Time Series: Homework 1

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Exercise 1.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function such that $|f(x)| \leq \|x\|^2$ for every $x \in \mathbb{R}^n$. Show that f is Fréchet differentiable in 0.

Solution: By hypothesis, $f(\vec{0}) = 0$

$$\begin{aligned}\lim_{\|x\|\to 0} \frac{f(x) - f(\vec{0})}{\|x\|} &\leq \lim_{\|x\|\to 0} \frac{\|x\|^2}{\|x\|} \\ &= 0.\end{aligned}$$

Thus, the linear function $\ell(x) = 0$ is the Fréchet derivative of f at 0.

Exercise 2.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$f(x, y) = \frac{xy^2}{x^2 + y^4}$$

if $(x, y) \neq (0, 0)$, otherwise 0.

a) Show that f has directional derivatives at the origin for every direction d.

b) Show that f is not Gâteaux differentiable at the origin.

c) Show that f is not continuous at the origin.

Solution Part (a)

The directional derivate of f at a direction $d = (x, y)$ at the origin is defined as

$$\begin{aligned} f'(\vec{0}; d) &= \lim_{h \to 0^+} \frac{f(hd) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{h^3 x y^2}{h^3 x^2 + h^5 y^4} \\ &= \lim_{h \to 0^+} \frac{x y^2}{x^2 + h^2 y^4} \\ \text{(L'Hospital)} &= \frac{y^2}{x} \end{aligned}$$

Solution Part (b)

$$\begin{aligned} f'(\vec{0}; e_1) &= \lim_{h \to 0^+} \frac{f(he_1) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{h \cdot 0^2}{h^3 + 0^4} \\ &= 0, \end{aligned}$$

$$\begin{aligned} f'(\vec{0}; e_2) &= \lim_{h \to 0^+} \frac{f(he_2) - f(\vec{0})}{h} \\ &= \lim_{h \to 0^+} \frac{0 \cdot h^2}{0^2 + h^5} \\ &= 0. \end{aligned}$$

If we take the direction $d = (1, 1)$.

$$f'(\vec{0}; d) = 1 \neq 0 = f'(\vec{0}; e_1) + f'(\vec{0}; e_2).$$

So f' is not a linear function, and thus, f is not Gateaux differentiable.

Solution Part (c)

In order to prove that the function is not continuous, take the curve

$$x(t) = t^2,\ y(t) = t,\ t \in (0, 1).$$

Then, if the function is continuous, then the following limit should be unique,

$$
\begin{aligned}
0 = \lim_{\|(x,y)\| \to 0} f(x,y) &= \lim_{t \to 0^+} f(x(t), y(t)) \\
&= \lim_{t \to 0^+} \frac{t^2 \cdot t^2}{t^4 + t^4} \\
&= \lim_{t \to 0^+} \frac{t^4}{2t^4} \\
&= \frac{1}{2}.
\end{aligned}
$$

But that would lead to a contradiction.

Exercise 3.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by

$$
f(x,y) = \frac{2y \exp(-x^{-2})}{y^2 + \exp(-2x^{-2})}
$$

if $(x,y) \neq (x,y)$ and 0 otherwise. Show that f is Gâteaux differentiable at the origin but it's not continuous.

Solution:

Let $d = (x,y)$ be the direction where we are differentiating. We want to show that $f'(\vec{0}, d)$ exists for any $d = xe_1 + ye_2 \in \mathbb{R}^2$ and it's linear

$$
\begin{aligned}
f'(\vec{0}; d) &= \lim_{h \to 0^+} \frac{f(\vec{0} + hd) - f(\vec{0})}{h} \\
&= \lim_{h \to 0^+} \frac{2hy \exp(-(hx)^{-2})}{h(h^2y^2 + \exp(-2(hx)^{-2}))} \\
&= \lim_{h \to 0^+} \frac{\exp(-(hx)^{-2})}{h^2 y + \exp(-(hx)^{-2})} \\
{\scriptstyle \text{(L'Hospital)}} &= 0.
\end{aligned}
$$

However, note that the function is not continuous. For instance, take the curve

$$
x(t) = \sqrt{\frac{-1}{\ln t}},\ y(t) = t,\ t \in (0,1)
$$

Note that if we assume that f is continuous, then the following limit should be unique

$$
\begin{aligned}
0 = \lim_{\|(x,y)\|\to 0} f(x,y) &= \lim_{t\to 0^+} f(x(t),y(t)) \\
&= \lim_{t\to 0^+} \frac{2te^{\ln t}}{t^2 + e^{2\ln t}} \\
&= \lim_{t\to 0^+} \frac{2t^2}{2t^2} \\
&= 1.
\end{aligned}
$$

But this would lead to a contradiction.

