Differential Geometry

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Exercise 1.

Build an atlas with 2 charts over the n-dimensional sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

Solution. Let ϵ_i be the i-th unitary vector from the standard basis, and let $R := \epsilon_{n+1}, -R = -\epsilon_{n+1}$ be the "north" and "south" poles of the sphere.
The atlas with two charts that we are going to build is the following. Let $U_+ := \mathbb{S}^n \backslash \{R\}$ and $U_- := \mathbb{S}^n \backslash \{-R\}$,
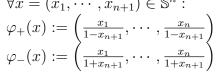
$$\mathcal{A} = \{(U_+, \ \varphi_+ : U_+ \mapsto \mathbb{R}^n), \ (U_-, \ \varphi_- : U_- \mapsto \mathbb{R}^n)\}$$

Where φ_+, φ_- are the stereographic projections of the sphere with respect to the poles $R, -R$:

$$\forall x = (x_1, \cdots, x_{n+1}) \in \mathbb{S}^n :$$
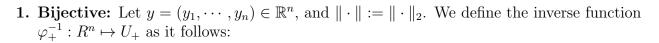$$\varphi_+(x) := \left(\frac{x_1}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}} \right)$$
$$\varphi_-(x) := \left(\frac{x_1}{1+x_{n+1}}, \cdots, \frac{x_n}{1+x_{n+1}} \right)$$

Note that $\forall x \in \mathbb{S}^n$

$$\varphi_-(x) = \left(\varphi_+ \circ \begin{bmatrix} \epsilon_1 & \cdots & \epsilon_n & -\epsilon_{n+1} \end{bmatrix} \right)(x)$$

So we only need to prove that φ_+ is a homeomorphism.

1. **Bijective:** Let $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, and $\| \cdot \| := \| \cdot \|_2$. We define the inverse function $\varphi_+^{-1} : R^n \mapsto U_+$ as it follows:

$$\varphi_+^{-1}(y) = \left(\frac{2y_1}{\|y\|^2 + 1}, \cdots, \frac{2y_n}{\|y\|^2 + 1}, \frac{\|y\|^2 - 1}{\|y\|^2 + 1} \right)$$

Note that

$$
\begin{aligned}
\|\varphi_+^{-1}(y)\|^2 &= \frac{1}{(\|y\|^2 + 1)^2} \left[(\|y\|^2 - 1)^2 + \sum_{i=1}^{n} 4y_i^2 \right] \\
&= \frac{1}{\|y\|^4 + 2\|y\|^2 + 1} \left[\|y\|^4 - 2\|y\|^2 + 1 + 4\|y\|^2 \right] \\
&= \frac{1}{\|y\|^4 + 2\|y\|^2 + 1} (\|y\|^4 + 2\|y\|^2 + 1) \\
&= 1
\end{aligned}
$$