

Time Series: Homework 4

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Exercise 3.19.

Suppose that $\{X_t\}$ and $\{Y_t\}$ are two zero-mean stationary processes with the same autocovariance function and that $\{Y_t\}$ is an ARMA(p, q) process. Show that $\{X_t\}$ must also be an ARMA(p, q) process. (Hint: If ϕ_1, \dots, ϕ_p are the AR coefficients for $\{Y_t\}$, show that $\{W_t := X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p}\}$ has an autocovariance function which is zero for lags $|h| > q$. Then apply Proposition 3.2.1 to $\{W_t\}$.)

Solution

Let $\gamma(\cdot)$ be the autocovariance function for both $\{X_t\}$ and $\{Y_t\}$.

If $\{Y_t\}$ is an **ARMA** (p, q) process, then there exists $\{Z_t\} \sim \mathbf{WN}(0, \sigma^2)$ (with $\sigma > 0$) and functions $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$, $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$ such that

$$\phi(B)X_t = \theta(B)Z_t.$$

To avoid ambiguity let $\theta_0 = 1$ and $\theta_q \neq 0$.

Now, define $W_t = \phi(B)X_t$ and $W'_t = \phi(B)Y_t$. Then note that $\mathbf{E}(W_t) = 0$ and $\mathbf{E}(W_t^2) < \infty$. In order to prove there exists an autocovariance function for $\{W_t\}$, note that $\{W'_t\}$ is stationary because $W'_t = \theta(B)Z_t$ is a sum of uncorrelated stationary random variables. Therefore, there exists an autocovariance function $\omega(\cdot) = \mathbf{cov}(W_t, W_{t+h})$, $\forall t \in \mathbb{Z}$, and from the following calculations we can deduce that the autocovariance function of the process

$\{W_t\}$ is ω too:

$$\begin{aligned}
\mathbf{cov}(W_t, W_{t+h}) &= \mathbf{E}[W_t W_{t+h}] \\
&= \sum_{i=0}^p \sum_{j=0}^p \phi_i \phi_j \mathbf{E}[X_{t-i} X_{t+h-j}] \\
&= \sum_{i=0}^p \sum_{j=0}^p \phi_i \phi_j \gamma(h-j+i) \\
&= \sum_{i=0}^p \sum_{j=0}^p \phi_i \phi_j \mathbf{E}[Y_{t-i} Y_{t+h-j}] \\
&= \mathbf{E}[W'_t W'_{t+h}] = \omega(h).
\end{aligned}$$

Then, note that for $|h| > q$, $\mathbf{E}[Z_{h-i} Z_{-j}] = 0$ for every $i, j \in \{0, \dots, q\}$ because in order to $\mathbf{E}[Z_{h-i} Z_{-j}] = \sigma^2 \neq 0$, it must happen that $h-i = -j \iff h = i-j$ but in this case $i-j \leq q$. Therefore,

$$\begin{aligned}
\omega(h) &= \mathbf{E}[W'_h W'_0] \\
&= \mathbf{E}[\theta(B) Z_h \theta(B) Z_0] \\
&= \sum_{i=0}^q \sum_{j=0}^q \theta_i \theta_j \mathbf{E}[Z_{h-i} Z_{-j}] \\
&= 0.
\end{aligned}$$

Finally, if $i, j \in \{0, \dots, q\}$, then $\mathbf{E}[Z_{q-i} Z_{-j}] \neq 0$ only if $i = q, j = 0$. Thus,

$$\omega(q) = \sum_{i=0}^q \sum_{j=0}^q \theta_i \theta_j \mathbf{E}[Z_{q-i} Z_{-j}] = \theta_q \theta_0 \sigma^2 \neq 0.$$

Proposition 3.2.1. If $\{W_t\}$ is a zero-mean stationary process with autocovariance function $\omega(\cdot)$ such that $\omega(h) = 0$ for $|h| > q$ and $\omega(q) \neq 0$, then $\{W_t\}$ is an $MA(q)$ process, i.e. there exists a white noise process $\{Z'_t\}$ such that

$$W_t = Z'_t + \theta_1 Z'_{t-1} + \dots + \theta_q Z'_{t-q}.$$

So from this proposition follows that X_t is and **ARMA** (p, q) process.

