Complex Analysis: Homework 11

Martín Prado

November 1, 2024

Universidad de los Andes – Bogotá Colombia

Theorem 1. **(Hurwitz's Theorem)** Let $U \subset \mathbb{C}$ be open and connected, and let f_n, $f : U \to \mathbb{C}$ be holomorphic functions. Suppose that $f_n \to f$ compactly. Suppose there exists $m \in \mathbb{N}_0$ such that for all $n \in \mathbb{N}$:

$$\text{the number of zeros of } f_n \text{ (counted with multiplicities) } \leq m.$$

Then f is constant or the number of zeros of f (counted with multiplicities) $\leq m$.

□

Theorem 2. If $(f_n)_{n\in\mathbb{N}}$ converges compactly to f, then $(f_n^{(k)})_{n\in\mathbb{N}}$ converges compactly to $f^{(k)}$.

Proof: Fix a compact set $K \subset U$. Then, U is an open metric space, we can find a slightly larger compact set by making a thickening of K with some $R > 0$:

$$K' = \overline{\bigcup_{z\in K} B_{2R}(z)}$$

in such way that $K \subset K'^{\circ} \subset K' \subset U$. Then, since K' is compact, any circle of radius R centered at $z \in K$: $\gamma = \partial B_R(z) \subset K'^{\circ}$ has length $2\pi R$. Also, for every $z \in K$ and $w \in \partial B_R(z)$, $|z - w| = R > 0$.

By compact convergence, since K' is a compact set, for any $\varepsilon > 0$ there exists N such that for $n \geq N$

$$|f_n(w) - f(w)| < \varepsilon, \quad \forall w \in K'.$$

Therefore, by Cauchy Integral Formula, for every $z \in K$

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &= \frac{k!}{2\pi}\left|\int_\gamma \frac{f_n(w) - f(w)}{(z-w)^{k+1}}dw\right| \\ &\leq \frac{k!}{2\pi}2\pi R \cdot \frac{\max_{w\in K'}|f_n(w) - f(w)|}{R^{k+1}} \\ (n \geq N) &< \frac{k!}{R^k}\cdot\varepsilon. \end{aligned}$$

R only depends in the set K we've chosen at the beginning, so $f_n^{(k)}$ compactly converges to $f^{(k)}$ for every $k \in \mathbb{N}$.

$\square$

Theorem 3. **(Ahlfors' Chapter 4.3.3 Theorem 11.)** Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. If $\varepsilon > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all a with $|a - w_0| < \delta$ the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \varepsilon$.

$\square$

Theorem 4. **(Vitali's Convergence Theorem)** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of locally bounded holomorphic functions. If $\lim_n f_n(z)$ exists for every $z \in V \subset U$ with V having an accumulation point, then $(f_n)_{n\in\mathbb{N}}$ converges compactly.

$\square$

Exercise 1.

Let $U \subseteq \mathbb{C}$ be an open set and $(f_n)_{n\in\mathbb{N}}$ be a sequences of holomorphic functions $U \to \mathbb{C}$. Suppose that $f_n \to f$ compactly and that f is not constant. Show that for every $z_0 \in U$ there exists a sequence $(z_n)_{n\in\mathbb{N}}$ and $N_0 \in \mathbb{N}$ with $\lim_{n\to\infty} z_n = z_0$ and $f_n(z_n) = f(z_0)$ for every $n \geq N_0$.

Solution:

Fix $z_0 \in U$ and without restriction assume that $f(z_0) = 0$, else repeat the following argument with $F(z) = f(z) - f(z_0)$.

Hypothesis 1: For the sake of contradiction suppose that for every sequence $(z_n)_{n\in\mathbb{N}}$, such that $z_n \to z_0$, there exists a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_k}(z_{n_k}) \neq 0$ for every $k \in \mathbb{N}$.

Note that f is holomorphic, and thus, its zeroes are isolated.

Claim 1: Let V be an open neighborhood of z_0 such that $\overline{V}$ doesn't contain any other zero of f different from z_0. Then, there must exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ for which f_{n_k} doesn't have any zero in V.

Proof: Otherwise, assume there exists $N \in \mathbb{N}$ such that f_n has at least a zero in V for every $n \geq N$, and thus, we can choose $z_n \in f_n^{-1}(\{0\}) \neq \emptyset$ for $n \geq N$ and build the following sequence

$$(w_n)_{n\in\mathbb{N}} = (\underbrace{z_0, z_0, \ldots, z_0}_{N-1 \text{ times}}, z_N, z_{N+1}, \ldots)$$

The sequence $(w_n)_{n\in\mathbb{N}}$ must converge to z_0, otherwise, if $w_n \to w \neq z_0$, then

$$f(w) = \lim_n f_n(w_n) = \lim_n f_n(z_n) = 0.$$

However that would contradict the fact that $\overline{V}$ doesn't contain any other zero of f. Since $f(w_n) = f(z_n) = 0$ for $n \geq N$, it follows that $(w_n)_{n\in\mathbb{N}}$ contradicts **Hypothesis 1** because it only has finely many $n \in \mathbb{N}$ for which $f_n(w_n) \neq 0$, and thus, **Claim 1** is proved.

