Complex Analysis: Homework 1

Martín Prado

August 14, 2024

Universidad de los Andes – Bogotá Colombia

Exercise 1.

Let $U \subseteq \mathbb{C}$ be an open set. Prove that U is connected if and only if it is path connected.

Solution:

$\Longleftarrow$: The following claims are basic results from topology. Assume that U is path connected, and for the sake of contradiction assume that U is not connected. Thus, there exists a separation A, B of the set U.

Now, let $z \in A, w \in B$ and $f : [0, 1] \to U$ be a continuous path such that $f(0) = z$ and $f(1) = w$.

Claim 1: The interval $[0, 1]$ is connected.

Claim 2: If I is connected and f is a continuous function, then $f(I)$ is connected.

With the first 2 claims we're saying that $f([0, 1])$ is a connected set.

Claim 3: If the sets A, B form a separation of U and if Y is a connected set, then Y lies entirely within either A or B.

With this last claim we'll reach a contradiction, because if $Y = f([0, 1])$, then either $z, w \in A$ or $z, w \in B$. This cannot be possible since A, B is a separation.

$\Longrightarrow$: Now assume that U is connected. The goal here is to prove that every path-connected component is both an open and a closed set, and thus, if there exists more than 1 path-connected component, there would exist a separation for U.

For this purpose fix $x \in U$, and define the relationship $y_1 \sim y_2$ for when there exists a continuous path that connects y_1 and y_2.

Claim 1: "$\sim$" defines an equivalence relationship, and thus, the set $U_x = \{y \in U \ : \ y \sim x\}$ is a well defined equivalence class set.

Claim 2: The open ball $B_\varepsilon(z)$ is convex for every $z \in \mathbb{C}$ and $\varepsilon > 0$. Thus, it's path connected since every convex combination of 2 elements is within the ball.

***U_x* is open:** Let $z \in U_x \subset U$ and let $\varepsilon > 0$ such that $B_\varepsilon(z) \subset U$ (it does exists because U is open). With the previous claim, we know that for any $y \in B_\varepsilon(z)$, $y \sim z$, and since $z \sim x$, we conclude from the transitivity of "$\sim$" that $y \sim x$. Thus, $B_\varepsilon(z) \subset U_x$.

***U_x* is closed:** Finally, for a similar reason, note that $U \backslash U_x$ is open. Let $z \in U \backslash U_x$ and let $\varepsilon > 0$ such that $B_\varepsilon(z) \subset U$. Since $z \not\sim x$ and $y \sim z$ for every $y \in B_\varepsilon(z)$, it follows that $y \not\sim x$. Therefore, $B_\varepsilon(z) \subset U \backslash U_x$.

If $U_x \subsetneq U$, then U_x and $U \backslash U_x$ would form a separation for U.

Exercise 2.

Part (a)

Let $z, w \in \mathbb{C}$ with $\overline{z}w \neq 1$, and $|z| \leq 1$ and $|w| \leq 1$. Prove that

$$\left| \frac{w - z}{1 - \overline{w}z} \right| \leq 1$$

with equality if and only if $|z| = 1$ or $|w| = 1$.

Solution:

In the first place, let $z = a + ib$, $w = x + iy$ and note that

$$\begin{aligned}
|w - z|^2 &= \overline{(w - z)}(w - z) \\
&= (\overline{w} - \overline{z})(w - z) \\
&= \overline{w}w - \overline{w}z - \overline{z}w + \overline{z}z \\
&= |w|^2 + |z|^2 - [(x - iy)(a + ib) + (x + iy)(a - ib)] \\
&= |w|^2 + |z|^2 - [(2ax + 2by) + i \cdot 0] \\
&= |w|^2 + |z|^2 - 2\text{Re}(\overline{w}z),
\end{aligned}$$

and similarly,

$$\begin{aligned}
|1 - \overline{w}z|^2 &= \overline{(1 - \overline{w}z)}(1 - \overline{w}z) \\
&= (1 - \overline{z}w)(1 - \overline{w}z) \\
&= 1 - \overline{w}z - \overline{z}w + |wz|^2 \\
&= 1 + |w|^2|z|^2 - 2\text{Re}(\overline{w}z)
\end{aligned}$$