Exercise 3.22.

If $X_t = Z_t - \theta Z_{t-1}$, $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ and $|\theta| < 1$, show from the prediction equations that the best linear predictor of X_{n+1} in $\overline{\text{sp}}\{X_1, \dots, X_n\}$ is

$$\hat{X}_{n+1} = \sum_{j=1}^n \phi_j X_{n+1-j},$$

where ϕ_1, \dots, ϕ_n satisfy the difference equations,

$$-\theta\phi_{j-1} + (1 + \theta^2)\phi_j - \theta\phi_{j+1} = 0, \quad 2 \leq j \leq n-1,$$

with boundary conditions,

$$(1 + \theta^2)\phi_n - \theta\phi_{n-1} = 0$$

and

$$(1 + \theta^2)\phi_1 - \theta\phi_2 = -\theta.$$

Solution

The prediction equations state that

$$\left\langle X_{n+1} - \sum_{j=1}^n \phi_j X_{n+1-j}, X_k \right\rangle = 0, \quad k = n, n-1, \dots, 1.$$

Equivalently, we know that if $\{X_t\}$ is a stationary process with autocovariance function $\gamma(\cdot)$, then $\phi = (\phi_1, \dots, \phi_n)^T$ is the solution for the equation

$$\Gamma_n \phi = \gamma_n,$$

where $\gamma_n = (\gamma(1), \dots, \gamma(n))^T$ and $\Gamma_n = [\gamma(i-j)]_{i,j=1}^n$. Note that

$$\begin{aligned} \gamma(0) &= \mathbf{E}[X_0 X_0] \\ &= \mathbf{E}[Z_0 Z_0] - \theta \mathbf{E}[Z_0 Z_{-1}] - \theta \mathbf{E}[Z_{-1} Z_0] + \theta^2 \mathbf{E}[Z_{-1} Z_{-1}] \\ &= \sigma^2(1 + \theta^2). \end{aligned}$$

$$\begin{aligned} \gamma(-1) &= \gamma(1) = \mathbf{E}[X_1 X_0] \\ &= \mathbf{E}[Z_1 Z_0] - \theta \mathbf{E}[Z_1 Z_{-1}] - \theta \mathbf{E}[Z_0 Z_0] + \theta^2 \mathbf{E}[Z_0 Z_{-1}]. \\ &= -\sigma^2 \theta. \end{aligned}$$

Then, for $a = (1 + \theta^2)$ and $b = -\theta$, we have the following system:

$$\sigma^2 \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_n \end{bmatrix} = \sigma^2 \begin{bmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The first line of this system says

$$(1 + \theta^2)\phi_1 - \theta\phi_2 = a\phi_1 + b\phi_2 = b = -\theta.$$

From the 2nd to $(n - 1)$ -th line we have

$$-\theta\phi_{j-1} + (1 + \theta^2)\phi_j - \theta\phi_{j+1} = b\phi_{j-1} + a\phi_j - b\phi_{j+1} = 0.$$

Finally, the n -th line gives us

$$(1 + \theta^2)\phi_n - \theta\phi_{n-1} = a\phi_n + b\phi_{n-1} = 0.$$

Exercise 3.23.

Use Definition 3.4.2 and the results of Problem 3.22 to determine the partial autocorrelation function of a moving average of order 1.