$\square$

Let $(f_{n_k})_{k\in\mathbb{N}}$ be the subsequence from **Claim 1**. Define $g_{n_k} = f_{n_k}|_V$ and $g = f|_V$. Note that g_{n_k} converges compactly to g because f_{n_k} converges uniformly to f when restricted to any compact $K \subset V$. However, for every $k \in \mathbb{N}$, g_{n_k} doesn't have any zero while g does have exactly one contradicting **Hurwitz's Theorem:**

The number of zeroes of g must be less or equal to the number of zeroes of g_{n_k} for each $k \in \mathbb{N}$.

Exercise 2.

Let $U \subseteq \mathbb{C}$ be an open set and $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions $U \to \mathbb{C}$. Suppose that $f_n \to f$ compactly and that f is not constant. Show:

(a) If there exists $W \subset \mathbb{C}$ such that $f_n(U) \subset W$ for every $n \in \mathbb{N}$, then $f(U) \subseteq W$ too.

(b) If all f_n are injective, then f is injective too.

(c) If all f_n are locally biholomorphic, then f is locally biholomorphic too.

Solution Item (a)

Let $W \subset \mathbb{C}$ such that $f_n(U) \subset W$.

For the sake of contradiction, assume that there exists $z_0 \in U$ such that $f(z_0) \notin W$. Now, use **Exercise 1** to obtain a sequence $(z_n)_{n\in\mathbb{N}} \to z_0$ such that $f_n(z_n) = f(z_0)$ for $n \geq N$ for some $N \in \mathbb{N}$. Since $f_N(z_N) \in W$, it follows that $f(z_0) = f_N(z_N) \in W$ which leads to a contradiction.

Solution Item (b)

Fix any $z_0 \in U$, define $g(z) = f(z) - f(z_0)$ and $g_n(z) = f_n(z) - f_n(z_0)$. Note that the injectivity of f_n implies that z_0 is the only zero of g_n. Then, by **Hurwitz's Theorem**, the number of zeroes of g is less or equal than one, and thus, it's injective too (because z_0 is arbitrary).

Solution Item (c)

For the sake of contradiction, assume that there exists $z_0 \in U$ for which f is not biholomorphic (injective) at any neighborhood of z_0.

Claim 1. $f'(z_0) = 0$.

Proof: The contrapositive of this claim is the inverse function theorem.

Inverse Function Theorem: If $f'(z_0) \neq 0$, then there exists a neighborhood of z_0 in U for which f is injective (locally holomorphic).

□

Claim 2. For every $w \in U$ $f_n'(w) \neq 0$ for all n.

Proof: For the sake of contradiction, assume otherwise. For some $w \in U$, if $f_n'(w) = 0$ for some n, then $g_n(z) = f_n(z) - f_n(w)$ has a zero of order two at w because $g^{(0)}(w) = g^{(1)}(w)$, and thus,

$$\begin{aligned} g_n(z) &= \sum_{k=2}^{\infty} \frac{g^{(k)}(z_0)}{k!}(z - z_0)^k \\ &= (z - z_0)^2 \sum_{k=0}^{\infty} \frac{g^{(k+2)}(z_0)}{(k+2)!}(z - z_0)^k \\ &= (z - z_0)^2 h(z), \quad h \in H(U). \end{aligned}$$

Therefore, since w is a zero of order 2 of g_n, by **Theorem 3**, for any $\varepsilon > 0$ sufficiently small, f_n is not injective at $B_\varepsilon(w) \subset U$, contradicting the assumption that f_n is locally biholomorphic.

Finally, by **Theorem 2** f_n' converges compactly to f', and thus, there's a contradiction with the fact that f' has more zeros than $(f_n')_{n\in\mathbb{N}}$ (which has none) and **Hurwitz's Theorem**.

Exercise 3.

(a) Let $R > 1$ and $f : B_R(0) \to \mathbb{C}$ be a holomorphic function with Taylor series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Suppose that $\|f\|_{B_1(0)}^2 := \int_{B_1(0)} |f(z)|^2 \, dz = M < \infty$. Prove that for every $0 < r < 1$

$$\|f\|_{B_r(0)}^2 = \pi \sum_{n=0}^{\infty} \frac{|c_n|^2 r^{2n+2}}{n+1} \quad \text{and} \quad |f(0)| \leq \frac{\|f\|_{B_1(0)}}{\sqrt{\pi}}.$$

(b) Let $U \subset \mathbb{C}$ be a bounded region and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions $U \to \mathbb{C}$. Suppose that there exists $C > 0$ such that

$$\|f_n\|_U < C, \quad n \in \mathbb{N}.$$

Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is locally bounded. Conclude that it contains a subsequence that converges uniformly on compact subsets of U.

