Complex Analysis: Homework 14

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Preliminaries

Theorem 1 (Parseval's Identity)**.** For any function $f \in L_2(-\pi, \pi)$, with Fourier series expansion

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}, \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

we have that

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Theorem 2 (Weierstrass factorization theorem)**.** Let $f : U \to \mathbb{C}$ be a holomorphic function, let m be the multiplicity of 0 as a zero of f and let $(a_n)_{n\in\mathbb{N}}$ be a (infinite) sequence with the zeros of f, each zero being in the sequence as many times as its multiplicity.

Now, let $(p_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that for any $r \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \left|\frac{r}{a_n}\right|^{p_n+1} < \infty.$$

It follows that if

$$E_p(z) = (1 - z) \cdot \exp\left(\underbrace{1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}}_{\approx -\log(1-z)}\right),$$

then, the function $z \mapsto \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ exists, has the same zeros with the same multiplicities as f and that there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right), \quad .$$

In particular, if $p_n = n$, this theorem holds.

Exercise 1.

Calculate

$$\sum_{n=1}^{\infty} n^{-4}.$$

Solution 1.

Note: A time ago I found an interesting solution that only required to know that $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$ and no calculus nor analysis. I won't prove some steps in this solution, the intended proof using complex analysis is at solution 2. If you want to skip it, just do it.

The first step is to define the following sequence

$$a_{i,j} = \frac{2}{i^3 j} + \frac{1}{i^2 j^2} + \frac{2}{i j^3}.$$

Then, by expanding the sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} - a_{i,i+j} - a_{i+j,j}$, there are multiple terms that are canceled:

$$\begin{array}{lllllll}
(\boldsymbol{a_{1,1}} - \overset{(1)}{a_{1,2}} - \overset{(2)}{a_{2,1}}) & + & (\overset{(1)}{a_{1,2}} - \overset{(3)}{a_{1,3}} - \overset{(7)}{a_{3,2}}) & + & (\overset{(3)}{a_{1,3}} - \overset{(5)}{a_{1,4}} - \cdots) & + & (\overset{(5)}{a_{1,4}} - \cdots \\
(\overset{(2)}{a_{2,1}} - \overset{(6)}{a_{2,3}} - \overset{(4)}{a_{3,1}}) & + & (\boldsymbol{a_{2,2}} - \overset{(9)}{a_{2,4}} - \overset{(10)}{a_{4,2}}) & + & (\overset{(6)}{a_{2,3}} - \cdots \qquad) & + & (\overset{(9)}{a_{2,4}} - \cdots \\
(\overset{(4)}{a_{3,1}} - \overset{(11)}{a_{3,4}} - \overset{(8)}{a_{4,1}}) & + & (\overset{(7)}{a_{3,2}} - \cdots \qquad) & + & (\boldsymbol{a_{3,3}} - \cdots \qquad) & + & (\overset{(11)}{a_{3,4}} - \cdots \\
(\overset{(8)}{a_{4,1}} - \cdots \qquad) & + & (\overset{(10)}{a_{4,2}} - \cdots \qquad) & & & &
\end{array}$$

Eventually, the only terms that survive are in the diagonal $i = j$ (proof required), so

$$\sum_{i,j \in \mathbb{N}^+} a_{i,j} - a_{i,i+j} - a_{i+j,j} = \sum_{n=1}^{\infty} a_{n,n} = \sum_{n=1}^{\infty} \frac{5}{n^4}.$$

On the other hand, note that after simplifying (proof required), we obtain that

$$a_{i,j} - a_{i,i+j} - a_{i+j,j} = \frac{2}{i^2 j^2},$$

so by Cauchy summation,

$$\sum_{i,j \in \mathbb{N}^+} a_{i,j} - a_{i,i+j} - a_{i+j,j} = \sum_{i,j \in \mathbb{N}^+} \frac{2}{i^2 j^2} = 2 \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right) \cdot \left(\sum_{j=1}^{\infty} \frac{1}{j^2} \right) = 2 \left(\frac{\pi^2}{6} \right)^2.$$

Finally, after putting everything together, we obtain that

$$\sum_{n=1}^{\infty} \frac{5}{n^4} = \sum_{i,j \in \mathbb{N}^+} a_{i,j} - a_{i,i+j} - a_{i+j,j} = \frac{\pi^4}{18}$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution 2.

