Stochastic Processes: Homework 4

Martín Prado

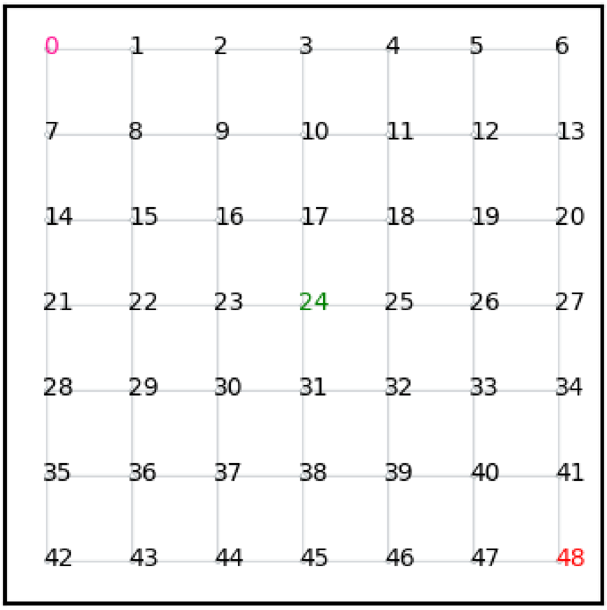
April 19, 2024

Universidad de los Andes – Bogotá Colombia

Exercise 1

Part (a)

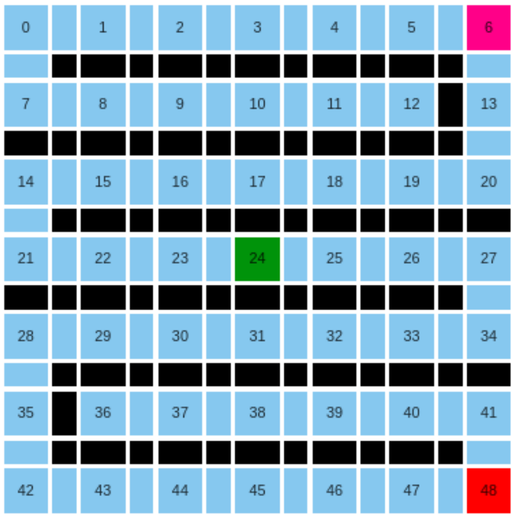
On a base labyrinth we define a $N \times N$ grid where we enumerate each position with the lexicographical ordering. In particular, for the case $N = 7$ we have the following ordering,



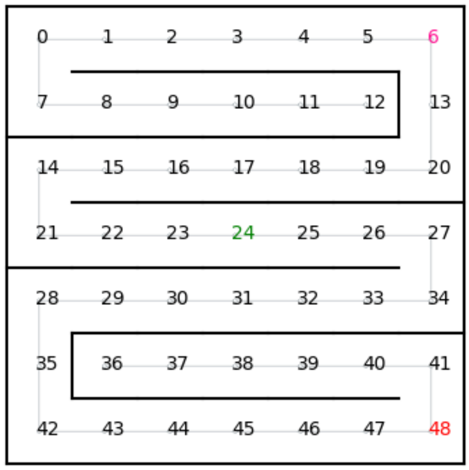
We put 2 exits on this graph, each one corresponds to an absorbing state. In the previous image, the absorbing states are 0 (pink) and 48 (red). Furthermore, unless we put a wall between to positions in the grid, each vertex on the position k is connected (if possible) with $k+1$ (right), $k-1$ (left), $k+N$ (up), $k-N$ (down).

For the walls I made a graphical interface that lets the user interact with which walls the labyrinth should have. You can find the [program clicking here]() The program lets the user:

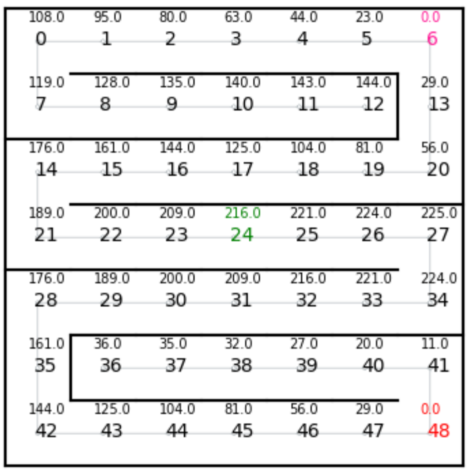
- Move the starting points and the exits.
- Change the size of the grid.
- See labels for the average time of absorption on each position.
- See the labels for the probability of absorption of Exit 1 vs Exit 2 on each position.



