}{d\tau} \simeq -\sum_\mu a_i^\mu E(A(\tau)|\mu) - \epsilon\phi_i \sim -\frac{\partial H}{\partial \phi_i}$$

where $E(A|\mu) \simeq \sum_j E(n_j) a_j^\mu = \sum_j \phi_j a^\mu$

if

$$H = \frac{1}{2}\sum_\mu E(A|\mu)^2 + \epsilon \sum_j \phi_j = H_0 + \epsilon N_{act}$$

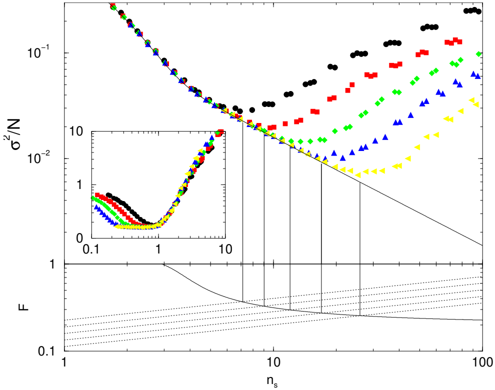
Variable N model: results



small N large N

- $n_s = N/P = 1/\alpha$, $\langle n_{\text{act}} \rangle = \frac{1}{P}\sum_i E(n_i)$
- saturation of the number of active speculators
- $H > 0$ as soon as $\epsilon > 0$ (exact solution)

Variable N model: results for larger N



small N large N

- large fluctuations $\longleftrightarrow$ small signal-to-noise ratio

Suitable mathematical formalism

Is $H_0 = 0$ doable?

$$\begin{aligned} E(A|\mu) &= 0 \\ \iff \Omega^\mu + \sum_i a_i^\mu E(n_i) &= 0 \end{aligned}$$

Set $\phi_i = E(n_i)$.

$$\Omega^\mu + \sum_i a_i^\mu \phi_i = 0 \qquad 0 \leq \phi_i \leq 1$$

P equations, N bounded variables $\Longrightarrow$ yes for $N = KP$, $K > 1$

MG and frustrated systems

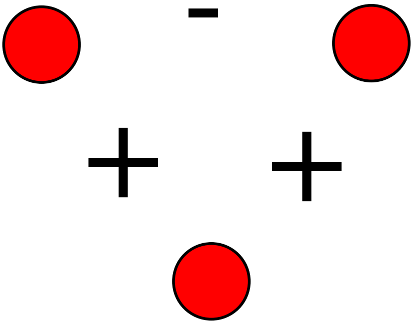
- Predictability $H = \frac{1}{P}\sum_\mu E(A|\mu)^2 + \epsilon \sum_i \phi_i$

$$H = \frac{1}{P}\sum_\mu \left[(\Omega^\mu)^2 + \sum_i \Omega^\mu a_i^\mu \phi_i + \sum_{i,j} a_i^\mu a_j^\mu \phi_i \phi_j \right] + \epsilon \sum_i \phi_i$$
$$= \frac{1}{P}\sum_\mu \left[(\Omega^\mu)^2 \right] + \sum_i (h_i + \epsilon)\phi_i + \sum_{i,j} J_{i,j}\phi_i\phi_j$$

- Random heterogeneity $\Longleftrightarrow$ random h_i and $J_{i,j}$
- Random h_i: cf random field Ising model
- Random $J_{i,j}$: frustrated system, spin-glass

Frustrated systems

- Frustration: friend of friend = friend ?



- Random $J_{i,j}$: if $s_i \in \{-1, +1\}$, 2^N configurations to test
- Special mathematical methods to deal with random $J_{i,j}$.
- Parisi: Nobel prize in Physics 2021 for spin-glass problems

Exact solution of Minority Games

Problem 1: how to compute minimum of H?

