Stochastic Processes: Homework 0

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Exercise 1

Consider a sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$ with $\mathbf{E}\,X_i = 0$ and $\mathbf{Var}\,X_i = 1$ for every $i \in \mathbb{N}$.

1. Show with th Law of Large Numbers that,

$$\lim_{n\to\infty} \|X_1, \ldots, X_n\|_2 - \sqrt{n} \to 0$$

   (a) in $\mathbb{P}$,

   (b) a.e.,

   (c) in distribution,

   (d) Show that if $X_i \in L^p$ for some $p > 1$, then it converges in L^q for every $q \in [1 \leq p)$.

2. Infer from the previous results that for

$$\mathrm{Law}(X_1, \ldots, X_n) \approx \mathrm{UNI}(\sqrt{n}\mathbb{S}^{n-1})$$

Solution Part 1

Exercise 2

Show that for every random variable $X \in L^2$,

$$\mathbf{E}\,|X - \mathbf{E}\,X|^2 \leq \mathbf{E}\,|X|^2$$

Exercise 3

Let $f : [0,1] \to \mathbb{R}$ be a continuos function. Show with the Law of Large Numbers that there exists a sequence of polynomial $(P_n)_{n\in\mathbb{N}}$ such that $\deg(P_n) = n$ and,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f(x) - P_n(x)| = 0.$$