Then, note that since $|z| \leq 1$, $|w| \leq 1$

$$1 + |w|^2|z|^2 - |w|^2 - |z|^2 = (1 - |w|^2)(1 - |z|^2) \geq 0.$$

Thus,

$$\begin{aligned}
& 1 + |w|^2|z|^2 - |w|^2 - |z|^2 \geq 0 \\
\iff \quad & |w|^2 + |z|^2 \leq 1 + |w|^2|z|^2 \\
\iff \quad & |w|^2 + |z|^2 - 2\mathrm{Re}(\overline{w}z) \leq 1 + |w|^2|z|^2 - 2\mathrm{Re}(\overline{w}z) \\
\iff \quad & \frac{|w|^2 + |z|^2 - 2\mathrm{Re}(\overline{w}z)}{1 + |w|^2|z|^2 - 2\mathrm{Re}(\overline{w}z)} \leq 1 \\
\iff \quad & \left| \frac{w - z}{1 - \overline{w}z} \right|^2 \leq 1 \\
\iff \quad & \left| \frac{w - z}{1 - \overline{w}z} \right| \leq 1.
\end{aligned}$$

From the previous chain of equations, note that we can change "$\leq$" for "$=$" without changing the implications. Therefore,

$$\left| \frac{w - z}{1 - \overline{w}z} \right| = 1 \iff (1 - |w|^2)(1 - |z|^2) = 0.$$

The right side is also equivalent to $|z| = 1$ or $|w| = 1$.

Part (b)

Let $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ be the open unit disc in $\mathbb{C}$. For a fixed $w \in \mathbb{D}$ define

$$F(z) = \frac{w - z}{1 - \overline{w}z} \quad \text{for } z \in \mathbb{C} \text{ with } \overline{w}z \neq 1.$$

Prove that

(i) F is holomorphic in $\mathbb{D}$ and $F(\mathbb{D}) \subseteq \mathbb{D}$.

(ii) $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ for $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective.

Solution:

(i) F is a rational function of order 1. According to the Ahlfors' book, the derivative of a rational function is

$$F'(z) = \left(\frac{P(z)}{Q(z)}\right)' = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q(z)^2} = \frac{-(1-\overline{w}z) + \overline{w}(w-z)}{(1-\overline{w}z)^2},$$

and it only exists when $Q(z) \neq 0$ which, by hypothesis, never occurs because $\overline{w}z \neq 1$. To prove $F(\mathbb{D}) \subseteq \mathbb{D}$ use the previous part of this exercise:

Since $|z| < 1$ and $|w| < 1$, it follows that (using the equivalences from part (a)):

$$|F(z)| = \left|\frac{w-z}{1-\overline{w}z}\right| < 1$$
$$\implies F(z) \in \mathbb{D}.$$

(ii) $F(0) = \dfrac{w-0}{1-\overline{w}0} = \dfrac{w}{1} = w$, and since $\overline{w}z \neq 1$, it follows that $1 - \overline{w}w \neq 0$. Therefore,

$$F(w) = \frac{w-w}{1-\overline{w}w} = 0.$$

(iii) It's explicitly given by part (a).

(iv) Let $z_1 \neq z_2$, but assume for the sake of contradiction that $F(z_1) = F(z_2)$

$$\begin{aligned}
& \frac{w-z_1}{1-\overline{w}z_1} = \frac{w-z_2}{1-\overline{w}z_2} \\
\iff \quad & (w-z_1)(1-\overline{w}z_2) = (w-z_2)(1-\overline{w}z_1) \\
\iff \quad & w - |w|^2 z_2 - z_1 + \overline{w}z_1z_2 = w - |w|^2 z_1 - z_2 + \overline{w}z_1z_2 \\
\iff \quad & |w|^2 z_2 + z_1 = |w|^2 z_1 + z_2 \\
\iff \quad & (|w|^2 - 1)z_2 = (|w|^2 - 1)z_1
\end{aligned}$$

