Time Series: Homework 6

Martín Prado

November 22, 2024
Universidad de los Andes – Bogotá Colombia

Exercise 4.2

Establish whether or not the following function is the autocovariance function of a stationary process:

$$\gamma(h) = \begin{cases} 1, & \text{if } h = 0, \\ -.5, & \text{if } h = \pm 2, \\ -.25, & \text{if } h = \pm 3, \\ 0, & \text{otherwise.} \end{cases}$$

Solution

Let $f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-inx} \gamma(n)$, that is

$$\begin{aligned} 2\pi f(x) &= -0.25(e^{-3ix} + e^{3ix}) - 0.5(e^{-2ix} + e^{2ix}) + e^{0ix} \\ &= -0.5\cos(3x) - \cos(2x) + 1 \end{aligned}$$

Finally, note that $f(0) = -0.5 - 1 + 1 = -0.5$ so according to **Corollary 4.3.2** the function γ doesn't correspond to an autocovariance function.

Exercise 4.3

If $0 < a < \pi$, use equation (4.3.1) to show that

$$\gamma(h) = \begin{cases} h^{-1}\sin(ah), & h = \pm 1, \pm 2, \ldots, \\ a, & h = 0, \end{cases}$$

is the autocovariance function of a stationary process $\{X_t, t = 0, \pm 1, \ldots\}$. What is the spectral density of $\{X_t\}$?

Solution

Note: For this exercise and exercise 4.9, the function $\ln : \mathbb{C}\backslash(-\infty, 0) \to \mathbb{C}$ is the main branch of the complex log function

Let $f(x) = \dfrac{1}{2\pi}\left(a + \sum\limits_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \dfrac{e^{-inx}\sin(an)}{n}\right)$ and note that since $n^{-1}\sin(an) = (-n)^1\sin(-an)$, it follows that,

$$2\pi f(x) = a + \sum_{n=1}^{\infty} \frac{\cos(nx)\sin(an)}{n}$$

Then, using the complex identity $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$, we obtain

$$\begin{aligned}
\cos(nx)\sin(an) &= \frac{i}{4}\left(e^{-ina} - e^{ina}\right)\left(e^{-inx} + e^{inx}\right) \\
&= \frac{i}{4}(e^{(-a-x)in} + e^{(-a+x)in} - e^{(a-x)in} - e^{(a+x)in})
\end{aligned}$$

Now, using the identity $\sum_{n=1}^{\infty} y^n/n = -\ln(1-y)$ for $|y| \leq 1$ and $y \neq -1$, we obtain with the function $y(z) = e^{iz}$ the following result:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\cos(nx)\sin(an)}{n} &= \frac{i}{4}\sum_{n=1}^{\infty} n^{-1}(e^{(-a-x)in} + e^{(-a+x)in} - e^{(a-x)in} - e^{(a+x)in}) \\
&= \frac{i}{4}\sum_{n=1}^{\infty} n^{-1}(y(-a-x)^n + y(-a+x)^n - y(a-x)^n - y(a+x)^n) \\
&= \frac{-i}{4}(\ln(1-y(-a-x)) + \ln(1-y(-a+x)) - \ln(1-y(a-x)) - \ln(1-y(a+x))) \\
&= \frac{-i}{4}\ln\left(\frac{(1-e^{(-a-x)i})}{(1-e^{(a+x)i})}\cdot\frac{(1-e^{(-a+x)i})}{(1-e^{(a-x)i})}\right).
\end{aligned}$$

Now, use the identity $(1-z)/(1+z^{-1}) = -z$ with $z = e^{(-a\pm x)i}$ to obtain

$$\begin{aligned}
\cdots &= \frac{-i}{4}\ln\left((-e^{(-a-x)i})(-e^{(-a+x)i})\right) \\
&= \frac{-i}{4}(-a-x-a+x)i = \frac{a}{2} \geq 0. \\
\implies f(x) &= \frac{1}{2\pi}(a + a/2) = \frac{3a}{4\pi} \geq 0.
\end{aligned}$$

Using **Corollary 4.3.2** we can conclude that γ is an autocovariance function for a process with spectral density $f(\lambda) = 3a/4\pi$.

