Complex Analysis: Homework 13

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Exercise 1.

Determine whether the following products converge:

$$\text{(a)} \prod_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n}\right), \quad \text{(b)} \prod_{n=1}^{\infty}\left(1+\frac{(-1)^n}{\sqrt{n}}\right).$$

Note: The first term in both products is 0, so we are going to skip and show that the tails from $n=2$ forward converge (or diverge) for both cases.

Solution Item (a)

We are going to prove that $\sum_{n=2}^{\infty}\log\left(1+\frac{(-1)^n}{n}\right)$ converges. In fact, if M is even,

$$\begin{aligned}\sum_{n=2}^{M}\log\left(1+\frac{(-1)^n}{n}\right) &= \sum_{k=1}^{M/2}\log\left(\frac{2k+1}{2k}\right)+\log\left(\frac{2k}{2k+1}\right)\\ &= \sum_{k=1}^{M/2}\log\left(\frac{2k+1}{2k}\cdot\frac{2k}{2k+1}\right)\\ &= 0,\end{aligned}$$

and if $M=2K+1$ is odd,

$$\begin{aligned}\sum_{n=2}^{M}\log\left(1+\frac{(-1)^n}{n}\right) &= \log\left(\frac{2K}{2K+1}\right)+\sum_{k=1}^{(M-1)/2}\log\left(\frac{2k+1}{2k}\right)+\log\left(\frac{2k}{2k+1}\right)\\ &= \log\left(\frac{2K}{2K+1}\right).\end{aligned}$$

Since log is continuous at 1 and $\frac{2K}{2K+1} \to 1$ when $K \to \infty$, we conclude that $\log\left(\frac{2K}{2K+1}\right)$ converges to 0. Since the entire log-series converges to 0, it must follow that

$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right) = \exp\left[\sum_{n=2}^{\infty}\log\left(1+\frac{(-1)^n}{n}\right)\right] = 1.$$

Solution Item (b)

Now, we are going to prove that $\sum_{n=2}^{\infty}\log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ diverges. In fact, we are going to show that the subsequence $\sum_{n=1}^{M}\log\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ with $M = 2k$ diverges to $-\infty$. Note that

$$\begin{aligned}
\log\left(1+\frac{1}{\sqrt{2n}}\right)+\log\left(1-\frac{1}{\sqrt{2n+1}}\right) &= \log\left[\left(1+\frac{1}{\sqrt{2n}}\right)\cdot\left(1-\frac{1}{\sqrt{2n+1}}\right)\right] \\
&= \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right]
\end{aligned}$$

By the limit comparison test $\log(1+x) = x - O(x^2) \approx x$ for x near to 0. Then, we can compare the series of $a_n = \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right]$ with the series of $b_n = \frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}$.

Remark: Note that for $n \geq 1$, $\sqrt{2n+1}-\sqrt{2n}-1 < 0$. The limit comparison test only applies if $a_n, b_n > 0$, but for our case,

$$a_n = \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right] < 0 \text{ and } b_n = \frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}} < 0$$

and thus, the same argument can be applied to $-a_n, -b_n > 0$ to conclude that $-\sum_n a_n$ diverges because $-\sum_n b_n$ does.

Since $a_n = \log(1+b_n)$ and $b_n \to 0$ when $n \to \infty$, by uniqueness of limit and then L-hôspital rule,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{1}{1+x} = 1.$$

Then, we can also compare b_n with $c_n = \frac{-1}{2n}$ because

$$\begin{aligned}
\frac{b_n}{c_n} &= \frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\cdot(-2n) \\
&= \underbrace{\sqrt{\frac{2n}{2n+1}}}_{\to 1} + \underbrace{\frac{2n}{\sqrt{2n+1}} - \sqrt{2n}}_{\to 0} \\
&\to 1, \quad n\to\infty
\end{aligned}$$

Finally, since $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{-1}{2n}$ diverges to $-\infty$, it follows that $\sum_{n=1}^{\infty} b_n$ diverges, and thus, $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$ too. Therefore,

$$\begin{aligned}\sum_{n=2}^{2k} \log\left(1+\frac{(-1)^n}{\sqrt{n}}\right) &= \sum_{n=1}^{k} \log\left[1+\frac{\sqrt{2n+1}-\sqrt{2n}-1}{\sqrt{2n}\cdot\sqrt{2n+1}}\right] \\ &= \sum_{n=1}^{k} a_n \to -\infty, \quad M \to \infty.\end{aligned}$$

That implies that $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$ has a subsequence that diverges to 0.

