Stochastic Processes: Homework 3

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Exercise 1

We use the lexicographic ordering to assign a numerical index to each of the $24 = 4!$ permutations. Let,

$$f : \mathbb{Z}_{24} \to S_4 \qquad f(x) =$$

$$\begin{array}{llll}
0 \mapsto [0,1,2,3] & 6 \mapsto [1,0,2,3] & 12 \mapsto [2,0,1,3] & 18 \mapsto [3,0,1,2] \\
1 \mapsto [0,1,3,2] & 7 \mapsto [1,0,3,2] & 13 \mapsto [2,0,3,1] & 19 \mapsto [3,0,2,1] \\
2 \mapsto [0,2,1,3] & 8 \mapsto [1,2,0,3] & 14 \mapsto [2,1,0,3] & 20 \mapsto [3,1,0,2] \\
3 \mapsto [0,2,3,1] & 9 \mapsto [1,2,3,0] & 15 \mapsto [2,1,3,0] & 21 \mapsto [3,1,2,0] \\
4 \mapsto [0,3,1,2] & 10 \mapsto [1,3,0,2] & 16 \mapsto [2,3,0,1] & 22 \mapsto [3,2,0,1] \\
5 \mapsto [0,3,2,1] & 11 \mapsto [1,3,2,0] & 17 \mapsto [2,3,1,0] & 23 \mapsto [3,2,1,0]
\end{array}$$

Part (a)

For the transition matrix, take for instance the base permutation, $[0, 1, 2, 3]$, which has index 0. The 3 possibilities according to the shuffling rules in this case are:

$$f(0) = [0,1,2,3] \to [1,0,2,3] = f(6),$$

$$f(0) = [0,1,2,3] \to [1,2,0,3] = f(8),$$

$$f(0) = [0,1,2,3] \to [1,2,3,0] = f(9).$$

Therefore for the 0-th row in the transition matrix,

$$\Pi_{0,6} = \Pi_{0,8} = \Pi_{0,9} = 1/3,$$

and the rest of the entries are 0. The complete transition matrix would be:

$$\Pi = \frac{1}{3} \cdot \begin{bmatrix}
0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1\\
1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1\\
1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0
\end{bmatrix}$$

Now, to evaluate the uniformity of n shuffles we are going to calculate, for multiple n's, the uniform error e_n,

$$e_n = \|U - \Pi^n\|_\infty = \max_{i,j}(|\Pi_{i,j} - 1/24|),$$

where U is a matrix where all entries are $1/24$. This matrix represents the transition matrix for perfectly uniform shuffles.

$$\begin{array}{lll}
e_1 = 29.16667\% & e_2 = 6.94444\% & e_3 = 3.24074\% \\
e_4 = 0.7716\% & e_5 = 0.36008\% & e_6 = 0.08573\% \\
e_7 = 0.04001\% & e_8 = 0.00953\% & e_9 = 0.00445\% \\
e_{10} = 0.00106\% & e_{11} = 0.00049\% & e_{12} = 0.00012\%
\end{array}$$

Part (b)

Now, for this new set of shuffling rules, we have the following transition matrix:

$$
\Pi = \frac{1}{4} \cdot \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

Then, with the same formula as the previous part, we have the following errors.