Button Graphical Interface



Labyrinth (All Labels Off)



Average Absorption Times Labels On

Absorption Probabilities Labels On

Part (b)

Let A be the graph $N^2 \times N^2$ matrix that describes all the allowed connections of the vertices in the labyrinth when there's no wall between them. In general,

$$A_{k,l} = \begin{cases} 1 & \text{if } l \in \{k+1, k-1, k+N, k-N\} \\ 0 & \text{otherwise (or when k and l are too far)} \end{cases}, \quad k, l \in \{0, 1, \ldots, N^2 - 1\}$$

Let $W = \{(k, l) \in W\}$ be a set of walls in the labyrinth, which is a subset of all possible edges allowed by the graph $A_{k,l}$. Then, we define our labyrinth graph with the following matrix

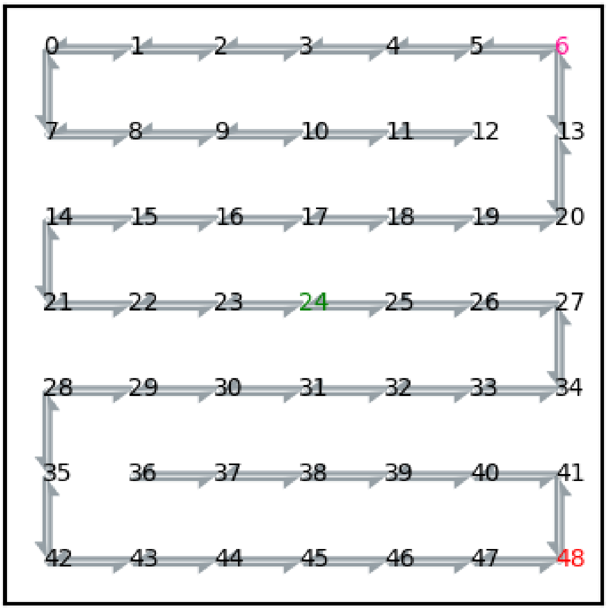
$$M_{k,l} = A_{k,l} - \mathbb{1}\{(k, l) \in W\},$$

which is the set of all the allowed edges that aren't obstructed by a wall. Let $m_k = \sum_{l=0}^{N^2} M_{k,l}$ be the sum of the entries of k-th row in the matrix M, and let k_1, k_2 be the indices for the 2 exits, the transition matrix for our random walk on the graph is defined by

$$\Pi_{k,l} = \frac{M_{k,l}}{m_k}, \ k, l \notin \{k_1, k_2\},$$

$$\Pi_{k_1,k_1} = \Pi_{k_2,k_2} = 1, \qquad \Pi_{k_1,l} = \Pi_{k_2,l} = 0, \ l \neq k_1, k_2 \text{ (respectively)}.$$

Without the walls, the graph of the previous example would look like:



Part (c)

Let $A = \{k_1, k_2\}$. For $l \in A$, the linear system for the vector $g_l = (P\{Z_{T_A} \mid X_0 = k\})_{k<N^2}$ is the following:

$$\begin{cases} \sum_{k=0}^{N^2} \Pi_{k,m} \cdot g_l(m) - g_l(k) & = 0, \quad \text{For } k \notin A \\ \qquad g_l(k) & = 0, \quad \text{For } k \in A,\ k \neq l \\ \qquad g_l(l) & = 1 \end{cases}$$

Then, we can express this system as follows

$$A \cdot g_l = b_l,$$

where, for $l \in A$, we define $b_l \in \mathbb{R}^{N^2}$ as

$$b_l(k) = \begin{cases} 1 & k = l \\ 0 & k \neq l. \end{cases}$$

If I_k is the k-th row of the identity matrix and Π_k the k-th row of Π, then the k-th row of A is defined as

$$A_k = \begin{cases} I_k & k \in A \\ \Pi_k - I_k & k \notin A \end{cases}$$

Thus, the probability of going from Start 24 to Exit 6 is 0.6 and Exit 48 is 0.4. Note that all the other probabilities starting from other positions also sum 1.0.

