# Complex Analysis: Homework 9

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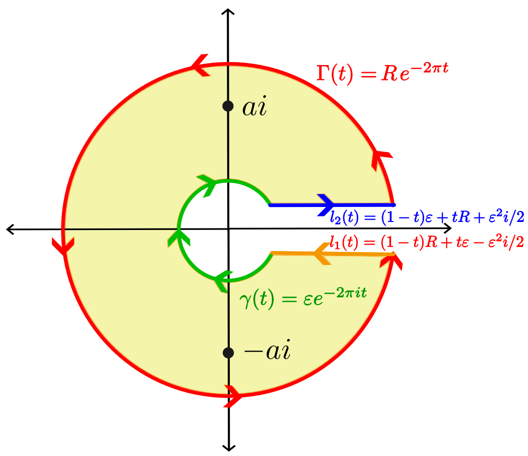
## Exercise 1.

Let $a > 0$. Calculate the following integrals:

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2}\, dx, \qquad \int_0^\infty \frac{\sqrt{x}}{(x^2+4)^2}\, dx.$$

### Solution




For both items of the exercise we are going to consider the following contour which consists of 4 paths. In the first place let $R, \varepsilon > 0$ such that $\pm ai \in B_R(0) \backslash B_\varepsilon(0)$ (for item (b) $a = 4$). Then, the closed path $\mathcal{C}$ we're going to integrate over is the concatenation of 4 paths:

$$\mathcal{C} = \Gamma + l_1 + \gamma + l_2.$$

$$\Gamma(t) = Re^{2\pi i t}, \quad l_1(t) = (1-t)R + t\varepsilon - \frac{\varepsilon^2 i}{2},$$

$$\gamma(t) = \varepsilon e^{-2\pi i t}, \quad l_2(t) = (1-t)\varepsilon + tR + \frac{\varepsilon^2 i}{2}.$$

Of course, without giving much more detail the start and finish points of $\Gamma$ and $\gamma$ are defined in such way that allow $\mathcal{C}$ to be the continuous closed curve as the left picture shows.

This is the outline for my solution:

- $\displaystyle\int_{\mathcal{C}} f(z)dz = \int_\Gamma f(z)dz + \int_{l_1} f(z)dz + \int_\gamma f(z)dz + \int_{l_2} f(z)dz.$

- In both items of this exercise, $f$ has poles at $z = \pm ai$ and $\mathcal{C}$ surrounds each pole exactly once, so by the residue theorem,

$$\int_{\mathcal{C}} f(z)dz = 2\pi i(\mathrm{Res}_f(ai) + \mathrm{Res}_f(ai)).$$

- Using $\left|\int_\alpha f(z)dz\right| \leq \mathrm{length}(\alpha) \cdot \max_{z\in\alpha} |f(z)|$, we can conclude that $\left|\int_\Gamma f(z)dz\right|$ and $\left|\int_\gamma f(z)dz\right|$ vanish when $R \to \infty$ and $\varepsilon \to 0$ respectively.

- I'm going to define $\sqrt{x} = \exp\left(\frac{1}{2}\log(z)\right)$, where the branch of log I'm considering is

$$\log(z) = \ln|z| + i\arg(z), \quad \arg(z) \in (0, 2\pi).$$

  This branch is defined for $\mathbb{C}\backslash\{x \in \mathbb{R} \, : \, x \geq 0\}$.

- Using the previous remark, we are also showing that as $R \to \infty$ and $\varepsilon \to 0$, for both items:

$$\int_{l_i} f(z)dz \to \int_0^\infty f(x)dx, \quad i = 1, 2.$$

  In fact, after putting everything together, we are showing that:

$$\int_0^\infty f(x)dx = \frac{1}{2}\int_{\mathcal{C}} f(z)dz = \pi i(\mathrm{Res}_f(ai) + \mathrm{Res}_f(ai)).$$

