¡¡¡¡¡¡¡ HEAD
======= ¿¿¿¿¿¿¿ 16d4ed7 (yes)

Time Series: Homework 1

Martín Prado

August 22, 2024
Universidad de los Andes – Bogotá Colombia

Exercise 1.1.

Suppose that $X_t = Z_t + \theta Z_{t-1},\ t = 1, 2, \ldots,$ where Z_0, Z_1, Z_2 are independent random variables, each with moment generating function $\mathbf{E} \exp(\lambda Z_i) = m(\lambda)$.

(a) Express the joint moment generating function $\mathbf{E} \exp\left(\sum_{i=1}^{n} \lambda_i X_i\right)$ in terms of the function $m(\cdot)$.

(b) Deduce from (a) that $\{X_t\}$ is strictly stationary.

Solution part (a)

Since $\{Z_t\}$ are independent, for $X_t = Z_t + \theta Z_{t-1}$, the moment generating function:

$$\begin{aligned}
\mathbf{E} \exp(\lambda X_t) &= \mathbf{E} \exp(\lambda(Z_t + \theta Z_{t-1})) \\
&= \mathbf{E} \exp(\lambda Z_t) \cdot \mathbf{E} \exp(\lambda \theta \mathbb{Z}_{t-1}) \\
&= m(\lambda) \cdot m(\theta \lambda)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i X_i &= \sum_{i=1}^{n} \lambda_i Z_i + \sum_{i=1}^{n} \lambda_i \theta Z_{i-1} \\
&= \sum_{i=1}^{n} \lambda_i Z_i + \sum_{i=0}^{n-1} \lambda_{i+1} \theta Z_i \\
&= \lambda_n Z_n + \left[\sum_{i=1}^{n-1} (\lambda_i + \theta \lambda_{i+1}) Z_i \right] + \lambda_1 \theta Z_0.
\end{aligned}$$

Therefore, using a similar argument as before

$$\mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right) = m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)$$

Solution part (b)

Let $(X_1,\ldots,X_n)'$ be a random vector in $\mathbb{R}^k$. The moment generating function of this vector is defined as follows,

$$M_{X_{1:n}}(\lambda_{1:n}) = \mathbf{E}\exp(\langle\lambda_{1:n}, X_{1:n}\rangle) = \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right),\ \lambda_{1:n}\in\mathbb{R}^n.$$

So, we know from the previous part that

$$\begin{aligned}
M_{X_{1:n}}(\lambda_{1:n}) &= m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)\\
&= \mathbf{E}\exp(\lambda_n Z_{n+h}) + \left[\prod_{i=1}^{n-1}\mathbf{E}\exp((\lambda_i+\theta\lambda_{i+1})Z_{i+h})\right] + \mathbf{E}\exp(\lambda_1\theta Z_h)\\
&= \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_{i+h}\right)\\
&= M_{X_{1+h:n+h}}(\lambda_{1:n})
\end{aligned}$$

Since the moment generating function of both $(X_1,\ldots,X_n)'$ and $(X_{1+h},\ldots,X_{n+h})'$ coincide, they have the same joint distribution. Thus, $\{X_t\}$ is strictly stationary.

Exercise 1.4.

If $m_t = \sum_{k=0}^{p} c_k t^k, t = 0, \pm 1, \ldots$, show that ∇m_t is a polynomial of degree $(p-1)$ in t and hence that $\nabla^{p+1} m_t = 0$.

