

# Complex Analysis: Homework 10

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Let  $B_j := B_{r_j}(z_j)$  ( $j = 0, 1, \dots, n$ ) be open disks with  $z_{j-1}, z_j \in B_{j-1} \cap B_j$  for all  $j = 1, \dots, n$ . Then,  $(B_0, B_1, \dots, B_n)$  is called a *chain of disks*. If  $f_j: B_j \rightarrow \mathbb{C}$  are holomorphic functions such that  $f_{j-1} = f_j$  on  $B_{j-1} \cap B_j$  for all  $j = 1, \dots, n$ , then  $f_n$  is called *the analytic extension of  $f_0$  along the chain of disks  $B_0, \dots, B_n$* .

## Exercise 1.

Let  $\mathcal{B} = (B_0, \dots, B_n)$  be a chain of disks and let  $f_0: B_0 \rightarrow \mathbb{C}$  be an analytic function. Suppose that  $f'_0$  has an analytic extension along  $\mathcal{B}$ . Prove that  $f_0$  also has an analytic extension along  $\mathcal{B}$ .

## Solution

According to the previous definition, let  $g_j$  be the analytic continuation of  $f'_0$  along the chain  $B_0, \dots, B_j$  ( $g_0 = f'_0$ ) until we have  $g_n$  with is the extension along  $\mathcal{B}$ .

Now, let  $w \in B_0$ , define the holomorphic function  $h_0(w) = \int_{\gamma_w} g_0(z) dz$  for a smooth path  $\gamma_w$  that starts at  $z_0$  and ends at  $w$ .

This function is well defined because if we take two different paths  $\gamma_w^{(1)}$  and  $\gamma_w^{(2)}$  that start at  $z_0$  and end at  $w$ , then  $\Gamma = \gamma_w^{(1)} + (-\gamma_w^{(2)})$  is a closed path in a simply connected domain  $B_0$ . Therefore, by *Cauchy Integral Formula*,

$$\int_{\Gamma} g_0(z) dz = 0 \implies \int_{\gamma_w^{(1)}} g_0(z) dz = \int_{\gamma_w^{(2)}} g_0(z) dz$$

Then, note that by the *Fundamental Theorem of Calculus*,  $f'_0(w) = h'_0(w)$  for any  $w \in B_0$ , and thus,  $f_0$  and  $h_0$  differ only by a constant:

$$f_0(w) = h_0(w) - h_0(z_0) + f_0(z_0).$$

For  $w \in B_1$ , define  $h_1(w) = \int_{\gamma_w} g_1(z) dz$  for any smooth path that starts at  $z_1$  and ends at  $w$ . For every  $w \in B_0 \cap B_1$ ,  $g_1(w) = g_0(w)$ , and thus, it follows that  $f'_0(w) = h'_1(w)$  so  $f_0$  differs from  $h_1$  only by a constant. Then, define for  $w \in B_1$

$$f_1(w) = h_1(w) - h_1(z_1) + f_0(z_1),$$

which coincides with  $f_0(w)$  for  $w \in B_0 \cap B_1$ .

Recursively, define for  $w \in B_j$ ,  $h_j(w) = \int_{\gamma_w} g_j(z) dz$  for any smooth path that starts at  $z_j$  and ends at  $w$ , to then define

$$f_j(w) = h_j(w) - h_j(z_j) + f_{j-1}(z_j).$$

Applying a similar argument to before, we can prove that  $f_j$  is well defined (using *Cauchy Integral Formula*) and that  $f_j$  coincides with  $f_{j-1}$  at  $B_{j-1} \cap B_j$  (using *Fundamental Theorem of Calculus*). This gives us the analytic extension  $f_n$  of  $f_0$  along  $\mathcal{B}$  we're looking for.

## Exercise 2.

Let  $U = B_1(0)$  and

$$f : U \rightarrow \mathbb{C}, \quad f(z) = \sum_{n=1}^{\infty} 2^{-n^2} z^{2^n}.$$

Prove that  $f$  has no analytic extension to any open set  $G$  with  $G \supsetneq U$ .