## Solution Item (a)

In this case, $f(z) = \frac{\sqrt{z}}{z^2+a^2}$ with poles of order 1 in $\pm ai$. In the first place, remember that we've chosen $R$ such that $R > a$, so in order to prove that $\left|\int_\Gamma f(z)dz\right| \to 0$ as $R \to \infty$, note that for $z \in \Gamma$, using the inverse triangle inequality

$$\frac{1}{|z^2 + a^2|} \leq \frac{1}{|z^2| - |a^2|} = \frac{1}{R^2 - a^2}$$

$$\begin{aligned}
\implies \left|\int_\Gamma f(z)dz\right| &\leq \underbrace{2\pi R}_{\leq \, \mathit{length}(\Gamma)} \cdot \underbrace{\frac{R^{1/2}}{R^2 - a^2}}_{\mathit{upper\ bound}} \\
&= 2\pi\frac{R^{3/2}}{R^2 - a^2} \\
&\to 0, \quad R \to \infty.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\implies \left|\int_\gamma f(z)dz\right| &\leq \underbrace{2\pi\varepsilon}_{\leq \, length(\gamma)} \cdot \underbrace{\frac{\varepsilon^{1/2}}{\varepsilon^2 - a^2}}_{upper\ bound} \\
&= 2\pi\frac{\varepsilon^{3/2}}{\varepsilon^2 - a^2} \\
&\to 2\pi\frac{0^{3/2}}{0^2 - a^2} = 0, \quad \varepsilon \to 0.
\end{aligned}
$$

Now, take into account that

$$
\sqrt{z} = \exp\left(\frac{1}{2}\ln|z|\right) \cdot \exp\left(\frac{i}{2}\arg(z)\right)
$$

Then, note that for $z \in l_2$, $\arg(z)$ converges to 0 because every point in $l_2$ approaches to the real line from above. Therefore,

$$
\begin{aligned}
\int_{l_2} \frac{\sqrt{z}}{z^2 + a^2}dz &= \int_{l_2} \frac{\exp\left(\frac{1}{2}\ln|z|\right)\exp\left(\frac{1}{2}\arg(z)\right)}{z^2 + a^2}dz \\
&\to \int_0^\infty \frac{\exp\left(\frac{1}{2}\ln|z|\right)\exp\left(0\right)}{z^2 + a^2}dz \\
&= \int_0^\infty \frac{\sqrt{x}}{x^2 + a^2}\, dx.
\end{aligned}
$$

On the other hand for $z \in l_1$, $\arg(z)$ converges to $2\pi$ because every point in $l_1$ approaches to the real line from below. Therefore,

$$
\begin{aligned}
\int_{l_1} \frac{\sqrt{z}}{z^2 + a^2}dz &= \int_{l_1} \frac{\exp\left(\frac{1}{2}\ln|z|\right)\exp\left(\frac{1}{2}\arg(z)\right)}{z^2 + a^2}dz \\
&\to -\int_0^\infty \frac{\exp\left(\frac{1}{2}\ln|z|\right)\exp\left(\pi i\right)}{z^2 + a^2}dz \\
&= \int_0^\infty \frac{\sqrt{x}}{x^2 + a^2}\, dx.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathrm{Res}_f(ai) &= \lim_{z \to ai}(z - ai)\frac{\sqrt{z}}{(z - a_i)(z + ai)} = \frac{\sqrt{a}(1 + i)\sqrt{2}^{-1}}{2ai}, \\
\mathrm{Res}_f(-ai) &= \lim_{z \to -ai}(z + ai)\frac{\sqrt{z}}{(z - a_i)(z + ai)} = \frac{\sqrt{a}(-1 + i)\sqrt{2}^{-1}}{-2ai}.
\end{aligned}
$$

Thus,

$$
\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2}\, dx = \frac{\pi i\sqrt{a}\sqrt{2}^{-1}}{2ai}(1 + i + 1 - i) = \frac{\pi}{\sqrt{2a}}.
$$

