Complex Analysis: Homework 8

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Exercise 1.

Let $R = \frac{P}{Q}$ with polynomials P and Q such that $Q(x) \neq 0$ for every $x \in \mathbb{R}$ and such that $\deg(Q) \geq \deg(P) + 1$. Show that $\lim\limits_{r\to\infty} \displaystyle\int_{-r}^{r} R(x)dx$ exists and express this limit in terms of the residues of R. Don't forget to formulate your assertion.

Solution: The assertion is that $\int_{-\infty}^{\infty} R(x)dx$ exists if the degree of Q is at least two units higher than the degree of P (I believe that there's a typo in the exercise). The result of this integral in terms of the residues is

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}\, R(z).$$

Let $\gamma_1(t) = -r(1-t) + rt$ and $\gamma_2(t) = re^{-t\pi i}$, both curves defined for $t \in [0,1]$. γ_1 is the line that goes from $-r$ to r and γ_2 is the semicircle that starts at r and ends at $-r$. It follows that $\gamma = \gamma_1 + \gamma_2$ is a closed curve, and if r is big enough for γ to contain all the poles of R, it follows by the Argument Principle that

$$\int_{\gamma} R(z)dz = \int_{-r}^{r} R(x)dx + \int_{\gamma_2} R(z)dz = 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}\, R(z).$$

Finally, if $\deg(Q) - \deg(P) \geq 2$, then $|R(z)|$ behaves asymptotically as $|z|^{-2}$, and thus, there exists $K > 0$ such that $|R(z)| \leq K|z|^{-2} = Kr^{-2}$ for $z \in \gamma_2$.

$$\left|\int_{\gamma_2} R(z)dz\right| \leq \int_{\gamma_2} |R(z)|dz \leq \underbrace{\pi r}_{\text{arc length}} \cdot \underbrace{Kr^{-2}}_{\text{upper bound}} \overset{r\to\infty}{\to} 0.$$

This proves the previous assertion.

Exercise 2.

Partial fractions of $(\sin \pi a)^{-2}$.

Let $n \in \mathbb{N}$ and let γ_n be the border of a rectangle with corners $n+\frac{1}{2}+in,\ -n-\frac{1}{2}+in,\ -n-\frac{1}{2}-in,\ n+\frac{1}{2}-in$. Let $a \in \mathbb{C}\backslash\mathbb{Z}$.

(a) Demonstrate that $\displaystyle\lim_{n\to\infty} \int_{\gamma_n} \frac{\pi \cot(\pi z)}{(z+a)^2} dz = 0.$

(b) Demonstrate that $\displaystyle\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2}.$

Exercise 3.

(a) Let $U \subset \mathbb{C}$ be a region, $g : U \to \mathbb{C}$ holomorphic, f meromorphic in U with zeros in $z_1, \ldots, z_n$ and poles in $p_1, \ldots, p_k$. Let γ be a closed null-homotopic in U and suppose that $\gamma \cap \{z_1, \ldots, z_n, p_1, \ldots, p_k\} = \emptyset$. Show that

$$\frac{1}{2\pi i} \int_\gamma g(z) \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{n} g(z_j) \mathrm{ord}(f, z_j) \mathrm{ind}_\gamma(z_j) - \sum_{j=1}^{k} g(p_j) \mathrm{ord}(f, p_j) \mathrm{ind}_\gamma(p_j).$$

(b) Let $U \subseteq \mathbb{C}$ be open, let $p \in \mathbb{C},\ R > 0$ such that $\overline{B_R(p)} \subset U$. Let $f : U \to \mathbb{C}$ be holomorphic and suppose that $f|_{B_R(p)}$ is injective. Let $V := \{f(z)\ :\ z \in B_R(p)\}$. Then, $f^{-1} : V \to B_R(p)$ is well defined. Show that

$$f^{-1}(q) = \frac{1}{2\pi i} \int_{\partial B_R(p)} \frac{z f'(z)}{f(z) - q} dz, \qquad q \in V.$$

Solution Item (a)

Remember that if for some function p,

$$f(z) = (z - z_0)^a p(z),$$

then,

$$\frac{f'(z)}{f(z)} = \frac{(z - z_0)^{a-1} p(z) + (z - z_0)^a p'(z)}{(z - z_0)^a p(z)} = \frac{a}{z - z_0} + \frac{p'(z)}{p(z)}.$$

Similarly, if for some function p,

$$f(z) = (z - p_0)^{-b} p(z),$$

then,
$$\frac{f'(z)}{f(z)} = \frac{-b(z-p_0)^{-b-1}p(z) + (z-p_0)^{-b}p'(z)}{(z-p_0)^{-b}p(z)} = -\frac{b}{z-p_0} + \frac{p'(z)}{p(z)}.$$