Solution

Definition 3.4.2. The partial autocorrelation $\alpha(k)$ of $\{X_t\}$ at lag k is

$$\alpha(k) = \phi_{kk}, \quad k \geq 1,$$

where ϕ_{kk} is uniquely determined by the following system

$$\begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} \phi_{n,1} \\ \phi_{n,2} \\ \phi_{n,3} \\ \vdots \\ \phi_{n,n} \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

the standard linear time elimination algorithm for tridiagonal systems gives us the following recursive solution

$$\begin{aligned} \phi_{n,n} &= \alpha_n = \alpha(n), \\ \phi_{n,i} &= \alpha_i - \beta_i \phi_{n,i+1} \quad i \leq n-1 \end{aligned}$$

with

$$\alpha_i = \begin{cases} \frac{b}{a}, & i = 1, \\ \frac{-b\alpha_{i-1}}{a - b\beta_{i-1}} & i \geq 2, \end{cases}, \quad \beta_i = \begin{cases} \frac{b}{a}, & i = 1, \\ \frac{b}{a - b\beta_{i-1}} & i \geq 2, \end{cases}.$$

We then have,

$$\alpha_2 = \frac{-b\alpha_1}{a - b\beta_1} = \frac{-b^2 a^{-1}}{a - b^2 a^{-1}} = \frac{b^2}{b^2 - a^2}$$

I calculated the first 10 coefficients for this solution

$$\begin{aligned}
\alpha_2 &= \frac{b^2}{b^2 - a^2} & \beta_2 &= \frac{ab}{a^2 - b^2} \\
\alpha_3 &= \frac{b^3}{a(a^2 - 2b^2)} & \beta_3 &= \frac{b(a^2 - b^2)}{a(a^2 - 2b^2)} \\
\alpha_4 &= -\frac{b^4}{a^4 - 3a^2b^2 + b^4} & \beta_4 &= \frac{ab(a^2 - 2b^2)}{a^4 - 3a^2b^2 + b^4} \\
\alpha_5 &= \frac{b^5}{a(a^4 - 4a^2b^2 + 3b^4)} & \beta_5 &= \frac{b(a^4 - 3a^2b^2 + b^4)}{a(a^4 - 4a^2b^2 + 3b^4)} \\
\alpha_6 &= -\frac{b^6}{a^6 - 5a^4b^2 + 6a^2b^4 - b^6} & \beta_6 &= \frac{ab(a^4 - 4a^2b^2 + 3b^4)}{a^6 - 5a^4b^2 + 6a^2b^4 - b^6} \\
\alpha_7 &= \frac{b^7}{a(a^6 - 6a^4b^2 + 10a^2b^4 - 4b^6)} & \beta_7 &= \frac{b(a^6 - 5a^4b^2 + 6a^2b^4 - b^6)}{a(a^6 - 6a^4b^2 + 10a^2b^4 - 4b^6)} \\
\alpha_8 &= -\frac{b^8}{a^8 - 7a^6b^2 + 15a^4b^4 - 10a^2b^6 + b^8} & \beta_8 &= \frac{ab(a^6 - 6a^4b^2 + 10a^2b^4 - 4b^6)}{a^8 - 7a^6b^2 + 15a^4b^4 - 10a^2b^6 + b^8} \\
\alpha_9 &= \frac{b^9}{a(a^8 - 8a^6b^2 + 21a^4b^4 - 20a^2b^6 + 5b^8)} & \beta_9 &= \frac{b(a^8 - 7a^6b^2 + 15a^4b^4 - 10a^2b^6 + b^8)}{a(a^8 - 8a^6b^2 + 21a^4b^4 - 20a^2b^6 + 5b^8)} \\
\alpha_{10} &= -\frac{b^{10}}{a^{10} - 9a^8b^2 + 28a^6b^4 - 35a^4b^6 + 15a^2b^8 - b^{10}} & \beta_{10} &= \frac{ab(a^8 - 8a^6b^2 + 21a^4b^4 - 20a^2b^6 + 5b^8)}{a^{10} - 9a^8b^2 + 28a^6b^4 - 35a^4b^6 + 15a^2b^8 - b^{10}}
\end{aligned}$$

