

Complex Analysis: Homework 9

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October 16, 2024

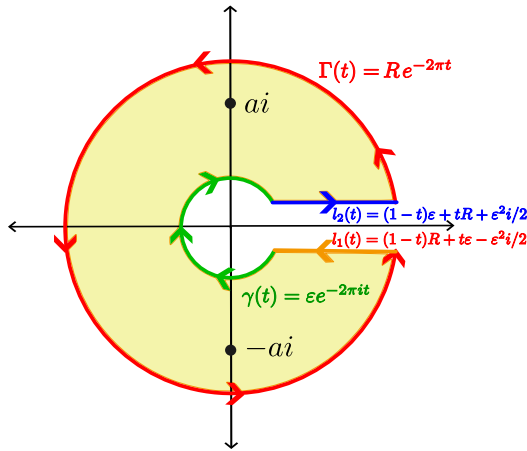
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Exercise 1.

Let $a > 0$. Calculate the following integrals:

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2} dx, \quad \int_0^\infty \frac{\sqrt{x}}{(x^2 + 4)^2} dx.$$

Solution



For both items of the exercise we are going to consider the following contour which consists of 4 paths. In the first place let $R, \varepsilon > 0$ such that $\pm ai \in B_R(0) \setminus B_\varepsilon(0)$ (for item (b) $a = 4$). Then, the closed path \mathcal{C} we're going to integrate over is the concatenation of 4 paths:

$$\mathcal{C} = \Gamma + l_1 + \gamma + l_2.$$

$$\Gamma(t) = Re^{2\pi it}, \quad l_1(t) = (1-t)R + t\varepsilon - \frac{\varepsilon^2 i}{2},$$

$$\gamma(t) = \varepsilon e^{-2\pi it}, \quad l_2(t) = (1-t)\varepsilon + tR + \frac{\varepsilon^2 i}{2}.$$

Of course, without giving much more detail the start and finish points of Γ and γ are defined in such way that allow \mathcal{C} to be the continuous closed curve as the left picture shows.

This is the outline for my solution:

$$\bullet \int_{\mathcal{C}} f(z) dz = \int_{\Gamma} f(z) dz + \int_{l_1} f(z) dz + \int_{\gamma} f(z) dz + \int_{l_2} f(z) dz.$$

- In both items of this exercise, f has poles at $z = \pm ai$ and \mathcal{C} surrounds each pole exactly once, so by the residue theorem,

$$\int_{\mathcal{C}} f(z)dz = 2\pi i(\text{Res}_f(ai) + \text{Res}_f(-ai)).$$

- Using $|\int_{\alpha} f(z)dz| \leq \text{length}(\alpha) \cdot \max_{z \in \alpha} |f(z)|$, we can conclude that $|\int_{\Gamma} f(z)dz|$ and $|\int_{\gamma} f(z)dz|$ vanish when $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ respectively.
- I'm going to define $\sqrt{x} = \exp(\frac{1}{2} \log(x))$, where the branch of \log I'm considering is

$$\log(z) = \ln|z| + i \arg(z), \quad \arg(z) \in (0, 2\pi).$$

This branch is defined for $\mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$.

- Using the previous remark, we are also showing that as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$, for both items:

$$\int_{l_i} f(z)dz \rightarrow \int_0^\infty f(x)dx, \quad i = 1, 2.$$

In fact, after putting everything together, we are showing that:

$$\int_0^\infty f(x)dx = \frac{1}{2} \int_{\mathcal{C}} f(z)dz = \pi i(\text{Res}_f(ai) + \text{Res}_f(-ai)).$$

Solution Item (a)