Now, we apply a similar idea as the proof of the Argument Principle, we can find an holomorphic function h with no zeroes in U that satisfies

$$f(z) = \frac{\prod_{i=1}^{n}(z-z_i)^{a_i}}{\prod_{j=1}^{k}(z-p_j)^{b_i}} \cdot h(z).$$

Then, it follows that
$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{a_i}{z-z_i} - \sum_{j=1}^{k} \frac{b_j}{z-p_j} + \frac{h'(z)}{h(z)},$$

so by multiplying g we obtain,

$$g(z)\frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{g(z)a_i}{z-z_i} - \sum_{j=1}^{k} \frac{g(z)b_j}{z-p_j} + g(z)\frac{h'(z)}{h(z)}.$$

Note that the function $g \cdot \frac{h'}{h}$ is holomorphic so $\int_\gamma g \cdot \frac{h'}{h} dz = 0$, and by Cauchy integral formula $\frac{1}{2\pi i}\int_\gamma \frac{g}{z-a} dz = g(a)\mathrm{ind}_\gamma(a)$. So finally,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz &= \sum_{i=1}^{n} a_i \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-z_i}dz - \sum_{j=1}^{k} b_j \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z-p_j}dz + \frac{1}{2\pi i}\int_\gamma g(z)\frac{h'(z)}{h(z)}dz \\ &= \sum_{i=1}^{n} a_i\ g(z_i)\mathrm{ind}_\gamma(z_i) - \sum_{j=1}^{k} b_j\ g(z_i)\mathrm{ind}_\gamma(z_i) + 0,\end{aligned}$$

and a_i, b_j are the multiplicities and orders of the respective zeroes and poles of f.

Solution Item (b)

For every $q \in V$ there exists a unique $z_q \in B_R(p)$ such that $f(z_q) = q$, so there exists a holomorphic function h such that $h(z) \neq 0$ for every $z \in B_R(p)$ and,

$$f(z) - q = (z - z_q)h(z).$$

From the previous item, we know that since

- $z \mapsto f(z) - q$ has no poles and only one zero at $z = z_q$ with multiplicity 1,
- $\partial B_R(p)$ surrounds q exactly once.

it follows that,

$$\frac{1}{2\pi i}\int_{\partial B_R(p)} g(z)\frac{(f(z)-q)'}{f(z)-q}dz = \frac{1}{2\pi i}\int_{\partial B_R(p)} g(z)\frac{f'(z)}{f(z)-q}dz = g(z_q)$$

Finally, by letting $g(z)=z$ we obtain

$$\frac{1}{2\pi i}\int_{\partial B_R(p)} g(z)\frac{f'(z)}{f(z)-q}dz = z_q = f^{-1}(q)$$

Exercise 4.

Let $\gamma = \partial(B_2(0\cap\{z\in\mathbb{C} \ : \ \mathrm{Im}(z)\geq 0\}))$. Calculate the following integrals

$$\text{(a)} \int_{\partial B_2(0)} \frac{1}{(\sin z)^2\cos z}dz, \qquad \text{(b)} \int_\gamma \frac{e^{\pi z}}{z^2+1}dz.$$

Solution Item (b)

Using partial fractions, we obtain

$$\frac{1}{1+z^2} = \frac{i}{2(z+i)} - \frac{i}{2(z-i)}.$$

Then,

$$\begin{aligned}
\int_\gamma \frac{e^{\pi z}}{z^2+1}dz &= \frac{2\pi i^2}{2}\int_\gamma \frac{e^{\pi z}}{z+i}dz - \frac{2\pi i^2}{2}\int_\gamma \frac{e^{\pi z}}{z-i}dz \\
&= \pi\int_\gamma \frac{e^{\pi z}}{z-i} - \pi\int_\gamma \frac{e^{\pi z}}{z+i} \\
&= \pi e^{\pi i} - 0 = -\pi.
\end{aligned}$$

The last equation follows from Cauchy Integral Formula. Note that the semicircle γ surrounds i once but doesn't surround $-i$. Therefore,

$$\int_\gamma \frac{e^{\pi z}}{z-i} = e^{\pi i}, \qquad \int_\gamma \frac{e^{\pi z}}{z+i} = 0$$

Exercise 5.

Determine all the values that $\displaystyle\int_\gamma \frac{1}{1+z^2}dz$ can take if γ is a closed path in $\mathbb{C}\backslash\{\pm i\}$.

Solution:

Using partial fractions, we obtain

$$\frac{1}{1+z^2} = \frac{i}{2(z+i)} - \frac{i}{2(z-i)}.$$

Then, by some nice integral formula,

$$\begin{aligned}\int_\gamma \frac{1}{1+z^2}dz &= 2\pi i\left(\frac{i}{2}\mathrm{ind}_\gamma(-i) - \frac{i}{2}\mathrm{ind}_\gamma(i)\right)\\ &= \pi\mathrm{ind}_\gamma(i) - \pi\mathrm{ind}_\gamma(-i) \in \pi\mathbb{Z}.\end{aligned}$$