Stochastic Processes: Homework 1

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Exercise 1

Show that, for a sequence $(X_n)_{n\in\mathbb{N}}$ of Bernoulli random variables, with $X_n \sim \mathrm{Be}(n, p_n)$ and $p_n \to 0$. If

$$\mathbf{P}\left\{X_n = k\right\} \overset{n\to\infty}{\longrightarrow} \frac{\lambda^k}{k!}e^{-\lambda}$$

then,

$$\lim_{n\to\infty} n \cdot p_n = \lambda$$

Exercise 2

Prove the following theorem for $\tau = 0$ and $\tau = \infty$.

Theorem 1. For any given X with F_X and $\overline{F}_X(x) = 1 - F_X(x)$, $x \in \mathbb{R}$ and

- $\tau \in [0, \infty]$
- $(u_n)_{n\in\mathbb{N}}$ a non-decreasing sequence.

Then the following are equivalent,

1. $\lim_{n\to\infty} \mathbf{P}(X_{n:n} \leq u_n) = e^{-\tau}$
2. $\lim_{n\to\infty} n\overline{F}_X(u_n) = \tau$.

Exercise 3

Exercise 4

1. Simulate 50 times, graph and compare the empirical distribution of $X_{i:6}$ for $i \in \{1,\dots,6\}$ with $X_k \sim U(0,1)$. Calculate the uniform distance $\|\cdot\|_\infty$ between the empirical and the real distributions.

2. Simulate 50 times, graph and compare the empirical distribution of $X_{i:6}$ for $i \in \{1,\dots,6\}$ with $X_k \in F$, where

$$F = \begin{cases} 0 & x \leq 1 \\ 1 - x^{-1} & x > 1. \end{cases}$$

   Calculate the uniform distance $\|\cdot\|_\infty$ between the empirical and the real distributions. Also calculate $f_{i:6}$, $F_{i:6}$ and determine which moments are finite.

Solution Part 1

We use the letter F for the uniform distribution, $F_{i:n}$ for the order distribution of the $i-th$ element of the sample, and $F^*_{i:n}$ for the empirical distribution obtained from sampling m times the $i-th$ order statistic. In fact, since we are working with the uniform distribution,

$$F_{i:n}(x) = \mathrm{Beta}_{(i,n-i+1)}(x).$$

The formula for the empirical distribution is the following:

$$F^*_{i:n} = \frac{1}{m}\sum_{k=1}^{m} \mathbb{1}\{X^{(k)}_{i:n} < x\}.$$

Finally, since $F_{i:n}(x)$ is monotonically increasing and $F^*_{i:n}(x)$ is piecewise constant, the formula for the uniform distance can be approximated by calculating M times the following formula

$$\|F^*_{i:n} - F_{i:n}\|_\infty \approx \max_{k\in\{1,\dots,M\}} |F_{i:n}(k/m) - F^*_{i:n}(k/m)|.$$

The exercise required $n = 6$ and $m = 50$, and I personally used $M = 5000$ for graphing and calculating the uniform distance.

