Complex Analysis: Homework 1

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Exercise 1.

Let $U \subseteq \mathbb{C}$ be an open set. Prove that U is connected if and only if it is path connected.

Solution:

$\implies$:

Exercise 2.

Part (a)

Let $z, w \in \mathbb{C}$ with $\overline{z}w \neq 1$, and $|z| \leq 1$ and $|w| \leq 1$. Prove that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| \leq 1$$

with equality if and only if $|z| = 1$ or $|w| = 1$.

Part (b)

Let $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ be the open unit disc in $\mathbb{C}$. For a fixed $w \in \mathbb{D}$ define

$$F(z) = \frac{w - z}{1 - \overline{w}z} \quad \text{for } z \in \mathbb{C} \text{ with } \overline{w}z \neq 1.$$

Prove that

(i) F is holomorphic in $\mathbb{D}$ and $F(\mathbb{D}) \subseteq \mathbb{D}$.

(ii) $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ for $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective.

Exercise 3.

Let $U := \{z \in \mathbb{C} \;:\; \operatorname{Im}(z) > 0\}$. Prove that $\Phi : \mathbb{D} \to U$, $\Phi(z) = i\frac{1-z}{1+z}$ is a bijection and calculate its inverse.

Exercise 4.

Let $U := \{z \in \mathbb{C} \;:\; \operatorname{Im}(z) > 0\}$ and let $\Psi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ for fixed $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

(a) Suppose that $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha\delta - \beta\gamma > 0$. Prove that $\Psi : U \to U$ is a bijection.

(b) Suppose that $\Psi : U \to U$ is a bijection. Prove tat the numbers $\alpha, \beta, \gamma, \delta$ can be chosen from $\mathbb{R}$.

Exercise 5.

Prove that $\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}}$. Formulate and prove the chain rule for the Wirtinger derivatives.