When $n \neq 0$, the n-th coefficient of the Fourier transform of $f(x) = x^2$ is

$$
\begin{aligned}
2\pi \cdot c_n &= \int_{-\pi}^{\pi} x^2 e^{-inx} dx \\
\left(\scriptstyle uv - \int v du\right) &= \underbrace{\left[x^2 \cdot \frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi}}_{A} + \frac{2}{in} \int_{-\pi}^{\pi} x e^{-inx} dx \\
\left(\scriptstyle uv - \int v du\right) &= A + \underbrace{\frac{2}{in} \left[x \cdot \frac{e^{-inx}}{-in}\right]_{-\pi}^{\pi}}_{B} + \underbrace{\frac{2}{(in)^2} \int_{-\pi}^{\pi} e^{-inx} dx}_{C}
\end{aligned}
$$

Then, since $e^{\pm in\pi} = (-1)^n$, it follows that

$$
A = \left(\pi^2 \cdot \frac{e^{-in\pi}}{-in}\right) - \left((-\pi)^2 \cdot \frac{e^{in\pi}}{-in}\right) = \frac{\pi^2}{-in}((-1)^n - (-1)^n) = 0
$$

$$
B = \frac{2}{in}\left(\pi \cdot \frac{e^{-in\pi}}{-in}\right) - \frac{2}{in}\left((-\pi) \cdot \frac{e^{in\pi}}{-in}\right) = \frac{2\pi}{n^2}((-1)^n + (-1)^n) = \frac{4\pi(-1)^n}{n^2}
$$

$$
\begin{aligned}
C &= \frac{2}{(in)^2} \int_{-\pi}^{\pi} e^{-inx} dx = \frac{2}{(in)^2} \cdot n \cdot \int_{\partial B_1(0)} dz = 0. \\
\implies c_n &= \frac{A + B + C}{2\pi} = \frac{0 + \frac{4\pi(-1)^n}{n^2} + 0}{2\pi} = \frac{2(-1)^n}{n^2} \\
\implies |c_n|^2 &= \frac{4}{n^4}
\end{aligned}
$$

When $n = 0$,

$$
2\pi \cdot c_0 = \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3} = 2\pi \cdot \frac{\pi^2}{3}.
$$

$$
\implies c_0 = \frac{\pi^2}{3} \implies |c_0|^2 = \frac{\pi^4}{9}.
$$

Then, using Parseval's Identity,

$$\begin{aligned}
\frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} &= |c_0|^2 + 2\sum_{n=1}^{\infty}|c_n|^2 \\
&= \sum_{-\infty}^{\infty}|c_n|^2 \\
\text{(Parseval's Identity)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|x^2|^2dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4dx \\
&= \frac{1}{2\pi}\cdot\frac{2\pi^5}{5} = \frac{\pi^4}{5}
\end{aligned}$$

Finally, from this equality, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{8}\cdot\frac{4}{45} = \frac{\pi^4}{90}.$$

Exercise 2.

Show that for any meromorphic function $h : \mathbb{C} \to \mathbb{C}$ the exists entire functions $f, g : \mathbb{C} \to \mathbb{C}$ with no common zeros such that $h = f/g$.

Solution

Note: Assume that the set of poles of h is infinite (and thus unbounded). Otherwise, let $p_1, \ldots, p_n$ be the poles with order $k_1, \ldots, k_n$. Then, $h(z) \cdot \prod_{j=1}^{n}(z - p_j)^{k_j}$ is a entire function with the same zeros as $h(z)$

In the first place, either 0 is a pole or $f(0)$ is defined. So let

$$m = \begin{cases} -\mathrm{ord}(f, 0) & \text{if } 0 \text{ is a pole,} \\ \mathrm{mult}(f, 0) & \text{if } 0 \text{ is a zero,} \\ 0 & \text{otherwise,} \end{cases}$$

and note that the map $z \mapsto z^{-m}h(z)$ has a removable singularity at 0.

