Complex Analysis: Homework 3

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Exercise 1.

(a) Calculate $\oint_{|z-1|=2} z^n \sin(z) dz$ for $n \in \mathbb{Z}$.

(b) For $n \in \mathbb{N}_0$ prove that

$$\int_{|z+2i|=3} \frac{1}{(z^2+\pi^2)^{n+1}} dz = \frac{-(2n)!}{(n!)^2}(2\pi)^{-2n}$$

Solution Part (a)

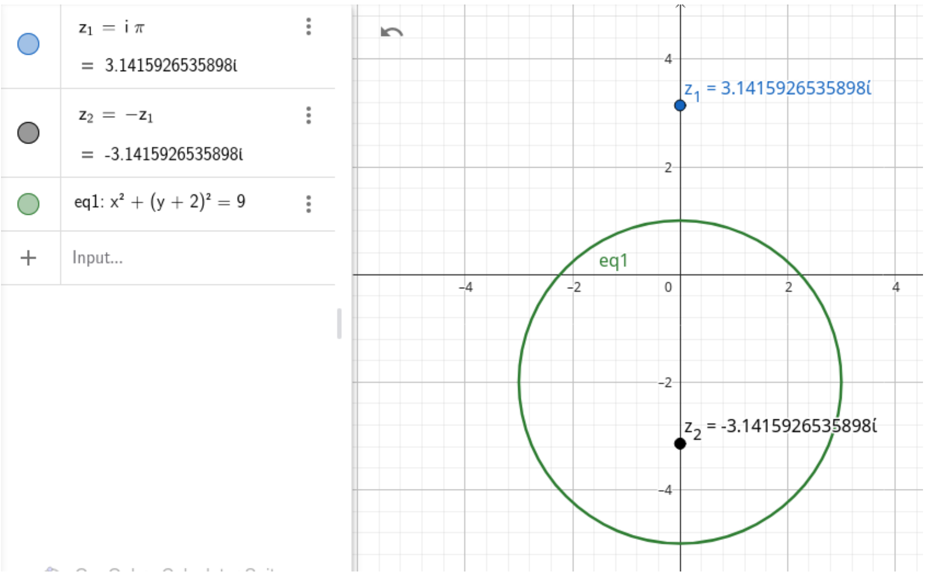
$z \mapsto z^n \sin(z)$ is an entire function with Taylor series

$$z^n \sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1+n}.$$

Therefore, using Cauchy's theorem, we assert that

$$\oint_{|z-1|=2} z^n \sin(z) dz = 0.$$

Solution Part (b)



$$
\begin{aligned}
\int_{|z+2i|=3} \frac{1}{z^2+\pi^2} dz &= \int_{|z+2i|=3} \frac{1}{(z+i\pi)(z-i\pi)} dz \\
&= \frac{-1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z+i\pi} - \frac{1}{z-i\pi} dz \\
&= \frac{-1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z+i\pi} + \frac{1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z-i\pi} dz \\
&= \frac{-1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z+i\pi} + 0 \\
&= \frac{-1}{2\pi i} \cdot 2\pi i \\
&= \frac{-(2\cdot 0)!}{(0)!^2} (2\pi)^{2\cdot 0}.
\end{aligned}
$$

Now, assume that the formula is true for $n-1$.

In the first place, one consequence of Cauchy's integral formula is that for a continuous function $\phi(z)$ continuous for $z \in \gamma$ for an arc γ,

$$F_n(z) =$$

Exercise 2.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that the exist $M, r > 0$ and $n \in \mathbb{N}$ such that $|f(z)| < M|z|^n$ for every $z \in \mathbb{C}$ for $|z| \geq r$. Show that f is a polynomial of degree at most n.

Observe that the case $n = 0$ is Liouville's theorem.

Solution:

For the case $n = 0$, we have Liouville's theorem because

$$\sup_{z\in\mathbb{C}}\{|f(z)|\} = \max(\sup_{|z|>r}\{|f(z)|\}, \sup_{|z|\leq r}\{|f(z)|\})$$

$$= \max(M, \max_{|z|\leq r}\{|f(z)|\}) < \infty.$$

It follows that $f(z)$ is bounded, and thus, a constant function by Liouville's theorem.

