Complex Analysis: Homework 10

Martín Prado

October 23, 2024

Universidad de los Andes – Bogotá Colombia

Let $B_j := B_{r_j}(z_j) \quad (j = 0, 1, \ldots, n)$ be open disks with $z_{j-1}, z_j \in B_{j-1} \cap B_j$ for all $j = 1, \ldots, n$. Then, $(B_0, B_1, \ldots, B_n)$ is called a *chain of disks.* If $f_j \colon B_j \to \mathbb{C}$ are holomorphic functions such that $f_{j-1} = f_j$ on $B_{j-1} \cap B_j$ for all $j = 1, \ldots, n$, then f_n is called *the analytic extension of f_0 along the chain of disks $B_0, \ldots, B_n$.*

Exercise 1.

Let $\mathcal{B} = (B_0, \ldots, B_n)$ be a chain of disks and let $f_0 : B_0 \to \mathbb{C}$ be an analytic function. Suppose that f_0' has an analytic extension along $\mathcal{B}$. Prove that f_0 also has an analytic extension along $\mathcal{B}$.

Solution

According to the previous definition, let g_j be the analytic continuation of f_0' along the chain $B_0, \ldots, B_j$ ($g_0 = f_0'$) until we have g_n with is the extension along $\mathcal{B}$.

Now, let $w \in B_0$, define the holomorphic function $h_0(w) = \int_{\gamma_w} g_0(z)dz$ for a smooth path γ_w that starts at z_0 and ends at w.

This function is well defined because if we take two different paths $\gamma_w^{(1)}$ and $\gamma_w^{(2)}$ that start at z_0 and end at w, then $\Gamma = \gamma_w^{(1)} + (-\gamma_w^{(2)})$ is a closed path in a simply connected domain B_0. Therefore, by *Cauchy Integral Formula*,

$$\int_\Gamma g_0(z)dz = 0 \implies \int_{\gamma_w^{(1)}} g_0(z)dz = \int_{\gamma_w^{(2)}} g_0(z)dz$$

Then, note that by the *Fundamental Theorem of Calculus*, $f_0'(w) = h_0'(w)$ for any $w \in B_0$, and thus, f_0 and h_0 differ only by a constant:

$$f_0(w) = h_0(w) - h_0(z_0) + f_0(z_0).$$

For $w \in B_1$, define $h_1(w) = \int_{\gamma_w} g_1(z)dz$ for any smooth path that starts at z_1 and ends at w. For every $w \in B_0 \cap B_1$, $g_1(w) = g_0(w)$, and thus, it follows that $f_0'(w) = h_1'(w)$ so f_0 differs from h_1 only by a constant. Then, define for $w \in B_1$

$$f_1(w) = h_1(w) - h_1(z_1) + f_0(z_1),$$

which coincides with $f_0(w)$ for $w \in B_0 \cap B_1$.

Recursively, define for $w \in B_j$, $h_j(w) = \int_{\gamma_w} g_j(z)dz$ for any smooth path that starts at z_j and ends at w, to then define

$$f_j(w) = h_j(w) - h_j(z_j) + f_{j-1}(z_j).$$

Applying a similar argument to before, we can prove that f_j is well defined (using *Cauchy Integral Formula*) and that f_j coincides with f_{j-1} at $B_{j-1} \cap B_j$ (using *Fundamental Theorem of Calculus*). This gives us the analytic extension f_n of f_0 along $\mathcal{B}$ we're looking for.

Exercise 2.

Let $U = B_1(0)$ and

$$f : U \to \mathbb{C}, \quad f(z) = \sum_{n=1}^{\infty} 2^{-n^2} z^{2^n}.$$

Prove that f has no analytic extension to any open set G with $G \supsetneq U$.

Hint: Prove that for every $n \in \mathbb{N}$ there exists a polynomial P_n such that

$$f\left(\mathrm{e}^{2\pi \mathrm{i}/2^n} z\right) = P_n(z) + f(z).$$

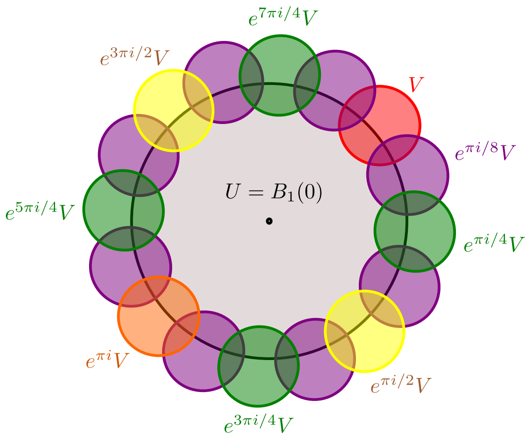
Solution

Assume, for the sake of contradiction, that there exists an open set G with $G \supsetneq U$ for which f can be analytically extended to a function $\tilde{f}$. In the first place, for every $m \in \mathbb{N}$, $\exp\left(2\pi i \frac{2^n}{2^m}\right) = 1$ for every $n \geq m$. Thus,