Let $Z = \{z \in \mathbb{C}\backslash\{0\} \ : \ h(z) = 0\}$ be the set of (non-zero) zeros and $P = \{z \in \mathbb{C}\backslash\{0\} \ : \lim_{w\to z}\frac{1}{h(w)} = 0\}$ be the set of (non-zero) poles of h. Note that $Z \cap P = \emptyset$ and both sets are countable because they have no isolated points (discrete). Now, define a sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ and $(p_n)_{n\in\mathbb{N}} \subset P$ where:

- If k is the multiplicity of $z \in P$ at h, then there are exactly k elements $z_{n_1}, \dots, z_{n_k}$ such that $z = z_{k_j}$ for $j \in \{1, \dots, k\}$.

- If k is the order of $z \in P$ at h, then there are exactly k elements $p_{n_1}, \dots, p_{n_k}$ such that $z = p_{k_j}$ for $j \in \{1, \dots, k\}$.

Since p_n is not bounded, it follows that for any $r \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $|p_n| = 2r$ for $n \geq N$, and thus,

$$\sum_{n=1}^{\infty} \left| \frac{r}{a_n} \right|^n \leq \sum_{n=1}^{N-1} \left| \frac{r}{a_n} \right|^n + \sum_{n=N}^{\infty} \frac{1}{2^n} < \infty,$$

so define the entire function $g : \mathbb{C} \to \mathbb{C}$, $g(z) = \prod_{n=1}^{\infty} E_n \left(\frac{z}{p_n} \right)$ with zeros at P. The multiplicity of any given zero of g is, by the Weierstrass factorization theorem, the same as its order as a pole of h. Therefore, the map $z \mapsto z^{-m} h(z) g(z)$ can be extended to an entire function f with zeros in Z. Then,

$$h(z) = z^m \cdot \frac{f(z)}{g(z)}.$$

The set of zeros of f is Z and the set of zeros of g is P so they don't intersect and don't contain 0.

Exercise 3.

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. For every $z \in \mathbb{C}$, let $m(z) \in \mathbb{N}_0$ denote the multiplicity of z as a zero of f. Prove that for every $k \in \mathbb{N}$, the following is equivalent:

(a) There exists a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $g^k = f$.

(b) $m(z) \in \mathbb{N}$ is divisible by k.

Solution

Since f is non-constant, then $k > 0$.

$\boldsymbol{(a)} \implies \boldsymbol{(b)}$: Let z_0 be a zero of f with multiplicity m, then, note that $f(z_0) = g(z_0)^k = 0$. Since the only solution of $w^k = 0$ is $w = 0$, it follows that $g^k(z_0) = 0$ if $g(z_0) = 0$. Now, z_0 is a zero of g with multiplicity n for some $n \in \mathbb{N}^+$, and thus, it follows that z_0 is a zero of $f = g^k$ with multiplicity $k \cdot n$. Therefore, $m = k \cdot n$.

$\boldsymbol{(b)} \implies \boldsymbol{(a)}$:

If the number of zeros is infinite, let $z_1, z_2, \ldots$ be the zeros (without multiplicities, so $z_j \neq z_k$ if $j \neq k$) of f. If the multiplicity of every zero is a multiple of k, then, $\operatorname{mult}(f, z_n) = k \cdot a_n$ for some $a_n \in \mathbb{N}$. Now make a sequence $(w_j)_{j\in\mathbb{N}}$, where for every $n \in \mathbb{N}$, there exists exactly a_n indices $(j_1, \ldots, j_{a_n})$ for which $z_n = w_{j_1} = \cdots = w_{j_{a_n}}$ and $f(w_j) = 0$ for every $j \in \mathbb{N}$. Since the zeros of f are unbounded, then $(w_j)_{j\in\mathbb{N}}$ is unbounded too. Therefore, using a similar argument to exercise 2,

$$\sum_{n=1}^{\infty} \left| \frac{r}{w_n} \right|^n < \infty, \quad \forall r \in \mathbb{R},$$

and thus, the function $h(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{w_n}\right)$ is an entire function where z_n is a zero of multiplicity a_n. Now, $h(z)^k$ is also an entire function where z_n is a zero of multiplicity $k \cdot a_n$, so f/h^k only has removable singularities and can be extended to an entire function with no zeros $e^r := \widetilde{f/h^k}$. The entire function we are looking for is $g : \mathbb{C} \to \mathbb{C},\ g(z) = h(z) \cdot e^{r(z)/k}$.

