# Time Series: Homework 1

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## Exercise 1.1.

Suppose that $X_t = Z_t + \theta Z_{t-1},\ t = 1, 2, \ldots,$ where $Z_0, Z_1, Z_2$ are independent random variables, each with moment generating function $\mathbf{E}\ \exp(\lambda Z_i) = m(\lambda)$.

(a) Express the joint moment generating function $\mathbf{E}\ \exp\left(\sum_{i=1}^n \lambda_i X_i\right)$ in terms of the function $m(\cdot)$.

(b) Deduce from (a) that $\{X_t\}$ is strictly stationary.

### Solution part (a)

Since $\{Z_t\}$ are independent, for $X_t = Z_t + \theta Z_{t-1}$, the moment generating function:

$$\begin{aligned}
\mathbf{E}\ \exp(\lambda X_t) &= \mathbf{E}\ \exp(\lambda(Z_t + \theta Z_{t-1})) \\
&= \mathbf{E}\ \exp(\lambda Z_t) \cdot \mathbf{E}\ \exp(\lambda \theta \mathbb{Z}_{t-1}) \\
&= m(\lambda) \cdot m(\theta\lambda)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^n \lambda_i X_i &= \sum_{i=1}^n \lambda_i Z_i + \sum_{i=1}^n \lambda_i \theta Z_{i-1} \\
&= \sum_{i=1}^n \lambda_i Z_i + \sum_{i=0}^{n-1} \lambda_{i+1} \theta Z_i \\
&= \lambda_n Z_n + \left[\sum_{i=1}^{n-1} (\lambda_i + \theta \lambda_{i+1}) Z_i\right] + \lambda_1 \theta Z_0.
\end{aligned}$$

Therefore, using a similar argument as before

$$\mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right) = m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)$$

### Solution part (b)

Let $(X_1,\ldots,X_n)'$ be a random vector in $\mathbb{R}^k$. The moment generating function of this vector is defined as follows,

$$M_{X_{1:n}}(\lambda_{1:n}) = \mathbf{E}\exp(\langle\lambda_{1:n}, X_{1:n}\rangle) = \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_i\right),\ \lambda_{1:n}\in\mathbb{R}^n.$$

So, we know from the previous part that

$$\begin{aligned}
M_{X_{1:n}}(\lambda_{1:n}) &= m(\lambda_n)\cdot\left[\prod_{i=1}^{n-1} m(\lambda_i+\theta\lambda_{i+1})\right]\cdot m(\lambda_1\theta)\\
&= \mathbf{E}\exp(\lambda_n Z_{n+h}) + \left[\prod_{i=1}^{n-1}\mathbf{E}\exp((\lambda_i+\theta\lambda_{i+1})Z_{i+h})\right] + \mathbf{E}\exp(\lambda_1\theta Z_h)\\
&= \mathbf{E}\exp\left(\sum_{i=1}^{n}\lambda_i X_{i+h}\right)\\
&= M_{X_{1+h:n+h}}(\lambda_{1:n})
\end{aligned}$$

Since the moment generating function of both $(X_1,\ldots,X_n)'$ and $(X_{1+h},\ldots,X_{n+h})'$ coincide, they have the same joint distribution. Thus, $\{X_t\}$ is strictly stationary.

## Exercise 1.2.

(a) Show that a linear filter $\{a_j\}$ passes an arbitrary polynomial of degree $k$ without distortion, i.e.

$$m_t = \sum_j a_j m_{t-j}$$

for all $k^{\text{th}}$ degree polynomials $m_t = c_0 + c_1 t + \cdots + c_k t^k$, if and only if

$$\begin{cases}\sum_j a_j = 1,\\ \sum_j j^r a_j = 0, \quad \text{for } r = 1,\ldots,k.\end{cases}$$

(b) Show that the Spencer 15-point moving average filter $\{a_j\}$ does not distort a cubic trend.