After substituting back again to $a = 1 + \theta^2$ and $b = -\theta$ we obtain

$$\begin{aligned}
\alpha_2 &= -\frac{\theta^2}{\theta^4 + \theta^2 + 1} & \beta_2 &= \frac{-\theta^3 - \theta}{\theta^4 + \theta^2 + 1} \\
\alpha_3 &= -\frac{\theta^3}{(\theta^2 + 1)(\theta^4 + 1)} & \beta_3 &= -\frac{\theta(\theta^4 + \theta^2 + 1)}{(\theta^2 + 1)(\theta^4 + 1)} \\
\alpha_4 &= -\frac{\theta^4}{\theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} & \beta_4 &= \frac{-\theta^7 - \theta^5 - \theta^3 - \theta}{\theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \\
\alpha_5 &= -\frac{\theta^5}{\theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} & \beta_5 &= \frac{-\theta^9 - \theta^7 - \theta^5 - \theta^3 - \theta}{\theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \\
\alpha_6 &= -\frac{\theta^6}{\theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} & \beta_6 &= \frac{-\theta^{11} - \theta^9 - \theta^7 - \theta^5 - \theta^3 - \theta}{\theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \\
\alpha_7 &= -\frac{\theta^7}{\theta^{14} + \theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} & \beta_7 &= \frac{-\theta^{13} - \theta^{11} - \theta^9 - \theta^7 - \theta^5 - \theta^3 - \theta}{\theta^{14} + \theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1}
\end{aligned}$$

Finally, I believe that the closed form formula for the autocorrelation function of $\{X_t\}$ is the following

$$\alpha(n) = \frac{-\theta^n}{\sum_{i=0}^n \theta^{2i}}$$

Exercise 3.24.

Let $\{X_t\}$ be the stationary solution of $\phi(B)X_t = \theta(B)Z_t$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$, and $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeroes. If A is any zero-mean random variable in L^2 which is uncorrelated with $\{X_t\}$ and if $|z_0| = 1$, show that the process $\{X_t + Az_0^t\}$ is a complex-valued stationary process (see Definition 4.1.1) and that $\{X_t + Az_0^t\}$ and $\{X_t\}$ both satisfy the equations $(1 - z_0 B)\phi(B)X_t = (1 - z_0 B)\theta(B)Z_t$.

Solution

Exercise 5.1.

Let $\{X_t\}$ be a stationary process with mean μ . Show that

$$P_{\overline{\text{sp}}|1 X_1, \dots, X_n} X_{n+h} = \mu + P_{\overline{\text{sp}}\{Y_1, \dots, Y_n\}} Y_{n+h},$$

where $\{Y_t\} = \{X_t - \mu\}$.

Solution

Let $\hat{X}_{n+h} = P_{\overline{\text{sp}}|1 X_1, \dots, X_n} X_{n+h}$ and $\hat{Y}_{n+h} = P_{\overline{\text{sp}}\{Y_1, \dots, Y_n\}} Y_{n+h}$. Then, from the definition of projection, note that for some $\{\psi_i\}_{i=1}^n$,

$$\hat{Y}_{n+h} = \sum_{i=1}^n \phi_i Y_{n+1-i}.$$

Therefore,

$$\mathbf{E}[X_{n+h} - \mu - \hat{Y}_{n+h}] = \mathbf{E}[X_{n+h} - \mu] - \sum_{i=1}^n \phi_i \mathbf{E}[Y_{n+1-i}] = 0$$

Finally, for every $k \in \{1, \dots, n\}$

$$\begin{aligned} \langle X_{n+h} - \mu - \hat{Y}_{n+h}, X_k \rangle &= \mathbf{E}[(X_{n+h} - \mu - \hat{Y}_{n+h})(X_k - \mu)] \\ &= \mathbf{E}[(Y_{n+h} - \hat{Y}_{n+h})(Y_k)] \\ &= 0, \end{aligned}$$

and, since $\mathbf{E}[1] = 1$,

$$\left\langle X_{n+h} - \mu - \hat{Y}_{n+h}, 1 \right\rangle = \mathbf{E}[(X_{n+h} - \mu - \hat{Y}_{n+h})(1 - 1)] = 0.$$

From the uniqueness of the projection, it follows that $\hat{X}_{n+h} = \mu + \hat{Y}_{n+h}$.