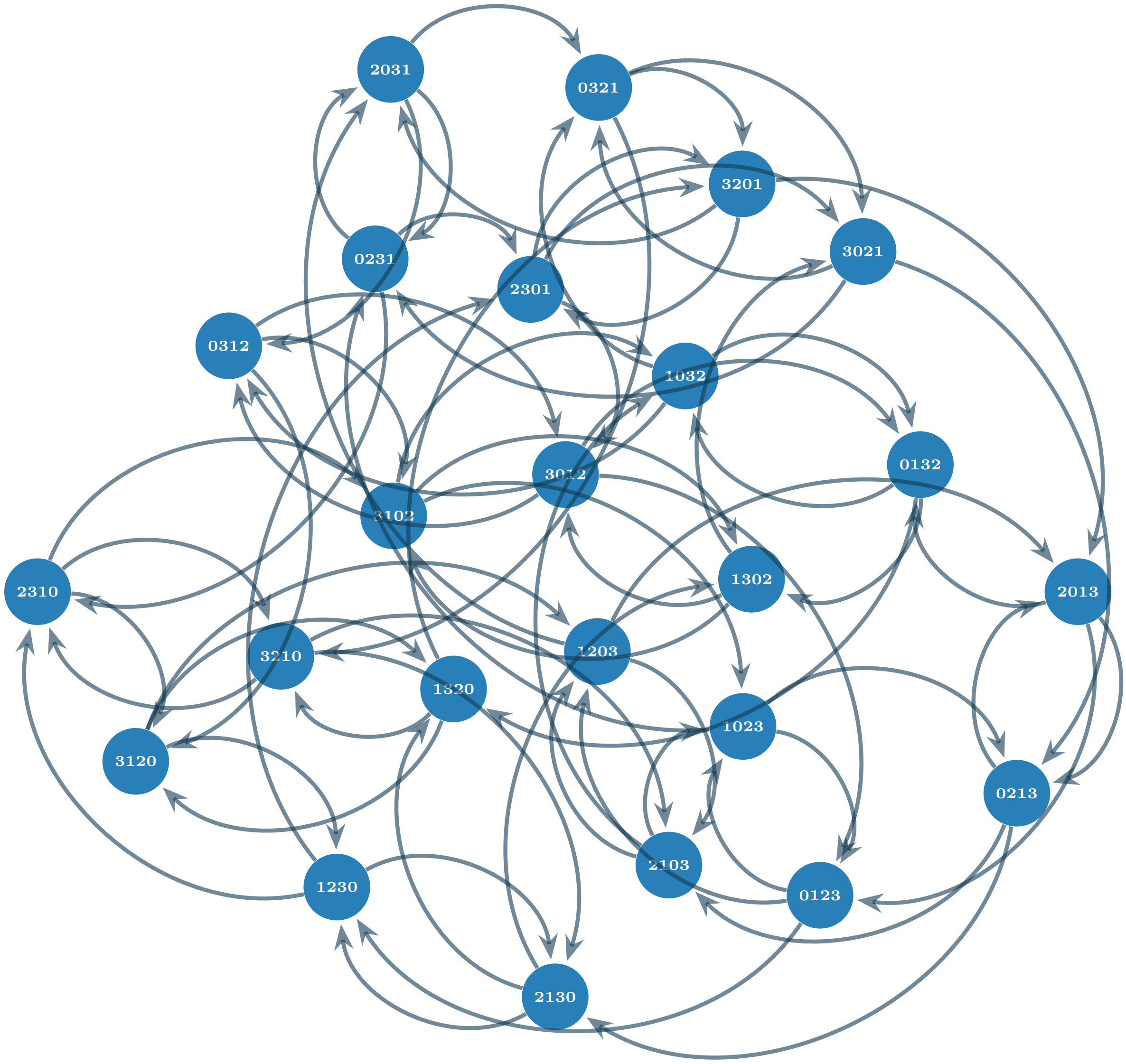
$$
\begin{array}{rrr}
e_1 = 20.83333\% & e_2 = 14.58333\% & e_3 = 9.89583\% \\
e_4 = 5.59896\% & e_5 = 5.20833\% & e_6 = 4.50033\% \\
e_7 = 4.2745\% & e_8 = 4.2394\% & e_9 = 4.19057\% \\
e_{10} = 4.1797\% & e_{11} = 4.1756\% & e_{12} = 4.16979\%
\end{array}
$$

The error seems to oscillate between $\pm 4.1667\%$. In fact, the error matrix seems to have the following behavior:

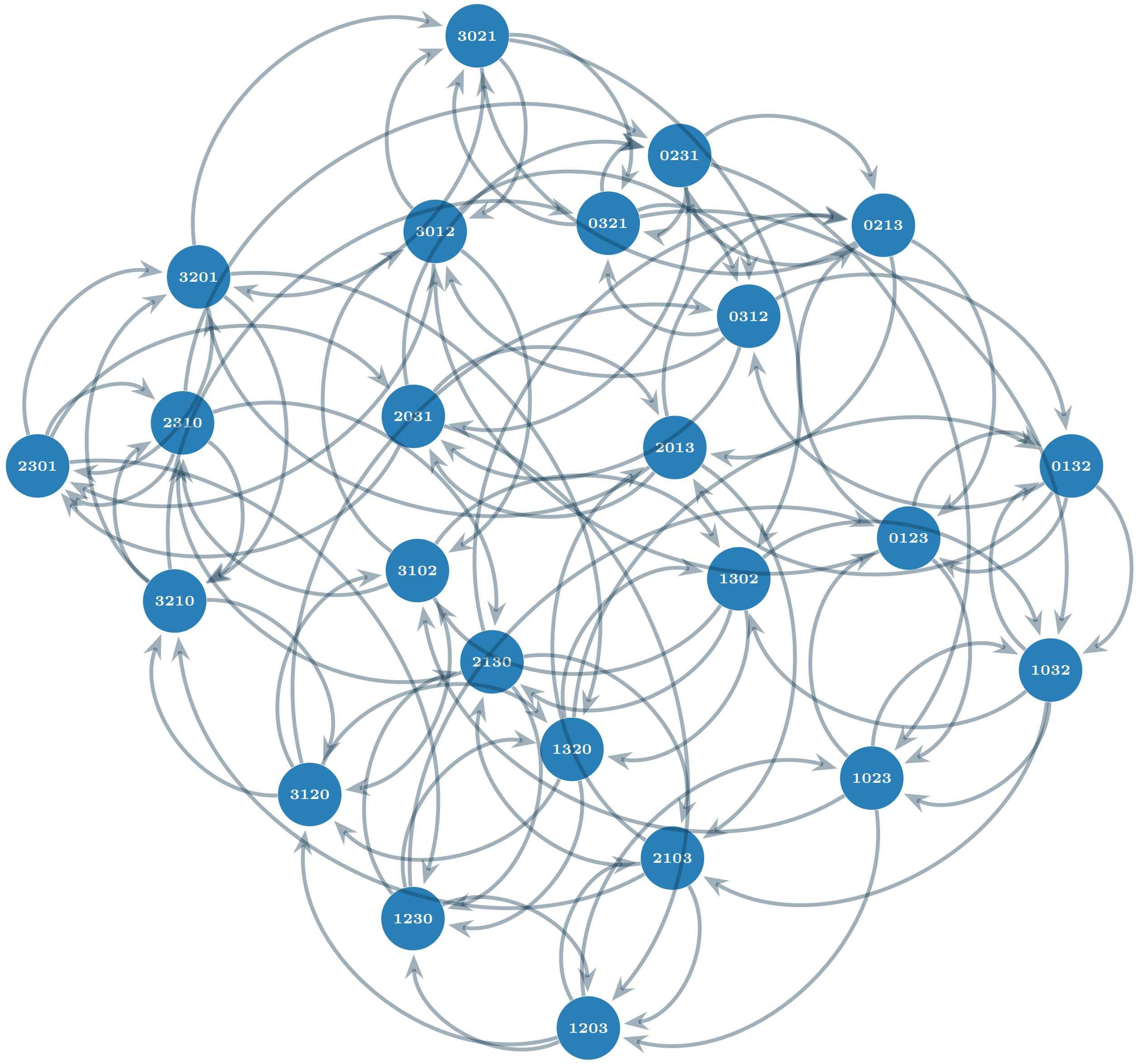
$$\Pi^n - U \approx -(\Pi^{n+1} - U),$$

and $|(\Pi - U)_{i,j}| \approx 4.1667\%$, and thus, the error never seems to converge to 0.