In this case, $f(z) = \frac{\sqrt{z}}{z^2 + a^2}$ with poles of order 1 in $\pm ai$. In the first place, remember that we've chosen R such that $R > a$, so in order to prove that $|\int_{\Gamma} f(z)dz| \rightarrow 0$ as $R \rightarrow \infty$, note that for $z \in \Gamma$, using the inverse triangle inequality

$$\begin{aligned} \frac{1}{|z^2 + a^2|} &\leq \frac{1}{|z^2| - |a^2|} = \frac{1}{R^2 - a^2} \\ \Rightarrow \left| \int_{\Gamma} f(z)dz \right| &\leq \underbrace{2\pi R}_{\leq \text{length}(\Gamma)} \cdot \underbrace{\frac{R^{1/2}}{R^2 - a^2}}_{\text{upper bound}} \\ &= 2\pi \frac{R^{3/2}}{R^2 - a^2} \\ &\rightarrow 0, \quad R \rightarrow \infty. \end{aligned}$$

Similarly,

$$\begin{aligned}
\Rightarrow \left| \int_{\gamma} f(z) dz \right| &\leq \underbrace{2\pi\varepsilon}_{\leq \text{length}(\gamma)} \cdot \underbrace{\frac{\varepsilon^{1/2}}{\varepsilon^2 - a^2}}_{\text{upper bound}} \\
&= 2\pi \frac{\varepsilon^{3/2}}{\varepsilon^2 - a^2} \\
&\rightarrow 2\pi \frac{0^{3/2}}{0^2 - a^2} = 0, \quad \varepsilon \rightarrow 0.
\end{aligned}$$

Now, take into account that

$$\sqrt{z} = \exp\left(\frac{1}{2} \ln |z|\right) \cdot \exp\left(\frac{i}{2} \arg(z)\right)$$

Then, note that for $z \in l_2$, $\arg(z)$ converges to 0 because every point in l_2 approaches to the real line from above. Therefore,

$$\begin{aligned}
\int_{l_2} \frac{\sqrt{z}}{z^2 + a^2} dz &= \int_{l_2} \frac{\exp\left(\frac{1}{2} \ln |z|\right) \exp\left(\frac{1}{2} \arg(z)\right)}{z^2 + a^2} dz \\
&\rightarrow \int_0^{\infty} \frac{\exp\left(\frac{1}{2} \ln |z|\right) \exp(0)}{z^2 + a^2} dz \\
&= \int_0^{\infty} \frac{\sqrt{x}}{x^2 + a^2} dx.
\end{aligned}$$

On the other hand for $z \in l_1$, $\arg(z)$ converges to 2π because every point in l_1 approaches to the real line from below. Therefore,

$$\begin{aligned}
\int_{l_1} \frac{\sqrt{z}}{z^2 + a^2} dz &= \int_{l_1} \frac{\exp\left(\frac{1}{2} \ln |z|\right) \exp\left(\frac{1}{2} \arg(z)\right)}{z^2 + a^2} dz \\
&\rightarrow - \int_0^{\infty} \frac{\exp\left(\frac{1}{2} \ln |z|\right) \exp(\pi i)}{z^2 + a^2} dz \\
&= \int_0^{\infty} \frac{\sqrt{x}}{x^2 + a^2} dx.
\end{aligned}$$

Finally,

$$\begin{aligned}
\text{Res}_f(ai) &= \lim_{z \rightarrow ai} (z - ai) \frac{\sqrt{z}}{(z - ai)(z + ai)} = \frac{\sqrt{a}(1+i)\sqrt{2}^{-1}}{2ai}, \\
\text{Res}_f(-ai) &= \lim_{z \rightarrow -ai} (z + ai) \frac{\sqrt{z}}{(z - ai)(z + ai)} = \frac{\sqrt{a}(-1+i)\sqrt{2}^{-1}}{-2ai}.
\end{aligned}$$

Thus,

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + a^2} dx = \frac{\pi i \sqrt{a} \sqrt{2}^{-1}}{2ai} (1 + i + 1 - i) = \frac{\pi}{\sqrt{2a}}.$$

Solution Item (b)

