Stochastic Processes: Homework 0

Martín Prado

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Universidad de los Andes – Bogotá Colombia

Exercise 1

Consider a sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}}$ with $\mathbf{E}\,X_i = 0$ and $\mathbf{Var}\,X_i = 1$ for every $i \in \mathbb{N}$.

1. Show with th Law of Large Numbers that,

$$\lim_{n\to\infty} \|X_1, \ldots, X_n\|_2 - \sqrt{n} \to 0$$

   (a) in $\mathbb{P}$,

   (b) a.e.,

   (c) in distribution,

   (d) Show that if $X_i \in L^p$ for some $p > 1$, then it converges in L^q for every $q \in [1 \leq p)$.

2. Infer from the previous results that for

$$\mathrm{Law}(X_1, \ldots, X_n) \approx \mathrm{UNI}(\sqrt{n}\mathbb{S}^{n-1})$$

Solution Part 1

Theorem 1 (Laws of Large Numbers)**.** Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that $\mathbf{E}\,X_i = \mu$ for every $i \in \mathbb{N}$, and let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then,

$$\lim_{n\to\infty} \mathbf{P}\{\|\overline{X_n} - \mu\| > \varepsilon\} = 0,\ \forall\varepsilon > 0. \qquad \text{(Weak Law of Large Numbers)}$$

$$\mathbf{P}\{\lim_{n\to\infty} \overline{X_n} \neq \mu\} = 0. \qquad \text{(Strong Law of Large Numbers)}$$

$\square$

Definition 1.1 (Convergence in probability)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We say that X_n converges to X in probability i.e. $X_n \xrightarrow{p} X$ when

$$\lim_{n\to\infty} \mathbf{P}\{\|X_n - X\| > \varepsilon\} = 0, \quad \forall \varepsilon > 0.$$

Definition 1.2 (Convergence almost everywhere)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. We say that X_n converges to X almost everywhere (or almost surely) i.e. $X_n \xrightarrow{a.e.} X$ when

$$\mathbf{P}\{\lim_{n\to\infty} X_n \neq X\} = 0$$

According to theorem 1 and the previous definitions, since $(X_i^2)_{n\in\mathbb{N}}$ is still a sequence of i.i.d. random variables,

$$\text{(a)} \quad \frac{1}{n}\|X_1,\dots,X_n\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathbf{E}\, X^2 = \mathbf{Var}\, X - \mathbf{E}\, X = 1$$

$$\text{(b)} \quad \frac{1}{n}\|X_1,\dots,X_n\|_2^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{a.e.} \mathbf{E}\, X^2 = \mathbf{Var}\, X - \mathbf{E}\, X = 1$$

Therefore,

$$\|X_1,\dots,X_n\|_2 - \sqrt{n} \xrightarrow{p} 0, \qquad \|X_1,\dots,X_n\|_2 - \sqrt{n} \xrightarrow{a.e.} 0.$$

Definition 1.3 (Convergence of distribution)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with probability distributions P_n. Let X a random variable with a probability distribution P. We say that X_n converges to X in distribution i.e. $X_n \xrightarrow{d} X$ if

$$\lim_{n\to\infty} \mathbf{E}\left[f(X_n)\right] = \mathbf{E}\left[f(X)\right]$$

for every bounded and continuous function $f : \mathcal{X} \to \mathbb{R}$.

Theorem 2. A direct consequence of Fatou's Lemma and Dominated Convergence is that,

$$X_n \xrightarrow{a.e.} X \implies X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X.$$

$\square$

Thus, this proves that

$$\text{(c)} \quad \|X_1, \ldots, X_n\|_2 - \sqrt{n} \xrightarrow{d} 0.$$

Definition 2.1 (Convergence in L_p)**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. For some $p \in [1, \infty)$, we say that X_n converges to X in L_p norm i.e. $X_n \overset{L_p}{\to} X$ if $\mathbf{E}\,|X_n|^p$ and $\mathbf{E}\,|X|^p$ exist, and

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^p = 0.$$

for every bounded and continuous function $f : \mathcal{X} \to \mathbb{R}$.

Theorem 3 (Exercise 1.d.)**.** Convergence in L_p implies convergence in L_q for every $q \in [1, p)$.