Exercise 4.8

Let $\{X_t\}$ and $\{Y_t\}$ be stationary zero-mean processes with spectral densities f_X and f_Y. If $f_X(\lambda) \le f_Y(\lambda)$ for all $\lambda \in [-\pi, \pi]$, show that

(a) $\Gamma_{n,Y} - \Gamma_{n,X}$ is a non-negative definite matrix, where $\Gamma_{n,Y}$ and $\Gamma_{n,X}$ are the covariance matrices of $\mathbf{Y} = (Y_1, \dots, Y_n)'$ and $\mathbf{X} = (X_1, \dots, X_n)'$, respectively, and

(b) $\mathrm{Var}(\mathbf{b}'\mathbf{X}) \le \mathrm{Var}(\mathbf{b}'\mathbf{Y})$ for all $\mathbf{b} = (b_1, \dots, b_n)' \in \mathbb{R}^n$.

Solution

Let γ_X, γ_Y be the respective autocovariance functions for X_t and Y_t. Then, we have that

$$f_Y(\lambda) - f_X(\lambda) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-ni\lambda}(\gamma_Y(n) - \gamma_X(n)) \ge 0,$$

so let $m = \inf_\lambda f_Y(\lambda) - f_X(\lambda) \ge 0$.

According to **Definition 1.5.1** a function $\kappa : \mathbb{Z} \to \mathbb{R}$ is non-negative definite if for any vector $a = (a_1, \dots, a_n)' \in \mathbb{R}^n$,

$$a'\Gamma_n a = \sum_{i,j=1}^{n} a_i \kappa(i-j) a_j \ge 0,$$

for the matrix $(\Gamma_n)_{i,j} = \kappa(i-j)$.

Then, let $\kappa(h) = \gamma_Y(h) - \gamma_X(h)$, let $(\Gamma_n)_{i,j} = \kappa(i-j)$ and similar to the proof of **Proposition 4.5.3**

$$\begin{aligned}
a'\Gamma_n a &= a'\Gamma_{n,Y} a - a'\Gamma_{n,X} a \\
&= \int_{-\pi}^{\pi} \left| \sum_{j=1}^{n} a_j e^{-ijv} \right|^2 f_Y(v) dv - \int_{-\pi}^{\pi} \left| \sum_{j=1}^{n} a_j e^{-ijv} \right|^2 f_X(v) dv \\
&= \int_{-\pi}^{\pi} \left| \sum_{j=1}^{n} a_j e^{-ijv} \right|^2 \underbrace{(f_Y(v) - f_X(v))}_{\ge m \ge 0} dv \\
&\ge \int_{-\pi}^{\pi} \sum_{j,k=1}^{n} a_j a_k e^{-i(k-j)v} m \, dv \\
&= 2\pi m \sum_{j=1}^{n} a_j^2 \ge 0.
\end{aligned}$$

Finally, note that $a'\Gamma_{n,Y} a - a'\Gamma_{n,X} a = \mathbf{Var}\,(a'Y_{1:n}) - \mathbf{Var}\,(a'X_{1:n}) \ge 0$ so not only $\Gamma_{n,Y} - \Gamma_{n,X}$ is non-negative definite, but also, $\mathbf{Var}\,(a'Y_{1:n}) \ge \mathbf{Var}\,(a'X_{1:n})$ for any vector $a \in \mathbb{R}^n$.

Exercise 4.9

Let $\{X_t\}$ be the process

$$X_t = A\cos(\pi t/3) + B\sin(\pi t/3) + Y_t$$

where $Y_t = Z_t + 2.5Z_{t-1}$, $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and A and B are uncorrelated $\mathcal{N}(0, v^2)$ random variables which are also uncorrelated with $\{Z_t\}$. Find the covariance function and the spectral distribution function of $\{X_t\}$.

