Holomorphic Functional Calculus

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1 Introduction

The goal of this project is to describe how the definition of holomorphic complex functions can be extended to linear operators in Banach spaces. For instance, if $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a *polynomial* defined for the complex numbers, then it's natural to define for a linear operator $T : X \to X$,

$$P(T) = a_0 I + a_1 T + \cdots + a_n T^n,$$
$$\underbrace{T^0 x := Ix = x}_{\text{identity operator}}, \quad \underbrace{T^k x = T(\cdots (T(x)))}_{k \text{ times composition}}, \ \forall x \in X.$$

Definition 1. A linear operator between two Banach spaces $T : X \to Y$ is called bounded if there exists $M > 0$ such that, for every $x \in X$,

$$\|Tx\|_Y \leq M\|x\|_X.$$

The space of these functions is called $L(X, Y)$, and it's a Banach space if Y is. Also, it can be shown that a linear operator is bounded if and only if is continuous. The operator's norm is defined as

$$\|T\| = \sup_{\|x\|=1} \|Tx\|_Y.$$

Now, for a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ with a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that absolutely converges for $|z| < R$ and a bounded linear operator T such that $\|T\| < R$, the operator $f(T) := \sum_{n=0}^{\infty} a_n T^n$ is well defined because

$$\forall x \in X: \ \|f(T)(x)\| \leq \sum_{n=0}^{\infty} |a_n| \|T^n x\| < \|x\| \sum_{n=0}^{\infty} |a_n| R^n < \infty.$$

However, the tools provided by the power series are limited, and thus, we require another method that allow us to use similar versions of the theorems

Definition 2. An open set $D \subset \mathbb{C}$ is said to be a *Cauchy domain* if these two conditions are met:

- D has a finite number of connected components.
- The boundary of D is composed of a finite number of simple closed rectifiable curves that don't intersect with each other.

Theorem 1 (Cauchy Integral Formula)**.** Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic function. Let D be a Cauchy domain such that $\overline{D} \subseteq U$. Then,

$$\int_C f(\lambda) d\lambda = 0,$$

where C denotes any positively oriented curve that encloses the boundary of D. Also, the n-th derivative of f, $f^{(n)}$ exists on D, and for $\lambda_0 \in D$,

$$f^{(n)}(\lambda_0) = \frac{n!}{2\pi i} \int_C \frac{f(\lambda)}{(\lambda - \lambda_0)^{n+1}} \, d\lambda.$$

□

Therefore, we find a more general way to define $f(T)$ using a version of the Cauchy Integral formula,

$$f(T) = \frac{1}{2\pi i} \int_C \frac{f(\lambda)}{\lambda - T} \, d\lambda,$$

where $1/(\lambda - T) := (\lambda I - T)^{-1}$ is called the *resolvent* of T at λ and it's defined only if the operator $(\lambda I - T)$ is *invertible*. The scope of this project is to formalize this particular definition of the Cauchy Formula, which requires the introduction of a more general notion of complex differentiation and integration for functions with values on Banach spaces.

Definition 3. The *resolvent set* of a linear operator $T : X \to X$ in a Banach space X is defined as

$$\rho(T) := \{\lambda \in \mathbb{C} \ : \ (\lambda I - T) \text{ is bijective}\}.$$

If $\lambda \in \rho(T)$, then the *resolvent of* T *at* λ is $R(\lambda, T) := (\lambda I - T)^{-1}$. Also, the complement of the resolvent set, $\sigma(T) := \mathbb{C} \backslash \rho(T)$ is called the *spectrum of* T.

2 Complex Derivatives for Vector Valued Functions

The goal of this section is to extend the notion of analyticity to a broader set of functions. We're basing this discussion mainly on Chapter V of Taylor and Lay (1986), but some definitions and proofs were adapted from Chapter 5 of Winklmeier (2013).

Let X be a complex Banach space, and let $f : U \subseteq \mathbb{C} \to X$ be a function that is defined on an open set U with values in X.

Definition 4. We say that $f : U \to X$ is *analytic* on U if for every $\lambda_0 \in U$, there exists an element of X called $f'(\lambda_0)$ such that

$$\lim_{\lambda \to \lambda_0} \left\| \frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} - f'(\lambda_0) \right\| = 0.$$

In order for the previous definition to be relevant with our problem, there must exist a bridge between functions from $\mathbb{C} \to X$ and holomorphic functions $\mathbb{C} \to \mathbb{C}$. The bridge in mind is the dual space of X, which provides an interesting way to translate analyticity in this context to complex differentiability.

