# Time Series: Homework 1

Martín Prado

August 20, 2024
Universidad de los Andes – Bogotá Colombia

## Exercise 1.1.

Suppose that $X_t = Z_t + \theta Z_{t-1},\ t = 1, 2, \ldots,$ where $Z_0, Z_1, Z_2$ are independent random variables, each with moment generating function $\mathbf{E}\exp(\lambda Z_i) = m(\lambda)$.

(a) Express the joint moment generating function $\mathbf{E}\exp\left(\sum_{i=1}^{n} \lambda_i X_i\right)$ in terms of the function $m(\cdot)$.

(b) Deduce from (a) that $\{X_t\}$ is strictly stationary.

### Solution part (a)

Since $\{Z_t\}$ are independent, for $X_t = Z_t + \theta Z_{t-1}$, the moment generating function:

$$\begin{aligned}
\mathbf{E}\exp(\lambda X_t) &= \mathbf{E}\exp(\lambda(Z_t + \theta Z_{t-1})) \\
&= \mathbf{E}\exp(\lambda Z_t) \cdot \mathbf{E}\exp(\lambda\theta \mathbb{Z}_{t-1}) \\
&= m(\lambda) \cdot m(\theta\lambda)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i X_i &= \sum_{i=1}^{n} \lambda_i Z_i + \sum_{i=1}^{n} \lambda_i \theta Z_{i-1} \\
&= \sum_{i=1}^{n} \lambda_i Z_i + \sum_{i=0}^{n-1} \lambda_{i+1} \theta Z_i \\
&= \lambda_n Z_n + \left[\sum_{i=1}^{n-1} (\lambda_i + \theta\lambda_{i+1}) Z_i\right] + \lambda_1 \theta Z_0.
\end{aligned}$$

Therefore, using a similar argument as before

$$\mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right) = m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)$$

### Solution part (b)

Let $(X_1,\ldots,X_n)'$ be a random vector in $\mathbb{R}^k$. The moment generating function of this vector is defined as follows,

$$M_{X_{1:n}}(\lambda_{1:n}) = \mathbf{E}\exp(\langle\lambda_{1:n}, X_{1:n}\rangle) = \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right),\ \lambda_{1:n}\in\mathbb{R}^n.$$

So, we know from the previous part that

$$\begin{aligned}
M_{X_{1:n}}(\lambda_{1:n}) &= m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)\\
&= \mathbf{E}\exp(\lambda_n Z_{n+h}) + \left[\prod_{i=1}^{n-1}\mathbf{E}\exp((\lambda_i+\theta\lambda_{i+1})Z_{i+h})\right] + \mathbf{E}\exp(\lambda_1\theta Z_h)\\
&= \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_{i+h}\right)\\
&= M_{X_{1+h:n+h}}(\lambda_{1:n})
\end{aligned}$$

Since the moment generating function of both $(X_1,\ldots,X_n)'$ and $(X_{1+h},\ldots,X_{n+h})'$ coincide, they have the same joint distribution. Thus, $\{X_t\}$ is strictly stationary.

## Exercise 1.4.

If $m_t = \sum_{k=0}^{p} c_k t^k, t = 0, \pm 1, \ldots,$ show that $\nabla m_t$ is a polynomial of degree $(p-1)$ in $t$ and hence that $\nabla^{p+1} m_t = 0$.

**Solution:**

$$\begin{aligned}
m_{t-1} &= \sum_{k=0}^{p} c_k (t-1)^k\\
&= \sum_{k=0}^{p} c_k \sum_{j=0}^{k}\binom{k}{j} t^j (-1)^{k-j}\\
&= \sum_{j=0}^{p} t^j \sum_{k=j}^{p}\binom{k}{j}(-1)^{k-j} c_k
\end{aligned}$$

The last line can be deduced from the following diagram

$$m_{t-1} = \sum_{k=0}^{p} c_k \sum_{j=0}^{k} \binom{k}{j} t^j (-1)^{k-j} =$$

$$\begin{array}{ccccccccc}
c_0 \binom{0}{0} t^0 (-1)^{0-0} & & & & & & & & \\
c_1 \binom{1}{0} t^0 (-1)^{1-0} & + & c_1 \binom{1}{1} t^1 (-1)^{1-1} & & & & & & \\
c_2 \binom{2}{0} t^0 (-1)^{2-0} & + & c_2 \binom{2}{1} t^1 (-1)^{2-1} & + & c_2 \binom{2}{2} t^2 (-1)^{2-2} & & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
c_p \binom{p}{0} t^0 (-1)^{p-0} & + & c_p \binom{p}{1} t^1 (-1)^{p-1} & + & c_p \binom{p}{2} t^2 (-1)^{p-2} & + & \cdots & + & c_p \binom{p}{p} t^p (-1)^{p-p} \\
= & & = & & = & & \cdots & & = \\
t^0 \sum_{k=0}^{p} c_k \binom{k}{0} (-1)^{k-0} & + & t^1 \sum_{k=1}^{p} c_k \binom{k}{1} (-1)^{k-1} & + & t^2 \sum_{k=2}^{p} c_k \binom{k}{2} (-1)^{k-2} & + & \cdots & + & t^p \sum_{k=p}^{p} c_k \binom{k}{p} (-1)^{k-p}
\end{array}$$

