Stochastic Processes: Homework 0

Martín Prado

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Universidad de los Andes – Bogotá Colombia

Exercise 1

Consider a sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$ with $\mathbf{E}\, X_i = 0$ and $\mathbf{Var}\, X_i = 1$ for every $i \in \mathbb{N}$.

1. Show with th Law of Large Numbers that,

$$\lim_{n\to\infty} \|X_1, \dots, X_n\|_2 - \sqrt{n} \to 0$$

   (a) in $\mathbb{P}$,

   (b) a.e.,

   (c) in distribution,

   (d) Show that if $X_i \in L^p$ for some $p > 1$, then it converges in L^q for every $q \in [1 \leq p)$.

2. Infer from the previous results that for

$$\mathrm{Law}(X_1, \dots, X_n) \approx \mathrm{UNI}(\sqrt{n}\mathbb{S}^{n-1})$$

Solution Part 1

Theorem 1 (Laws of Large Numbers)**.** Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that $\mathbf{E}\, X_i = \mu$ for every $i \in \mathbb{N}$, and let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^n X_i$. Then,

$$\lim_{n\to\infty} \mathbf{P}\{\|\overline{X_n} - \mu\| > \varepsilon\} = 0,\ \forall \varepsilon > 0. \qquad \text{(Weak Law of Large Numbers)}$$

$$\mathbf{P}\{\lim_{n\to\infty} \overline{X_n} \neq \mu\} = 0. \qquad \text{(Strong Law of Large Numbers)}$$

$\square$

Definition 1.1 (Convergence in probability)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We say that X_n converges to X in probability i.e. $X_n \xrightarrow{p} X$ when

$$\lim_{n\to\infty} \mathbf{P}\{\|X_n - X\| > \varepsilon\} = 0, \quad \forall \varepsilon > 0.$$

Definition 1.2 (Convergence almost everywhere)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We say that X_n converges to X almost everywhere (or almost surely) i.e. $X_n \xrightarrow{a.e.} X$ when

$$\mathbf{P}\{\lim_{n\to\infty} X_n \neq X\} = 0$$

According to theorem 1 and the previous definitions, since $(X_i^2)_{n\in\mathbb{N}}$ is still a sequence of i.i.d. random variables,

$$\text{(a)} \quad \frac{1}{n}\|X_1, \dots, X_n\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathbf{E}\, X^2 = \mathbf{Var}\, X - \mathbf{E}\, X = 1$$

$$\text{(b)} \quad \frac{1}{n}\|X_1, \dots, X_n\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{a.e.} \mathbf{E}\, X^2 = \mathbf{Var}\, X - \mathbf{E}\, X = 1$$

Therefore,

$$\|X_1, \dots, X_n\|_2 - \sqrt{n} \xrightarrow{p} 0, \qquad \|X_1, \dots, X_n\|_2 - \sqrt{n} \xrightarrow{a.e.} 0.$$

Definition 1.3 (Convergence of distribution)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with probability distributions P_n. Let X a random variable with a probability distribution P. We say that X_n converges to X in distribution i.e. $X_n \xrightarrow{d} X$ if

$$\lim_{n\to\infty} \mathbf{E}\left[f(X_n)\right] = \mathbf{E}\left[f(X)\right]$$

for every bounded and continuous function $f : \mathcal{X} \to \mathbb{R}$.

Theorem 2. A direct consequence of Fatou's Lemma and Dominated Convergence is that,

$$X_n \xrightarrow{a.e.} X \implies X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X.$$

$\square$

Thus, this proves that

$$\text{(c)} \quad \|X_1, \ldots, X_n\|_2 - \sqrt{n} \xrightarrow{d} 0.$$

Definition 2.1 (Convergence in L_p)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. For some $p \in [1, \infty)$, we say that X_n converges to X in L_p norm i.e. $X_n \overset{L_p}{\to} X$ if $\mathbf{E}\,|X_n|^p$ and $\mathbf{E}\,|X|^p$ exist, and

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^p = 0$$

for every bounded and continuous function $f : \mathcal{X} \to \mathbb{R}$.

Theorem 3 (Exercise 1.d.)**.** Convergence in L_p implies convergence in L_q for every $q \in [1, p)$.

Proof. □

Exercise 2

Show that for every random variable $X \in L^2$,

$$\mathbf{E}\,|X - \mathbf{E}\,X|^2 \le \mathbf{E}\,|X|^2$$

Exercise 3

Let $f : [0, 1] \to \mathbb{R}$ be a continuos function. Show with the Law of Large Numbers that there exists a sequence of polynomial $(P_n)_{n\in\mathbb{N}}$ such that $\deg(P_n) = n$ and,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f(x) - P_n(x)| = 0.$$