Stochastic Processes: Homework 1

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Exercise 1

Show that, for a sequence $(X_n)_{n\in\mathbb{N}}$ of Bernoulli random variables, with $X_n \sim \mathrm{Be}(n, p_n)$ and $p_n \to 0$. If

$$\mathbf{P}\left\{X_n = k\right\} \overset{n\to\infty}{\longrightarrow} \frac{\lambda^k}{k!}e^{-\lambda}$$

then,

$$\lim_{n\to\infty} n \cdot p_n = \lambda$$

Solution

Using what we obtained from the other direction of the proof,

$$\mathbf{P}(X_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k} = \binom{n}{k} \frac{1}{n^k}(n \cdot p_n)^k (1 - \frac{n \cdot p_n}{n})^{n-k}.$$

For $k = 0$, we have

$$\lim_n \mathbf{P}(X_n = 0) = \binom{n}{0}\frac{1}{n^0}(n \cdot p_n)^0 \left(1 - \frac{n \cdot p_n}{n}\right)^n = (1-p_n)^n = e^{-\lambda}.$$

Taking logarithms at both side will leave us with

$$\lim_n n \ln(1-p_n) = -\lambda.$$

Using Taylor's expansion, we can assert that

$$-\ln(1-p_n) = p_n + o(p_n).$$

Finally,

$$-\lim_n n\ln(1-p_n) = \lim_n n(p_n + o(p_n)) = \lambda.$$

Since p_n dominates over $o(p_n)$, we can conclude that

$$\lim_n n \cdot p_n = \lambda.$$

Exercise 2

Prove the following theorem for $\tau = 0$ and $\tau = \infty$.

Theorem 1. For any given X with F_X and $\overline{F}_X(x) = 1 - F_X(x)$. Also let $M_n = X_{n:n}$. For $x \in \mathbb{R}$, if

- $\tau \in [0, \infty]$
- $(u_n)_{n\in\mathbb{N}}$ a non-decreasing sequence,

then the following items are equivalent,

1. $\lim_{n\to\infty} \mathbf{P}(M_n \leq u_n) = e^{-\tau}$
2. $\lim_{n\to\infty} n\overline{F}_X(u_n) = \tau$.

Solution $\tau = 0$

- (2) $\implies$ (1) If $n \cdot \overline{F}(u_n) \to 0$, then $\overline{F}(u_n) = o(1/n)$. Therefore,

$$\begin{aligned}\lim_n \mathbf{P}(M_n \leq u_n) = \lim_n F(u_n)^n = \lim_n \left(1 - \overline{F}(u_n)\right)^n \\ = \lim_n \left(1 - o\left(\tfrac{1}{n}\right)\right)^n\end{aligned}$$

  What this means is that for every $\varepsilon > 0$, we will eventually have that

$$\tfrac{-\varepsilon}{n} \leq -\overline{F}(u_n) \leq \tfrac{\varepsilon}{n}.$$

  In particular, for every $\varepsilon > 0$,

$$e^{-\varepsilon} \leq \lim_n \left(1 - \tfrac{\varepsilon}{n}\right)^n \leq \lim_n \left(1 - \overline{F}(u_n)\right)^n \leq \lim_n \left(1 + \tfrac{\varepsilon}{n}\right)^n = e^{\varepsilon}.$$

Therefore, by making ε go to 0 we would obtain,

$$1 \leq \lim_n \left(1 - \overline{F}(u_n)\right)^n \leq 1.$$

- (1) $\implies$ (2) Now, the hypothesis says that

$$\mathbf{P}(M_n \leq u_n) = \lim_n \left(1 - \overline{F}(u_n)\right)^n = 1.$$

To prove that $\overline{F}(u_n) \to 0$, we use the same argument from the original proof. If $\liminf_n \overline{F}(u_n) = \alpha > 0$, then there exists a subsequence $u_{n_k} \subset u_n$ such that

$$1 = \lim_k \left(1 - \overline{F}(u_{n_k})\right)^{n_k} \leq \lim_k (1 - \alpha)^{n_k} = 0.$$

With that in mind, we take logarithm at both sides to obtain

$$\lim_n n \cdot \ln((1 - \overline{F}(u_n))) = 0.$$

From Taylor's formula, $-\ln(1 - x) = x + o(x)_{x \to 0+}$. Thus,

$$-0 = -\lim_n n \cdot \ln((1 - \overline{F}(u_n))) = \lim_n n\overline{F}(u_n)$$

