Complex Analysis: Homework 13

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November 15, 2024
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Exercise 1.

Determine whether the following products converge:

$$\text{(a)} \prod_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n}\right), \quad \text{(b)} \prod_{n=1}^{\infty}\left(1+\frac{(-1)^n}{\sqrt{n}}\right).$$

Note: The first term in both products is 0, so we are going to skip and show that the tails from $n=2$ forward converge (or diverge) for both cases.

Solution Item (a)

We are going to prove that $\sum_{n=2}^{\infty}\log\left(1+\frac{(-1)^n}{n}\right)$ converges. In fact, if M is even,

$$\begin{aligned}\sum_{n=2}^{M}\log\left(1+\frac{(-1)^n}{n}\right) &= \sum_{k=1}^{M/2}\log\left(\frac{2k+1}{2k}\right)+\log\left(\frac{2k}{2k+1}\right)\\ &= \sum_{k=1}^{M/2}\log\left(\frac{2k+1}{2k}\cdot\frac{2k}{2k+1}\right)\\ &= 0,\end{aligned}$$

and if $M=2K+1$ is odd,

$$\begin{aligned}\sum_{n=2}^{M}\log\left(1+\frac{(-1)^n}{n}\right) &= \log\left(\frac{2K}{2K+1}\right)+\sum_{k=1}^{(M-1)/2}\log\left(\frac{2k+1}{2k}\right)+\log\left(\frac{2k}{2k+1}\right)\\ &= \log\left(\frac{2K}{2K+1}\right).\end{aligned}$$

Since log is continuous at 1 and $\frac{2K}{2K+1} \to 1$ when $K \to \infty$, we conclude that $\log\left(\frac{2K}{2K+1}\right)$ converges to 0. Since the entire log-series converges to 0, it must follow that

$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right) = \exp\left[\sum_{n=2}^{\infty}\log\left(1+\frac{(-1)^n}{n}\right)\right] = 1.$$

Solution Item (b)

Now, we are going to prove that $\sum_{n=2}^{\infty}\log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ diverges. We are going to be restricted to the even case to show that this sum diverges. Note that

$$\begin{aligned}\log\left(1+\frac{1}{\sqrt{2n}}\right)+\log\left(1-\frac{1}{\sqrt{2n+1}}\right) &= \log\left[\left(1+\frac{1}{\sqrt{2n}}\right)\cdot\left(1-\frac{1}{\sqrt{2n+1}}\right)\right]\\ &= \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right]\end{aligned}$$

By the limit comparison test $\log(1+x) = x - O(x^2) \approx x$ for x near to 0. Then, we can compare the series of $a_n = \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right]$ with the series of $b_n = \frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}$.

Remark: Note that for $n \geq 1$, $\sqrt{2n+1}-\sqrt{2n}-1 < 0$. The limit comparison test only applies if $a_n, b_n > 0$, but for our case,

$$a_n = \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right] < 0 \text{ and } b_n = \frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}} < 0$$

and thus, the same argument can be applied to $-a_n, -b_n > 0$ to conclude that $-\sum_n a_n$ diverges because $-\sum_n b_n$ does.

Since $a_n = \log(1+b_n)$ and $b_n \to 0$ when $n \to \infty$, by uniqueness of limit and then L-hôspital rule,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{1}{1+x} = 1.$$

Then, we can also compare b_n with $c_n = \frac{-1}{2n}$ because

$$\begin{aligned}\frac{b_n}{c_n} &= \frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\cdot(-2n)\\ &= \underbrace{\sqrt{\frac{2n}{2n+1}}}_{\to 1}+\underbrace{\frac{2n}{\sqrt{2n+1}}-\sqrt{2n}}_{\to 0}\end{aligned}$$

$$\to 1, \quad n\to\infty$$

Finally, since $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{-1}{2n}$ diverges to $-\infty$, it follows that $\sum_{n=1}^{\infty} b_n$ diverges, and thus, $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$ too. Therefore,

$$\begin{aligned}\sum_{n=2}^{2M} \log\left(1+\frac{(-1)^n}{\sqrt{n}}\right) &= \sum_{n=1}^{M} \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right] \\ &= \sum_{n=1}^{M} a_n \to -\infty, \quad M\to\infty.\end{aligned}$$

That implies that $\prod_{n=2}^{\infty} \log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ has a subsequence that diverges to 0.

Exercise 2.

Prove the class theorem: Let (X,d) be a compact metric space and let $g_n : X \to \mathbb{C}$ be continuous functions such that $\sum_{n=1}^{\infty} |g_n|$ converges uniformly. Define $f_n : X \to \mathbb{C}$ by

$$f_n(x) = \prod_{j=1}^{n} (1+g_j(x)).$$

We already know that for every $x \in X$, the product $\prod_{n=1}^{\infty}(1+g_j(x))$ is absolutely convergent. Then

$$f : X \to \mathbb{C}, \quad f(x) := \lim_{n\to\infty} f_n(x)$$

is well-defined.

Show that $f_n \to f$ uniformly and that there exists $N \in \mathbb{N}$ such that for all $x \in X$,

$$f(x) = 0 \iff g_n(x) = -1 \text{ for some } n \le N.$$

Exercise 3.

Let $U \subset \mathbb{C}$ be open and let $g_n : U \to \mathbb{C}$ be holomorphic functions such that $\sum_{n=1}^{\infty} |g_n|$ converges compactly in U. Define

$$f_n(x) = \prod_{j=1}^{n} (1+g_j(x)).$$

(a) Show that $(f_n)_{n\in\mathbb{N}}$ converges compactly to a holomorphic function $f : U \to \mathbb{C}$.

(b) Let $z_0 \in U$. Show that $f(z_0) = 0$ if and only if there exists $j \in \mathbb{N}$ such that $g_j(z_0) = -1$, that there are finitely many such j, and that the order of the zero z_0 for f is equal to the sum of the multiplicities of z_0 as a zero of all the functions $1+g_j$.

Exercise 4.

Let $U \subset \mathbb{C}$ be a region, let $f_n : U \to \mathbb{C}$ be holomorphic functions, and assume that $\prod_{j=1}^{\infty} f_n$ converges absolutely and compactly in U. Show that

$$\frac{f'}{f} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j}$$

where the sum on the right side converges compactly in its domain.