

# Complex Analysis: Homework 4

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## Exercise 1.

Let  $f$  be an entire function and let  $\zeta = e^{2\pi i/n}$  for some  $n \in \mathbb{N}$ . Suppose that  $f(\zeta z) = f(z)$  for every  $z \in \mathbb{C}$ . Show that there exists an entire function  $g$  such that  $f(z) = g(z^n)$  for every  $z \in \mathbb{C}$ .

### Solution:

Since  $f$  is entire, there's exists  $\{c_k\}$  such that

$$f(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Then, since  $f(\zeta z) = f(z)$ , we have that

$$\sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} c_k \zeta^k z^k.$$

When  $k$  is not a multiple of  $n$ , we have that  $\zeta^k \neq 1$ . Therefore, by uniqueness of the power series expansion,

$$c_k = \zeta^k c_k \implies c_k = 0, \quad k \not\equiv 0 \pmod{n}.$$

Finally, we can rewrite the series as follows

$$f(z) = \sum_{k=0}^{\infty} c_{nk} z^{nk} = \sum_{k=0}^{\infty} c_{nk} (z^n)^k.$$

For  $g(z) = \sum_{k=0}^{\infty} c_{nk} z^k$ ,  $f(z) = g(z^n)$ .

## Exercise 2.

- (a) Let  $U \subset \mathbb{C}$  be a region and  $K \subset U$  a compact subset with non empty interior  $K^\circ$ . Let  $f : U \rightarrow \mathbb{C}$  be an holomorphic function with  $|f|$  constant of the boundary of  $K$ . Show that  $f$  is constant or has a zero in  $K^\circ$ .

- (b) Let  $U \subset \mathbb{C}$ ,  $z_0 \in U$ ,  $\varepsilon > 0$  such that the closed ball  $\overline{B_\varepsilon(z_0)}$  is a subset of  $U$ . Let  $f : U \rightarrow \mathbb{C}$  be holomorphic with  $|f(z_0)| < \min\{|f(z)| : |z - z_0| = \varepsilon\}$ . Show that  $f$  has a zero in  $B_\varepsilon(z_0)$ .

### Solution Part (a)

In the first place, if  $|f| = 0$  on  $\partial K$ , then for  $g : z \mapsto 0$ ,  $f|_{\partial K} = g|_{\partial K}$ . Since  $\partial K$  has accumulation points (otherwise  $U$  is not open), using identity's theorem we conclude that  $f = g$  on  $U$ .

Now, assume that  $f$  doesn't have any zero on  $K^\circ$  and that  $|f| \neq 0$  on the boundary of  $K$ . Then,  $g = 1/f$  is an holomorphic function defined in  $K$ , and since  $K$  is compact,  $|f|$  has a maximum in  $K$ . We have two possible cases

- The maximum of  $|f|$  is attained at  $K^\circ$ , so by maximum principle  $f$  is constant.
- The maximum of  $|f|$  is attained at  $\partial K$ , so the minimum of  $1/|f|$  is attained at  $\partial K$ , and thus,  $1/|f|$  attains its maximum at  $K^\circ$ . Again, by maximum principle  $1/f$  is constant and so  $f$  too.

### Solution Part (b)

Assume that  $f$  doesn't have a zero in  $B_\varepsilon(z_0)$ , we want to prove that  $|f(z_0)| \geq \min_{\partial B_\varepsilon(z_0)} |f(z)|$ .

- If there's a zero in  $\partial B_\varepsilon(z_0)$ , then we won, so suppose that  $\min_{\partial B_\varepsilon(z_0)} |f(z)| > 0$ .
- If  $f$  is constant, then we also won. So assume it's not.

