Stochastic Processes: Homework 0

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Exercise 1

Consider a sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$ with $\mathbf{E}\, X_i = 0$ and $\mathbf{Var}\, X_i = 1$ for every $i \in \mathbb{N}$.

1. Show with th Law of Large Numbers that,

$$\lim_{n\to\infty} \|X_1, \ldots, X_n\|_2 - \sqrt{n} \to 0$$

   (a) in $\mathbb{P}$,

   (b) a.e.,

   (c) in distribution,

   (d) Show that if $X_i \in L^p$ for some $p > 1$, then it converges in L^q for every $q \in [1 \leq p)$.

2. Infer from the previous results that for

$$\text{Law}(X_1, \ldots, X_n) \approx \text{UNI}(\sqrt{n}\mathbb{S}^{n-1})$$

Solution Part 1

Theorem 1 (Laws of Large Numbers)**.** Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that $\mathbf{E}\, X_i = \mu$ for every $i \in \mathbb{N}$, and let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then,

$$\lim_{n\to\infty} \mathbf{P}\{\|\overline{X_n} - \mu\| > \varepsilon\} = 0,\ \forall\varepsilon > 0. \qquad \text{(Weak Law of Large Numbers)}$$

$$\mathbf{P}\{\lim_{n\to\infty} \overline{X_n} \neq \mu\} = 0. \qquad \text{(Strong Law of Large Numbers)}$$

Definition 1.1 (Convergence in probability)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We say that X_n converges to X in probability i.e. $X_n \xrightarrow{p} X$ when

$$\lim_{n\to\infty} \mathbf{P}\{|X_n - X| > \varepsilon\} = 0, \quad \forall \varepsilon > 0.$$

Definition 1.2 (Convergence almost everywhere)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We say that X_n converges to X almost everywhere (or) i.e. $X_n \xrightarrow{a.e.} X$ when

$$\mathbf{P}\{\lim_{n\to\infty} X_n \neq X\} = 0$$

Remark.

According to theorem 1 and the previous definitions, since X_i^2

$$\text{(a)} \quad \frac{1}{n}\|X_1, \ldots, X_n\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \to \mathbf{E}\, X^2 = \mathbf{Var}\, X - \mathbf{E}\, X = 1$$
$$\text{(b)} \quad \frac{1}{n}\|X_1, \ldots, X_n\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \to \mathbf{E}\, X^2 = \mathbf{Var}\, X - \mathbf{E}\, X = 1$$

Exercise 2

Show that for every random variable $X \in L^2$,

$$\mathbf{E}\,|X - \mathbf{E}\, X|^2 \leq \mathbf{E}\,|X|^2$$

Exercise 3

Let $f : [0,1] \to \mathbb{R}$ be a continuos function. Show with the Law of Large Numbers that there exists a sequence of polynomial $(P_n)_{n\in\mathbb{N}}$ such that $\deg(P_n) = n$ and,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f(x) - P_n(x)| = 0.$$