Solution Autocovariance

For **Exercise 1.7(c)** I proved that for $Z_1, Z_2 \sim \mathcal{N}(0, \sigma^2)$, the process $Z_1\cos(ct) + Z_2\sin(ct)$ has autocovariance function $h \mapsto \sigma^2\cos(ch)$. Therefore, the process

$$A\cos(\pi t/3) + B\sin(\pi t/3)$$

is an stationary process with autocovariance function $\gamma_1(h) = v^2\cos(\pi h/3)$.

On the other hand, according to Example 3.1.1, the autocovariance function for a **MA** (q) process $Y_t = \theta(B)Z_t$ is

$$\gamma_2(h) = \mathbb{1}_{|h|\leq q}\sigma^2 \sum_{j=0}^{q-|h|} \theta_j\theta_{j+h}.$$

Thus, for the **MA** (1) process $Y_t = Z_t + 2.5Z_{t-1}$,

$$\gamma_2(0) = \sigma^2(1 + 2.5^2) = 7.25\sigma^2, \quad \gamma_2(1) = 2.5\sigma^2.$$

Finally, in **Exercise 1.11** we proved that the autocovariance of the sum of uncorrelated stationary processes is the sum of their respective autocovariance functions. Thus, the autocovariance function for X_t is

$$\gamma(h) = \gamma_1(h) + \gamma_2(h).$$

Solution Spectral Distribution

According to equation (4.2.3), the spectral distribution F_1 of γ_1 must satisfy

$$\gamma_1(h) = \int_{-\pi}^{\pi} e^{ihv}dF(v).$$

Let $u = \pi t/3$. Note that we cannot take the spectral density

$$\begin{aligned} f_1(\lambda) &= \frac{1}{2\pi}v^2 \sum_{n=-\infty}^{\infty} e^{-in\lambda}\cos(nu) \\ &= \frac{v^2}{2\pi} + \frac{v^2}{\pi}\sum_{n=1}^{\infty}\cos(n\lambda)\cos(nu) = \infty \end{aligned}$$

because it diverges when $\lambda = \pm u$. Instead, since γ is non-definite, F must exist, so compute the spectral distribution by taking the antiderivative of every term directly:

$$
\begin{aligned}
F(\lambda) &= \frac{v^2}{2\pi} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{e^{-in\lambda}}{-in} \cos(nu) \\
&= \frac{v^2}{2\pi} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{-2ni} \left(e^{-i(\lambda-u)n} + e^{-i(\lambda+u)n} \right) \\
{\scriptstyle (y(z)\,=\,e^{iz})} \quad &= \frac{-v^2}{4\pi i} \sum_{n=1}^{\infty} \frac{1}{n} (y(-\lambda+u)^n + y(-\lambda-u)^n - y(\lambda-u)^n - y(\lambda+u)^n)
\end{aligned}
$$

Then, using the identity $\sum_{n=1}^{\infty} y^n/n = -\ln(1-y)$ for $|y| \leq 1$ and $y \neq -1$, we obtain

$$
\begin{aligned}
\cdots &= \frac{-v^2}{4\pi i}(-\ln(1-y(-\lambda+u)) - \ln(1-y(-\lambda-u)) + \ln(1-y(\lambda-u)) + \ln(1-y(\lambda+u))) \\
&= \frac{-v^2}{4\pi i} \ln \left(\frac{(1-e^{(-\lambda+u)i})}{(1-e^{(\lambda-u)i})} \cdot \frac{(1-e^{(-\lambda-u)i})}{(1-e^{(\lambda+u)i})} \right)
\end{aligned}
$$

