

# Stochastic Processes: Homework 1

Martín Prado

November 2023

Universidad de los Andes – Bogotá Colombia

## Exercise 1

Show that, for a sequence  $(X_n)_{n \in \mathbb{N}}$  of Bernoulli random variables, with  $X_n \sim \text{Be}(n, p_n)$  and  $p_n \rightarrow 0$ . If

$$\mathbf{P}\{X_n = k\} \xrightarrow{n \rightarrow \infty} \frac{\lambda^k}{k!} e^{-\lambda}$$

then,

$$\lim_{n \rightarrow \infty} n \cdot p_n = \lambda$$

## Solution

Using what we obtained from the other direction of the proof,

$$\mathbf{P}(X_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \binom{n}{k} \frac{1}{n^k} (n \cdot p_n)^k \left(1 - \frac{n \cdot p_n}{n}\right)^{n-k}.$$

For  $k = 0$ , we have

$$\lim_n \mathbf{P}(X_n = 0) = \binom{n}{0} \frac{1}{n^0} (n \cdot p_n)^0 \left(1 - \frac{n \cdot p_n}{n}\right)^n = (1 - p_n)^n = e^{-\lambda}.$$

Taking logarithms at both side will leave us with

$$\lim_n n \ln(1 - p_n) = -\lambda.$$

Using Taylor's expansion, we can assert that

$$-\ln(1 - p_n) = p_n + o(p_n).$$

Finally,

$$-\lim_n n \ln(1 - p_n) = \lim_n n(p_n + o(p_n)) = \lambda.$$

Since  $p_n$  dominates over  $o(p_n)$ , we can conclude that

$$\lim_n n \cdot p_n = \lambda.$$

## Exercise 2

Prove the following theorem for  $\tau = 0$  and  $\tau = \infty$ .

**Theorem 1.** For any given  $X$  with  $F_X$  and  $\bar{F}_X(x) = 1 - F_X(x)$ . Also let  $M_n = X_{n:n}$ . For  $x \in \mathbb{R}$ , if

- $\tau \in [0, \infty]$
- $(u_n)_{n \in \mathbb{N}}$  a non-decreasing sequence,

then the following items are equivalent,

1.  $\lim_{n \rightarrow \infty} \mathbf{P}(M_n \leq u_n) = e^{-\tau}$
2.  $\lim_{n \rightarrow \infty} n \bar{F}_X(u_n) = \tau$ .

### Solution $\tau = 0$

- (2)  $\implies$  (1) If  $n \cdot \bar{F}(u_n) \rightarrow 0$ , then  $\bar{F}(u_n) = o(1/n)$ . Therefore,

$$\begin{aligned} \lim_n \mathbf{P}(M_n \leq u_n) &= \lim_n F(u_n)^n = \lim_n (1 - \bar{F}(u_n))^n \\ &= \lim_n \left(1 - o\left(\frac{1}{n}\right)\right)^n \end{aligned}$$

What this means is that for every  $\varepsilon > 0$ , we will eventually have that

$$\frac{-\varepsilon}{n} \leq -\bar{F}(u_n) \leq \frac{\varepsilon}{n}.$$

In particular, for every  $\varepsilon > 0$ ,

$$e^{-\varepsilon} \leq \lim_n \left(1 - \frac{\varepsilon}{n}\right)^n \leq \lim_n (1 - \bar{F}(u_n))^n \leq \lim_n \left(1 + \frac{\varepsilon}{n}\right)^n = e^{\varepsilon}.$$

Therefore, by making  $\varepsilon$  go to 0 we would obtain,

$$1 \leq \lim_n (1 - \bar{F}(u_n))^n \leq 1.$$

- (1)  $\implies$  (2) Now, the hypothesis says that

$$\mathbf{P}(M_n \leq u_n) = \lim_n (1 - \bar{F}(u_n))^n = 1.$$

To prove that  $\bar{F}(u_n) \rightarrow 0$ , we use the same argument from the original proof. If  $\liminf_n \bar{F}(u_n) = \alpha > 0$ , then there exists a subsequence  $u_{n_k} \subset u_n$  such that

