Complex Analysis: Homework 12

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Exercise 1.

Let $U \subseteq \mathbb{C}$ be an open, connected, and bounded set with closure $\overline{U}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $\overline{U} \to \mathbb{C}$ whose restrictions to U are holomorphic. Suppose the sequence converges uniformly on $\overline{U} \setminus U$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $\overline{U}$.

Solution

Since (f_n) converges uniformly at $\partial U = \overline{U} \backslash U$, $\sup_{z\in\partial U} |f_n(z) - f_m(z)| \to 0$ as $n, m \to \infty$.

Define $g_{n,m} = f_n - f_m$ and note that by the Maximum Modulus Principle, since $g_{n,m}$ is a holomorphic function at U and U is bounded, it follows that $g_{n,m}$ attains its maximum at the boundary of U:

$$\max_{z\in\overline{U}} |g_{n,m}(z)| \leq \max_{z\in\partial U} |g_{n,m}(z)| \to 0.$$

Therefore, (f_n) is a Cauchy sequence of functions in the metric space $X = (H(\overline{U}), \|\cdot\|_\infty)$. Now, since $(H(\overline{U}), \|\cdot\|_\infty)$ is a closed subspace of $(C(\overline{U}), \|\cdot\|_\infty)$ which is a Banach space because every function in $C(\overline{U})$ is bounded, it follows that X is a Banach space too, and thus, (f_n) converges uniformly in $\overline{U}$.

Exercise 2.

Let $U_1 = \{z \in \mathbb{C} \;:\; \mathrm{Im}(z) > 0\}$, $U_2 = \mathbb{C}\backslash(-\infty, 0]$ and $\mathbb{E} = \{z \in \mathbb{C} \;:\; |z| < 1\}$, find biholomorphic functions

$$f : U_1 \to \mathbb{E}, \qquad g : U_2 \to \mathbb{E}.$$

Solution Item (a)

For the first homework we proved that the Cayley transform $f^{-1} : \mathbb{E} \to U_1,\ f^{-1}(z) = i\frac{1-z}{1+z}$ is a bijection with inverse function

$$f(z) = \frac{i-z}{i+z}.$$

Since the only singularity of f is outside its domain, it follows that f is holomorphic too.

Solution Item (b)

Remark. Note that the 2 solutions for $z^2 = a^2$ are: a and $e^{i\pi}a$. If $\mathrm{Re}(a) \neq 0$, then one of the 2 solutions is in $U_3 := \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 0\}$ and the other in $e^{i\pi}U_3 = \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) < 0\}$ (the other half plane). Therefore, the branch of $\sqrt{\cdot} = z \mapsto \sqrt{|z|} \cdot \exp(i \arg(z)/2)$ defined for $\arg(z) \in (-\pi, \pi)$ is a biholomorphic map between U_2 and U_3.

In the first place, note that by taking $g_1 = \sqrt{\cdot}$ as the branch of the square root defined for $\arg(z) \in (-\pi, \pi)$, we obtain a biholomorphic function from U_2 to $\{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 0\}$. Now, rotate 90° with the function $g_2(z) = iz$ to obtain U_1 and finally apply f to obtain $\mathbb{E}$:

$$g(z) = f \circ g_2 \circ g_1(z) = \frac{i - i\sqrt{z}}{i + i\sqrt{z}} = \frac{1-\sqrt{z}}{1+\sqrt{z}}.$$

Exercise 3.

Find a biholomorphic function $f : \{z \in \mathbb{C} \ : \ |z| < 1,\ \mathrm{Re}(z) > 0\} \to \{z \in \mathbb{C} \ : \ |z| < 1\}$.

Solution

Let $V_1 := \{z \in \mathbb{C} \ : \ |z| < 1,\ \mathrm{Re}(z) > 0\}$ be the right half (unit) disk.

Claim 1. Let $h : \mathbb{E} \to U_1$ be the Cayley transform $h(z) = i\frac{1-z}{1+z}$ from the previous exercise. Then, $h|_{V_2}$ is a bijection between the upper half disk $V_2 := \{z \in \mathbb{C} \ : \ |z| < 1,\ \mathrm{Im}(z) > 0\}$ and the first quadrant $V_3 := \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$.