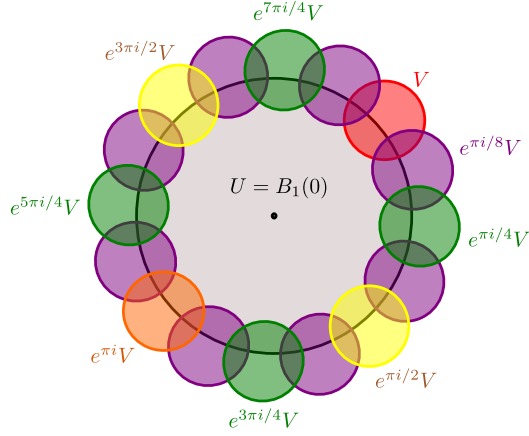
*Hint:* Prove that for every  $n \in \mathbb{N}$  there exists a polynomial  $P_n$  such that

$$f\left(e^{2\pi i/2^n} z\right) = P_n(z) + f(z).$$

## Solution

Assume, for the sake of contradiction, that there exists an open set  $G$  with  $G \supsetneq U$  for which  $f$  can be analytically extended. In the first place, for every  $m \in \mathbb{N}$ ,  $\exp\left(2\pi i \frac{2^n}{2^m}\right) = 1$  for every  $n \geq m$ . Thus,

$$\begin{aligned} f\left(e^{2\pi i/2^m} z\right) &= \sum_{n=1}^{\infty} 2^{-n^2} \exp\left(2\pi i \frac{2^n}{2^m}\right) z^{2^n} \\ &= \sum_{n=m}^{\infty} 2^{-n^2} z^{2^n} + \sum_{n=1}^{m-1} 2^{-n^2} \exp\left(2\pi i \frac{2^n}{2^m}\right) z^{2^n} \\ &= \sum_{n=m}^{\infty} 2^{-n^2} z^{2^n} + \sum_{n=1}^{m-1} 2^{-n^2} \left( z^{2^n} - z^{2^n} + \exp\left(2\pi i \frac{2^n}{2^m}\right) z^{2^n} \right) \\ &= \underbrace{\sum_{n=1}^{\infty} 2^{-n^2} z^{2^n}}_{f(z)} + \underbrace{\sum_{n=1}^{m-1} 2^{-n^2} z^{2^n} \left( \exp\left(2\pi i \frac{2^n}{2^m}\right) - 1 \right)}_{P_m(z)} \end{aligned}$$



Then, this implies that if  $f(z)$  exists, then  $f(e^{\theta i}z)$  exists for any rotation of  $z$  by  $\theta = 2\pi \frac{k}{2^m}$  radians for  $k \in \{1, \dots, 2^m - 1\}$ .

If there exists an open set  $V \subset G$  such that  $V \cap \partial U \neq \emptyset$ , we can find a suitable  $m$  for which the union of the rotations by  $2\pi \frac{k}{2^m}$  can cover  $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ . Look the picture on the left for reference.

$$W = \bigcup_{k=0}^{2^m-1} e^{2\pi \frac{k}{2^m}} V \supseteq \partial B_1(0).$$

Finally, this implies that since  $W$  is open and  $1 \in W$ , there exists  $\varepsilon > 0$  such that  $1 + \varepsilon \in W$ . Thus, there exists  $z \in V$  and  $\theta = 2\pi \frac{k}{2^m}$  such that  $1 + \varepsilon = e^{i\theta}z$ . Therefore,  $f(1 + \varepsilon) = f(e^{i\theta}z) = f(z) + P_m(z)$  exists, but this contradicts the fact that (by the ratio test):

$$f(1 + \varepsilon) = \sum_{n=1}^{\infty} 2^{-n^2} (1 + \varepsilon)^{2^n} \text{ diverges to } \infty.$$

### Exercise 3.

Let  $U = B_1(0)$ . Find an analytic continuation to the largest possible region for

$$f : U \rightarrow \mathbb{C}, \quad f(z) = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^n.$$

*Hint:* Consider  $f(w^2)$ .

