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Hilbert Spaces

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Contents

1 Inner product spaces

Throughout this section the symbol $\Bbbk$ will always stand for the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

definition[]Given a $\Bbbk$-vector space H a *sesquilinear form* on H is a map

$$\langle -, - \rangle : H \times H \to \Bbbk$$

fulfilling the following two axioms:

1.1 Definition libel=(SF0),leftmirgin=* The map $\langle -, - \rangle$ is *linear* in its first coordinate which means that

$$\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle \quad \text{and} \quad \langle rv, w \rangle = r \langle v, w \rangle$$

for all $v, w, v_1, v_2 \in H$ and $r \in \Bbbk$.

liibel=(SF0),leftmiirgiin=* The map $\langle -, - \rangle$ is *conjugate-linear* in its second coordinate which means that

$$\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle \quad \text{and} \quad \langle v, rw \rangle = \overline{r} \langle v, w \rangle$$

for all $v, w, v_1, v_2 \in H$ and $r \in \Bbbk$.

A sesquilinear form $\langle -, - \rangle$ on H is called a *hermitian form* if the following holds true:

(SF3) The map $\langle -, - \rangle$ is *conjugate symmetric* which means that

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

for all $v, w \in H$.

One says that a hermitian form $\langle -, - \rangle$ on H is *positive semidefinite* if

(SF4) $\langle v, v \rangle \geq 0$ for all $v \in H$,

and *positive definite* if it is positive semidefinite and if the following non-degeneracy condition is fulfilled:

(SF5) For all $v \in H$ the relation $\langle v, v \rangle = 0$ holds true if and only if $v = 0$.

By an *inner product* on H one understands a positive definite hermitian form defined on H. A $\Bbbk$-vector space H together with an inner product $\langle -, - \rangle : H \times H \to \Bbbk$ is called an *inner product space* or a *pre-Hilbert space*.

remark[]If the underlying ground field $\Bbbk$ coincides with the field of real numbers, a sesquilinear form is by definition the same as a bilinear form, and a hermitian form the same as a symmetric bilinear form.

examples

1.2 Remark **1.3 Examples** []

1. The vector space $\mathbb{R}^n$ with the *euclidean inner product* $\langle v, w\rangle = \sum_{i=1}^n v_i w_i$ for $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ is a real inner product space.
2. The vector space $\mathbb{C}^n$ with the sesquilinear form $\langle v, w\rangle = \sum_{i=1}^n v_i \overline{w}_i$, $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ is a complex inner product space.
3. $L^2(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is measurable and } \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty\right\}$ with
$$\langle f, g\rangle = \int_{\mathbb{R}^n} f\overline{g} d\mu$$
an inner product when modding out $L^2(\mathbb{R}^n)$ by the kernel.
4. The set $l^2 = \{(z_i \in \mathbb{C}^\omega \mid \sum_{i=1}^\infty |z_i|^2 < \infty\}$ of square summable sums of complex numbers is a complex inner product space with inner product
$$\langle z, w\rangle = \sum_{i=1}^\infty z_i \overline{w}_i \ .$$

Observe that an inner product defined on a vector spaces defines a norm by
$$\|\psi\| = \sqrt{\langle \psi, \psi\rangle}$$
Pulling out scalars is trivial and is left as an exercise. Consider the following:

proposition[Cauchy-Schwartz inequality] For any inner product space $(\mathcal{H}, \langle \cdot, \cdot\rangle)$
$$|\langle v, w\rangle| \leq \|v\|\|w\|$$

proof

1.4 Proposition *Proof.* []Notice if either v or w are 0, the proof is trivial, hence we can assume both are nonzero. Setting
$$\lambda = -\frac{\langle v, w\rangle}{\|v\|^2}$$
observe
$$\begin{aligned} 0 \leq \|\lambda v + w\| &= (\langle \lambda v + w, \lambda v + w\rangle)^{1/2} \\ &= (\lambda^2\langle v, v\rangle + \lambda\langle v, w\rangle + \overline{\lambda}\langle w, v\rangle + \langle w, w\rangle)^{1/2} \end{aligned}$$
and
$$\begin{aligned} \|\lambda v + w\|^2 &= |\lambda|^2\|v\|^2 + 2\mathrm{Re}\lambda\langle v, w\rangle + \|w\|^2 \\ &= \frac{|\langle v, w\rangle|^2}{\|v\|^2} - 2\frac{|\langle v, w\rangle|^2}{\|v\|^2} + \|w\|^2 \\ &= \|w\|^2 - \frac{|\langle v, w\rangle|^2}{\|v\|^2} \end{aligned}$$