- $N \to \infty$ limit
- Predictability H_N minimised: $H_N \equiv$cost function
- Partition function

$$Z(\{a_i,\Omega\}) = \int_0^{+1} \prod_i d\phi_i \, e^{-\beta H(\{a_i,\Omega\},\{\phi_i\})}$$

- Minimisation

$$\begin{aligned}\min_{m_i} H(\{a_i,\Omega\},\{\phi_i\}) &= \lim_{\beta\to\infty} -\frac{1}{\beta}\log Z(\{a_i,\Omega\}) \\ &= \lim_{\beta\to\infty} -\frac{1}{\beta}\log\left[e^{-\beta \min H}\prod_i d\phi_i \, e^{-\beta(H-\min H)}\right.\end{aligned}$$

Exact solution of the MG

Problem 2: average over heterogeneity

- Mathematically

$$H_N(\{a_i\},\{\phi_i\}) = \frac{1}{P}\sum_\mu E(A|\mu)^2 = \frac{1}{P}\sum_\mu(\Omega^\mu + \sum_i a_i^\mu E(\phi_i))^2$$

  $\{a_i, \Omega\}$: random heterogeneity

- Minimum of H_N depends on $\{a_i, \Omega\}$
- Compute $E_{\{a_i,\Omega\}}(\min_{\{m_i\}} H)$: average over heterogeneity

$$\begin{aligned}\tilde{H}_{\min} &= \lim_{N\to\infty} \min_{\phi_i} E_{\{a_i\}} H_N(\{a_i,\Omega\},\{\phi_i\}) \\ &= \lim_{\beta\to\infty} -\frac{1}{\beta} \lim_{N\to\infty} E_{\{a_i,\Omega\}}[\log Z(\{a_i,\Omega\})]\end{aligned}$$

Exact solution of MG: replica trick

- $E(\log Z)$: generally impossible to compute
- Trick:

$$E(\log Z) = \lim_{n \to 0} \frac{E(Z^n) - 1}{n}$$

- What is $E_{\{a_i, \Omega\}}(Z^n)$?
- Z^n : same agents, n duplicates of ϕ_i, $\phi_{i,c}$, $c = 1, \cdots, n$
- Compute now

$$E_{\{a_i, \Omega\}}\left[e^{-\frac{\beta}{P}(\Omega^\mu + \sum_j a_j^\mu \phi_{j,c})^2}\right]$$

- Gaussian integrals, doable.

Exact solution of the standard MG

Eventually (after about 6 A4 pages of calculus)

- Predictability, $\alpha = P/N > \alpha_c = 0.3374\dots$

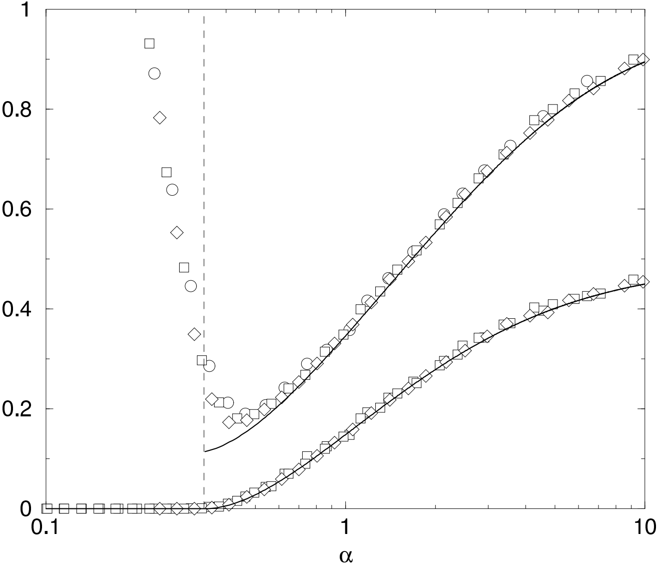
$$\tilde{H}_0 = \frac{1+Q}{2(1+\chi)^2}$$

$$Q = \frac{1}{N}\sum_i \phi_i^2$$

- No predictability $\iff \chi = \infty$: phase transition.
- $\alpha_c = \mathrm{erf}\left[\sqrt{\left|\log[\sqrt{\pi}(2-\alpha_c)]\right|}\right]$
- Fluctuations

$$\frac{\sigma^2}{N} \to \tilde{H}_0 + \frac{1-Q}{2}$$

Exact solution of Minority Games



Dynamical solutions of interacting agents

- H minimised: stationary state, static approach
- Exact dynamical solutions known: De Dominicis generating functionals
- From N dynamical equations to 1 effective agent equation, with complex time structure
- See Coolen book "Mathematical theory of minority games".

