Physique des marchés
modèles d'agents III

damien.challet@centralesupelec.fr
[https://s.42l.fr/PhM]

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Payoffs of producers

- Decomposition $A_{tot}(t) = A_{prod} + A_{spec}$

$$A_{prod}(t) = \Omega_{\mu(t)} \quad A_{spec}(t) = \sum_{i=1}^{N_s} n_i(t) a_{i,\mu(t)}$$

- Total payoff of producers

$$G_{prod}(t) = -\Omega_{\mu(t)} A_{tot}(t)$$

- Average payoff: denote $\frac{1}{P}\sum_\mu E(X|\mu) \equiv \overline{X}$,

$$E(G_{prod}) = \overline{G_{prod}} = -\overline{\Omega A_{total}}$$

- Without speculators

$$E(G_{prod}) = -\overline{\Omega^2} < 0$$

- With speculators, if $A_{spec} = -\Omega_\mu \ \ \forall\mu$

$$E(G_{prod}) = -\overline{\Omega\,[A_{spec} + \Omega]} = 0$$

Payoffs of speculators

- Total payoff of speculators

$$
\begin{aligned}
G_{spec}(t) &= -A_{spec}(t)A_{tot}(t) \\
&= -A_{spec}(t)\Omega_{\mu(t)} - [A_{spec}(t)]^2
\end{aligned}
$$

- Taking averages

$$
E(G_{spec}) = \underbrace{-\overline{A_{spec}\Omega}}_{\lesseqgtr 0} \underbrace{-E(A^2_{spec})}_{\leq 0}
$$

What do agents minimize?

- Each agent has a single, simple strategy, does not care about global behaviour

$$U_i(t+1) = U_i(t) - a_{i,\mu(t)} A_{total}(t) - \epsilon$$

- Continuous time approximation

$$\begin{aligned}
\frac{dU_i}{dt} &= -\overline{a_i A_{total}(t)} - \epsilon + \eta_i(t) \\
&= -\frac{1}{2}\frac{\partial}{\partial \phi_i} H_\epsilon(t) + \eta_i(t) \\
\phi_i &= E(n_i)
\end{aligned}$$

where

$$\begin{aligned}
H_\epsilon &= \overline{(\Omega + a_i \phi_i)^2} + 2\epsilon \sum_i \phi_i \\
&= H_0 + 2\epsilon E(N_{act})
\end{aligned}$$

Left-over predictability from costs and benchmarks

- Predictability in the stationary state
  $\simeq$ minimum performance neeeded $\simeq$ transaction costs

- In the MG with variable number of speculators, the agents minimize

$$H_\epsilon = H_0 + 2\epsilon E(N_{active})$$

- Assuming no active agents at the beginning, H_0 is minimized first until it is comparable with $2\epsilon E(N_{active})$
- Remaining predictability

$$H_0 \simeq 2\epsilon E(N_{active})$$

Optimal strategy and agent impact

- A trader that observes $E(A_{total}|\mu)$ should play

$$a_\mu^{(opt,naive)} = -\mathrm{sign}[E(A|\mu)]$$

- Naively,

$$E(gain^{(opt,naive)}) = \frac{1}{P}\sum_\mu |E(A|\mu)| \sim H_0$$

- Will play if $H_{total} > \epsilon$

- Practically, *ceteribus paribus*

$$E(gain^{(opt,real)}) = -\frac{1}{P}\sum_\mu a_\mu^{(opt)} E(A + a_\mu^{(opt)}|\mu) \sim H_0 - 1$$

- When $H_0 < 1$, $a_\mu^{(opt,naive)}$ is wrong

The role of self-impact

- Generically, in all MGs

$$gain_i(t) = -a_i(t)A(t) = -a_i(t)A_{-i}(t) - a_i^2(t)$$
$$A_{-i} = \sum_{j \neq i} a_j$$

- In variable-N MGs

$$gain_i(t) = -a_i(t)n_i(t)A(t) = \underbrace{-a_i(t)A_{-i}(t)}_{\text{virtual gain}} - \underbrace{n_i(t)}_{\text{self impact}}$$
$$A_{-i} = \sum_{j \neq i} a_j n_j$$