## Solution Part (a)

$\Longrightarrow$ Let $\{a_j\}$ such that, for any $k^{\text{th}}$ degree polynomial $m_t = c_0 + c_1 t + \cdots + c_k t^k$

$$m_t = \sum_j a_j m_{t-j}.$$

Let $c_0 = 1$ and $c_k = 0$ for $k > 0$ to see that

$$m_t = 1 = \sum_j a_j \cdot 1.$$

Now, note that for $c_1 = 1$ and $c_k = 0$ for $k \neq 1$,

$$m_0 = 0 = \sum_j a_j m_{-j} = -\sum_j a_j j$$

$$\Longrightarrow \ 0 = \sum_j a_j j$$

In general, for the polynomials with $c_{2i+1} = 1$ and $c_k = 0$ for $k \neq 2i+1$,

$$m_0 = 0 = -\sum_j a_j j^{2i+1}$$

$$\Longrightarrow \ 0 = \sum_j a_j j^{2i+1}$$

and for $c_{2i} = 1$ and $c_k = 0$ for $k \neq 2i$,

$$m_0 = 0 = \sum_j a_j (-j)^{2i} = \sum_j a_j j^{2i}$$

Therefore,

$$\sum_j j^r a_j = 0, \quad r \in \mathbb{N}^+.$$

$\Longleftarrow$ Let $a_i$ with $\sum_j a_j = 1$ and $\sum_j j^r a_j = 0$ for $r = 1, \ldots, k$. Note that for any polynomial of degree $k$ $m_t$,

$$\begin{aligned}\sum_j a_j m_{-j} &= \sum_j a_j \sum_{r=0}^{k} c_r (-j)^r \\ &= \sum_{r=0}^{k} c_r (-1)^r \underbrace{\sum_j a_j j^r}_{r \neq 0 \implies 0} = c_0 = m_0\end{aligned}$$

Similarly, for the $n$-th derivative of $m_t$,

$$\begin{aligned}\sum_j a_j m_{-j}^{(n)} &= \sum_j a_j \sum_{r=0}^{k} c_r \frac{r!}{(r-n)!} (-j)^{r-n} \\ &= \sum_{r=0}^{k} c_r \frac{(-1)^{r-n} r!}{(r-n)!} \underbrace{\sum_j a_j j^{r-n}}_{r \neq n \implies 0} = n! c_n = m_0^{(n)}\end{aligned}$$

Finally, using Taylor's theorem

$$
\begin{aligned}
m_t &= \sum_{n=0}^{k} \frac{m_0^{(n)}}{n!} t^n \\
&= \sum_{n=0}^{k} \sum_{j} a_j \sum_{r=0}^{k} c_r \frac{r!}{n!(n-r)!} t^n (-j)^{n-r} \\
&= \sum_{j} \sum_{n=0}^{k} a_j \sum_{r=0}^{k} c_r \frac{r!}{n!(n-r)!} t^n (-j)^{n-r} \\
\text{(Cauchy-Product)} \quad &= \sum_{j} a_j \sum_{r=0}^{k} c_r \sum_{n=0}^{r} \frac{r!}{n!(n-r)!} t^n (-j)^{n-r} \\
\text{(Binomial-Theorem)} \quad &= \sum_{j} a_j \sum_{r=0}^{k} c_r (t-j)^r \\
&= \sum_{j} a_j m_{t-j}.
\end{aligned}
$$

## Solution Part (b)

Using the lemma we proved in the previous part,

$$\sum_{j=-7}^{7} a_j = \frac{1}{320}(-3-6-5+3+21+46+67+74+67+46+21+3-5-6-3) = \frac{320}{320}.$$

```python
sma = np.array([-3,-6,-5,3,21,46,67,74,67,46,21,3,-5,-6,-3]) * 1/320
sum(sma)
```

[15] ✓ 0.0s Python

```
0.9999999999999999
```

$$\sum_{j=-7}^{7} a_j j^r = 0$$

```python
for r in range(1,4):
    condition = sma.T@np.array([(j-7)**r for j in range(15)])
    print(f"r = {r}:", condition)
```

[27] ✓ 0.0s Python

```
r = 1: -1.3877787807814457e-17
r = 2: 0.0
r = 3: 4.440892098500626e-16
```

## Exercise 1.3.

Suppose that $m_t = c_0 + c_1 t + c_2 t^2,\ t = 0, \pm 1, \ldots$

(a) Show that

$$m_t = \sum_{i=-2}^{2} a_i m_{t+i} = \sum_{i=-3}^{3} b_i m_{t+i}, \qquad t = 0, \pm 1, \ldots,$$

where $a_2 = a_{-2} = -\frac{3}{35}$, $a_1 = a_{-1} = \frac{12}{35}$, $a_0 = \frac{17}{35}$, and $b_3 = b_{-3} = -\frac{2}{21}$, $b_2 = b_{-2} = \frac{3}{21}$, $b_1 = b_{-1} = \frac{6}{21}$, $b_0 = \frac{7}{21}$.

