# Stochastic Processes: Homework 6

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## Exercise 2

Consider two independent Poisson processes with different parameters

(a) The sum of these processes, Which type of process is? Prove it.

(b) The difference between these processes, Which type of process is? Prove it.

### Solution Part (a)

The first process can be described by the random variable $N_1(t) \sim \text{Pois}(\lambda_1 t)$ and the second by $N_2(t) \sim \text{Pois}(\lambda_2 t)$. As we proved in previous assignments, the sum of 2 Poisson random variables is a Poisson random variable with the sum of the parameters. Thus,

$$N_1(t) + N_2(t) \sim \text{Pois}(\lambda_1 t + \lambda_2 t) = \text{Pois}((\lambda_1 + \lambda_2)t).$$

Therefore, it is a Poisson process with parameter $\lambda_1 + \lambda_2$.

### Solution Part (b)

The difference between 2 Poisson random variables has the Skellam distribution:

$$
\begin{aligned}
\mathbf{P}\{N_1(t) - N_2(t) = n\} &= \sum_{k=0}^{n} \mathbf{P}\{N_1(t) = k\}\mathbf{P}\{N_2(t) = n-k\} \\
&= \sum_{k=0}^{n} \left(\frac{e^{-\lambda_1 t}\lambda_1^k}{k!}\right)\left(\frac{e^{-\lambda_2 t}\lambda_2^{n-k}}{(n-k)!}\right) \\
&= e^{-(\lambda_1+\lambda_2)t} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)t} \left(\frac{\lambda_1}{\lambda_2}\right)^{n/2} I_{|n|}(2\sqrt{\lambda_1\lambda_2})
\end{aligned}
$$

Where $I_{|n|}(x)$ is the modified Bessel function of the first kind.

## Exercise 3

Show the Law of Large Numbers and the Central Limit Theorem for the Poisson Process

### Solution

The Law of Large numbers states that for a sequence of independent random variables $N_i \sim \text{Pois}(\lambda)$,

$$
\frac{1}{n}\sum_{i=1}^{n} N_i \to \lambda \quad \text{as } n \to \infty. \ a.e.
$$

As a matter of fact, $\sum_{i=1}^{n} N_i \sim \text{Pois}(\lambda n)$. Therefore, for the Poisson process $N(t) \sim \text{Pois}(\lambda t)$,

$$
\frac{N(n)}{n} \to \lambda \quad \text{as } n \to \infty. \ a.e.
$$

Now, for every rational $t = p/q$, if we re-parametrize $N(p/q) = N'(p) \sim \text{Pois}(p\frac{\lambda}{q})$ and denote $N'(p) = \sum_{i=1}^{p} N_i'$ where $N_i' \sim \text{Pois}(\lambda/q)$ we will obtain

$$
\frac{N(p/q)}{p/q} = q\frac{1}{p}\sum_{i=1}^{p} N_i' \to q\frac{\lambda}{q} = \lambda,
$$

For every $q \in \mathbb{N}$ $\mathbb{N}/q$, the sequence $\frac{N(p/q)}{p/q}$ converges almost everywhere to $\lambda$. Therefore, we can assert that for every increasing sequence of rational numbers $(t_n)_{n\in\mathbb{N}}$, the sequence $\frac{t_n}{t_n}$ converges almost everywhere to $\lambda$.

Finally, $N(t)/t \to \lambda$ a.e. in a dense subset (rational $t$), thus, $N(t)/t$ converges almost everywhere to $\lambda$.

Let $N(t) \sim \text{Pois}(\lambda t)$, define $\overline{N}(t) = \frac{N(t)-\lambda t}{\sqrt{\lambda t}}$

$$
\begin{aligned}
M_{\overline{N}(t)}(x) &= \mathbf{E}\left[\exp\left(x\frac{N(t)-\lambda t}{\sqrt{\lambda t}}\right)\right] \\
&= \exp(-x\sqrt{\lambda t}) \cdot \mathbf{E}\left[\exp\left(x\frac{N(t)}{\sqrt{\lambda t}}\right)\right] \\
&= \exp(-x\sqrt{\lambda t}) \cdot M_{N(t)}(x/\sqrt{\lambda t}) \\
&= \exp(-x\sqrt{\lambda t}) \cdot \exp\left(\lambda t(e^{x/\sqrt{\lambda t}} - 1)\right) \\
&= \exp\left[-x\sqrt{\lambda t} + \lambda t\left(\sum_{k=1}^{\infty} \frac{x^k}{k!(\lambda t)^{k/2}}\right)\right] \\
&= \exp\left[-x\sqrt{\lambda t} + x\sqrt{\lambda t} + x^2/2 + o(t^{-1/2})\right] \\
&\overset{t\to\infty}{\longrightarrow} e^{x^2/2} = M_Z(x)
\end{aligned}
$$

Where $Z \sim N(0,1)$. Therefore, the normalized process converges to the standard normal distribution when the time goes to infinity.

## Exercise 4

If $\pi_t$ is the random variable associated with the number of renovations that occur at a given time $t$. Then, its inverse function, takes as input the given number of renovations and returns the time they take. Since $\pi_t$ is the counting variable for exponential renovation times, then

$$\mu_t = \sum_{i=1}^{\pi_t} X_i,$$

where $X_i \sim \text{Exp}(\lambda)$, and thus, $\mu_t \sim \text{Gamma}(\pi_t, \lambda)$. This is a compound Poisson process.

## Exercise 5

For a random variable $X \sim \text{Geo}(p)$ with the geometric distribution

$$\begin{aligned}
\mathbf{P}\{X \geq n + m \mid X \geq m\} &= \frac{\mathbf{P}\{X \geq n + m \ \wedge \ X \geq m\}}{\mathbf{P}\{X \geq m\}} \\
&= \frac{\mathbf{P}\{X \geq n + m\}}{\mathbf{P}\{X \geq m\}} \\
&= \frac{(1-p)^{n+m}}{(1-p)^m} \\
&= (1-p)^n = \mathbf{P}\{X \geq n\}
\end{aligned}$$

This proves that the process is memoryless. Now, we define a sequence of i.i.d. random variables $(X_i)_{i\in\mathbb{N}} \sim \text{Geo}(p)$. Also, denote the $k$-th renovation time as $T_k = \sum_{i=1}^{k} X_i$, which has a negative binomial distribution since it's the sum of geometric random variables.

$$T_k \sim \text{BinNeg}(k, p).$$

For the trial $n$, we define the counting variable $N(n)$ as follows,

$$N(t) = \max\{n \in \mathbb{N} \ : \ T_n \leq t\}$$

This variable counts the number of renovations that occur in $n$-trials. Thus, it has the binomial distribution

$$N(t) \sim \text{Bin}(n, p)$$