Complex Analysis: Homework 7

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Exercise 1.

Calculate the principal part at 0 of the functions

$$f(z) = \frac{(\sin z)^2}{\sin(z^2)}, \qquad g(z) = \frac{1 - z^2}{z(1 - \cos(z^2))}.$$

Solution Item (a)

The Taylor series of $(\sin z)^2$ is by trigonometric identities,

$$\begin{aligned}
(\sin z)^2 &= \frac{1 - \cos(2z)}{2} \\
&= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} z^{2n} \\
&= z^2 - \frac{2^3 z^4}{4!} + \frac{2^5 z^6}{6!} + O(z^8)
\end{aligned}$$

The Taylor series of $\sin(z^2)$ is by substitution

$$\begin{aligned}
\sin(z^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{(2n+1)!} \\
&= z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + O(z^{14})
\end{aligned}$$

Then,

$$\begin{aligned}\lim_{z\to 0}\frac{(\sin z)^2}{\sin(z^2)} &= \lim_{z\to 0}\frac{z^2-\frac{2^3}{4!}z^4+O(z^6)}{z^2+O(z^6)}\\ &= \lim_{z\to 0}\frac{1+O(z^2)}{1+O(z^4)} = 1.\end{aligned}$$

Therefore, the principal part of the series is 0 because the series has a removable singularity at 0.

Solution Item (b)

The Taylor series expansion of $z(1-\cos(z^2))$ is the following

$$\begin{aligned}z(1-\cos(z^2)) &= z\left(1-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}z^{2n+1}\\ &= \frac{1}{2!}z^5-\frac{1}{4!}z^9+O(z^{13})\end{aligned}$$

Therefore,

$$g(z)=\frac{1-z^2}{\frac{1}{2}z^5+O(z^9)},$$

so it's clear that $g(z)$ has a pole of order 5 at $z=0$. In fact, if

$$g(z)=\sum_{n=-5}^{\infty}a_n z^n,$$

then $a_{-5}=\lim_{z\to 0}z^5 g(z)=\lim_{z\to 0}\frac{1-z^2}{\frac{1}{2}+O(z^4)}=2$. Then, for the next coefficients

$$\begin{aligned}a_{-4} &= \lim_{z\to 0}z^4(g(z)-a_{-5}z^{-5})\\ &= \lim_{z\to 0}z^4(g(z)-2z^{-5})\\ &= \lim_{z\to 0}z^4\frac{1-z^2-2z^{-5}(\frac{1}{2!}z^5+O(z^9))}{\frac{1}{2}z^5+O(z^9)}\\ &= \lim_{z\to 0}\frac{-z^2+O(z^4)}{\frac{1}{2}z+O(z^5)}=0\end{aligned}$$

$$
\begin{aligned}
a_{-3} &= \lim_{z\to 0} z^3(g(z) - a_{-5}z^{-5} - a_{-4}z^{-4}) \\
&= \lim_{z\to 0} z^3(g(z) - 2z^{-5}) \\
&= \lim_{z\to 0} z^3 \frac{1 - z^2 - 2z^{-5}(\frac{1}{2!}z^5 + O(z^9))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{-z^2 + O(z^4)}{\frac{1}{2}z^2 + O(z^4)} = -2
\end{aligned}
$$

$$
\begin{aligned}
a_{-2} &= \lim_{z\to 0} z^2(g(z) - a_{-5}z^{-5} - a_{-4}z^{-4} - a_{-3}z^{-3}) \\
&= \lim_{z\to 0} z^2(g(z) - 2z^{-5} + 2z^{-3}) \\
&= \lim_{z\to 0} z^2 \frac{1 - z^2 + (-2z^{-5} + 2z^{-3})(\frac{1}{2!}z^5 - \frac{1}{4!}z^9 + O(z^{13}))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{1 - z^2 + (-1 + \frac{2}{4!}z^4 + O(z^8)) + (z^2 - \frac{2}{4!}z^6 + O(z^{10}))}{\frac{1}{2}z^3 + O(z^7)} \\
&= \lim_{z\to 0} \frac{\frac{1}{12}z^4 + O(z^6)}{\frac{1}{2}z^3 + O(z^7)} = 0.
\end{aligned}
$$