Solution Item (a)

For $z \in \partial B_r(0)$

$$\begin{aligned}
|f(z)|^2 = \overline{f(z)} f(z) &= \left(\sum_{n=0}^{\infty} \overline{c_n z^n}\right) \cdot \left(\sum_{n=0}^{\infty} c_n z^n\right) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \overline{c_n} c_m \overline{z}^n z^m \\
(z = re^{i\theta}) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \overline{c_n} c_m r^{m+n} e^{i\theta(m-n)}
\end{aligned}$$

Then, $\int_0^{2\pi} e^{i\theta(n-m)} d\theta = 2\pi$ when $n = m$ and 0 otherwise. Thus,

$$\begin{aligned}
\|f\|_{B_r(0)}^2 &= \int_{B_r(0)} |f(z)|^2 dz \\
&= \int_0^r \int_0^{2\pi} f(re^{i\theta}) r \, d\theta \, dr \\
&= \int_0^r \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \overline{c_n} c_m r^{m+n+1} \int_0^{2\pi} e^{i\theta(n-m)} d\theta dr \\
&= \int_0^r \sum_{n=0}^{\infty} |c_n|^2 r^{2n+1} 2\pi dr \\
&=^{(\star)} 2\pi \sum_{n=0}^{\infty} |c_n|^2 \int_0^r r^{2n+1} dr \\
&= \pi \sum_{n=0}^{\infty} \frac{|c_n|^2 r^{2n+2}}{n+1}.
\end{aligned}$$

The equality on $(\star)$ is obtained using Tonelli's theorem with the fact that $2\pi|c_n|^2 r^{2n+1} \geq 0$ for $r \geq 0$ and $n \in \mathbb{N}$. On the other hand,

$$|f(0)|^2 = |c_0|^2 \leq |c_0|^2 + \sum_{n=1}^{M} \frac{|c_n|}{n+1} \leq \frac{\|f\|_{B_1(0)}^2}{\pi} < \infty.$$

Therefore, $|f(0)| \leq \frac{\|f\|_{B_1(0)}}{\sqrt{\pi}}$.

Solution Item (b)

Note: For this solution, I'm assuming that

$$\|g\|_U = \|g|_U\|_2 = \left(\iint_U |g(z)|^2 dA \right)^{1/2}$$

Once we show that for every $z \in U$ there exists a neighborhood $V \subseteq U$ of z and a constant $K > 0$ for which

$$|f_n(w)| \leq K, \quad \forall w \in V,\ \forall n \in \mathbb{N}$$

we are done because we can then use **Montel's theorem** to find a compactly convergent subsequence. Now, we're going to prove that the sequence is locally bounded with the following claim:

Claim 1: Let $z_0 \in U$ and let $g : U \to \mathbb{C}$ be a holomorphic function. Now, let $0 < \alpha < \beta$ such that $\overline{B_\beta(z_0)} \subseteq U$, then, there exists $M > 0$ that only depends on α, β such that

$$\sup_{z \in B_\alpha(z_0)} |g(z)| = \|g|_{B_\alpha(z_0)}\|_\infty \leq M \|g|_{B_\beta(z_0)}\|_2 = M \left(\iint_{B_\beta(z_0)} |g(z)|^2 dA \right)^{1/2}$$

Proof: Fix $z \in B_\alpha(z_0)$ and let $\gamma_r(\theta) = z + re^{i\theta},\ \theta \in [0, 2\pi]$. By Cauchy Integral formula, since $\gamma_r \subseteq U$ for $r \in [0, \beta - \alpha)$, it follows that

$$\begin{aligned}
|g(z)|^2 = |g^2(z)| &= \left| \frac{1}{2\pi i} \int_{\gamma_r} \frac{g^2(w)}{w - z} dw \right| \\
&= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{g^2(z + re^{i\theta}) \cdot ire^{i\theta}}{z + re^{i\theta} - z} d\theta \right| \\
&= \left| \frac{1}{2\pi} \int_0^{2\pi} g^2(z + re^{i\theta}) d\theta \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} |g(z + re^{i\theta})|^2 d\theta.
\end{aligned}$$

