Complex Analysis: Homework 7

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Exercise 1.

Calculate the principal part at 0 of the functions

$$f(z) = \frac{(\sin z)^2}{\sin(z^2)}, \qquad g(z) = \frac{1 - z^2}{z(1 - \cos(z^2))}.$$

Solution Item (a)

Note: After working a little more on the problem, I think I found a way to calculate the limits in this problem without using Big O notation.

Let $f_1(z) = \sin(z)/z$ and $f_2(z) = \sin(z^2)/z^2$. In the first place,

$$\lim_{z\to 0} z f_1(z) = \lim_{z\to 0} \frac{z}{z}\sin(z) = 0,$$

so 0 is a removable singularity for f_1, which is extended to the following holomorphic function (by uniqueness of the Taylor series expansion)

$$\tilde{f}_1 = \begin{cases} f_1(z) & z \neq 0, \\ 1 & z = 0. \end{cases} \qquad \tilde{f}_1(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}.$$

For a similar reason, $f_2(z) = f_1(z^2)$ has a removable singularity at 0, and by continuity of the map $z \mapsto z^2$, the continuous extension for f_2 is $\tilde{f}_2 = z \mapsto \tilde{f}_1(z^2)$. Finally,

$$\lim_{z\to 0} \frac{(f_1(z))^2}{f_2(z)} = \lim_{z\to 0} \frac{(\tilde{f}_1(z))^2}{\tilde{f}_2(z)} = \frac{(\tilde{f}_1(0))^2}{\tilde{f}_1(0^2)} = \frac{1^2}{1} = 1.$$

Therefore, the principal part of the function is 0.

Alternate Solution Item (a)

The Taylor series of $(\sin z)^2$ is by trigonometric identities,

$$\begin{aligned}(\sin z)^2 &= \frac{1-\cos(2z)}{2} \\ &= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} z^{2n} \\ &= z^2 - \frac{2^3 z^4}{4!} + \frac{2^5 z^6}{6!} + O(z^8)\end{aligned}$$

The Taylor series of $\sin(z^2)$ is by substitution

$$\begin{aligned}\sin(z^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{(2n+1)!} \\ &= z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + O(z^{14})\end{aligned}$$

Then,

$$\begin{aligned}\lim_{z\to 0} \frac{(\sin z)^2}{\sin(z^2)} &= \lim_{z\to 0} \frac{z^2 - \frac{2^3}{4!} z^4 + O(z^6)}{z^2 + O(z^6)} \\ &= \lim_{z\to 0} \frac{1 + O(z^2)}{1 + O(z^4)} = 1.\end{aligned}$$

Therefore, the principal part of the series is 0 because the series has a removable singularity at 0.

Solution Item (b)

Note: If found a very confortable way to calculate the principal part using Big-O notation. However, for the sake of this homework, I believe that this method requires further justifications. For now I'll define the notation as follows:

$$f(z) = O(g(z)) \ \text{ as } \ z \to 0$$

when there exists $K > 0$ and $\varepsilon > 0$ such that

$$|f(z)| \leq M|h(z)|, \ \forall z \in B_\varepsilon(z).$$

Then, (using real analysis) for every series with the form $f(z) = \sum_{n=k}^{\infty} a_n z^n$,

$$f(z) = O(z^k),$$

and thus,

$$\lim_{z\to 0} a_0 + f(z) = a_0.$$

The Taylor series expansion of $z(1-\cos(z^2))$ is the following

$$\begin{aligned}
z(1-\cos(z^2)) &= z\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}\right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} z^{2n+1} \\
&= \frac{1}{2!} z^5 - \frac{1}{4!} z^9 + O(z^{13}).
\end{aligned}$$

Therefore,

$$g(z) = \frac{1-z^2}{\frac{1}{2}z^5 + O(z^9)},$$

so it's clear that $g(z)$ has a pole of order 5 at $z=0$. In fact, if

$$g(z) = \sum_{n=-5}^{\infty} a_n z^n,$$

then $a_{-5} = \lim_{z\to 0} z^5 g(z) = \lim_{z\to 0} \frac{1-z^2}{\frac{1}{2}+O(z^4)} = 2$. Then, for the next coefficients

$$\begin{aligned}
a_{-4} &= \lim_{z\to 0} z^4 (g(z) - a_{-5} z^{-5}) \\
&= \lim_{z\to 0} z^4 (g(z) - 2z^{-5}) \\
&= \lim_{z\to 0} z^4 \frac{1 - z^2 - 2z^{-5}(\frac{1}{2!}z^5 + O(z^9))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{-z^2 + O(z^4)}{\frac{1}{2}z + O(z^5)} = 0,
\end{aligned}$$