$$1 = \lim_k (1 - \bar{F}(u_{n_k}))^{n_k} \leq \lim_k (1 - \alpha)^{n_k} = 0.$$

With that in mind, we take logarithm at both sides to obtain

$$\lim_n n \cdot \ln((1 - \bar{F}(u_n))) = 0.$$

From Taylor's formula,  $-\ln(1 - x) = x + o(x)_{x \rightarrow 0+}$ . Thus,

$$-0 = -\lim_n n \cdot \ln((1 - \bar{F}(u_n))) = \lim_n n \bar{F}(u_n)$$

**Solution**  $\tau = \infty$

- (2)  $\implies$  (1)  $n \cdot \bar{F}(u_n) \rightarrow \infty$  is equivalent to

$$\lim_n \frac{1/\bar{F}(u_n)}{n} = 0.$$

Which by definition means that  $\bar{F}(u_n)^{-1} = o(\frac{1}{n})$ . This implies that for every  $\varepsilon > 0$ ,

$$\bar{F}(u_n)^{-1} \leq \varepsilon n, \text{ (eventually)}$$

$$\implies -\bar{F}(u_n) \leq -\frac{\varepsilon'}{n}$$

$$\begin{aligned} \lim_n \mathbf{P}(M_n \leq u_n) &= \lim_n F(u_n)^n = \lim_n (1 - \bar{F}(u_n))^n \\ &\leq \lim_n (1 - \frac{\varepsilon}{n})^n = e^{-\varepsilon}. \end{aligned}$$

By making  $\varepsilon \rightarrow \infty$ , we can conclude that

$$\lim_n \mathbf{P}(M_n \leq u_n) = 0.$$

- (1)  $\implies$  (2) Now, note that

$$\lim_n n \cdot \ln((1 - \bar{F}(u_n))) = -\infty.$$

Using Taylor's polynomial like we did in the case  $\tau = 0$ , will give us that

$$\lim_n n \cdot (\bar{F}(u_n) + o(\bar{F}(u_n))) = \infty.$$

Since, by definition,  $\bar{F}(u_n)$  dominates over  $o(\bar{F}(u_n))$ , we can conclude that

$$\lim_n n \cdot \bar{F}(u_n) = \infty.$$

### Exercise 3

For the following theorem, give two examples where they satisfy item 2 and two examples where they don't.

**Theorem 2.** Let  $F$  be a probability distribution and  $\bar{F}(x) = 1 - F(x)$ . Then, define

$$x_F = F^{-1}(x) = \inf\{t > 0 \mid F(t) = 1\}.$$

If these two conditions are satisfied:

- $x_F \leq \infty$ ,
- $\tau \in (0, \infty)$ ,

Then, the following items are equivalent

1. There exists a sequence  $(u_n)_{n \in \mathbb{N}}$  such that

$$\lim_n n \cdot \bar{F}(u_n) = \tau.$$

- 2.

$$\lim_{x \rightarrow x_F -} \frac{\bar{F}(x)}{\bar{F}(x-)}.$$

Note that if  $F$  is discrete and  $x_F = \infty$ , then item 2. is equivalent to

$$\lim_{n \rightarrow \infty} \frac{f(n)}{\bar{F}(n)} = 0.$$

## Solution Good Examples

1. **Zeta Distribution:** For this distribution, let  $s \in (1, \infty)$  and  $k \in \mathbb{Z}^+$ . Then, for  $X \sim \text{Zeta}(s)$ ,

$$f(k) = \frac{1}{k^s \zeta(s)},$$

$$F(k) = \mathbf{P}\{X \leq k\} = \frac{\sum_{i=1}^k i^{-s}}{\zeta(s)} = \frac{\sum_{i=1}^k i^{-s}}{\sum_{i=1}^{\infty} i^{-s}}.$$

Thus,

$$\bar{F}(k) = 1 - \frac{\sum_{i=1}^k i^{-s}}{\sum_{i=1}^{\infty} i^{-s}} = \frac{\sum_{i=k+1}^{\infty} i^{-s}}{\sum_{i=1}^{\infty} i^{-s}} = \frac{\sum_{i=k+1}^{\infty} i^{-s}}{\zeta(s)}$$

Asymptotically,

$$\bar{F}(k) \sim \int_k^{\infty} x^{-s} dx = \frac{1}{s-1} k^{-s+1}.$$

Therefore,

$$\frac{f(k)}{\bar{F}(k)} \sim \frac{1}{s-1} \frac{k^{-s}}{k^{-s+1}} \sim k^{-1} \rightarrow 0$$