Use the identity $(1-x)/(1-x^{-1}) = -x$ with $x = e^{(-\lambda\pm u)}$ to obtain

$$
\begin{aligned}
\cdots &= \frac{-v^2}{4\pi i}(\ln(e^{(-\lambda+u)i}) + \ln(e^{(-\lambda-u)i})) \\
&= \frac{-v^2}{4\pi i}(-2\lambda)i = \frac{v^2}{2\pi}\lambda.
\end{aligned}
$$

On the other hand, in **Example 4.4.1.** we proved that the spectral density for any **MA** (1) process $Y_t = Z_t + \theta Z_{t-1}$ is

$$
f_2(\lambda) = \frac{\sigma^2}{2\pi}(1 + 2\theta\cos(\lambda) + \theta^2).
$$

For our case, $\theta = 2.5$, so

$$
f_2(\lambda) = \frac{\sigma^2}{2\pi}(5\cos(\lambda) + 7.25),
$$

and thus,

$$
F_2(\lambda) = \frac{\sigma^2}{2\pi}(5\sin(\lambda) + 7.25\lambda).
$$

Finally, by linearity of the integral we obtain that the spectral density of X_t is

$$
F(\lambda) = F_1(\lambda) + F_2(\lambda) = \frac{1}{2\pi}(5\sigma^2\sin(\lambda) + (7.25\sigma^2 + v^2)\lambda).
$$

Exercise 12

Let $\{X_t\}$ denote the Wölfer sunspot numbers (Example 1.1.5) and let $\{Y_t\}$ denote the mean-corrected series, $Y_t = X_t - 46.93$, $t = 1, \ldots, 100$. The following AR(2) model for $\{Y_t\}$ is obtained by equating the theoretical and sample autocovariances at lags 0, 1, and 2:

$$Y_t - 1.317Y_{t-1} + 0.634Y_{t-2} = Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, 289.3).$$

(These estimated parameter values are called ”Yule–Walker” estimates and can be found using the program PEST, option 3.) Determine the spectral density of the fitted model and find the frequency at which it achieves its maximum value. What is the corresponding period? (The spectral density of any ARMA process can be computed numerically using the program PEST, option 5.)

Solution

The spectral density of an ARMA process according to **Theorem 4.4.2** is

$$f(\lambda) = \frac{\sigma^2}{2\pi} \left| \frac{\theta(e^{-\lambda})}{\phi(e^{-i\lambda})} \right|^2,$$

and in our case, $\theta(z) = 1$ and $\psi(z) = 1 - 1.317z - 0.634z^2$ with $\sigma^2 = 289.3$ so

$$f_Y(\lambda) = \frac{289.3}{2\pi} \frac{1}{|1 - 1.317e^{-i\lambda} - 0.634e^{-2i\lambda}|^2}.$$

To find λ for which f is maximized we must minimize $|1 + az + vz^2|^2$ for $a = -1.317$, $b = 0.634$ and $z = x + iy$ with the restriction $G(x, y) = |z|^2 = x^2 + y^2 = 1$.

$$F(z) = \left\| \begin{bmatrix} 1 + ax + bx^2 - by^2 \\ ay + 2bxy \end{bmatrix} \right\|^2 = (1 + ax + bx^2 - by^2)^2 + (ay + 2bxy)^2.$$

