Complex Analysis: Homework 5

Martín Prado

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Universidad de los Andes – Bogotá Colombia

Exercise 1.

Let $D := \{z \in \mathbb{C} : |z| < 1\}$. For the following function determine the type of singularity at 0. If it is a removable singularity, determine the continuous extension of the function; If it's a pole, determine the principal part of its Laurent series at 0; If it's an essential singularity, determine $\{f(z) \ : \ 0 < |z| < \varepsilon\}$ for $\varepsilon > 0$.

$$f : D \to \mathbb{C}, \quad f(z) = \frac{1}{1 - e^z} \qquad g : D \to \mathbb{C}, \quad g(z) = e^{\frac{1}{z}}$$

$$h : D \to \mathbb{C}, \quad h(z) = \cos\frac{1}{z} \qquad k : D \to \mathbb{C}, \quad k(z) = \frac{\sin z}{z}$$

Solution:

- For f, note that for $z \in \mathbb{R}$,

$$e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!} = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

so it follows that

$$\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots = 1.$$

Therefore, since 0 is a zero of multiplicity 1 in $e^z - 1$, it follows that 0 is a pole of order 1 in $f(z)$. The Laurent series of f has the following form

$$f(z) = \frac{\lambda}{z} + h(z) = \frac{\lambda}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

- For g, note that $e^{1/z}$ is not bounded on any punctured neighborhood around 0 because

$$\lim_{t \to 0^+} |e^{1/(t+0i)}| = \infty.$$

On the other hand, the limit of the module doesn't exactly diverges to infinity because

$$\lim_{t\to 0^+} |e^{1/(-t+0i)}| = 0.$$

Therefore, 0 is neither a removable singularity nor a pole, which implies that 0 is an essential singularity.

Note that the map $z \mapsto 1/z$ makes every punctured ball $B_\varepsilon^\bullet(0)$ go to $\mathbb{C}\backslash\overline{B_\varepsilon^{-1}(0)}$.

Now, consider $w = |w|e^{i\theta} \in \mathbb{C}\backslash\{0\}$. Then, with $z_0 = \ln|w| + i(\theta + 2k\pi)$ with k big enough so $z_0 \in \mathbb{C}\backslash\overline{B_\varepsilon^{-1}(0)}$, we can see that $w = e^{z_0}$ for $0 < |1/z_0| < \varepsilon$. However, e^z doesn't have zeroes, so it follows that

$$\{g(z) \ : \ 0 < |z| < \varepsilon\} = g(B_\varepsilon^\bullet(0)) = \exp(\mathbb{C}\backslash\overline{B_\varepsilon^{-1}(0)}) = \mathbb{C}\backslash\{0\}$$

- For h, note that cos is entire, so we can use the Taylor series to see that 0 is an essential singularity
$$\cos(z^{-1}) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n}}{(2n)!}.$$
Therefore, there doesn't exist k such that $z^k \cos z^{-1}$ has a removable singularity at 0. Now, for the image of the punctured neighborhood, I can say that by Great Picard's theorem, $\{g(z) \ : \ 0 < |z| < \varepsilon\}$ is either $\mathbb{C}$ or $\mathbb{C}\backslash\{z_0\}$ for some $z_0 \in \mathbb{C}$, but I don't know how to find it.

- For k, according to Riemann's removable singularity theorem, k has a removable singularity at a if $\lim_{z\to a}(z-a)f(z) = 0$. For our case, we have that
$$\lim_{z\to 0} z\frac{\sin z}{z} = \lim_{z\to 0} \sin(z) = \sin(0) = 0.$$
Since the limit of removable singularities is unique, we use a known fact from real analysis, for $t \in \mathbb{R}$
$$\lim_{z\to 0} \frac{\sin z}{z} = \lim_{t\to 0^+} \frac{\sin(t+0i)}{t+0i} = 1$$

Exercise 2.

Let $U \subset \mathbb{C}$ be an open set, $z_0 \in U$ and $f : U\backslash\{z_0\} \to \mathbb{C}$ holomorphic. Show that e^f doesn't have a pole in z_0.

