

# Complex Analysis: Homework 7

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## Exercise 1.

Calculate the principal part at 0 of the functions

$$f(z) = \frac{(\sin z)^2}{\sin(z^2)}, \quad g(z) = \frac{1 - z^2}{z(1 - \cos(z^2))}.$$

### Solution Item (a)

The Taylor series of  $(\sin z)^2$  is by trigonometric identities,

$$\begin{aligned} (\sin z)^2 &= \frac{1 - \cos(2z)}{2} \\ &= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} z^{2n} \\ &= z^2 - \frac{2^3 z^4}{4!} + \frac{2^5 z^6}{6!} + O(z^8) \end{aligned}$$

The Taylor series of  $\sin(z^2)$  is by substitution

$$\begin{aligned} \sin(z^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{(2n+1)!} \\ &= z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + O(z^{14}) \end{aligned}$$

Then,

$$\begin{aligned}\lim_{z \rightarrow 0} \frac{(\sin z)^2}{\sin(z^2)} &= \lim_{z \rightarrow 0} \frac{z^2 - \frac{2^3}{4!}z^4 + O(z^6)}{z^2 + O(z^6)} \\ &= \lim_{z \rightarrow 0} \frac{1 + O(z^2)}{1 + O(z^4)} = 1.\end{aligned}$$

Therefore, the principal part of the series is 0 because the series has a removable singularity at 0.

### Solution Item (b)

The Taylor series expansion of  $z(1 - \cos(z^2))$  is the following

$$\begin{aligned}z(1 - \cos(z^2)) &= z \left( 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} z^{2n+1} \\ &= \frac{1}{2!} z^5 - \frac{1}{4!} z^9 + O(z^{13})\end{aligned}$$

Therefore,

$$g(z) = \frac{1 - z^2}{\frac{1}{2}z^5 + O(z^9)},$$

so it's clear that  $g(z)$  has a pole of order 5 at  $z = 0$ . In fact, if

$$g(z) = \sum_{n=-5}^{\infty} a_n z^n,$$

then  $a_{-5} = \lim_{z \rightarrow 0} z^5 g(z) = \lim_{z \rightarrow 0} \frac{1 - z^2}{\frac{1}{2} + O(z^4)} = 2$ . Then, for the next coefficients

$$\begin{aligned}a_{-4} &= \lim_{z \rightarrow 0} z^4 (g(z) - a_{-5} z^{-5}) \\ &= \lim_{z \rightarrow 0} z^4 (g(z) - 2z^{-5}) \\ &= \lim_{z \rightarrow 0} z^4 \frac{1 - z^2 - 2z^{-5}(\frac{1}{2!}z^5 + O(z^9))}{\frac{1}{2}z^5 + O(z^9)} \\ &= \lim_{z \rightarrow 0} \frac{-z^2 + O(z^4)}{\frac{1}{2}z + O(z^5)} = 0\end{aligned}$$

$$\begin{aligned}
a_{-3} &= \lim_{z \rightarrow 0} z^3 (g(z) - a_{-5}z^{-5} - a_{-4}z^{-4}) \\
&= \lim_{z \rightarrow 0} z^3 (g(z) - 2z^{-5}) \\
&= \lim_{z \rightarrow 0} z^3 \frac{1 - z^2 - 2z^{-5}(\frac{1}{2!}z^5 + O(z^9))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z \rightarrow 0} \frac{-z^2 + O(z^4)}{\frac{1}{2}z^2 + O(z^4)} = -2
\end{aligned}$$

$$\begin{aligned}
a_{-2} &= \lim_{z \rightarrow 0} z^2 (g(z) - a_{-5}z^{-5} - a_{-4}z^{-4} - a_{-3}z^{-3}) \\
&= \lim_{z \rightarrow 0} z^2 (g(z) - 2z^{-5} + 2z^{-3}) \\
&= \lim_{z \rightarrow 0} z^2 \frac{1 - z^2 + (-2z^{-5} + 2z^{-3})(\frac{1}{2!}z^5 - \frac{1}{4!}z^9 + O(z^{13}))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z \rightarrow 0} \frac{1 - z^2 + (-1 + \frac{2}{4!}z^4 + O(z^8)) + (z^2 - \frac{2}{4!}z^6 + O(z^{10}))}{\frac{1}{2}z^3 + O(z^7)} \\
&= \lim_{z \rightarrow 0} \frac{\frac{1}{12}z^4 + O(z^6)}{\frac{1}{2}z^3 + O(z^7)} = 0.
\end{aligned}$$