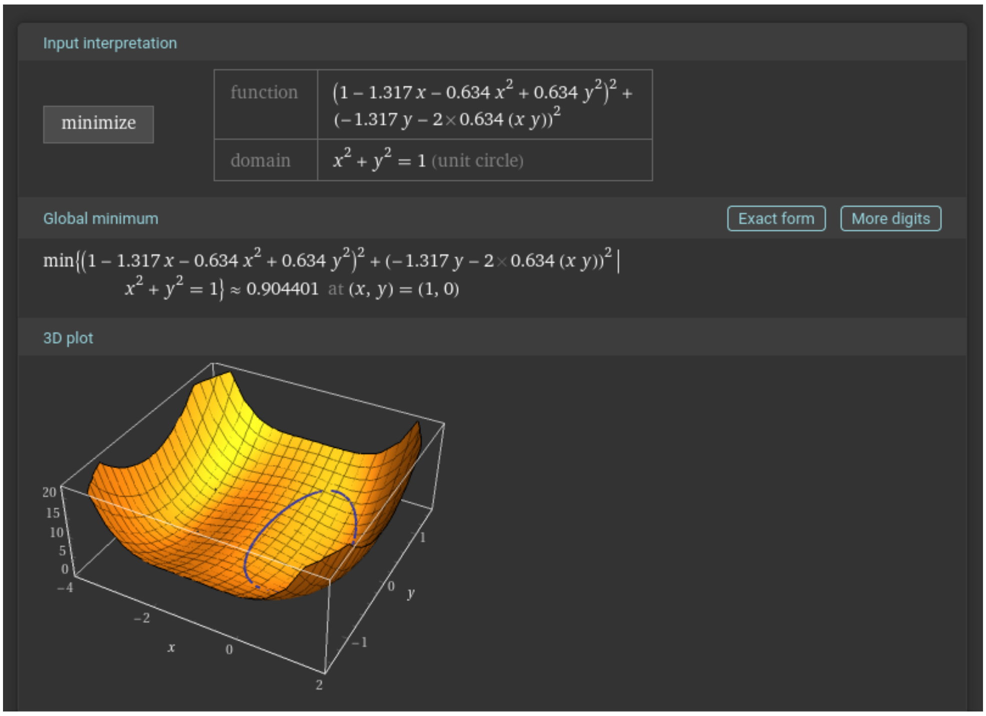
With Lagrange multipliers we obtain that the minimum of this function F at the restriction $G(x, y) = 1$ is

$$\min_{|z|=1} F(z) = 0.904401 = (1 - 1.317 - 0.634)^2$$

at $e^{-i\lambda} = z = 1 + 0i$, that is when $\lambda = 0$. Thus,

$$\max_{\lambda \in [-\pi, \pi]} f(\lambda) = \frac{289.3}{0.904401 \cdot 2\pi} \approx 50.9105$$

and the frequency λ that maximizes f is $\lambda_{\max} = 0$. Therefore, the period is undefined.



Exercise 18

Let $\{X_t\}$ be any stationary series with continuous spectral density f such that $0 \leq f(\lambda) \leq K$ and $f(\pi) \neq 0$.

Let $f_n(\lambda)$ denote the spectral density of the differenced series $\{(1-B)^n X_t\}$.

(a) Express $f_n(\lambda)$ in terms of $f_{n-1}(\lambda)$ and hence evaluate $f_n(\lambda)$.

(b) Show that $f_n(\lambda)/f_n(\pi) \to 0$ as $n \to \infty$ for each $\lambda \in [0, \pi)$.

(c) What does (b) suggest regarding the behaviour of the sample-paths of $\{(1-B)^n X_t\}$ for large values of n?.

(d) Plot $\{(1-B)^n X_t\}$ for $n = 1,\ 2,\ 3$ and 4 , where X_t, $t = 1, \ldots, 100$ are the Wolfer sunspot numbers (Example 1.1.S). Do the realizations exhibit the behavior expected from (c)? Notice the dependence of the sample variance on the order of differencing. (The graphs and the sample variances can be found using the program PEST.)

Solution Part (a)

Using **Theorem 4.4.1** we obtain that if $Y_t = (1-B)X_t$, then the spectral distribution of Y is

$$
\begin{aligned}
F_Y(\lambda) &= \int_{-\pi}^{\lambda} \left|1 - e^{-it}\right|^2 f_X(t)dt \\
&= \int_{-\pi}^{\lambda} (1 - e^{-it})\overline{(1 - e^{-it})} f_X(t)dt \\
&= \int_{-\pi}^{\lambda} (1 - e^{-it})(1 - e^{it}) f_X(t)dt \\
&= \int_{-\pi}^{\lambda} (2 - (e^{-it} + e^{it})) f_X(t)dt \\
&= \int_{-\pi}^{\lambda} (2 - 2\cos(t)) f_X(t)dt
\end{aligned}
$$

