Convex Optimization: Homework 1

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Exercise 1.

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function such that $|f(x)| \leq \|x\|^2$ for every x.

Exercise 2.

(a) Let $u(x, y) = x^3 - 3xy^2$. Find all the entire functions f such that $u = \mathrm{Re}(f)$.

(b) Let $v(x, y) = x^2 + y^2$. Find all the entire functions f such that $v = \mathrm{Im}(f)$.

(c) Let $U \subseteq \mathbb{C}$ be a region and let $f, g : U \to \mathbb{C}$ be holomorphic functions such that $f(U) \subset \mathbb{R}$ and $g(U) \subset \mathbb{S}^1 := \{z \in \mathbb{C} \ : \ |z| = 1\}$. Prove that f and g are constant.

Exercise 3.

(a) $\exp(z + w) = \exp(z)\exp(w)$.

(b) $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.

(c) $|\exp(z)| = 1$ if and only if $z \in i\mathbb{R}$.

(d) $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathbb{C}$.

(e) $\cos(z + 2\pi) = \cos(z)$ and $\sin(z + 2\pi) = \sin(z)$ for all $z \in \mathbb{C}$.

(f) $\cos(z) = 0$ or $\sin(z) = 0 \implies z \in \mathbb{R}$.

(g) For every $x \in \mathbb{R}$, $\lim_{t\to\pm\infty} |\cos(x+it)| = \infty$ and $\lim_{t\to\pm\infty} |\sin(x+it)| = \infty$. The limit is uniform in x.

Exercise 4.

Prove that

(a) $\sum_{n=1}^{\infty} nz^n$ does not converge to any point for $z \in \mathbb{S}^1$.

(b) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges to every point for $z \in \mathbb{S}^1$

(c) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges to every point for $z \in \mathbb{S}^1$, except for 1.

Exercise 5.

A subset $S \subset \mathbb{N}$ is in *arithmetic progression* if there exists $a, d \in \mathbb{N}$ such that

$$S = \{a + nd \ : \ n \in \mathbb{N}_0\}.$$

The number d is called the difference of the progression. Prove that $\mathbb{N}$ cannot be partitioned in a finite number greater than 1 of arithmetic progressions with different differences.