Definition 5. Let X be a Banach space. We define $X' = L(X, \mathbb{C})$ as the dual space of X, which consists of all bounded operators from X to $\mathbb{C}$. The elements of the dual space are called *functionals*. Similarly to how we treat operators in general, for $x' \in X'$ and $y \in X$, we use the notation

$$x'y = x'(y).$$

For $x' \in X'$, the norm $\|x'\|$ is defined as the operator's norm of x'.

Remark. Since functionals are continuous linear functions, it follows that for an analytic function f and $x' \in X$,

$$\lim_{\lambda \to \lambda_0} \frac{x'f(\lambda_0) - x'f(\lambda)}{\lambda_0 - \lambda} = x' \left(\lim_{\lambda \to \lambda_0} \frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} \right) = x'f'(\lambda_0).$$

Therefore, if f is analytic, then $x'f := x' \circ f : U \to \mathbb{C}$ is an holomorphic function.

In function analysis, we often refer to a *weak* version of a property when the property holds when evaluated using functionals in the dual space. For example, a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is said to *weakly converge* to $x \in X$ if, for every $x' \in X$, the sequence $x'x_n$ converges to $x'x$. With the previous remark, we could introduce a weak version of analyticity when $x'f'(\lambda_0)$ exists for every $\lambda_0 \in U$ and $x' \in X'$. However, part of what makes everything work here is that this weak notion is equivalent to analyticity.

Theorem 2. If $x'f : U \to \mathbb{C}$ is holomorphic for every $x' \in X$, then $f : U \to X$ is analytic.

Proof. Since X is a complete space, it suffices to prove that for every $\lambda_0 \in U$,

$$\frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} - \frac{f(\lambda_0) - f(\mu)}{\lambda_0 - \mu} \to 0, \quad \text{when } \lambda, \mu \to \lambda_0 \text{ on any direction.}$$

Before proceeding, we must state a theorem that is important for this proof.

Lemma 1 (Uniform Boundness Principle)**.** Let A be a subset of a linear space X with a norm $\|\cdot\|$. If, for every $x' \in X'$, $x'A = \{x'a \ : \ a \in A\}$ is bounded, then there exists $M > 0$ such that $\|a\| \leq M$ for every $a \in A$. In other words, if $x'A$ is bounded for every $x' \in X$, then A is bounded too. □

Let $r > 0$ and let $C := \overline{B_r(\lambda_0)}$ be a positively oriented circe centered at λ_0 with radius r. Since $x'f := x' \circ f$ is a holomorphic function (and thus continuous), it follows that $x'f(C)$ is a bounded set for every $x' \in X'$. Therefore, by the Uniform Boundness Principle, there exists $M > 0$ such that

$$\|f(\lambda)\|_X \leq M, \quad \lambda \in C.$$

Now, for $\lambda, \mu \in B_{r/2}(\lambda_0)$, Cauchy Integral Formula give us

$$x'f(\lambda) = \frac{1}{2\pi i} \int_C \frac{x'f(\zeta)}{\zeta - \lambda} d\zeta,$$

so we have that,

$$\begin{aligned} x' \left[\frac{f(\lambda_0) - f(\lambda)}{\lambda_0 - \lambda} - \frac{f(\lambda_0) - f(\mu)}{\lambda_0 - \mu} \right] = \quad & \frac{1}{2\pi i} \int_C \frac{x'f(\zeta)}{(\lambda_0 - \lambda)(\zeta - \lambda_0)} - \frac{x'f(\zeta)}{(\lambda_0 - \lambda)(\zeta - \lambda)} d\zeta \\ & + \frac{1}{2\pi i} \int_C \frac{x'f(\zeta)}{(\lambda_0 - \mu)(\zeta - \lambda_0)} - \frac{x'f(\zeta)}{(\lambda_0 - \mu)(\zeta - \mu)} d\zeta \\ = \quad & \frac{1}{2\pi i} \int_C \frac{(\lambda - \mu) x'f(\zeta)}{(\zeta - \lambda_0)(\zeta - \lambda)(\zeta - \mu)} d\zeta. \end{aligned}$$

Now note that since $|\zeta - \lambda| \geq r/2$, $|\zeta - \mu| \geq r/2$ and $|\zeta - \lambda_0| = r$, it follows that

$$\left|\frac{1}{2\pi i}\int_C \frac{(\lambda-\mu)x'f(\zeta)}{(\zeta-\lambda_0)(\zeta-\lambda)(\zeta-\mu)}d\zeta\right| \leq \frac{4|\lambda-\mu|}{2\pi r^3}\sup_{\lambda\in C}|x'f(\lambda)|\cdot\underbrace{\operatorname{len}(C)}_{2\pi r}$$

$$\leq \frac{4|\lambda-\mu|}{r^2}M\|x'\|.$$

The last inequality follows from the fact that $\|x'y\| \leq \|x'\|\|y\|$. Again, before concluding this proof, we require another theorem from functional analysis.