1. If $n_i(t) = 0$, over-estimation of strategy value
2. If $n_i(t) = 1$, actual gain $=$ virtual gain -1

Who wins, who loses

- $\phi_i = 1$: always plays, average gain $> \epsilon$
- $0 < \phi_i < 1$, plays sometimes, average gain $< \epsilon$
- $\phi_i = 0$: never plays

(see TP)

Less naive agents

- Practictioners know that they have an impact (slippage)
- Some agents may be able to add slippage κ_i

$$U_i(t+1) = U_i(t) - a_{i,\mu(t)} A(t) - \epsilon - \kappa_i \times [1 - n_i(t)]$$

- Claim: if all the speculators have a $\kappa_i > 0$, they minimize

$$\sigma^2 = E(A^2)$$

(see TP)

Back to basics: speculative gain

Round trip gain

$$\begin{aligned} g &= a_i(t)[p(t'+1) - p(t+1)] \\ &= a_i(t)[p(t') - p(t)] + a_i(t)A(t') - a_i(t)A(t) \\ &= \underbrace{a_i(t)[p(t') - p(t)]}_{\text{delayed majority game}} \underbrace{- a_i(t')A(t')}_{\text{impact}} \underbrace{- a_i(t)A(t)}_{\text{impact}} \end{aligned}$$

- MG: only impact
- \$-game: only delayed majority game

$$U_i(t+1) = U_i(t) + a_{i,\mu(t-1)}A(t)$$

$-game

(Sornette and Andersen 2003) [paper]

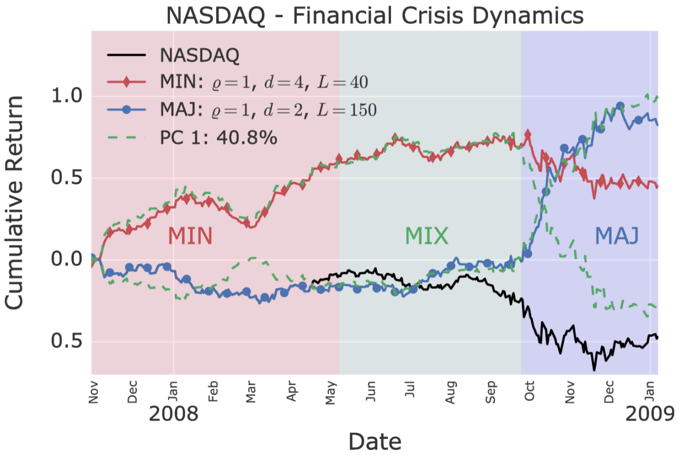
$$U_i(t+1) = U_i(t) + a_{i,\mu(t-1)} A(t)$$

- $A(t+1)$ comes from $\sum_i a_{i,\mu(t+1)} n_{i,t+1}$
- $\mu(t)$ cannot be random

1-step markets: MG or $-game?

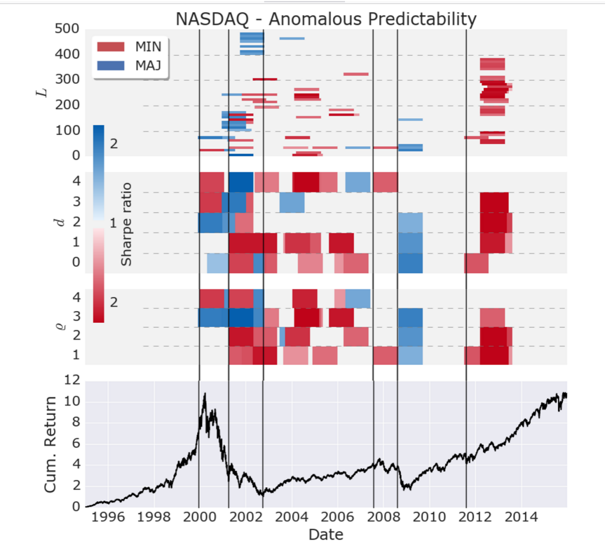
Fiévet and Sornette (2018) [paper]

- Define $\mu(t) = \text{sign}\{A(t-1), \cdots, A(t-\rho)\}$
- Fit MG and $-game models to real market data



Markets: MG or $-games

Fiévet and Sornette (2018) [paper]



What does trigger agent activity?