(b) Suppose that $X_t = m_t + Z_t$ where $\{Z_t, t = 0, \pm 1, \ldots\}$ is an independent set of normal random variables, each with mean 0 and variance $\sigma^2$. Let $U_t = \sum_{i=-2}^{2} a_i X_{t+i}$ and $V_t = \sum_{i=-3}^{3} b_i X_{t+i}$.

(i) Find the means and variances of $U_t$ and $V_t$.

(ii) Find the correlations between $U_t$ and $V_t$.

(iii) Which of the two filtered series $\{U_t\}$ and $\{V_t\}$ would you expect to be smoother in appearance?

### Solution Part (a)

In the first place, note that since $a_i = a_{-i}$ and $b_i = b_{-j}$, it follows that

$$\sum_j a_j m_{t+j} = \sum_j a_j m_{t-j},$$
$$\sum_j b_j m_{t+j} = \sum_j b_j m_{t-j}.$$

The same goes for $U_t$ and $V_t$.

We use the lemma from the previous exercise to prove that the filters $\{a_j\}$ and $\{b_j\}$ don't distort quadratic polynomials.

```
a = np.array([-3, 12, 17, 12, -3]) * 1/35
b = np.array([-2, 3, 6, 7, 6, 3, -2]) * 1/21
```

$$\sum_{j=-2}^{2} a_j j^r = \begin{cases} 1, & r = 0 \\ 0, & r = 1, 2 \end{cases}$$

```python
for r in range(0,3):
    condition = a.T@np.array([(j-2)**r for j in range(5)])
    print(f"r = {r}:", condition)
```

```
r = 0: 1.0000000000000002
r = 1: 0.0
r = 2: 0.0
```

$$\sum_{j=-3}^{3} b_j j^r = \begin{cases} 1, & r = 0 \\ 0, & r = 1, 2 \end{cases}$$

```python
for r in range(0,3):
    condition = b.T@np.array([(j-3)**r for j in range(7)])
    print(f"r = {r}:", condition)
```

```
r = 0: 0.9999999999999999
r = 1: 0.0
r = 2: 0.0
```

Therefore,

$$m_t = \sum_{j=-2}^{2} a_j j^r = \sum_{j=-3}^{3} b_j j^r.$$

## Solution Part (b)

**Item (i):** By linearity,

$$\begin{aligned} U_t = \sum_{i=-2}^{2} a_i X_{t-i} &= \sum_{i=-2}^{2} a_i m_{t-i} + \sum_{i=-2}^{2} a_i Z_{t-i} \\ &= m_t + \sum_{i=-2}^{2} a_i Z_{t-i} \end{aligned}$$

$$\implies \mathbf{E}\, U_t = \mathbf{E}\, m_t + \sum_{i=-2}^{2} a_i \mathbf{E}\, Z_{t-i} = m_t + 0.$$

$$\implies \mathbf{Var}\, U_t = \mathbf{Var}\, m_t + \sum_{i=-2}^{2} a_i \mathbf{Var}\, Z_{t-i} = 0 + \sigma^2 \sum_{i=-2}^{2} a_i = \sigma^2.$$

If we do the same with $V_t$, then we obtain $\mathbf{E}\, V_t = m_t$ and $\mathbf{Var}\, V_t = \sigma^2$

**Item (ii):**

$$
\begin{aligned}
\sigma^2 \mathbf{corr}\,(U_t, V_t) &= \mathbf{E}\,[(U_t - m_t)(V_t - m_t)] \\
&= \mathbf{E}\,(U_t V_t) - m_t \mathbf{E}\,U_t - m_t \mathbf{E}\,V_t + \mathbf{E}\,m_t^2 \\
&= \mathbf{E}\left[\left(m_t + \sum_{i=-2}^{2} a_i Z_{t-i}\right)\left(m_t + \sum_{i=-3}^{3} b_i Z_{t-i}\right)\right] - m_t^2 \\
&= \mathbf{E}\left(m_t^2 + m_t \sum_{i=-2}^{2} a_i Z_{t-i} + m_t \sum_{i=-3}^{3} b_i Z_{t-i} + \sum_{i=-2}^{2} a_i Z_{t-i} \sum_{i=-3}^{3} b_i Z_{t-i}\right) - m_t^2 \\
&= m_t + 0 + 0 + \mathbf{E}\left(\sum_{i=0}^{4} a_{i-2} Z_{t-i+2} \sum_{i=0}^{6} b_{j-3} Z_{t-j+3}\right) - m_t^2 \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_{j-2} b_{k-j-3} \mathbf{E}\,Z_{t-j+2} Z_{t-k+j+3}
\end{aligned}
$$