Lemma 2 (Hahn-Banach Theorem)**.** If $y \neq 0 \in X$, then there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'y = \|x\|$. In particular, if for every $x' \in X'$, $x'y = 0$, then $y = 0$. □

Finally, we conclude that there exists some $x' \in X'$ with $\|x'\| = 1$ and

$$\left\|\frac{f(\lambda_0)-f(\lambda)}{\lambda_0-\lambda} - \frac{f(\lambda_0)-f(\mu)}{\lambda_0-\mu}\right\| = \left|x'\left(\frac{f(\lambda_0)-f(\lambda)}{\lambda_0-\lambda} - \frac{f(\lambda_0)-f(\mu)}{\lambda_0-\mu}\right)\right|$$

$$\leq \frac{4M}{r^2}\cdot|\lambda-\mu| \to 0, \quad \lambda,\mu\to\lambda_0.$$

so we conclude that f is analytic. □

From the previous discussion it is relatively easy to see that all notions of analyticity for maps $U \subseteq \mathbb{C} \to L(X,Y)$ are equivalent.

Corollary 2.1. Let X, Y be Banach spaces and let $U \subseteq \mathbb{C}$ be an open set. For every $\lambda \in U$, let $A_\lambda \in L(X,Y)$. The following statements are equivalent:

(a) For $x \in X$, $y' \in Y'$, the mapping $\lambda \mapsto y'(A_\lambda x)$ is analytic (*weak analyticity*).

(b) For $x \in X'$, the mapping $\lambda \mapsto A_\lambda x$ is analytic (*strong analyticity*).

(c) The mapping $\lambda \mapsto A_\lambda$ is analytic (*analytic in the operator norm*).

Proof. (a) $\iff$ (b) follows from Theorem 2. (c) $\implies$ (b) follows from the definition:

$$\left\|\frac{A_{\lambda_0}x - A_\lambda x}{\lambda_0-\lambda}\right\| \leq \left\|\frac{A_{\lambda_0}-A_\lambda}{\lambda_0-\lambda}\right\|\|x\| \to 0, \quad \lambda\to\lambda_0,\ \forall x\in X.$$

Finally, for (a) $\implies$ (c), suppose that $\lambda \mapsto y'A_\lambda x$ is analytic (and thus continuous) for every $x \in X$. Then, since C is a compact set, by the Uniform Boundness Principle there exists $M_x > 0$ such that

$$\sup_{\lambda\in C}|A_\lambda x| \leq M_x.$$

We need the following theorem to proceed,

Lemma 3 (Banach-Steinhaus theorem)**.** Let X be a Banach space and Y a normed space. Let $F \subset L(X,Y)$ be a family of operators such that

$$\forall x\in X,\ \exists M_x > 0 \quad \forall f\in F\ \|f(x)\| \leq M_x.$$

Then, there exists $M > 0$ such that

$$\|f\| < M \quad \forall f \in F$$

□

Using this lemma, we obtain $M > 0$ such that

$$\sup_{\lambda \in C} |A_\lambda x| \leq M\|x\|, \quad \forall x \in X.$$

Finally, similar to the proof of the previous theorem,

$$\left\| \frac{A_{\lambda_0}x - A_\lambda x}{\lambda_0 - \lambda} \right\| \leq \frac{4|\lambda - \mu|}{r^2} \sup_{\lambda \in C} |A_\lambda x|.$$

Thus,

$$\left\| \frac{A_{\lambda_0} - A_\lambda}{\lambda_0 - \lambda} \right\| \leq \frac{4|\lambda - \mu|}{r^2} M \to 0 \quad \lambda, \mu \to \lambda_0.$$

□

3 Cauchy Integral Formula for Banach Spaces

In this section we're going to discuss the generalization of the integral formula for functions with values in Banach spaces. The discussion of Bochner integrals is based entirely on section 1.2 of Hytönen et al. (2016), but the version of the Cauchy Integral formula is adapted from section V.1 of Taylor and Lay (1986) and section 9.5 of Taylor (2012). For this section, we assume that (S, Σ, μ) is a measure space and X is a complex Banach space

Definition 6 (Simple functions)**.** For $A \subset S$, let $\mathbb{1}_A : S \to \{0, 1\}$ be the *indicator function* of the set A, that is,

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

For a sequence $x_1, \ldots, x_n \in X$ and a sequence of measurable sets $A_1, \ldots, A_n$, the function $f : S \to X$,

$$f(x) = \sum_{k=1}^{n} \mathbb{1}_{A_k}(x) \cdot x_k,$$

is called a *simple function*.

