Stochastic Processes: Homework 5

Martín Prado

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Universidad de los Andes – Bogotá Colombia

Exercise 1

Consider the following transition matrix

$$\Pi = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \end{pmatrix}$$

(a) Build the random dynamical system that parametrizes this Markov Chain.

(b) Why does a unique invariant distribution exist?

(c) Calculate the invariant distribution π and verify it is reversible.

(d) Verify that it is strongly irreducible. Which is the exponent m?

(e) The random walk on $\mathbb{Z}$ with independent increments $\frac{2}{3}\delta_{-1} + \frac{1}{3}\delta_1$ is not strongly irreducible. Explain the difference between this case and the case of Π

(f) Determine the time $\mathbf{E}\left[T_i^r\right]$ for $i = 0, \ldots, 4$.

(g) Calculate the convergence rate

$$\mu\Pi^n \to \pi, \qquad n \to \infty$$

Solution Part (a)

$$f(i, \theta) = (i+1) \bmod 5 \cdot \mathbb{1}_{[0,\frac{1}{3})}(\theta) + (i-1) \bmod 5 \cdot \mathbb{1}_{[\frac{1}{3},1]}(\theta).$$

Solution Part (b)

The ergodic theorem for Markov Chains states that a strongly irreducible Markov chain has only one invariant distribution π. (See item (d))

Solution Part (c)

We must find π that satisfies

$$\pi\Pi = \pi,$$

$$\sum_{i=0}^{4} \pi_i = 1$$

The symmetry of the graph that is spanned by this matrix hints that all the entries of π must be equal to $\frac{1}{5}$. In fact, after calculating the left eigenvectors of Π we find that the only one with eigenvalue 1 is

$$\pi = \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right].$$

Note that it's not reversible, take for instance $i = 0, j = 1$,

$$\frac{1}{5} \cdot \frac{1}{3} = \pi(0) \cdot \Pi_{0,1} \neq \pi(1) \cdot \Pi_{1,0} = \frac{1}{5} \cdot \frac{2}{3}$$

Solution Part (d)

$$\Pi^2 = \begin{bmatrix} \frac{4}{9} & 0 & \frac{1}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & 0 & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & 0 & \frac{4}{9} & 0 & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} & 0 & \frac{4}{9} & 0 \\ 0 & \frac{1}{9} & \frac{4}{9} & 0 & \frac{4}{9} \end{bmatrix},$$

$$\Pi^3 = \begin{bmatrix} 0 & \frac{2}{9} & \frac{8}{27} & \frac{1}{27} & \frac{4}{9} \\ \frac{4}{9} & 0 & \frac{2}{9} & \frac{8}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{4}{9} & 0 & \frac{2}{9} & \frac{8}{27} \\ \frac{8}{27} & \frac{1}{27} & \frac{4}{9} & 0 & \frac{2}{9} \\ \frac{2}{9} & \frac{8}{27} & \frac{1}{27} & \frac{4}{9} & 0 \end{bmatrix}$$

$$\Pi^4 \begin{bmatrix} \frac{8}{27} & \frac{16}{81} & \frac{8}{81} & \frac{32}{81} & \frac{1}{81} \\ \frac{1}{81} & \frac{8}{27} & \frac{16}{81} & \frac{8}{81} & \frac{32}{81} \\ \frac{32}{81} & \frac{1}{81} & \frac{8}{27} & \frac{16}{81} & \frac{8}{81} \\ \frac{8}{81} & \frac{32}{81} & \frac{1}{81} & \frac{8}{27} & \frac{16}{81} \\ \frac{16}{81} & \frac{8}{81} & \frac{32}{81} & \frac{1}{81} & \frac{8}{27} \end{bmatrix}$$

The exponent is $m = 4$.

Solution Part (e)

Let S_n be the sum of n increments of this random walk. Note that if n is odd, then S_n is too and viceversa. Therefore,

$$S_n \equiv n \mod 2,$$

and thus, odd states are not accesible from even n's and viceversa. On the other hand, since $2\mathbb{Z}_5 \simeq \mathbb{Z}_5$, one can access both even and odd states from odd n's and viceversa. From the previous part is also easy to see that for $n = 4$ one can for any $i, j \in S$ from i to j in 4 steps with probability greater than 0.