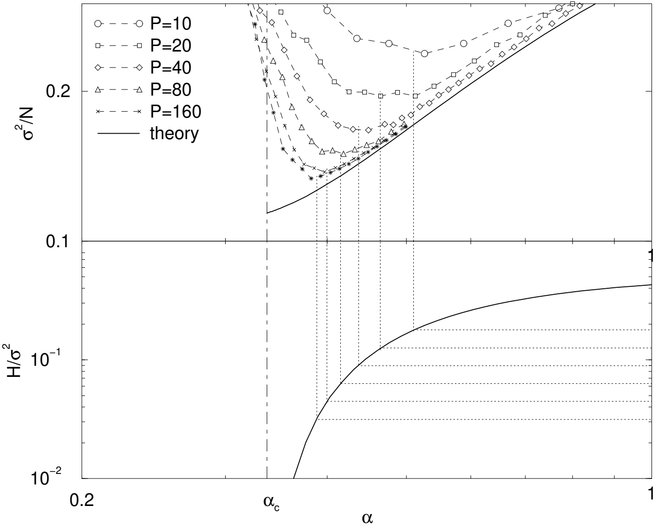
Why solution exact?

- Mathematically complex (non-linear, dynamical, heterogeneity) N-agent model
- Exact solutions generically in 1, 2, ∞ dimensional models
- Payoff

$$-a_i A$$

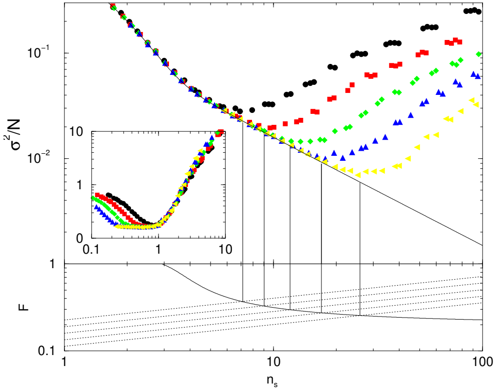
- Everybody interacts with everybody else through aggregate quantity
- Mean-field type of interaction
- When is a modified MG still mean-field?

Signal-to-noise transition: standard MG



Signal-to-noise transition: variable-N MG

$F =$ signal ratio of the strategies of the agents



Summary so far

- Agents minimise predictability by learning
- When *exploitable* predictability is too small $\to$ explosion of volatility
- Exploitable means

$$\frac{\text{signal}}{\text{noise}} \simeq \frac{H}{\sigma} \text{ large enough}$$

- Signal OF THE STRATEGIES of the agents

Learning $\rightarrow$ instability

Pazelt and Pawelzik: "criticality of adaptive control dynamics" (2011)

- Signal $y_t = \alpha y_{t-1} + \beta_{t-1}$,
  - α unknown
  - $\beta \sim \mathcal{N}(0, \sigma^2)$
- The agents learn α from the last m time steps and try to cancel y_{t+1}
- Example: $m = 2$, minimize $E(y_{t+1}^2 | y_{t,}, y_{t-1}) \rightarrow$

$$\tilde{\alpha}_{t+1} = \frac{y_t}{y_{t-1}} + \tilde{\alpha}_t$$

- The signal becomes

$$y_{t+1} = (\alpha - \tilde{\alpha}_{t+1}) y_t + \beta_t = -\frac{y_t}{y_{t-1}} \beta_{t-1} + \beta_t$$

- One shows that

$$P(y > |r|) \propto \frac{1}{|r|^m}$$

Suppression d'un signal