Thus, by the Fundamental Theorem of Calculus

$$
f_(Y)(\lambda) = (2 - 2\cos(\lambda)) f_X(\lambda),
$$

and therefore,

$$
f_n(\lambda) = 2^n(1 - \cos(\lambda))^n f_X(\lambda).
$$

Solution Part (b)

From the previous result, it follows that since $\lambda \in [0, \pi)$, $(1 - \cos(\lambda)) < 2$, and thus,

$$
\frac{f_n(\lambda)}{f_n(\pi)} = \frac{2^n(1 - \cos(\lambda))^n}{2^n(1 - (-1))^2} \frac{f_X(\lambda)}{f_X(\pi)} = \frac{(1 - \cos(\lambda))^n}{2^n} \frac{f_X(\lambda)}{f_X(\pi)} \to 0.
$$

Solution Part (c)

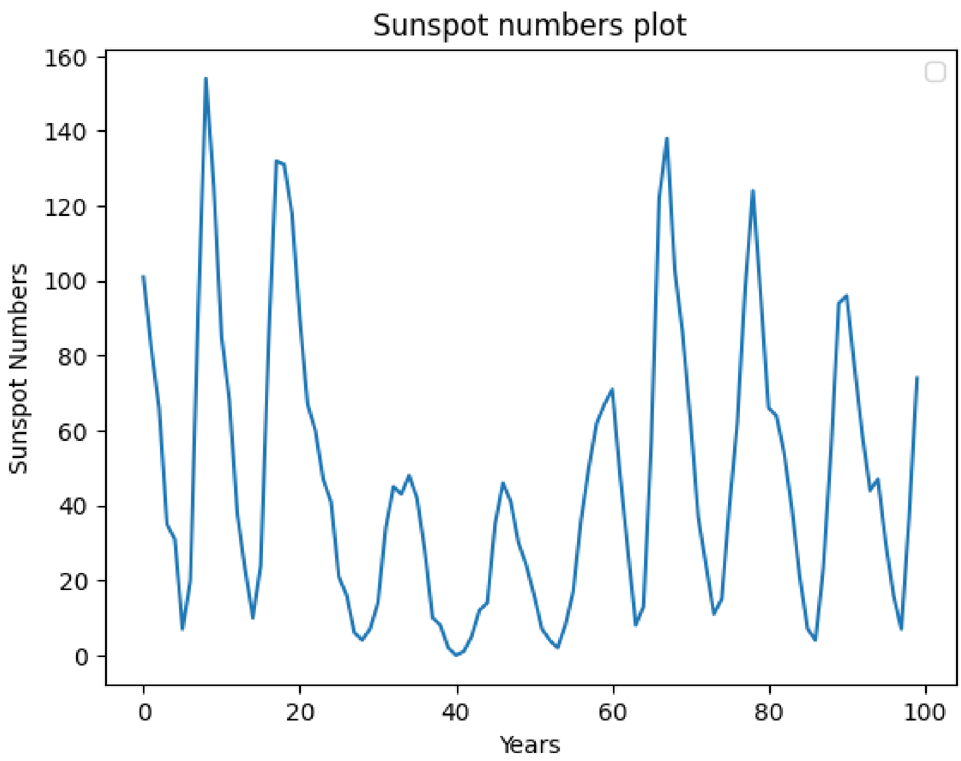
The function f_n is maximized at $\lambda = \pi$ with $f_n(\pi) = O(4^n)$ while at any other frequency, $f_n(\lambda) = O(2^n(1 - \cos(\lambda))^n)$, and $f_n(0) = 0$ minimizes the function. In fact, according to **Proposition 4.5.3** $0 \leq \mathbf{Var}\left((I - B)^n X_t\right) \leq 2\pi 4^n f_X(\pi)$. So the values of $\nabla^n X_t$ might get exponentially larger and the period exponentially shorter when $n \to \infty$.