Now,  $1/f$  is an holomorphic function defined at  $\overline{B_\varepsilon(z_0)}$ . Since  $1/f$  is not constant, by maximum principle, it attains its maximum modulus at  $\partial B_\varepsilon(z_0)$ . Therefore,

$$\begin{aligned} 1/|f(z_0)| &\leq \max_{\partial B_\varepsilon} |1/f(z)| \\ \implies |f(z_0)| &\geq \min_{\partial B_\varepsilon} |f(z)| \end{aligned}$$

as we intended.

### Exercise 3.

Let  $U \subset \mathbb{C}$  be open and connected,  $f : U \rightarrow X$  be a non constant holomorphic function and  $N := \{z \in \mathbb{C} : f(z) = 0\}$ . Show that  $N$  is closed and discrete in  $U$ .

**Solution:**

To see that  $w \in \mathbb{C} \setminus N$  is open, take  $w \in \mathbb{C}$  such that  $f(w) = \omega \neq 0$ . Then, there exists for some  $\varepsilon > 0$ , an open ball  $B_\varepsilon(\omega) \subset X$  that doesn't contain 0. Finally, using the continuity of  $f$ ,  $f^{-1}(B_\varepsilon(\omega))$  is an open set that contains  $w$  which is inside  $\mathbb{C} \setminus N$ .

Now, assume that  $N$  has some limit point  $z_0$ . Also let  $g : z \mapsto 0$ , and note that  $f = g$  on  $N$ . Using identity's theorem we conclude that  $f(z) = g(z) = 0$  for every  $z \in U$ , contradicting the fact that  $f$  is not constant.

## Exercise 4.

Let  $U \subset \mathbb{C}$  be open and bounded, without isolated points of the frontier, and let  $M \subset U$  be a subset without accumulation points in  $U$ . Show that every biholomorphic function  $f : U \setminus M \rightarrow U \setminus M$  has a biholomorphic extension  $g : U \rightarrow U$ .

**Solution:** Let  $g = f^{-1}$ .

Since  $M$  is discrete in  $U$ , for every  $w \in M$ , there exists  $\varepsilon_w$  such that  $M \cap B_{\varepsilon_w}(w) = \{w\}$  and  $B_{\varepsilon_w}(w) \subset U$ . Since  $f$  and  $g$  images are bounded ( $U \setminus M$  is bounded), then  $f, g$  are bounded on the set  $B_{\varepsilon_w}^\bullet(w) = B_{\varepsilon_w}(w) \setminus \{w\}$ . Therefore, using Riemann's removable singularity criterion,  $w$  is a removable singularity for every  $w \in M$  for both  $f$  and  $f^{-1}$ .

Let  $\tilde{f}$  and  $\tilde{g}$  be the extensions for  $f$  and  $g$  respectively. Let  $h_1 = \tilde{f} \circ \tilde{g}$  and  $h_2 = \tilde{g} \circ \tilde{f}$ . For  $z \in U \setminus M$ ,

$$h_1(z) = f \circ g(z) = z = g \circ f(z) = h_2(z).$$

Now, to prove that  $h_1, h_2$  are defined for every  $z \in U$ , we must prove that  $\tilde{f}(w), \tilde{g}(w) \in U$  for  $w \in M$ . For the sake of contradiction assume it's not. We know that

$$\tilde{f}(B_{\varepsilon_w}^\bullet(w)) = f(B_{\varepsilon_w}^\bullet(w)) \subset U \setminus M \subset U.$$

Also,  $\overline{B_{\varepsilon_w}^\bullet(w)} = \overline{B_{\varepsilon_w}(w)}$ , so by continuity of  $f$

$$\implies \tilde{f}(B_{\varepsilon_w}(w)) \subset \tilde{f}(\overline{B_{\varepsilon_w}(w)}) = \tilde{f}(\overline{B_{\varepsilon_w}^\bullet(w)}) \subset \overline{f(B_{\varepsilon_w}^\bullet(w))} \subset \overline{U}$$

Therefore,  $\tilde{f}(w) \in \overline{U}$  and since we assumed (for the contradiction) that  $\tilde{f}(w) \notin U$ , it follows that  $\tilde{f}(w) \in \partial U$ . However, note that