2. **Gauss-Kuzmin Distribution:** For this distribution,

$$f(k) = -\log_2 \left[ 1 - \frac{1}{(k+1)^2} \right].$$

$$\bar{F}(k) = \log_2 \left( \frac{k+2}{k+1} \right)$$

Thus,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\bar{F}(k)}{\bar{F}(k-1)} &= \lim_{k \rightarrow \infty} \frac{\log(k+2) - \log(k+1)}{\log(k+1) - \log(k)} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{1}{k+2} - \frac{1}{k+1}}{\frac{1}{k+1} - \frac{1}{k}} \\ &= \lim_{k \rightarrow \infty} \frac{(k+1) - (k+2)}{(k+2) \cdot (k+1)} \cdot \frac{(k+1) \cdot (k)}{(k) - (k+1)} \\ &= \lim_{k \rightarrow \infty} \frac{-k^2 - k}{-k^2 - 3k - 2} = 1 \end{aligned}$$

## Solution Bad Examples

1. **Log Distribution:** For  $0 < p < 1$ , we define:

$$F(k) = 1 + \frac{\beta_k}{\ln(1-p)}, \quad \overline{F}(k) = \frac{\beta_k}{-\ln(1-p)},$$

$$f(k) = \frac{-p}{\ln(1-p)(1-p)},$$

where

$$\beta_k = \beta(p; k+1, 0) = \int_0^p t^k (1-t)^{-1} dt,$$

$$\frac{f(k)}{\overline{F}(k)} = \frac{p}{(1-p)} \cdot \frac{1}{\beta_k}.$$

The series expansion of  $\beta_k$  is the following

$$\beta_k = \frac{p^{k+1}}{k+1} + \frac{p^{k+2}}{k+2} + \frac{p^{k+3}}{k+3} + \cdots = \sum_{i=k+1}^{\infty} \frac{p^i}{i}.$$

This series converges for every  $k \in \mathbb{N}$  since  $p < 1$  (ratio test). Also, the sequence  $\beta_k$  is decreasing because less positive terms are being summed when  $k$  is increased, and thus,  $\lim_k \beta_k = 0$ . Therefore,

$$\lim_k \frac{p}{(1-p)} \cdot \frac{1}{\beta_k} = \infty.$$

## Failed Attempts

1. **Zipf-Mandelbrot Distribution:** Note that what played in favor for the last example was the heaviness of the tails. In this next case, something similar will happen. For  $N \in \mathbb{Z}^+, q \geq 0, s > 0$ , a random variable  $X_{N,q,s}$  with the Zipf-Mandelbrot distribution has the following properties,

$$f(k) = f_{N,q,s}(k) = \frac{1}{(k+q)^s} \cdot \frac{1}{H_{N,q,s}},$$

where

$$H_{N,q,s} = \sum_{i=1}^N \frac{1}{(i+q)^s}.$$

As a matter of fact, when  $q = 0$  and  $N \rightarrow \infty$ ,  $X_{N,0,s}$  converges in distribution to a zeta distribution. Now,

$$F(k) = \frac{H_{k,q,s}}{H_{N,q,s}}$$

and thus, title

$$\overline{F}(k) = \frac{\sum_{i=k+1}^N (i+q)^{-s}}{H_{N,q,s}}.$$

Asymptotically,

$$\overline{F}(k) \sim \int_k^N (x+q)^{-s} dx = \frac{1}{s-1} (k^{1-s} - N^{1-s}).$$

Hmmmm, it seems that it has finite support.

2. **Beta Negative Binomial:** For  $\alpha, \beta > 0 \in \mathbb{R}$  and  $r \in \mathbb{N}$ , define

$$f(k) = \binom{r+k-1}{k} \frac{B(\alpha+r, \beta+k)}{B(\alpha, \beta)},$$

where

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Then, by Stirling's approximation we have that

$$f(k) \sim \frac{\Gamma(\alpha+r)}{\Gamma(r)B(\alpha, \beta)} \frac{k^{r-1}}{(\beta+k)^{r+\alpha}}$$