Solution Part (d)

In order to plot the diffs of the sunspot numbers' sample we use the following Python code

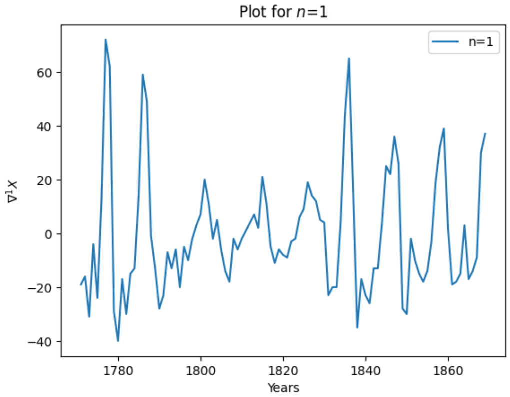
```
import numpy as np
import matplotlib.pyplot as plt
```

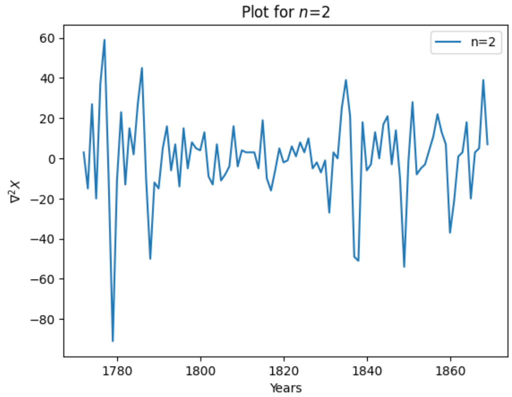
```
X = np.array([
    101, 82, 66, 35, 31, 7, 20, 92, 154, 125,
    85, 68, 38, 23, 10, 24, 83, 132, 131, 118,
    90, 67, 60, 47, 41, 21, 16, 6, 4, 7,
    14, 34, 45, 43, 48, 42, 28, 10, 8, 2,
    0, 1, 5, 12, 14, 35, 46, 41, 30, 24,
    16, 7, 4, 2, 8, 17, 36, 50, 62, 67,
    71, 48, 28, 8, 13, 57, 122, 138, 103, 86,
    63, 37, 24, 11, 15, 40, 62, 98, 124, 96,
    66, 64, 54, 39, 21, 7, 4, 23, 55, 94,
    96, 77, 59, 44, 47, 30, 16, 7, 37, 74
])
years = np.arange(1770,1870)
plt.plot(X)
```