$$\begin{aligned}
a_{-1} &= \lim_{z \rightarrow 0} z (g(z) - a_{-5}z^{-5} - a_{-4}z^{-4} - a_{-3}z^{-3} - a_{-2}z^{-2}) \\
&= \lim_{z \rightarrow 0} z (g(z) - 2z^{-5} + 2z^{-3}) \\
&= \lim_{z \rightarrow 0} z \frac{1 - z^2 + (-2z^{-5} + 2z^{-3})(\frac{1}{2!}z^5 - \frac{1}{4!}z^9 + O(z^{13}))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z \rightarrow 0} \frac{1 - z^2 + (-1 + \frac{2}{4!}z^4 + O(z^8)) + (z^2 - \frac{2}{4!}z^6 + O(z^{10}))}{\frac{1}{2}z^4 + O(z^8)} \\
&= \lim_{z \rightarrow 0} \frac{\frac{1}{12}z^4 + O(z^6)}{\frac{1}{2}z^4 + O(z^8)} = \frac{1}{6}.
\end{aligned}$$

Finally, the principal part is

$$\frac{2}{z^5} - \frac{2}{z^3} + \frac{1}{6z}.$$

## Exercise 2.

Let  $M \subset \mathbb{C}$  be a finite set and let  $f : \mathbb{C} \setminus M \rightarrow \mathbb{C}$  be holomorphic.

(a) Show that  $g(z) = z^{-2}f(z^{-1})$  is holomorphic at  $B_\varepsilon(0) \setminus \{0\}$  for  $\varepsilon > 0$  sufficiently small.

(b) Show that  $\text{Res}_0 g = \sum_{c \in \mathbb{C}} \text{Res}_c f$ .

(c) Calculate  $\int_{\partial B_1(0)} \frac{5z^6 + 4}{2z^7 + 1} dz$ .

### Solution Item (a)

For some  $R > 0$ ,  $M \subset B_R(0)$ , so it follows that  $f$  is holomorphic at  $\mathbb{C} \setminus \overline{B_R(0)}$ . Then, the map  $z \mapsto f(z^{-1})$  is holomorphic at  $B_{1/R}(0) \setminus \{0\}$ , so it follows that  $g$  is also holomorphic at  $B_{1/R}(0) \setminus \{0\}$ .

### Solution Item (b)

### Solution Item (c)

Let

$$f(z) = \frac{5z^6 + 4}{2z^7 + 1},$$

and let,

$$\begin{aligned} g(z) &= z^{-2}f(z^{-1}) \\ &= \frac{1}{z^2} \frac{5z^{-6} + 4}{2z^{-7} + 1} \\ &= \frac{1}{z^2} \frac{4z^7 + 5}{z^7 + 2} \\ &= \frac{4z^6 + 5}{z^8 + 2z}. \end{aligned}$$

Finally, since all the zeroes of  $2z^7 + 1$  are in  $\partial B_{2^{-1/7}}(0) \subset \text{int} \partial B_1(0)$ , it follows that

$$\frac{1}{2\pi i} \int_{\partial B_1(0)} \frac{5z^6 + 4}{2z^7 + 1} dz = \sum_{c \in \mathbb{C}} \text{Res}_c f = \text{Res}_0 g,$$

and

$$\begin{aligned}\operatorname{Res}_0 g &= \lim_{z \rightarrow 0} z g(z) \\ &= \lim_{z \rightarrow 0} \frac{4z^6 + 5}{z^7 + 2} = \frac{5}{2}, \\ \implies \int_{\partial B_1(0)} \frac{5z^6 + 4}{2z^7 + 1} dz &= 5\pi i\end{aligned}$$

### Exercise 3.

Calculate the following integrals with complex analysis methods

$$(a) \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 6x^2 + 13} dx$$

$$(b) \int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx$$

$$(c) \int_0^{\infty} \frac{\sin x}{x} dx$$

### Solution Item (a)

The difference of the degrees between the denominator and numerator is 2, so we can use the following method

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 6x^2 + 13} dx = 2\pi i \sum_{\operatorname{Im}(z_0) > 0} \operatorname{Res}_{z=z_0} \left( \frac{z^2}{z^4 + 6z^2 + 13} \right)$$

The function  $z^4 + 6z^2 + 13$  has a zero with multiplicity 1 at

$$a = \sqrt[4]{13} \cos \left( \frac{1}{2} \left( \tan^{-1} \left( \frac{2}{3} \right) - \pi \right) \right) - i \sqrt[4]{13} \sin \left( \frac{1}{2} \left( \tan^{-1} \left( \frac{2}{3} \right) - \pi \right) \right)$$

It also has multiplicity 1 zeroes at  $-\bar{a}, \bar{a}, -a$ , but the only ones at the upper half plane are  $a$  and  $-\bar{a}$ .