Solution:

$$\begin{aligned}
m_{t-1} &= \sum_{k=0}^{p} c_k (t-1)^k\\
&= \sum_{k=0}^{p} c_k \sum_{j=0}^{k}\binom{k}{j} t^j (-1)^{k-j}\\
&= \sum_{j=0}^{p} t^j \sum_{k=j}^{p}\binom{k}{j}(-1)^{k-j} c_k
\end{aligned}$$

The last line can be deduced from the following diagram

$$m_{t-1} = \sum_{k=0}^{p} c_k \sum_{j=0}^{k} \binom{k}{j} t^j (-1)^{k-j} =$$

$$\begin{array}{ccccccccc}
c_0\binom{0}{0}t^0(-1)^{0-0} & & & & & & & & \\
c_1\binom{1}{0}t^0(-1)^{1-0} & + & c_1\binom{1}{1}t^1(-1)^{1-1} & & & & & & \\
c_2\binom{2}{0}t^0(-1)^{2-0} & + & c_2\binom{2}{1}t^1(-1)^{2-1} & + & c_2\binom{2}{2}t^2(-1)^{2-2} & & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
c_p\binom{p}{0}t^0(-1)^{p-0} & + & c_p\binom{p}{1}t^1(-1)^{p-1} & + & c_p\binom{p}{2}t^2(-1)^{p-2} & + \cdots + & & & c_p\binom{p}{p}t^p(-1)^{p-p} \\
= & & = & & = & & \cdots & & = \\
t^0\sum_{k=0}^{p} c_k\binom{k}{0}(-1)^{k-0} & + & t^1\sum_{k=1}^{p} c_k\binom{k}{1}(-1)^{k-1} & + & t^2\sum_{k=2}^{p} c_k\binom{k}{2}(-1)^{k-2} & + \cdots + & & & t^p\sum_{k=p}^{p} c_k\binom{k}{p}(-1)^{k-p}
\end{array}$$

Thus, for $j = p$, the coefficient that accompanies t^p is $\binom{p}{p}(-1)^{p-p}c_p = c_p$. So it follows that

$$\begin{aligned}
\nabla m_t = &\quad \sum_{j=0}^{p} c_j t^j \quad - \quad \sum_{j=0}^{p} t^j \sum_{k=j}^{p} \binom{k}{j}(-1)^{k-j} c_k \\
&= c_p t^p + \sum_{j=0}^{p-1} c_j t^j - c_p t^p - \sum_{j=0}^{p-1} t^j \sum_{k=j}^{p} \binom{k}{j}(-1)^{k-j} c_k \\
&= \sum_{j=0}^{p-1} t^j \cdot \left[c_j - \sum_{k=j}^{p} \binom{k}{j}(-1)^{k-j} c_k \right],
\end{aligned}$$

which is a $(p-1)$-degree polynomial. Now, note that

$$\nabla^n m_t = (I - B)^n (m_t).$$

One can inductively show that $\nabla^n m_t$ has degree $p - n$ for any polynomial m_t of degree p. We proved the base case previously, so assume that $\nabla^{n-1} m_t$ has degree $p - n + 1$. Then, define $d_j = [\nabla^{n-1} m_t]_{t^j}$ as the coefficient that accompanies t^j.

Since we proved that $(I - B)$ reduces by one the degree of any polynomial, it follows that $(I - B)\nabla^{n-1} m_t$ has degree $(p - n + 1) - 1 = p - n$. This can be verified with the following calculation:

$$\begin{aligned}
\nabla^n m_t &= (I-B)(I-B)^{n-1}m_t \\
&= (I-B)\nabla^{n-1}m_t \\
&= \nabla\left(\sum_{k=0}^{p-n+1} d_k t^k\right) \\
&= \sum_{j=0}^{p-n} t^j \cdot \left[d_j - \sum_{k=j}^{p-n+1}\binom{k}{j}(-1)^{k-j}d_k\right].
\end{aligned}$$

Finally, $\nabla^p m_t$ is polynomial of degree 0, and thus, it's a constant function $f_t = K$. Therefore,

$$\begin{aligned}
\nabla^{p+1}m_t &= (I-B)(\nabla^p m_t) \\
&= (I-B)(Kt^0) \\
&= K - BK \\
&= K - K = 0.
\end{aligned}$$

The backwards shift operator evaluated on a constant is the same constant since $f_t = f_{t-1} = K$ for a constant function f_t.