Since $|w| < 1$, the last part can only happen if $z_1 = z_2$. Thus, F is injective. On the other hand, in order to prove that F is surjective, we must find for every $v \in \mathbb{D}$, a complex number $z \in \mathbb{D}$ such that $v = F(z)$:

$$\begin{aligned}
& v = \frac{w-z}{1-\overline{w}z} \\
\iff \quad & v(1-\overline{w}z) = w - z \\
\iff \quad & z - v\overline{w}z = w - v \\
\iff \quad & z = \frac{w-v}{1-\overline{w}v} = F(v)
\end{aligned}$$

This surprisingly implies that $F^{-1}(z) = F(z)$. Note that, from the fact $F(\mathbb{D}) \subset \mathbb{D}$, we can finally conclude that $z \in \mathbb{D}$ as we intended. Thus, F is also surjective.

Exercise 3.

Let $U := \{z \in \mathbb{C} \ : \ \text{Im}(z) > 0\}$. Prove that $\Phi : \mathbb{D} \to U$, $\Phi(z) = i\dfrac{1-z}{1+z}$ is a bijection and calculate its inverse.

Solution: First, we are going to prove that Φ is injective. Let $z_1 \neq z_2 \in \mathbb{D}$, and assume for the sake of contradiction that $\Phi(z_1) = \Phi(z_2)$. Then,

$$
\begin{aligned}
& i\frac{1-z_1}{1+z_1} = i\frac{1-z_2}{1+z_2} \\
\iff \quad & (1-z_1)(1+z_2) = (1+z_1)(1-z_2) \\
\iff \quad & 1 - z_1 + z_2 - z_1 z_2 = 1 + z_1 - z_2 - z_1 z_2 \\
\iff \quad & 2z_1 = 2z_2.
\end{aligned}
$$

Which proves that Φ is injective.

Now, in order to prove that Φ is surjective, let $w = x + iy \in U$. The goal is to find $z \in \mathbb{D}$ such that $w = f(z)$:

$$
\begin{aligned}
\iff \quad & w = i\frac{1-z}{1+z} \\
\iff \quad & w(1+z) = i - iz \\
\iff \quad & iz + wz = i - w \\
\iff \quad & z = \frac{i-w}{i+w}
\end{aligned}
$$

Now, it is left to prove that $z \in \mathbb{D}$, or equivalently, that $\text{Im}(z) > 0$:

$$
\begin{aligned}
z = \frac{-x + i(1-y)}{x + i(1+y)} &= \frac{-x + i(1-y)}{x + i(1+y)} \cdot \frac{x - i(1+y)}{x - i(1+y)} \\
&= \frac{(1 - x^2 - y^2) + i(2x)}{x^2 + (1+y)^2}.
\end{aligned}
$$

So it follows that,

$$
\begin{aligned}
\text{Re}(z) &= \frac{1 - x^2 - y^2}{x^2 + y^2 + 2y + 1}, \\
\text{Im}(z) &= \frac{2x}{x^2 + y^2 + 2y + 1}.
\end{aligned}
$$

I'm supposed to prove that $\text{Im}(z) > 0$. However, from the previous steps, I made more clear that $\text{Re}(z) = \frac{1-|w|^2}{x^2+(1+y)^2} > 0$, because $|w|^2 < 1$ and the denominator is a positive real number. I have reviewed the previous steps and I cannot see the mistake. I'm completely sure that the inverse function is

$$\Phi^{-1}(w) = \frac{i-w}{i+w},$$

and with computational brute force I can prove it (for $w \in \mathbb{C}\backslash\{-i\}$):

```python
phi = lambda z : I*(1-z)/(1+z)
phi_inv = lambda w : (I - w)/(I + w)

print(phi(phi_inv(x+I*y)).simplify())
print(phi_inv(phi(x+I*y)).simplify())
```

```
x + I*y
x + I*y
```

In an ideal world I'd have gotten $\text{Im}(z) = \frac{1-x^2-y^2}{x^2+y^2+2y+1} = \frac{1-|w|^2}{x^2+(1+y)^2} > 0$. But I honestly don't know what's wrong with my procedure.

Exercise 4.

Let $U := \{z \in \mathbb{C} \,:\, \text{Im}(z) > 0\}$ and let $\Psi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ for fixed $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

(a) Suppose that $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha\delta - \beta\gamma > 0$. Prove that $\Psi : U \to U$ is a bijection.

