Holomorphic Functional Calculus

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1 Introduction

The goal of this project is to describe how the definition of holomorphic complex functions can be extended to operators in Banach spaces. For instance, if $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a *polynomial* defined for the complex numbers, then it's natural to define for a linear operator $T : X \to X$,

$$P(T) = a_0 I + a_1 T + \cdots + a_n T^n,$$
$$\underbrace{T^0 x := Ix = x}_{\text{identity operator}}, \quad \underbrace{T^k x = T(\cdots(T(x)))}_{k \text{ times composition}}, \ \forall x \in X.$$

Now, for a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ with a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that absolutely converges for $|z| < R$ and a bounded linear operator T such that $\|Tx\| < R\|x\|$ for any $x \in X$, the operator $f(T) := \sum_{n=0}^{\infty} a_n T^n$ is well defined because

$$\forall x \in X : \ \|f(T)(x)\| \leq \sum_{n=0}^{\infty} |a_n| \|T^n x\| < \|x\| \sum_{n=0}^{\infty} |a_n| R^n < \infty. \tag{$\star$}$$

There are some notions of spectral theory that will be useful for extending this definition to a broader set of functions. For example, for a holomorphic function $f : U \to \mathbb{C}$ in a domain U one might define $f(T)$ using a version of the Cauchy Integral formula for some curve $\gamma \in U$,

$$f(T) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\lambda - T} \, d\lambda,$$

where $1/(\lambda - T) := (\lambda I - T)^{-1}$ is called the *resolvent* of T at λ and it's defined if the operator $(\lambda I - T)$ is *regular* (invertible). The resolvent is defined for every $\lambda \in \mathbb{C}$ except for a closed set $\sigma(T)$ called the *spectrum* of T. These definitions are the cornerstone of this project, so let's take some time to work on them.

Definition 1. The *resolvent set* of a linear operator $T : X \to X$ in a Banach space X is defined as

$$\rho(T) := \{\lambda \in \mathbb{C} \ : \ (\lambda I - T) \text{ is bijective}\}.$$

If $\lambda \in \rho(T)$, then the *resolvent of* T *at* λ is $R(\lambda, T) := (\lambda I - T)^{-1}$. $\sigma(T) := \mathbb{C} \backslash \rho(T)$ is called the *spectrum of* T and is the complement of the resolvent set.

The following theorem is used some times in the following propositions, but the proof might be out of the scope of this document, so we're just going to state it.

Theorem 1. For a Banach space X and a bounded operator $T : X \to X$, that is $\|Tx\| \leq M\|x\|$ for some $M > 0$, the *resolvent set* $\rho(T) \subset \mathbb{C}$ is an open set, and thus, the spectrum $\sigma(T)$ is closed.

□

Definition 2. Let $\sigma_p(T) := \{\lambda \in \mathbb{C} \ : \ (\lambda I - T) \text{ is not injective}\}$. This set may be called the *point spectrum* or the *eigenvalues* of T and it is a subset of $\sigma(T)$.

Note that for an operator T and $x \neq y \in X$, $T(x) = T(y) \iff T(x-y) = 0$. Therefore T is injective if and only if $\ker(T) := \{x \ : \ Tx = 0\} = \{0\}$.

Remark. If X has a basis with n-elements $\{x_1, \ldots, x_n\}$ and $\ker(T) = \{0\}$, then, for such given basis, let M be the $n \times n$ matrix representation of T. It follows that M is full rank, and therefore, the set with the n columns of M is linearly independent. This set of columns is the vector representation of $\{Tx_1, \ldots, Tx_n\} \subset X$ which is also linearly independent, and thus,

$$\dim TX = n = \dim X \implies TX = X.$$

In the finite dimensional case, a linear operator $T : X \to X$ is bijective if and only if T is injective, so $\sigma_p(T) = \sigma(T)$. However, if X is infinite dimensional, then that might not be necessarily the case.

