Complex Analysis: Homework 2

Martín Prado

August 21, 2024
Universidad de los Andes – Bogotá Colombia

Exercise 1.

Find all the points $z \in \mathbb{C}$ where the following functions are differentiable and find the largest open set U where they are holomorphic.

(a) $f(z) = \overline{z}$

(b) $f(x + iy) = x^2 + y^2 + i(x^2 - y^2)$

Solution Part (a)

$$f(x + iy) = x - iy = u(x, y) + iv(x, y)$$

Then,

$$\frac{\partial u}{\partial x}(x, y) = 1, \qquad \frac{\partial v}{\partial x}(x, y) = 0,$$
$$\frac{\partial u}{\partial y}(x, y) = 0, \qquad \frac{\partial v}{\partial y}(x, y) = -1,$$

All the partial derivatives exists and are continuous on any $(x, y) \in \mathbb{R}^2$, and thus, the function is differentiable. However, the Cauchy-Riemann equations are a requirement for f to be complex-differentiable. Therefore, since $\partial u/\partial x \neq \partial v/\partial y$ on all points, the largest open set where it's holomorphic is $U = \emptyset$.

Solution Part (b)

In this case,

$$u(x, y) = x^2 + y^2, \quad v(x, y) = x^2 - y^2,$$

and the respective partial derivatives are

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = -2y.$$

For differentiability in $\mathbb{R}^2$, the argument is again that the partial derivatives exist and are continuous. For complex-differentiability, the function is holomorphic only when $2x = -2y$. Thus, the largest open set is again $U = \emptyset$.

Exercise 2.

(a) Let $u(x, y) = x^3 - 3xy^2$. Find all the entire functions f such that $u = \mathrm{Re}(f)$.

(b) Let $v(x, y) = x^2 + y^2$. Find all the entire functions f such that $v = \mathrm{Im}(f)$.

(c) Let $U \subseteq \mathbb{C}$ be a region and let $f, g : U \to \mathbb{C}$ be holomorphic functions such that $f(U) \subset \mathbb{R}$ and $g(U) \subset \mathbb{S}^1 := \{z \in \mathbb{C} \ : \ |z| = 1\}$. Prove that f and g are constant.

Solution Part (a)

$$\frac{\partial v}{\partial y}(x, y) = \frac{\partial u}{\partial x}(x, y) = 3x^2 - 3y^2,$$
$$-\frac{\partial v}{\partial x}(x, y) = \frac{\partial u}{\partial y}(x, y) = -6xy.$$

The solutions for these partial equations are

$$v(x, y) = \int 3x^2 - 3y^2 \, dy = 3x^2y - y^3 + K_1(x),$$
$$v(x, y) = \int 6xy \, dx = 3x^2y + K_2(y).$$

Therefore,

$$v(x, y) = 3x^2y - y^3 + K_1(y) = 3x^2y + K_2(x)$$
$$\implies K_1(y) = y^3 + K_2(x)$$

This can only happen if K_1 is a constant $K \in \mathbb{C}$ and $K_2(y) = y^3 + K$. Thus,

$$v(x, y) = 3x^2y - y^3 + K.$$

Finally, the family of entire functions that satisfy the initial condition are:

$$f_K(x, y) = x^3 - 3xy^2 + i(3x^2y - y^3 + K), \quad K \in \mathbb{C}$$
$$= (x + iy)^3 + iK.$$

Solution Part (b)

$$-\frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial x}(x,y) = 2x,$$
$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 2y.$$

The solutions for these partial equations are

$$u(x,y) = \int -2x\;dy = -2xy + K_1(x),$$

$$u(x,y) = \int 2y\;dx = 2xy + K_2(y).$$

Therefore,

$$u(x,y) = -2xy + K_1(y) = 2xy + K_2(x)$$
$$\implies K_1(y) - K_2(x) = 4xy.$$

However, this cannot be possible for any entire function because the previous would imply that the functions K_1, K_2 are not well defined:

$$\frac{\partial K_1}{\partial y}(y) = 4x,$$

$$\frac{\partial K_2}{\partial y}(x) = -4y.$$

Thus, there doesn't exist any entire function with the initial conditions.

