Stochastic Processes: Homework 2

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Exercise 1

Let $f(x) = \frac{1}{2}\sin(x)\mathbb{1}_{[0,\pi]}(x)$.

1. For $n = 4$ **calculate** the first moment of the order statistic $X_{i:n}$ for every $i \in 1, \ldots, 4$.
2. **Sketch a telling drawing** with the original density, the densities $f_{i:4}$ and $\mathbf{E}\left[X_{i:4}\right]$

Solution Part 1

In the first place, for $x \in [0, \pi]$

$$\begin{aligned} F(x) &= \frac{1}{2}\int_{-\infty}^{x} \sin(t)\mathbb{1}(t)dt = \frac{1}{2}\int_{0}^{x} \sin(t)dt \\ &= \cos(x)\Big|_{0}^{x} = \frac{1}{2}(1-\cos(x)) \cdot \mathbb{1}_{[0,\pi]}(x) \end{aligned}$$

Thus,

$$\overline{F}(x) = \frac{1}{2}(1+\cos(x)) \cdot \mathbb{1}_{[0,\pi]}(x),$$

and,

$$\begin{aligned} f_{i:n}(x) &= i \cdot \binom{n}{i} f(x) \cdot F^{i-1}(x) \cdot \overline{F}^{n-i}(x) \\ &= i \cdot \binom{n}{i} \frac{1}{2^n} \cdot \sin(x) \cdot (1-\cos(x))^{i-1} \cdot (1+\cos(x))^{n-i}. \end{aligned}$$

For simplicity, define $C_{i:n} = i\binom{n}{i}2^{-n}$. Now let $n = 4$. In order the calculate the exact expected value formula, we must use the following angle identities,

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x), \quad \cos^2(x) = \frac{1}{2}(1+\cos(2x)),$$

and the result of these integrals

$$\int x\sin(kx)dx = \frac{\sin(kx) - kx\cos(kx)}{k^2}.$$

From now on I'm going to simplify the trigonometric expressions using the TR8 algorithm provided by the package sympy. Also, use the package to integrate and evaluate the final expression. As always, the code is included with this document.

- $\boldsymbol{i = 1:}$

$$C_{1:4}^{-1} \cdot f_{1:4} = \frac{7\sin(x)}{4} + \frac{7\sin(2x)}{4} + \frac{3\sin(3x)}{4} + \frac{\sin(4x)}{8},$$

$$C_{1:4}^{-1} \cdot \mathbf{E}[X_{1:4}] = C_{1:4}^{-1} \cdot \int_0^\pi x f_{1:4} dx.$$

$$= \begin{array}{c} -\frac{7x\cos(x)}{4} - \frac{7x\cos(2x)}{8} - \frac{x\cos(3x)}{4} - \frac{x\cos(4x)}{32} \\ \frac{7\sin(x)}{4} + \frac{7\sin(2x)}{16} + \frac{\sin(3x)}{12} + \frac{\sin(4x)}{128} \end{array}\Bigg|_0^\pi = \frac{35\pi}{32}.$$

- $\boldsymbol{i = 2:}$

$$C_{2:4}^{-1} \cdot f_{2:4} = \frac{3\sin(x)}{4} + \frac{\sin(2x)}{4} - \frac{\sin(3x)}{4} - \frac{\sin(4x)}{8},$$

$$C_{2:4}^{-1} \cdot \mathbf{E}[X_{2:4}] = C_{2:4}^{-1} \cdot \int_0^\pi x f_{2:4} dx.$$

$$= \begin{array}{c} -\frac{3x\cos(x)}{4} - \frac{x\cos(2x)}{8} + \frac{x\cos(3x)}{12} + \frac{x\cos(4x)}{32} \\ \frac{3\sin(x)}{4} + \frac{\sin(2x)}{16} - \frac{\sin(3x)}{36} - \frac{\sin(4x)}{128} \end{array}\Bigg|_0^\pi = \frac{55\pi}{96}.$$