Definition 7 (Bochner Integral)**.** For a simple function $f(x) = \sum_{k=1}^{n} \mathbb{1}_{A_k}(x) \cdot x_k$, we define the *Bochner integral* as

$$\int_S f \, d\mu := \sum_{k=1}^{n} \mu(A_k) \cdot x_k \in X.$$

Now, for a measurable function $f : S \to X$, we say that f is *Bochner integrable* with respect to μ, if there exists a sequence of simple functions $\{f_n\}_{n \in \mathbb{N}}$ such that two conditions are met:

- $\lim_{n \to \infty} f_n(s) = f(s)$ for almost every $s \in S$. In particular, we say that f is *strongly measurable* if f satisfies this property for some sequence of simple functions.

- $\lim_{n \to \infty} \int_S \|f - f_n\| \, d\mu = 0.$

In such case, since X is complete, the sequence $\int_S f_n\, d\mu$ converges to what we call the *Bochner integral of* f,

$$\int_S f\, d\mu = \lim_{n\to\infty} \int_S f_n\, d\mu,$$

and this integral is an element of X. There's another characterization of Bochner integrability in the following theorem.

Theorem 3. A strongly measurable function f is Bochner integrable if and only if

$$\int_S \|f\|\, d\mu < \infty.$$

Proof. Proposition 1.2.2. Hytönen et al. (2016). □

Remark. From the previous theorem, some of the properties of Riemann and Lebesgue integrals immediately transfer for the Bochner integral. For Bochner integrable functions $f, g : S \to X$ and $\lambda \in \mathbb{C}$,

- $\displaystyle\int_S f + \lambda g\, d\mu = \int_S f\, d\mu + \lambda \int_S g\, d\mu$. (Linearity)
- $\displaystyle\left\| \int_S f\, d\mu \right\| \leq \int_S \|f\|\, d\mu$. (Triangle inequality)
- If $f = g$ almost everywhere, then $\displaystyle\int_S f\, d\mu = \int_S g\, d\mu$.
- $\displaystyle\int_S \mathbb{1}_A f\, d\mu = \int_A f|_A\, d\mu|_A$. (Truncation)

Finally, there's one last theorem we require before returning to the discussion of operators.

Theorem 4. Let $f : S \to X$ be Bochner integrable. If $X_0 \subset X$ is a closed subspace such that $f(s) \in X_0$ for almost every $s \in S$, then $\int_S f\, d\mu \in X_0$.

Proof. Proposition 1.2.3. of Hytönen et al. (2016). □

From this theorem, we can conclude that if $T : X \to y$ is a bounded linear operator between Banach spaces and f is Bochner integrable, then

$$T \int_S f\, d\mu = \int_S Tf\, d\mu.$$

Furthermore, as we stated at the start of the document, we want to use the Cauchy Formula in a broader set of operators. The operators of interested at the moment are called closed operators.

Definition 8 (Closed linear operator)**.** A linear operator $T : \mathcal{D}(T) \to Y$ defined on a subspace $\mathcal{D}(T) \subseteq X$ (the *domain* of T) and taking values in a Banach space Y, is said to be *closed* if its graph is a closed subspace of $X \times Y$. The *graph* of T is defined as follows,

$$G(T) := \{(x, Tx)\ :\ x \in \mathcal{D}(T)\} \subseteq X \times Y.$$

Remark. There are some important facts about closed operators:

- For a closed linear operator T, $\mathcal{D}(T)$ is a Banach space with respect to the graph norm:

$$\|x\|_{\mathcal{D}(T)} := \|x\| + \|Tx\|.$$

- An equivalent definition for a closed operator is a linear operator T that satisfies the following: for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$, if $(x_n)_{n\in\mathbb{N}}$ and $(Tx_n)_{n\in\mathbb{N}}$ converge in X and Y respectively, then, there exists $x_0 \in \mathcal{D}(T)$ such that

$$x_0 := \lim_{n\to\infty} x_n, \quad \text{and} \quad \lim_{n\to\infty} Tx_n = Tx_0.$$

- Any bounded operator is closed. Also, the *closed graph theorem* asserts that for a closed linear operator $T : \mathcal{D}(T) \subseteq X \to Y$, if $\mathcal{D}(T) = X$, then T is bounded, so whether a closed operator is bounded depends on $\mathcal{D}(T)$.