$$
\begin{aligned}
a_{-1} &= \lim_{z\to 0} z(g(z) - a_{-5}z^{-5} - a_{-4}z^{-4} - a_{-3}z^{-3} - a_{-2}z^{-2}) \\
&= \lim_{z\to 0} z(g(z) - 2z^{-5} + 2z^{-3}) \\
&= \lim_{z\to 0} z \frac{1 - z^2 + (-2z^{-5} + 2z^{-3})(\frac{1}{2!}z^5 - \frac{1}{4!}z^9 + O(z^{13}))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{1 - z^2 + (-1 + \frac{2}{4!}z^4 + O(z^8)) + (z^2 - \frac{2}{4!}z^6 + O(z^{10}))}{\frac{1}{2}z^4 + O(z^8)} \\
&= \lim_{z\to 0} \frac{\frac{1}{12}z^4 + O(z^6)}{\frac{1}{2}z^4 + O(z^8)} = \frac{1}{6}.
\end{aligned}
$$

Finally, the principal part is

$$\frac{2}{z^5} - \frac{2}{z^3} + \frac{1}{6z}.$$

Exercise 2.

Let $M \subset \mathbb{C}$ be a finite set and let $f : \mathbb{C}\backslash M \to \mathbb{C}$ be holomorphic.

(a) Show that $g(z) = z^{-2}f(z^{-1})$ is holomorphic at $B_\varepsilon(0)\backslash\{0\}$ for $\varepsilon > 0$ sufficiently small.

(b) Show that $\text{Res}_0 g = \sum_{c\in\mathbb{C}} \text{Res}_c f$.

(c) Calculate $\displaystyle\int_{\partial B_1(0)} \frac{5z^6+4}{2z^7+1} dz$.

Solution Item (a)

For some $R > 0$, $M \subset B_R(0)$, so it follows that f is holomorphic at $\mathbb{C}\backslash\overline{B_R(0)}$. Then, the map $z \mapsto f(z^{-1})$ is holomorphic at $B_{1/R}(0)\backslash\{0\}$, so it follows that g is also holomorphic at $B_{1/R}(0)\backslash\{0\}$.

Solution Item (b)

Solution Item (c)

Let

$$f(z) = \frac{5z^6+4}{2z^7+1},$$

and let,

$$\begin{aligned} g(z) &= z^{-2}f(z^{-1}) \\ &= \frac{1}{z^2}\frac{5z^{-6}+4}{2z^{-7}+1} \\ &= \frac{1}{z^2}\frac{4z^7+5z}{z^7+2} \\ &= \frac{4z^6+5}{z^8+2z}. \end{aligned}$$

Finally, since all the zeroes of $2z^7+1$ are in $\partial B_{2^{-1/7}}(0) \subset \text{int}\partial B_1(0)$, it follows that

$$\frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{5z^6+4}{2z^7+1} dz = \sum_{c\in\mathbb{C}} \text{Res}_c f = \text{Res}_0 g,$$

and

$$\begin{aligned}\mathrm{Res}_0 g &= \lim_{z\to 0} z g(z)\\ &= \lim_{z\to 0} \frac{4z^6+5}{z^7+2} = \frac{5}{2},\\ \implies \int_{\partial B_1(0)} \frac{5z^6+4}{2z^7+1} dz &= 5\pi i\end{aligned}$$

Exercise 3.

Calculate the following integrals with complex analysis methods

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+6x^2+13} dx$

(b) $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1} dx$

(c) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x} dx$

Solution Item (a)

The difference of the degrees between the denominator and numerator is 2, so we can use the following method

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+6x^2+13} dx = 2\pi i \sum_{\mathrm{Im}(z_0)>0} \mathrm{Res}_{z=z_0}\left(\frac{z^2}{z^4+6z^2+13}\right)$$

The function z^4+6z^2+13 has a zero with multiplicity 1 at

$$a = \sqrt[4]{13}\cos\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right) - i\sqrt[4]{13}\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)$$

It also has multiplicity 1 zeroes at $-\overline{a}, \overline{a}, -a$, but the only ones at the upper half plane are a and $-\overline{a}$.