Solution: If f can be extended at z_0, then by continuity of the exp function, e^f can also be extended at z_0 (uniqueness of the limit).

If z_0 is a essential singularity, then (by Casorati-Weierstrass theorem) $f(U\backslash\{z_0\})$ is dense in $\mathbb{C}$ so we can choose $a, b \in f(U\backslash\{z_0\})$ such that $e^a \neq e^b$. Also, there exist two sequences

$(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq U\backslash\{z_0\}$ with $(x_n)_{n\in\mathbb{N}} \overset{n\to\infty}{\to} z_0$ and $(y_n)_{n\in\mathbb{N}} \overset{n\to\infty}{\to} z_0$, such that $f(x_n) \overset{n\to\infty}{\to} a$ and $f(y_n) \overset{n\to\infty}{\to} b$. By continuity of exp,

$$\lim_{n\to\infty} e^{f(x_n)} = e^a \neq e^b = \lim_{n\to\infty} e^{f(y_n)}.$$

So z_0 is an essential singularity for e^f too.

Now, assume that f has a pole at z_0, and let $n \geq 1$ be the order of that pole. Then, for some neighborhood of z_0, f has a Laurent series

$$f(z) = \sum_{k=-n}^{\infty} a_k(z-z_0)^k.$$

Therefore,

$$f'(z) = \sum_{k=-n}^{\infty} ka_k(z-z_0)^{k-1} = \sum_{k=-n-1}^{\infty} (k+1)a_{k+1}(z-z_0)^k,$$

so $f'(z)$ has a pole of order $n+1$. Now, suppose that e^f has a pole of order m at z_0,

$$e^{f(z)} = \sum_{k=-m}^{\infty} b_k(z-z_0)^k,$$

and by the same logic $(e^f)'$ has a pole of order $m+1$. However, using the chain rule,

$$\begin{aligned}
(e^f)'(z) &= f'(z)e^{f(z)} \\
&= \sum_{k=-n-1}^{\infty} (k+1)a_{k+1}(z-z_0)^k \cdot \sum_{k=-m}^{\infty} b_k(z-z_0)^k \\
&= \sum_{k=-m-n-1}^{\infty} c_k(z-z_0)^k,
\end{aligned}$$

where $c_k = \sum_{l=-m-n-1}^{k}(l+m+1)a_{l+m+1}b_{k-l-m}$ is the coefficient of the Cauchy product between the two series. Even if I made a mistake, the important part is that $c_{-m-n-1} = (-n)a_{-n}b_{-m} \neq 0$, and thus, $(e^f)'$ has a pole of order $m+n+1 \neq m+1$ which leads to a contradiction.

Exercise 3.

Determine the Laurent series of $f(z) = \frac{1}{z(z-1)(z-2)}$ in the regions $U_1 := \{0 < |z| < 1\}$, $U_2 := \{1 < |z| < 2\}$, $U_3 := \{|z| > 2\}$

Solution:

The partial fraction decomposition is the following

$$
\begin{aligned}
f(z) = \frac{1}{z(z-1)(z-2)} &= \frac{1}{2z} + \frac{1}{1-z} - \frac{1}{2(2-z)} \\
&= \frac{1}{2z} + \frac{1}{1-z} - \frac{1}{4}\frac{1}{1-z/2}
\end{aligned}
$$

For $0 < |z| < 1$, the functions $1/(1-z)$ and $1/(1-z/2)$ have convergent power series. Therefore, the Laurent series is the following

$$
\begin{aligned}
f(z) &= \frac{1}{2z} + \sum_{k=0}^{\infty} z^k - \frac{1}{4}\sum_{k=0}^{\infty} \frac{z^k}{2^k} \\
&= \frac{1}{2z} + \sum_{k=0}^{\infty} z^k \left(1 - \frac{1}{2^{k+2}}\right) \\
&= \frac{1}{2z} + \frac{3}{4} + \frac{7z}{8} + \frac{15z^2}{16} + \cdots \\
&= \sum_{k=-1}^{\infty} z^k \left(1 - \frac{1}{2^{k+2}}\right)
\end{aligned}
$$