$$\begin{aligned}
a_{-3} &= \lim_{z\to 0} z^3 (g(z) - a_{-5} z^{-5} - a_{-4} z^{-4}) \\
&= \lim_{z\to 0} z^3 (g(z) - 2z^{-5}) \\
&= \lim_{z\to 0} z^3 \frac{1 - z^2 - 2z^{-5}(\frac{1}{2!}z^5 + O(z^9))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{-z^2 + O(z^4)}{\frac{1}{2}z^2 + O(z^4)} = -2,
\end{aligned}$$

$$
\begin{aligned}
a_{-2} &= \lim_{z\to 0} z^2(g(z) - a_{-5}z^{-5} - a_{-4}z^{-4} - a_{-3}z^{-3}) \\
&= \lim_{z\to 0} z^2(g(z) - 2z^{-5} + 2z^{-3}) \\
&= \lim_{z\to 0} z^2 \frac{1 - z^2 + (-2z^{-5} + 2z^{-3})(\frac{1}{2!}z^5 - \frac{1}{4!}z^9 + O(z^{13}))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{1 - z^2 + (-1 + \frac{2}{4!}z^4 + O(z^8)) + (z^2 - \frac{2}{4!}z^6 + O(z^{10}))}{\frac{1}{2}z^3 + O(z^7)} \\
&= \lim_{z\to 0} \frac{\frac{1}{12}z^4 + O(z^6)}{\frac{1}{2}z^3 + O(z^7)} = 0,
\end{aligned}
$$

$$
\begin{aligned}
a_{-1} &= \lim_{z\to 0} z(g(z) - a_{-5}z^{-5} - a_{-4}z^{-4} - a_{-3}z^{-3} - a_{-2}z^{-2}) \\
&= \lim_{z\to 0} z(g(z) - 2z^{-5} + 2z^{-3}) \\
&= \lim_{z\to 0} z \frac{1 - z^2 + (-2z^{-5} + 2z^{-3})(\frac{1}{2!}z^5 - \frac{1}{4!}z^9 + O(z^{13}))}{\frac{1}{2}z^5 + O(z^9)} \\
&= \lim_{z\to 0} \frac{1 - z^2 + (-1 + \frac{2}{4!}z^4 + O(z^8)) + (z^2 - \frac{2}{4!}z^6 + O(z^{10}))}{\frac{1}{2}z^4 + O(z^8)} \\
&= \lim_{z\to 0} \frac{\frac{1}{12}z^4 + O(z^6)}{\frac{1}{2}z^4 + O(z^8)} = \frac{1}{6}.
\end{aligned}
$$

Finally, the principal part is

$$\frac{2}{z^5} - \frac{2}{z^3} + \frac{1}{6z}.$$

Exercise 2.

Let $M \subset \mathbb{C}$ be a finite set and let $f : \mathbb{C}\backslash M \to \mathbb{C}$ be holomorphic.

(a) Show that $g(z) = z^{-2}f(z^{-1})$ is holomorphic at $B_\varepsilon(0)\backslash\{0\}$ for $\varepsilon > 0$ sufficiently small.

(b) Show that $\mathrm{Res}_0 g = \sum_{c\in\mathbb{C}} \mathrm{Res}_c f$.

(c) Calculate $\displaystyle\int_{\partial B_1(0)} \frac{5z^6 + 4}{2z^7 + 1} dz$.

Solution Item (a)

For some $R > 0$, $M \subset B_R(0)$, so it follows that f is holomorphic at $\mathbb{C}\backslash\overline{B_R(0)}$. Then, the map $z \mapsto f(z^{-1})$ and the map $z \mapsto z^{-2}$ are holomorphic at $B_{1/R}(0)\backslash\{0\}$. So finally, $g : z \mapsto z^{-2}f(z^{-1})$ is holomorphic at $B_{1/R}(0)\backslash\{0\}$.