Part (d)

For the average absorption time we have the following linear system

$$\begin{cases} h_A(k) - \sum_{k=0}^{N^2} \Pi_{k,l} \cdot h_A(l) & = 1, \quad \text{For } k \notin A \\ h_A(k) & = 0, \quad \text{For } k \in A. \end{cases}$$

Then, we can express this system as

$$B \cdot h_A = b_A,$$

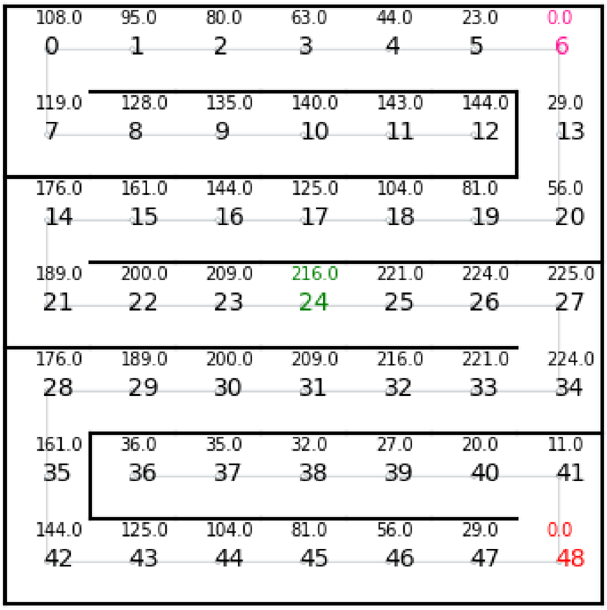
where, b_A is defined as

$$b_A(k) = \begin{cases} 1 & k \notin A \\ 0 & k \in A \end{cases},$$

and the rows of B as

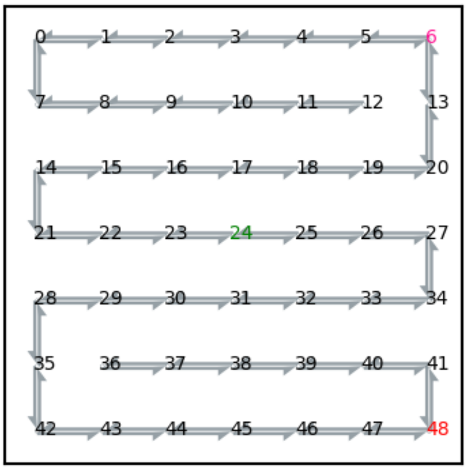
$$B_k = \begin{cases} I_k - \Pi_k & k \notin A \\ I_k & k \in A. \end{cases}$$

After solving the system for the previous example we obtain

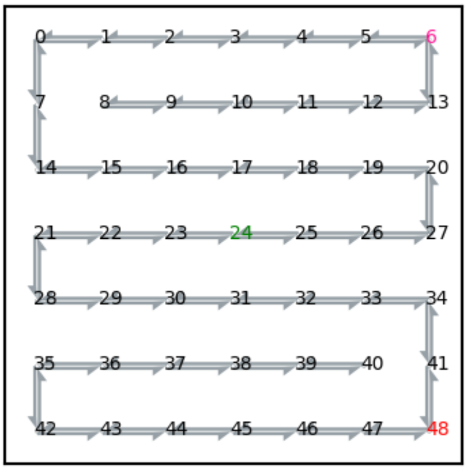


Part (e)

If we keep the 2 exits at $\{6, 48\}$ but we reflect the walls over both x, y axis we obtain the following labyrinth:



Original Graph



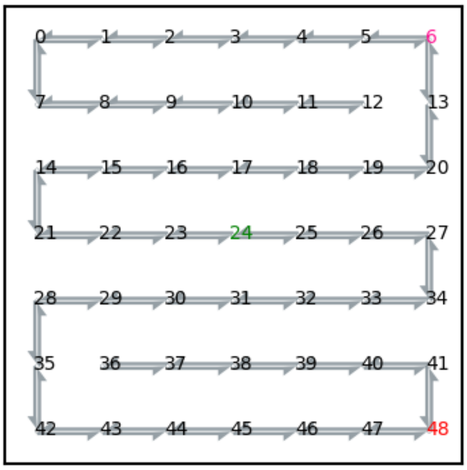
Reflected Graph

Now, the absorption probabilities should be the following

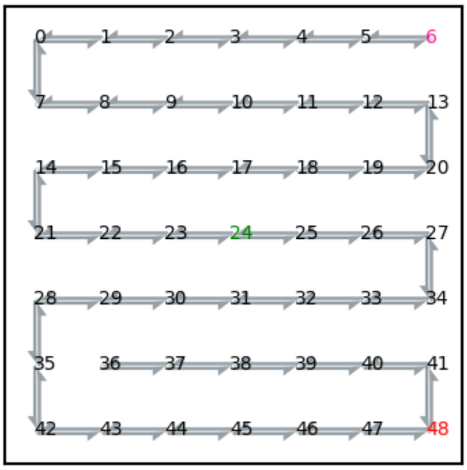
Thus, according to the previous image, the probability of going from Start 24 to Exit 6 is 0.4 and Exit 48 is 0.6. Which is the oposite to the other labyrinth.

Part (f) - Double Time

We remove from the original graph the edge $(6, 13)$ and add the edge $(12, 13)$. Then, the distance between Start 24 and Exit 6 is doubled. For some reason, this also doubles the expected time to reach one of the exits. I believe that this is associated to the fact that every point connected to both exits has to pass through Start 24, but I'm not sure.

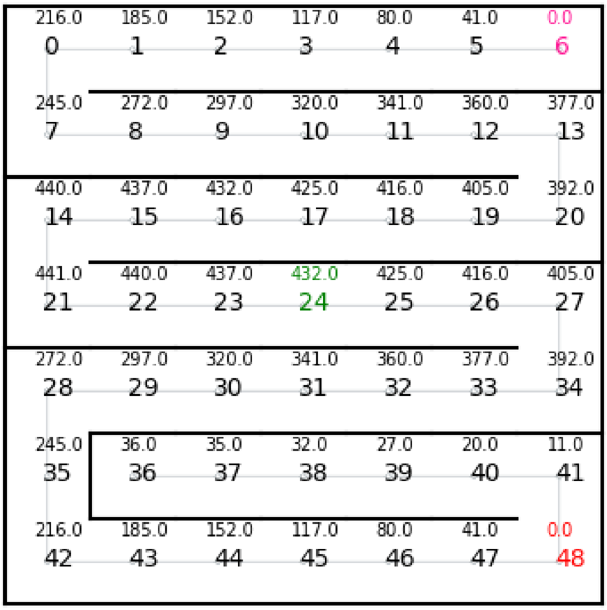


Original Graph



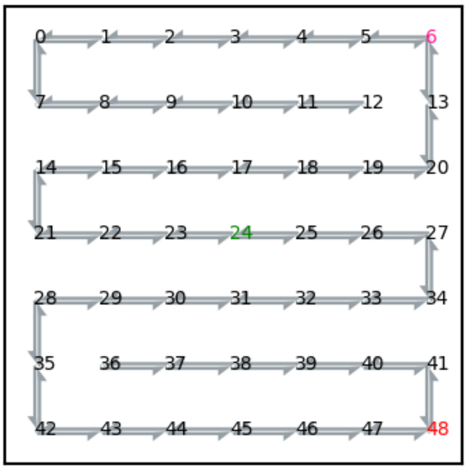
Double Time Graph

Now, this new graph takes in average double the time of the original time to reach one of the exits