Then,

$$\begin{aligned}
\operatorname{Res}_a f(z) &= \lim_{z \rightarrow a} (z - a) \frac{z^2}{(z - a)(z + a)(z - \bar{a})(z + \bar{a})} \\
&= \frac{a^2}{2a(2i\operatorname{Im}(a))(2\operatorname{Re}(a))} \\
&= \frac{-ia}{8\operatorname{Im}(a)\operatorname{Re}(a)}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}_{-\bar{a}} f(z) &= \lim_{z \rightarrow -\bar{a}} (z + \bar{a}) \frac{z^2}{(z - a)(z + a)(z - \bar{a})(z + \bar{a})} \\
&= \frac{\bar{a}^2}{(-2\operatorname{Re}(a))(-2i\operatorname{Im}(a))(-2\bar{a})} \\
&= \frac{-i\bar{a}}{8\operatorname{Im}(a)\operatorname{Re}(a)}
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 6x^2 + 13} dx &= \operatorname{Res}_a f(z) + \operatorname{Res}_{-\bar{a}} f(z) \\
&= \frac{-ia - i\bar{a}}{8\operatorname{Im}(a)\operatorname{Re}(a)} \\
&= \frac{i(-2\operatorname{Re}(a))}{8\operatorname{Im}(a)\operatorname{Re}(a)} \\
&= \frac{-i}{4\operatorname{Im}(a)} \\
&= \frac{i}{4\sqrt[4]{13} \sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right) - \pi\right)\right)}
\end{aligned}$$

So it follows that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 6x^2 + 13} dx = \frac{-2\pi}{4\sqrt[4]{13} \sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right) - \pi\right)\right)} \approx 0.8643$$

Input

$$\frac{-2\pi}{4\sqrt[4]{13} \sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right) - \pi\right)\right)}$$

$\tan^{-1}(x)$  is the inverse tangent function

Exact Result

$$-\frac{\pi \csc\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right) - \pi\right)\right)}{2\sqrt[4]{13}}$$

(result in radians)

Decimal approximation More digits

0.8643314998746620415510133329799156368152394651865508561031229358...

(result in radians)

and this coincides with the real result

Definite integral More digits Step-by-step solution

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 6x^2 + 13} dx = \frac{1}{2} \sqrt{\frac{1}{2}(\sqrt{13} - 3)} \pi \approx 0.8643$$

Indefinite integral Approximate form Step-by-step solution

## Solution Item (b)

This integral has the form  $\int_0^\infty x^\alpha R(x) dx$  where  $0 < \alpha = 1/2 < 1$  and  $R(x) = O(x^{-2})$  without any poles at the origin. Therefore, by using the substitution  $x = t^2$ ,  $dx = 2t dt$ , we obtain

$$\begin{aligned} \int_0^\infty x^\alpha R(x) dx &= 2 \int_0^\infty t^{2\alpha+1} R(t^2) dt \\ &= \int_{-\infty}^\infty t^{2\alpha+1} R(t^2) dt \\ &= \int_{-\infty}^\infty \frac{t^2}{t^4 + 1} dt \end{aligned}$$

It follows that since the difference between the degrees of the denominator and numerator is two,

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 1} dx = \sum_{\text{Im}(z_0) > 0} \text{Res}_{z=z_0} \left( \frac{z^2}{z^4 + 1} \right)$$

The polynomial  $z^4 + 1$  has a root of multiplicity 1 at

$$a = \frac{1+i}{\sqrt{2}}$$

and also has roots at  $-a, \bar{a}, -\bar{a}$ , from which only  $a$  and  $-\bar{a}$  are in the upper half plane. Using the same logic as the previous item (because it's the exact same case only changing

the value of  $a$ ),

$$\begin{aligned}\operatorname{Res}_a f(z) &= \lim_{z \rightarrow a} (z - a) \frac{z^2}{(z - a)(z + a)(z - \bar{a})(z + \bar{a})} \\ &= \frac{-ia}{8\operatorname{Im}(a)\operatorname{Re}(a)}\end{aligned}$$

$$\begin{aligned}\operatorname{Res}_{-\bar{a}} f(z) &= \lim_{z \rightarrow -\bar{a}} (z + \bar{a}) \frac{z^2}{(z - a)(z + a)(z - \bar{a})(z + \bar{a})} \\ &= \frac{-i\bar{a}}{8\operatorname{Im}(a)\operatorname{Re}(a)}\end{aligned}$$