Exercise 1.7.

Let $Z_t, t = 0, \pm 1, \ldots,$ be independent normal random variables each with mean 0 and variance σ^2 and let a, b and c be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.

(a) $X_t = a + bZ_t + cZ_{t-1}$,

(c) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$,

(e) $X_t = Z_t \cos(ct) + Z_{t-1}\sin(ct)$

Note: I assumed by mistake that $\sigma^2 = 1$. However, in all of the equations on the following solution, the σ^2 term can always be factorized without altering the truth value of the propositions.

Solution part (a)

Using the linearity of the expected value and the variance (Z_t's are independent)

$$\mathbf{E}X_t = a + b\mathbf{E}Z_t + c\mathbf{E}Z_{t-1} = a$$

$$\mathbf{Var}(X_t) = b^2\mathbf{Var}(Z_t) + c^2\mathbf{Var}(Z_{t-1}) = b^2 + c^2$$

$$\implies \mathbf{E}|X_t|^2 = \mathbf{Var}(X_t) + (\mathbf{E}X_t)^2 = a^2 + b^2 + c^2 < \infty$$

Now for the autocovariance function,

$$\begin{aligned}\gamma_X(r,s) &= \mathbf{E}[(X_r - a)(X_s - a)] \\ &= \mathbf{E}[(bZ_r + cZ_{r-1})]\mathbf{E}[(bZ_s + cZ_{s-1})] \\ &= b^2\mathbf{E}Z_rZ_s + bc\mathbf{E}Z_rZ_{s-1} + bc\mathbf{E}Z_{r-1}Z_s + c^2\mathbf{E}Z_{r-1}Z_{s-1}.\end{aligned}$$

There are two cases where γ_X is not zero, and that's because $\mathbf{E}Z_rZ_s = 1 \iff r = s$:

$$\begin{aligned}\gamma_X(t,t) &= b^2\mathbf{E}Z_tZ_t + 2bc\mathbf{E}Z_tZ_{t-1} + c^2\mathbf{E}Z_{t-1}Z_{t-1} \\ &= b^2\mathbf{E}Z_t^2 + c^2\mathbf{E}Z_{t-1}^2 \\ &= b^2 + c^2,\end{aligned}$$

and then, by symmetry of γ,

$$\begin{aligned}\gamma_X(t,t+1) = \gamma_X(t,t-1) &= b^2\mathbf{E}Z_tZ_{t-1} + bc\mathbf{E}Z_tZ_{t-2} + bc\mathbf{E}Z_{t-1}Z_{t-1} + c^2\mathbf{E}Z_{t-1}Z_{t-2} \\ &= bc\mathbf{E}Z_{t-1}^2 \\ &= bc.\end{aligned}$$

On the other hand, for $|h| > 1$,

$$t \neq t+h, \quad t \neq t+h-1, \quad t-1 \neq t+h, \quad t-1 \neq t+h-1$$

$$\begin{aligned}\implies \gamma_X(t,t+h) &= b^2\mathbf{E}Z_tZ_{t+h} + bc\mathbf{E}Z_tZ_{t+h-1} + bc\mathbf{E}Z_{t-1}Z_{t+h} + c^2\mathbf{E}Z_{t-1}Z_{t+h-1} \\ &= 0.\end{aligned}$$

Finally, note that γ_X is only dependent on the difference $r-s$, and thus, X_t is a stationary process with autocovariance function

$$\gamma(h) = \begin{cases} b^2+c^2 & h=0, \\ bc & h=1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution part (b)

Again, using the linearity of expectation and variance,

$$\mathbf{E}X_t = \cos(ct)\mathbf{E}Z_1 + \sin(ct)\mathbf{E}Z_2 = 0,$$

$$\begin{aligned} \mathbf{E}|X_t|^2 = \mathbf{Var}(X_t) &= \cos^2(ct)\mathbf{Var}Z_1 + \sin^2(ct)\mathbf{Var}Z_2 \\ &= \cos^2(ct) + \sin^2(ct) \\ &= 1. \end{aligned}$$

