Time Series: Homework 1

Martín Prado

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Universidad de los Andes – Bogotá Colombia

Exercise 1.1.

Suppose that $X_t = Z_t + \theta Z_{t-1},\ t = 1, 2, \ldots,$ where Z_0, Z_1, Z_2 are independent random variables, each with moment generating function $\mathbf{E}\exp(\lambda Z_i) = m(\lambda)$.

(a) Express the joint moment generating function $\mathbf{E}\exp\left(\sum_{i=1}^{n} \lambda_i X_i\right)$ in terms of the function $m(\cdot)$.

(b) Deduce from (a) that $\{X_t\}$ is strictly stationary.

Solution part (a)

Since $\{Z_t\}$ are independent, for $X_t = Z_t + \theta Z_{t-1}$, the moment generating function:

$$
\begin{aligned}
\mathbf{E}\exp(\lambda X_t) &= \mathbf{E}\exp(\lambda(Z_t + \theta Z_{t-1})) \\
&= \mathbf{E}\exp(\lambda Z_t) \cdot \mathbf{E}\exp(\lambda\theta \mathbb{Z}_{t-1}) \\
&= m(\lambda) \cdot m(\theta\lambda)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i X_i &= \sum_{i=1}^{n} \lambda_i Z_i + \sum_{i=1}^{n} \lambda_i \theta Z_{i-1} \\
&= \sum_{i=1}^{n} \lambda_i Z_i + \sum_{i=0}^{n-1} \lambda_{i+1} \theta Z_i \\
&= \lambda_n Z_n + \left[\sum_{i=1}^{n-1} (\lambda_i + \theta\lambda_{i+1}) Z_i\right] + \lambda_1 \theta Z_0.
\end{aligned}
$$

Therefore, using a similar argument as before

$$\mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right) = m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)$$

Solution part (b)

Let $(X_1,\dots,X_n)'$ be a random vector in $\mathbb{R}^k$. The moment generating function of this vector is defined as follows,

$$M_{X_{1:n}}(\lambda_{1:n}) = \mathbf{E}\exp(\langle\lambda_{1:n}, X_{1:n}\rangle) = \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right),\ \lambda_{1:n}\in\mathbb{R}^n.$$

So, we know from the previous part that

$$\begin{aligned}
M_{X_{1:n}}(\lambda_{1:n}) &= m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)\\
&= \mathbf{E}\exp(\lambda_n Z_{n+h}) + \left[\prod_{i=1}^{n-1}\mathbf{E}\exp((\lambda_i+\theta\lambda_{i+1})Z_{i+h})\right] + \mathbf{E}\exp(\lambda_1\theta Z_h)\\
&= \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_{i+h}\right)\\
&= M_{X_{1+h:n+h}}(\lambda_{1:n})
\end{aligned}$$

Since the moment generating function of both $(X_1,\dots,X_n)'$ and $(X_{1+h},\dots,X_{n+h})'$ coincide, they have the same joint distribution. Thus, $\{X_t\}$ is strict stationary.

Exercise 1.4.

If $m_t = \sum_{k=0}^{p} c_k t^k, t = 0, \pm 1, \dots$, show that ∇m_t is a polynomial of degree $(p-1)$ in t and hence that $\nabla^{p+1} m_t = 0$.

Solution:

$$\begin{aligned}
m_{t-1} &= \sum_{k=0}^{p} c_k (t-1)^k\\
&= \sum_{k=0}^{p} c_k \sum_{j=0}^{k}\binom{k}{j} t^j (-1)^{k-j}\\
&= \sum_{j=0}^{p} t^j \sum_{k=j}^{p}\binom{k}{j}(-1)^{k-j} c_k
\end{aligned}$$

The last line can be deduced from the following diagram

$$m_{t-1} = \sum_{k=0}^{p} c_k \sum_{j=0}^{k} \binom{k}{j} t^j (-1)^{k-j} =$$

$$\begin{array}{ccccccccc}
c_0\binom{0}{0}t^0(-1)^{0-0} & & & & & & & & \\
c_1\binom{1}{0}t^0(-1)^{1-0} & + & c_1\binom{1}{1}t^1(-1)^{1-1} & & & & & & \\
c_2\binom{2}{0}t^0(-1)^{2-0} & + & c_2\binom{2}{1}t^1(-1)^{2-1} & + & c_2\binom{2}{2}t^2(-1)^{2-2} & & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
c_p\binom{p}{0}t^0(-1)^{p-0} & + & c_p\binom{p}{1}t^1(-1)^{p-1} & + & c_p\binom{p}{2}t^2(-1)^{p-2} & + & \cdots & + & c_p\binom{p}{p}t^p(-1)^{p-p} \\
= & & = & & = & & \cdots & & = \\
t^0\sum_{k=0}^{p} c_k\binom{k}{0}(-1)^{k-0} & + & t^1\sum_{k=1}^{p} c_k\binom{k}{1}(-1)^{k-1} & + & t^2\sum_{k=2}^{p} c_k\binom{k}{2}(-1)^{k-2} & + & \cdots & + & t^p\sum_{k=p}^{p} c_k\binom{k}{p}(-1)^{k-p}
\end{array}$$

Thus, for $j = p$, the coefficient that accompanies t^p is $\binom{p}{p}(-1)^{p-p}c_p = c_p$. So it follows that

$$\begin{aligned}
\nabla m_t = & \quad \sum_{j=0}^{p} c_j t^j \quad - \quad \sum_{j=0}^{p} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \\
= & \; c_p t^p + \sum_{j=0}^{p-1} c_j t^j \; - \; c_p t^p - \sum_{j=0}^{p-1} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \\
= & \sum_{j=0}^{p-1} t^j \cdot \left[c_j - \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \right],
\end{aligned}$$

which is a $(p-1)$-degree polynomial. Now, note that

$$\nabla^n m_t = (I - B)^n (m_t).$$

One can inductively that $\nabla^n m_t$ has degree $p - n$ for any polynomial m_t of degree p. We proved the base case previously, so assume that $\nabla^{n-1} m_t$ has degree $p - n + 1$. Then, define $d_j = [\nabla^{n-1} m_t]_{t^j}$ as the coefficient that accompanies t^j.

Since we proved that $(I - B)$ reduces by one the degree of any polynomial, it follows that $(I - B)\nabla^{n-1} m_t$ has degree $(p - n + 1) - 1 = p - n$. This can be verified with the following calculation:

$$\begin{aligned}
\nabla^n m_t &= (I-B)(I-B)^{n-1} m_t \\
&= (I-B)\nabla^{n-1} m_t \\
&= \nabla \left(\sum_{k=0}^{p-n+1} d_k t^k \right) \\
&= \sum_{j=0}^{p-n} t^j \cdot \left[d_j - \sum_{k=j}^{p-n+1} \binom{k}{j} (-1)^{k-j} d_k \right].
\end{aligned}$$

Finally, $\nabla^p m_t$ is polynomial of degree 0, and thus, it's a constant function $f_t = K$. Therefore,

$$\begin{aligned}
\nabla^{p+1} m_t &= (I-B)(\nabla^p m_t) \\
&= (I-B)(Kt^0) \\
&= K - BK \\
&= K - K = 0.
\end{aligned}$$

The backwards shift operator evaluated on a constant is the same constant since $f_t = f_{t-1} = K$ for a constant function f_t.

Exercise 1.7.

Let $Z_t, t = 0, \pm 1, \ldots,$ be independent normal random variables each with mean 0 and variance σ^2 and let a, b and c be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.

(a) $X_t = a + bZ_t + cZ_{t-1}$,

(c) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$,

(e) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

Solution part (a)