Thus, we can approximate the tail of the distribution with the following integral:

$$C = \frac{\Gamma(\alpha+r)}{\Gamma(r)B(\alpha, \beta)},$$

$$\begin{aligned} \overline{F}(n) &\sim C \int_n^\infty \frac{x^{r-1}}{(x-\beta)^{r+\alpha}} dx \\ &= C \frac{x^r \cdot {}_2F_1(1, 1-a; r+1; \frac{x}{b})}{br(x-\beta)^{r+a-1}} \end{aligned}$$

Honestly, I tried many things to make a direct proof of the polynomial behavior of the tails. However, I couldn't find any literature on this topic. The argument I'm trying to make is that while  $f(n) \sim Kn^{1-\alpha}$ , the tail should behave like a higher degree polynomial, and thus,

$$\lim_n \frac{f(n)}{\overline{F}(n)} = 0.$$

This argument could make sense in the case where  $r = \beta = \alpha = 1$ , because

$$\begin{aligned} f(k) &\sim \binom{k}{k} \frac{B(2, k+1)}{B(1, 1)} = \frac{1}{(k+1)(k+2)} = \text{Cau}(k+1) \\ \overline{F}(k) &\sim \end{aligned}$$

## Exercise 4

1. Simulate 50 times, graph and compare the empirical distribution of  $X_{i:6}$  for  $i \in \{1, \dots, 6\}$  with  $X_k \sim U(0, 1)$ . Calculate the uniform distance  $\|\cdot\|_\infty$  between the empirical and the real distributions.
2. Simulate 50 times, graph and compare the empirical distribution of  $X_{i:6}$  for  $i \in \{1, \dots, 6\}$  with  $X_k \in F$ , where

$$F = \begin{cases} 0 & x \leq 1 \\ 1 - x^{-1} & x > 1. \end{cases}$$

Calculate the uniform distance  $\|\cdot\|_\infty$  between the empirical and the real distributions. Also calculate  $f_{i:6}$ ,  $F_{i:6}$  and determine which moments are finite.

### Solution Part 1

We use the letter  $F$  for the uniform distribution,  $F_{i:n}$  for the order distribution of the  $i$ -th element of the sample, and  $F_{i:n}^*$  for the empirical distribution obtained from sampling  $m$  times the  $i$ -th order statistic. In fact, since we are working with the uniform distribution,

$$F_{i:n}(x) = \text{Beta}_{(i, n-i+1)}(x).$$

The formula for the empirical distribution is the following:

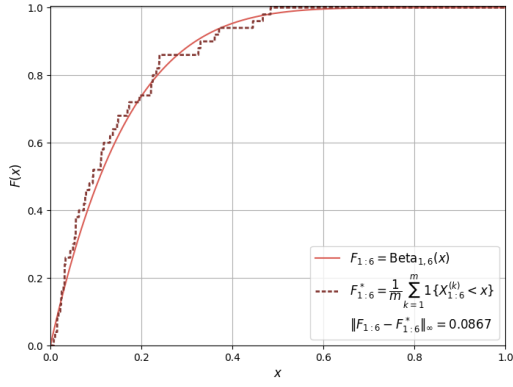
$$F_{i:n}^* = \frac{1}{m} \sum_{k=1}^m \mathbf{1}\{X_{i:n}^{(k)} < x\}.$$

Finally, since  $F_{i:n}(x)$  is monotonically increasing and  $F_{i:n}^*(x)$  is piecewise constant, the formula for the uniform distance can be approximated by calculating  $M$  times the following formula

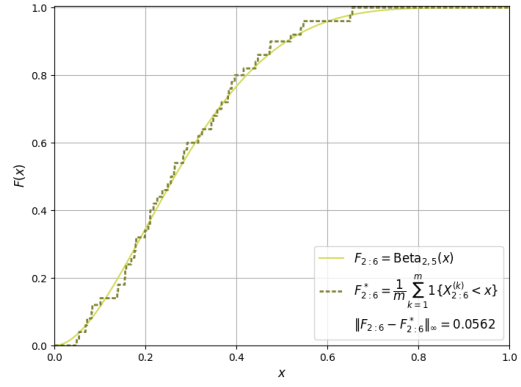
$$\|F_{i:n}^* - F_{i:n}\|_\infty \approx \max_{k \in \{1, \dots, M\}} |F_{i:n}(k/m) - F_{i:n}^*(k/m)|.$$

The exercise required  $n = 6$  and  $m = 50$ , and I personally used  $M = 5000$  for graphing and calculating the uniform distance.