Exercise 4.

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by

$$f(x,y) = \frac{x^3 y}{x^4 + y^2}$$

if $(x,y) \neq (x,y)$ and 0 otherwise. Show that f is Gâteaux differentiable at the origin but it's not Fréchet differentiable.

Solution:

The directional derivative is:

$$
\begin{aligned}
f'(\vec{0}; d) &= \lim_{h\to 0^+} \frac{f(hd) - f(\vec{0})}{h} \\
&= \lim_{h\to 0^+} \frac{h^4}{h^5 + h^3} \\
&= \lim_{h\to 0^+} \frac{h}{h^2 + 1} = 0.
\end{aligned}
$$

And the partial derivatives evaluated at the origin,

$$
\begin{aligned}
f'(\vec{0}; e_1) &= \lim_{h\to 0^+} \frac{f(he_1) - f(\vec{0})}{h} \\
&= \lim_{h\to 0^+} \frac{h^3 \cdot 0}{h^5 + 0^2} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
f'(\vec{0}; e_2) &= \lim_{h \to 0^+} \frac{f(he_2) - f(\vec{0})}{h} \\
&= \lim_{h \to 0^+} \frac{0 \cdot h}{0^4 + h^3} \\
&= 0.
\end{aligned}
$$

Thus, it follows that f' is linear and the Gâteaux derivative exists. On the other hand, in order to prove that f is not Fréchet differentiable at the origin, assume for the sake of contradiction that it is. Then, we know that since the limit is unique,

$$
\lim_{\|v\| \to 0} \frac{f(v) - f(\vec{0}) - \ell(v)}{\|v\|} = 0,
$$

where $\ell(v) = f'(\vec{0}; v) = 0$. However, by taking the curve

$$
x(t) = t, \quad y(t) = t^2.
$$

Then, using the uniform convergence provided by the Fréchet differentiability,

$$
\begin{aligned}
0 = \lim_{\|v\| \to 0} \frac{f(v) - f(\vec{0}) - \ell(v)}{\|v\|} &= \lim_{t \to 0} \frac{f(x(t), y(t)) - f(\vec{0}) - l(x(t), y(t))}{\|(x(t), y(t))\|} \\
&= \lim_{t \to 0} \frac{\frac{t^3 t^2}{t^4 + t^4}}{\sqrt{t^2 + t^4}} \\
&= \lim_{t \to 0} \frac{t^5}{2t^5 \sqrt{1 + t^2}} \\
&= \frac{1}{2} \lim_{t \to 0} \frac{1}{\sqrt{1 + t^2}} \\
&= \frac{1}{2}.
\end{aligned}
$$

Therefore, a contradiction.

Exercise 5.

Calculate the first 2 derivatives of the function $f(X) = \det(X)$ for a $n \times n$ symmetric matrix X.

Solution:

We know that for $f : \mathrm{Sym}^n_{++} \mapsto \mathbb{R}$, $f(X) = \ln(\det(X))$, $Df(X) = X^{-1}$. Also, the inner product here is defined as

$$
\langle A, B \rangle = \mathrm{tr}(A^* B)
$$

Using the chain rule,

$$D(g \circ f) = (Dg \circ f) \circ Df$$

Thus, by defining $h(X) = \det(X) = \exp(\ln(\det(X)))$, we obtain

$$\begin{aligned} Dh(X) &= (D\exp)(\ln\det(X)) \cdot (D\ln\det)(X) \\ &= \exp(\ln\det(X)) \cdot X^{-1} \\ &= e^{f(X)} \cdot Df(X). \end{aligned}$$

For the second derivative, we know that $Hf = D(Df)(X)(Y) = -X^{-1}YX^{-1}$. Therefore, the Taylor approximation is

$$f(Z) = f(X) + \left\langle X^{-1}, (Z - X)\right\rangle - \frac{1}{2}\left\langle X^{-1}(Z - X)X^{-1}, (Z - X)\right\rangle + o(\|Z - X\|^2).$$

With this definition, we can use the chain rule on $D(Dh)$.

$$\begin{aligned} D(Dh)(X)(Y) &= D(e^{f(X)} \cdot Df(X))(Y) \\ &= [Df(X) \cdot e^{f(X)} \cdot Df(X)](Y) + e^{f(X)} \cdot D(Df)(X)(Y) \\ &= e^{f(X)}(X^{-2}Y - X^{-1}YX^{-1}) \end{aligned}$$