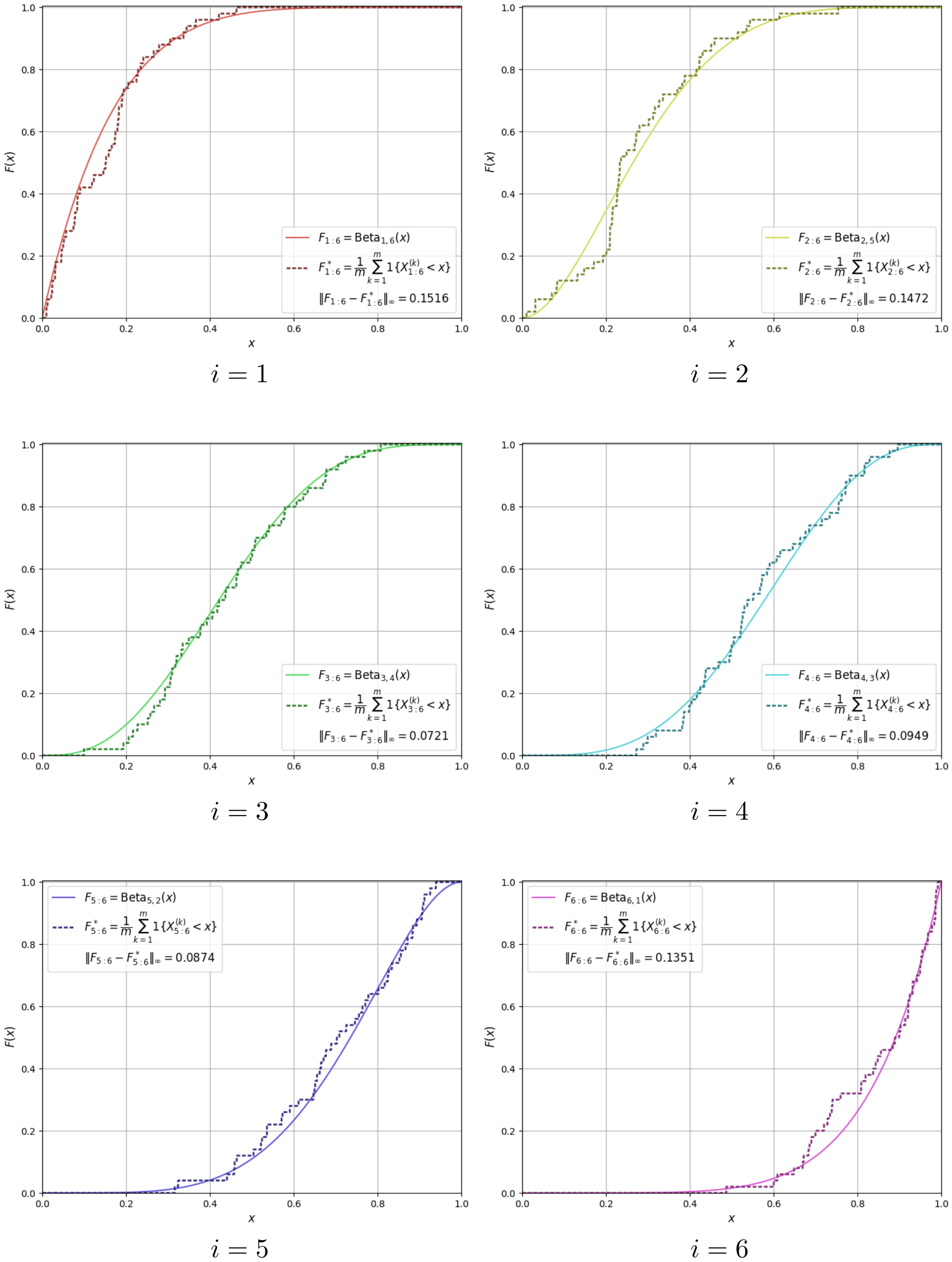

Figure 1: Simulation of the six order statistics for 6 samples of the Uniform Distribution

Solution Part 2

The distribution F from part 2 is a type I Pareto distribution

$$\mathrm{Pareto}_{(\alpha,\sigma)}(x) = \begin{cases} 0 & x \leq \sigma \\ 1 - \left(\frac{\sigma}{x}\right)^{\alpha} & x > \sigma. \end{cases}$$

with parameters $\alpha = \sigma = 1$. Therefore, the condition required for it to have the k-moment is that $k < \alpha$. Since $\alpha = 1$, it doesn't have any finite moments. Using the formula for the i-th statistic, we obtain