Proof: It's clear from the first homework that the restriction to V_2 of the Cayley transform is injective. Now, let $w = x + iy \in V_3$ with $x, y \in \mathbb{R}^+$. We want to show that $h^{-1}(w) =$

$\frac{i-w}{i+w} \in V_2$, that is, $\text{Im}(h^{-1}(w)) > 0$ (because $h^{-1}(w) \in B_1(0)$). Note that

$$\begin{aligned} h^{-1}(w) &= \frac{i-w}{i+w} = \frac{i-iy-x}{i+iy+x} \\ &= \frac{i(1-y)-x}{i(1+y)+x} \cdot \frac{i(1+y)-x}{i(1+y)-x} \\ &= \frac{1-x^2-y^2}{(1+y)^2+x^2} + i\frac{2x}{(1+y)^2+x^2} \end{aligned}$$

Therefore, $\text{Im}(h^{-1}(w)) = \frac{2x}{(1+y)^2+x^2} > 0$ so $h|_{V_2}$ is surjective.

Now, $z \mapsto z^2$ is a bijection between V_3 and the upper half plane $V_4 := \{z \in \mathbb{C} \ : \ \text{Im}(z) > 0\}$. Finally, from the previous exercise we know that the Inverse Cayley transform maps V_4 to the unit disk $V_5 := B_0(1)$. The map we're looking for is the following

$$V_1 \xrightarrow{z \mapsto iz} V_2 \xrightarrow{z \mapsto h(z)} V_3 \xrightarrow{z \mapsto z^2} V_4 \xrightarrow{z \mapsto h^{-1}(z)} V_5$$

$$f(z) = h^{-1}\left((h(iz))^2\right).$$

Since f is a composition of biholomorphic functions, it follows that f is biholomorphic too.

Exercise 4.

Let $E = \{z \in \mathbb{C} : |z| < 1\}$, and let $f : E \to E$ be a biholomorphic function. Prove that there exist $\alpha \in \mathbb{R}$ and $z_0 \in E$ such that

$$f(z) = e^{i\alpha}\frac{z-z_0}{1-\overline{z_0}z}.$$

Solution

In the first homework we proved that for $z_0 \in E$, the function

$$F(z) = \frac{z_0 - z}{1-\overline{z_0}z}$$

is a biholomorphism from the unit disk E to itself. Also, we proved that $F^{-1}(z) = F(z)$ for every $z \in E$.

Now, let $z_0 = f(0)$ and define F accordingly. Then, define $g = f \circ F : E \to E$ with $g(0) = 0$ and $|g(z)| \leq 1$. By Schwarz Lemma, it follows that $|g(z)| \leq |z|$.

Also, since $g^{-1} = F^{-1} \circ f^{-1} = F \circ f^{-1}$ exists and satisfies $g^{-1}(0) = 0$ and $|g^{-1}(z)| \leq 1$ too, it follows by Schwarz lemma that $|g^{-1}(z)| \leq |z|$. Now, for every $z \in E$

$$|z| = |g^{-1}(w)| \leq |w| = |g(z)| \leq |z|.$$

Therefore, by Schwarz lemma, again, it follows that $g(z) = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$. Finally,

$$f(z) = g \circ F^{-1}(z) = g \circ F(z) = e^{i\alpha} \frac{z_0 - z}{1 - \overline{z_0} z}.$$

Exercise 5.

Does there exist a homeomorphic function $\mathbb{C} \to \mathbb{E}$?

Solution

Note that the function $f(x) = \frac{x}{1-|x|}$ is a bijection between $(0,1)$ and $(0,\infty)$ with inverse $f^{-1}(x) = \frac{x}{1+|x|}$. In fact, if $x < y$, then

$$1 - |x| > 1 - |y| \implies \frac{1}{1-|x|} < \frac{1}{1-|y|} \implies f(x) < f(y)$$

On the other hand, for $x \in (0,\infty)$, since $|x| = x$ and $\left|\frac{x}{1+x}\right| = \frac{x}{1+x}$, it follows that

$$f(f^{-1}(x)) = f\left(\frac{x}{1+x}\right) = \frac{\frac{x}{1+x}}{1 - \frac{x}{1+x}} = \frac{x}{1-x} = \frac{\frac{x}{1+x}}{\frac{1+x-x}{1+x}} = x.$$

Therefore, f is surjective too.

Now, when we extend to the complex plane with $|\cdot|$ being the module function,

$$f(e^{i\theta}x) = \frac{e^{i\theta}x}{1 - |e^{i\theta}x|} = e^{i\theta}\frac{x}{1-|x|} = e^{i\theta} f(x).$$

Therefore, $f(z)$ is a bijection between $e^{i\theta}(0,1)$ and $e^{i\theta}(0,\infty)$ and $f(0) = 0$. Therefore, after joining all the domains, we obtain that

$$f(z) = \frac{z}{1-|z|}$$

is a bijection between $B_1(0)$ and $\mathbb{C}$ and it's continuous because is a composition of continuous functions. Therefore, f is a homeomorphism.