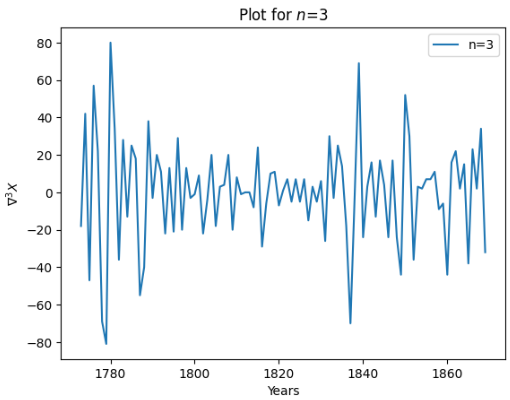


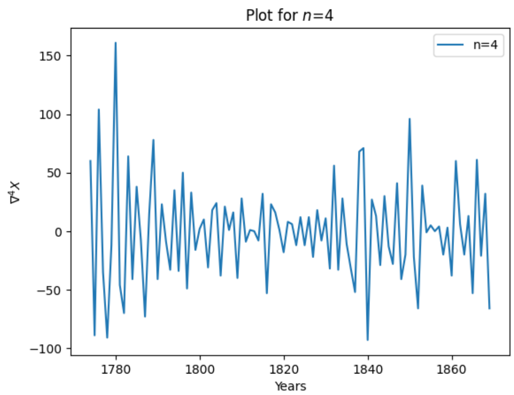

```
from functools import reduce
I = lambda X : X[1:]
B = lambda X : X[:-1]

for n in range(1,5):
    years_reduced = reduce(lambda x, _: I(x), range(n), years)
    I_minus_B_n_times = reduce(lambda x, _: I(x)-B(x), range(n), X)

    plt.figure()
    plt.plot(years_reduced, I_minus_B_n_times, label=f'n={n}')
    plt.xlabel("Years")
    plt.ylabel(rf"$\nabla^{{{n}}} X$")
```

```
    plt.title(f"Plot for $n$={n}")
    plt.legend()
    plt.show()
```



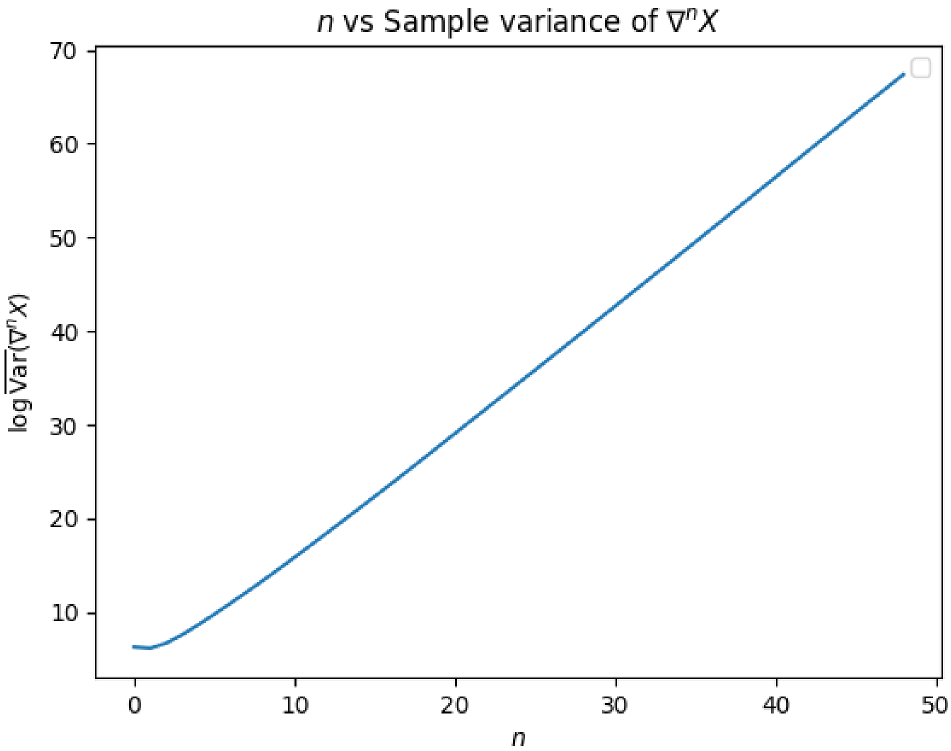






Now, we'll see that the sample variance of $(I - B)^n X$ increases exponentially with the following code

```
variance = []
for n in range(1,50,1):
    I_minus_B_n_times = reduce(lambda x, _: I(x)-B(x), range(n), X)
    variance.append(np.var(I_minus_B_n_times, ddof=1))
plt.plot(np.log(variance))
plt.xlabel("$n$")
plt.ylabel(rf"$\log \overline{{\text{{Var}}}}(\nabla^n X)$")
plt.title(rf"$n$ vs Sample variance of $\nabla^n X$")
plt.legend()
plt.show()
```



n vs Sample variance of $\nabla^n X$
70
60
50
40
30
20
10
$\log \overline{\mathrm{Var}}(\nabla^n X)$
0
10
20
30
40
50
n