Theorem 5 (Hille's theorem)**.** Let $f : S \to X$ be Bochner integrable and let $T : \mathcal{D}(T) \subseteq X \to Y$ be a closed linear operator. Suppose that $f(S) \subseteq \mathcal{D}(T)$ and the function $Tf : S \to Y$ is Bochner integrable too. Then, $\int_S f \, d\mu \in \mathcal{D}(T)$ and,

$$T \int_S f \, d\mu = \int_S Tf \, d\mu.$$

Proof. Theorem 1.2.4. Hytönen et al. (2016) □

Now, define $f : U \to X$ to be an analytic function from an open set $U \subset \mathbb{C}$ with values in $\mathcal{D}(T) \subseteq X$ and S to be a complex rectifiable curve $C \subseteq U$. Note that since f is continuous and C is a compact set, it follows that

$$\int_C \|f(\lambda)\| d\lambda \leq \sup_{\lambda\in C} \|f(\lambda)\| + \mathrm{len}(C) < \infty.$$

Therefore, by Theorem 3, every analytic function is Bochner integrable on a rectifiable curve. The version of Hille's theorem we're looking for is:

$$T \int_C f(\lambda) \, d\lambda = \int_C Tf(\lambda) \, d\lambda.$$

A direct application of Hille's Theorem is the generalization of Cauchy Integral Formula for analytic functions with values over a Banach space. Let $y := \int_C f(\lambda)d\lambda$. Using Corollary 2.1, we have that, for every $x' \in X'$, $x'f$ is a holomorphic function, and thus, by Hille's Theorem and Cauchy Integral Formula,

$$x'y = x' \left(\int_C f(\lambda)d\lambda \right) = \int_C x'f(\lambda)d\lambda = 0.$$

Therefore, by the Hahn-Banach Theorem, since $x'y = 0$ for every $x' \in X'$, it follows that $y = 0$. This is the proof of the first part of the following theorem,

Theorem 6 (General Cauchy Integral Formula)**.** Let $U \subset \mathbb{C}$ be an open set, X a Banach space and $f : U \to X$ an analytic function. Let D be a Cauchy domain such that $\overline{D} \subseteq U$. Then,

$$\int_{\partial D} f(\lambda)d\lambda = 0,$$

where ∂D denotes a curve that encloses the boundary of D. Also, the n-th derivative of f, $f^{(n)}$ exists on D, and for $\lambda \in D$,

$$f^{(n)}(\lambda) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - \lambda)^{n+1}} \, d\zeta.$$

4 The Resolvent

To conclude the discussion we are going to describe some properties of the resolvent operator. Similar to the discussion in Taylor and Lay (1986), for this section, we use the notation $T_\lambda = (\lambda I - T)$ and $R_\lambda = R(\lambda; T)$. Using Lemma 3.33 from Winklmeier (2013), we can assert that if $T : X \to X$ is a closed operator and $\lambda \in \rho(T)$, then R_λ is bounded. So from this point forward, assume that $T : X \to X$ is a closed linear operator.

In the first place, for $S \in L(X)$ note that if $\|S\| < 1$, the Neumann series gives us

$$(I - S)^{-1} = \sum_{n=0}^{\infty} S^n.$$

Now, for $\lambda_0 \in \rho(T)$ and $\lambda \in \mathbb{C}$, we have that since

$$T_\lambda = T_{\lambda_0} - (\lambda_0 - \lambda)I = (I - (\lambda_0 - \lambda)R_{\lambda_0})T_{\lambda_0}.$$

So if we choose λ such that $\|(\lambda_0 - \lambda)R_{\lambda_0}\| < 1$, that is $|\lambda_0 - \lambda| < \|R_{\lambda_0}\|^{-1}$, then we can invert the previous expression using the Neumann series to obtain

$$\begin{aligned} R_\lambda &= R_{\lambda_0}(I - (\lambda_0 - \lambda)R_{\lambda_0})^{-1} \\ &= R_{\lambda_0} \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}^n \\ &= \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}^{n+1}. \end{aligned}$$

This proves that for $\lambda_0 \in \rho(T)$, the ball of radius $\|R_{\lambda_0}\|^{-1}$ is contained at $\rho(T)$, and thus, the resolvent set is open. On the other hand, note that if T is bounded and $|\lambda| > \|T\|$, then $\|\lambda^{-1}T\| < 1$, so it follows that

$$T_\lambda = \lambda(I - \lambda^{-1}T) \text{ is invertible.}$$

Therefore, $\sigma(T) \subseteq \overline{B_{\|T\|}(0)}$ implying that $\sigma(T)$ is compact when $T \in L(X)$.