Solution Part (f)

Theorem 3.83 states that if π is the only invariant distribution, then

$$\pi(i) = \frac{1}{\mathbf{E}\left[T_i^r\right]}$$

Therefore, $\mathbf{E}\left[T_i^r\right] = 5$ for every $i \in S$.

Solution Part (g)

Let X_n^μ be the random variable associated with $\mu\Pi^n$ theorem 3.68 states that for

$$\alpha = \sum_{j \in S} \min_{i \in S} \Pi^m(i,j) = \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} = \frac{5}{81},$$

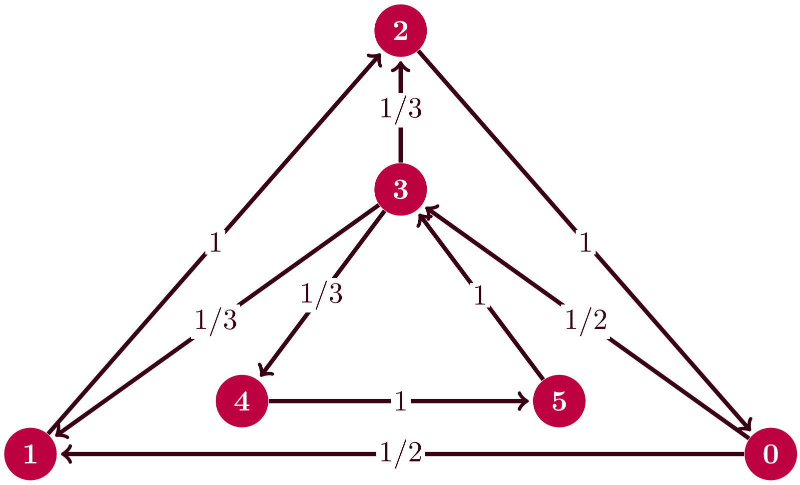
we have

$$\sup_{A \subset S} |P\{X_n^\mu \in A\} - \pi(A)| \leq (1-\alpha)^{\lfloor \frac{n}{m} \rfloor} = \left(\tfrac{76}{81}\right)^{\lfloor \frac{n}{4} \rfloor}.$$

Thus, X_n^μ converges in distribution to π at an exponential rate.

Exercise 2

Consider the following 6 vertex graph and a random walk over these states



(a) Show that this random walk is irreducible

(b) Build the linear system for the arrival times for each state and solve them.

(c) Infer the invariant measure.

(d) If $X_0 = 0$, does the marginal law converges to the invariant measure?

Solution (a)

The transition matrix is

$$\Pi = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

If one follows the path:

$$1 \to 2 \to 0 \to 3 \to 4 \to 5 \to 3 \to 1.$$

It's possible to create a path from i to j in less than 7 steps, for any i, j. Furthermore, the system is also strongly irreducible, because the first m for which $\Pi^m(i,j) > 0$ for every $i, j \in \mathbb{S}$ is $m = 11$:

$$\Pi^{10} = \begin{bmatrix} \frac{1}{3} & \frac{43}{216} & \frac{2}{27} & \frac{1}{3} & \frac{1}{216} & \frac{1}{18} \\ \frac{1}{36} & \frac{5}{18} & \frac{7}{18} & \frac{1}{6} & \frac{5}{36} & 0 \\ \frac{7}{18} & \frac{5}{72} & \frac{1}{3} & \frac{1}{72} & \frac{1}{18} & \frac{5}{36} \\ \frac{13}{108} & \frac{5}{18} & \frac{25}{108} & \frac{5}{18} & \frac{1}{12} & \frac{1}{108} \\ \frac{5}{18} & \frac{1}{9} & \frac{7}{18} & \frac{1}{36} & \frac{1}{12} & \frac{1}{9} \\ \frac{7}{18} & \frac{4}{27} & \frac{13}{108} & \frac{1}{4} & \frac{1}{108} & \frac{1}{12} \end{bmatrix}$$