Example 1. In this example we're going to show a case of an operator in an infinite dimensional space for which $\sigma_p(T) \neq \sigma(T)$. Define $X = \ell^2(\mathbb{N} \to \mathbb{C})$ as the set of sequences $x = (x_1, x_2, \ldots)$ that satisfy

$$\|x\|_2 = \sum_{n=1}^{\infty} |x_n|^2 < \infty, \quad x_n \in \mathbb{C}.$$

Now, define the left shift operator as follows

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

and note that for any $x = (x_1, x_2, \ldots) \in X$,

$$\|Lx\|_2 = \sum_{n=2}^{\infty} |x_n|^2 \leq \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_2.$$

Then, note that the function $f_\lambda(z) = \frac{1}{\lambda - z}$ has a power series given by the geometric series when $|\lambda| > 1$:

$$f_\lambda(z) = \frac{\lambda^{-1}}{1 - \lambda^{-1} z} = \sum_{n=0}^{\infty} \lambda^{-n-1} z^n,$$

and thus, since $\|\lambda^{-1} Lx\| \leq \lambda^{-1}\|x\|$, the resolvent of L at λ given by the $(\star)$ equation is

$$R(\lambda, L) = (\lambda I - L)^{-1} = f_\lambda(L) = \sum_{n=0}^{\infty} \lambda^{-n-1} L^n, \quad |\lambda| > 1$$

$$\implies \sigma(T) \subseteq \{\lambda \in \mathbb{C} \,:\, |\lambda| \leq 1\}.$$

On the other hand, note that $(\lambda I - L)x = 0$ if and only if for every $n \in \mathbb{N}$,

$$(\lambda I - L)x = (\lambda x_1 - x_2, \lambda x_2 - x_3, \lambda x_3 - x_4, \ldots) = 0$$

$$\iff x_n = \lambda x_{n-1} \iff x_n = \lambda^{n-1} x_1.$$

Therefore, since $p_\lambda = (1, \lambda, \lambda^2, \ldots) \in X$ only if $|\lambda| < 1$, it follows that

$$\ker(\lambda I - L) = \begin{cases} \operatorname{span}\{p_\lambda\} & |\lambda| < 1 \\ \{0\} & |\lambda| \geq 1 \end{cases}$$

$$\implies \sigma_p(L) = \{\lambda \in \mathbb{C} \,:\, |\lambda| < 1\}.$$

Finally, since $\sigma(L)$ is a closed set and

$$\sigma_p(L) = \{\lambda \in \mathbb{C} \,:\, |\lambda| < 1\} \subseteq \sigma(L) \subseteq \{\lambda \in \mathbb{C} \,:\, |\lambda| \leq 1\},$$

it follows that $\sigma(L) = \{\lambda \in \mathbb{C} \,:\, |\lambda| \leq 1\}$ which has more elements than $\sigma_p(L)$.

2 Spectral Theory in Finite Dimensional Spaces

We're going to begin with the case where X is a finite dimensional Banach space of complex numbers and $T : X \to X$ is an operator. According to the remark in the previous section for this chapter we are allowed to call $\sigma(T)$ the eigenvalues of T. For $\lambda \in \sigma(T)$ there exists multiple solutions $x \in X$, $x \neq 0$ for the equation $(\lambda I - T)x = 0$. These solutions are called the *eigenvectors* and the set that contains all the solutions for the previous equation: $N_\lambda(T) := \ker(\lambda I - T)$ is called the *geometric eigenspace* in λ.

Also, for a positive integer v define the set $N_\lambda^v := \ker((\lambda I - T)^v)$ and note that if $n = \dim(X)$, then $N_\lambda^v \subseteq N_\lambda^{v+1} \subseteq N_\lambda^n = N_\lambda^{n+1}$. Let $v(\lambda) \leq n$ be a positive integer such that

$$N_\lambda^{v(\lambda)-1} \subsetneq N_\lambda^{v(\lambda)} = N_\lambda^{v(\lambda)+1}$$

and define the set $A_\lambda := N_\lambda^{v(\lambda)}$ which is called the *algebraic eigenspace* in λ. Note that since $N_\lambda^0 = \ker(I) = \{0\}$ and $N_\lambda^1 \supsetneq \{0\}$ only when $\lambda \in \sigma(T)$, it follows that

$$v(\lambda) > 0 \iff \lambda \in \sigma(T).$$

The dimension of the eigenspaces in λ are called *geometric multiplicity* and *algebraic multiplicity* of λ respectively. The following examples will illustrate how this concepts work.

Example 2. ($v(\lambda)$ and the algebraic multiplicity in λ are not the same)

Let $X = \mathbb{C}^3$ and define T by its matrix representation:

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $\sigma(T) = \{0\}$, $N_0 = \operatorname{span}\{(1,0,0)', (0,1,0)'\}$ and since $T^2 = 0$, $A_0 = N_0^2 = \operatorname{span}\{(1,0,0)', (0,1,0)', (0,0,1)\}$. Therefore, $v(0) = 2$ and the algebraic multiplicity in 0 is 3.