Proof. Hölder inequality states that for $a, b \in [1, \infty]$ such that $a^{-1} + b^{-1} = 1$ and random variables A, B,

$$\|AB\|_1 = \mathbf{E}\,|AB| \leq (\mathbf{E}\,|A|^a)^{a^{-1}} \left(\mathbf{E}\,|B|^b\right)^{b^{-1}} = \|A\|_a \|B\|_b.$$

By letting

$$\begin{aligned} A &= |X_n - X|^q, \quad & B &= 1, \\ a &= \frac{p}{q}, & b &= \frac{p}{p-q}, \end{aligned}$$

we obtain,

$$\begin{aligned} \mathbf{E}\,|X_n - X|^q &= \mathbf{E}\,||X_n - X|^q| \\ &\leq \left(\mathbf{E}\,|X_n - X|^{q\cdot p/q}\right)^{q/p} \cdot \left(\mathbf{E}\,|1|^{p/(p-q)}\right)^{(p-q)/p} \\ &= (\mathbf{E}\,|X_n - X|^p)^{q/p}. \end{aligned}$$

The hypothesis of $q \in [1, p)$ is used on the fact that, if $q \geq p$, then $p - q \leq 0$, and if $q < 1$, then the L_q norm wouldn't be defined. Finally, since

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^p = 0,$$

it follows that

$$\lim_{n\to\infty} \mathbf{E}\,|X_n - X|^q \leq \lim_{n\to\infty} (\mathbf{E}\,|X_n - X|^p)^{q/p} = 0.$$

$\square$

Solution Part 2

In the first place note that the function $\|\cdot\|_2$ is invariant under rotations because for any orthogonal (rotation) matrix $O \in \{M : M^T M = I\}$,

$$\begin{aligned}\|OX\|_2 &= \langle OX, OX\rangle^{1/2} \\ &= \left\langle X, O^T OX\right\rangle^{1/2} \\ &= \langle X, X\rangle^{1/2} \\ &= \|X\|_2\end{aligned}.$$

Therefore, for any vector $Z_n \sim \mathcal{N}(0, I_n)$ the density formula, which only depends on $\|Z_n\|_2$, is invariant under rotations. Thus, for any $n \times n$ rotation matrix O and measurable set $A \subset \mathbb{R}^2$,

$$P(Z_n \in A) = P(OZ_n \in A).$$

Then, by normalizing everything, we would obtain that for every measurable subset of the $(n-1)$-sphere $A \subset \mathbb{S}^{n-1}$,

$$P(Z_n/\|Z_n\|_2 \in A) = P(OZ_n/\|Z_n\|_2 \in A) = P(OZ_n/\|OZ_n\|_2 \in A).$$

So it follows that $Z_n/\|Z_n\|_2$ is uniformly distributed over the $(n-1)$-sphere.

On the other hand, according to the Central Limit Theorem,

$$\overline{Z_n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mathbf{E}\, X_i)/(\sqrt{\mathbf{Var}\, X_i}) \xrightarrow{d} Z_n.$$

According to the exercise's statement, $\mathbf{E}\, X_i = 0$ and $\mathbf{Var}\, X_i = 1$. Thus,

$$\overline{Z_n} = \frac{\sqrt{n}}{n} \sum_{i=1}^{n} X_i = \sqrt{n} \cdot \overline{X_n}.$$

Also, using the a.e. convergence from the part 1, we can make the following approximation,

$$\|Z_n\|_2 \approx \sqrt{n}.$$

Finally, it follows that

$$\overline{X_n} = \frac{\overline{Z_n}}{\sqrt{n}} \approx \frac{\overline{Z_n}}{\|Z_n\|_2} \xrightarrow{d} \frac{Z_n}{\|Z_n\|_2},$$

which will let us conclude that the distribution of $\overline{X_n}$ which is $\text{Law}(X_1, \ldots, X_n)$ might be similar to a uniform distribution on a $(n-1)$-sphere.

Exercise 2

Show that for every random variable $X \in L^2$,

$$\mathbf{E}\,[\,|X - \mathbf{E}\,X|^2\,] \leq \mathbf{E}\,[\,|X|^2\,].$$

Solution

Since $|x|^2 = x^2$, it follows that,

$$\begin{aligned}
\mathbf{E}\,[\,|X - \mathbf{E}\,X|^2\,] &= \mathbf{E}\,(X - \mathbf{E}\,X)^2 \\
&= \mathbf{E}\,[X^2] - E[X]^2 \\
&= \mathbf{E}\,[\,|X|^2\,] - E[X]^2 \\
&\leq \mathbf{E}\,[\,|X|^2\,].
\end{aligned}$$

Exercise 3

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show with the Law of Large Numbers that there exists a sequence of polynomial $(P_n)_{n\in\mathbb{N}}$ such that $\deg(P_n) = n$ and,

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f(x) - P_n(x)| = 0.$$

Solution

The goal of this solution is to prove the Stone-Weierstrass approximation theorem using the Law of Large Numbers. I will use Hoeffding's inequality in a similar fashion its used in the Glivenko-Cantelli theorem's proof.