## Solution Item (b)

In this case, $f(z) = \frac{\sqrt{z}}{(z^2+a^2)^2}$ with poles of order 2 at $\pm ai$ where $a = 2$. Using the exact same argument from the previous item,

$$\frac{1}{|(z^2+a^2)^2|} \leq \frac{1}{|z^4| - 2|a||z^2| - |a^2|} = \frac{1}{R^4 - 2aR^2 - a^2}$$

$$\implies \left| \int_\Gamma f(z) dz \right| \leq \underbrace{2\pi R}_{\leq \, length(\Gamma)} \cdot \underbrace{\frac{R^{1/2}}{R^4 - 2aR^2 - a^2}}_{upper\ bound}$$

$$= 2\pi \frac{R^{3/2}}{R^4 - 2aR^2 - a^2}$$

$$\to 0, \quad R \to \infty.$$

For $\gamma$:

$$\implies \left| \int_\gamma f(z) dz \right| \leq \underbrace{2\pi \varepsilon}_{\leq \, length(\gamma)} \cdot \underbrace{\frac{\varepsilon^{1/2}}{\varepsilon^4 - 2a\varepsilon^2 - a^2}}_{upper\ bound}$$

$$= 2\pi \frac{\varepsilon^{3/2}}{\varepsilon^4 - 2a\varepsilon^2 - a^2}$$

$$\to 0, \quad \varepsilon \to 0.$$

For $l_2$, as $\varepsilon \to 0$, $\arg(z) \to 0$, so

$$\int_{l_2} \frac{\sqrt{z}}{(z^2+a^2)^2} dz \to \int_0^\infty \frac{\exp\left(\frac{1}{2}\ln|z|\right)\exp(0)}{(z^2+a^2)^2} dz = \int_0^\infty \frac{\sqrt{x}}{(x^2+a^2)^2} dx.$$

For $l_1$, as $\varepsilon \to 0$, $\arg(z) \to 2\pi$, so

$$\int_{l_1} \frac{\sqrt{z}}{(z^2+a^2)^2} dz \to -\int_0^\infty \frac{\exp\left(\frac{1}{2}\ln|z|\right)\exp(\pi i)}{(z^2+a^2)^2} dz = \int_0^\infty \frac{\sqrt{x}}{(x^2+a^2)^2} dx.$$

Therefore, since

$$\mathrm{Res}_f(ai) = \lim_{z\to ai} \frac{d}{dz} \frac{\sqrt{z}(z-ai)^2}{(z-a_i)^2(z+ai)^2} = \lim_{z\to ai} \frac{ai - 3z}{2\sqrt{z}(z+ai)^3} = \frac{ai - 3ai}{2\sqrt{ai}(2ai)^3} = \frac{(ai)^{-5/2}}{8},$$

$$\mathrm{Res}_f(-ai) = \lim_{z\to -ai} \frac{d}{dz} \frac{\sqrt{z}(z+ai)^2}{(z-a_i)^2(z+ai)^2} = \lim_{z\to -ai} \frac{-ai - 3z}{2\sqrt{z}(z-ai)^3} = \frac{-ai + 3ai}{2\sqrt{-ai}(-2ai)^3} = \frac{(-ai)^{-5/2}}{8},$$

it follows that

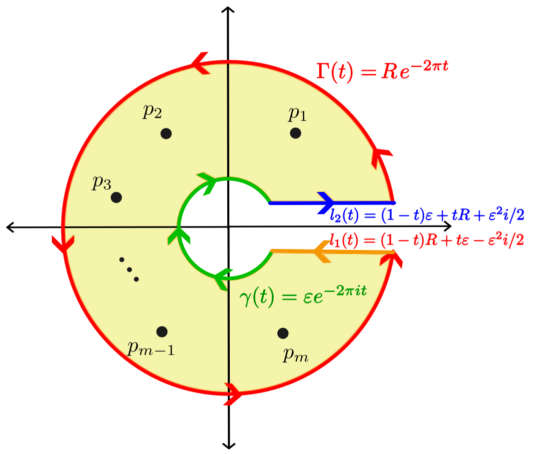
$$\int_0^\infty \frac{\sqrt{x}}{(x^2+a^2)^2} dx = \pi i \left( \frac{(ai)^{-5/2}}{8} + \frac{(-ai)^{-5/2}}{8} \right) = \frac{\pi}{4\sqrt{2a^5}} = \frac{\pi}{32}.$$

## Exercise 2.

Let $P, Q$ be polynomials with $Q(x) \neq 0$ for every $x \geq 0$ and $\deg Q \geq 2 + \deg P$ and let $F = \frac{P}{Q}$. Express $\displaystyle\int_0^\infty F(x)dx$ in terms of the residues of $\log(\cdot)F(\cdot)$ where log is the branch of the complex logarithm defined in $\mathbb{C}\backslash\{r \in \mathbb{R} \, : \, r \geq 0\}$.