Theorem 7. The resolvent of a closed linear operator is open. Also, for $\lambda_0 \in \rho(T)$, if $|\lambda_0 - \lambda| < \|R_{\lambda_0}\|^{-1}$, then $\lambda \in \rho(T)$. □

In the next theorem, we are going to prove that the mapping $\lambda \mapsto R_\lambda$ is analytic in its domain.

Theorem 8. For $y \in X$ and $x' \in X'$, the mapping $\lambda \mapsto x'R_\lambda y$ is holomorphic.

Proof. Let λ_0. Then, note that the Neumann series provided by Theorem 7 is absolutely convergent for $\lambda \in B_{\|R_{\lambda_0}\|^{-1}}(\lambda_0)$. Thus, by continuity of x',

$$\begin{aligned} x'R_\lambda y &= x' \left(\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R_{\lambda_0}^{n+1} y \right) \\ &= \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \underbrace{x' R_{\lambda_0}^{n+1} y}_{a_n} \\ &= \sum_{n=0}^{\infty} a_n (\lambda_0 - \lambda)^n. \end{aligned}$$

$\square$

Corollary 8.1. The following consequences follow from the previous theorem. For $\lambda \in \rho(T)$,

- The mapping $\lambda \mapsto f(\lambda)R_\lambda$ is analytic (as an operator in $L(X)$).
- For a holomorphic function f, the mapping $\lambda \mapsto f(\lambda)R_\lambda$ is analytic (as an operator in $L(X)$).

$\square$

Closing statements

From this discussion we justified the methods to calculate the Cauchy Integral for closed operators, when $\sigma(T)$ is contained at a domain D where f is holomorphic,

$$\int_{\partial D} f(\lambda)R_\lambda \, d\lambda, \quad \sigma(T) \subsetneq D.$$

In fact, since we proved that, for a bounded operator T, the spectrum is bounded, we can find $R > 0$ such that $R(\lambda; T)$ is defined,

$$R(\lambda; T) = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}, \quad |\lambda| > R.$$

Now, for any function f that is holomorphic over an open set containing $\sigma(T)$ and C any rectifiable closed curve that encloses $\sigma(T)$ where f is defined, the proposed definition for $f(T)$,

$$f(T) := \frac{1}{2\pi i} \int_C f(\lambda)R_\lambda \, d\lambda,$$

makes sense. There are many important repercussions that require further investigation. For example, the next consequence of this integral formula would be the following theorem

Theorem 9. Let $T \in L(X)$. If f, g are two holomorphic function defined on an open set that contains $\sigma(T)$ and $f(\lambda) = g(\lambda)$ for every $\lambda \in U$ for an open set U contained in their domains, then $f(T) = g(T)$. $\square$

For unbounded closed operators, since $\sigma(T)$ might not be bounded, more details have to be taken into consideration. A plausible alternative is to only consider a set of functions $\mathscr{F}(f)$ that satisfies the following conditions:

- For $f \in \mathscr{F}(T)$, f is holomorphic on its domain $U \subset \mathbb{C}$ which is an open set that contains $\sigma(T)$.
- The complement of U is compact.
- $f(\lambda)$ is bounded and the limit $f(\infty) = \lim_{|\lambda| \to \infty} f(\lambda)$ exists.

In fact, for $f \in \mathscr{F}(T)$ with $f : U \to \mathbb{C}$, if D is an unbounded Cauchy domain such that $\overline{D} \subset U$, then ∂B is compact and the Cauchy Integral Formula would hold as follows

$$f(\lambda_0) = f(\infty) + \int_{\partial D} \frac{f(\lambda)}{\lambda_0 - \lambda}\, d\lambda.$$

Therefore, according to the previous discussion, the appropriate definition of $f(T)$ would be the following

$$f(T) := f(\infty) + \int_{\partial D} f(\lambda) R_\lambda \, d\lambda.$$

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