Complex Analysis: Homework 2

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Exercise 1.

Find all the points $z \in \mathbb{C}$ where the following functions are differentiable and find the largest open set U where they are holomorphic.

(a) $f(z) = \overline{z}$

(b) $f(x + iy) = x^2 + y^2 + i(x^2 - y^2)$

Solution Part (a)

$$f(x + iy) = x - iy = u(x, y) + iv(x, y)$$

Then,

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= 1, \qquad & \frac{\partial v}{\partial x}(x, y) &= 0, \\ \frac{\partial u}{\partial y}(x, y) &= 0, \qquad & \frac{\partial v}{\partial y}(x, y) &= -1, \end{aligned}$$

All the partial derivatives exists and are continuous on any $(x, y) \in \mathbb{R}^2$, and thus, the function is differentiable. However, the Cauchy-Riemann equations are a requirement for f to be complex-differentiable. Therefore, since $\partial u/\partial x \neq \partial v/\partial y$ on all points, the largest open set where it's holomorphic is $U = \emptyset$.

Solution Part (b)

In this case,

$$u(x, y) = x^2 + y^2, \quad v(x, y) = x^2 - y^2,$$

and the respective partial derivatives are

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = -2y.$$

For differentiability in $\mathbb{R}^2$, the argument is again that the partial derivatives exist and are continuous. For complex-differentiability, the function is holomorphic only when $2x = -2y$. Thus, the largest open set is again $U = \emptyset$.

Exercise 2.

(a) Let $u(x, y) = x^3 - 3xy^2$. Find all the entire functions f such that $u = \mathrm{Re}(f)$.

(b) Let $v(x, y) = x^2 + y^2$. Find all the entire functions f such that $v = \mathrm{Im}(f)$.

(c) Let $U \subseteq \mathbb{C}$ be a region and let $f, g : U \to \mathbb{C}$ be holomorphic functions such that $f(U) \subset \mathbb{R}$ and $g(U) \subset \mathbb{S}^1 := \{z \in \mathbb{C} \,:\, |z| = 1\}$. Prove that f and g are constant.

Solution Part (a)

$$\frac{\partial v}{\partial y}(x, y) = \frac{\partial u}{\partial x}(x, y) = 3x^2 - 3y^2,$$
$$-\frac{\partial v}{\partial x}(x, y) = \frac{\partial u}{\partial y}(x, y) = -6xy.$$

The solutions for these partial equations are

$$v(x, y) = \int 3x^2 - 3y^2 \, dy = 3x^2y - y^3 + K_1(x),$$

$$v(x, y) = \int 6xy \, dx = 3x^2y + K_2(y).$$

Therefore,

$$v(x, y) = 3x^2y - y^3 + K_1(y) = 3x^2y + K_2(x)$$
$$\implies K_1(y) = y^3 + K_2(x)$$

This can only happen if K_1 is a constant $K \in \mathbb{C}$ and $K_2(y) = y^3 + K$. Thus,

$$v(x, y) = 3x^2y - y^3 + K.$$

Finally, the family of entire functions that satisfy the initial condition are:

$$f_K(x, y) = x^3 - 3xy^2 + i(3x^2y - y^3 + K), \quad K \in \mathbb{C}$$
$$= (x + iy)^3 + iK.$$

Solution Part (b)

$$-\frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial x}(x,y) = 2x,$$
$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 2y.$$

The solutions for these partial equations are

$$u(x,y) = \int -2x \; dy = -2xy + K_1(x),$$

$$u(x,y) = \int 2y \; dx = 2xy + K_2(y).$$

Therefore,

$$u(x,y) = -2xy + K_1(y) = 2xy + K_2(x)$$
$$\implies K_1(y) - K_2(x) = 4xy.$$

However, this cannot be possible for any entire function because the previous would imply that the functions K_1, K_2 are not well defined:

$$\frac{\partial K_1}{\partial y}(y) = 4x,$$

$$\frac{\partial K_2}{\partial y}(x) = -4y.$$

Thus, there doesn't exist any entire function with the initial conditions.

Solution Part (c)

Let $z = x + iy$. For f, note that if $f(x,y) = u(x,y) + i(v,y)$, then $v(x,y) = 0$ for $z \in U$. Then, using the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

However, $\nabla u(x,y) = (0,0)$ if and only if u is a constant function, and thus, f is a constant function too.

For g, we can make a variable substitution to the polar coordinates. $x(r,\theta) = r\cos(\theta)$ and $y(r,\theta) = r\sin(\theta)$b

Exercise 3.

(a) $\exp(z+w) = \exp(z)\exp(w)$.

(b) $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.

(c) $|\exp(z)| = 1$ if and only if $z \in i\mathbb{R}$.

(d) $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathbb{C}$.

(e) $\cos(z + 2\pi) = \cos(z)$ and $\sin(z + 2\pi) = \sin(z)$ for all $z \in \mathbb{C}$.

(f) $\cos(z) = 0$ or $\sin(z) = 0 \implies z \in \mathbb{R}$.