Moving averages crossing [source]

- Alternative view point: μ is a signal, e.g., crossing between two moving averages

Back to basics: speculative gain

Round trip gain

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- MG: only impact
- \$-game: delayed majority game
- Keep both?

Agent-based speculation model with holding periods

Challet (2008) [link]

- Collection of possible signals $\boldsymbol{\mu} = \{\mu_1, \cdots, \mu_P\}$

- At each time step, the market is in a unique state μ_t

- Agent i
  1. recognizes two signals $\boldsymbol{\mu}_{\mathrm{i}} = \{\mu_{i,1}, \mu_{i,2}\} \subset \boldsymbol{\mu}$;
  2. may change her position only when $\mu_t \in \boldsymbol{\mu}_{\mathrm{i}}$;
  3. computes the performance $\mu_{i,1} \to \mu_{i,2}$ and $\mu_{i,2} \to \mu_{i,1}$
  4. If e.g. $\mu_t = \mu_{i,1}$,
     - open/close position depending on performance $\mu_{i,1} \to \mu_{i,2}$
  5. If $\mu_t \notin \{\mu_{i,1}, \mu_{i,2}\}$, do nothing (hold or wait)

Performance measure and activity criterion

- Same as in variable-N MG:

$$|E(\sum A|\mu_1 \to \mu_2)| > \epsilon$$

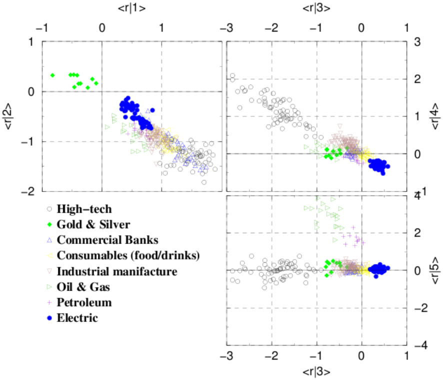
- Without producers, no active speculators

- Add producers: fixed biases Ω_μ

$$A_{total}(t) = A_{spec}(t) + \Omega_{\mu(t)}$$

Fixed conditional biases Ω_μ

Marsili (2002) [paper]: they exist



Dynamics of market states

- Dynamics for μ: uniform transition probabilities $W_{\mu \to \nu}$ do not produce inter-pattern predictability

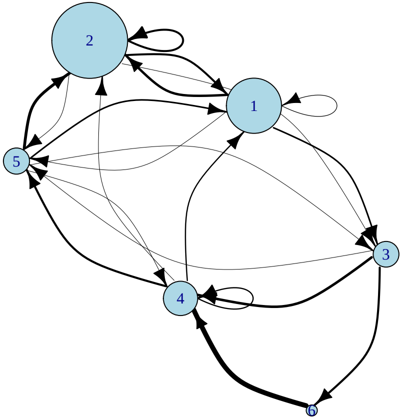
- [Hendricks, Gebbie and Wilcox (2015)], 1h periods

		state_{t+1}					
		1	**2**	**3**	**4**	**5**	**6**
	1	0.13	0.49	0.32	0.00	0.06	0.00
	2	0.41	0.41	0.09	0.00	0.09	0.00
state_t	**3**	0.00	0.00	0.00	0.52	0.05	0.43
	4	0.25	0.07	0.00	0.25	0.43	0.00
	5	0.32	0.59	0.05	0.05	0.00	0.00
	6	0.00	0.00	0.00	1.00	0.00	0.00