In this case, $f(z) = \frac{\sqrt{z}}{(z^2+a^2)^2}$ with poles of order 2 at $\pm ai$ where $a = 2$. Using the exact same argument from the previous item,

$$\begin{aligned} \frac{1}{|(z^2+a^2)^2|} &\leq \frac{1}{|z^4|-2|a||z^2|-|a^2|} = \frac{1}{R^4-2aR^2-a^2} \\ \Rightarrow \left| \int_{\Gamma} f(z)dz \right| &\leq \underbrace{2\pi R}_{\leq \text{length}(\Gamma)} \cdot \underbrace{\frac{R^{1/2}}{R^4-2aR^2-a^2}}_{\text{upper bound}} \\ &= 2\pi \frac{R^{3/2}}{R^4-2aR^2-a^2} \\ &\rightarrow 0, \quad R \rightarrow \infty. \end{aligned}$$

For γ :

$$\begin{aligned} \Rightarrow \left| \int_{\gamma} f(z)dz \right| &\leq \underbrace{2\pi\varepsilon}_{\leq \text{length}(\gamma)} \cdot \underbrace{\frac{\varepsilon^{1/2}}{\varepsilon^4-2a\varepsilon^2-a^2}}_{\text{upper bound}} \\ &= 2\pi \frac{\varepsilon^{3/2}}{\varepsilon^4-2a\varepsilon^2-a^2} \\ &\rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

For l_2 , as $\varepsilon \rightarrow 0$, $\arg(z) \rightarrow 0$, so

$$\int_{l_2} \frac{\sqrt{z}}{(z^2+a^2)^2} dz \rightarrow \int_0^\infty \frac{\exp(\frac{1}{2}\ln|z|)\exp(0)}{(z^2+a^2)^2} dz = \int_0^\infty \frac{\sqrt{x}}{(x^2+a^2)^2} dx.$$

For l_1 , as $\varepsilon \rightarrow 0$, $\arg(z) \rightarrow 2\pi$, so

$$\int_{l_1} \frac{\sqrt{z}}{(z^2+a^2)^2} dz \rightarrow - \int_0^\infty \frac{\exp(\frac{1}{2}\ln|z|)\exp(\pi i)}{(z^2+a^2)^2} dz = \int_0^\infty \frac{\sqrt{x}}{(x^2+a^2)^2} dx.$$

Therefore, since

$$\text{Res}_f(ai) = \lim_{z \rightarrow ai} \frac{d}{dz} \frac{\sqrt{z}(z-ai)^2}{(z-ai)^2(z+ai)^2} = \lim_{z \rightarrow ai} \frac{ai-3z}{2\sqrt{z}(z+ai)^3} = \frac{ai-3ai}{2\sqrt{ai}(2ai)^3} = \frac{(ai)^{-5/2}}{8},$$

$$\text{Res}_f(-ai) = \lim_{z \rightarrow -ai} \frac{d}{dz} \frac{\sqrt{z}(z+ai)^2}{(z-ai)^2(z+ai)^2} = \lim_{z \rightarrow -ai} \frac{-ai-3z}{2\sqrt{z}(z-ai)^3} = \frac{-ai+3ai}{2\sqrt{-ai}(-2ai)^3} = \frac{(-ai)^{-5/2}}{8},$$

it follows that

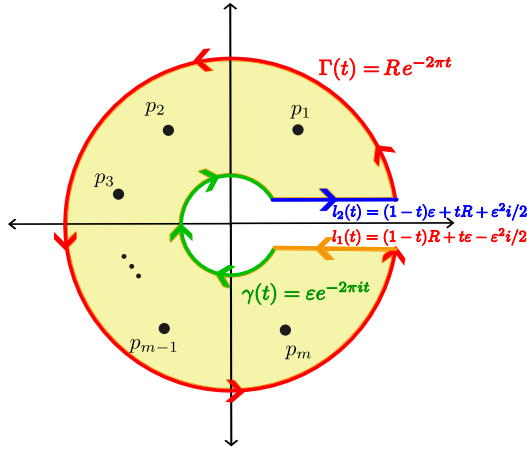
$$\int_0^\infty \frac{\sqrt{x}}{(x^2+a^2)^2} dx = \pi i \left(\frac{(ai)^{-5/2}}{8} + \frac{(-ai)^{-5/2}}{8} \right) = \frac{\pi}{4\sqrt{2}a^5} = \frac{\pi}{32}.$$

Exercise 2.

