Complex Analysis: Homework 4

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September 2, 2024

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Exercise 1.

Let f be an entire function and let $\zeta = e^{2\pi i/n}$ for some $n \in \mathbb{N}$. Suppose that $f(\zeta z) = f(z)$ for every $z \in \mathbb{C}$. Show that there exists an entire function g such that $f(z) = g(z^n)$ for every $z \in \mathbb{C}$.

Exercise 2.

(a) Let $U \subset \mathbb{C}$ be a region and $K \subset U$ a compact subset with non empty interior K°. Let $f : U \to \mathbb{C}$ be an holomorphic function with $|f|$ constant of the boundary of K. Show that f is constant or has a zero in K°.

(b) Let $U \subset \mathbb{C}$, $z_0 \in U$, $\varepsilon > 0$ such that the closed ball $\overline{B_\varepsilon(z_0)}$ is a subset of U. Let $f : U \to \mathbb{C}$ be holomorphic with $|f(z_0)| < \min\{|f(z)| \ : \ |z - z_0| = \varepsilon\}$. Show that f has a zero in $B_\varepsilon(z_0)$.

Exercise 3.

Let $U \subset \mathbb{C}$ be open and connected, $f : U \to X$ be a non constant holomorphic function and $N := \{z \in \mathbb{C} \ : \ f(z) = 0\}$. Show that N is closed and discrete in U.

Solution:

To see that N is closed, take $w \in \mathbb{C}$ such that $f(w) = \omega \neq 0$. Assuming that X is a vector space with some metric, there exists for some $\varepsilon > 0$, an open ball $B_\varepsilon(\omega)$ that doesn't contain 0. Then, by continuity of f, $f^{-1}(B_\varepsilon(\omega)) \subset \mathbb{C} \backslash N$ is an open set that contains w.

Now, assume that N has some limit point z_0. Also let $g : z \mapsto 0$, and note that $f = g$ on N. Using identity's theorem we conclude that there exists $\varepsilon > 0$ such that $f(z) = g(z) = 0$ for every $z \in B_\varepsilon(z_0)$. Therefore, Since f is entire, all of the Taylor series coefficients of f must be 0 to coincide with the Taylor series of g, contradicting the fact that f is not constant.

Exercise 4.

Let $U \subset \mathbb{C}$ be open and bounded, without isolated points of the frontier, and let $M \subset U$ be a subset without accumulation points in U. Show that every biholomorphic function $f : U \backslash M \to U \backslash M$ has a biholomorphic extension $g : U \to U$.

Exercise 5.

Let $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ be a power series with convergence radius $R \in (0, \infty)$. Show that f has at least a singular point in the frontier of the convergence disk.