For the autocovariance function,

$$\begin{aligned} \gamma_X(r,s) &= \mathbf{E}[(\cos(cr)Z_1 + \sin(cr)Z_2)(\cos(cs)Z_1 + \sin(cs)Z_2)] \\ &= \cos(cr)\cos(cs)\mathbf{E}Z_1^2 + \cos(cr)\sin(cs)\mathbf{E}Z_1Z_2 \\ &+ \sin(cr)\cos(cs)\mathbf{E}Z_2Z_1 + \sin(cr)\sin(cs)\mathbf{E}Z_2^2 \\ &= \cos(cr)\cos(cs) + \sin(cr)\sin(cs) \\ &= \cos(c(r-s)), \end{aligned}$$

which is only dependent of the value $r-s$, and thus, $\{X_t\}$ is stationary. The autocovariance function can then be defined as

$$\gamma(h) = \cos(c(h))$$

Solution part (c)

$$\mathbf{E}X_t = \cos(ct)\mathbf{E}Z_t + \sin(ct)\mathbf{E}Z_t = 0,$$

$$\begin{aligned} \mathbf{E}|X_t|^2 = \mathbf{Var}(X_t) &= \cos^2(ct)\mathbf{Var}Z_t + \sin^2(ct)\mathbf{Var}Z_{t-1} \\ &= \cos^2(ct) + \sin^2(ct) \\ &= 1. \end{aligned}$$

Now, we can prove that $\{X_t\}$ is not stationary by taking the case when $r - s = 1$,

$$\begin{aligned}\gamma_X(t, t-1) &= \mathbf{E}[(\cos(ct)Z_t + \sin(ct)Z_{t-1})(\cos(c(t-1))Z_{t-1} + \sin(c(t-1))Z_{t-2})] \\ &= \cos(ct)\cos(c(t-1))\mathbf{E}Z_t\mathbf{E}Z_{t-1} + \cos(ct)\sin(c(t-1))\mathbf{E}Z_tZ_{t-2} \\ &+ \sin(ct)\cos(c(t-1))\mathbf{E}Z_{t-1}Z_{t-1} + \sin(ct)\sin(c(t-1))\mathbf{E}Z_{t-1}Z_{t-2} \\ &= \sin(ct)\cos(c(t-1))\end{aligned}$$

This case depends on the value of t (unless c is a multiple of π). For example, if $c = \pi/2$, then

$$\gamma_X(1, 0) = \sin(\pi/2)\cos(0) = 1,$$
$$\gamma_X(2, 1) = \sin(\pi)\cos(\pi/2) = 0.$$

Therefore, $\{X_t\}$ is not stationary.

Exercise 1.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function such that $|f(x)| \leq \|x\|^2$ for every $x \in \mathbb{R}^n$. Show that f is Fréchet differentiable in 0.

Solution: By hypothesis, $f(\vec{0}) = 0$

$$\begin{aligned}\lim_{\|x\|\to 0} \frac{f(x) - f(\vec{0})}{\|x\|} &\leq \lim_{\|x\|\to 0} \frac{\|x\|^2}{\|x\|} \\ &= 0.\end{aligned}$$

Thus, the linear function $\ell(x) = 0$ is the Fréchet derivative of f at 0.

Exercise 2.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$f(x, y) = \frac{xy^2}{x^2 + y^4}$$

if $(x, y) \neq (0, 0)$, otherwise 0.

a) Show that f has directional derivatives at the origin for every direction d.

b) Show that f is not Gâteaux differentiable at the origin.

c) Show that f is not continuous at the origin.