$$\begin{aligned}
f\left(e^{2\pi i/2^m} z\right) &= \sum_{n=1}^{\infty} 2^{-n^2} \exp\left(2\pi i \frac{2^n}{2^m}\right) z^{2^n} \\
&= \sum_{n=m}^{\infty} 2^{-n^2} z^{2^n} + \sum_{n=1}^{m-1} 2^{-n^2} \exp\left(2\pi i \frac{2^n}{2^m}\right) z^{2^n} \\
&= \sum_{n=m}^{\infty} 2^{-n^2} z^{2^n} + \sum_{n=1}^{m-1} 2^{-n^2} \left(z^{2^n} - z^{2^n} + \exp\left(2\pi i \frac{2^n}{2^m}\right) z^{2^n}\right) \\
&= \underbrace{\sum_{n=1}^{\infty} 2^{-n^2} z^{2^n}}_{f(z)} + \underbrace{\sum_{n=1}^{m-1} 2^{-n^2} z^{2^n} \left(\exp\left(2\pi i \frac{2^n}{2^m}\right) - 1\right)}_{P_m(z)}
\end{aligned}$$

Now, since $\tilde{f}$ coincides with f on U, it follows (uniqueness of power series expansion) that for every z in G,

$$\tilde{f}(e^{2\pi i/2^m} z) = \tilde{f}(z) + P_m(z)$$



Then, this implies that if $\tilde{f}(z)$ is defined, then $\tilde{f}(e^{\theta i} z)$ can be defined for any rotation of z by $\theta = 2\pi \frac{k}{2^m}$ radians for $k \in \{1, \ldots, 2^m - 1\}$.

If there exists an open set $V \subset G$ such that $V \cap \partial U \neq \emptyset$, we can find a suitable m for which the union of the rotations by $2\pi \frac{k}{2^m}$ can cover $\partial U = \{z \in \mathbb{C} : |z| = 1\}$. Look the picture on the left for reference.

$$W = \bigcup_{k=0}^{2^m-1} e^{2\pi \frac{k}{2^m}} V \supseteq \partial B_1(0).$$

Finally, this implies that since W is open and $1 \in W$, there exists $\varepsilon > 0$ such that $G \supseteq B_{1+\varepsilon}(0)$. Since the power series expansion of $\tilde{f}$ is the same for f, this would imply that the radius of convergence of the power series of $\tilde{f}$ is strictly greater than 1. But this is a contradiction to the fact that the ratio test gives us radius of convergence equal to 1 for f.

Exercise 3.

Let $U = B_1(0)$. Find an analytic continuation to the largest possible region for

$$f : U \to \mathbb{C}, \quad f(z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^n.$$

Hint: Consider $f(w^2)$.

Solution

For $|w| < 1$ we have absolute convergence, and thus, using the geometric series,

$$\begin{aligned} f(z) = f(w^2) &= \sum_{n=0}^{\infty} (-1)^n (2n+1) w^{2n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{d}{dz} w^{2n+1} \\ &= \frac{d}{dz} \left(\frac{w}{-1 - w^2} \right) \\ &= \frac{1 - w^2}{(1 + w^2)^2} = \frac{1 - z}{(1 + z)^2}. \end{aligned}$$

Now, the function f can be analytically extended to the function $g(z) = \frac{1-z}{(1+z)^2}$ on $z \in \mathbb{C} \backslash \{-1\}$. Note that f cannot be extended further, otherwise g could be extended to another function at $z = -1$ but that would contradict the fact that g has pole at $z = -1$ with order 2.

Exercise 4.

Let X be a metric space. A sequence $(f_n)_{n \in \mathbb{N}}$ of functions $U \to \mathbb{C}$ is called continuously convergent if for every convergent sequence $(x_n)_{n \in \mathbb{N}} \subset X$, the limit $\lim_{n \to \infty} f_n(x_n)$ exists.

(a) Let X be a metric space and $(f_n)_{n \in \mathbb{N}}$ a sequence of functions in X that converges continuously. Prove that $f : X \to \mathbb{C}, \quad f(x) = \lim_{n \to \infty} f_n(x_n)$ is well-defined (i.e., it is independent of the chosen sequence $(x_n)_{n \in \mathbb{N}}$) and that f is continuous (even if the f_n are not).

(b) Let $U \subseteq \mathbb{C}$ be open and $(f_n)_{n \in \mathbb{N}}$ a sequence of functions in U. Prove that the following are equivalent:

(i) $(f_n)_{n \in \mathbb{N}}$ converges compactly to a function $f \in C(U)$.

(ii) $(f_n)_{n \in \mathbb{N}}$ converges continuously.

In particular, a continuously converging sequence of holomorphic functions converges to a holomorphic function.