$$F_{i:n} =$$

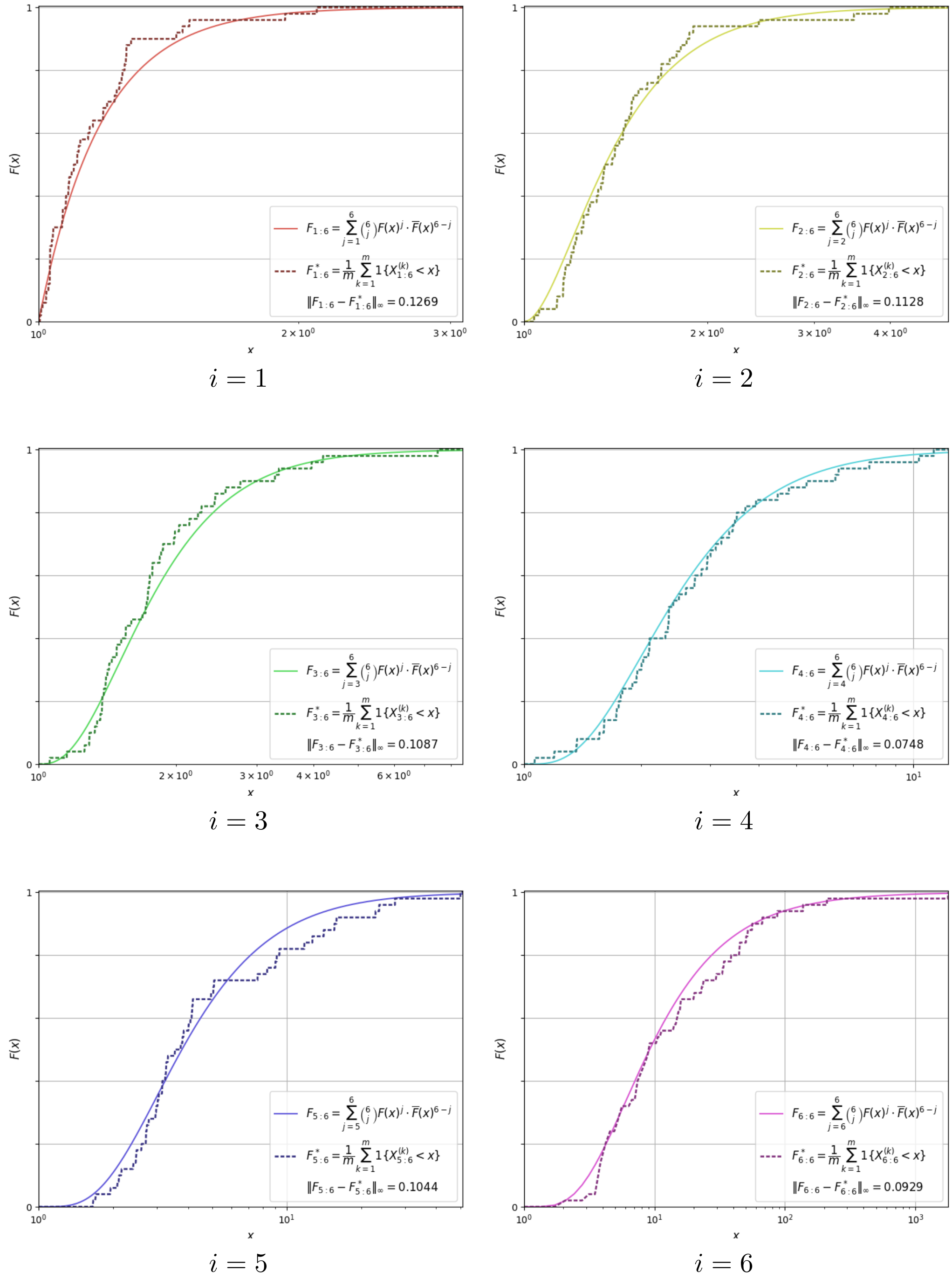

Figure 2: Simulation of the six order statistics for 6 samples of the Pareto Distribution