Solution Part (c)

Let $z = x + iy$. For f, note that if $f(x,y) = u(x,y) + i(v,y)$, then $v(x,y) = 0$ for $z \in U$. Then, using the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

However, $\nabla u(x,y) = (0,0)$ if and only if u is a constant function, and thus, f is a constant function too.

For g, we can make a variable substitution to the polar coordinates. $x(r,\theta) = r\cos(\theta)$ and $y(r,\theta) = r\sin(\theta)$

Exercise 3.

(a) $\exp(z+w) = \exp(z)\exp(w)$.

(b) $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.

(c) $|\exp(z)| = 1$ if and only if $z \in i\mathbb{R}$.

(d) $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathbb{C}$.

(e) $\cos(z + 2\pi) = \cos(z)$ and $\sin(z + 2\pi) = \sin(z)$ for all $z \in \mathbb{C}$.

(f) $\cos(z) = 0$ or $\sin(z) = 0 \implies z \in \mathbb{R}$.

(g) For every $x \in \mathbb{R}$, $\lim_{t\to\pm\infty} |\cos(x+it)| = \infty$ and $\lim_{t\to\pm\infty} |\sin(x+it)| = \infty$. The limit is uniform in x.

Solution Part (a)

The Cauchy product of 2 series implies that

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \cdot \frac{n!}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\
&= \exp(z+w)
\end{aligned}$$

Solution Part (b)

For every complex number z, there exists an additive inverse $(-z)$ such that

$$z + (-z) = 0.$$

Thus, if it was the case that there exists $z \in \mathbb{C}$ such that $e^z = 0$, then, using part (a),

$$1 = e^0 = e^{z+(-z)} = e^z e^{-z} = 0,$$

and this would lead to a contradiction.

Solution Part (c)

$\Longleftarrow$ **:** We are going to prove Euler's formula. Let $z = iy,\ y \in \mathbb{R}$,

$$\begin{aligned}
\exp(iy) &= \frac{(iy)^0}{0!} + \frac{(iy)^1}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\
&= \left(\frac{(iy)^0}{0!} + \frac{(iy)^2}{2!} + \cdots\right) + i\left(\frac{i^0 y^1}{1!} + \frac{i^2 y^3}{3!} + \cdots\right) \\
&= \sum_{n=0}^{\infty} i^{2n} \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} i^{2n} \frac{y^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} \\
&= \cos(y) + i\sin(y).
\end{aligned}$$

Therefore, we have $\cos^2(y) + \sin^2(y) = 1$ for $y \in \mathbb{R}$, and thus,

$$|\exp(iy)| = \sqrt{\cos^2(y) + \sin^2(y)} = 1$$

$\Longrightarrow$ **:** Let $z = x + iy,\ x, y \in \mathbb{R}$ such that $|\exp(z)| = 1$. Then, using part (a), $|\exp(z)| = |\exp(x)||\exp(iy)|$. Then, using the previous implication, we know that $|\exp(iy)| = 1$. Therefore, $|\exp(z)| = |\exp(x)| = \exp(x) = 1$, but for real numbers, the only solution for $\exp(x) = 1$ is $x = 0$.

Exercise 4.

Prove that

(a) $\sum_{n=1}^{\infty} nz^n$ does not converge to any point for $z \in \mathbb{S}^1$.

(b) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges to every point for $z \in \mathbb{S}^1$

(c) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges to every point for $z \in \mathbb{S}^1$, except for 1.

Exercise 5.

A subset $S \subset \mathbb{N}$ is in *arithmetic progression* if there exists $a, d \in \mathbb{N}$ such that

$$S = \{a + nd \ : \ n \in \mathbb{N}_0\}.$$

The number d is called the difference of the progression. Prove that $\mathbb{N}$ cannot be partitioned in a finite number greater than 1 of arithmetic progressions with different differences.