Then,

$$\begin{aligned}
\mathrm{Res}_a f(z) &= \lim_{z\to a}(z-a)\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})} \\
&= \frac{a^2}{2a(2i\mathrm{Im}(a))(2\mathrm{Re}(a))} \\
&= \frac{-ia}{8\mathrm{Im}(a)\mathrm{Re}(a)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}_{-\overline{a}} f(z) &= \lim_{z\to -\overline{a}}(z+\overline{a})\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})} \\
&= \frac{\overline{a}^2}{(-2\mathrm{Re}(a))(-2i\mathrm{Im}(a))(-2\overline{a})} \\
&= \frac{-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)}
\end{aligned}$$

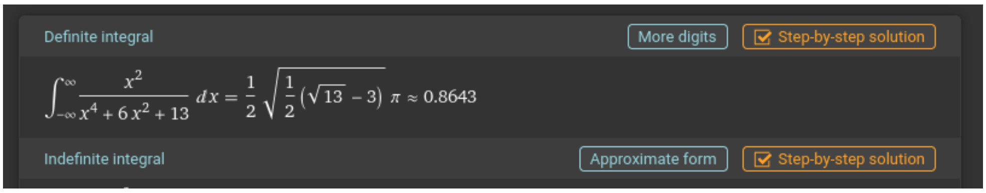
Finally,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{x^2}{x^4+6x^2+13}dx &= \mathrm{Res}_a f(z) + \mathrm{Res}_{-\overline{a}} f(z) \\
&= \frac{-ia-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)} \\
&= \frac{i(-2\mathrm{Re}(a))}{8\mathrm{Im}(a)\mathrm{Re}(a)} \\
&= \frac{-i}{4\mathrm{Im}(a)} \\
&= \frac{i}{4\sqrt[4]{13}\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)}
\end{aligned}$$

So it follows that

$$\int_{-\infty}^{\infty}\frac{x^2}{x^4+6x^2+13}dx = \frac{-2\pi}{4\sqrt[4]{13}\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)} \approx 0.8643$$

and this coincides with the real result



Solution Item (b)

This integral has the form $\int_0^\infty x^\alpha R(x)$ where $0 < \alpha = 1/2 < 1$ and $R(x) = O(x^{-2})$ without any poles at the origin. Therefore, by using the substitution $x = t^2$, $dx = 2tdt$, we obtain

$$\begin{aligned}\int_0^\infty x^\alpha R(x)dx &= 2\int_0^\infty t^{2\alpha+1}R(t^2)dt \\ &= \int_{-\infty}^\infty t^{2\alpha+1}R(t^2)dt \\ &= \int_{-\infty}^\infty \frac{t^2}{t^4+1}dt\end{aligned}$$

It follows that since the difference between the degrees of the denominator and numerator is two,

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \sum_{\mathrm{Im}(z_0)>0} \mathrm{Res}_{z=z_0}\left(\frac{z^2}{z^4+1}\right)$$

The polynomial z^4+1 has a root of multiplicity 1 at

$$a = \frac{1+i}{\sqrt{2}}$$

and also has roots at $-a, \overline{a}, -\overline{a}$, from which only a and $-\overline{a}$ are in the upper half plane. Using the same logic as the previous item (because it's the exact same case only changing the value of a),

$$\begin{aligned}\mathrm{Res}_a f(z) &= \lim_{z\to a}(z-a)\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})} \\ &= \frac{-ia}{8\mathrm{Im}(a)\mathrm{Re}(a)}\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{-\overline{a}} f(z) &= \lim_{z\to -\overline{a}}(z+\overline{a})\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})} \\ &= \frac{-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)}\end{aligned}$$