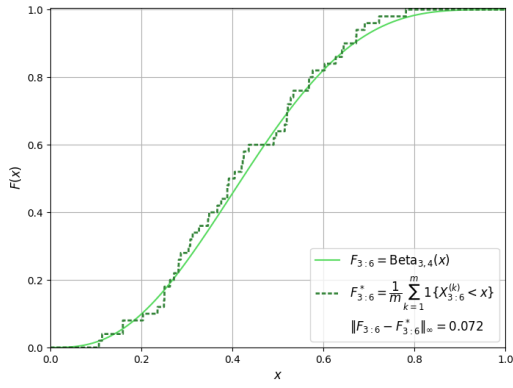




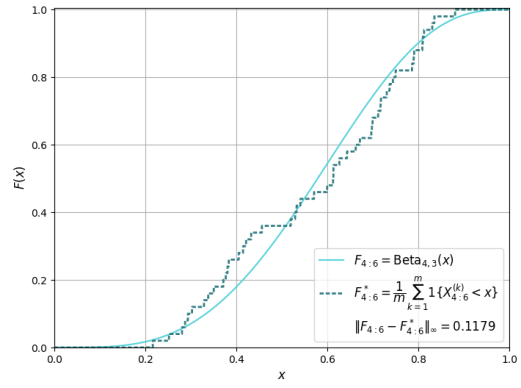
$i = 1$



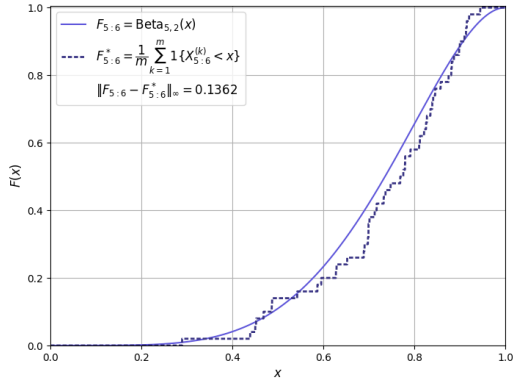
$i = 2$



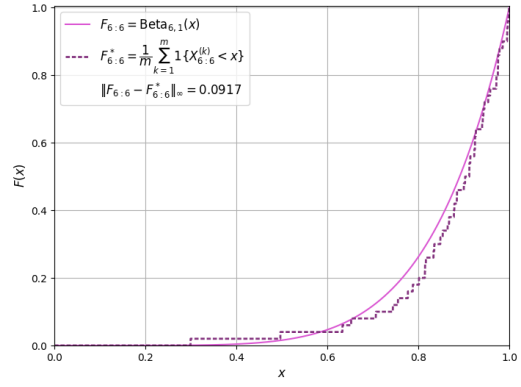
$i = 3$



$i = 4$



$i = 5$



$i = 6$

Figure 1: Simulation of the six order statistics for 6 samples of the Uniform Distribution

## Solution Part 2

The distribution  $F$  from part 2 is a type I Pareto distribution

$$\text{Pareto}_{(\alpha, \sigma)}(x) = \begin{cases} 0 & x \leq \sigma \\ 1 - \left(\frac{\sigma}{x}\right)^\alpha & x > \sigma. \end{cases}$$

with parameters  $\alpha = \sigma = 1$ . Therefore, the condition required for it to have the  $k$ -moment is that  $k < \alpha$ . Since  $\alpha = 1$ , it doesn't have any finite moments. Using the formula for the  $i$ -th statistic, we obtain

$$\begin{aligned} F_{i:n}(x) &= \sum_{j=i}^n \binom{n}{j} F^j(x) \overline{F}^{n-j}(x) \\ &= \sum_{j=i}^n \binom{n}{j} (1 - x^{-1})^j x^{-n+j}. \end{aligned}$$

Also,

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{i} \cdot \binom{n}{i} f(x) F^{i-1}(x) \overline{F}^{n-i}(x) \\ &= \frac{1}{i} \cdot \binom{n}{i} x^{-2} (1 - x^{-1})^{i-1} x^{i-n} \end{aligned}$$

Thus, the formula for the  $k$ -moment is the following

$$E[X^k] = \frac{1}{i} \cdot \binom{n}{i} \int_1^\infty \underbrace{x^{i+k} x^{-2-n}}_{\text{dominant}} (1 - x^{-1})^{i-1} dx$$

This last integral converges if  $n + 2 - i - k > 1$ , thus, the only moments available are when  $k < n - 1 - i$ .