For the case $1 < |z| < 2$, $1/(1-z/2)$ has a power series expansion but $1/(1-z)$ doesn't. Instead, we use $1/(1-z) = \dfrac{1}{z(1-1/z)}$:

$$
\begin{aligned}
f(z) &= \frac{1}{2z} - \frac{1}{z}\frac{1}{1-1/z} - \frac{1}{4}\frac{1}{1-z/2} \\
&= \frac{1}{2z} - \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} - \frac{1}{4}\sum_{k=0}^{\infty} \frac{z^k}{2^k} \\
&= \sum_{k=2}^{\infty} \frac{-1}{z^k} - \frac{1}{2z} - \sum_{k=0}^{\infty} \frac{z^k}{2^{k+2}}
\end{aligned}
$$

Finally, for the case $|z| > 2$, neither $1/(1-z/2)$ nor $1/(1-z)$ have geometric series expansions, but $1/(1-z) = \dfrac{-1}{z(1-1/z)}$ and $\dfrac{-1}{(z/2)(1-2/z)}$

$$
\begin{aligned}
f(z) &= \frac{1}{2z} - \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{4(z/2)}\frac{1}{1-2/z} \\
&= \frac{1}{2z} - \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} + \frac{1}{4(z/2)}\sum_{k=0}^{\infty}\frac{2^k}{z^k} \\
&= \frac{1}{2z} - \sum_{k=1}^{\infty}\frac{1}{z^k} + \sum_{k=1}^{\infty}\frac{2^{k-2}}{z^k} \\
&= \frac{1}{z}\left(\frac{1}{2} - 1 + \frac{1}{2}\right) + \sum_{k=2}^{\infty}\frac{1}{z^k}(2^{k-2}-1) \\
&= \sum_{k=2}^{\infty}\frac{1}{z^k}(2^{k-2}-1)
\end{aligned}
$$

Exercise 4.

Let $U \subset \mathbb{C}$ be an open set that contains $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$. Let $f : U \backslash \{1\} \to \mathbb{C}$ be an holomorphic function with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ at 0. Suppose that f has a simple pole at 1. Prove that $\lim_{n\to\infty} \dfrac{a_n}{a_{n+1}} = 1$.

Solution:

If there is a simple pole at 1, then, for $\lambda \neq 0$, f can be written as follows

$$
f(z) = \frac{\lambda}{1-z} + h(z),
$$

where $h : U \to \mathbb{C}$ is holomorphic with Taylor series $h(z) = \sum_{k=0}^{\infty} b_k z^k$. Then, when $|z| < 1$, we have that

$$
f(z) = \lambda\sum_{k=0}^{\infty} z^k + \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} z^k\left(\lambda + b_k\right).
$$

The Taylor series expansion is unique, and thus, it follows that

$$
a_n = \lambda + b_n,
$$

Note that h is holomorphic at 1, so h doesn't have any singularity at $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$. Thus, the radius of convergence of the Taylor series, which we proved previously that is the distance from the center of the series to the nearest non-removable singularity, is greater than 1. It follows that, $|h(1)| = |\sum_{k=0}^{\infty} b_k| \leq \sum_{k=0}^{\infty} |b_k| < \infty$ so we conclude that $b_n \overset{n\to\infty}{\to} 0$.

Finally,

$$
\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\lambda + b_{k+1}}{\lambda + b_k} = \frac{\lambda}{\lambda} = 1.
$$

Exercise 5.

What can be concluded from the previous exercise if

(a) The pole of f is not at 1 but at $e^{i\phi}$ for some $\phi \in \mathbb{R}$.

(b) The pole is of order $k \geq 1$.