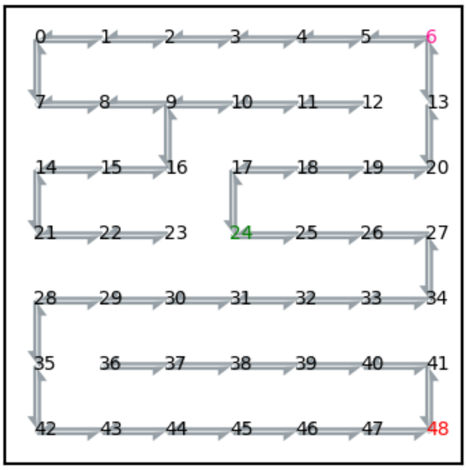


Part (f) - Half Time

We removed the edges $\{(23, 24), (16, 17)\}$ and added the edges $\{(9, 16), (17, 24)\}$. With the same idea, this halved the distance between Start 24 and Exit 6.

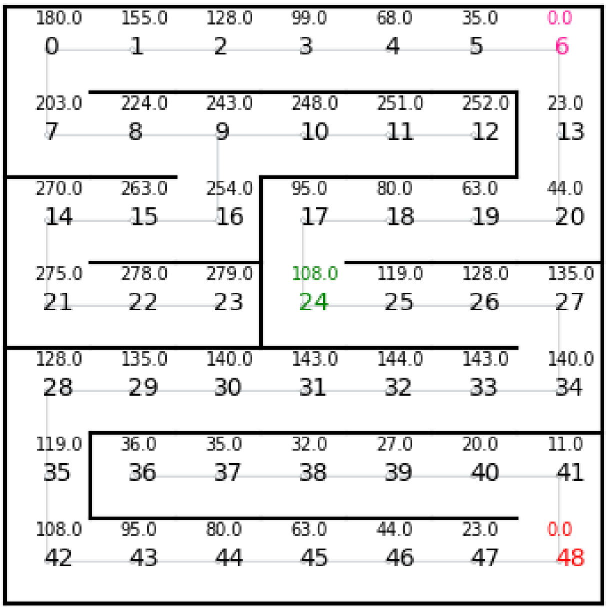


Original Graph



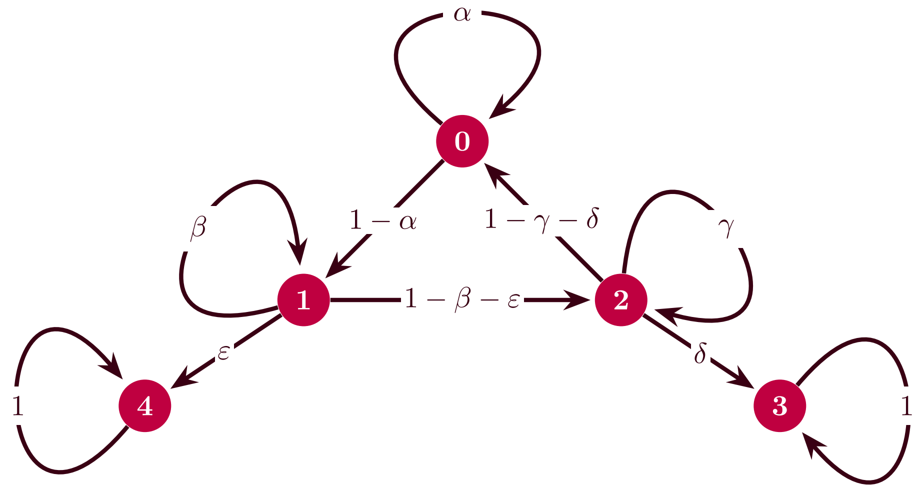
Half Time Graph

Now, this new graph takes in average half the time of the original to reach one of the exits



Exercise 2

Part (a)



$$\Pi = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 & 0 \\ 0 & \beta & 1-\beta-\varepsilon & 0 & \varepsilon \\ 1-\gamma-\delta & 0 & \gamma & \delta & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Part (b)

It can be the case that $A, B, C \in M$ if $\alpha, \beta, \gamma = 1$ respectively. Otherwise, $M = \{D, E\}$. However, in every case the maximum state should always be E.