$$\to W_{\mu \to \nu} \neq \frac{1}{P}$$

Dynamics of market states

[Hendricks, Gebbie and Wilcox (2015)], 1h periods



Relevant predictability

- Contemporaneous predictability: not exploitable

$$H = \frac{1}{P} \sum_{\mu} E(A_t|\mu_t)^2$$

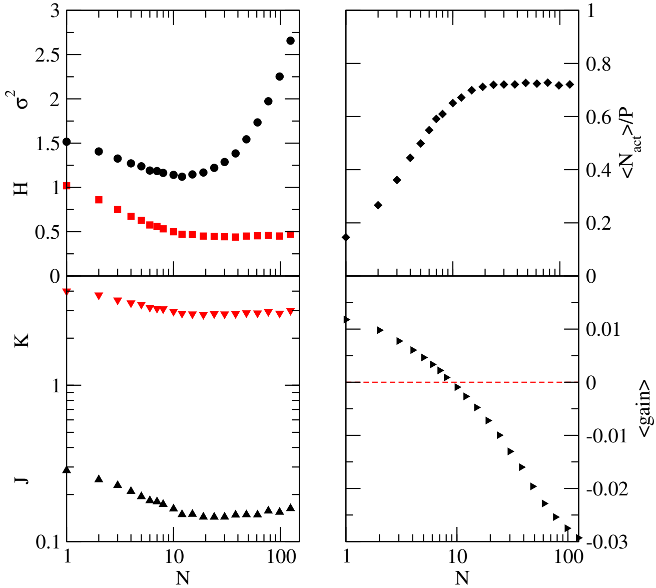
- Predictability associated to strategies

$$J = \frac{2}{P(P-1)} \sum_{\mu,\nu} E(\sum A_{t+1}|\mu_t \to \nu)^2$$

- Naive predictability

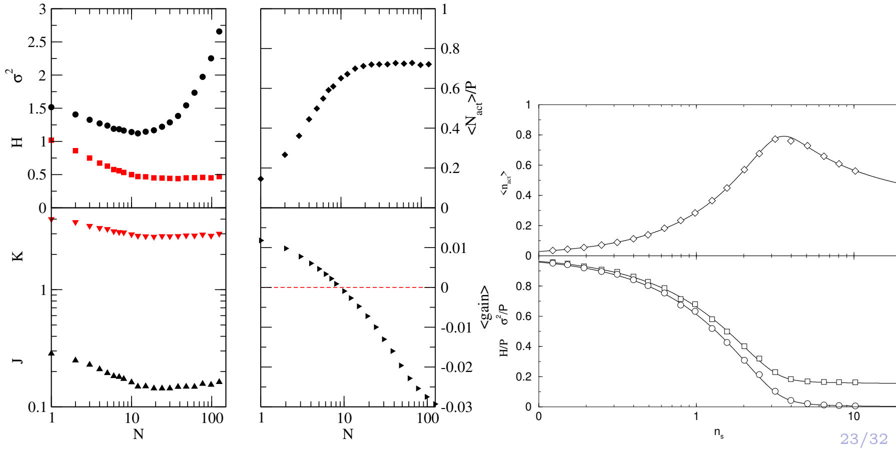
$$K = \frac{2}{P(P-1)} \sum_{\mu,\nu} E(\sum A_t|\mu_t \to \nu)^2$$

Model: results ($\epsilon = 0.05$)



Model: results

- J and K: real and naive predictability between patterns
- Exploitable predictability is decreased by speculation
- H, gain, N_{active} and $\sigma^2 = E(A^2)$: similar to MG



Can inter-pattern predictability disappear?

- Without speculators

$$H = \overline{E(A|\mu)^2} = \overline{\Omega_\mu^2} = \sigma_\Omega^2$$

- Hypothesis: circular history

$$\mu_t = t \,\mathrm{MOD}\, P$$

- Hypothesis: holding period of 1 step

$\rightarrow$ there are P groups of agents
$\rightarrow$ at each time step, one group of agents opens and one closes their position

Predictability in a circular world

- Hypothesis: circular history

$$\mu_t = t \,\mathrm{MOD}\, P$$

- Hypothesis: holding period of 1 step

  $\rightarrow$ there are P groups of agents
  $\rightarrow$ at each time step, one group of agents opens and one closes their position

- One average, for a given μ

$$\Omega_\mu + A_{Open,\mu} + A_{Close,\mu} = 0$$

P equations, P variables $\rightarrow$ $H = 0$ is possible.