- $\boldsymbol{i = 3:}$

$$C_{3:4}^{-1} \cdot f_{3:4} = \frac{3\sin(x)}{4} - \frac{\sin(2x)}{4} - \frac{\sin(3x)}{4} + \frac{\sin(4x)}{8},$$

$$C_{3:4}^{-1} \cdot \mathbf{E}[X_{3:4}] = C_{3:4}^{-1} \cdot \int_0^\pi x f_{3:4} dx.$$

$$= \begin{array}{c} -\frac{3x\cos(x)}{4} + \frac{x\cos(2x)}{8} + \frac{x\cos(3x)}{12} - \frac{x\cos(4x)}{32} \\ \frac{3\sin(x)}{4} - \frac{\sin(2x)}{16} - \frac{\sin(3x)}{36} + \frac{\sin(4x)}{128} \end{array}\Bigg|_0^\pi = \frac{73\pi}{96}.$$

- $\boldsymbol{i = 4}$ **:**

$$C_{4:4}^{-1} \cdot f_{4:4} = \frac{7\sin(x)}{4} - \frac{7\sin(2x)}{4} + \frac{3\sin(3x)}{4} - \frac{\sin(4x)}{8},$$

$$C_{4:4}^{-1} \cdot \mathbf{E}\left[X_{4:4}\right] = C_{4:4}^{-1} \cdot \int_0^\pi x f_{4:4} dx.$$

$$= \begin{array}{c} \frac{7x\cos(x)}{4} + \frac{7x\cos(2x)}{8} - \frac{x\cos(3x)}{4} + \frac{x\cos(4x)}{32} \\ \frac{7\sin(x)}{4} - \frac{7\sin(2x)}{16} + \frac{\sin(3x)}{12} - \frac{\sin(4x)}{128} \end{array} \Bigg|_0^\pi = \frac{93\pi}{32}.$$

Therefore,

$$\mathbf{E}\left[X_{1:4}\right] = \frac{35\pi}{128}, \quad \mathbf{E}\left[X_{2:4}\right] = \frac{55\pi}{128},$$
$$\mathbf{E}\left[X_{3:4}\right] = \frac{73\pi}{128}, \quad \mathbf{E}\left[X_{4:4}\right] = \frac{93\pi}{128}.$$

Solution Part 2

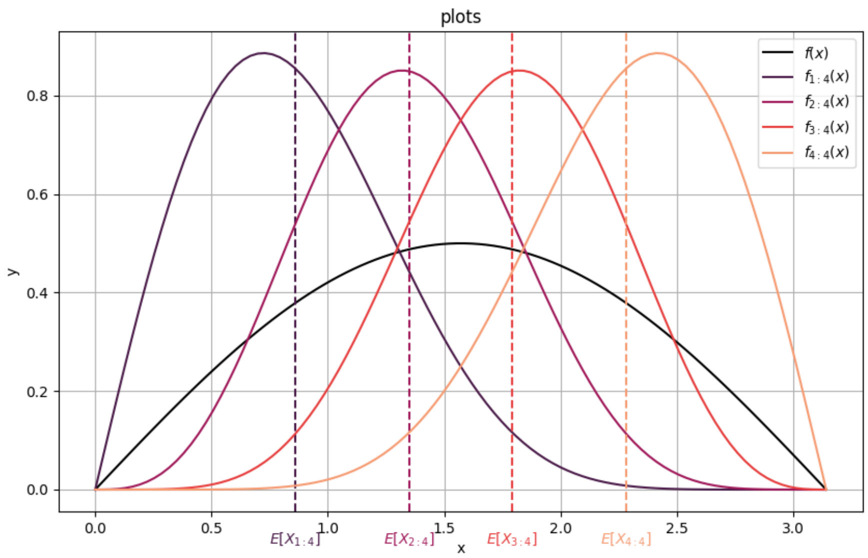

Figure 1: This is a floating figure with an image.

Exercise 2

1. **Formulate** the criterion that characterizes the existence of an extreme distribution (in terms of $\overline{F}$).