Now, for the general case, note that since f is entire, it has a power series around 0

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

if $|f(z)| < M|z|^n$, then for $R > r$

$$\begin{aligned}
|a_k| &\leq \left|\frac{1}{2\pi i}\right| \oint_{|z|=R} \frac{|f(z)|}{|z|^{n+1}} dz \\
&< \frac{1}{2\pi} \oint_{|z|=R} \frac{M|z|^k}{|z|^{n+1}} dz \\
&\leq \frac{1}{2\pi} \underbrace{2\pi R}_{\text{arc lenght}} \cdot \underbrace{\frac{M}{R^{n-k+1}}}_{\text{function max}} \\
&= \frac{M}{R^{n-k}}
\end{aligned}$$

By letting $R \to \infty$ we conclude that, for $k \geq n+1$, $a_k = 0$. Therefore,

$$f(z) = \sum_{k=0}^{n} a_k z^k,$$

which is a polynomial of degree at most n.

Exercise 3.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Show that either the range of f is dense in $\mathbb{C}$ or f is constant.

(b) Suppose that $\text{Re}(f)$ is bounded. Show that f is constant.

Solution Part (a)

Assume that $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Then, there exists $w_0 \in \mathbb{C}$ and $\varepsilon > 0$ such that $B_\varepsilon(w_0) \cap f(\mathbb{C}) = \emptyset$. This implies that $f(\mathbb{C}) \subseteq \mathbb{C} \backslash B_\varepsilon(w_0)$.

Now, consider the function $\phi(w) = \frac{\varepsilon}{w - w_0}$ which takes every point in the complement of $B_\varepsilon(w_0)$ inside the closed disk $B_1(0)$. That is because, if $|w - w_0| \geq \varepsilon$, then

$$|\phi(w)| = \frac{\varepsilon}{|w - w_0|} \leq \frac{\varepsilon}{\varepsilon} = 1.$$

It follows that $\phi \circ f$ is entire because $f(z) \neq w_0$ for every $z \in \mathbb{C}$ and it's bounded because $\phi \circ f(\mathbb{C}) \subseteq \phi(\mathbb{C} \backslash B_\varepsilon(z_0)) = B_1(0)$. Finally, if $\phi \circ f(z) = K$, then

$$f(z) = \frac{K}{\varepsilon} + w_0,$$

so f is a constant function.

Solution Part (b)

Let $f(z) = u(z) + iv(z)$, where $u, v : \mathbb{C} \to \mathbb{R}$ and $u(z) \leq M$ for every $z \in \mathbb{C}$. Then, we use Euler's formula,

$$e^{f(z)} = e^{u(z)}(\cos(v(z)) + i \sin(v(z))).$$

Note that since u is bounded by M, $e^{u(z)} \leq e^M$. On the other hand, $\cos(z) + i \sin(z)$ is on the unit circle (for $z \in \mathbb{R}$). Therefore,

$$|e^{f(z)}| \leq e^M$$

This implies that $\exp \circ f$ is a constant function $e^{f(z)} = K$, $K \neq 0$. Then,

$$\begin{aligned} \frac{d}{dz} e^{f(z)} &= 0 \\ \implies f'(z) e^{f(z)} &= 0 \\ \implies f'(z) K &= 0 \\ \implies f'(z) &= 0. \end{aligned}$$

Therefore, f is a constant function too.

Exercise 4.

Let $U \subseteq \mathbb{C}$ be a region, $z_0 \in U$ and $R > 0$ such that $B_R(z_0) \subseteq U$. Let $f : U \to \mathbb{C}$ be holomorphic with a Taylor series $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ centered around z_0. For $0 < r < R$ define $M(r) := \sup_{|z-z_0|=r} |f(z)|$.

(a) Show that for every $n \in \mathbb{N}_0$ and $0 < r < R$

$$c_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})e^{-int} dt.$$

(b) Show that for every $0 < r < R$

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|^2 dt \leq M(r)^2.$$