### Solution

We use the exact same outline for the previous exercise. We use the same contour with the same log branch. The only difference is that this time we choose $R$ and $\varepsilon$ for the contour to surround all the zeroes $\{p_1, \ldots, p_m\}$ of $Q$ exactly once




$$\mathcal{C} = \Gamma + l_1 + \gamma + l_2.$$

$$\Gamma(t) = Re^{2\pi i t}, \quad l_1(t) = (1-t)R + t\varepsilon - \frac{\varepsilon^2 i}{2},$$

$$\gamma(t) = \varepsilon e^{-2\pi i t}, \quad l_2(t) = (1-t)\varepsilon + tR + \frac{\varepsilon^2 i}{2}.$$

Also,

$$\log(z) = \ln|z| + i\arg(z), \quad \arg(z) \in (0, 2\pi).$$

So we want to prove that as $R \to \infty$ and $\varepsilon \to 0$,

$$2\pi i \sum_{z \in \mathbb{C}\backslash\mathbb{R}^{\geq 0}} \operatorname*{Res}_{\log(\cdot)F(\cdot)}(z) = \int_{\mathcal{C}} \frac{\log(z)P(z)}{Q(z)} dz = -2\pi i \int_0^\infty \frac{P(x)}{Q(x)} dx.$$

Using real analysis, we can get the following bound: $|F(z)| \leq K\dfrac{1}{R^2}$ for some $K > 0$ and $z \in \Gamma$. Also, $|\log(z)| = |\ln|z| + i\arg(z)| \leq \ln|R| + 2\pi$. Therefore,

$$\left|\int_\Gamma F(z)dz\right| \leq \underbrace{2\pi R}_{\leq \, length(\Gamma)} \cdot \underbrace{\frac{K(\ln R + 2\pi)}{R^2}}_{upper\ bound}$$
$$\to 0, \quad R \to \infty.$$

Now, note that since $Q(0) \neq 0$, it follows that if $Q(z) = q_z z^m + \cdots + q_0$, then $q_0 \neq 0$. Then, for $z \in \gamma$, the inverse triangle inequality states that

$$|Q(z)| \geq |z||q_m z^{m-1} + \cdots + q_1| - q_1$$
$$\implies |Q(z)| \geq O(\varepsilon) - q_0, \quad \text{as } \varepsilon \to 0$$

Since $\lim_{x\to 0} x \ln x = 0$, we conclude that

$$\left|\int_\gamma F(z)dz\right| \leq \underbrace{2\pi\varepsilon}_{\leq \, length(\gamma)} \cdot \underbrace{\frac{(\ln\varepsilon + 2\pi)|P(z)|}{|Q(z)|}}_{upper\ bound}$$
$$\to \frac{\lim_{\varepsilon\to 0} \varepsilon \ln\varepsilon}{0 - q_0} = 0, \quad \varepsilon \to 0.$$

For $l_2$, as $\varepsilon \to 0$, $\arg(z) \to 0$, so

$$\int_{l_2} \frac{\log(z)P(z)}{Q(z)}dz \to \int_0^\infty \frac{(\ln|z| + 0)P(z)}{Q(z)}dz = \int_0^\infty \frac{\ln(x)P(x)}{Q(x)}dx.$$

For $l_1$, as $\varepsilon \to 0$, $\arg(z) \to 2\pi$, so

$$\int_{l_1} \frac{\log(z)P(z)}{Q(z)}dzdz \to \int_\infty^0 \frac{(\ln|z| + 2\pi)P(z)}{Q(z)}dz = -\int_0^\infty \frac{\ln(x)P(x)}{Q(x)}dx - 2\pi i \int_0^\infty \frac{P(x)}{Q(x)}dx.$$

Finally,

$$\int_{\mathcal{C}} \frac{\log(z)P(z)}{Q(z)}dz = \int_0^\infty \frac{\ln(x)P(x)}{Q(x)}dx - \int_0^\infty \frac{\ln(x)P(x)}{Q(x)}dx - 2\pi i \int_0^\infty \frac{P(x)}{Q(x)}dx$$
$$= -2\pi i \int_0^\infty \frac{P(x)}{Q(x)}dx.$$