2. **Determine and justify** by this criterion if the following distribution has an extreme distribution. The cumulative distribution function is given by

$$F(n) := 1 - \frac{C}{(n+1)^{\ln(n+1)}}, \quad n \in \mathbb{N}.$$

3. In the case that it has a limit distribution, **argue** about which should be the limit distribution.

Solution Part 1

The criterion states that for every distribution F that satisfies the following condition:

$$\lim_{x \to x_F-} \frac{\overline{F}(x)}{\overline{F}(x-)} = 1,$$

then, there exist sequences d_n and c_n such that, for the extreme $M_n = X_{n:n}$,

$$\frac{M_n - d_n}{c_n} \xrightarrow{d} H,$$

for a non-degenerate distribution H. Furthermore, for the discrete case, the condition would be that

$$\lim_{n \to \infty} \frac{\overline{F}(n)}{\overline{F}(n-1)} = 1,$$

which is equivalent to

$$\lim_{n \to \infty} \frac{f(n)}{\overline{F}(n)} = 0.$$

Solution Part 2

$$\overline{F}(n) = \frac{C}{(n+1)^{\ln(n+1)}}.$$

$$
\begin{aligned}
\ln\left(\lim_{n\to\infty}\frac{\overline{F}(n)}{\overline{F}(n-1)}\right) &= \lim_{n\to\infty}\left(\ln\overline{F}(n)-\ln\overline{F}(n-1)\right)\\
&= \lim_{x\to\infty}\ln^2(x+1)-\ln^2(x)\\
&= \lim_{x\to\infty}\left(\ln(x+1)+\ln(x)\right)\cdot\left(\ln(x+1)-\ln(x)\right)\\
&= \lim_{x\to\infty}\frac{\ln\frac{x+1}{x}}{\ln^{-1}(x^2+x)}\\
\text{(L'Hôpital)} &= \lim_{x\to\infty}\frac{-(x^2+x)^{-1}}{-(2x+1)(x^2+x)^{-1}\ln^{-2}(x^2+x)}\\
&= \lim_{x\to\infty}\frac{\ln^2(x^2+x)}{2x+1}\\
\text{(L'Hôpital)} &= \lim_{x\to\infty}\frac{(2x+1)(x^2+x)^{-1}}{2}\\
&= \lim_{x\to\infty}\frac{2x+1}{2(x^2+x)} = 0.
\end{aligned}
$$

This implies that

$$
\lim_{n\to\infty}\frac{\overline{F}(n)}{\overline{F}(n-1)} = \exp\left[\ln\left(\lim_{n\to\infty}\frac{\overline{F}(n)}{\overline{F}(n-1)}\right)\right] = e^0 = 1.
$$

Solution Part 3

It's clear that F is not in the domain of attraction of the Weibull since its extreme isn't bounded. In order to see it cannot be either Fréchet, let's take a look to the continuous case,

$$
\overline{G}(x) = \frac{C}{x^{\ln(x)}}\cdot\mathbb{1}_{[1,\infty)}(x) = e^{\ln^2(x)}\mathbb{1}_{[1,\infty)}(x)
$$

$$
\begin{aligned}
\ln\left(\lim_{x\to\infty}\frac{\overline{G}(\lambda\cdot x)}{\overline{G}(x)}\right) &= \lim_{x\to\infty}\ln^2(\lambda x)-\ln^2(x)\\
&= \lim_{x\to\infty}\left(\ln(\lambda x)+\ln(x)\right)\cdot\left(\ln(\lambda x)-\ln(x)\right)\\
&= \lim_{x\to\infty}\ln(\lambda x^2)\ln(\lambda)\\
&= \infty.
\end{aligned}
$$

Thus, it's clear that $G \in DAM(\Lambda)$ since it's not regular. Finally, note that since $F(n) = G(\lceil x \rceil)$ for $x \in (n-1, n]$, it follows that F and G have equivalent tails, and thus, F inherits the same domain of attraction that G has.

Exercise 3

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables with $X_1 \sim \mu$,

$$M_n := \max\{X_1, \ldots, X_n\}, \quad \text{and} \quad N_n := \min\{X_1, \ldots, X_n\}.$$

1. For $\mu = \text{Gamma}(\alpha, \beta)$, $\alpha, \beta > 0$, **determine and justify** the extreme distribution of (M_n) and (N_n).

2. For $\mu = \text{Beta}(\alpha, \beta)$, $\alpha, \beta > 0$, that is $f(x) = C_{\alpha,\beta} x^{\alpha-1}(1-x)^{\beta-1}$, $\alpha, \beta > 0$, **determine and justify** the extreme distribution of (M_n) and (N_n).