Example 3. (The geometric and algebraic multiplicities are not the same)

Let $X = \mathbb{C}^3$ and define T by its matrix representation:

$$T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $\sigma(T) = \{i, 0\}$ with $N_i = \text{span}\{(1,0,0)'\}$ and $N_0 = \text{span}\{(0,1,0)'\}$ so the geometric multiplicity of both eigenvalues is 1. However, after taking $(\lambda I - T)^2$ we obtain

$$(iI - T)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad (0I - T)^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $N_i^2 = N_i$ and $N_0^2 = \text{span}\{(0,1,0)', (0,0,1)'\}$ implying that $v(i) = 1$ and $v(0) = 2$.

This also shows that the algebraic multiplicity in i is 1 while the algebraic multiplicity in 0 is 2. In this case the algebraic multiplicities of the eigenvalues sum to the space's dimension and the direct sum of the algebraic eigenspaces gives us X.

In general, we can prove that X is decomposed by the direct sum of the algebraic eigenspaces of any operator $T : X \to X$. But, before that, we want to show a important theorem that states that for any polynomial $P : \mathbb{C} \to \mathbb{C}$ and an operator T, the operator given by $P(T)$ is identically zero if and only if for every $\lambda \in \sigma(T)$, the multiplicity of λ as a zero of P coincides with the index $v(\lambda)$.

Theorem 2. For a complex polynomial P and an operator $T : X \to X$ the following conditions are equivalent:

(a) $P(T) = 0$

(b) λ is a zero of P with multiplicity $v(\lambda)$ for every $\lambda \in \sigma(T)$.

Proof. **Preliminaries:** Since we're assuming that X is finitely dimensional, let $n = \dim X$ and let $\{x_1, \ldots, x_n\}$ be a basis of X. Then, for every $k = 1, \ldots, n$, the set $\{x_k, Tx_k, \ldots, T^n x_k\}$ with $n+1$ elements is linearly dependent, so there exists a non-zero and non-constant polynomial S_k such that $S_k(T)x_k = 0$. Define the operator $R = S_1 \cdot S_2 \cdots S_n$ which satisfies $R(T)x_k$ for every $k = 1, \ldots, n$, and thus, since every $x \in X$ can be written as a linear combination of basis elements $x = a_1x_1 + \cdots + a_nx_n$. It follows that for any operator T there always exists a non-zero polynomial $R : \mathbb{C} \to \mathbb{C}$ such that $R(T)x = 0$ for every $x \in X$.

Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be the zeros of the polynomial R so we can factorize them as follows

$$R(z) = \beta \prod_{j=1}^{m} (\lambda_j - z)^{m_j},$$

that way $R(T) = \beta \prod_{j=1}^{m} (\lambda_j I - T)^{m_j}$ and it doesn't matter the order that we multiply these factors because after expanding we should get the same expression:

$$(aI + T)(bI + T) = abI + (a + b)T + T^2 = (bI + T)(aI + T).$$

There are two possible scenarios for λ_j:

- If $\lambda_j \in \rho(T)$, then $(\lambda_j I - T)$ invertible. So after reordering the set of zeros of R in such way $\{\lambda_1, \ldots, \lambda_p\} \subset \sigma(T)$ and $\{\lambda_{p+1}, \ldots, \lambda_m\} \subset \rho(T)$ we obtain

$$R(T) = \beta \underbrace{\prod_{j=p+1}^{m} (\lambda_j I - T)^{m_j}}_{R_1(T)} \times \underbrace{\prod_{j=1}^{p} (\lambda_j I - T)^{m_j}}_{R_2(T)}.$$

  Now, note that for every $x \in X$, $R(T)(x) = R_1(T)R_2(T)(x) = 0$ and $R_1(T)$ is invertible because is the product of invertible operators. Therefore, since $R_1(T)y = 0$ if and only if $y = 0$, it follows that $R_2(T) = 0$.

- For $\lambda_j \in \sigma(T)$ and $x \in X$ if $(\lambda_j I - T)^{m_j} x = 0$, then $x \in A_\lambda(T)$ so it follows that $(\lambda_j I - T)^{v(\lambda_j)} x = 0$. Therefore, if we define

$$R_3(T) = \prod_{j=1}^{p} (\lambda_j I - T)^{v(\lambda_j)},$$

  then $R_3(T) = 0$.

(b) $\implies$ ***(a)***: If every $\lambda \in \sigma(T)$ is a zero of P with multiplicity $v(\lambda)$, then P is divisible by R_3 and thus, for some polynomial Q, $P(T) = R_3(T)Q(T) = 0$.

