Complex Analysis: Homework 6

Martín Prado

September 25, 2024

Universidad de los Andes – Bogotá Colombia

Exercise 1.

Let P be a polynomial of degree n and $R > 0$ such that $|z| < R$ for every z with $P(z) = 0$. Define $\gamma : [0, 1] \to \mathbb{C},\ \gamma(t) = Re^{2\pi t}$. Calculate $\oint_\gamma \frac{P'}{P} dz$.

Solution:

If $P(z) = \lambda(z - z_1) \cdots (z - z_n)$, then by product rule

$$
\begin{aligned}
P'(z) = \quad & \lambda \cdot 1 \cdot (z - z_2)(z - z_3) \cdots (z - z_n) \\
& + \lambda(z - z_1) \cdot 1 \cdot (z - z_3) \cdots (z - z_n) \\
& + \lambda(z - z_1)(z - z_2) \cdot 1 \cdot \cdots (z - z_n) \\
& + \qquad\qquad \vdots \\
& + (z - z_1)(z - z_2) \cdots (z - z_{n-1}) \cdot 1. \\
= \quad & \sum_{k=1}^{n} \lambda \prod_{j \neq k} (z - z_j) \\
= \quad & \sum_{k=1}^{n} \frac{P(z)}{z - z_k}.
\end{aligned}
$$

Therefore,

$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k}.$$

Finally, z_k is inside the circle we're evaluating the integral, so by residue's theorem,

$$\oint_\gamma \frac{P'}{P} dz = \sum_{k=1}^{n} \oint_\gamma \frac{1}{z - z_k} dz = n2\pi i.$$

Exercise 2.

Determine all the biholomorphic functions $\mathbb{C} \to \mathbb{C}$. **Hint.** Suppose that f is a biholomorphic function $\mathbb{C} \to \mathbb{C}$. Consider $f(1/z)$

Solution:

Since f is entire in $\mathbb{C}$, it has the following Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then,

$$f(1/z) = \sum_{n=0}^{\infty} a_n \frac{1}{z^n}.$$

If we have an infinite number of n for which $a_n \neq 0$, then 0 is an essential singularity, otherwise, f would be a polynomial. So assume that it's the case that 0 is an essential singularity of $g(z) = f(1/z)$.

By Picard's theorem, for some suitable $z_0 \in \mathbb{C}$,

$$\mathbb{C}\backslash\{z_0\} \subseteq g(B_\varepsilon(0)^\bullet) = f(\mathbb{C}\backslash\overline{B_{\varepsilon^{-1}}(0)}),\ \forall\varepsilon > 0.$$

However, since f is bijective, this would imply, $f(\overline{B_{\varepsilon^{-1}}(0)}) \subseteq \{z_0\}$, but this would contradict injectivity.

Therefore, f is a polynomial. Let n be the degree of f, then, $f(z) = \lambda(z - z_1)\cdots(z - z_n)$. However, the injectivity of f implies $n = 1$, otherwise for some $w \in \mathbb{C}$ we would obtain multiple solutions for $f(z) = 0$.

It might also happen that $z_1 = \cdots = z_n$, so $f = (z - z_1)^n$. Then there exists n different solutions $\zeta_k = e^{2\pi i k/n},\ k \leq n$ for the equation $f(z+z_1) = 1$, so $\zeta_k + z_1,\ k \leq n$ are n different solutions for $f(z) = 1$ contradicting injectivity again.

Finally, all the entire biholomorphic functions are degree 1.

Exercise 3.

Let f be a meromorphic function in $\mathbb{C}$. It's said that f is meromorphic at ∞ is the function $z \mapsto g(z) := f(1/z)$ is meromorphic at a neighborhood of 0.

(a) Show that a rational function is meromorphic at $\mathbb{C}$ and at ∞.

(b) Show that a meromorphic function at $\mathbb{C}$ and at ∞ is a rational function.

Solution Item (a)

Let $P(z)$ and $Q(z)$ be polynomials such that

$$f(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{j=0}^{n} a_j z^j}{\sum_{j=0}^{m} b_j z^j} = \lambda \frac{(z-\alpha_1)\cdots(z-\alpha_n)}{(z-\beta_1)\cdots(z-\beta_m)},$$

where $\alpha_1, \ldots, \alpha_n$ are the n roots of $P(z)$ and $\beta_1, \ldots, \beta_m$ the m roots of $Q(z)$ (some of them could be the repeated), and $\alpha_i \neq \beta_j$ without restriction. Then, the singularities of f are located at $\beta_1, \ldots, \beta_m$ and they are non-essential because these are the zeroes of a m degree polynomial that can be written as the sum of meromorphic functions:

$$\frac{1}{Q(z)} = \frac{Q_1(z)}{(z-\beta_{k_1})^{m_1}} + \cdots + \frac{Q_l(z)}{(z-\beta_{k_l})^{m_l}}, \quad m_1 + \cdots + m_l = m, \quad Q_i \text{ is a polynomial}$$

$$\implies \operatorname{ord}(1/Q; z) \leq \max_{i=1,\ldots,l}(m_i), \ \forall z \in \mathbb{C}.$$