3. For μ such that for $\alpha > 0$,

$$F(x) = \begin{cases} 0, & \text{for } x < 1, \\ \frac{\ln(x)}{x^\alpha}, & \text{for } x \geq 1. \end{cases}$$

determine and justify the extreme distribution of (M_n) and (N_n)

Solution Part 1

The gamma distribution with parameters $\alpha, \beta > 0$ has the following density function:

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x^2}, \; x \in [0, \infty).$$

It's clear that the Gamma distribution has extreme distribution convergence since it's absolutely continuous. The extreme distribution cannot be Weibull because the population maximum $x_F = \infty$. Now, we are going to prove that the tails are not regular, and thus, the extremes do not converge to Fréchet (using L'Hôpital rule):

$$\begin{aligned}
\lim_{x \to \infty} \frac{\overline{F}(\lambda \cdot x)}{\overline{F}(x)} &= \lim_{x \to \infty} \frac{\lambda \cdot f(\lambda x)}{f(x)} \\
&= \lim_{x \to \infty} \frac{\lambda^\alpha x^{\alpha-1} e^{-\beta \lambda^2 x^2}}{x^{\alpha-1} e^{-\beta x^2}} \\
&= \lim_{x \to \infty} \lambda^\alpha \exp\left(-\beta x^2 (1 - \lambda^2)\right) \\
&= 0, \; \forall \lambda > 0.
\end{aligned}$$

Therefore, the only option left for M_n is to be in the Gumbell's domain of attraction, that is $F \in DAM(\Lambda)$.

Now, for N_n, we are going to use the von Mises condition to prove that F is in the domain of attraction of the Weibull distribution. In the first place, note that the population minimum x_F equals 0 and we are approaching it from the right. Thus, after reflecting the whole distribution over the y axis, we obtain

$$\begin{aligned} f'(x) &= (\alpha-1)x^{\alpha-2}e^{-\beta x^2} - x^{\alpha-1}\cdot 2\beta x e^{-\beta x^2} \\ &= x^{\alpha-2}e^{-\beta x^2}((\alpha-1)-2\beta x^2) \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0^-} \frac{xf(x)}{F(x)} &= \lim_{x\to 0^-} \frac{xf'(x)+f(x)}{f(x)} \\ &= \lim_{x\to 0^-} \frac{x^{\alpha-1}e^{-\beta x^2}((\alpha-1)-2\beta x^2)}{x^{\alpha-1}e^{-\beta x^2}} + \frac{f(x)}{f(x)} \\ &= \lim_{x\to 0^-} (\alpha-1) - 2\beta x^2 + 1 = \alpha. \end{aligned}$$

Thus, $\overline{F}(-\cdot) \in DAM(\Psi_\alpha)$.

Solution Part 2

Both M_n and N_n will be in the domain of attraction of Weibull. For M_n the von Mises condition is satisfied on the following limit:

$$\begin{aligned} f'(x) &= (\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} - (\beta-1)x^{\alpha-1}(1-x)^{\beta-2} \\ &= x^{\alpha-2}(1-x)^{\beta-2}((\alpha-1)(1-x)-(\beta-1)x) \end{aligned}$$

$$
\begin{aligned}
\lim_{x\to 1^+}\frac{(1-x)f(x)}{\overline{F}(x)} &= \lim_{x\to 1^+}\frac{(1-x)f'(x)-f(x)}{-f(x)}\\
&= \lim_{x\to 1^+}\frac{(1-x)\cdot x^{\alpha-2}(1-x)^{\beta-2}((\alpha-1)(1-x)-(\beta-1)x)}{-x^{\alpha-1}(1-x)^{\beta-1}}+1\\
&= \lim_{x\to 1^+} -(\alpha-1)\underbrace{\frac{1-x}{x}}_{\to 0}+(\beta-1)+1\\
&= \beta-1+1=\beta.
\end{aligned}
$$