Solution $\tau = \infty$

- (2) $\implies$ (1) $n \cdot \overline{F}(u_n) \to \infty$ is equivalent to

$$\lim_n \frac{1/\overline{F}(u_n)}{n} = 0.$$

Which by definition means that $\overline{F}(u_n)^{-1} = o(\frac{1}{n})$. This implies that for every $\varepsilon > 0$,

$$\overline{F}(u_n)^{-1} \leq \varepsilon n, \text{ (eventually)}$$

$$\implies -\overline{F}(u_n) \leq -\frac{\varepsilon'}{n}$$

$$\begin{aligned} \lim_n \mathbf{P}(M_n \leq u_n) &= \lim_n F(u_n)^n = \lim_n \left(1 - \overline{F}(u_n)\right)^n \\ &\leq \lim_n (1 - \tfrac{\varepsilon}{n})^n = e^{-\varepsilon}. \end{aligned}$$

By making $\varepsilon \to \infty$, we can conclude that

$$\lim_n \mathbf{P}(M_n \leq u_n) = 0.$$

- (1) $\implies$ (2) Now, note that

$$\lim_n n \cdot \ln((1 - \overline{F}(u_n))) = -\infty.$$

Using Taylor's polynomial like we did in the case $\tau = 0$, will give us that

$$\lim_n n \cdot (\overline{F}(u_n) + o(\overline{F}(u_n))) = \infty.$$

Since, by definition, $\overline{F}(u_n)$ dominates over $o(\overline{F}(u_n))$, we can conclude that

$$\lim_n n \cdot \overline{F}(u_n) = \infty.$$

Exercise 3

For the following theorem, give two examples where they satisfy item 2 and two examples where they don't.

Theorem 2. Let F be a probability distribution and $\overline{F}(x) = 1 - F(x)$. Then, define

$$x_F = F^{-1}(x) = \inf\{t > 0 \mid F(t) = 1\}.$$

If these two conditions are satisfied:

- $x_F \leq \infty$,
- $\tau \in (0, \infty)$,

Then, the following items are equivalent

1. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\lim_n n \cdot \overline{F}(u_n) = \tau.$$

2.

$$\lim_{x \to x_F-} \frac{\overline{F}(x)}{\overline{F}(x-)}.$$

Note that if F is discrete and $x_F = \infty$, then item 2. is equivalent to

$$\lim_{n\to\infty} \frac{f(n)}{\overline{F}(n)} = 0.$$

Solution Good Examples

1.

Solution Bad Examples

1. **Log Distribution:** For $0 < p < 1$, we define:

$$F(k) = 1 + \frac{\beta_k}{\ln(1-p)}, \quad \overline{F}(k) = \frac{\beta_k}{-\ln(1-p)},$$

$$f(k) = \frac{-p}{\ln(1-p)(1-p)},$$

where

$$\beta_k = \beta(p; k+1, 0) = \int_0^p t^k (1-t)^{-1} dt,$$

$$\frac{f(k)}{\overline{F}(k)} = \frac{p}{(1-p)} \cdot \frac{1}{\beta_k}.$$

The series expansion of β_k is the following

$$\beta_k = \frac{p^{k+1}}{k+1} + \frac{p^{k+2}}{k+2} + \frac{p^{k+3}}{k+3} + \cdots = \sum_{i=k+1}^{\infty} \frac{p^i}{i}.$$

This series converges for every $k \in \mathbb{N}$ since $p < 1$ (ratio test). Also, the sequence β_k is decreasing because less positive terms are being summed when k is increased, and thus, $\lim_k \beta_k = 0$. Therefore,

$$\lim_k \frac{p}{(1-p)} \cdot \frac{1}{\beta_k} = \infty.$$

2. **Beta Negative Binomial:** For $\alpha, \beta > 0 \in \mathbb{R}$ and $r \in \mathbb{N}$, define

$$f(k) = \binom{r+k-1}{k} \frac{B(\alpha+r, \beta+k)}{B(\alpha, \beta)},$$

where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Then, by Stirling's approximation we have that

$$f(k) \sim \frac{\Gamma(\alpha+r)}{\Gamma(r)B(\alpha, \beta)} \frac{k^{r-1}}{(\beta+k)^{r+\alpha}}$$

Thus, we can approximate the tail of the distribution with the following integral:

$$C = \frac{\Gamma(\alpha + r)}{\Gamma(r)B(\alpha, \beta)},$$

$$\begin{aligned}
\overline{F}(n) &\sim C \int_n^\infty \frac{x^{r-1}}{(x-\beta)^{r+\alpha}} dx \\
&= -C \frac{x^r \cdot {}_2F_1(1, 1-a; r+1; \frac{x}{b})}{br(x-\beta)^{r+a-1}}
\end{aligned}$$