Therefore, by vector calculus, for every $z \in B_\alpha(z_0)$,

$$\begin{aligned}
2\pi|g(z)|^2 \cdot \frac{(\beta-\alpha)^2}{2} &\leq \int_0^{2\pi} |g(z+re^{i\theta})|^2 d\theta \cdot \int_0^{(\beta-\alpha)} r\ dr \\
&= \int_0^{2\pi}\int_0^{(\beta-\alpha)} |g(z+re^{i\theta})|^2 r\ drd\theta \\
&= \iint_{B_{\beta-\alpha}(z)} |g(z)|^2\, dA \\
{\scriptstyle (B_{\beta-\alpha}(z)\,\subseteq\, B_\beta(z_0))} &\leq \iint_{B_\beta(z_0)} |g(z)|^2\, dA = \|g|_{B_\beta(z_0)}\|_2^2.
\end{aligned}$$

Thus, it follows that if $M = \frac{1}{\sqrt{\pi}(\beta-\alpha)}$, then

$$\|g|_{B_\alpha(z_0)}\|_\infty \leq M\|g|_{B_\beta(z_0)}\|_2.$$

$\square$

Finally, fix any $z_0 \in U$ and let $\varepsilon > 0$ such that $\overline{B_\varepsilon(z_0)} \subset U$ (it exists because U is open). Then, by the previous claim, let $M := \frac{2}{\sqrt{\pi}\varepsilon}$ ($\alpha = \varepsilon/2,\ \beta = \varepsilon$), so for every $n \in \mathbb{N}$,

$$\|f_n|_{B_{\varepsilon/2}(z_0)}\|_\infty \leq M\|f_n|_{B_\varepsilon(z_0)}\|_2 \leq M\|f_n|_U\|_2 \leq M \cdot C.$$

Hence, $(f_n)_{n\in\mathbb{N}}$ is locally bounded.

Exercise 4.

Let $U \subset \mathbb{C}$ be an open and connected set, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions that is locally bounded in $U \to \mathbb{C}$. Suppose that there exists a point $z_0 \in U$ such that the sequence $(f_n^{(k)}(z_0))_{n\in\mathbb{N}}$ converges for every $k \in \mathbb{N}_0$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges compactly.

Solution

By Montel's theorem, there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ that converges compactly to a holomorphic function f. Then, by **Theorem 2**, $f_{n_j}^{(k)}$ converges compactly to $f^{(k)}$, and thus,

$$\begin{aligned}
\lim_n f_n^{(k)}(z_0) &= \lim_j f_{n_j}^{(k)}(z_0) = f^{(k)}(z_0) \\
\implies &\lim_{n,m} |f_n^{(k)}(z_0) - f_m^{(k)}(z_0)| = 0.
\end{aligned}$$

Let $R > 0$ such that $\overline{B_R(z_0)} \subset U$ and

$$|f_n(z)| \leq M, \quad \forall z \in \overline{B_R(z_0)},\ n \in \mathbb{N}.$$

Then,

$$
\begin{aligned}
|f_n^{(k)}(z_0) + f_m^{(k)}(z_0)| &\leq |f_n^{(k)}(z_0)| + |f_m^{(k)}(z_0)| \\
&\leq 2 \left| \frac{k!}{2\pi i} \int_{\partial B_R(z_0)} \frac{M}{(z-z_0)^{k+1}} dz \right| \\
&\leq k! \frac{2M}{R^k}
\end{aligned}
$$

Therefore, for $z \in B_{R/2}$ and any $N \in \mathbb{N}$

$$
\begin{aligned}
\lim_{n,m} |f_n(z) - f_m(z)| &\leq \lim_{n,m} \left| \sum_{k=0}^{N} \frac{f_n^{(k)}(z_0) - f_m^{(k)}(z_0)}{k!} (z-z_0)^k \right| + \lim_{n,m} \left| \sum_{k=N+1}^{\infty} \frac{f_n^{(k)}(z_0) - f_m^{(k)}(z_0)}{k!} (z-z_0)^k \right| \\
&\leq \sum_{k=0}^{N} \frac{\lim_{n,m} \left| f_n^{(k)}(z_0) - f_m^{(k)}(z_0) \right|}{k!} (R/2)^k + \sum_{k=N+1}^{\infty} \frac{2M}{R^k} (R/2)^k \\
&\leq \quad 0 \quad + \quad 2M \frac{2^{-N-1}}{2^{-1}} \\
&= \quad M 2^{-N+1}.
\end{aligned}
$$

Therefore, $\lim_{n,m} |f_n(z) - f_m(z)| = 0$ because N can be arbitrarily large.

Finally, for every $z \in B_{R/2}(z_0)$, the sequence $(f_n(z))_{n \in \mathbb{N}}$ is a Cauchy sequence with a convergent subsequence $(f_{n_j}(z))_{j \in \mathbb{N}}$. Therefore, $(f_n(z))_{n \in \mathbb{N}}$ converges for every $z \in B_{R/2}(z_0)$ which has accumulation points, so by **Vitali's Theorem**, $(f_n)_{n \in \mathbb{N}}$ converges compactly.