Solution Item (b)

Let $\gamma(t) = (1/r) \cdot e^{it}$ for $r > R$. Then,

$$\begin{aligned}
\mathrm{Res}_0 g &= \frac{1}{2\pi i}\int_\gamma g(z)dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(1/z)}{z^2}dz \\
(u(z) = 1/z) &= \frac{1}{2\pi i}\int_{u\circ\gamma} \frac{f(u)}{u^2}dz \\
(dz = -du/u^2) &= \frac{1}{2\pi i}\int_{u\circ\gamma} -f(u)du.
\end{aligned}$$

Now, note that $u \circ \gamma(t) = re^{-it}$, so the orientation of the circle is inverted, and

$$\frac{1}{2\pi i}\int_{u\circ\gamma} -f(u)du = \frac{1}{2\pi i}\int_{-u\circ\gamma} f(u)du = \int_{\partial B_r(0)} f(z)dz.$$

Then, since $B_r(0) \supseteq B_R(0)$, and $B_R(0)$ contains all the singularities of f, it follows that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{u\circ\gamma} -f(u)du &= \int_{\partial B_r(0)} f(z)dz \\
&= \sum_{c\in B_r(0)} \mathrm{Res}_c f \\
&= \sum_{c\in B_R(0)} \mathrm{Res}_c f \\
&= \sum_{c\in\mathbb{C}} \mathrm{Res}_c f.
\end{aligned}$$

Solution Item (c)

Let

$$f(z) = \frac{5z^6 + 4}{2z^7 + 1},$$

and let,

$$
\begin{aligned}
g(z) &= z^{-2} f(z^{-1}) \\
&= \frac{1}{z^2} \frac{5z^{-6} + 4}{2z^{-7} + 1} \\
&= \frac{1}{z^2} \frac{4z^7 + 5z}{z^7 + 2} \\
&= \frac{4z^6 + 5}{z^8 + 2z}.
\end{aligned}
$$

Finally, since all the zeroes of $2z^7 + 1$ are in $\partial B_{2^{-1/7}}(0) \subset \text{int}\partial B_1(0)$, it follows that

$$
\frac{1}{2\pi i} \int_{\partial B_1(0)} \frac{5z^6 + 4}{2z^7 + 1} dz = \sum_{c \in \mathbb{C}} \text{Res}_c f = \text{Res}_0 g,
$$

and

$$
\begin{aligned}
\text{Res}_0 g &= \lim_{z \to 0} z g(z) \\
&= \lim_{z \to 0} \frac{4z^6 + 5}{z^7 + 2} = \frac{5}{2}, \\
\implies \int_{\partial B_1(0)} & \frac{5z^6 + 4}{2z^7 + 1} dz = 5\pi i
\end{aligned}
$$

Exercise 3.

Calculate the following integrals with complex analysis methods

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 6x^2 + 13} dx$

(b) $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx$

(c) $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x} dx$

Solution Item (a)

The difference of the degrees between the denominator and numerator is 2, so we can use the following method

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+6x^2+13}dx = 2\pi i \sum_{\mathrm{Im}(z_0)>0} \mathrm{Res}_{z=z_0}\left(\frac{z^2}{z^4+6z^2+13}\right)$$

The function z^4+6z^2+13 has a zero with multiplicity 1 at

$$a = \sqrt[4]{13}\cos\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right) - i\sqrt[4]{13}\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)$$

It also has multiplicity 1 zeroes at $-\overline{a}, \overline{a}, -a$, but the only ones at the upper half plane are a and $-\overline{a}$.

Then,

$$\begin{aligned}
\mathrm{Res}_a f(z) &= \lim_{z\to a}(z-a)\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})} \\
&= \frac{a^2}{2a(2i\mathrm{Im}(a))(2\mathrm{Re}(a))} \\
&= \frac{-ia}{8\mathrm{Im}(a)\mathrm{Re}(a)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}_{-\overline{a}} f(z) &= \lim_{z\to -\overline{a}}(z+\overline{a})\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})} \\
&= \frac{\overline{a}^2}{(-2\mathrm{Re}(a))(-2i\mathrm{Im}(a))(-2\overline{a})} \\
&= \frac{-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)}
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{x^2}{x^4+6x^2+13}dx &= \mathrm{Res}_a f(z) + \mathrm{Res}_{-\overline{a}} f(z) \\
&= \frac{-ia-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)} \\
&= \frac{i(-2\mathrm{Re}(a))}{8\mathrm{Im}(a)\mathrm{Re}(a)} \\
&= \frac{-i}{4\mathrm{Im}(a)} \\
&= \frac{i}{4\sqrt[4]{13}\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)}
\end{aligned}$$

So it follows that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+6x^2+13}dx = \frac{-2\pi}{4\sqrt[4]{13}\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)} \approx 0.8643$$

