# Stochastic Processes: Homework 5

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## Exercise 1

Consider the following transition matrix

$$\Pi = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \end{pmatrix}$$

(a) Build the random dynamical system that parametrizes this Markov Chain.

(b) Why does a unique invariant distribution exist?

(c) Calculate the invariant distribution $\pi$ and verify it is reversible.

(d) Verify that it is strongly irreducible. Which is the exponent $m$?

(e) The random walk on $\mathbb{Z}$ with independent increments $\frac{2}{3}\delta_{-1} + \frac{1}{3}\delta_1$ is not strongly irreducible. Explain the difference between this case and the case of $\Pi$

(f) Determine the time $\mathbf{E}\left[T_i^r\right]$ for $i = 0, \ldots, 4$.

(g) Calculate the convergence rate

$$\mu\Pi^n \to \pi, \qquad n \to \infty$$

### Solution Part (a)

$$f(i,\theta) = (i+1) \bmod 5 \cdot \mathbb{1}_{[0,\frac{1}{3})}(\theta) + (i-1) \bmod 5 \cdot \mathbb{1}_{[\frac{1}{3},1]}(\theta).$$

## Solution Part (b)

The ergodic theorem for Markov Chains states that a strongly irreducible Markov chain has only one invariant distribution $\pi$. (See item (d))

## Solution Part (c)

We must find $\pi$ that satisfies

$$\pi\Pi = \pi,$$
$$\sum_{i=0}^{4} \pi_i = 1$$

The symmetry of the graph that is spanned by this matrix hints that all the entries of $\pi$ must be equal to $\frac{1}{5}$. In fact, after calculating the left eigenvectors of $\Pi$ we find that the only one with eigenvalue 1 is

$$\pi = \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right].$$

## Solution Part (d)

$$\Pi^2 = \begin{bmatrix} \frac{4}{9} & 0 & \frac{1}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & 0 & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & 0 & \frac{4}{9} & 0 & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} & 0 & \frac{4}{9} & 0 \\ 0 & \frac{1}{9} & \frac{4}{9} & 0 & \frac{4}{9} \end{bmatrix},$$

$$\Pi^3 = \begin{bmatrix} 0 & \frac{2}{9} & \frac{8}{27} & \frac{1}{27} & \frac{4}{9} \\ \frac{4}{9} & 0 & \frac{2}{9} & \frac{8}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{4}{9} & 0 & \frac{2}{9} & \frac{8}{27} \\ \frac{8}{27} & \frac{1}{27} & \frac{4}{9} & 0 & \frac{2}{9} \\ \frac{2}{9} & \frac{8}{27} & \frac{1}{27} & \frac{4}{9} & 0 \end{bmatrix}$$

$$\Pi^4 \begin{bmatrix} \frac{8}{27} & \frac{16}{81} & \frac{8}{81} & \frac{32}{81} & \frac{1}{81} \\ \frac{1}{81} & \frac{8}{27} & \frac{16}{81} & \frac{8}{81} & \frac{32}{81} \\ \frac{32}{81} & \frac{1}{81} & \frac{8}{27} & \frac{16}{81} & \frac{8}{81} \\ \frac{8}{81} & \frac{32}{81} & \frac{1}{81} & \frac{8}{27} & \frac{16}{81} \\ \frac{16}{81} & \frac{8}{81} & \frac{32}{81} & \frac{1}{81} & \frac{8}{27} \end{bmatrix}$$

The exponent is $m = 4$.

## Solution Part (e)

Let $S_n$ be the sum of $n$ increments of this random walk. Note that if $n$ is odd, then $S_n$ is too and viceversa. Therefore,

$$S_n \equiv n \mod 2,$$

and thus, odd states are not accesible from even $n$'s and viceversa. On the other hand, since $2\mathbb{Z}_5 \simeq \mathbb{Z}_5$, one can access both even and odd states from odd $n$'s and viceversa. From the previous part is also easy to see that for $n = 4$ one can for any $i, j \in S$ from $i$ to $j$ in 4 steps with probability greater than 0.

## Solution Part (f)

Theorem 3.83 states that if $\pi$ is the only invariant distribution, then

$$\pi(i) = \frac{1}{\mathbf{E}\left[T_i^r\right]}$$

Therefore, $\mathbf{E}\left[T_i^r\right] = 5$ for every $i \in S$.

## Solution Part (g)

Let $X_n^\mu$ be the random variable associated with $\mu\Pi^n$ theorem 3.68 states that for

$$\alpha = \sum_{j \in S} \min_{i \in S} \Pi^m(i,j) = \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} = \frac{5}{81},$$

we have

$$\sup_{A \subset S} |P\{X_n^\mu \in A\} - \pi(A)| \leq (1-\alpha)^{\lfloor \frac{n}{m} \rfloor} = \left(\tfrac{76}{81}\right)^{\lfloor \frac{n}{4} \rfloor}.$$

Thus, $X_n^\mu$ converges in distribution to $\pi$ at an exponential rate.