Solution Part (a)

The directional derivate of f at a direction $d = (x, y)$ at the origin is defined as

$$\begin{aligned} f'(\vec{0}; d) &= \lim_{h \to 0^+} \frac{f(hd) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{h^3 x y^2}{h^3 x^2 + h^5 y^4} \\ &= \lim_{h \to 0^+} \frac{x y^2}{x^2 + h^2 y^4} \\ \text{(L'Hospital)} &= \frac{y^2}{x} \end{aligned}$$

Solution Part (b)

$$\begin{aligned} f'(\vec{0}; e_1) &= \lim_{h \to 0^+} \frac{f(he_1) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{h \cdot 0^2}{h^3 + 0^4} \\ &= 0, \end{aligned}$$

$$\begin{aligned} f'(\vec{0}; e_2) &= \lim_{h \to 0^+} \frac{f(he_2) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{0 \cdot h^2}{0^2 + h^5} \\ &= 0. \end{aligned}$$

If we take the direction $d = (1, 1)$.

$$f'(\vec{0}; d) = 1 \neq 0 = f'(\vec{0}; e_1) + f'(\vec{0}; e_2).$$

So f' is not a linear function, and thus, f is not Gateaux differentiable.

Solution Part (c)

In order to prove that the function is not continuous, take the curve

$$x(t) = t^2,\ y(t) = t,\ t \in (0,1).$$

Then, if the function is continuous, then the following limit should be unique,

$$\begin{aligned} 0 = \lim_{\|(x,y)\|\to 0} f(x,y) &= \lim_{t\to 0^+} f(x(t), y(t)) \\ &= \lim_{t\to 0^+} \frac{t^2 \cdot t^2}{t^4 + t^4} \\ &= \lim_{t\to 0^+} \frac{t^4}{2t^4} \\ &= \frac{1}{2}. \end{aligned}$$

But that would lead to a contradiction.

Exercise 3.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by

$$f(x,y) = \frac{2y\exp(-x^{-2})}{y^2 + \exp(-2x^{-2})}$$

if $(x,y) \neq (x,y)$ and 0 otherwise. Show that f is Gâteaux differentiable at the origin but it's not continuous.

Solution:

Let $d = (x,y)$ be the direction where we are differentiating. We want to show that $f'(\vec{0}, d)$ exists for any $d = xe_1 + ye_2 \in \mathbb{R}^2$ and it's linear

$$\begin{aligned} f'(\vec{0}; d) &= \lim_{h\to 0^+} \frac{f(\vec{0} + hd) - f(\vec{0})}{h} \\ &= \lim_{h\to 0^+} \frac{2hy\exp(-(hx)^{-2})}{h(h^2y^2 + \exp(-2(hx)^{-2}))} \\ &= \lim_{h\to 0^+} \frac{\exp(-(hx)^{-2})}{h^2y + \exp(-(hx)^{-2})} \\ {\scriptstyle\text{(L'Hospital)}} &= 0. \end{aligned}$$

However, note that the function is not continuous. For instance, take the curve

$$x(t) = \sqrt{\frac{-1}{\ln t}},\ y(t) = t,\ t \in (0,1)$$

Note that if we assume that f is continuous, then the following limit should be unique

$$\begin{aligned} 0 = \lim_{\|(x,y)\| \to 0} f(x,y) &= \lim_{t \to 0^+} f(x(t), y(t)) \\ &= \lim_{t \to 0^+} \frac{2te^{\ln t}}{t^2 + e^{2 \ln t}} \\ &= \lim_{t \to 0^+} \frac{2t^2}{2t^2} \\ &= 1. \end{aligned}$$

But this would lead to a contradiction.

Exercise 4.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by

$$f(x,y) = \frac{x^3 y}{x^4 + y^2}$$

if $(x,y) \neq (x,y)$ and 0 otherwise. Show that f is Gâteaux differentiable at the origin but it's not Fréchet differentiable.