Now, for N_n, after reflecting the whole distribution, we obtain:

$$
\begin{aligned}
\lim_{x\to 0^-}\frac{xf(x)}{F(x)} &= \lim_{x\to 0^-}\frac{xf'(x)+f(x)}{f(x)}\\
&= \lim_{x\to 0^-}\frac{x\cdot x^{\alpha-2}(1-x)^{\beta-2}((\alpha-1)(1-x)-(\beta-1)x)}{x^{\alpha-1}(1-x)^{\beta-1}}+1\\
&= \lim_{x\to 0^-}(\alpha-1)-(\beta-1)\underbrace{\frac{x}{1-x}}_{\to 0}+1\\
&= \alpha-1+1=\alpha.
\end{aligned}
$$

For M_n is Weibull with parameter β and for N_n is Weibull with parameter α.

$$F\in DAM(\Psi_\beta),\quad \overline{F}(-\cdot)\in DAM(\Psi_\alpha).$$

Solution Part 3

The provided distribution function cannot be a distribution function since is not monotonically increasing, as in fact, it goes to 0 when $x\to\infty$. I'm going to assume that it's the density function, but even then, we would encounter some problems. After some tinkering, I believe the correct density and distribution functions should be

$$f(x)=\alpha^2\frac{\ln(x)}{x^{\alpha+1}}\mathbb{1}_{[1,\infty)}(x),$$

$$F(x)=1-\frac{\alpha\ln(x)+1}{x^\alpha}\mathbb{1}_{[1,\infty)}(x).$$

For M_n we are using the von Mises condition to prove it is in the domain of attraction of Fréchet:

$$
\begin{aligned}
\lim_{x\to\infty} \frac{xf(x)}{\overline{F}(x)} &= \lim_{x\to\infty} \alpha^2 \frac{\ln(x)}{x^\alpha} \cdot \frac{x^\alpha}{\alpha\ln(x)+1} \\
&= \lim_{x\to\infty} \frac{\alpha^2\ln(x)}{\alpha\ln(x)+1} \\
\text{(L'Hôpital)} &= \lim_{x\to\infty} \frac{\alpha^2 x^{-1}}{\alpha x^{-1}} \\
&= \alpha > 0.
\end{aligned}
$$

Thus, $F \in DAM(\Phi_\alpha)$. Now, for N_n we are going to use the von Mises condition for the Weibull domain of attraction. After re-centering and reflecting over the y axis we obtain,

$$
\begin{aligned}
\lim_{x\to 1-} \frac{(x-1)f(x)}{F(x)} &= \lim_{x\to 1-} \frac{\alpha^2(x-1)\ln(x)}{x^{\alpha+1}} \cdot \frac{x^\alpha}{x^\alpha - \alpha\ln(x)+1} \\
&= \lim_{x\to 1-} \frac{1}{x} \cdot \frac{\alpha^2\ln(x)x - \ln(x)}{x^\alpha - \alpha\ln(x) - 1} \\
&= \frac{1}{1} \cdot \frac{0}{1-1}.
\end{aligned}
$$

After applying L'Hôpital rule,

$$
f'(x) = \frac{\alpha^2(1-(a+1)\ln(x))}{x^{\alpha+2}}
$$

$$
\begin{aligned}
\lim_{x\to 1-} \frac{(x-1)f(x)}{F(x)} &= \lim_{x\to 1-} \frac{(x-1)f'(x)+f(x)}{f(x)} \\
&= \lim_{x\to 1-} \frac{(x-1)\alpha^2(1-(a+1)\ln(x))}{x^{\alpha+2}} \cdot \frac{x^{\alpha+1}}{\alpha^2\ln(x)} + 1 \\
&= \lim_{x\to 1-} \frac{x-1}{x} \cdot \frac{1-(\alpha+1)\ln(x)}{\ln(x)} + 1 \\
&= \lim_{x\to 1-} \frac{(x-1)}{x\ln(x)} - (\alpha+1)\frac{(x-1)\ln(x)}{x\ln(x)} + 1 \\
\text{(L'Hôpital)} &= \lim_{x\to 1-} \frac{1}{\ln(x)+1} - (\alpha+1)\frac{1+\ln(x)-x^{-1}}{\ln(x)+1} + 1 \\
&= \frac{1}{0+1} - (\alpha+1)\frac{1+0-1}{0+1} + 1 \\
&= 2.
\end{aligned}
$$

In the end we obtained that N_n is in the domain of attraction of the Weibull, that is $\overline{F}(-\cdot) \in DAM(\Psi_2)$