If the number of zeros is finite, then, let $z_1, \ldots, z_n$ be the zeros with multiplicities $k \cdot a_1, \ldots, k \cdot a_n$. Now, define $h(z) = (z - z_1)^{a_1} \cdots (z - z_n)^{a_n}$ and procede similarly to the previous argument to define $g = h \cdot e^{r/k}$.

Exercise 4.

Let $0 \neq p \in \mathbb{C}$. Prove: For every $\varepsilon > 0$ and $c \in \mathbb{C}$, there exists an entire function g such that $g(p) = c$ and $|g(z)| < \varepsilon$ for every $|z| \leq |p|/2$.

Solution

If $c = 0$, then the function $g(z) = 0$ satisfies both conditions.

Fix $\varepsilon > 0$ and $c \in \mathbb{C} \backslash \{0\}$.

The function $z \mapsto c \cdot \frac{z}{p}$ does satisfy the first condition but might not satisfy the second. So we are looking for a function $g(z) = c \cdot \frac{z}{p} \cdot f(z)$ such that $f(p) = 1$ and

$$\left| \frac{cz}{p} \cdot f(z) \right| < \varepsilon, \quad |z| \leq \frac{|p|}{2}$$

$$\iff |f(z)| < \frac{\varepsilon |p|}{|c||z|}, \quad |z| \leq \frac{|p|}{2}. \tag{$\star$}$$

A natural candidate is a function h such that $h(p) = 0$ and $f = e^h$. Then, note that

$$|f(z)| = |e^{h(z)}| = \exp(\operatorname{Re}(h(z)))$$

so if $\mathrm{Re}(h(z)) + \ln|z/p| < \ln\left(\frac{\varepsilon}{|c|}\right)$ for $|z/p| \leq \frac{1}{2}$, then $(\star)$ holds. In fact, we can relax the condition by noting that $\ln|z/p| \leq \ln(1/2)$ for $|z/p| \leq \frac{1}{2}$, so the following condition is enough for our purpose:

$$\mathrm{Re}(h(z)) < \ln\left(\frac{2\varepsilon}{|c|}\right), \quad |z| \leq \frac{|p|}{2}.$$

Finally, since $z \mapsto \mathrm{Re}(1 - z/p)$ is a continuous function, there exists an upper bound $M \in \mathbb{R}$ for $z \in \overline{B_{|p|/2}(0)}$:

$$\forall z \in \overline{B_{|p|/2}(0)} : \quad \mathrm{Re}(1 - z/p) < M \implies \mathrm{Re}\left(\frac{1 - z/p}{M} \cdot \ln\left(\frac{2\varepsilon}{|c|}\right)\right) < \ln\left(\frac{2\varepsilon}{|c|}\right)$$

Finally, define $h(z) := \frac{1-z/p}{M} \cdot \ln\left(\frac{2\varepsilon}{|c|}\right)$ to obtain a function $f = e^h$ that satisfies $(\star)$. The function $g(z) = \frac{czf(z)}{p}$ satisfies $g(p) = c$ because $f(p) = 1$ and satisfies $|g(z)| < \varepsilon$ for $|z| \leq |p|/2$ because f satisfies $(\star)$.

Note that for $|z|/|p| \leq 1/2$,

$$\mathrm{Re}(1 - z/p) \leq |\mathrm{Re}(1 - z/p)| \leq |1 - z/p| \leq 1 + |z|/|p| < \frac{3}{2} + \varepsilon.$$

Thus, $M = 3/2 + \varepsilon$ should work.

Exercise 5.

Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of distinct points such that $\lim_{n\to\infty} |a_n| = \infty$. Let $(w_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence. Find an entire function f such that $f(a_n) = w_n$ for all $n \in \mathbb{N}$.

Hint: $f = \sum f_n$, where $f_n(a_1) = f_n(a_2) = \cdots = f_n(a_{n-1}) = 0$.

Solution

Without restriction, assume that $0 \notin (a_n)_{n\in\mathbb{N}}$. Otherwise, translate every point in (a_n) by a constant so 0 is not in the sequence anymore.

In the first place, if $\lim_{n\to\infty} |a_n| = \infty$, then $(a_n)_{n\in\mathbb{N}}$ doesn't have accumulation points. Then, by Weierstrass factorization theorem, the function

$$h(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

is entire and only has zeros of multiplicity 1 at a_n for every $n \in \mathbb{N}$. Note that $h(z)/(z - a_n)$ can be extended to an entire function that doesn't vanish at a_n. Therefore,

$$h'(a_n) = \lim_{z\to a_n} \frac{h(z) - \overbrace{h(a_n)}^{0}}{z - a_n} \neq 0.$$

Now, we must introduce the Mittag-Leffler theorem in order to proceed

Theorem 4 Chapter 5.2 (Ahlfors') Let $(b_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty}|b_n| = \infty$, and let $P_n(z)$ be polynomials such that $P_n(z) = 0$ for every $n \in \mathbb{N}$. Then, there exists a function F with poles at b_n and the corresponding principal part is $P_n\left(\frac{1}{z-b_n}\right)$. In particular, one can find polynomials p_n and an entire function g for which

$$F(z) = g(z) + \sum_{n=1}^{\infty} P_n\left(\frac{1}{z-b_n}\right) - p_n(z).$$

$P_n(1/(z-b_n))$ is holomorphic for $|z| < |b_n|$ so it has a Taylor series at the origin. If we define p_n as the partial Taylor sum with m_n coefficients and $M_n = \max_{|z|\leq|b_n|/2}|P_n(z)|$, then we obtain the following approximation error bound:

$$\left|P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right| \leq 2M_n\left(\frac{2|z|}{|b_n|}\right)^{m_n+1}, \quad |z| \leq |b_n|/4$$

One can choose m_n in such way that $2^{m_n} \geq M_n 2^n$. . ☐

For our case, I want $P_n(z) = z$ so

$$F(z) = \sum_{n=1}^{\infty}\left(\frac{w_n/h'(a_n)}{z-a_n} - p_n(z)\right)$$

and $p_n = -\frac{w_n}{h'(a)a_n}\sum_{k=0}^{m_n}(z/a_n)^k$ (partial geometric series of $\frac{w_n/h'(a_n)}{z-a_n}$) with an appropriate sequence of powers $(m_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ for which $\sum_{n=1}^{M}\frac{w_n/h'(a_n)}{z-a_n} - p_n(z)$ converges absolutely when $M \to \infty$ so F is well defined. Then,

$$\begin{aligned}
\lim_{z\to a_n} F(z)\cdot h(z) &= \lim_{z\to a_n}\left(\frac{w_n/h'(a_n)}{z-a_n} - p_n\right)\cdot h(z) + \sum_{k\neq n}^{\infty}\left(\frac{w_k/h'(a_k)}{z-a_k} - p_k(z)\right)\cdot h(z) \\
&= \lim_{z\to a_n}\left(\frac{w_n/h'(a_n)}{z-a_n} - p_n\right)\cdot h(z) + 0 \\
&= \lim_{z\to a_n}\frac{w_n/h'(a_n)}{z-a_n}\cdot h(z) - 0 \\
&= \lim_{z\to a_n}\frac{w_n}{h'(a_n)}\frac{h(z) - \overbrace{h(a_n)}^{0}}{z-a_n} \\
&= w_n\cdot\frac{h'(a_n)}{h'(a_n)} = w_n.
\end{aligned}$$

Finally, $F\cdot h$ only has removable singularities so it can be extended to an entire function f that satisfies $f(a_n) = w_n$.