Solution:

The directional derivative is:

$$\begin{aligned} f'(\vec{0}; d) &= \lim_{h \to 0^+} \frac{f(hd) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{h^4}{h^5 + h^3} \\ &= \lim_{h \to 0^+} \frac{h}{h^2 + 1} = 0. \end{aligned}$$

And the partial derivatives evaluated at the origin,

$$
\begin{aligned}
f'(\vec{0}; e_1) &= \lim_{h \to 0^+} \frac{f(he_1) - f(\vec{0})}{h} \\
&= \lim_{h \to 0^+} \frac{h^3 \cdot 0}{h^5 + 0^2} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
f'(\vec{0}; e_2) &= \lim_{h \to 0^+} \frac{f(he_2) - f(\vec{0})}{h} \\
&= \lim_{h \to 0^+} \frac{0 \cdot h}{0^4 + h^3} \\
&= 0.
\end{aligned}
$$

Thus, it follows that f' is linear and the Gâteaux derivative exists. On the other hand, in order to prove that f is not Fréchet differentiable at the origin, assume for the sake of contradiction that it is. Then, we know that since the limit is unique,

$$
\lim_{\|v\| \to 0} \frac{f(v) - f(\vec{0}) - \ell(v)}{\|v\|} = 0,
$$

where $\ell(v) = f'(\vec{0}; v) = 0$. However, by taking the curve

$$
x(t) = t, \quad y(t) = t^2.
$$

Then, using the uniform convergence provided by the Fréchet differentiability,

$$
\begin{aligned}
0 = \lim_{\|v\| \to 0} \frac{f(v) - f(\vec{0}) - \ell(v)}{\|v\|} &= \lim_{t \to 0} \frac{f(x(t), y(t)) - f(\vec{0}) - l(x(t), y(t))}{\|(x(t), y(t))\|} \\
&= \lim_{t \to 0} \frac{\frac{t^3 t^2}{t^4 + t^4}}{\sqrt{t^2 + t^4}} \\
&= \lim_{t \to 0} \frac{t^5}{2t^5 \sqrt{1 + t^2}} \\
&= \frac{1}{2} \lim_{t \to 0} \frac{1}{\sqrt{1 + t^2}} \\
&= \frac{1}{2}.
\end{aligned}
$$

Therefore, a contradiction.

Exercise 5.

Calculate the first 2 derivatives of the function $f(X) = \det(X)$ for a $n \times n$ symmetric matrix X.

Solution:

We know that for $f : \mathrm{Sym}^n_{++} \mapsto \mathbb{R},\ f(X) = \ln(\det(X)),\ Df(X) = X^{-1}$. Also, the inner product here is defined as

$$\langle A, B \rangle = \mathrm{tr}(A^* B)$$

Using the chain rule,

$$D(g \circ f) = (Dg \circ f) \circ Df$$

Thus, by defining $h(X) = \det(X) = \exp(\ln(\det(X)))$, we obtain

$$\begin{aligned} Dh(X) &= (D\exp)(\ln\det(X)) \cdot (D\ln\det)(X) \\ &= \exp(\ln\det(X)) \cdot X^{-1} \\ &= e^{f(X)} \cdot Df(X). \end{aligned}$$

For the second derivative, we know that $Hf = D(Df)(X)(Y) = -X^{-1}YX^{-1}$. Therefore, the Taylor approximation is

$$f(Z) = f(X) + \left\langle X^{-1}, (Z - X) \right\rangle - \frac{1}{2} \left\langle X^{-1}(Z - X)X^{-1}, (Z - X) \right\rangle + o(\|Z - X\|^2).$$

With this definition, we can use the chain rule on $D(Dh)$.

$$\begin{aligned} D(Dh)(X)(Y) &= D(e^{f(X)} \cdot Df(X))(Y) \\ &= [Df(X) \cdot e^{f(X)} \cdot Df(X)](Y) + e^{f(X)} \cdot D(Df)(X)(Y) \\ &= e^{f(X)}(X^{-2}Y - X^{-1}YX^{-1}) \end{aligned}$$