From this, it follows that there exists $k \in \mathbb{N}$ such that for some $\varepsilon > 0$ and $z \in B_\varepsilon(0)$

$$f(z) = \sum_{j=-k}^{\infty} c_j z^j.$$

Finally, note that $Q(z)f(z) = P(z)$, and $P(1/z), Q(1/z)$ are meromorphic because they have a Laurent series with finite of non-zero coefficients, (assume W.L.O.G that $a_n, b_m \neq 0$)

$$P(1/z) = \sum_{j=0}^{n} a_j z^{-j} = \sum_{j=-n}^{0} a_{-j} z^j$$

$$Q(1/z) = \sum_{j=0}^{m} b_j z^{-j} = \sum_{j=-m}^{0} b_{-j} z^j$$

so there must exist $K \in \mathbb{N}$ such that $c_j = 0$ for every $j > K$. Otherwise, $f(1/z)$ has an essential singularity at 0 because it has an infinite number of non-zero coefficients in the Laurent series. Then, $Q(1/z)f(1/z)$ also has an infinite number of non-zero coefficients (because $b_m \neq 0$), but since $P(1/z)$ only has finite, we would get a contradiction. So, we have that

$$f(z) = \sum_{j=-k}^{K} c_j z^j$$

$$\implies f(1/z) = \sum_{j=-K}^{k} c_{-j} z^j,$$

so $f(1/z)$ is meromorphic at zero at $B_\varepsilon(0)$.

Solution Item (b)

If f is meromorphic at $\mathbb{C}$, then for some $k \in \mathbb{N}$ and $\varepsilon_1 > 0$

$$f(z) = \sum_{j=-k}^{\infty} c_j z^j, \ z \in B_{\varepsilon_1}(0).$$

On the other hand, if $f(z)$ is meromorphic at ∞, then $f(1/z)$ is meromorphic at 0, so there exists $K \in \mathbb{N}$ and $\varepsilon_2 > 0$ such that

$$f(1/z) = \sum_{j=-K}^{\infty} c_{-j} z^j = \sum_{j=-\infty}^{K} c_j z^{-j}, \ z \in B_{\varepsilon_2}(0).$$

By mixing both results together, we have that for $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$

$$f(z) = \sum_{j=-k}^{K} c_j z^j, \ z \in B_{\varepsilon}(0)$$

which can be expanded to obtain a rational function, and later be extended to the rest of the complex plane (minus the roots of the denominator) using identity theorem.

Exercise 4.

Let $0 \le r < R$, $z_0 \in \mathbb{C}$ and let f be a holomorphic function in the ring $A = \{r < |z - z_0| < R\}$ with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$. Suppose that f has an antiderivative in A. Show that $c_{-1} = 0$.

Solution:

Let $\gamma \subseteq A$ be the following curve $\gamma(t) = \rho e^{2\pi i t}$ for $\rho \in (r, R)$, and let g_1 be function such that $f(z) = g_1'(z)$ for $z \in A$.

Now let $h(z) = \sum_{n=-\infty, n\neq -1}^{\infty} c_n (z - z_0)^n$ which has antiderivative

$$g_2(z) = \sum_{n=-\infty, n\neq -1}^{\infty} \frac{c_n}{n+1} (z - z_0)^{n+1}.$$

Both functions $h(z)$ and $g_2(z)$ are defined in A because $h(z) = f(z) - c_{-1}(z - z_0)^{-1}$, and by absolute convergence, for every $\rho \in (r, R)$ and $|z - z_0| = \rho$,

$$|g_2(z)| \le \sum_{n=-\infty, n\neq -1}^{\infty} \frac{c_n}{n+1} \rho^{n+1} \le \rho \sum_{n=-\infty, n\neq -1}^{\infty} c_n \rho^n < \infty.$$

Finally, note that if f has antiderivative in A, then $f - h$ has antiderivative too in A, which is $g_1 - g_2$. However,

$$f(z) - h(z) = \frac{c_{-1}}{z - z_0},$$

and $(z - z_0)^{-1}$ doesn't have antiderivative at $A \subseteq \mathbb{C} \backslash \{z_0\}$, so it must be the case that $c_{-1} = 0$.