Stochastic Processes: Homework 2

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Exercise 1

Let $f(x) = \frac{1}{2}\sin(x)\mathbb{1}_{[0,\pi]}(x)$.

1. For $n = 4$ **calculate** the first moment of the order statistic $X_{i:n}$ for every $i \in 1, \ldots, 4$.
2. **Sketch a telling drawing** with the original density, the densities $f_{i:4}$ and $\mathbf{E}[X_{i:4}]$

Solution Part 1

In the first place, for $x \in [0, \pi]$

$$F(x) = \frac{1}{2}\int_{-\infty}^{x} \sin(t)\mathbb{1}(t)dt = \frac{1}{2}\int_{0}^{x} \sin(t)dt$$
$$= \cos(x)\Big|_0^x = \frac{1}{2}(1 - \cos(x)) \cdot \mathbb{1}_{[0,\pi]}(x)$$

Thus,

$$\overline{F}(x) = \frac{1}{2}(1 + \cos(x)) \cdot \mathbb{1}_{[0,\pi]}(x),$$

and,

$$f_{i:n}(x) = i \cdot \binom{n}{i} f(x) \cdot F^{i-1}(x) \cdot \overline{F}^{n-i}(x)$$
$$= i \cdot \binom{n}{i} \frac{1}{2^n} \cdot \sin(x) \cdot (1 - \cos(x))^{i-1} \cdot (1 + \cos(x))^{n-i}.$$

For simplicity, define $C_{i:n} = i\binom{n}{i}2^{-n}$. Now let $n = 4$. In order the calculate the exact expected value formula, we must use the following angle identities,

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x), \quad \cos^2(x) = \frac{1}{2}(1 + \cos(2x)),$$

and the result of these integrals

$$\int x\sin(kx)dx = \frac{\sin(kx) - kx\cos(kx)}{k^2}.$$

From now on I'm going to simplify the trigonometric expressions using the TR8 algorithm provided by the package sympy. Also, use the package to integrate and evaluate the final expression. As always, the code is included with this document.

- $\boldsymbol{i = 1:}$

$$C_{1:4}^{-1} \cdot f_{1:4} = \frac{7\sin(x)}{4} + \frac{7\sin(2x)}{4} + \frac{3\sin(3x)}{4} + \frac{\sin(4x)}{8},$$

$$C_{1:4}^{-1} \cdot \mathbf{E}\left[X_{1:4}\right] = C_{1:4}^{-1} \cdot \int_0^\pi x f_{1:4} dx.$$

$$= \left. \begin{array}{r} -\frac{7x\cos(x)}{4} - \frac{7x\cos(2x)}{8} - \frac{x\cos(3x)}{4} - \frac{x\cos(4x)}{32} \\ \frac{7\sin(x)}{4} + \frac{7\sin(2x)}{16} + \frac{\sin(3x)}{12} + \frac{\sin(4x)}{128} \end{array} \right|_0^\pi = \frac{35\pi}{32}.$$

- $\boldsymbol{i = 2:}$

$$C_{2:4}^{-1} \cdot f_{2:4} = \frac{3\sin(x)}{4} + \frac{\sin(2x)}{4} - \frac{\sin(3x)}{4} - \frac{\sin(4x)}{8},$$

$$C_{2:4}^{-1} \cdot \mathbf{E}\left[X_{2:4}\right] = C_{2:4}^{-1} \cdot \int_0^\pi x f_{2:4} dx.$$

$$= \left. \begin{array}{r} -\frac{3x\cos(x)}{4} - \frac{x\cos(2x)}{8} + \frac{x\cos(3x)}{12} + \frac{x\cos(4x)}{32} \\ \frac{3\sin(x)}{4} + \frac{\sin(2x)}{16} - \frac{\sin(3x)}{36} - \frac{\sin(4x)}{128} \end{array} \right|_0^\pi = \frac{55\pi}{96}.$$

- $\boldsymbol{i = 3:}$

$$C_{3:4}^{-1} \cdot f_{3:4} = \frac{3\sin(x)}{4} - \frac{\sin(2x)}{4} - \frac{\sin(3x)}{4} + \frac{\sin(4x)}{8},$$

$$C_{3:4}^{-1} \cdot \mathbf{E}\left[X_{3:4}\right] = C_{3:4}^{-1} \cdot \int_0^\pi x f_{3:4} dx.$$

$$= \left. \begin{array}{r} -\frac{3x\cos(x)}{4} + \frac{x\cos(2x)}{8} + \frac{x\cos(3x)}{12} - \frac{x\cos(4x)}{32} \\ \frac{3\sin(x)}{4} - \frac{\sin(2x)}{16} - \frac{\sin(3x)}{36} + \frac{\sin(4x)}{128} \end{array} \right|_0^\pi = \frac{73\pi}{96}.$$