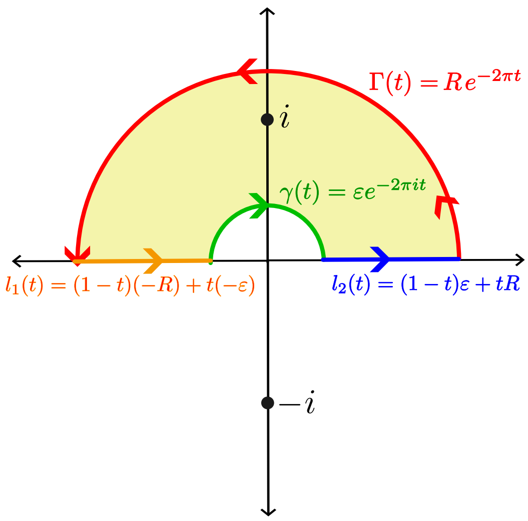
Then, it follows that

$$\int_0^\infty \frac{P(x)}{Q(x)}dx = \sum_{z\in\mathbb{C}\setminus\mathbb{R}_{\geq 0}} \operatorname*{Res}_{\log(\cdot)F(\cdot)}(z).$$

## Exercise 3.

Show that $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}dx = 0.$

### Solution




Similarly to the first exercise. Let $R, \varepsilon > 0$ such that $\pm i \in B_R(0) \backslash B_\varepsilon(0) \cap \{z \in C \ : \ \text{Im}(z) \geq 0\}$. Then, the closed path $\mathcal{C}$ we're going to integrate over is the concatenation of 4 paths:

$$\mathcal{C} = \Gamma + l_1 + \gamma + l_2.$$

$$\Gamma(t) = Re^{\pi i t}, \quad l_1(t) = (1-t)(-R) + t(-\varepsilon),$$

$$\gamma(t) = \varepsilon e^{-\pi i t}, \quad l_2(t) = (1-t)\varepsilon + tR.$$

In this case I can say that, for each curve, $t \in [0, 1]$.

Now we are going to use a similar idea to exercise 1,

$$\int_{\mathcal{C}} f(z)dz = \int_\Gamma f(z)dz + \int_{l_1} f(z)dz + \int_\gamma f(z)dz + \int_{l_2} f(z)dz.$$

However, in this case we're going to use the branch of $\log = \ln|\cdot| + i\arg(\cdot)$ defined on $\mathbb{C}\backslash\{-ix \ : \ x \geq 0\}$ ($\arg(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$). Note that for $z \in \Gamma$ and $z \in \gamma$, $\arg(z) \leq \pi$. Therefore, we can use this bound

$$\left| \int_\Gamma \frac{\ln|z| + i\arg(z)}{z^2+1} \right| \leq \underbrace{\pi R}_{length(\Gamma)} \underbrace{\frac{\ln R + \pi}{R^2 - 1}}_{upper\ bound} \to 0, \quad \text{as } R \to \infty.$$

Also, $\lim_{x \to 0} x\ln(x) = 0$, and thus,

$$\left| \int_\gamma \frac{\ln|z| + i\arg(z)}{z^2+1} \right| \leq \underbrace{\pi \varepsilon}_{length(\Gamma)} \underbrace{\frac{\ln \varepsilon + \pi}{\varepsilon^2 - 1}}_{upper\ bound} \to 0, \quad \text{as } \varepsilon \to 0.$$

Now, for the line integrals, it's clear for $l_2$ that

$$\int_{l_2} \frac{\log(z)}{z^2+1} = \int_\varepsilon^R \frac{\ln x}{x^2+1}dx \to \int_0^\infty \frac{\ln x}{x^2+1}dx, \quad \text{as } \varepsilon \to 0 \text{ and } R \to \infty.$$