For the uniform norm I did the same as the previous part.

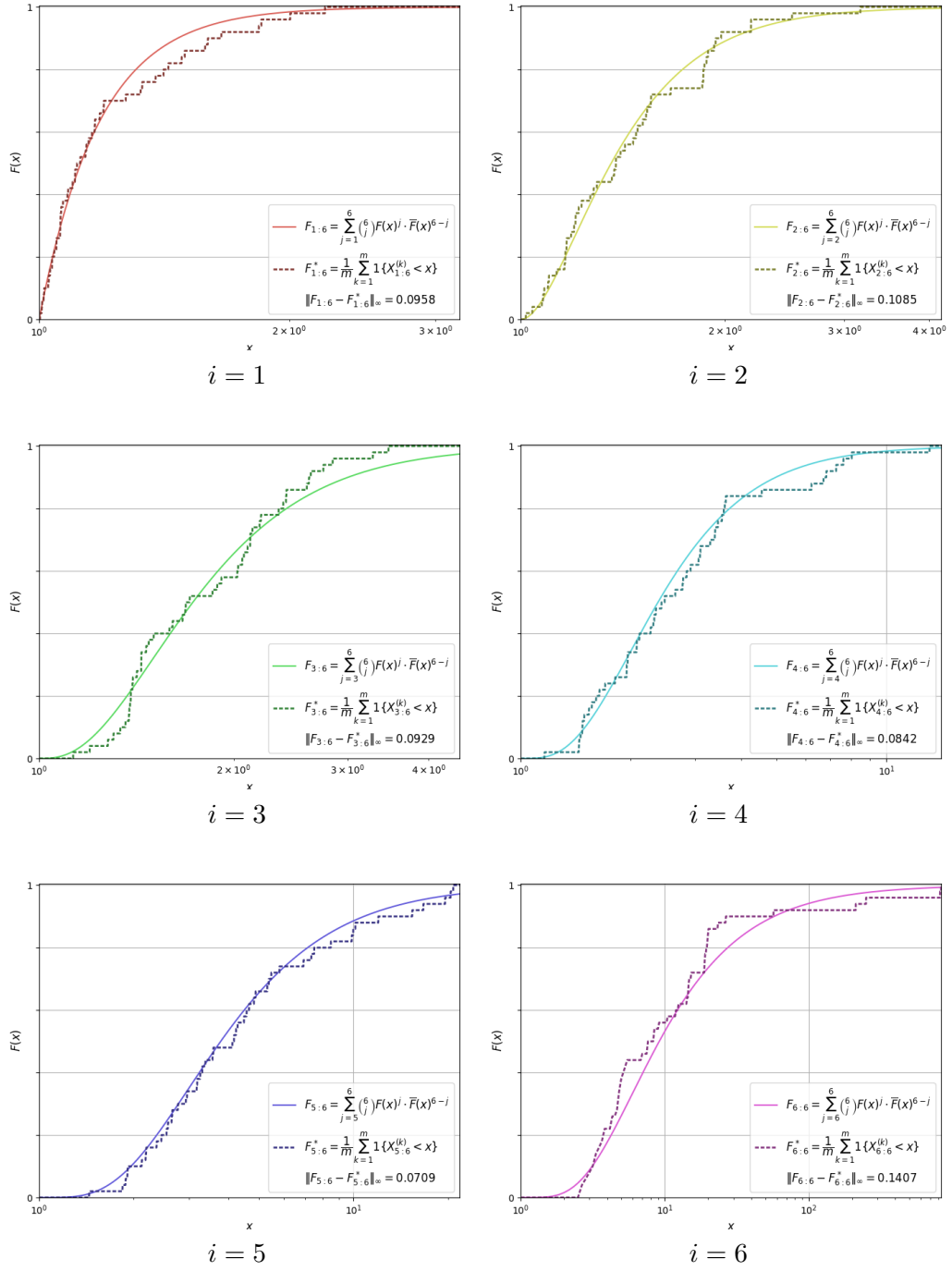


Figure 2: Simulation of the six order statistics for 6 samples of the Pareto Distribution