Hence

$$\|v\|^2\|\lambda v + w\|^2 + |\langle v, w\rangle|^2 = \|v\|^2\|w\|^2 \geq |\langle v, w\rangle|^2$$

Therefore the proposition gives us the triangle inequality:

proposition[]

proof

1.5 Proposition *Proof.* []

$$\begin{aligned}
\|\phi + \psi\| &= (\langle \psi, \psi\rangle + \langle \psi, \phi\rangle + \langle \phi, \psi\rangle + \langle \phi, \phi\rangle)^{1/2} \\
\|\phi + \psi\|^2 &= \|\psi\|^2 + \|\phi\|^2 + |\langle \phi, \psi\rangle| + |\langle \psi, \phi\rangle| \\
&\leq \|\psi\|^2 + \|\phi\|^2 + 2\|\psi\|\|\phi\|
\end{aligned}$$

definition[]A Hilbert space is a Banach space with an inner product where the norm on this space is induced from the inner product.

example

1.6 Definition 1.7 Example []

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. $L^2(\mathbb{R}^{kn})$
4. l^2

where $\mathbb{R}^{kn}$ denotes the space of k-particles in n-space.

theorem[] Let H be a normed $\Bbbk$-vector space. Then, the norm $\|\cdot\|$ comes from an inner product if and only if the the following *parallelogram identity* holds true:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

proof

1.8 Theorem *Proof.* []For the forward direction, observe

$$\begin{aligned}
\langle x + y, x + y\rangle + \langle x - y, x - y\rangle &= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&\quad + \langle x, y\rangle + \langle y, x\rangle - \langle x, y\rangle - \langle y, x\rangle
\end{aligned}$$

The parallelogram identity gives the following two identities:

lemma[]

$$\begin{aligned}
\frac{1}{2}(\|x + y\|^2 + \|x - y\|^2) &= \|x\|^2 + \|y\|^2 \\
2\|x\|^2 &= \|x + y\|^2 + \|x - y\|^2 - 2\|y\|^2
\end{aligned}$$

One of the issues with infinite-dimensional analysis is that a subspace of a Banach space may not have a complement. Fortunately, the situation in Hilbert space theory is not so grim. In order to prove that every subspace of a Hilbert space admits an orthogonal complement, we need a couple propositions.

proposition

1.9 Lemma 1.10 Proposition []The inner product is continuous; that is, for any convergent sequences of vectors $v_n \to v$ and $w_n \to w$

$$\lim_{n\to\infty} \langle v_n, w_n \rangle = \langle v, w \rangle$$

proof[]Consider

$$\begin{aligned} |\langle v_n, w_n \rangle - \langle v, w \rangle| &= |\langle v_n, w_n \rangle - \langle v_n, w \rangle + \langle v_n, w \rangle - \langle v, w \rangle| \\ &\leq |\langle v_n, w_n \rangle - \langle v_n, w \rangle| + |\langle v_n, w \rangle - \langle v, w \rangle| \\ &= |\langle v_n, w_n - w \rangle| + |\langle v_n - v, w \rangle| \\ &\leq \|v_n\| \|w_n - w\| + \|w\| \|v_n - v\| \end{aligned}$$

Since all terms are nonnegative and both $v_n \to v$, $w_n \to w$, the desired limit exists.

theorem

Proof. **1.11 Theorem** []Every closed convex nonempty subset E of a Hilbert space $\mathcal{H}$ has a unique element of minimal norm.

proof[]Let $d = \inf\{\|v\| : v \in E\}$ which is a non-negative real number. We claim there exists a unique $v_0 \in E$ where $\|v_0\| = d$. For uniqueness, consider two vectors v_0, v_1 satisfying the desired property, and let $v = \frac{1}{2}(v_0 + v_1)$ be the midpoint. Then

$$\|v\| = \frac{1}{2}\|v_0 + v_1\| \leq \frac{1}{2}(\|v_0\| + \|v_1\|) = d$$

Since each vector satisfies the minimality of the norm, we have

$$\|v\|^2 + \left\| \frac{1}{2}(v_0 - v_1) \right\| = 2\left\| \frac{v_0}{2} \right\|^2 + 2\left\| \frac{v_1}{2} \right\|^2 = d^2$$

hence

$$\left| \frac{v_0}{2} - \frac{v_1}{2} \right| \leq d^2 - \|v\|^2 = 0$$

Proving $v_0 = v_1$.