### Solution

For  $|w| < 1$  we have absolute convergence, and thus, using the geometric series,

$$\begin{aligned} f(z) &= f(w^2) = \sum_{n=0}^{\infty} (-1)^n (2n+1) w^{2n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{d}{dz} w^{2n+1} \\ &= \frac{d}{dz} \left( \frac{w}{-1 - w^2} \right) \\ &= \frac{1 - w^2}{(1 + w^2)^2} = \frac{1 - z}{(1 + z)^2}. \end{aligned}$$

Now, the function  $f$  can be analytically extended to the function  $g(z) = \frac{1-z}{(1+z)^2}$  on  $z \in \mathbb{C} \setminus \{-1\}$ . Note that  $f$  cannot be extended further, otherwise  $g$  could be extended to another function at  $z = -1$  but that would contradict the fact that  $g$  has pole at  $z = -1$  with order 2.

## Exercise 4.

Let  $X$  be a metric space. A sequence  $(f_n)_{n \in \mathbb{N}}$  of functions  $U \rightarrow \mathbb{C}$  is called continuously convergent if for every convergent sequence  $(x_n)_{n \in \mathbb{N}} \subset X$ , the limit  $\lim_{n \rightarrow \infty} f_n(x_n)$  exists.

- (a) Let  $X$  be a metric space and  $(f_n)_{n \in \mathbb{N}}$  a sequence of functions in  $X$  that converges continuously. Prove that  $f : X \rightarrow \mathbb{C}$ ,  $f(x) = \lim_{n \rightarrow \infty} f_n(x_n)$  is well-defined (i.e., it is independent of the chosen sequence  $(x_n)_{n \in \mathbb{N}}$ ) and that  $f$  is continuous (even if the  $f_n$  are not).
- (b) Let  $U \subseteq \mathbb{C}$  be open and  $(f_n)_{n \in \mathbb{N}}$  a sequence of functions in  $U$ . Prove that the following are equivalent:
  - (i)  $(f_n)_{n \in \mathbb{N}}$  converges compactly to a function  $f \in C(U)$ .
  - (ii)  $(f_n)_{n \in \mathbb{N}}$  converges continuously.

In particular, a continuously converging sequence of holomorphic functions converges to a holomorphic function.

## Solution Item (a)

Let  $(x_n)_{n \in \mathbb{N}} \subseteq U$  be any sequence that converge to  $x \in U$ . Assume for the sake of contradiction that  $f_n(x_n) \rightarrow c \neq f(x)$ , then, define  $y_n = (x_1, x, x_3, x, \dots)$  and note that  $y_n \rightarrow x$  so  $f_n(y_n)$  has a limit. However,  $f_{2n+1}(y_{2n+1}) \rightarrow c$  and  $f_{2n}(y_{2n}) \rightarrow f(x)$  which contradicts the fact the limit of  $f_n(y_n)$  exists. Therefore, for every  $(x_n) \rightarrow x$  it must happen that  $f_n(x_n) \rightarrow f(x)$ , so  $f$  is well defined.

In order to prove that  $f(x)$  is continuous we want to show that for every  $(x_n) \rightarrow x$ ,  $f(x_n) \rightarrow f(x)$ . In the first place, note that for every  $m \in \mathbb{N}$

$$|f(x) - f(x_n)| \leq |f(x) - f_m(x_n)| + |f_m(x_n) - f(x_n)|.$$

Then, fix  $\varepsilon > 0$  and note that since  $f_m \rightarrow f$  pointwise, it follows that for every  $n \in \mathbb{N}$  there exists  $N_n$  such that

$$|f_m(x_n) - f(x_n)| < \varepsilon, \quad \forall m \geq N_n.$$

So define a subsequence  $(f_{m_n})_{n \in \mathbb{N}}$  such that  $m_n \geq N_n$  and  $m_n > m_{n-1}$ . This way,

$$|f(x) - f(x_n)| < |f(x) - f_{m_n}(x_n)| + \varepsilon.$$

Then, define the following sequence

$$y_k = \begin{cases} x_1 & k \in [0, m_1] \\ x_n & k \in [m_n - m_{n-1}, m_n] \end{cases}$$