Input

$$\frac{-2\pi}{4\sqrt[4]{13}\,\sin\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)}$$

$\tan^{-1}(x)$ is the inverse tangent function

Exact Result

$$-\frac{\pi\csc\left(\frac{1}{2}\left(\tan^{-1}\left(\frac{2}{3}\right)-\pi\right)\right)}{2\sqrt[4]{13}}$$

(result in radians)

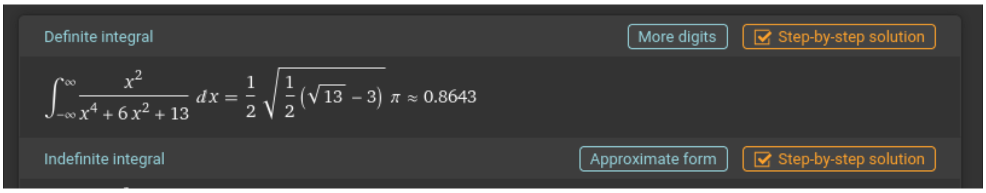
Decimal approximation

More digits

0.86433149987466204155101333297991563681523946518655085610312293 58...

(result in radians)

and this coincides with the real result



Definite integral

More digits | Step-by-step solution

$$\int_{-\infty}^{\infty}\frac{x^2}{x^4+6x^2+13}\,dx = \frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{13}-3\right)}\,\pi \approx 0.8643$$

Indefinite integral

Approximate form | Step-by-step solution

Solution Item (b)

This integral has the form $\int_0^\infty x^\alpha R(x)$ where $0 < \alpha = 1/2 < 1$ and $R(x) = O(x^{-2})$ without any poles at the origin. Therefore, by using the substitution $x = t^2$, $dx = 2tdt$, we obtain

$$\begin{aligned}\int_0^\infty x^\alpha R(x)dx &= 2\int_0^\infty t^{2\alpha+1}R(t^2)dt\\ &= \int_{-\infty}^\infty t^{2\alpha+1}R(t^2)dt\\ &= \int_{-\infty}^\infty \frac{t^2}{t^4+1}dt\end{aligned}$$

It follows that since the difference between the degrees of the denominator and numerator is two,

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \sum_{\mathrm{Im}(z_0)>0} \mathrm{Res}_{z=z_0}\left(\frac{z^2}{z^4+1}\right)$$

The polynomial z^4+1 has a root of multiplicity 1 at

$$a = \frac{1+i}{\sqrt{2}}$$

and also has roots at $-a, \overline{a}, -\overline{a}$, from which only a and $-\overline{a}$ are in the upper half plane. Using the same logic as the previous item (because it's the exact same case only changing the value of a),

$$\begin{aligned}
\mathrm{Res}_a f(z) &= \lim_{z\to a}(z-a)\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})}\\
&= \frac{-ia}{8\mathrm{Im}(a)\mathrm{Re}(a)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}_{-\overline{a}} f(z) &= \lim_{z\to -\overline{a}}(z+\overline{a})\frac{z^2}{(z-a)(z+a)(z-\overline{a})(z+\overline{a})}\\
&= \frac{-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)}
\end{aligned}$$

So finally,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{t^2}{t^4+1}dt &= \mathrm{Res}_a f(z) + \mathrm{Res}_{-\overline{a}} f(z)\\
&= \frac{-ia-i\overline{a}}{8\mathrm{Im}(a)\mathrm{Re}(a)}\\
&= \frac{i(-2\mathrm{Re}(a))}{8\mathrm{Im}(a)\mathrm{Re}(a)}\\
&= \frac{-i}{4\mathrm{Im}(a)}\\
&= \frac{-i}{4\sqrt{2}},
\end{aligned}$$

and thus,

$$\int_0^{\infty}\frac{\sqrt{x}}{x^2+1}dx = 2\int_{-\infty}^{\infty}\frac{t^2}{t^4+1}dt = \frac{\pi}{\sqrt{2}},$$

which coincides with the real result

Solution Item (c)

We have that

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}dx &= \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}dx\\ &= \int_0^\infty \frac{e^{ix}}{2ix}dx - \int_0^\infty \frac{e^{-ix}}{2ix}dx\\ &= \int_0^\infty \frac{e^{ix}}{2ix}dx + \int_{-\infty}^0 \frac{e^{ix}}{2ix}dx\\ &= \int_{-\infty}^\infty \frac{e^{ix}}{2ix}dx\end{aligned}$$