For $z \in l_1$ note that $\arg(z) = \pi$. Then, we make a substitution: $z = -u$ to obtain,

$$\begin{aligned}
\int_{l_1} \frac{\log(z)}{z^2+1} &= \int_{-R}^{-\varepsilon} \frac{\ln z + i\overbrace{\arg(z)}^{\pi}}{z^2+1}dz \\
&= \int_R^\varepsilon \frac{-\ln(-u) + i\pi}{(-u)^2+1}du \\
&= \int_\varepsilon^R \frac{\ln u + \cancelto{0}{\ln(-1)} + i\pi}{u^2+1}du \\
&\to \int_0^\infty \frac{\ln x}{x^2+1}dx + \int_0^\infty \frac{\pi i}{x^2+1}dx, \quad \text{as } \varepsilon \to 0 \text{ and } R \to \infty.
\end{aligned}$$

Using the substitution $x = \tan(u)$ we obtain $\displaystyle\int \frac{1}{x^2+1}dx = \arctan(x)$. Then, since $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$ and $\arctan(0) = 0$,

$$\int_0^\infty \frac{\pi i}{x^2+1}dx = \frac{\pi^2 i}{2}.$$

Finally, using residue theorem. The contour $\mathcal{C}$ surrounds $i$ exactly once, and thus,

$$2\pi i \cdot \mathrm{Res}_f(i) = \int_{\mathcal{C}} \frac{\log z}{z^2+1}dz = 2\int_0^\infty \frac{\ln(x)}{x^2+1}dx + \frac{\pi^2 i}{2}.$$

and since $z = i$ is a simple pole

$$\mathrm{Res}_f(i) = \lim_{z\to i}(z-i)\frac{\log(z)}{(z-i)(z+i)} = \frac{\log(i)}{2i} = \frac{\pi i/2}{2i}$$

$$\Longrightarrow \int_0^\infty \frac{\ln(x)}{x^2+1}dx = \pi i(\mathrm{Res}_f(i)) - \frac{\pi^2 i}{4} = \frac{\pi^2 i}{4} - \frac{\pi^2 i}{4} = 0.$$

## Exercise 4.

Let $\gamma : [0,1] \to \mathbb{C}$ be a path, and let $(f_t, U_t)_{t\in[0,1]}$ be an analytic continuation along $\gamma$. For $t \in [0,1]$, let $R(t)$ be the radius of convergence of the Taylor series of $f_t$ centered at $\gamma(t)$. Prove that either $R(t) = \infty$ for all $t$, or that $R : [0,1] \to (0,\infty)$ is continuous.

**Solution:**

For the sake of contradiction assume that $R(t) < \infty$ and that there exists a discontinuity of $R$ at $t_0$. Then, define the Taylor series of $f_{t_0}$ centered at $\gamma(t_0)$

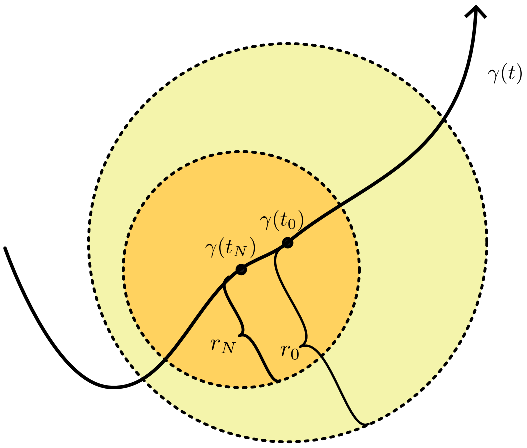
$$f_{t_0}(z) = \sum_{n=0}^\infty a_n(z-\gamma(t_0))^n.$$

Since $t_0$ is a discontinuity, there exists $\{t_n\}_{n=1}^\infty \subset [0,1]$ such that $\lim_n t_n = t_0$, but the limit for $r_n = R(t_n)$ doesn't coincide with $r_0$:

$$\lim_n \underbrace{R(t_n)}_{r_n} =: r \neq r_0 := R(t_0).$$

There are 2 possible scenarios:

- **If $r < r_0$,** then fix $0 < \varepsilon \leq r_0 - r$, let $N_1 \in \mathbb{N}$ such that $|\gamma(t_n) - \gamma(t_0)| < \varepsilon/4$ for every $n \geq N_1$ and let $N_2 \in \mathbb{N}$ such that $|r - r_n| < \varepsilon/4$ for every $n \geq N_2$.