Since  $y_n \rightarrow x$ , it follows that  $f_n(y_n) \rightarrow f(x)$ . Then,  $y_{m_n} = x_n$ , so  $f_{m_n}(x_n) = f_{m_n}(y_{m_n}) \rightarrow f(x)$ . So for every  $\varepsilon > 0$  we can find  $N \in \mathbb{N}$  such that  $|f(x) - f_{m_n}(x_n)| < \varepsilon$  for  $n \geq N$ . So finally,

$$\begin{aligned} |f(x) - f(x_n)| &< 2\varepsilon, \quad \forall n \geq N \\ f(x_n) &\rightarrow f(x), \quad \forall (x_n)_{n \in \mathbb{N}} \subset U \end{aligned}$$

### Solution Item (b)

**(i)  $\implies$  (ii):** In the previous item we showed that if  $f_n$  converges continuously, then there exists a continuous function  $f$ , such that  $f_n(x) \rightarrow f(x)$  for each  $x \in U$ . Now, for the sake of contradiction assume that there exists a compact set  $K \subset U$  such that  $f_n$  doesn't converges uniformly to  $f$  when restricted to  $K$ . The statement for *Not uniformly convergent* is the following:

$$\exists \varepsilon > 0 : \forall M \in \mathbb{N}, \exists n \geq M, \exists x_n \in K : |f_n(x_n) - f(x_n)| \geq \varepsilon.$$

Fix  $\varepsilon > 0$ ,  $(x_n)_{n \in \mathbb{N}} \subset K$  from this definition and define a subsequence  $(f_{n_k})$  that satisfies  $|f_{n_k}(y_k) - f(y_k)| \geq \varepsilon$  for every  $k \in \mathbb{N}$  (with  $y_k = x_{n_k}$ ). Note that there exists a convergent subsequence  $(y_{k_j}) \rightarrow y$  because  $K$  is compact. Then, from the following inequality,

$$\varepsilon \leq |f_{n_{k_j}}(y_{k_j}) - f(y_{k_j})| \leq |f_{n_{k_j}}(y_{k_j}) - f(y)| + |f(y) - f(y_{k_j})|,$$

note that from the fact that  $f_n$  converges continuously to  $f$  it follows that

- $|f_{n_{k_j}}(y_{k_j}) - f(y)| \rightarrow 0$  because we proved that  $|f_j(z_j) - f(z)| \rightarrow 0$  whenever  $z_j \rightarrow z$ .
- $|f(y) - f(y_{k_j})| \rightarrow 0$  because  $f$  is continuous.

Therefore,  $\varepsilon \leq |f_{n_{k_j}}(y_{k_j}) - f(y_{k_j})| \rightarrow 0$  is a contradiction.

**(ii)  $\implies$  (i):** Now assume that for every compact set  $K \subset U$ ,  $f_n$  converges uniformly to a continuous function  $f$  when restricted to  $K$ . We want to prove that for every sequence  $(x_n) \rightarrow x$ ,  $f_n(x_n)$  converges.

Note that for any compact set  $K$  that contains  $(x_n)$ , the following inequality holds,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|.$$

- Since  $f_n$  converges uniformly to  $f$ , it follows that  $|f_n(x_n) - f(x_n)| \rightarrow 0$  as  $n \rightarrow \infty$ .
- Since  $f$  is continuous  $|f(x_n) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

Therefore,  $|f_n(x_n) - f(x)| \rightarrow 0$  concluding the proof.

## Final Step

Let  $(f_n)$  a sequence of holomorphic functions that continuously converges to  $f$ . We know that  $f$  is continuous so we can integrate  $f$ . Now, take any closed curve  $\gamma$  and note that from item (b),  $f_n$  converges uniformly to  $f$  when restricted to  $\gamma$  (which is compact). Thus,

$$\int_{\gamma} f dz = \int_{\gamma} \lim_n f_n dz = \lim_n \int_{\gamma} f_n dz = 0.$$

Finally, by *Morera's theorem* we conclude that  $f$  is holomorphic.