$$\Pi^{11} = \begin{bmatrix} \frac{2}{27} & \frac{5}{18} & \frac{67}{216} & \frac{2}{9} & \frac{1}{9} & \frac{1}{216} \\ \frac{7}{18} & \frac{5}{72} & \frac{1}{3} & \frac{1}{72} & \frac{1}{18} & \frac{5}{36} \\ \frac{1}{3} & \frac{43}{216} & \frac{2}{27} & \frac{1}{3} & \frac{1}{216} & \frac{1}{18} \\ \frac{25}{108} & \frac{11}{72} & \frac{10}{27} & \frac{5}{72} & \frac{5}{54} & \frac{1}{12} \\ \frac{7}{18} & \frac{4}{27} & \frac{13}{108} & \frac{1}{4} & \frac{1}{108} & \frac{1}{12} \\ \frac{13}{108} & \frac{5}{18} & \frac{25}{108} & \frac{5}{18} & \frac{1}{12} & \frac{1}{108} \end{bmatrix}$$

Solution (b)

For this part, we define a matrix $\Pi^{(j)}$ which is the matrix Π but the j-th column is only 0's. This way, we obtain

$$(\Pi^{(j)}\mu_j)_i = \sum_{l \neq j} \Pi(i,j)\mu_j(l)$$

The linear system we are going to solve is

$$(I - \Pi^{(j)})\mu_j = \mathbb{1}_6.$$

By doing this, we obtain

$$\mu_0 = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 4 \\ 6 \\ 5 \end{bmatrix}, \mu_1 = \begin{bmatrix} \frac{10}{3} \\ \frac{16}{3} \\ \frac{13}{3} \\ \frac{14}{3} \\ \frac{20}{3} \\ \frac{17}{3} \end{bmatrix}, \mu_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 3 \\ 5 \\ 4 \end{bmatrix}, \mu_3 = \begin{bmatrix} 4 \\ 6 \\ 5 \\ \frac{16}{3} \\ 2 \\ 1 \end{bmatrix}, \mu_4 = \begin{bmatrix} 18 \\ 20 \\ 19 \\ 14 \\ 16 \\ 15 \end{bmatrix}, \mu_5 = \begin{bmatrix} 19 \\ 21 \\ 20 \\ 15 \\ 1 \\ 16 \end{bmatrix}.$$

It's clear that in a good result, $\mu_3(4) = 2$ because from 4 you can only go to 3 in exactly 2 steps with probability 1. In a similar fashion,

$$\mu_0(2) = \mu_2(1) = \mu_3(5) = \mu_5(4) = 1,$$

because in those cases there exists an arrow with probability 1 of transitioning from i to j.

Solution (c)

The invariant π is the solution to the system

$$\mu\Pi = \mu, \quad \sum_{i\in\mathbb{S}} \mu_i = 1, \quad \mu_i > 0,\ \forall i \in \mathbb{S}.$$

This solution is

$$\pi = \begin{bmatrix} \frac{1}{4} & \frac{3}{16} & \frac{1}{4} & \frac{3}{16} & \frac{1}{16} & \frac{1}{16} \end{bmatrix}$$

It is a probability measure and it satisfies $\pi\Pi = \pi$.

Solution (d)

According to the ergodic theorem, for irreducible Markov chains, let m such that $\Pi^m(i,j) > 0$ for every $i, j \in \mathbb{S}$ and let

$$\alpha = \sum_j \inf_i \Pi^m(i,j)$$

Then, for every measure $\mu \in \mathcal{P}(\mathbb{S})$

$$\sup_{A\subset\mathbb{S}} |\mu\Pi^n(A) - \pi(A)| \le (1-\alpha)^{\lfloor \frac{n}{m} \rfloor}.$$

Therefore, if $\mu = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mu\Pi^n$ will converge to π at an exponential rat because we proved that the process is strongly irreducible.

Exercise 3

Consider a ramification process with a geometric reproduction distribution with parameter $\frac{1}{2}$.

(a) What are the entries of the rows of the transition matrix

(b) Is it irreducible?