Thus, for $j = p$, the coefficient that accompanies $t^p$ is $\binom{p}{p}(-1)^{p-p} c_p = c_p$. So it follows that

$$\begin{aligned}
\nabla m_t = & \sum_{j=0}^{p} c_j t^j - \sum_{j=0}^{p} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \\
= & \, c_p t^p + \sum_{j=0}^{p-1} c_j t^j - c_p t^p - \sum_{j=0}^{p-1} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \\
= & \sum_{j=0}^{p-1} t^j \cdot \left[ c_j - \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \right],
\end{aligned}$$

which is a $(p-1)$-degree polynomial. Now, note that

$$\nabla^n m_t = (I - B)^n (m_t).$$

One can inductively show that $\nabla^n m_t$ has degree $p - n$ for any polynomial $m_t$ of degree $p$. We proved the base case previously, so assume that $\nabla^{n-1} m_t$ has degree $p - n + 1$. Then, define $d_j = [\nabla^{n-1} m_t]_{t^j}$ as the coefficient that accompanies $t^j$.

Since we proved that $(I - B)$ reduces by one the degree of any polynomial, it follows that $(I - B)\nabla^{n-1} m_t$ has degree $(p - n + 1) - 1 = p - n$. This can be verified with the following calculation:

$$
\begin{aligned}
\nabla^n m_t &= (I-B)(I-B)^{n-1} m_t \\
&= (I-B)\nabla^{n-1} m_t \\
&= \nabla\left(\sum_{k=0}^{p-n+1} d_k t^k\right) \\
&= \sum_{j=0}^{p-n} t^j \cdot \left[d_j - \sum_{k=j}^{p-n+1} \binom{k}{j}(-1)^{k-j} d_k\right].
\end{aligned}
$$

Finally, $\nabla^p m_t$ is polynomial of degree 0, and thus, it's a constant function $f_t = K$. Therefore,

$$
\begin{aligned}
\nabla^{p+1} m_t &= (I-B)(\nabla^p m_t) \\
&= (I-B)(Kt^0) \\
&= K - BK \\
&= K - K = 0.
\end{aligned}
$$

The backwards shift operator evaluated on a constant is the same constant since $f_t = f_{t-1} = K$ for a constant function $f_t$.

## Exercise 1.7.

Let $Z_t, t = 0, \pm 1, \ldots,$ be independent normal random variables each with mean 0 and variance $\sigma^2$ and let $a, b$ and $c$ be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.

(a) $X_t = a + bZ_t + cZ_{t-1}$,

(c) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$,

(e) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

**Note:** I assumed by mistake that $\sigma^2 = 1$. However, in all of the equations on the following solution, the $\sigma^2$ term can always be factorized without altering the truth value of the propositions.

## Solution part (a)

Using the linearity of the expected value and the variance ($Z_t$'s are independent)

$$\mathbf{E}X_t = a + b\mathbf{E}Z_t + c\mathbf{E}Z_{t-1} = a$$

$$\mathbf{Var}(X_t) = b^2\mathbf{Var}(Z_t) + c^2\mathbf{Var}(Z_{t-1}) = b^2 + c^2$$

$$\implies \mathbf{E}|X_t|^2 = \mathbf{Var}(X_t) + (\mathbf{E}X_t)^2 = a^2 + b^2 + c^2 < \infty$$

Now for the autocovariance function,

$$\begin{aligned}
\gamma_X(r,s) &= \mathbf{E}[(X_r - a)(X_s - a)] \\
&= \mathbf{E}[(bZ_r + cZ_{r-1})]\mathbf{E}[(bZ_s + cZ_{s-1})] \\
&= b^2\mathbf{E}Z_rZ_s + bc\mathbf{E}Z_rZ_{s-1} + bc\mathbf{E}Z_{r-1}Z_s + c^2\mathbf{E}Z_{r-1}Z_{s-1}.
\end{aligned}$$