1. First,  $\tilde{f}(B_{\varepsilon_w}(w)) = f(B_{\varepsilon_w}^\bullet(w)) \dot{\cup} \{\tilde{f}(w)\}$  (disjoint union). Therefore, since  $U$  is open and  $f(B_{\varepsilon_w}^\bullet(w)) \subset U = U^\circ$ , it follows that  $f(B_{\varepsilon_w}^\bullet(w)) \cap \partial U = \emptyset$ . Thus,  $\partial U \cap \tilde{f}(B_{\varepsilon_w}(w)) = \{\tilde{f}(w)\}$
2. By Open Mapping Theorem,  $\tilde{f}(B_{\varepsilon_w}(w))$  is an open set. Therefore,  $\{\tilde{f}(w)\}$  is an isolated point of the boundary of  $U$  (contradiction).

So  $\tilde{f}(w) \in U$  for every  $w \in U$ . The same argument applies for  $\tilde{g}$ , so  $h_1$  and  $h_2$  are defined for every  $z \in U$ . By identity theorem, it follows that  $h_1(z) = z = h_2(z)$  for every  $z \in U$  implying that  $\tilde{f}$  and  $\tilde{g}$  are each other's inverse functions.

## Exercise 5.

Let  $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$  be a power series with convergence radius  $R \in (0, \infty)$ . Show that  $f$  has at least a singular point in the frontier of the convergence disk.

### Solution:

For the sake of contradiction assume that  $f$  has no singularities at  $\partial B_R(z_0)$  and can be extended to an holomorphic function  $\tilde{f}$  at  $\overline{B_R(z_0)}$ . Then, for every  $w \in \partial B_R(z_0)$ , there exists  $\varepsilon_w$  such that there exists a power series  $g_w(z) = \sum_{j=0}^{\infty} a_{w,j}(z - w)^j$  around  $B_{\varepsilon_w}(w)$  that coincides with  $\tilde{f}$  in  $\overline{B_R(z_0)} \cap B_{\varepsilon_w}(w)$ .

Since  $\partial B_R(z_0)$  is compact, there exists

$$\varepsilon = \frac{\min\{\varepsilon_w : w \in \partial B_R(z_0)\}}{2} > 0.$$

The set  $\overline{B_{R+\varepsilon}(z_0)} \setminus B_R(z_0)$  is compact and it is covered by the open balls  $\{B_{\varepsilon_w}(w) : w \in \partial B_R(z_0)\}$ . Therefore, (by the definition of compact set) there exists  $w_1, \dots, w_n$  for  $n \in \mathbb{N}$  such that  $\overline{B_{R+\varepsilon}(z_0)} \setminus B_R(z_0)$  is covered by  $\{B_{\varepsilon_{w_k}}(w_k) : k \leq n\}$

Using Identity's theorem  $\tilde{f}$  can again be extended to a function  $\tilde{f}_1$  that coincides with  $g_{w_1}$  in  $\overline{B_R(z_0)} \cup B_{\varepsilon_{w_1}}(w_1)$ . Then, we recursively apply Identity's theorem to extend  $\tilde{f}_k$  to a function  $\tilde{f}_{k+1}$  that coincides with  $g_{w_{k+1}}$  in  $\overline{B_R(z_0)} \cup \bigcup_{j=1}^{k+1} B_{\varepsilon_{w_j}}(w_j)$  for every  $k < n$ .

Finally,  $\tilde{f}_n$  extends  $f$  to the ball  $B_{R+\varepsilon}(z_0) \subset \overline{B_R(z_0)} \cup \bigcup_{j=1}^n B_{\varepsilon_{w_j}}(w_j)$ . However, this is a contradiction to the fact that  $R$  is the greatest number for which  $f(z)$  converges for every  $|z - z_0| < R$ .