(a) $\implies$ ***(b)***: Now let $P(T) = 0$ and

$$P(z) = \alpha \prod_{j=1}^{q} (\lambda_j - z)^{\alpha_j}.$$

Using the same argument as before, one can ignore the factors $(\lambda_j - z)$ if $\lambda_j \in \rho(T)$, so assume without restriction that $\{\lambda_1, \ldots, \lambda_q\} \subseteq \sigma(T)$. On the other hand, let $\lambda_0 \in \sigma(T)$ and note that there exists $y \neq 0$ for which $Ty = \lambda_0 y$. Since

$$0 = P(T)y = P(\lambda_0)y, \; y \neq 0 \implies P(\lambda_0) = 0,$$

it follows that $\lambda_0 \in \{\lambda_1, \ldots, \lambda_q\}$, and thus, $\sigma(T) \subseteq \{\lambda_1, \ldots, \lambda_q\}$. It is left to prove that $\alpha_j \geq v(\lambda_j)$ for every $j = 1, \ldots, q$, so for the sake of contradiction assume $\alpha_j < v(\lambda_j)$ for some $j = 1, \ldots q$. Assume without restriction that $j = 1$ and note that $N_{\lambda_1}^{\alpha_1} \subsetneq N_{\lambda_1}^{\alpha_1+1}$ so there exists $x_1 \in X$ such that

$$(\lambda_1 I - T)^{\alpha_1+1} x_1 = 0, \quad y_1 = (\lambda_1 I - T)^{\alpha_1} x_1 \neq 0.$$

Now, let Q be a polynomial such that $P(z) = Q(z)(\lambda_1 - z)^{a_1}$ and $Q(\lambda_1) \neq 0$. Finally, since $Ty_1 = \lambda_1 y_1$, it follows that

$$P(T)x_1 = Q(T)(\lambda_1 I - T)^{\alpha_1} x_1 = Q(T)y_1 = \underbrace{Q(\lambda_1)}_{\neq 0} \cdot \underbrace{y_1}_{\neq 0} \neq 0,$$

which leads to a contradiction with the fact that $P(T) = 0$. Therefore, every $\lambda \in \sigma(T)$ is a zero of multiplicity $v(\lambda)$ of the polynomial P. $\square$

From this theorem there are multiple and important consequences. In the proof we also showed that there always exists a non-constant polynomial R for which $R(T) = 0$. This polynomial, as the theorem implies must satisfy that each eigenvalue is a zero. Since non-constant polynomials have a finite number of zeros, but at least one, it follows that the number of eigenvalues in a finite dimensional space is finite and greater that 0.

Corollary 2.1. For a finite dimensional space X, the spectrum of an operator is non-empty and finite.

$\square$

Now, if we replace P in the theorem with the difference between two polynomials we obtain the following corollary:

Corollary 2.2. If P, Q are polynomials, then $P(T) = Q(T)$ if for every $\lambda \in \sigma(T)$, λ is a zero of multiplicity $v(\lambda)$ of the polynomial $P - Q$.

$\square$

In fact, if for two polynomials P, Q, all their derivatives coincide at the spectrum of T, that is

$$P^{(m)}(\lambda) = Q^{(m)}(\lambda), \quad \lambda \in \sigma(T),\ m < v(\lambda),$$

then they define the same operator $P(T) = Q(T)$. Furthermore, by generalizing this notion to a holomorphic function f, we have that $f(T)$ is characterized only by the values of f and some of its derivatives at the spectrum of T.

Let $\mathscr{F}(T)$ be the family of all functions that are holomorphic at some open set containing $\sigma(T)$. For each function we can interpolate a polynomial P such that

$$f^{(m)}(\lambda) = P^{(m)}(\lambda), \quad \lambda \in \sigma(T),\ m < v(\lambda),$$

and with the previous corollary, we can be sure that there are no ambiguities if we define $f(T) = P(T)$ because any other polynomial that satisfies these equations would yield the same result. The following theorem immediately follows from the previous discussion.

Theorem 3. If f, g are functions in $\mathscr{F}(T)$ and α, β are complex numbers, then

(a) $\alpha f + \beta g \in \mathscr{F}(T)$ and is defined as $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$.

(b) $f \cdot g \in \mathscr{F}(T)$, is defined as $(f \cdot g)(T) = f(T) \cdot g(T)$, and also this implies $f(T) \cdot g(T) = g(T) \cdot f(T)$.