Exercise 2.

Prove the class theorem: Let (X, d) be a compact metric space and let $g_n : X \to \mathbb{C}$ be continuous functions such that $\sum_{n=1}^{\infty} |g_n|$ converges uniformly. Define $f_n : X \to \mathbb{C}$ by

$$f_n(x) = \prod_{j=1}^{n}(1+g_j(x)).$$

We already know that for every $x \in X$, the product $\prod_{n=1}^{\infty}(1+g_j(x))$ is absolutely convergent. Then

$$f : X \to \mathbb{C}, \quad f(x) := \lim_{n\to\infty} f_n(x)$$

is well-defined.

Show that (a) $f_n \to f$ uniformly and (b) that there exists $N \in \mathbb{N}$ such that for all $x \in X$,

$$f(x) = 0 \iff g_n(x) = -1 \text{ for some } n \leq N.$$

Solution Item (a)

Claim 1: for any real number x, $x + 1 \leq e^x$.

Proof: The function $F(x) = e^x - x - 1$ has derivative $F'(x) = e^x - 1$ which has a critical point at $x = 0$. The function is convex because $F''(x) = e^x > 0$ and thus, $x = 0$ is a global minimum of F. Since $F(0) = 0$, it follows that for any $x \in \mathbb{R}$, $F(x) \geq F(0) = 0$, and thus, $e^x - x - 1 \geq 0$.

. $\square$

Claim 2: For an absolutely convergent sequence $(a_n)_{n\in\mathbb{N}}$, that is $\sum_{n=1}^{\infty} |a_n| < \infty$,

$$\left|\prod_{k=1}^{n}(1+a_k) - 1\right| \leq \prod_{k=1}^{n}(1+|a_k|) - 1.$$

Proof: By the triangle inequality, any polynomial $P(a_1, \ldots, a_n)$ satisfies

$$|P(a_1, \ldots, a_n)| \leq P(|a_1|, \ldots, |a_n|).$$

Therefore, by taking the polynomial $P_n(a_1, \ldots, a_n) = \prod_{k=1}^{n}(1 + a_k) - 1$, we obtain the desired result.

. $\square$

Claim 3: For an absolutely convergent sequence $(a_n)_{n\in\mathbb{N}}$,

$$\left|\prod_{k=1}^{\infty}(1 + a_k) - 1\right| \leq \exp\left(\sum_{k=1}^{\infty}|a_k|\right) - 1.$$

Proof: we know that both $\lim_n \prod_{k=1}^{n}(1 + a_k)$ and $\lim_n \sum_{k=1}^{n}|a_k|$ exist from the hypothesis that a_n is absolutely convergent. Use **Claim 2** and **Claim 1** to conclude that

$$\left|\prod_{k=1}^{n}(1 + a_k) - 1\right| \leq \prod_{k=1}^{n}(1 + |a_k|) - 1 \leq \exp\left(\sum_{k=1}^{n}|a_k|\right) - 1.$$

Therefore, after taking limits on both sides we obtain the desired result. In fact, this exact same argument also works on the tails:

$$\left|\prod_{k=N+1}^{\infty}(1 + a_k) - 1\right| \leq \exp\left(\sum_{k=N+1}^{\infty}|a_k|\right) - 1, \quad \forall N \in \mathbb{N}.$$

. $\square$

Now, note that we can factorize the first n product terms of f_n from f

$$f(z) - f_n(z) = \left(\frac{f(z)}{f_n(z)} - 1\right) \cdot f_n(z) = \underbrace{\left(\prod_{j=n+1}^{\infty}(1 + g_j(z)) - 1\right)}_{(1)} \cdot \underbrace{\left(\prod_{j=1}^{n} 1 + g_j(z)\right)}_{(2)}.$$

For (1) apply **Claim 3** to conclude that

$$\left|\prod_{j=n+1}^{\infty}(1 + g_j(z)) - 1\right| \leq \exp\left(\sum_{k=n+1}^{\infty}|g_j(z)|\right) - 1.$$

Since $\sum_{k=n+1}^{\infty}|g_j(z)|$ converges uniformly to 0 (tail of a uniformly convergent sequence), for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty}|g_j(z)| < \varepsilon$ for every $z \in X,\ n \geq N$, and thus,

$$\exp\left(\sum_{k=n+1}^{\infty}|g_j(z)|\right) - 1 < \underbrace{e^{\varepsilon} - 1}_{\approx 0} \quad \forall z \in X,\ n \geq N.$$

so it follows that $\prod_{j=n+1}^{\infty}(1 + g_j(z)) - 1$ converges uniformly to 0.

