Complex Analysis: Homework 3

Martín Prado

August 28, 2024

Universidad de los Andes – Bogotá Colombia

Exercise 1.

(a) Calculate $\oint_{|z-1|=2} z^n \sin(z) dz$ for $n \in \mathbb{Z}$.

(b) For $n \in \mathbb{N}_0$ prove that

$$\int_{|z+2i|=3} \frac{1}{(z^2+\pi^2)^{n+1}} dz = \frac{-(2n)!}{(n!)^2} (2\pi)^{-2n}$$

Solution Part (a)

When $n \geq 0$, $z \mapsto z^n \sin(z)$ is an entire function with Taylor series

$$z^n \sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1+n}.$$

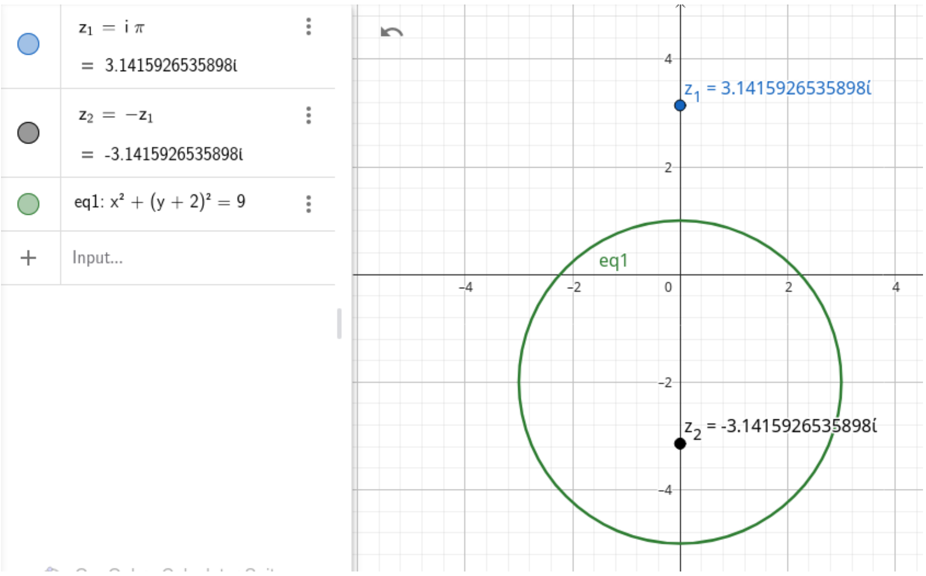
Therefore, using Cauchy's theorem, we assert that

$$\oint_{|z-1|=2} z^n \sin(z) dz = 0.$$

Finally, for the negative case, let $n \in \mathbb{Z}^+$ and note that by using Cauchy formula we obtain

$$\int_{|z-1|=2} \frac{\sin(z)}{z^n} dz = \frac{2\pi i}{n!} (\sin)^{(n-1)}(0) = \begin{cases} 0, & n \equiv 1,3 \mod 4 \\ 1, & n \equiv 2 \mod 4 \\ -1, & n \equiv 0 \mod 4. \end{cases}$$

Solution Part (b)



Let $f(z) = \dfrac{1}{(z - i\pi)^{n+1}} = (z - i\pi)^{-(n+1)}$ for $z \in \mathbb{C}\backslash\{i\pi\}$, and note (from the image above) that f is analytic on the disk $\{z \in \mathbb{C} \;:\; |z + 2i| \leq 3\}$. Therefore, we can use Cauchy's formula to conclude that

$$\frac{f^{(n)}(-i\pi) \cdot 2\pi i}{n!} = \int_{|z+2i|=3} \frac{f(z)}{(z + i\pi)^{n+1}} dz = \int_{|z+2i|=3} \frac{1}{(z^2 + \pi^2)^{n+1}} dz.$$

Then,

$$\begin{aligned}
f^{(1)}(z) &= (-(n+1))(z - i\pi)^{-(n+2)} \\
f^{(2)}(z) &= (-(n+1))(-(n+2))(z - i\pi)^{-(n+3)} \\
\vdots \quad & \quad \vdots \\
f^{(n)}(z) &= (-(n+1)) \cdots (-2n)(z - i\pi)^{-(2n+1)} \\
&= \frac{2n!}{n!}(z - i\pi)^{-(2n+1)}
\end{aligned}$$