(c) If f is a polynomial $f(z) = \sum_{n=0}^{m} a_n z^n$, then $f(T) = \sum_{n=0}^{m} a_n T^n$.

(d) $f(T) = 0$ if and only if

$$f^{(m)}(\lambda) = 0, \quad \lambda \in \sigma(T),\ m < v(\lambda).$$

$\square$

Let $\lambda_0 \in \mathbb{C}$ and define $e_{\lambda_0}(z)$ to be a function that is equal to one at a neighborhood of λ_0 and zero at a neighborhood of each point in $\sigma(T)\backslash\{\lambda_0\}$ (the neighborhoods do not intersect).

The function e_{λ_0} is in fact holomorphic at an open set that contains $\sigma(T)$ although this set is not connected.

Now define $E(\lambda_0) = e_\lambda(T)$, and apply the previous theorem to the following proposition

Theorem 4. For the operator $E(\cdot)$ defined previously and $\lambda_0, \lambda_1 \in \mathbb{C}$:

(a) $E(\lambda_0) = 0$ if and only if $\lambda_0 \in \rho(T)$.

(b) $E(\lambda_0)^2 = \mathbb{E}(\lambda_0)$.

(c) $E(\lambda_0)E(\lambda_1) = 0$ if $\lambda_0 \neq \lambda_1$.

(d) $I = \sum_{\lambda \in \sigma(T)} E(\lambda)$.

Proof. Note that e_{λ_0} is locally constant so all of its derivatives are zero, so according to the previous theorem, the only thing that matters is whether $e_{\lambda_0}(\lambda) = 0$ for every $\lambda \in \sigma(T)$ or not.

(a) If $\lambda_0 \in \rho(T)$, then the neighborhood V_0 that contains λ_0 doesn't contain any eigenvalue of T, and thus, $e_{\lambda_0}(\lambda) = \mathbb{1}_{\lambda \in V_0} = 0$ for every $\lambda \in \sigma(T)$. If $E(\lambda_0) = 0$, then $e_{\lambda_0}(\lambda) = \mathbb{1}_{\lambda \in V_0} = 0$ for every $\lambda \in \sigma(T)$, and thus, all the eigenvalues are outside of V_0, implying that λ_0 is not an eigenvalue.

(b) The function e_{λ_0} only takes two values $\{0, 1\}$ and the squares of both values are equal to themselves. Thus, $e^2_{\lambda_0}(\lambda) = e_{\lambda_0}(\lambda)$ for every $\lambda_0, \lambda \in \mathbb{C}$ and $e^2_{\lambda_0}$ is also locally constant, so it follows that $e^2_{\lambda_0}(T) = e_{\lambda_0}(T)$.

(c) If $\lambda_0 \in \rho(T)$ or $\lambda_1 \in \rho(T)$, then $E(\lambda_0)E(\lambda_1) = 0$ by item (a), so assume without restriction that $\lambda_0, \lambda_1 \in \sigma(T)$. Note that $e_{\lambda_1}(\lambda_0) = e_{\lambda_0}(\lambda_1) = 0$ and $e_{\lambda_0}(\lambda) = e_{\lambda_1}(\lambda) = 0$ for any other eigenvalue of T different from λ_0 and λ_1. Finally, $e_{\lambda_0}(z) \cdot e_{\lambda_1}(z)$ is identically 0 for any $z \in \sigma(T)$ and all the derivatives are 0, so it follows that, $e_{\lambda_0}(T) \cdot e_{\lambda_1}(T) = 0$.

(d) Let $V_1, \ldots, V_q$ be the disjoint neighborhoods for $\lambda_1, \ldots, \lambda_q$ for which $e_{\lambda_j}(\lambda) = \mathbb{1}_{\lambda \in V_j}$. Then, since all the sets are disjoint every eigenvalue is exactly in one of this sets, so $f(\lambda) := \sum_{j=1}^{q} e_{\lambda_j}(\lambda) = 1$ for every $\lambda \in \sigma(T)$. Therefore, since $I = 1(T)$, and both f and 1 are locally constant functions, it follows that $\sum_{j=1}^{q} e_{\lambda_j}(T) = I$.

□

Now, let $\sigma(T) = \{\lambda_1, \ldots, \lambda_q\}$ and let $X_i = E(\lambda_i)X$. From item (b) of the previous theorem, it follows that $X_i \cap X_j = \{0\}$, from item (c) it follows that $E(\lambda_i)X_i = X_i$ and from item (d) it follows that $X = X_1 + \cdots + X_q$. Therefore,

Corollary 4.1. $X = X_1 \oplus \cdots \oplus X_q$.