Solution Item (a)

Let $(x_n)_{n\in\mathbb{N}} \subseteq U$ be any sequence that converge to $x \in U$. Assume for the sake of contradiction that $f_n(x_n) \to c \neq f(x)$, then, define $y_n = (x_1, x, x_3, x, \ldots)$ and note that $y_n \to x$ so $f_n(y_n)$ has a limit. However, $f_{2n+1}(y_{2n+1}) \to c$ and $f_{2n}(y_{2n}) \to f(x)$ which contradicts the fact the limit of $f_n(y_n)$ exists. Therefore, for every $(x_n) \to x$ it must happen that $f_n(x_n) \to f(x)$, so f is well defined.

In order to prove that $f(x)$ is continuous we want to show that for every $(x_n) \to x$, $f(x_n) \to f(x)$. In the first place, note that for every $m \in \mathbb{N}$

$$|f(x) - f(x_n)| \leq |f(x) - f_m(x_n)| + |f_m(x_n) - f(x_n)|.$$

Then, fix $\varepsilon > 0$ and note that since $f_m \to f$ pointwise, it follows that for every $n \in \mathbb{N}$ there exists N_n such that

$$|f_m(x_n) - f(x_n)| < \varepsilon, \qquad \forall m \geq N_n.$$

So define a subsequence $(f_{m_n})_{n\in\mathbb{N}}$ such that $m_n \geq N_n$ and $m_n > m_{n-1}$. This way,

$$|f(x) - f(x_n)| < |f(x) - f_{m_n}(x_n)| + \varepsilon.$$

Then, define the following sequence

$$y_k = \begin{cases} x_1 & k \in [0, m_1] \\ x_n & k \in [m_n - m_{n-1}, m_n] \end{cases}$$

Since $y_n \to x$, it follows that $f_n(y_n) \to f(x)$. Then, $y_{m_n} = x_n$, so $f_{m_m}(x_n) = f_{m_n}(y_{m_n}) \to f(x)$. So for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $|f(x) - f_{m_m}(x_n)| < \varepsilon$ for $n \geq N$. So finally,

$$|f(x) - f(x_n)| < 2\varepsilon, \qquad \forall n \geq N$$

$$f(x_n) \to f(x), \quad \forall (x_n)_{n\in\mathbb{N}} \subset U$$

Solution Item (b)

(i) $\implies$ (ii): In the previous item we showed that if f_n converges continuously, then there exists a continuous function f, such that $f_n(x) \to f(x)$ for each $x \in U$. Now, for the sake of contradiction assume that there exists a compact set $K \subset U$ such that f_n doesn't converges uniformly to f when restricted to K. The statement for *Not uniformly convergent* is the following:

$$\exists \varepsilon > 0 \ : \ \forall M \in \mathbb{N}, \exists n \geq M, \exists x_n \in K \ : \ |f_n(x_n) - f(x_n)| \geq \varepsilon.$$

Fix $\varepsilon > 0$, $(x_n)_{n\in\mathbb{N}} \subset K$ from this definition and define a subsequence (f_{n_k}) that satisfies $|f_{n_k}(y_k) - f(y_k)| \geq \varepsilon$ for every $k \in \mathbb{N}$ (with $y_k = x_{n_k}$). Note that there exists a convergent subsequence $(y_{k_j}) \to y$ because K is compact. Then, from the following inequality,

$$\varepsilon \leq |f_{n_{k_j}}(y_{k_j}) - f(y_{k_j})| \leq |f_{n_{k_j}}(y_{k_j}) - f(y)| + |f(y) - f(y_{k_j})|,$$

note that from the fact that f_n converges continuously to f it follows that

- $|f_{n_{k_j}}(y_{k_j}) - f(y)| \to 0$ because we proved that $|f_j(z_j) - f(z)| \to 0$ whenever $z_j \to z$.

- $|f(y) - f(y_{k_j})| \to 0$ because f is continuous.

Therefore, $\varepsilon \leq |f_{n_{k_j}}(y_{k_j}) - f(y_{k_j})| \to 0$ is a contradiction.

(ii) $\implies$ (i): Now assume that for every compact set $K \subset U$, f_n converges uniformly to a continuous function f when restricted to K. We want to prove that for every sequence $(x_n) \to x$, $f_n(x_n)$ converges.

Note that for any compact set K that contains (x_n), the following inequality holds,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|.$$

- Since f_n converges uniformly to f, it follows that $|f_n(x_n) - f(x_n)| \to 0$ as $n \to \infty$.

- Since f is continuous $|f(x_n) - f(x)| \to 0$ as $n \to \infty$.

Therefore, $|f_n(x_n) - f(x)| \to 0$ concluding the proof.

Final Step

Let (f_n) a sequence of holomorphic functions that continuously converges to f. We know that f is continuous so we can integrate f. Now, take any closed curve γ and note that from item (b), f_n converges uniformly to f when restricted to γ (which is compact). Thus,

$$\int_\gamma f dz = \int_\gamma \lim_n f_n dz = \lim_n \int_\gamma f_n dz = 0.$$

Finally, by *Morera's theorem* we conclude that f is holomorphic.