(b) Suppose that $\Psi : U \to U$ is a bijection. Prove that the numbers $\alpha, \beta, \gamma, \delta$ can be chosen from $\mathbb{R}$.

Solution Part (a)

To prove it's injective, assume for the sake of contradiction that there exists $z_1 \neq z_2 \in \mathbb{C}$ such that $\Psi(z_1) = \Psi(z_2)$. Then,

$$
\begin{aligned}
& & \frac{\alpha z_1+\beta}{\gamma z_1+\delta} &= \frac{\alpha z_2+\beta}{\gamma z_2+\delta} \\
&\iff & (\alpha z_1+\beta)(\gamma z_2+\delta) &= (\alpha z_2+\beta)(\gamma z_1+\delta) \\
&\iff & \alpha\gamma z_1 z_2+\alpha\delta z_1+\beta\gamma z_2+\beta\delta &= \alpha\gamma z_1 z_2+\alpha\delta z_2+\beta\gamma z_1+\beta\delta \\
&\iff & \alpha\delta z_1+\beta\gamma z_2 &= \alpha\delta z_2+\beta\gamma z_1 \\
&\iff & (\alpha\delta-\beta\gamma)z_1 &= (\alpha\delta-\beta\gamma)z_2
\end{aligned}
$$

Since, by hypothesis, $\alpha\delta-\beta\gamma>0$ it would follow that $z_1=z_2$. Therefore, Ψ is injective.

To prove it's surjective, we are going to prove that for any $w\in U$, there exists $z\in U$ such that $w=\Psi(z)$:

$$
\begin{aligned}
&\iff & w &= \frac{\alpha z+\beta}{\gamma z+\delta} \\
&\iff & w(\gamma z+\delta) &= \alpha z+\beta \\
&\iff & \gamma wz+\delta w &= \alpha z+\beta \\
&\iff & \gamma wz-\alpha z &= \beta-\delta w \\
&\iff & z &= \frac{\beta-\delta w}{\gamma w-\alpha}
\end{aligned}
$$

It's left to prove that $\mathrm{Im}(z)>0$. Let $w=x+iy$, where, by hypothesis $y>0$

$$
\begin{aligned}
z &= \frac{\beta-\delta x-\delta iy}{\gamma x+\gamma iy-\alpha} \\
&= \frac{\beta-\delta x-i\delta y}{\gamma x-\alpha+i\gamma y}\cdot\frac{\gamma x-\alpha-i\gamma y}{\gamma x-\alpha-i\gamma y} \\
&= \frac{(\beta-\delta x-i\delta y)(\gamma x-\alpha-i\gamma y)}{(\gamma x-\alpha)^2+(\gamma y)^2}
\end{aligned}
$$

Thus, by letting $R = (\gamma x - \alpha)^2 + (\gamma y)^2 \in \mathbb{R}^{\geq 0}$,

$$\begin{aligned} R \cdot \operatorname{Im}(z) &= (\alpha - \gamma x)\delta y - (\beta - \delta x)\gamma y \\ &= (\alpha\delta - \gamma\delta x - \beta\gamma + \gamma\delta x)y \\ &= (\alpha\delta - \beta\gamma)y \\ &> 0. \end{aligned}$$

Solution Part (b)

If $\Psi : z \mapsto \dfrac{\alpha z + \beta}{\gamma z + \delta}$ is bijective, then, from the previous part we know that

$$\Psi^{-1}(w) = \frac{\beta - \delta w}{\gamma w - \alpha}$$

By hypothesis Ψ is defined for any $z \in U$, and Ψ^{-1} is also defined for any $w \in U$. However, both functions have complex poles in $-\delta/\gamma$ and α/γ respectively. Therefore,

$$\frac{-\delta}{\gamma}, \frac{\alpha}{\gamma} \notin U.$$

On the other hand, by evaluating at 0 (which is not in U):

$$\Psi(0) = \frac{\beta}{\delta},\ \Psi^{-1}(0) = \frac{-\beta}{\alpha},$$

so it follows that

$$\frac{\beta}{\delta}, \frac{-\beta}{\alpha} \notin U.$$

From here I don't know how to procede, but I believe that, using the conjugate of the function, there's a way to prove that

$$\frac{-\delta}{\gamma}, \frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \frac{-\beta}{\alpha} \in \mathbb{R}.$$

With this, we can then make a choosing of $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Exercise 5.