Graph Drawings Part (a)



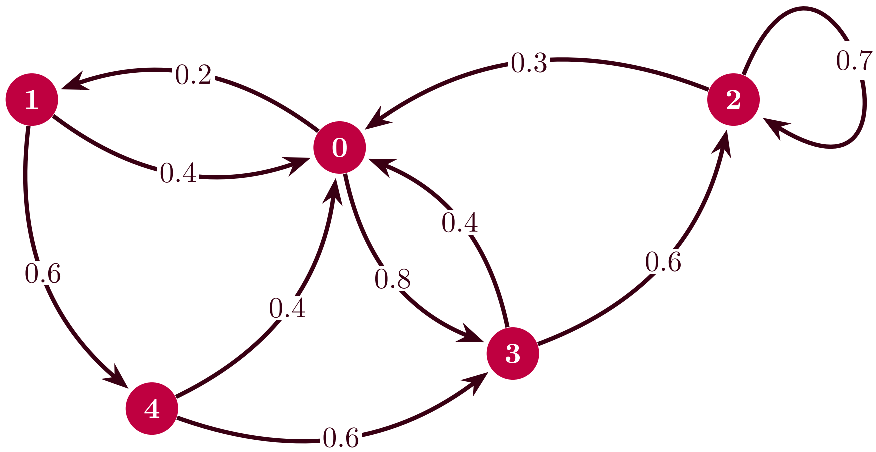
Graph Drawings Part (b)



Exercise 2

$$\Pi = \frac{1}{10} \cdot \begin{bmatrix} 0 & 2 & 0 & 8 & 0 \\ 4 & 0 & 0 & 0 & 6 \\ 3 & 0 & 7 & 0 & 0 \\ 4 & 0 & 6 & 0 & 0 \\ 4 & 0 & 0 & 6 & 0 \end{bmatrix}$$

Part (a)



Part (b)

Let $\Theta \sim \mathrm{UNI}[0,1]$, the function f for the dynamical system that is defined by the previous transition matrix is:

$$\begin{aligned}
f(0,\Theta) &= 1 \cdot \mathbb{1}_{[0,2)}(10\Theta) + 3 \cdot \mathbb{1}_{[2,10]}(10\Theta), \\
f(1,\Theta) &= 0 \cdot \mathbb{1}_{[0,4)}(10\Theta) + 4 \cdot \mathbb{1}_{[4,10]}(10\Theta), \\
f(2,\Theta) &= 0 \cdot \mathbb{1}_{[0,3)}(10\Theta) + 2 \cdot \mathbb{1}_{[3,10]}(10\Theta), \\
f(3,\Theta) &= 0 \cdot \mathbb{1}_{[0,4)}(10\Theta) + 2 \cdot \mathbb{1}_{[4,10]}(10\Theta), \\
f(4,\Theta) &= 0 \cdot \mathbb{1}_{[0,4)}(10\Theta) + 3 \cdot \mathbb{1}_{[4,10]}(10\Theta).
\end{aligned}$$

Part (c)

Now, with the formula

$$\mathbf{E}\left[T_4 \mid X_0 = i\right] = x_i = 1 + \sum_{j \neq i} \Pi_{i,j} x_j,$$

we get the following linear system for the expected value of the time of the first return to 4,

$$\begin{aligned}
10x_0 &= 10 + 2x_1 + 8x_3 \\
10x_1 &= 10 + 4x_0 + 6x_4 \\
10x_2 &= 10 + 3x_0 + 7x_2 \\
10x_3 &= 10 + 4x_0 + 6x_2 \\
10x_4 &= 0.
\end{aligned}$$

The equivalent system to solve is

$$\begin{bmatrix} 10 & -2 & 0 & -8 & 0 \\ -4 & 10 & 0 & 0 & -6 \\ -3 & 0 & 10 & 0 & 0 \\ -4 & 0 & -6 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 0 \end{bmatrix}.$$