- $\boldsymbol{i=4:}$

$$C_{4:4}^{-1}\cdot f_{4:4}=\frac{7\sin\left(x\right)}{4}-\frac{7\sin\left(2x\right)}{4}+\frac{3\sin\left(3x\right)}{4}-\frac{\sin\left(4x\right)}{8},$$

$$C_{4:4}^{-1}\cdot\mathbf{E}\left[X_{4:4}\right]=C_{4:4}^{-1}\cdot\int_0^{\pi}xf_{4:4}dx.$$

$$=\left.\begin{array}{c}\frac{7x\cos\left(x\right)}{4}+\frac{7x\cos\left(2x\right)}{8}-\frac{x\cos\left(3x\right)}{4}+\frac{x\cos\left(4x\right)}{32}\\ \frac{7\sin\left(x\right)}{4}-\frac{7\sin\left(2x\right)}{16}+\frac{\sin\left(3x\right)}{12}-\frac{\sin\left(4x\right)}{128}\end{array}\right|_0^{\pi}=\frac{93\pi}{32}.$$

Therefore,

$$\mathbf{E}\left[X_{1:4}\right]=\frac{35\pi}{128},\quad \mathbf{E}\left[X_{2:4}\right]=\frac{55\pi}{128},$$
$$\mathbf{E}\left[X_{3:4}\right]=\frac{73\pi}{128},\quad \mathbf{E}\left[X_{4:4}\right]=\frac{93\pi}{128}.$$

Solution Part 2

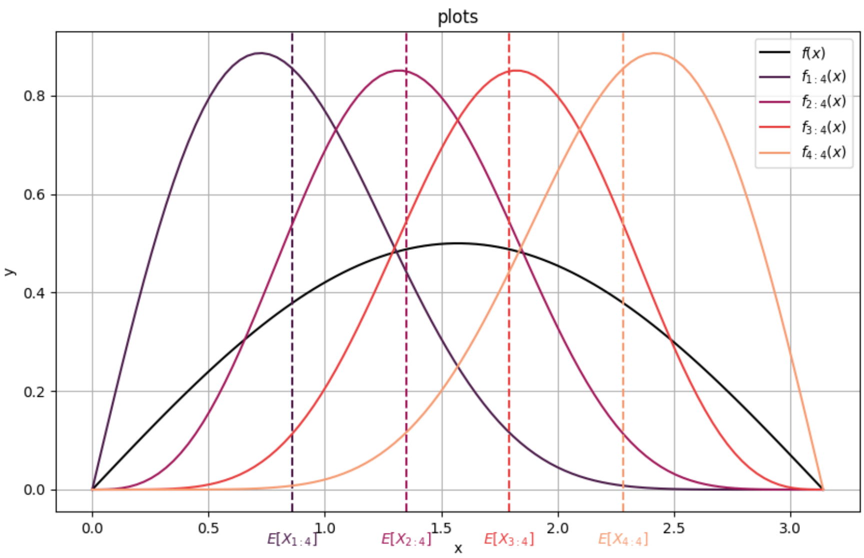

Figure 1: This is a floating figure with an image.

Exercise 2

1. **Formulate** the criterion that characterizes the existence of an extreme distribution (in terms of $\overline{F}$).

2. **Determine and justify** by this criterion if the following distribution has an extreme distribution. The cumulative distribution function is given by

$$F(n) := 1 - \frac{C}{(n+1)^{\ln(n+1)}}, \quad n \in \mathbb{N}.$$

3. In the case that it has a limit distribution, **argue** about which should be the limit distribution.

Exercise 3

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $X_1 \sim \mu$,

$$M_n := \max\{X_1, \ldots, X_n\}, \quad \text{and} \quad N_n := \min\{X_1, \ldots, X_n\}.$$

1. For $\mu = \text{Gamma}(\alpha, \beta),\ \alpha, \beta > 0$, **determine and justify** the extreme distribution of (M_n) and (N_n).

2. For $\mu = \text{Beta}(\alpha, \beta),\ \alpha, \beta > 0$, that is $f(x) = C_{\alpha,\beta} x^{\alpha-1}(1-x)^{\beta-1},\ \alpha, \beta > 0$, **determine and justify** the extreme distribution of (M_n) and (N_n).

3. For μ such that for $\alpha > 0$,

$$F(x) = \begin{cases} 0, & \text{for } x < 1, \\ \frac{\ln(x)}{x^\alpha}, & \text{for } x \geq 1. \end{cases}$$

determine and justify the extreme distribution of (M_n) and (N_n)