Complex Analysis: Homework 4

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September 4, 2024

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Exercise 1.

Let f be an entire function and let $\zeta = e^{2\pi i/n}$ for some $n \in \mathbb{N}$. Suppose that $f(\zeta z) = f(z)$ for every $z \in \mathbb{C}$. Show that there exists an entire function g such that $f(z) = g(z^n)$ for every $z \in \mathbb{C}$.

Solution:

Since f is entire, there's exists $\{c_k\}$ such that

$$f(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Then, since $f(\zeta z) = f(z)$, we have that

$$\sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} c_k \zeta^k z^k.$$

When k is not a multiple of n, we have that $\zeta^k \neq 1$. Therefore, by uniqueness of the power series expansion,

$$c_k = \zeta^k c_k \implies c_k = 0, \; k \not\equiv 0 \mod n.$$

Now, we can rewrite the series as follows

$$f(z) = \sum_{k=0}^{\infty} c_{nk} z^{nk} = \sum_{k=0}^{\infty} c_{nk} (z^n)^k.$$

For $g(z) = \sum_{k=0}^{\infty} c_{nk} z^k$, $f(z) = g(z^n)$. To show that g is absolutely convergent in all $\mathbb{C}$ note that for every $R \in \mathbb{R}^+$ and $|z| < R$,

$$|f(z)| \leq \sum_{k=0}^{\infty} c_{nk} |z|^{nk} < \sum_{k=0}^{\infty} c_{nk} R^{nk} < \infty.$$

Therefore, for every $|z| < R^n \in (0, \infty)$,

$$|g(z)| \leq \sum_{k=0}^{\infty} c_{nk}|z|^k < \sum_{k=0}^{\infty} c_{nk}R^{nk} < \infty.$$

That shows that g exists and is entire everywhere.

Exercise 2.

(a) Let $U \subset \mathbb{C}$ be a region and $K \subset U$ a compact subset with non empty interior K°. Let $f : U \to \mathbb{C}$ be an holomorphic function with $|f|$ constant of the boundary of K. Show that f is constant or has a zero in K°.

(b) Let $U \subset \mathbb{C}$, $z_0 \in U$, $\varepsilon > 0$ such that the closed ball $\overline{B_\varepsilon(z_0)}$ is a subset of U. Let $f : U \to \mathbb{C}$ be holomorphic with $|f(z_0)| < \min\{|f(z)| \ : \ |z - z_0| = \varepsilon\}$. Show that f has a zero in $B_\varepsilon(z_0)$.

Solution Part (a)

In the first place, if $|f| = 0$ on ∂K, then for $g : z \mapsto 0$, $f|_{\partial K} = g|_{\partial K}$. Since ∂K has accumulation points (otherwise U is not open), using identity's theorem we conclude that $f = g$ on U.

Now, assume that f doesn't have any zero on K° and that $|f| \neq 0$ on the boundary of K. Then, $g = 1/f$ is an holomorphic function defined in K, and since K is compact, $|f|$ has a maximum in K. We have two possible cases

- The maximum of $|f|$ is attained at K°, so by maximum principle f is constant.
- The maximum of $|f|$ is attained at ∂K, so the minimum of $1/|f|$ is attained at ∂K, and thus, $1/|f|$ attains its maximum at K°. Again, by maximum principle $1/f$ is constant and so f too.

Solution Part (b)

Assume that f doesn't have a zero in $B_\varepsilon(z_0)$, we want to prove that $|f(z_0)| \geq \min_{\partial B_\varepsilon(z_0)} |f(z)|$.

- If there's a zero in $\partial B_\varepsilon(z_0)$, then we won, so suppose that $\min_{\partial B_\varepsilon(z_0)} |f(z)| > 0$.
- If f is constant, then we also won. So assume it's not.

Now, $1/f$ is an holomorphic function defined at $\overline{B_\varepsilon(z_0)}$. Since $1/f$ is not constant, by maximum principle, it attains it maximum modulus at $\partial B_\varepsilon(z_0)$. Therefore,

$$1/|f(z_0)| \leq \max_{\partial B_\varepsilon} |1/f(z)|$$

$$\implies |f(z_0)| \geq \min_{\partial B_\varepsilon} |f(z)|$$

as we intended.

Exercise 3.

Let $U \subset \mathbb{C}$ be open and connected, $f : U \to X$ be a non constant holomorphic function and $N := \{z \in \mathbb{C} \ : \ f(z) = 0\}$. Show that N is closed and discrete in U.

Solution:

To see that $w \in \mathbb{C} \backslash N$ is open, take $w \in \mathbb{C}$ such that $f(w) = \omega \neq 0$. Then, there exists for some $\varepsilon > 0$, an open ball $B_\varepsilon(\omega) \subset X$ that doesn't contain 0. Finally, using the continuity of f, $f^{-1}(B_\varepsilon(\omega))$ is an open set that contains w which is inside $\mathbb{C}\backslash N$.

$$N \text{ has no accumulation points} \iff N \text{ is closed and } N \text{ is discrete.}$$

Now, assume that N has some accumulation point z_0. Also let $g : z \mapsto 0$, and note that $f = g$ on N. Using identity's theorem we conclude that $f(z) = g(z) = 0$ for every $z \in U$, contradicting the fact that f is not constant.

Exercise 4.

Let $U \subset \mathbb{C}$ be open and bounded, without isolated points of the frontier, and let $M \subset U$ be a subset without accumulation points in U. Show that every biholomorphic function $f : U\backslash M \to U\backslash M$ has a biholomorphic extension $g : U \to U$.

Solution: Let $g = f^{-1}$.