In the first place, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random i.i.d. Bernoulli variables, each with probability $x \in [0,1]$. Define $\overline{X_n} = \sum_{i=1}^{n} X_i$, and note that $\overline{X_n} \sim \mathrm{Bi}(n,x)$. The polynomial we are going to use in this proof is the Bernstein polynomial:

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) = \mathbf{E}\left[f\left(\frac{\overline{X_n}}{n}\right)\right].$$

Now, in conjunction with the Heine-Cantor theorem, we use the fact that f is a continuous on a compact set to say that f is uniformly continuous. Therefore, for $\varepsilon/2 > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon/2, \quad \text{whenever } |x - y| \leq \delta, \ \forall x, y \in [0, 1].$$

We rewrite our expression as follows,

$$\begin{aligned} |f(x) - P_n(x)| &= \left| f(x) - \mathbf{E} \left[f\left(\tfrac{\overline{X_n}}{n} \right) \right] \right| \\ &= \left| \mathbf{E} \left[f(x) - f\left(\tfrac{\overline{X_n}}{n} \right) \right] \right|. \end{aligned}$$

Then, we separate the expression in two cases.

$$1 = \mathbb{1}\left\{ |x - \tfrac{\overline{X_n}}{n}| \leq \delta \right\}(x) + \mathbb{1}\left\{ |x - \tfrac{\overline{X_n}}{n}| > \delta \right\}(x).$$

For the case $|x - \frac{\overline{X_n}}{n}| \leq \delta$, use uniform continuity:

$$\forall n \in \mathbb{N}: \ \mathbf{E} \left[\left(f(x) - f\left(\tfrac{\overline{X_n}}{n} \right) \right) \cdot \mathbb{1}\left\{ |x - \tfrac{\overline{X_n}}{n}| \leq \delta \right\} \right] < \frac{\varepsilon}{2}. \tag{1}$$

For the case $|x - \frac{\overline{X_n}}{n}| > \delta$, define $M = \|f\|_\infty = \max_{x \in [0,1]} |f(x)|$,

$$\begin{aligned} &\forall n \in \mathbb{N}: \mathbf{E} \left[\left(f(x) - f\left(\tfrac{\overline{X_n}}{n} \right) \right) \cdot \mathbb{1}\left\{ |x - \tfrac{\overline{X_n}}{n}| > \delta \right\} \right] \\ &\leq 2M \ \cdot \ \mathbf{E} \left[\mathbb{1}\left\{ |x - \tfrac{\overline{X_n}}{n}| > \delta \right\} \right] \\ &= 2M \ \cdot \ \mathbf{P}\left\{ |x - \tfrac{\overline{X_n}}{n}| > \delta \right\}. \end{aligned}$$

Since $x = \mathbf{E}\, X_n$ for every $n \in \mathbb{N}$, we can use Hoeffding inequality to assert that

$$\forall x \in [0, 1]: \ \mathbf{P}\left\{ |x - \tfrac{\overline{X_n}}{n}| > \delta \right\} \leq 2e^{-2n\delta^2}. \tag{2}$$

Then, choose n_0 big enough to satisfy $2e^{-2n\delta^2} \leq \frac{\varepsilon}{4M}$ for every $n > n_0$. Finally, combine (1), (2) to obtain,

$$\forall \varepsilon > 0, \ \exists n_0: \ |f(x) - P_n(x)| < \frac{\varepsilon}{2} + \frac{2M\varepsilon}{4M} = \varepsilon, \ \forall x \in [0, 1], \ \forall n > n_0.$$

Thus,

$$\sup_{x \in [0,1]} |f(x) - P_n(x)| < \varepsilon, \ \forall n > n_0.$$

This implies that

$$\lim_{n \to \infty} \sup_{x \in [0,1]} |f(x) - P_n(x)| = 0.$$