Part (c)

I don't understand if the problem ask for the minimum state in all these five cases:

1. $m = A$
2. $m = B$
3. $m = C$
4. $m = D$

Or if there's a typo and it's asking for the maximum, which should always be: 5. $m = E$.
On every case we should encounter the same linear system, given by:

$$\begin{cases} \sum_{k=0}^{N^2} \Pi_{k,m} \cdot g_l(m) - g_l(k) & = 0, \quad \text{For } k \notin A \\ g_l(k) & = 0, \quad \text{For } k \in A,\ k \neq l \\ g_l(l) & = 1 \end{cases}$$

Which is equivalent to solving g_m in

$$A \cdot g_m = b_m,$$

where

$$A_k = \begin{cases} I_k & k \in M \\ \Pi_k - I_k & k \notin M \end{cases}, \qquad b_m(k) = \begin{cases} 1 & k = m \\ 0 & k \neq m. \end{cases}$$

Part (d)

We solve the system under the cases of wether α, β, γ equal 1 or not.

Caso $m = D$:

Then, we assume $\alpha, \beta, \gamma \neq 1$. In such case, $M = \{D, E\}$ and

$$g_D = \begin{bmatrix} \dfrac{\delta\left(\beta+\varepsilon-1\right)}{\delta\left(\beta+\varepsilon-1\right)+\gamma\varepsilon-\varepsilon} \\ \dfrac{\delta\left(\beta+\varepsilon-1\right)}{\delta\left(\beta+\varepsilon-1\right)+\gamma\varepsilon-\varepsilon} \\ \dfrac{\delta\left(\beta-1\right)}{\delta\left(\beta+\varepsilon-1\right)+\gamma\varepsilon-\varepsilon} \\ 1 \\ 0 \end{bmatrix} \qquad g_E = \begin{bmatrix} \dfrac{\varepsilon\left(\gamma-1\right)}{\beta\delta-\delta+\varepsilon\left(\delta+\gamma-1\right)} \\ \dfrac{\varepsilon\left(\gamma-1\right)}{\beta\delta-\delta+\varepsilon\left(\delta+\gamma-1\right)} \\ \dfrac{\varepsilon\left(\delta+\gamma-1\right)}{\beta\delta-\delta+\varepsilon\left(\delta+\gamma-1\right)} \\ 0 \\ 1 \end{bmatrix}$$

```python
sp.simplify((solution(3)+solution(4)))
```

[76] ✓ 1.9s Python

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1.0 \\ 1.0 \end{bmatrix}$$

$$g_D + g_E = 1_5$$

Caso $m = C$:

Then, $\gamma = 1$ and $M = \{C, D, E\}$. And so, the symbolic calculator gives us

$$g_C = \begin{bmatrix} \frac{\beta+\varepsilon-1}{\beta-1} \\ \frac{\beta+\varepsilon-1}{\beta-1} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} -\frac{\varepsilon}{\beta-1} \\ -\frac{\varepsilon}{\beta-1} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Caso $m = B$:

- $M = \{B, D, E\}$

$$g_B = \begin{bmatrix} 1 \\ 1 \\ \frac{\delta+\gamma-1}{\gamma-1} \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ 0 \\ -\frac{\delta}{\gamma-1} \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- $M = \{B, C, D, E\}$

$$g_B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Caso $m = A$:

- $M = \{A, D, E\}$

$$g_A = \begin{bmatrix} 1 \\ \frac{\beta\delta + \beta\gamma - \beta + \delta\varepsilon - \delta + \gamma\varepsilon - \gamma - \varepsilon + 1}{\beta\gamma - \beta - \gamma + 1} \\ \frac{\delta + \gamma - 1}{\gamma - 1} \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ \frac{\delta\left(-\beta - \varepsilon + 1\right)}{\beta\gamma - \beta - \gamma + 1} \\ -\frac{\delta}{\gamma - 1} \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} 0 \\ -\frac{\varepsilon}{\beta - 1} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- $M = \{A, C, D, E\}$