We have a simple pole at $x = 0$ and $R(\infty) = 0$, so we can apply the following formula

$$\begin{aligned}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx &= \int_{-\infty}^\infty R(x)e^{ix}dx\\ &= 2\pi i \sum_{\mathrm{Im}(z_0)>0} \mathrm{Res}_{z=z_0} R(z)e^{iz} + \pi i \sum_{\mathrm{Im}(z_0)=0} \mathrm{Res}_{z=z_0} R(z)e^{iz}\\ &= \pi i \mathrm{Res}_{z=0}\frac{e^{iz}}{z} = \pi i.\end{aligned}$$

Finally,

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx = \frac{\pi}{2}$$

Exercise 4.

(a) Let γ be a closed curve in $\mathbb{C}\backslash\{0\}$. Let $n \in \mathbb{N}$ and $p : \mathbb{C} \to \mathbb{C}$, $p(z) = z^n$. Show that $\mathrm{ind}_{p\circ\gamma}(0) = n\, \mathrm{ind}_\gamma(0)$.

(b) Let $U \subset \mathbb{C}$ be open and connected, $c \in U$ and γ be a closed curve in $U\backslash\{c\}$ such that $\mathrm{int}(\gamma) \subset D$. Para a biholomorphic function $f : U \to f(U)$ show that

$$\mathrm{ind}_\gamma(c) = \mathrm{ind}_{f\circ\gamma}(f(c))$$

Solution Part (a)

Without restriction, let $\gamma : [0, 2\pi] \to \mathbb{C}\backslash\{0\}$. Then, by the argument principle,

$$\begin{aligned}
\operatorname{ind}_{p\circ\gamma}(0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(p\circ\gamma)'(t)}{p\circ\gamma(t)}dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)(p'\circ\gamma)(t)}{p\circ\gamma(t)}dt \\
&= \frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}dz \\
&= \sum_{i=1}^{n} \operatorname{ind}_\gamma(a_i); \qquad p(a_i) = 0.
\end{aligned}$$

Since 0 is a zero of multiplicity n for p, it follows that $a_1 = \cdots = a_n = 0$, and thus,

$$\operatorname{ind}_{p\circ\gamma}(0) = n \cdot \operatorname{ind}_\gamma(0).$$

Solution Part (b)

For every $c \in U$, c is the only element in $f^{-1}(\{f(c)\})$. Therefore, for the function $g(z) = f(z) - f(c)$, c is the unique solution for the equation $g(z) = 0$. So by following similar steps as the previous item,

$$\begin{aligned}
\operatorname{ind}_{f\circ\gamma}(f(c)) &= \operatorname{ind}_{g\circ\gamma}(0) \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{(g\circ\gamma)'(t)}{g\circ\gamma(t)}dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)(g'\circ\gamma)(t)}{g\circ\gamma(t)}dt \\
&= \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}dz \\
&= \sum_{i=1}^{n} \operatorname{ind}_\gamma(a_i); \qquad g(a_i) = 0 \\
&= \operatorname{ind}_\gamma(c).
\end{aligned}$$

Exercise 5.

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Suppose that for every $a \in \mathbb{C}$, at least one coefficient in the Taylor series of f in a is vanished. Show that f is a polynomial

Solution:

We previously proved that the set of zeroes of a non-zero holomorphic function is discrete and closed, so it has to be countable (because uncountable sets have accumulation points in $\mathbb{R}^n$).

Now assume for the sake of contradiction that f is not a polynomial, so the k-th derivative $f^{(k)}$ is always non-zero, and thus, has a countable set of zeroes. So it follows that $(f^{(k)})^{-1}(\{0\})$ is a countable set, and thus,

$$\bigcup_{k=0}^{\infty}(f^{(k)})^{-1}(\{0\}) \text{ is countable too.}$$

Also note that since for every $a \in \mathbb{C}$ there exists $n \in \mathbb{N}$ such that $a_n = 0$ for the series

$$f(z) = \sum_{k=0} a_k(z-a)^k,$$

so it follows that for such n, $f^{(n)}(a) = 0$, and thus, for every $a \in \mathbb{C}$

$$a \in (f^{(n)})^{-1}(\{0\}) \subset \bigcup_{k=0}^{\infty}(f^{(k)})^{-1}(\{0\})$$

$$\implies \mathbb{C} \subset \bigcup_{k=0}^{\infty}(f^{(k)})^{-1}(\{0\}),$$

but that would imply that $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ is countable, which is false.