On the other hand, for (2), since $h := \sum_{j=1}^{\infty} |g_j|$ is the uniform limit of continuous functions on a compact set, it follows that there exists $M > 0$ such that $h(z) < M$ for every $z \in X$. In fact, since $|g_j| \geq 0$, it follows that the sequence $h_n := \sum_{j=1}^{n} |g_j|$ is increasing and $h_n \leq h < M$. Then, by **Claim 1**

$$\begin{aligned}\left|\prod_{j=1}^{n} 1 + g_j(z)\right| &= \prod_{j=1}^{n} |1 + g_j(z)| \\ &\leq \prod_{j=1}^{n} 1 + |g_j(z)| \\ &\leq \exp\left(\sum_{j=1}^{n} |g_j(z)|\right) \\ &< e^M \quad \forall z \in X.\end{aligned}$$

Finally, (1) converges uniformly to 0 and (2) is uniformly bounded, so it follows that $|f_n(z) - f(z)|$ converges uniformly to 0.

Solution Item (b)

Since X is compact and f is holomorphic, the number of zeros of f is finite, otherwise, the set of zeros would have an accumulation point on X. If $\{z_1, \ldots, z_p\}$ is the set of zeros of f, the goal is to find $\forall j \leq p$, N_j for which $g_{N_j}(z_j) = -1$ and then take $N = \max_j N_j$.

Let z be one of those zeros and assume for the sake of contradiction that $g_j(z) \neq -1$ for every $j \in \mathbb{N}$. Then, we would obtain a contradiction with the fact that for a convergent product $\prod_{j=1}^{\infty} a_j$ to be zero, one of the elements in the sequence is zero (otherwise, the product must diverge to 0), so let $a_j = (1 + g_j(z))$ to obtain the contradiction. Therefore, for every j, there exists $N_j \in N$ for which $g_{N_j}(z_j) = -1$.

If $g_n(z) = -1$ for some $n \leq N$, then it's clear from the pointwise convergence that z is a zero of f:

$$f(z) = (1 - g_n(z)) \times \prod_{j \neq n} (1 - g_j(z)).$$

Exercise 3.

Let $U \subset \mathbb{C}$ be open and let $g_n : U \to \mathbb{C}$ be holomorphic functions such that $\sum_{n=1}^{\infty} |g_n|$ converges compactly in U. Define

$$f_n(x) = \prod_{j=1}^{n} (1 + g_j(x)).$$

(a) Show that $(f_n)_{n\in\mathbb{N}}$ converges compactly to a holomorphic function $f : U \to \mathbb{C}$.

(b) Let $z_0 \in U$. Show that $f(z_0) = 0$ if and only if there exists $j \in \mathbb{N}$ such that $g_j(z_0) = -1$, that there are finitely many such j, and that the order of the zero z_0 for f is equal to the sum of the multiplicities of z_0 as a zero of all the functions $1 + g_j$.

Solution Part (a)

Let $K \subseteq U$ be a compact set. Then, $\sum_{n=1}^{\infty} |g_n|$ converges uniformly in K, so apply exercise 2 to conclude that f_n converges uniformly to a function f_K in K. By Weierstrass theorem, f_K must be holomorphic.

Now for any $z \in U$ let $K_z \subset U$ be a compact set that contains z. Define $f(z) = f_{K_z}(z)$. f is well defined because if we take another compact set K'_z that contains z, then by uniqueness of limit, $f_{K_z}(w) = f_{K_z \cap K'_z}(w) = f_{K'_z}(w)$ for every $w \in K_z \cap K'_z$.

Fix $z \in U$. We want to show that there exists a neighborhood of z for which f is holomorphic. Since the choosing of the compact set K_z doesn't affect the value of $f(z)$, let $\varepsilon > 0$ such that $\overline{B_\varepsilon(z)} \subset U$ and let $K_w = \overline{B_\varepsilon(z)}$ for every $w \in B_\varepsilon(z)$. With this choosing of $K_w = K_z$ we get that $f(w) = f_{K_z}(w)$, and since f_{K_z} is holomorphic, it follows that f is holomorphic at $B_\varepsilon(z)$. Thus, f is holomorphic.