$$g_A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_B = \begin{bmatrix} 0 \\ \frac{\beta + \varepsilon - 1}{\beta - 1} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} 0 \\ -\frac{\varepsilon}{\beta - 1} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- $M = \{A, B, D, E\}$

$$g_A = \begin{bmatrix} 1 \\ 0 \\ \frac{\delta + \gamma - 1}{\gamma - 1} \\ 0 \\ 0 \end{bmatrix}, \qquad g_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ 0 \\ -\frac{\delta}{\gamma - 1} \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- $M = \{A, B, C, D, E\}$

$$g_A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad g_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad g_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad g_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In all of the cases, the sum of vectors gives 1 on every entry.

Part (e)

δ dictates how probable is to end the random walk going from C to D while ε dictates how probable is to end the random walk going from B to E. Therefore, g_D increases and g_E decreases when we increase δ, while the oposite happens when we increase ε.

Exercise 3

Part (a)

For the average absorption time we have the following linear system

$$\begin{cases} h_A(k) - \sum_{k=0}^{N^2} \Pi_{k,l} \cdot h_A(l) & = 1, \quad \text{For } k \notin A \\ h_A(k) & = 0, \quad \text{For } k \in A. \end{cases}$$

which can be expressed as

$$B \cdot h_M = b_M$$

where, b_M is defined as

$$b_M(k) = \begin{cases} 1 & k \notin M \\ 0 & k \in M \end{cases},$$

and the rows of B as

$$B_k = \begin{cases} I_k - \Pi_k & k \notin M \\ I_k & k \in M. \end{cases}$$

Part (b)

- $M = \{D, E\}$

$$h_M = \begin{bmatrix} \frac{\alpha\beta+\alpha\gamma+\alpha\varepsilon-2\alpha+\beta\gamma-2\beta-2\gamma-\varepsilon+3}{(\alpha-1)(\beta\delta+\delta\varepsilon-\delta+\gamma\varepsilon-\varepsilon)} \\ \frac{\alpha\beta+\alpha\gamma+\alpha\varepsilon-2\alpha+\beta\delta+\beta\gamma-2\beta+\delta\varepsilon-\delta+\gamma\varepsilon-2\gamma-2\varepsilon+3}{(\alpha-1)(\beta\delta+\delta\varepsilon-\delta+\gamma\varepsilon-\varepsilon)} \\ \frac{\alpha\beta+\alpha\delta+\alpha\gamma-2\alpha+\beta\delta+\beta\gamma-2\beta-2\delta-2\gamma+3}{(\alpha-1)(\beta\delta+\delta\varepsilon-\delta+\gamma\varepsilon-\varepsilon)} \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{C, D, E\}$

$$h_M = \begin{bmatrix} -\frac{\alpha+\beta-2}{(\alpha-1)(\beta-1)} \\ -\frac{1}{\beta-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{B, D, E\}$

$$h_M = \begin{bmatrix} -\frac{1}{\alpha-1} \\ 0 \\ -\frac{\alpha+\delta+\gamma-2}{(\alpha-1)(\gamma-1)} \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{B, C, D, E\}$

$$h_M = \begin{bmatrix} -\frac{1}{\alpha-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{A, D, E\}$

$$h_M = \begin{bmatrix} 0 \\ -\frac{\beta+\gamma+\varepsilon-2}{(\beta-1)(\gamma-1)} \\ -\frac{1}{\gamma-1} \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{A, C, D, E\}$

$$h_M = \begin{bmatrix} 0 \\ -\frac{1}{\beta-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{A, B, D, E\}$

$$h_M = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\gamma-1} \\ 0 \\ 0 \end{bmatrix}$$

- $M = \{A, B, C, D, E\}$

$$h_M = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Part (c)

If $\alpha = \beta = \gamma = 0$, then $M = \{D, E\}$. In the first place, note that this would imply that the probability of going from A to B is 1, and thus, $h_M(A) = h_M(B) + 1$. On the other hand, the fact that we cannot repeat the same state on the walk and we tend to go more often to other states implies that the average time should decrease.