Solution (a):

Let $\zeta \in \{z \in \mathbb{C} \ : \ |z| = 1\}$ be the simple pole of f in U. Then, for $\lambda \neq 0$

$$f(z) = \frac{\lambda}{\zeta - z} + h(z) = \frac{\lambda}{\zeta(1 - z/\zeta)} + h(z),$$

where, again, h is holomorphic at U and has a Taylor series $h(z) = \sum_{k=0}^{\infty} b_k z^k$ with radius of convergence strictly greater than one, and thus, $b_n \to 0$ as $n \to \infty$. Also, when $|z| < 1$,

$$\frac{\lambda}{\zeta(1 - z/\zeta)} = \frac{\lambda}{\zeta} \sum_{k=0}^{\infty} \zeta^{-k} z^k = \lambda \sum_{k=0}^{\infty} \zeta^{-k-1} z^k.$$

Put everything together to obtain:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (b_n - \lambda \zeta^{-n-1}) z^n.$$

By uniqueness of the power series expansion

$$a_n = b_n - \lambda \zeta^{-n-1}.$$

Also note that $|b_n \zeta^n| = |b_n| \to 0$ so it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{b_{n+1} - \lambda\zeta^{-n-2}}{b_n - \lambda\zeta^{-n-1}} = \frac{b_{n+1}\zeta^{n+2} - \lambda}{b_n \zeta^{n+2} - \lambda\zeta^{-1}} = \frac{-\lambda}{-\lambda\zeta^{-1}} = \zeta.$$

Solution (b):

If there's a pole of order 2 at ζ, then, there exists $\lambda_1, \lambda_2 \neq 0$ such that

$$f(z) = \frac{\lambda_2}{\zeta^2(1 - z/\zeta)^2} + \frac{\lambda_1}{\zeta(1 - z/\zeta)} + h(z),$$

where, $h(z) = \sum_{n=0}^{\infty} b_n z^n$ has the same properties we mentioned before ($b_n \zeta^n \overset{n\to\infty}{\to} 0$). Note that

$$\frac{d}{dz} \frac{1}{1 - z/\zeta} = \frac{1}{(1 - z/\zeta)^2},$$

so the power series expansion of $\frac{1}{(1-z/\zeta)^2}$ when $|z| < 1$ is

$$\frac{1}{(1 - z/\zeta)^2} = \sum_{n=0}^{\infty} \zeta^{-n} \frac{dz^n}{dz} = \sum_{n=0}^{\infty} n \zeta^{-n} z^{n-1} = \sum_{n=0}^{\infty} (n+1) \zeta^{-n-1} z^n.$$

It follows that

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n z^n \\
&= \frac{\lambda_2}{\zeta^2} \sum_{n=0}^{\infty} \zeta^{-n-1}(n+1)z^n + \frac{\lambda_1}{\zeta} \sum_{n=0}^{\infty} \zeta^{-n} z^n + \sum_{k=0}^{\infty} b_n z^n \\
&= \sum_{n=0}^{\infty} \lambda_2 \zeta^{-n-3}(n+1)z^n + \sum_{n=0}^{\infty} \lambda_1 \zeta^{-n-1} z^n + \sum_{k=0}^{\infty} b_n z^n \\
&\implies a_n = \lambda_2 \zeta^{n-3}(n+1) + \lambda_1 \zeta^{n-1} + b_n
\end{aligned}
$$

Then,

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\lambda_2 \zeta^{-n-4}(n+2) + \lambda_1 \zeta^{-n-2} + b_{n+1}}{\lambda_2 \zeta^{-n-3}(n+1) + \lambda_1 \zeta^{-n-1} + b_n} \\
&= \lim_{n\to\infty} \frac{\lambda_2 \zeta (n+2)}{\lambda_2 (n+1)} \\
&= \zeta.
\end{aligned}
$$

Now, for any $k \geq 1$, note that when $|z| < 1$

$$
\begin{aligned}
\frac{\lambda_k}{\zeta^k} \cdot \frac{1}{(1 - z/\zeta)^k} &= \frac{\lambda_k}{\zeta^k} \cdot \frac{1}{(k-1)!} \cdot \frac{d^{k-1}}{dz^{k-1}} \frac{1}{1-z} \\
&= \frac{\lambda_k}{\zeta^k (k-1)!} \sum_{n=0}^{\infty} \underbrace{(n+k-1)\cdots(n+1)}_{=(n+k-1)!/n!} z^n \zeta^{-n-k+1} \\
&= \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda_k \zeta^{-n-2k+1} z^n
\end{aligned}
$$