Finally, f_n converges compactly to f because for any compact set K, f_n converges uniformly to f_K, and again, by uniqueness of limit, $f(z) = f_K(z)$ for every $z \in K$.

Solution Part (b)

If $f(z_0) = 0$ for $z_0 \in U$, then there exists $j \in \mathbb{N}$ for which $1 + g_j(z) = 0$, otherwise we would get the same contradiction we formulated at exercise 2(b).

If for some reason there exists an infinite number of $j \in \mathbb{N}$ for which $1 + g_j(z_0) = 0$, then for every $n \in \mathbb{N}$, the tail $\prod_{j=n+1}^{\infty}(1 + g_j(z_0)) = 0 \not\to 1$ (doesn't converge to 1), so the product doesn't converge to f according to the definition.

Now, let $J = \{j_1, \dots, j_q\}$ be the set of indices for which $g_{j_i}(z_0) = -1$. Then, for each of this indices, we can factorize the zeros of order say m_i to obtain $1 + g_{j_i}(z) = (z - z_0)^{m_i} \cdot h_i(z)$ for some holomorphic function that doesn't vanish at z_0. Then,

$$\begin{aligned} f(z) &= \prod_{j\in J}(1 + g_j(z)) \times \prod_{j\notin J}(1 + g_j(z)) \\ &= (z - z_0)^{m_1+\cdots+m_q} \cdot \prod_{i=1}^{q} h_i(z) \times \prod_{j\in\mathbb{N}\setminus J}(1 + g_j(z)). \end{aligned}$$

From the definition of h_i, we know that $\prod_{i=1}^{q} h_i(z_0) \neq 0$ and we defined J in such way that there are no indices outside of J for which $1 + g_j(z_0) = 0$. Therefore, f has a zero of order $m_1 + \cdots + m_q$ at z_0.

Exercise 4.

Let $U \subset \mathbb{C}$ be a region, let $f_n : U \to \mathbb{C}$ be holomorphic functions, and assume that $\prod_{j=1}^{\infty} f_j$ converges absolutely and compactly in U. Show that

$$\frac{f'}{f} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j}$$

where the sum on the right side converges compactly in its domain.

Solution

f is holomorphic because is the compact limit of holomorphic functions (Weierstrass theorem).

Let $g_n = \prod_{j=1}^{n} f_j$, since $(g_n)_{n\in\mathbb{N}}$ is a sequence of holomorphic functions that converges compactly to f, by Weierstrass' theorem, the sequence of derivatives $(g_n')_{n\in\mathbb{N}}$ also converge compactly to f'. Now, by the product rule

$$\begin{aligned} \frac{g_n'}{g_n} &= \frac{\sum_{k=1}^{n} f_k' \times \prod_{j\neq k}^{n} f_j}{\prod_{j=1}^{n} f_j} \\ &= \sum_{k=1}^{n} f_k' \times \frac{\prod_{j\neq k}^{n} f_j}{\prod_{j=1}^{n} f_j} \\ &= \sum_{k=1}^{n} \frac{f_k'}{f_k}. \end{aligned}$$

Then domain of $\frac{f'}{f}$ is U_0 which is equal to U excluding the zeros of f (which are isolated). For every $n \in \mathbb{N}$, g_n cannot have more zeros than f, otherwise there would be a contradiction with

$$\underbrace{f(z)}_{\neq 0} = \underbrace{g_n(z)}_{=0} \times \underbrace{\prod_{j=n+1}^{\infty} f_j(z)}_{<\infty}.$$

Finally, $\frac{g_n'}{g_n}$ is defined on U_0, and in every compact set of U_0, $\frac{1}{g_n}$ converges compactly to $\frac{1}{f}$ because:

For any $K \subset U_0$, there exists a constant $M > 0$ for which $|g_n(z)|, |f(z)| < M$ for every $z \in K$, and thus,

$$\sup_{z \in K} \left| \frac{1}{g_n(z)} - \frac{1}{f(z)} \right| = \sup_{z \in K} \left| \frac{g_n(z) - f(z)}{g_n(z) f(z)} \right| < \frac{1}{M^2} \cdot \sup_{z \in K} |g_n(z) - f(z)| \to 0.$$

Therefore, $\frac{g_n'}{g_n}$ converges compactly to $\frac{f'}{f}$.