Since M is discrete in U, for every $w \in M$, there exists ε_w such that $M \cap B_{\varepsilon_w}(w) = \{w\}$ and $B_{\varepsilon_w}(w) \subset U$. Since f and g images are bounded ($U\backslash M$ is bounded), then f, g are bounded on the set $B^\bullet_{\varepsilon_w}(w) = B_{\varepsilon_w}(w)\backslash\{w\}$. Therefore, using Riemann's removable singularity criterion, w is a removable singularity for every $w \in M$ for both f and f^{-1}.

Let $\tilde{f}$ and $\tilde{g}$ be the extensions for f and g respectively. Let $h_1 = \tilde{f} \circ \tilde{g}$ and $h_2 = \tilde{g} \circ \tilde{f}$. For $z \in U\backslash M$,

$$h_1(z) = f \circ g(z) = z = g \circ f(z) = h_2(z).$$

Now, to prove that h_1, h_2 are defined for every $z \in U$, we must prove that $\tilde{f}(w), \tilde{g}(w) \in U$ for $w \in M$. For the sake of contradiction assume it's not ($\exists w \in M,\ \tilde{f}(w) \notin U$). We know that

$$\tilde{f}(B^{\bullet}_{\varepsilon_w}(w)) = f(B^{\bullet}_{\varepsilon_w}(w)) \subset U \backslash M \subset U.$$

Also, $\overline{B^{\bullet}_{\varepsilon_w}(w)} = \overline{B_{\varepsilon_w}(w)}$, so by continuity of f

$$\implies \tilde{f}(B_{\varepsilon_w}(w)) \subset \tilde{f}(\overline{B_{\varepsilon_w}(w)}) = \tilde{f}(\overline{B^{\bullet}_{\varepsilon_w}(w)}) \subset \overline{f(B^{\bullet}_{\varepsilon_w}(w))} \subset \overline{U}$$

Therefore, $\tilde{f}(w) \in \overline{U}$ and since we assumed that $\tilde{f}(w) \notin U$, it follows that $f(w) \in \partial U$. However, note that

1. First, $\tilde{f}(B_{\varepsilon_w}(w)) = f(B^{\bullet}_{\varepsilon_w}(w)) \dot{\cup} \{\tilde{f}(w)\}$ (disjoint union). Therefore, since U is open and $f(B^{\bullet}_{\varepsilon_w}(w)) \subset U = U^{\circ}$, it follows that $f(B^{\bullet}_{\varepsilon_w}(w)) \cap \partial U = \emptyset$. Thus, $\partial U \cap \tilde{f}(B_{\varepsilon_w}(w)) = \{\tilde{f}(w)\}$

2. By Open Mapping Theorem, $\tilde{f}(B_{\varepsilon_w}(w))$ is an open set. Therefore, $\{\tilde{f}(w)\}$ is an isolated point of the boundary of U (contradiction).

So $\tilde{f}(w) \in U$ for every $w \in U$. The same argument applies for $\tilde{g}$, so h_1 and h_2 are defined for every $z \in U$. By identity theorem, it follows that $h_1(z) = z = h_2(z)$ for every $z \in U$ implying that $\overline{f}$ and $\overline{g}$ are each other's inverse functions.

Exercise 5.

Let $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ be a power series with convergence radius $R \in (0, \infty)$. Show that f has at least a singular point in the frontier of the convergence disk.

Solution:

For the sake of contradiction assume that f has no singularities at $\partial B_R(z_0)$ and can be extended to an holomorphic function $\tilde{f}$ at $\overline{B_R(z_0)}$. Then, for every $w \in \partial B_R(z_0)$, there exists ε_w such that there exists a power series $g_w(z) = \sum_{j=0}^{\infty} a_{w,j}(z - w)^j$ around $B_{\varepsilon_w}(w)$ that coincides with $\tilde{f}$ in $\overline{B_R(z_0)} \cap B_{\varepsilon_w}(w)$.

Since $\partial B_R(z_0)$ is compact, there exists

$$\varepsilon = \frac{\min\{\varepsilon_w \ : \ w \in \partial B_R(z_0)\}}{2} > 0.$$

The set $\overline{B_{R+\varepsilon}(z_0)} \backslash B_R(z_0)$ is compact and it is covered by the open balls $\{B_{\epsilon_w}(w) \ : \ w \in \partial B_R(z_0)\}$. Therefore, (by the definition of compact set) there exists $w_1, \dots, w_n$ for $n \in \mathbb{N}$ such that $\overline{B_{R+\varepsilon}(z_0)} \backslash B_R(z_0)$ is covered by $\{B_{\epsilon_{w_k}}(w_k) \ : \ k \leq n\}$

Using Identity's theorem $\tilde{f}$ can again be extended to a function $\tilde{f}_1$ that coincides with g_{w_1} in $\overline{B_R(z_0)} \cup B_{\varepsilon_{w_1}}(w_1)$. Then, we recursively apply Identity's theorem to extend $\tilde{f}_k$ to a function $\tilde{f}_{k+1}$ that coincides with $g_{w_{k+1}}$ in $\overline{B_R(z_0)} \cup \bigcup_{j=1}^{k+1} B_{\varepsilon_{w_j}}(w_j)$ for every $k < n$.

Finally, $\tilde{f}_n$ extends f to the ball $B_{R+\varepsilon}(z_0) \subset \overline{B_R(z_0)} \cup \bigcup_{j=1}^n B_{\varepsilon_{w_j}}(w_j)$. However, this is a contradiction to the fact that R is the greatest number for which $f(z)$ converges for every $|z - z_0| < R$.