Finally, by putting everything together, we obtain

$$\begin{aligned}
\int_{|z+2i|=3} \frac{1}{(z^2 + \pi^2)^{n+1}} dz &= \frac{f^{(n)}(-i\pi) \cdot 2\pi i}{n!} \\
&= \frac{2n! \cdot (2\pi i)}{(n!)^2 \cdot (-2\pi i)^{2n+1}} \\
&= \frac{-(2n)!}{(n!)^2}(2\pi)^{-2n}.
\end{aligned}$$

Also, I made the case $n = 0$ using partial fractions:

$$\begin{aligned}
\int_{|z+2i|=3} \frac{1}{z^2+\pi^2} dz &= \int_{|z+2i|=3} \frac{1}{(z+i\pi)(z-i\pi)} dz \\
&= \frac{-1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z+i\pi} - \frac{1}{z-i\pi} dz \\
&= \frac{-1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z+i\pi} + \frac{1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z-i\pi} dz \\
&= \frac{-1}{2\pi i} \int_{|z+2i|=3} \frac{1}{z+i\pi} + 0 \\
&= \frac{-1}{2\pi i} \cdot 2\pi i \\
&= \frac{-(2\cdot 0)!}{(0)!^2} (2\pi)^{2\cdot 0}.
\end{aligned}$$

Exercise 2.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that the exist $M, r > 0$ and $n \in \mathbb{N}$ such that $|f(z)| < M|z|^n$ for every $z \in \mathbb{C}$ for $|z| \geq r$. Show that f is a polynomial of degree at most n.

Observe that the case $n = 0$ is Liouville's theorem.

Solution:

For the case $n = 0$, we have Liouville's theorem because

$$\begin{aligned}
\sup_{z\in\mathbb{C}}\{|f(z)|\} &= \max(\sup_{|z|>r}\{|f(z)|\}, \sup_{|z|\leq r}\{|f(z)|\}) \\
&= \max(M, \max_{|z|\leq r}\{|f(z)|\}) < \infty.
\end{aligned}$$

It follows that $f(z)$ is bounded, and thus, a constant function by Liouville's theorem.

Now, for the general case, note that since f is entire, it has a power series around 0

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

if $|f(z)| < M|z|^n$, then for $R > r$

$$\begin{aligned} |a_k| &\leq \left|\frac{1}{2\pi i}\right| \oint_{|z|=R} \frac{|f(z)|}{|z|^{n+1}} dz \\ &< \frac{1}{2\pi} \oint_{|z|=R} \frac{M|z|^k}{|z|^{n+1}} dz \\ &\leq \frac{1}{2\pi} \underbrace{2\pi R}_{\text{arc lenght}} \cdot \underbrace{\frac{M}{R^{n-k+1}}}_{\text{function max}} \\ &= \frac{M}{R^{n-k}}. \end{aligned}$$

Then, by letting $R \to \infty$ we conclude that, for $k \geq n+1$, $a_k = 0$. Therefore,

$$f(z) = \sum_{k=0}^{n} a_k z^k,$$

which is a polynomial of degree at most n.

Exercise 3.

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

(a) Show that either the range of f is dense in $\mathbb{C}$ or f is constant.

(b) Suppose that $\mathrm{Re}(f)$ is bounded. Show that f is constant.

Solution Part (a)

Assume that $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Then, there exists $w_0 \in \mathbb{C}$ and $\varepsilon > 0$ such that $B_\varepsilon(w_0) \cap f(\mathbb{C}) = \emptyset$. This implies that $f(\mathbb{C}) \subseteq \mathbb{C} \backslash B_\varepsilon(w_0)$.

Now, consider the function $\phi(w) = \frac{\varepsilon}{w - w_0}$ which takes every point in the complement of $B_\varepsilon(w_0)$ inside the closed disk $B_1(0)$. That is because, if $|w - w_0| \geq \varepsilon$, then

$$|\phi(w)| = \frac{\varepsilon}{|w - w_0|} \leq \frac{\varepsilon}{\varepsilon} = 1.$$

It follows that $\phi \circ f$ is entire because $f(z) \neq w_0$ for every $z \in \mathbb{C}$ and it's bounded because $\phi \circ f(\mathbb{C}) \subseteq \phi(\mathbb{C} \backslash B_\varepsilon(z_0)) = B_1(0)$. Finally, if $\phi \circ f(z) = K$, then

$$f(z) = \frac{K}{\varepsilon} + w_0,$$

so f is a constant function.