So finally,

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{t^2}{t^4+1}dt &= \mathrm{Res}_a f(z) + \mathrm{Res}_{-\overline{a}} f(z) \\ &= \frac{-ia-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)} \\ &= \frac{i(-2\mathrm{Re}(a))}{8\mathrm{Im}(a)\mathrm{Re}(a)} \\ &= \frac{-i}{4\mathrm{Im}(a)} \\ &= \frac{-i}{4\sqrt{2}},\end{aligned}$$

and thus,

$$\int_0^{\infty}\frac{\sqrt{x}}{x^2+1}dx = 2\int_{-\infty}^{\infty}\frac{t^2}{t^4+1}dt = \frac{\pi}{\sqrt{2}},$$

which coincides with the real result

Solution Item (c)

We have that

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}dx &= \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}dx\\ &= \int_0^\infty \frac{e^{ix}}{2ix}dx - \int_0^\infty \frac{e^{-ix}}{2ix}dx\\ &= \int_0^\infty \frac{e^{ix}}{2ix}dx + \int_{-\infty}^0 \frac{e^{ix}}{2ix}dx\\ &= \int_{-\infty}^\infty \frac{e^{ix}}{2ix}dx\end{aligned}$$

We have a simple pole at $x = 0$ and $R(\infty) = 0$, so we can apply the following formula

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx &= \int_{-\infty}^\infty R(x)e^{ix}dx\\ &= 2\pi i \sum_{\mathrm{Im}(z_0)>0} \mathrm{Res}_{z=z_0} R(z)e^{iz} + \pi i \sum_{\mathrm{Im}(z_0)=0} \mathrm{Res}_{z=z_0} R(z)e^{iz}\\ &= \pi i \mathrm{Res}_{z=0} \frac{e^{iz}}{z} = \pi i.\end{aligned}$$

Finally,

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx = \frac{\pi}{2}$$

Exercise 4.

(a) Let γ be a closed curve in $\mathbb{C}\backslash\{0\}$. Let $n \in \mathbb{N}$ and $p : \mathbb{C} \to \mathbb{C},\ p(z) = z^n$. Show that $\mathrm{ind}_{p\circ\gamma}(0) = n\ \mathrm{ind}_\gamma(0)$.

(b) Let $U \subset \mathbb{C}$ be open and connected, $c \in U$ and γ be a closed curve in $U\backslash\{c\}$ such that $\mathrm{int}(\gamma) \subset D$. Para a biholomorphic function $f : U \to f(U)$ show that

$$\mathrm{ind}_\gamma(c) = \mathrm{ind}_{f\circ\gamma}(f(c))$$

Exercise 5.

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Suppose that for every $a \in \mathbb{C}$, at least one coefficient in the Taylor series of f in a is vanished. Show that f is a polynomial

Solution:

We previously proved that the set of zeroes of a non-zero holomorphic function is discrete and closed, so it has to be countable (because uncountable sets have accumulation points in $\mathbb{R}^n$).

Now assume for the sake of contradiction that f is not a polynomial, so the k-th derivative $f^{(k)}$ is always non-zero, and thus, has a countable set of zeroes. So it follows that $(f^{(k)})^{-1}(\{0\})$ is a countable set, and thus,

$$\bigcup_{k=0}^{\infty}(f^{(k)})^{-1}(\{0\}) \text{ is countable too.}$$

Also note that since for every $a \in \mathbb{C}$ there exists $n \in \mathbb{N}$ such that $a_n = 0$ for the series

$$f(z) = \sum_{k=0} a_k(z-a)^k,$$

so it follows that $f^{(n)}(a) = 0$, and thus, for every $a \in \mathbb{C}$

$$a \in (f^{(n)})^{-1}(\{0\}) \subset \bigcup_{k=0}^{\infty}(f^{(k)})^{-1}(\{0\})$$

$$\implies \mathbb{C} \subset \bigcup_{k=0}^{\infty}(f^{(k)})^{-1}(\{0\}),$$

but that would imply that $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ is countable, which is false.