Exercise 5.5.

Let $\{X_t\}$ be the **MA**(1) process of Example 5.2.1. If $|\theta| < 1$, show that as $n \rightarrow \infty$,

- (a) $\|X_n - \hat{X}_n - Z_n\| \rightarrow 0$,
- (b) $v_n \rightarrow \sigma^2$,
- (c) $\theta_{n1} \rightarrow \theta$. (Note that $\theta = E(X_{n+1}Z_n)\sigma^{-2}$ and $\theta_{n1} = v_{n-1}^{-1}E(X_{n+1}(X_n - \hat{X}_n))$.)

Solution

In the first place, there exists $K \geq 0$ such that

$$\begin{aligned} \|X_n - \hat{X}_n - Z_n\| &= \|\theta Z_{n-1} - \theta(X_{n-1} - \hat{X}_{n-1})\| \\ &= |\theta| \|X_{n-1} - \hat{X}_{n-1} - Z_{n-1}\| \\ &= \vdots \\ &= |\theta|^{n-1} \|X_0 - \hat{X}_0 - Z_0\| = |\theta|^{n-1} K \end{aligned}$$

Therefore, since $|\theta| < 1$, it follows that $\lim_n \|X_n - \hat{X}_n - Z_n\| = 0$.

Now, I'm going to prove using induction that

$$v_n = \sigma^2 \frac{\sum_{i=0}^n \theta^{2i}}{\sum_{i=0}^{n-1} \theta^{2i}}.$$

The base case is true because $v_0 = \sigma^2(1 + \theta^2)$. For the inductive step,

$$\begin{aligned}
v_{n+1} &= [1 + \theta^2 - \theta^2 \sigma^2 v_n^{-1}] \sigma^2 \\
&= \left[1 + \theta^2 - \theta^2 \frac{\sum_{i=0}^{n-1} \theta^{2i}}{\sum_{i=0}^n \theta^{2i}} \right] \sigma^2 \\
&= \sigma^2 \frac{(1 + \theta^2) \sum_{i=0}^n \theta^{2i} - \theta^2 \sum_{i=0}^{n-1} \theta^{2i}}{\sum_{i=0}^n \theta^{2i}} \\
&= \sigma^2 \frac{\sum_{i=0}^n \theta^{2i} + \sum_{i=1}^{n+1} \theta^{2i} - \sum_{i=1}^n \theta^{2i}}{\sum_{i=0}^n \theta^{2i}} \\
&= \sigma^2 \frac{\sum_{i=0}^{n+1} \theta^{2i}}{\sum_{i=0}^n \theta^{2i}}.
\end{aligned}$$

Therefore, using the partial geometric series expansion ($\theta^2 < 1$), we know that

$$v_n = \sigma^2 \frac{\frac{1 - \theta^{2n+2}}{1 - \theta^2}}{\frac{1 - \theta^{2n}}{1 - \theta^2}} = \sigma^2 \frac{1 - \theta^{2n+2}}{1 - \theta^{2n}},$$

and we know v_n is decreasing because since $\theta^2 < 1$ that implies

$$\begin{aligned}
&\theta^{2n+2} < \theta^{2n} \\
\implies &1 - \theta^{2n+2} < 1 - \theta^{2n} \\
\implies &\frac{v_{n+1}}{v_n} = \frac{1 - \theta^{2n+2}}{1 - \theta^{2n+4}} < 1
\end{aligned}$$

This also implies that r_n is decreasing. As a matter of fact, this proves that $v_n \rightarrow \sigma^2$ and $r_n \rightarrow 1$ as $n \rightarrow \infty$ because $\theta^{2n} \rightarrow 0$.

Finally, since $v_n \rightarrow \sigma^2$, it follows that

$$\theta_{n1} = \theta \frac{\sigma^2}{v_{n-1}} \rightarrow \theta,$$

as $n \rightarrow \infty$.