(c) Which are the recurrent states and the transient states?

(d) Build a recursion system for the invariant measure and solve it.

(e) Is this invariant measure reversible?

(f) Using the transition matrix, which is the distribution for the extinction time if we start from the invariant distribution.

Solution (a)

The set of states in this process is the population numbers. Thus, if we are at the state n, then we have n individuals in our population. The descendance of this population is the sum of the descendance of each member in the population, each is distributed with $\sim \text{Geo}(1/2)$ and are independent from each other.

$$S_n = X_1 + \cdots + X_n$$

The sum of n independent random variables with the geometric distribution has the negative binomial distribution

$$\Pi(n,k) = \mathbf{P}\{S_n = k\} = \binom{k+n-1}{k}\left(\frac{1}{2}\right)^{n+k},\ k \in \mathbb{N}.$$

In particular, note that the row corresponding to the state $n = 0$ is

$$\Pi(0,i) = \begin{cases} 1, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution (b)

No, the process is not irreducible because 0 is an absorbing state.

Solution (c)

The only recurrent state is 0 because it's an absorbing state, and thus,

$$p_0 = P\{\exists n,\ X_n = 0 \mid X_0 = 0\} = 1.$$

For the other states, $i \neq 0$

$$p_i = P\{\exists n,\ X_n = i \mid X_0 = i\} < 1$$

because $\mathbf{P}\{S_n = 0\} > 0$, and in every step there's a risk of extinction. Thus, the transient states are the ones different from 0.

Solution (d)

According to the entries of the matrix we calculated on item (a), the recurrence obtained from the system $\mu\Pi = \mu$ is the following

$$\sum_{n \in \mathbb{S}} \mu(n)\Pi(n,k) = \mu(k)$$

$$= \sum_{n \in \mathbb{S}} \binom{n+k-1}{k} \left(\frac{1}{2}\right)^{n+k} \mu(n) = \mu(k).$$

For $k = 0$,

$$\begin{aligned} \mu(0) &= \sum_{n \in \mathbb{S}} \left(\frac{1}{2}\right)^{n} \mu(n) \\ &= \mu(0) + \sum_{\substack{n \in \mathbb{S} \\ n \neq 0}} \left(\frac{1}{2}\right)^{n} \mu(n) \end{aligned}$$

This can only happen if $\mu(0) = 1$ and $\mu(n) = 0$ for $n \neq 0$. Therefore, we define the invariant distribution π as

$$\pi(i) = \begin{cases} 1, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution (e)

No it isn't, for $j \neq 0$ note that

$$\pi(0) \cdot \Pi(0, j) = \Pi(0, j) \neq 0 = \pi(j) \cdot \Pi(j, 0)$$

Solution (f)

The distribution of the extinction time is 1 with probability 1 because we start already extinct according to π.

Exercise 4

Consider the system of the i.i.d. current age of a machine. Using the analytical criterion of recurrence and transience determine a criterion in term of the transition probabilities for when every state is recurrent or transient.

Solution

Assume that $\Pi(n, 0) < 1$ for every $n \in \mathbb{N}$. If there exists N such that, for every $n > N$, $\Pi(n, 0) = 0$. Then $\Pi(n, n+1) = 1$, and thus, in this case every state is transient because there's a path of probability greater than 0 from the state 0 to the state $N+1$ and eventually we would get there with probability 1.

The other case is when we assume that $\Pi(n, 0) > 0$ for infinitely many n. I'm aware that by the analytical criterion there could be condition when this case happens and every state is transient. In fact, it could be that if $\Pi(n, 0) \to 0$ at an exponential rate maybe every state is a transient state. However, I'm not sure how can that be proven (or disproven).

Exercise 5

Consider the inverse Ehrenfest model. Let N be the total number of particles inside 2 compartments and k be the number of particles in compartment 1. The probability that compartment 1 loses a particle is $\frac{N-k}{N}$ and the probability that it gains a particle is $\frac{k}{N}$

(a) Model this system, show the transition matrix.

(b) Is it irreducible?