Let P, Q be polynomials with $Q(x) \neq 0$ for every $x \geq 0$ and $\deg Q \geq 2 + \deg P$ and let $F = \frac{P}{Q}$. Express $\int_0^\infty F(x)dx$ in terms of the residues of $\log(\cdot)F(\cdot)$ where \log is the branch of the complex logarithm defined in $\mathbb{C} \setminus \{r \in \mathbb{R} : r \geq 0\}$.

Solution

We use the exact same outline for the previous exercise. We use the same contour with the same \log branch. The only difference is that this time we choose R and ε for the contour to surround all the zeroes $\{p_1, \dots, p_m\}$ of Q exactly once



$$\mathcal{C} = \Gamma + l_1 + \gamma + l_2.$$

$$\Gamma(t) = Re^{2\pi it}, \quad l_1(t) = (1-t)R + t\varepsilon - \frac{\varepsilon^2 i}{2},$$

$$\gamma(t) = \varepsilon e^{-2\pi it}, \quad l_2(t) = (1-t)\varepsilon + tR + \frac{\varepsilon^2 i}{2}.$$

Also,

$$\log(z) = \ln|z| + i \arg(z), \quad \arg(z) \in (0, 2\pi).$$

So we want to prove that as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$,

$$2\pi i \sum_{z \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}} \text{Res}_{\log(\cdot)F(\cdot)}(z) = \int_{\mathcal{C}} \frac{\log(z)P(z)}{Q(z)} dz = -2\pi i \int_0^\infty \frac{P(x)}{Q(x)} dx.$$

Using real analysis, we can get the following bound: $|F(z)| \leq K \frac{1}{R^2}$ for some $K > 0$ and $z \in \Gamma$. Also, $|\log(z)| = |\ln|z| + i \arg(z)| \leq \ln|R| + 2\pi$. Therefore,

$$\left| \int_{\Gamma} F(z) dz \right| \leq \underbrace{2\pi R}_{\leq \text{length}(\Gamma)} \cdot \underbrace{\frac{K(\ln R + 2\pi)}{R^2}}_{\text{upper bound}} \rightarrow 0, \quad R \rightarrow \infty.$$

Now, note that since $Q(0) \neq 0$, it follows that if $Q(z) = q_m z^m + \dots + q_0$, then $q_0 \neq 0$. Then,

for $z \in \gamma$, the inverse triangle inequality states that

$$\begin{aligned} |Q(z)| &\geq |z||q_m z^{m-1} + \cdots + q_1| - q_1 \\ \implies |Q(z)| &\geq O(\varepsilon) - q_0, \quad \text{as } \varepsilon \rightarrow 0 \end{aligned}$$

Since $\lim_{x \rightarrow 0} x \ln x = 0$, we conclude that

$$\begin{aligned} \left| \int_{\gamma} F(z) dz \right| &\leq \underbrace{2\pi\varepsilon}_{\leq \text{length}(\gamma)} \cdot \underbrace{\frac{(\ln \varepsilon + 2\pi)|P(z)|}{|Q(z)|}}_{\text{upper bound}} \\ &\rightarrow \frac{\lim_{\varepsilon \rightarrow 0} \varepsilon \ln \varepsilon}{0 - q_0} = 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

For l_2 , as $\varepsilon \rightarrow 0$, $\arg(z) \rightarrow 0$, so

$$\int_{l_2} \frac{\log(z)P(z)}{Q(z)} dz \rightarrow \int_0^{\infty} \frac{(\ln|z| + 0)P(z)}{Q(z)} dz = \int_0^{\infty} \frac{\ln(x)P(x)}{Q(x)} dx.$$