There are two cases where $\gamma_X$ is not zero, and that's because $\mathbf{E}Z_rZ_s = 1 \iff r = s$:

$$\begin{aligned}
\gamma_X(t,t) &= b^2\mathbf{E}Z_tZ_t + 2bc\mathbf{E}Z_tZ_{t-1} + c^2\mathbf{E}Z_{t-1}Z_{t-1} \\
&= b^2\mathbf{E}Z_t^2 + c^2\mathbf{E}Z_{t-1}^2 \\
&= b^2 + c^2,
\end{aligned}$$

and then, by symmetry of $\gamma$,

$$\begin{aligned}
\gamma_X(t,t+1) = \gamma_X(t,t-1) &= b^2\mathbf{E}Z_tZ_{t-1} + bc\mathbf{E}Z_tZ_{t-2} + bc\mathbf{E}Z_{t-1}Z_{t-1} + c^2\mathbf{E}Z_{t-1}Z_{t-2} \\
&= bc\mathbf{E}Z_{t-1}^2 \\
&= bc.
\end{aligned}$$

On the other hand, for $|h| > 1$,

$$t \neq t+h, \quad t \neq t+h-1, \quad t-1 \neq t+h, \quad t-1 \neq t+h-1$$

$$\begin{aligned}
\implies \gamma_X(t,t+h) &= b^2\mathbf{E}Z_tZ_{t+h} + bc\mathbf{E}Z_tZ_{t+h-1} + bc\mathbf{E}Z_{t-1}Z_{t+h} + c^2\mathbf{E}Z_{t-1}Z_{t+h-1} \\
&= 0.
\end{aligned}$$

Finally, note that $\gamma_X$ is only dependent on the difference $r-s$, and thus, $X_t$ is a stationary process with autocovariance function

$$\gamma(h) = \begin{cases} b^2 + c^2 & h = 0, \\ bc & h = 1, \\ 0 & \text{otherwise.} \end{cases}$$

## Solution part (b)

Again, using the linearity of expectation and variance,

$$\mathbf{E}X_t = \cos(ct)\mathbf{E}Z_1 + \sin(ct)\mathbf{E}Z_2 = 0,$$

$$\begin{aligned} \mathbf{E}|X_t|^2 = \mathbf{Var}(X_t) &= \cos^2(ct)\mathbf{Var}Z_1 + \sin^2(ct)\mathbf{Var}Z_2 \\ &= \cos^2(ct) + \sin^2(ct) \\ &= 1. \end{aligned}$$

For the autocovariance function,

$$\begin{aligned} \gamma_X(r,s) &= \mathbf{E}[(\cos(cr)Z_1 + \sin(cr)Z_2)(\cos(cs)Z_1 + \sin(cs)Z_2)] \\ &= \cos(cr)\cos(cs)\mathbf{E}Z_1^2 + \cos(cr)\sin(cs)\mathbf{E}Z_1Z_2 \\ &+ \sin(cr)\cos(cs)\mathbf{E}Z_2Z_1 + \sin(cr)\sin(cs)\mathbf{E}Z_2^2 \\ &= \cos(cr)\cos(cs) + \sin(cr)\sin(cs) \\ &= \cos(c(r-s)), \end{aligned}$$

which is only dependent of the value $r-s$, and thus, $\{X_t\}$ is stationary. The autocovariance function can then be defined as

$$\gamma(h) = \cos(c(h))$$

## Solution part (c)

$$\mathbf{E}X_t = \cos(ct)\mathbf{E}Z_t + \sin(ct)\mathbf{E}Z_t = 0,$$

$$\begin{aligned} \mathbf{E}|X_t|^2 = \mathbf{Var}(X_t) &= \cos^2(ct)\mathbf{Var}Z_t + \sin^2(ct)\mathbf{Var}Z_{t-1} \\ &= \cos^2(ct) + \sin^2(ct) \\ &= 1. \end{aligned}$$

Now, we can prove that $\{X_t\}$ is not stationary by taking the case when $r - s = 1$,

$$\begin{aligned}
\gamma_X(t, t-1) &= \mathbf{E}[(\cos(ct)Z_t + \sin(ct)Z_{t-1})(\cos(c(t-1))Z_{t-1} + \sin(c(t-1))Z_{t-2})] \\
&= \cos(ct)\cos(c(t-1))\mathbf{E}Z_t\mathbf{E}Z_{t-1} + \cos(ct)\sin(c(t-1))\mathbf{E}Z_tZ_{t-2} \\
&+ \sin(ct)\cos(c(t-1))\mathbf{E}Z_{t-1}Z_{t-1} + \sin(ct)\sin(c(t-1))\mathbf{E}Z_{t-1}Z_{t-2} \\
&= \sin(ct)\cos(c(t-1))
\end{aligned}$$

This case depends on the value of $t$ (unless $c$ is a multiple of $\pi$). For example, if $c = \pi/2$, then

$$\gamma_X(1, 0) = \sin(\pi/2)\cos(0) = 1,$$
$$\gamma_X(2, 1) = \sin(\pi)\cos(\pi/2) = 0.$$

Therefore, $\{X_t\}$ is not stationary.