(c) Starting on a state $i \in \{1, \dots, N-1\}$ write the linear system of the probabilities of absorption on the extremes. Calculate it for $N = 7$.

(d) For $N = 7$ calculate the absorption times on $\{0, 7\}$

Solution (a)

The system is given by the following transition matrix

$$\Pi(i, j) = \begin{cases} \frac{N-i}{N} & j = i - 1 \\ \frac{i}{N} & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for the system to make sense, $\Pi(0, 0) = 1$ and $\Pi(N, N) = 1$. Making $0, N$ absorbing states.

Solution (b)

It is not irreducible because we have 2 absorbing states.

Solution (c)

In general, the system for the absorption probabilities is, for the set of absorbing states A,

$$\mathbf{P}\{Z_{T_A} = l | Z_0 = k\} = g_l(i) = \sum_{j \in \mathbb{S}} \Pi(i,j) g_l(j)$$

which is the solution for $(I - \Pi) \cdot g_l = 0$, or equivalently the vectors in eigenspace associated with the eigenvalue 1. The multiplicity of the eigenvalue 1 is equal to $|A|$. To obtain the solution we're looking for, we must solve the following system.

Let $m = |A|$ and $W = \text{span}\{v_1, \dots, v_m\}$ be solution space for $I - \Pi$. Additionally, let I_k be the k-th row of the Identity matrix. The solution $g_l \in W$ we are looking for, must satisfy the frontier conditions. For $k \in A$,

$$g_l(k) = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

Which is equivalent to solving for $k_2, \dots, k_m \in A \backslash \{l\}$

$$\begin{bmatrix} I_l \\ \hline I_{k_2} \\ \hline \vdots \\ \hline I_{k_m} \end{bmatrix} \cdot g_l = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

This way, we can find $x_1, \dots, x_m$ such

$$g_l = x_1 \cdot v_1 + \cdots + x_m v_m = \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

and satisfy the frontier conditions. That is to solve $x_1, \dots, x_m$ in the system

$$\left(\begin{bmatrix} I_l \\ \hline I_{k_2} \\ \hline \vdots \\ \hline I_{k_m} \end{bmatrix} \cdot \left[\begin{array}{c|c|c} v_1 & \cdots & v_m \end{array}\right] \right) \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$$

Returning to our case, for $N = 7$,

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{6}{7} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{7} & 0 & \frac{2}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{7} & 0 & \frac{3}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{7} & 0 & \frac{4}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{7} & 0 & \frac{5}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 0 & \frac{6}{7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also $A = \{i,\ P(i,i) = 1\} = \{0, 7\}$. If we apply the previous algorithm we can conclude that

$$g_0 = \begin{bmatrix} 1.0 \\ 0.984375 \\ 0.890625 \\ 0.65625 \\ 0.34375 \\ 0.109375 \\ 0.015625 \\ 0 \end{bmatrix}, \qquad g_N = \begin{bmatrix} 0 \\ 0.015625 \\ 0.109375 \\ 0.34375 \\ 0.65625 \\ 0.890625 \\ 0.984375 \\ 1.0 \end{bmatrix}$$

Solution (d)

Now, for the absorption times, let $h_A(k) = \mathbf{E}\left[T_A \mid X_0 = k\right]$. Then, we have the following linear system

$$h_A(k) = 1 + \sum_{j \in \mathbb{S}} \Pi(k,j) \cdot h_A(j), \qquad k \in \mathbb{S} \backslash A$$
$$h_A(k) = 0, \qquad k \in A$$

Which is the solution of the following system

$$M \cdot h_A = b$$

Where the k-th row of the matrix M is

$$M_k = \begin{cases} (I - \Pi)_k, & k \in \mathbb{S} \backslash A \\ I_k, & k \in A \end{cases}$$

and,

$$b_k \begin{cases} 1, & k \in \mathbb{S} \backslash A \\ 0, & k \in A \end{cases}$$

This gives us the solution

$$h_A = \begin{bmatrix} 0 \\ 1.5167 \\ 3.6167 \\ 5.3667 \\ 5.3667 \\ 3.6167 \\ 1.5167 \\ 0 \end{bmatrix}$$