Prove that $\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}}$. Formulate and prove the chain rule for the Wirtinger derivatives.

Solution:

Let $z = x + iy$ and $f(z) = u(z) + iv(z)$, where u, v are real functions, using the definition of the Wirtinger derivative,

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
\implies \overline{\frac{\partial f}{\partial z}} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)
\end{aligned}$$

Additionally, $\overline{f}(z) = u(z) + i(-v(z))$. Thus,

$$\begin{aligned}
\frac{\partial \overline{f}}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial(-v)}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial(-v)}{\partial x} + \frac{\partial u}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
&= \overline{\frac{\partial f}{\partial z}}
\end{aligned}$$

Chain Rule

In order to simplify the following expressions, call $F_x = \frac{\partial f}{\partial x} \circ g$, $F_y = \frac{\partial f}{\partial y} \circ g$, $g_x = \frac{\partial g}{\partial x}$ and $g_y = \frac{\partial g}{\partial y}$.

For the chain rule, the formulation is:

$$\frac{\partial f \circ g}{\partial z} = \frac{\partial g}{\partial z} \cdot \left(\frac{\partial f}{\partial z} \circ g\right) + \frac{\partial \overline{g}}{\partial z} \cdot \left(\frac{\partial f}{\partial \overline{z}} \circ g\right)$$

To prove this, note that

$$
\begin{aligned}
\frac{\partial f\circ g}{\partial z} &= \frac{1}{2}\left(\frac{\partial f\circ g}{\partial x}(z) - i\frac{\partial f\circ g}{\partial y}\right)\\
&= \frac{1}{2}\left(\frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial x}\circ g - i\frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial y}\circ g\right)\\
(\star) &= \frac{1}{2}\left(g_xF_x - ig_yF_y\right)
\end{aligned}
$$

and on the other hand, Also, note that

$$
\frac{\partial\overline{g}}{\partial z} = \overline{\frac{\partial g}{\partial\overline{z}}} = \overline{(g_x+ig_y)} = \overline{g_x} - i\overline{g_y}
$$

$$
\begin{aligned}
\frac{\partial g}{\partial z}\cdot\left(\frac{\partial f}{\partial z}\circ g\right) + \frac{\partial\overline{g}}{\partial z}\cdot\left(\frac{\partial f}{\partial\overline{z}}\circ g\right) &= \frac{1}{4}(g_x-ig_y)\cdot(F_x-iF_y) + \frac{1}{4}(\overline{g_x}-i\overline{g_y})\cdot(F_x+iF_y)\\
&= \frac{1}{4}\left[g_xF_x - ig_xF_y - ig_yF_x + g_yF_y\right]\\
&+ \frac{1}{4}\left[\overline{g_x}F_x + i\overline{g_x}F_y - i\overline{g_y}F_x - \overline{g_y}F_y\right]\\
&= \frac{1}{4}\left[g_x + \overline{g_x} - ig_y - i\overline{g_y}\right]F_x\\
&+ \frac{1}{4}\left[-ig_x + i\overline{g_x} + g_y - \overline{g_y}\right]F_y\\
&= \frac{1}{4}\left[2\mathrm{Re}(g_x) - 2i\mathrm{Re}(g_y)\right]F_x\\
&+ \frac{1}{4}\left[2\mathrm{Im}(g_x) + 2i\mathrm{Im}(g_y)\right]F_y\\
\text{(Cauchy-Riemann)} &= \frac{1}{4}\left[2\mathrm{Re}(g_x) + 2i\mathrm{Im}(g_x)\right]F_x\\
&+ \frac{1}{4}\left[2\mathrm{Im}(g_x) - 2i\mathrm{Re}(g_y)\right]F_y\\
&= \frac{1}{2}g_xF_x - \frac{i}{2}g_yF_y\\
(\star) &= \frac{\partial f\circ g}{\partial z}
\end{aligned}
$$