Then, let $K \geq 1$ the order of the pole of f at ζ,

$$
\begin{aligned}
f(z) &= \sum_{k=1}^{K} \frac{\lambda_k}{(\zeta - z)^k} + h(z) \\
&= \sum_{k=1}^{K} \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda_k \zeta^{-n-2k+1} z^n + \sum_{n=0}^{\infty} b_n z^n.
\end{aligned}
$$

So it follows that

$$
a_n = b_n + \sum_{k=1}^{K} \binom{n+k-1}{n} \lambda_k \zeta^{-n-2k+1}
$$

Finally, $b_n \to 0$ and $\binom{n+K-1}{n}$ dominates the expression since is the polynomial of n with greatest degree, so it follows that

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{b_{n+1}+\sum_{k=1}^{K}\binom{n+k}{n+1}\lambda_k\zeta^{-n-2k}}{b_n+\sum_{k=1}^{K}\binom{n+k-1}{n}\lambda_k\zeta^{-n-2k+1}} \\ &= \lim_{n\to\infty} \frac{n^K\lambda_k\zeta}{n^K\lambda_k} = \zeta.\end{aligned}$$

Exercise 6.

Let $U \subset \mathbb{C}$ be an open set, $z_0 \in G$, $\tilde{G} = G\backslash\{z_0\}$, $f, g : \tilde{G} \to \mathbb{C}$ holomorphic and z_0 be a pole of f and g. Let

$$\operatorname{ord}(f, z_0) = \text{order of the pole of } f \text{ at } z_0 \text{ if } z_0 \text{ is a pole.}$$

Show that z_0 is a non-essential singularity of $f+g$, fg and, if $g(z) \neq 0$ for every $z \in \tilde{G}$, $\dfrac{f}{g}$ and that the following formulas are valid:

(a) $\operatorname{ord}(f+g; z_0) \leq \max\{\operatorname{ord}(f; z_0), \operatorname{ord}(g; z_0)\}$.

(b) $\operatorname{ord}(fg, z_0) = \operatorname{ord}(f; z_0) + \operatorname{ord}(g; z_0)$

(c) $\operatorname{ord}\left(\frac{f}{g}; z_0\right) = \operatorname{ord}(f; z_0) - \operatorname{ord}(g; z_0)$ if $\operatorname{ord}(f; z_0) > \operatorname{ord}(g; z_0)$.

Solution Part (a)

Let m be the order of z_0 at f and n at g, without restriction $m \leq n$. Then, there exist holomorphic functions h_1 and h_2 such that

$$f(z) = \sum_{k=0}^{n} \frac{a_k}{(z-z_0)^k} + h_1(z),$$

$$g(z) = \sum_{k=0}^{m} \frac{b_k}{(z-z_0)^k} + h_2(z).$$

Then,

$$f(z)+g(z) = \sum_{k=m+1}^{n} \frac{a_k}{(z-z_0)^k} + \sum_{k=0}^{m} \frac{a_k+b_k}{(z-z_0)^k} + h_1(z) + h_2(z),$$

so it follows that n is the maximum possible order of $f+g$ at z_0 (some coefficients could cancel if $m = n$ and $a_k = b_k$ for some $k \leq m$).

Solution Part (b)

Again, let m be the order of z_0 at f and n at g. Then, there exists h_1, h_2 holomorphic functions that don't cancel at z_0, such that

$$f(z) = \frac{h_1(z)}{(z - z_0)^n}, \qquad g(z) = \frac{h_2(z)}{(z - z_0)^m}.$$

Then,

$$f(z)g(z) = \frac{h_1(z)h_2(z)}{(z - z_0)^{n+m}},$$

where z_0 is a pole of order $n + m$, because $h_1(z_0)h_2(z_0) \neq 0$.

Solution Part (c)

If $g(z)$ doesn't have zeros in $\tilde{G}$, then $1/g(z) = \frac{(z-z_0)^m}{h_2(z)}$ is defined in all $\tilde{G}$, so it follows that

$$f(z)/g(z) = \frac{h_1(z)/h_2(z)}{(z - z_0)^{n-m}}.$$

Since $h_2(z) \neq 0$ in G because $h_2(z_0) \neq 0$, it follows that $f(z)/g(z)$ has a pole of order $n - m$ at z_0.