Solution Part (b)

Let $f(z) = u(z) + iv(z)$, where $u, v : \mathbb{C} \to \mathbb{R}$ and $u(z) \leq M$ for every $z \in \mathbb{C}$. Then, we use Euler's formula,

$$e^{f(z)} = e^{u(z)}(\cos(v(z)) + i\sin(v(z))).$$

Note that since u is bounded by M, $e^{u(z)} \leq e^M$. On the other hand, $\cos(z) + i\sin(z)$ is on the unit circle (for $z \in \mathbb{R}$). Therefore,

$$|e^{f(z)}| \leq e^M$$

This implies that $\exp \circ f$ is a constant function $e^{f(z)} = K$, $K \neq 0$. Then,

$$\begin{aligned} \frac{d}{dz}e^{f(z)} &= 0 \\ \implies f'(z)e^{f(z)} &= 0 \\ \implies f'(z)K &= 0 \\ \implies f'(z) &= 0. \end{aligned}$$

Therefore, f is a constant function too.

Exercise 4.

Let $U \subseteq \mathbb{C}$ be a region, $z_0 \in U$ and $R > 0$ such that $B_R(z_0) \subseteq U$. Let $f : U \to \mathbb{C}$ be holomorphic with a Taylor series $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ centered around z_0. For $0 < r < R$ define $M(r) := \sup_{|z-z_0|=r} |f(z)|$.

(a) Show that for every $n \in \mathbb{N}_0$ and $0 < r < R$

$$c_n r^n = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})e^{-int}dt.$$

(b) Show that for every $0 < r < R$

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2 dt \leq M(r)^2.$$

Solution Part (a)

For $0 < r < R$, c_n is defined as the n-th Taylor's series coefficient,

$$\begin{aligned}
c_n = \frac{f^{(n)}(z_0)}{n!} &= \frac{n!}{n!2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}dz \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{(z_0+re^{it}-z_0)^{n+1}}d(z_0+re^{it}) \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{r^{n+1}e^{it(n+1)}}ire^{it}dt \\
&= \frac{1}{r^n 2\pi}\int_0^{2\pi} f(z_0+re^{it})e^{-int}dt.
\end{aligned}$$

Solution Part (b)

$$\begin{aligned}
\int_0^{2\pi} |f(z_0+re^{it})|^2 dt &= \int_0^{2\pi} \overline{f(z_0+re^{it})}\cdot f(z_0+re^{it})dt \\
&= \int_0^{2\pi} \sum_{n=0}^{\infty} \overline{c_n r^n e^{int}} \cdot \sum_{n=0}^{\infty} c_n r^n e^{int} dt \\
&= \int_0^{2\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \overline{c_k r^k e^{ikt}} c_{n-k} r^{n-k} e^{i(n-k)t} dt \\
&=^* \sum_{n=0}^{\infty}\sum_{k=0}^{n} \overline{c_k} c_{n-k} r^n \int_0^{2\pi} e^{-ikt}e^{i(n-k)t}dt.
\end{aligned}$$

Then, note that if $n-2k \neq 0$, then $t \mapsto e^{i(n-2k)t}$ is an entire function. Thus, by Cauchy integral theorem,

$$\int_0^{2\pi} e^{-ikt}e^{i(n-k)t}dt = \int_0^{2\pi} e^{i(n-2k)t}dt = \begin{cases} 2\pi, & n = 2k \\ 0, & n \neq 2k. \end{cases}$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi} |f(z_0+re^{it})|^2 dt &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \overline{c_k} c_{n-k} r^n \int_0^{2\pi} e^{i(n-2k)t}dt \\
&= \sum_{k=0}^{\infty} \overline{c_k} c_{2k-k} r^{2k} \cdot 2\pi \\
&= 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2k}.
\end{aligned}$$

Finally, note that $M(r) = \sup\limits_{|z-z_0|=r} |f(z)| = \sup\limits_{t\in[0,2\pi]} |f(z_0 + re^{it})|$. Then, by using the integral inequality we conclude

$$\int_0^{2\pi} |f(z_0 + re^{it})|^2 dt \leq \underbrace{2\pi}_{\text{arc lenght}} \cdot \underbrace{M(r)^2}_{\text{function max}}$$