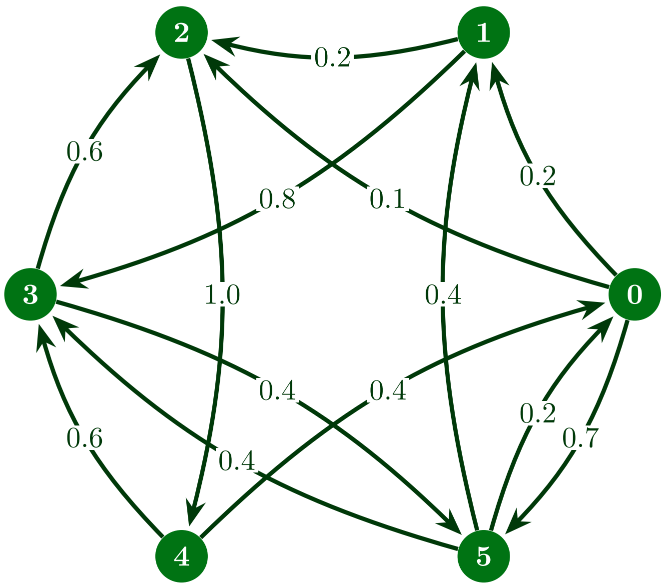
Then,

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5.44 \\ 3.18 \\ 2.63 \\ 4.75 \\ 0 \end{bmatrix}$$

Exercise 3

$$\Pi = \frac{1}{10} \cdot \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 7 \\ 0 & 0 & 2 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 6 & 0 & 0 & 4 \\ 4 & 0 & 0 & 6 & 0 & 0 \\ 2 & 4 & 0 & 4 & 0 & 0 \end{bmatrix}$$

Part (a)



Part (b)

Let $\Theta \sim \mathrm{UNI}[0,1]$, the function f for the dynamical system that is defined by the previous transition matrix is:

$$
\begin{aligned}
f(0,\Theta) &= 1 \cdot \mathbb{1}_{[0,2)}(10\Theta) + 2 \cdot \mathbb{1}_{[2,3]}(10\Theta) + 5 \cdot \mathbb{1}_{[3,10]}(10\Theta), \\
f(1,\Theta) &= 2 \cdot \mathbb{1}_{[0,2)}(10\Theta) + 3 \cdot \mathbb{1}_{[2,10]}(10\Theta), \\
f(2,\Theta) &= 4, \\
f(3,\Theta) &= 2 \cdot \mathbb{1}_{[0,6)}(10\Theta) + 5 \cdot \mathbb{1}_{[6,10]}(10\Theta), \\
f(4,\Theta) &= 0 \cdot \mathbb{1}_{[0,4)}(10\Theta) + 3 \cdot \mathbb{1}_{[4,10]}(10\Theta), \\
f(0,\Theta) &= 0 \cdot \mathbb{1}_{[0,2)}(10\Theta) + 1 \cdot \mathbb{1}_{[2,6]}(10\Theta) + 3 \cdot \mathbb{1}_{[6,10]}(10\Theta),
\end{aligned}
$$

Part (c)

Similar to the previous part, we get the following linear system for the expected time of return to state 4:

$$
\begin{aligned}
10x_0 &= 10 + 2x_1 + 1x_2 + 7x_5 \\
10x_1 &= 10 + 2x_2 + 8x_3 \\
10x_2 &= 10 + 10x_4 \\
10x_3 &= 10 + 6x_2 + 4x_5 \\
10x_4 &= 0 \\
10x_5 &= 10 + 2x_0 + 4x_1 + 4x_3
\end{aligned}
.
$$

Then,

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5.63 \\ 4.17 \\ 1 \\ 3.71 \\ 0 \\ 5.27 \end{bmatrix}$$