So finally,

$$\begin{aligned}\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{t^2}{t^4 + 1} dt &= \operatorname{Res}_a f(z) + \operatorname{Res}_{-\bar{a}} f(z) \\ &= \frac{-ia - i\bar{a}}{8\operatorname{Im}(a)\operatorname{Re}(a)} \\ &= \frac{i(-2\operatorname{Re}(a))}{8\operatorname{Im}(a)\operatorname{Re}(a)} \\ &= \frac{-i}{4\operatorname{Im}(a)} \\ &= \frac{-i}{4\sqrt{2}},\end{aligned}$$

and thus,

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx = 2 \int_{-\infty}^{\infty} \frac{t^2}{t^4 + 1} dt = \frac{\pi}{\sqrt{2}},$$

which coincides with the real result

Definite integral
More digits

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx = \frac{\pi}{\sqrt{2}} \approx 2.22144$$

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## Solution Item (c)

We have that

$$\begin{aligned}
 \int_0^\infty \frac{\sin x}{x} dx &= \int_0^\infty \frac{e^{ix} - e^{-ix}}{2ix} dx \\
 &= \int_0^\infty \frac{e^{ix}}{2ix} dx - \int_0^\infty \frac{e^{-ix}}{2ix} dx \\
 &= \int_0^\infty \frac{e^{ix}}{2ix} dx + \int_{-\infty}^0 \frac{e^{ix}}{2ix} dx \\
 &= \int_{-\infty}^\infty \frac{e^{ix}}{2ix} dx
 \end{aligned}$$

We have a simple pole at  $x = 0$  and  $R(\infty) = 0$ , so we can apply the following formula

$$\begin{aligned}
 \int_{-\infty}^\infty \frac{e^{ix}}{x} dx &= \int_{-\infty}^\infty R(x) e^{ix} dx \\
 &= 2\pi i \sum_{\text{Im}(z_0) > 0} \text{Res}_{z=z_0} R(z) e^{iz} + \pi i \sum_{\text{Im}(z_0) = 0} \text{Res}_{z=z_0} R(z) e^{iz} \\
 &= \pi i \text{Res}_{z=0} \frac{e^{iz}}{z} = \pi i.
 \end{aligned}$$

Finally,

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix}}{x} dx = \frac{\pi}{2}$$



Definite Integral More digits

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2} \approx 1.5708$$

## Exercise 4.

- Let  $\gamma$  be a closed curve in  $\mathbb{C} \setminus \{0\}$ . Let  $n \in \mathbb{N}$  and  $p : \mathbb{C} \rightarrow \mathbb{C}$ ,  $p(z) = z^n$ . Show that  $\text{ind}_{p \circ \gamma}(0) = n \text{ind}_\gamma(0)$ .
- Let  $U \subset \mathbb{C}$  be open and connected,  $c \in U$  and  $\gamma$  be a closed curve in  $U \setminus \{c\}$  such that  $\text{int}(\gamma) \subset D$ . Para a biholomorphic function  $f : U \rightarrow f(U)$  show that

$$\text{ind}_\gamma(c) = \text{ind}_{f \circ \gamma}(f(c))$$

## Exercise 5.

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function. Suppose that for every  $a \in \mathbb{C}$ , at least one coefficient in the Taylor series of  $f$  in  $a$  is vanished. Show that  $f$  is a polynomial

### Solution:

We previously proved that the set of zeroes of a non-zero holomorphic function is discrete and closed, so it has to be countable (because uncountable sets have accumulation points in  $\mathbb{R}^n$ ).

Now assume for the sake of contradiction that  $f$  is not a polynomial, so the  $k$ -th derivative  $f^{(k)}$  is always non-zero, and thus, has a countable set of zeroes. So it follows that  $(f^{(k)})^{-1}(\{0\})$  is a countable set, and thus,

$$\bigcup_{k=0}^{\infty} (f^{(k)})^{-1}(\{0\}) \text{ is countable too.}$$

Also note that since for every  $a \in \mathbb{C}$  there exists  $n \in \mathbb{N}$  such that  $a_n = 0$  for the series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k,$$

so it follows that  $f^{(n)}(a) = 0$ , and thus, for every  $a \in \mathbb{C}$

$$a \in (f^{(n)})^{-1}(\{0\}) \subset \bigcup_{k=0}^{\infty} (f^{(k)})^{-1}(\{0\})$$

$$\implies \mathbb{C} \subset \bigcup_{k=0}^{\infty} (f^{(k)})^{-1}(\{0\}),$$

but that would imply that  $\mathbb{C} = \mathbb{R} \times \mathbb{R}$  is countable, which is false.