For l_1 , as $\varepsilon \rightarrow 0$, $\arg(z) \rightarrow 2\pi$, so

$$\int_{l_1} \frac{\log(z)P(z)}{Q(z)} dz \rightarrow \int_{\infty}^0 \frac{(\ln|z| + 2\pi)P(z)}{Q(z)} dz = - \int_0^{\infty} \frac{\ln(x)P(x)}{Q(x)} dx - 2\pi i \int_0^{\infty} \frac{P(x)}{Q(x)} dx.$$

Finally,

$$\begin{aligned} \int_C \frac{\log(z)P(z)}{Q(z)} dz &= \int_0^{\infty} \frac{\ln(x)P(x)}{Q(x)} dx - \int_0^{\infty} \frac{\ln(x)P(x)}{Q(x)} dx - 2\pi i \int_0^{\infty} \frac{P(x)}{Q(x)} dx \\ &= -2\pi i \int_0^{\infty} \frac{P(x)}{Q(x)} dx. \end{aligned}$$

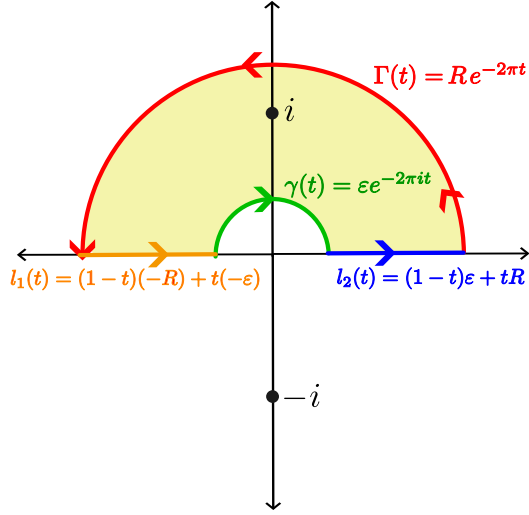
Then, it follows that

$$\int_0^{\infty} \frac{P(x)}{Q(x)} dx = \sum_{z \in \mathbb{C} \setminus \mathbb{R} \geq 0} \text{Res}_{\log(\cdot)F(\cdot)}(z).$$

Exercise 3.

Show that $\int_0^{\infty} \frac{\ln x}{1+x^2} dx = 0$.

Solution



Similarly to the first exercise. Let $R, \varepsilon > 0$ such that $\pm i \in B_R(0) \setminus B_\varepsilon(0) \cap \{z \in C : \text{Im}(z) \geq 0\}$. Then, the closed path \mathcal{C} we're going to integrate over is the concatenation of 4 paths:

$$\mathcal{C} = \Gamma + l_1 + \gamma + l_2.$$

$$\Gamma(t) = Re^{\pi it}, \quad l_1(t) = (1-t)(-R) + t(-\varepsilon),$$

$$\gamma(t) = \varepsilon e^{-\pi it}, \quad l_2(t) = (1-t)\varepsilon + tR.$$

In this case I can say that, for each curve, $t \in [0, 1]$.

Now we are going to use a similar idea to exercise 1,

$$\int_{\mathcal{C}} f(z) dz = \int_{\Gamma} f(z) dz + \int_{l_1} f(z) dz + \int_{\gamma} f(z) dz + \int_{l_2} f(z) dz.$$

However, in this case we're going to use the branch of $\log = \ln|\cdot| + i \arg(\cdot)$ defined on $\mathbb{C} \setminus \{-ix : x \geq 0\}$ ($\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$). Note that for $z \in \Gamma$ and $z \in \gamma$, $\arg(z) \leq \pi$. Therefore, we can use this bound

$$\left| \int_{\Gamma} \frac{\ln|z| + i \arg(z)}{z^2 + 1} dz \right| \leq \underbrace{\pi R}_{\text{length}(\Gamma)} \underbrace{\frac{\ln R + \pi}{R^2 - 1}}_{\text{upper bound}} \rightarrow 0, \quad \text{as } R \rightarrow \infty.$$