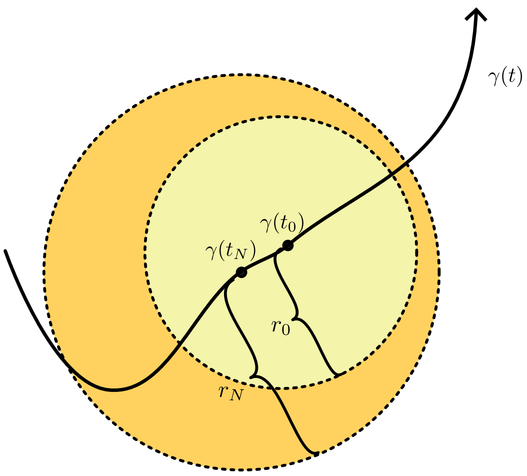

Then, define $N = \max(N_1, N_2)$ and note that for every $y \in B_{r_N}(\gamma(t_N))$,

$$\begin{aligned} |y - \gamma(t_0)| &= |y - \gamma(t_N) + \gamma(t_N) - \gamma(t_0)| \\ &\leq |y - \gamma(t_N)| + |\gamma(t_N) - \gamma(t_0)| \\ &< r_N + \frac{\varepsilon}{4} \\ &<^{(\star)} r + \frac{\varepsilon}{2} \\ &\leq r_0 - \frac{\varepsilon}{2} \leq r_0. \end{aligned}$$

$(\star)$: If $r_N > r$, then use $r_N \leq r + \varepsilon/4$, else, for the case $r_N \leq r \leq r + \varepsilon/4$ the inequality is still true.

Therefore, $y \in B_{r_0}(\gamma(t_0))$. Now, using Identity theorem, we can conclude that the Taylor series of $f_N$ can be extended to $B_{r_N+\varepsilon/2}(\gamma(t_N)) \subset B_{r_0}(\gamma(t_0))$ meaning $r_N$ is not the actual radius of convergence of $f_N$.

- **If $\boldsymbol{r > r_0}$,** then the principle is similar, fix $0 < \varepsilon \leq r - r_0$, let $N_1 \in \mathbb{N}$ such that $|\gamma(t_n) - \gamma(t_0)| < \varepsilon/4$ for every $n \geq N_1$ and let $N_2 \in \mathbb{N}$ such that $|r - r_n| < \varepsilon/4$ for every $n \geq N_2$




Then, define $N = \max(N_1, N_2)$ and note that for every $y \in B_{r_0}(\gamma(t_0))$,

$$\begin{aligned} |y - \gamma(t_N)| &= |y - \gamma(t_0) + \gamma(t_0) - \gamma(t_N)| \\ &\leq |y - \gamma(t_0)| + |\gamma(t_0) - \gamma(t_N)| \\ &< r_0 + \frac{\varepsilon}{4} \\ &\leq (r - \varepsilon) + \frac{\varepsilon}{4} \\ &<^{(\star)} r_N - \frac{\varepsilon}{2} \leq r_N. \end{aligned}$$

Again, this implies we can extend $f_{t_0}$ to the ball $B_{r_0+\varepsilon/2}(\gamma(t_0))$ implying that $r_0 = R(t_0)$ is not the actual convergence radius at $\gamma(t_0)$.

## Exercise 5.

Let $U \subset \mathbb{C} \setminus \{0\}$ be an open set, and suppose there exists a path in $U$ such that $\text{ind}_\gamma(0) = 1$. Prove that there is no holomorphic $n$-th root in $U$ for $n \geq 2$.

### Solution

Let $f(z) = z$ and $n \geq 2$ such that there exists a holomorphic function $g$ that satisfies for every $z \in U$ the equation

$$g^n(z) = f(z) = z.$$

Then, note that for every path in $U$, by the Argument Principle,

$$\begin{aligned}
\operatorname{ind}_\gamma(0) &= \operatorname{ind}_{\gamma\circ f}(0) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{ng'(z)g^{n-1}(z)}{g^n(z)} \\
&= n\cdot \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}}_{\in\mathbb{Z}} = nk, \quad k\in\mathbb{Z}.
\end{aligned}$$

Finally, we proved that $\operatorname{ind}_\gamma(0) \neq 0$ because $n \geq 2$.