Differential Geometry

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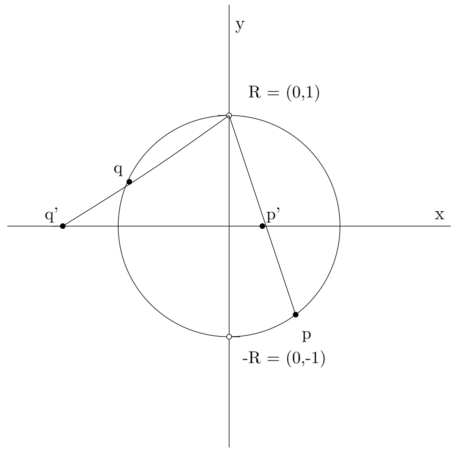
February 3, 2024

Exercise 1.

Build an atlas with 2 charts over the n-dimensional sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

Solution. Let ϵ_i be the i-th unitary vector from the standard basis, and let $R := \epsilon_{n+1}, -R = -\epsilon_{n+1}$ be the "north" and "south" poles of the sphere.
The atlas with two charts that we are going to build is the following. Let $U_+ := \mathbb{S}^n \backslash \{R\}$ and $U_- := \mathbb{S}^n \backslash \{-R\}$,

$$\mathcal{A} = \{(U_+,\ \varphi_+ : U_+ \mapsto \mathbb{R}^n),\quad (U_-,\ \varphi_- : U_- \mapsto \mathbb{R}^n)\}$$



Where φ_+, φ_- are the stereographic projections of the sphere with respect to the poles $R, -R$:

$$\begin{aligned}
&\forall x = (x_1, \cdots, x_{n+1}) \in \mathbb{S}^n : \\
&\varphi_+(x) := \left(\tfrac{x_1}{1-x_{n+1}}, \cdots, \tfrac{x_n}{1-x_{n+1}}\right) \\
&\varphi_-(x) := \left(\tfrac{x_1}{1+x_{n+1}}, \cdots, \tfrac{x_n}{1+x_{n+1}}\right)
\end{aligned}$$

Note that $\forall x \in \mathbb{S}^n$

$$\varphi_-(x) = \left(\varphi_+ \circ \begin{bmatrix}\epsilon_1 & \cdots & \epsilon_n & -\epsilon_{n+1}\end{bmatrix}\right)(x)$$

So we only need to prove that φ_+ is a homeomorphism.

1. **Bijective:** Let $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, and $\|\cdot\| := \|\cdot\|_2$. We define the inverse function $\varphi_+^{-1} : R^n \mapsto U_+$ as it follows:

$$\varphi_+^{-1}(y) = \left(\frac{2y_1}{\|y\|^2+1}, \cdots, \frac{2y_n}{\|y\|^2+1}, \frac{\|y\|^2-1}{\|y\|^2+1}\right)$$

Note that

$$\begin{aligned}
\|\varphi_+^{-1}(y)\|^2 &= \frac{1}{(\|y\|^2+1)^2}\left[(\|y\|^2-1)^2 + \sum_{i=1}^n 4y_i^2\right] \\
&= \frac{1}{\|y\|^4+2\|y\|^2+1}\left[\|y\|^4 - 2\|y\|^2 + 1 + 4\|y\|^2\right] \\
&= \frac{1}{\|y\|^4+2\|y\|^2+1}(\|y\|^4+2\|y\|^2+1) \\
&= 1
\end{aligned}$$