On the other hand, note that for every $\lambda \neq \lambda_i \in \sigma(T)$, λ is a zero of multiplicity $v(\lambda)$ of the function e_{λ_i} and λ_i is a zero of order $v(\lambda_i)$ of the function $z \mapsto (\lambda_i - z)^{v(\lambda_i)}$. Thus,

$$(\lambda_i I - T)^{v(\lambda_i)} E(\lambda_i) = 0.$$

This relation shows that $(\lambda_i I - T)^{v(\lambda_i)} E(\lambda_i)X = (\lambda_i I - T)^{v(\lambda_i)} X_i = \{0\}$, and thus, since for every $y \in X_i$, $(\lambda_i I - T)^{v(\lambda_i)} y = 0$, it follows that $X_i \subseteq N_{\lambda_i}^{v(\lambda_i)} = A_{\lambda_i}$. The other inclusion is part of the following theorem

Theorem 5. For $\lambda \in \sigma(T)$,

$$E(\lambda)X = N_\lambda^{v(\lambda)} = A_\lambda.$$

Proof. In the previous paragraph we proved that $E(\lambda)X \subseteq A_\lambda$. Now, in the previous corollary we stated that

$$X = \bigoplus_{\lambda \in \sigma(T)} E(\lambda)X,$$

so in order to show the other inclusion, we only need to prove that $A_\lambda \cap A_\mu = \{0\}$ for $\lambda \neq \mu$, $\lambda, \mu \in \sigma(T)$. Suppose for the sake of contradiction that there exists $x \in A_\lambda \cap A_\mu = N_\lambda^{v(\lambda)} \cap N_\mu^{v(\mu)}$ such that $x \neq 0$. Let $\alpha < v(\lambda)$ be a integer that satisfies

$$z := (\lambda I - T)^\alpha x \neq 0, \quad (\lambda I - T)^{\alpha+1} x = 0.$$

Thus, we have:

- First, it is clear that we can conmute the factors to obtain

$$(\lambda I - T)^{v(\mu)} z = (\lambda I - T)^{v(\mu)} (\lambda I - T)^\alpha x = (\lambda I - T)^\alpha \underbrace{(\lambda I - T)^{v(\mu)} x}_{x \in A_\mu} = 0.$$

- On the other hand, since $(\lambda I - T)z = (\lambda I - T)^{\alpha+1} x = 0$, it follows that $Tz = \lambda z$, and thus,

$$(\lambda I - T)^{v(\mu)} z = (\lambda I - \mu I)^{v(\mu)} z = \underbrace{(\lambda - \mu)^{v(\mu)}}_{\neq 0} \underbrace{z}_{\neq 0} \neq 0.$$

This is a contradiction, so $x = 0$. □

As we originally intended to show, the algebraic eigenspaces of T give us a direct sum decomposition of the space X, and thus, the sum of the algebraic multiplicities give us the dimension of X. We also proved that $E(\lambda)$ is a projection to each eigenspace, so the functions of T can be expressed as follows

Theorem 6. For $f \in \mathscr{F}(T)$,

$$f(T) = \sum_{\lambda \in \sigma(T)} \sum_{k=0}^{v(\lambda)-1} \frac{(T - \lambda I)^k}{k!} f^{(k)}(\lambda) E(\lambda).$$

Proof. The function $g \in \mathscr{F}(T)$, $g(z) = \sum_{\lambda \in \sigma(T)} \sum_{k=0}^{v(\lambda)-1} \frac{(z-\lambda)^k}{k!} f^{(k)}(\lambda) e_\lambda(z)$ interpolates f and its derivatives at each point $\lambda \in \sigma(T)$. In fact, for $\lambda_0 \in \sigma(T)$ and $m = 1, \ldots, v(\lambda) - 1$,

$$e_\lambda(\lambda_0) = 0,\ \lambda \neq \lambda_0, \quad e_\lambda(\lambda_0) = 1,\ \lambda = \lambda_0,$$

and thus,

$$g^{(m)}(\lambda_0) = \sum_{k=m}^{v(\lambda)-1} \frac{(\lambda_0 - \lambda_0)^{k-m}}{(k-m)!} f^{(k)}(\lambda_0) = f^{(m)}(\lambda_0).$$

Finally, by Theorem 3, it follows that

$$g(T) = f(T).$$

□

To end this chapter, we're going to introduce the notion of contour integral