(g) For every $x \in \mathbb{R}$, $\lim_{t\to\pm\infty} |\cos(x+it)| = \infty$ and $\lim_{t\to\pm\infty} |\sin(x+it)| = \infty$. The limit is uniform in x.

Solution Part (a)

The Cauchy product of 2 series implies that

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \cdot \frac{n!}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\
&= \exp(z+w)
\end{aligned}$$

Solution Part (b)

For every complex number z, there exists an additive inverse $(-z)$ such that

$$z + (-z) = 0.$$

Thus, if it was the case that there exists $z \in \mathbb{C}$ such that $e^z = 0$, then, using part (a),

$$1 = e^0 = e^{z+(-z)} = e^z e^{-z} = 0,$$

and this would lead to a contradiction.

Solution Part (c)

$\Longleftarrow$ **:** We are going to prove Euler's formula for the power series definition. Let $z = iy,\ y \in \mathbb{R}$,

$$\begin{aligned}
\exp(iy) &= \frac{(iy)^0}{0!} + \frac{(iy)^1}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\
&= \left(\frac{(iy)^0}{0!} + \frac{(iy)^2}{2!} + \cdots \right) + i \left(\frac{i^0 y^1}{1!} + \frac{i^2 y^3}{3!} + \cdots \right) \\
&= \sum_{n=0}^{\infty} i^{2n} \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} i^{2n} \frac{y^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} \\
&= \cos(y) + i \sin(y).
\end{aligned}$$

Note that from this and part (a), it follows that for $x, y \in \mathbb{R}$,

$$e^{x+iy} = e^x(\cos(y) + i \sin(y))$$

Therefore, we have $\cos^2(y) + \sin^2(y) = 1$ for $y \in \mathbb{R}$, and thus,

$$|\exp(iy)| = \sqrt{\cos^2(y) + \sin^2(y)} = 1$$

$\Longrightarrow$ **:** Let $z = x + iy,\ x, y \in \mathbb{R}$ such that $|\exp(z)| = 1$. Then, using part (a), $|\exp(z)| = |\exp(x)||\exp(iy)|$. Then, using the previous implication, we know that $|\exp(iy)| = 1$. Therefore, $|\exp(z)| = |\exp(x)| = \exp(x) = 1$, but for real numbers, the only solution for $\exp(x) = 1$ is $x = 0$.

Solution Part (d)

According to Ahlfors' book, the definition of the cosine and sine functions are:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Thus,

$$\begin{aligned}
\cos^2(z) + \sin^2(z) &= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\
&= \frac{1}{4}e^{2iz} + \frac{e^{iz-iz}}{2} + \frac{1}{4}e^{-2iz} - \frac{1}{4}e^{2iz} + \frac{e^{iz-iz}}{2} - \frac{1}{4}e^{-2iz} \\
&= e^{iz-iz} = 1.
\end{aligned}$$

Solution Part (e)

Using Euler's formula, we know that $e^{-2\pi i} = e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$. Now, with the same identity and part (a),

$$\begin{aligned}
\cos(z + 2\pi) &= \frac{e^{iz+i2\pi} + e^{-iz-i2\pi}}{2} \\
&= \frac{e^{iz}e^{2\pi i} + e^{-iz}e^{-2\pi i}}{2} \\
&= \frac{e^{iz} + e^{-iz}}{2} = \cos(z),
\end{aligned}$$

$$\begin{aligned}
\sin(z + 2\pi) &= \frac{e^{iz+i2\pi} - e^{-iz-i2\pi}}{2i} \\
&= \frac{e^{iz}e^{2\pi i} - e^{-iz}e^{-2\pi i}}{2i} \\
&= \frac{e^{iz} - e^{-iz}}{2i} = \sin(z).
\end{aligned}$$

Solution Part (f)

Let $z = x + iy$. Either $\cos(z) = 0$ or $\sin(z) = 0$ (or both). So let's start with the case $\cos(z) = 0$. Using general Euler's formula,

$$e^{iz} = \cos(z) + i\sin(z) = i\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$$

Remember that for $x \in \mathbb{R}$, $|e^{ix}| = 1$, and $|e^x| = e^x$. Now,

$$\begin{aligned}
\implies \quad e^{-iz} &= -e^{iz} \\
\implies \quad e^{y-ix} &= -e^{-y+ix} \\
\implies |e^{-ix}||e^{y}| &= |-1||e^{ix}||e^{-y}| \\
\implies \quad e^{y} &= e^{-y} \\
\implies \quad e^{2y} &= 1 \\
\implies \quad y &= \mathrm{Im}(z) = 0.
\end{aligned}$$

For the case when $\sin(z) = 0$, the argument is a similar one:

$$e^{iz} = \cos(z) + i\sin(z) = \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\begin{aligned}
\implies \quad e^{-iz} &= e^{iz} \\
\implies \quad e^{y-ix} &= e^{-y+ix} \\
\implies |e^{-ix}||e^{y}| &= |e^{ix}||e^{-y}| \\
\implies \quad e^{y} &= e^{-y} \\
\implies \quad e^{2y} &= 1 \\
\implies \quad y &= \mathrm{Im}(z) = 0.
\end{aligned}$$