Predictability in a non-deterministic world

- Hypothesis: possibly random history
- Hypothesis: holding period of any number of steps

  $\rightarrow$ there are $P(P-1)/2$ groups of agents
  $\rightarrow$ at each time step, one group of agents opens and one closes their position
- One average, for a given μ

$$\Omega_\mu + A_{Open,\mu} + A_{Close,\mu} = 0$$

P equations, $P(P-1)/2$ variables $\rightarrow$ $H = 0$ is possible.

Relevant predictability

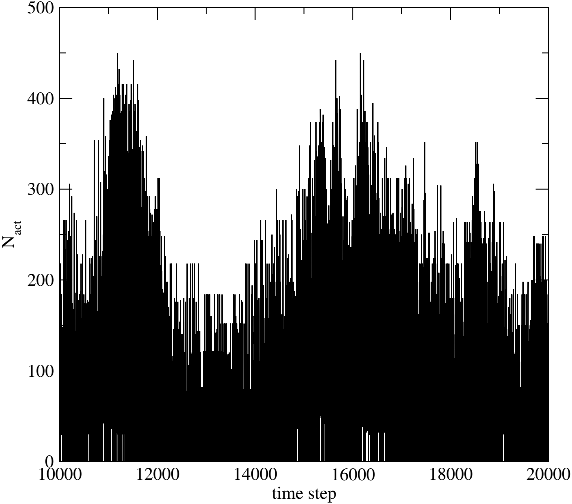
- Hypothesis: deterministic μ
- Unique path between two states
- Given the strategies, the relevant predictability is

$$J = \frac{2}{P(P-1)} \sum_{\mu,\nu} E\left(\sum A|\mu \to \nu\right)^2$$

- $J = 0$: $P(P-1)/2$ equations, $P(P-1)/2$ variables
- $\epsilon > 0 \Rightarrow J \neq 0$

Results: activity memory

Long memory of $N_{active} \propto Volume \propto volatility$ (subordination)



Strategy usage and long memory of activity

Bouchaud *et al.* (2001) On a universal mechanism for long-range volatility correlations [link]

- N agents
- agent i active if $n_i = \theta[U_i(t)]$, where $U_i(t)$ follows a random walk
- Theorem: volume $V(t)$

$$V(t) = \sum_i n_i$$

  has a long memory
- Proof: n_i has a long memory: see persistence properties of random walks.
- Corollary: activity in any agent-based model in which the volume is mostly modulated by approximately random events has a long memory

Strategy usage and long memory of activity

Variogram of V

$$v(V) = E([V(t) - V(t')]^2)$$

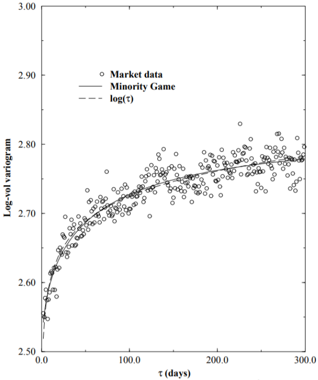


Figure 5: Variogram of the log-volatility, $\langle \log^2(\sigma_t/\sigma_{t+\tau})\rangle$ as a function of τ, averaged over 17 different stock indices (American, European, Asian). The full line is the MG result, with again both axis rescaled and a constant added to account for the presence of 'white noise' trading. The dashed line is the prediction of the multifractal model of [7], and is nearly indistinguishable from the MG result.

Summary

1. Agents use strategies to detect and exploit predictability
2. Predictability decreases w.r.t the strategies they use
   $\rightarrow$ minority game w.r.t. strategies
   $\rightarrow$ may increase the predictability of other strategies
3. Signal-to-noise ratio transitions to herding and large price fluctuations
4. Variable investment:
   long-range memory of activity $\equiv$ volatility

Contrôle: état d'esprit

- Pourquoi les marchés minimisent la prévisibilité
- Comment le font-ils? Modèle?
- Qui gagne et pourquoi?
- Qui est le pigeon?
- Comment penser la dynamique de la prévisibilité?
- Apprentissage et impact?
- Comment les marchés deviennent-ils instables?
- A quoi servent les stratégies de trading?
- Que se passe-t-il si les agents trouvent de nouvelles stratégies?