Also, $\lim_{x \rightarrow 0} x \ln(x) = 0$, and thus,

$$\left| \int_{\gamma} \frac{\ln|z| + i \arg(z)}{z^2 + 1} dz \right| \leq \underbrace{\pi \varepsilon}_{\text{length}(\Gamma)} \underbrace{\frac{\ln \varepsilon + \pi}{\varepsilon^2 - 1}}_{\text{upper bound}} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Now, for the line integrals, it's clear for l_2 that

$$\int_{l_2} \frac{\log(z)}{z^2 + 1} dz = \int_{\varepsilon}^R \frac{\ln x}{x^2 + 1} dx \rightarrow \int_0^{\infty} \frac{\ln x}{x^2 + 1} dx, \quad \text{as } \varepsilon \rightarrow 0 \text{ and } R \rightarrow \infty.$$

For $z \in l_1$ note that $\arg(z) = \pi$. Then, we make a substitution: $z = -u$ to obtain,

$$\begin{aligned} \int_{l_1} \frac{\log(z)}{z^2 + 1} dz &= \int_{-R}^{-\varepsilon} \frac{\ln z + i \arg(z)}{z^2 + 1} dz \\ &= \int_R^{\varepsilon} \frac{-\ln(-u) + i\pi}{(-u)^2 + 1} du \\ &= \int_{\varepsilon}^R \frac{\ln u + \ln(-1) + i\pi}{u^2 + 1} du \\ &\rightarrow \int_0^{\infty} \frac{\ln x}{x^2 + 1} dx + \int_0^{\infty} \frac{\pi i}{x^2 + 1} dx, \quad \text{as } \varepsilon \rightarrow 0 \text{ and } R \rightarrow \infty. \end{aligned}$$

Using the substitution $x = \tan(u)$ we obtain $\int \frac{1}{x^2 + 1} dx = \arctan(x)$. Then, since $\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$ and $\arctan(0) = 0$,

$$\int_0^\infty \frac{\pi i}{x^2 + 1} dx = \frac{\pi^2 i}{2}.$$

Finally, using residue theorem. The contour \mathcal{C} surrounds i exactly once, and thus,

$$2\pi i \cdot \text{Res}_f(i) = \int_{\mathcal{C}} \frac{\log z}{z^2 + 1} dz = 2 \int_0^\infty \frac{\ln(x)}{x^2 + 1} dx + \frac{\pi^2 i}{2}.$$

and since $z = i$ is a simple pole

$$\begin{aligned} \text{Res}_f(i) &= \lim_{z \rightarrow i} (z - i) \frac{\log(z)}{(z - i)(z + i)} = \frac{\log(i)}{2i} = \frac{\pi i/2}{2i} \\ \implies \int_0^\infty \frac{\ln(x)}{x^2 + 1} dx &= \pi i (\text{Res}_f(i)) - \frac{\pi^2 i}{4} = \frac{\pi^2 i}{4} - \frac{\pi^2 i}{4} = 0. \end{aligned}$$

Exercise 4.

Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a path, and let $(f_t, U_t)_{t \in [0, 1]}$ be an analytic continuation along γ . For $t \in [0, 1]$, let $R(t)$ be the radius of convergence of the Taylor series of f_t centered at $\gamma(t)$. Prove that either $R(t) = \infty$ for all t , or that $R : [0, 1] \rightarrow (0, \infty)$ is continuous.

Solution:

Exercise 5.

Let $U \subset \mathbb{C} \setminus \{0\}$ be an open set, and suppose there exists a path in U such that $\text{ind}_\gamma(0) = 1$. Prove that there is no holomorphic n -th root in U for $n \geq 2$.

Solution

If there exists an