Solution Part (g)

$$\begin{aligned}
|\cos(x+it)| &= \left|\frac{e^{ix-t} + e^{-ix+t}}{2}\right| \\
&= \left|\frac{e^{ix}e^{-t} + e^{-ix+t}e^{t}}{2}\right| \\
(\triangle\text{-ineq}) &\geq \frac{1}{2}\left||e^{ix}||e^{-t}| - |e^{-ix}||e^{t}|\right| \\
&= \frac{1}{2}\left||e^{-t}| - |e^{t}|\right|
\end{aligned}$$

$$\begin{aligned}
|\sin(x+it)| &= \left|\frac{e^{ix-t}-e^{-ix+t}}{2i}\right| \\
&= \left|\frac{e^{ix}e^{-t}-e^{-ix+t}e^{t}}{2i}\right| \\
(\triangle\text{-ineq}) &\geq \frac{1}{2}\left||e^{ix}||e^{-t}|-|e^{-ix}||e^{t}|\right| \\
&= \frac{1}{2}\left||e^{-t}|-|e^{t}|\right|
\end{aligned}$$

Note that the inequalities of both cases are satisfied for all $x \in \mathbb{R}$. Either e^t or e^{-t} converges to ∞ (and the other to 0), when $t \to \pm\infty$. Therefore, $\forall \varepsilon > 0$, there exists N such that for every $x \in \mathbb{R}$:

$$\frac{1}{\varepsilon} < \frac{1}{2}\left||e^{-t}|-|e^{t}|\right| \begin{array}{l} \leq |\cos(x+it)| \\ \leq |\sin(x+it)| \end{array} \qquad \forall t > N.$$

So the limit is uniform on x.

Exercise 4.

Prove that

(a) $\sum_{n=1}^{\infty} nz^n$ does not converge to any point for $z \in \mathbb{S}^1$.

(b) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges to every point for $z \in \mathbb{S}^1$

(c) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges to every point for $z \in \mathbb{S}^1$, except for 1.

Solution Part (a)

If a series $\sum_{n=1}^{\infty} a_n$ converges, then $|a_n| \to 0$. Note that $|nz^n| = n$ if $|z| = 1$, which as a matter of fact diverges to infinity. Thus, the series $\sum_{n=1}^{\infty} nz_n$ doesn't converge.

Solution Part (b)

$\mathbb{S}^1 \subseteq \mathbb{C}$ is a complete space, that implies that absolute convergent series are convergent. The series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is absolutely convergent because

$$\sum_{n=1}^{\infty}\left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|z^n|}{|n^2|} = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Therefore, the original series is also convergent.

Solution Part (c)

It's a known fact from real analysis that the series $\sum_{n=1}^{\infty} 1/n$ diverges.

So let $z \in \mathbb{S}^1 \backslash \{1\}$, and let $a_n = \frac{1}{n}$ and $B_k = \sum_{n=0}^{k} b_n$, where $b_n = z^n$. The summation by part formula states that

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n \\
&= \frac{1}{N} \sum_{n=0}^{N} z^n - \frac{1}{M} \sum_{n=0}^{M} z^n - \sum_{n=M}^{N-1} \frac{-1}{n(n+1)} \sum_{k=0}^{n} z^k \ . \\
&= \frac{1}{N} \frac{1 - z^N}{1 - z} - \frac{1}{M} \frac{1 - z^M}{1 - z} + \sum_{n=M}^{N-1} \frac{1}{n^2 + n} \frac{1 - z^n}{1 - z}
\end{aligned}$$

Then,

$$\sum_{n=1}^{N} \frac{z^n}{n} = \frac{1}{N} \frac{1 - z^N}{1 - z} - 1 + \frac{1}{1 - z} \sum_{n=1}^{N-1} \frac{1 - z^n}{n^2 + n} \ ,$$

On the other hand,

Exercise 5.

A subset $S \subset \mathbb{N}$ is in *arithmetic progression* if there exists $a, d \in \mathbb{N}$ such that

$$S = \{a + nd \ : \ n \in \mathbb{N}_0\}.$$

The number d is called the difference of the progression. Prove that $\mathbb{N}$ cannot be partitioned in a finite number greater than 1 of arithmetic progressions with different differences.

Solution:

Let $A_i = \{a_i + nd_i\}$ with $d_i \neq d_j$ if $i \neq j$. Assume that there exists a finite partition of $\mathbb{N}^+$ $\{A_1, \ldots, A_N\}$. W.L.O.G. assume that $d_1 < d_2 < \cdots < d_N$. Then,

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} z^{a_1 + nd_1} + \cdots + \sum_{n=0}^{\infty} z^{a_N + nd_N}.$$

This power series converges when $|z| < 1$:

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1 - z} = \sum_{i=1}^{N} \frac{z^{a_i}}{1 - z^{d_i}}$$

Note that $\frac{1}{1-z}$ has a pole of multiplicity 1 at $z = 1$. However, the right-hand side of the equation has a pole of multiplicity $d_N > 1$ at $z = 1$ and this would give us a contradiction.