Then, $\mathbf{E}\,Z_{t-j+2}Z_{t-j+k+3} = \sigma^2$ when $t - j + 3 = t - k + j + 3$ and 0 otherwise, and that is when $2j = k$. Thus,

$$
\begin{aligned}
\mathbf{corr}\,(U_t, V_t) &= \frac{\sigma^2}{\sigma^2} \sum_{k/2 \in \mathbb{N}_0} \sum_{j=k/2}^{k/2} a_{k-2} b_{k-j-3} \\
&= \sum_{k=0}^{\infty} a_{k-2} b_{k-3} \\
&= a_{-2}b_{-3} + a_{-1}b_{-2} + a_0 b_{-1} + a_1 b_0 + a_2 b_1 \\
&\approx 0.286
\end{aligned}
$$

**Item (iii):** $V_t$ should be smoother because $\{b_t\}$ is a weighted average of more elements and $b_0$ weights a 3rd of the total average. On the other hand, $a_0$ weights almost half of this sum, so $Z_t$ has more influence in $U_t$ than in $V_t$.

## Exercise 1.4.

If $m_t = \sum_{k=0}^{p} c_k t^k, t = 0, \pm 1, \ldots$, show that $\nabla m_t$ is a polynomial of degree $(p-1)$ in $t$ and hence that $\nabla^{p+1} m_t = 0$.

**Solution:**

$$
\begin{aligned}
m_{t-1} &= \sum_{k=0}^{p} c_k (t-1)^k \\
&= \sum_{k=0}^{p} c_k \sum_{j=0}^{k} \binom{k}{j} t^j (-1)^{k-j} \\
&= \sum_{j=0}^{p} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k
\end{aligned}
$$

The last line can be deduced from the following diagram

$$
m_{t-1} = \sum_{k=0}^{p} c_k \sum_{j=0}^{k} \binom{k}{j} t^j (-1)^{k-j} =
$$

$$
\begin{array}{ccccccccc}
c_0\binom{0}{0}t^0(-1)^{0-0} & & & & & & & & \\
c_1\binom{1}{0}t^0(-1)^{1-0} & + & c_1\binom{1}{1}t^1(-1)^{1-1} & & & & & & \\
c_2\binom{2}{0}t^0(-1)^{2-0} & + & c_2\binom{2}{1}t^1(-1)^{2-1} & + & c_2\binom{2}{2}t^2(-1)^{2-2} & & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
c_p\binom{p}{0}t^0(-1)^{p-0} & + & c_p\binom{p}{1}t^1(-1)^{p-1} & + & c_p\binom{p}{2}t^2(-1)^{p-2} & + \cdots + & & & c_p\binom{p}{p}t^p(-1)^{p-p} \\
= & & = & & = & & \cdots & & =
\end{array}
$$

$$
t^0 \sum_{k=0}^{p} c_k \binom{k}{0} (-1)^{k-0} + t^1 \sum_{k=1}^{p} c_k \binom{k}{1} (-1)^{k-1} + t^2 \sum_{k=2}^{p} c_k \binom{k}{2} (-1)^{k-2} + \cdots + t^p \sum_{k=p}^{p} c_k \binom{k}{p} (-1)^{k-p}
$$

Thus, for $j = p$, the coefficient that accompanies $t^p$ is $\binom{p}{p}(-1)^{p-p} c_p = c_p$. So it follows that

$$
\begin{aligned}
\nabla m_t = &\quad \sum_{j=0}^{p} c_j t^j \quad - \quad \sum_{j=0}^{p} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \\
&= c_p t^p + \sum_{j=0}^{p-1} c_j t^j - c_p t^p - \sum_{j=0}^{p-1} t^j \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \\
&= \sum_{j=0}^{p-1} t^j \cdot \left[ c_j - \sum_{k=j}^{p} \binom{k}{j} (-1)^{k-j} c_k \right],
\end{aligned}
$$

which is a $(p-1)$-degree polynomial. Now, note that

$$
\nabla^n m_t = (I - B)^n (m_t).
$$

One can inductively show that $\nabla^n m_t$ has degree $p-n$ for any polynomial $m_t$ of degree $p$. We proved the base case previously, so assume that $\nabla^{n-1} m_t$ has degree $p-n+1$. Then, define $d_j = [\nabla^{n-1} m_t]_{t^j}$ as the coefficient that accompanies $t^j$.

Since we proved that $(I-B)$ reduces by one the degree of any polynomial, it follows that $(I-B)\nabla^{n-1} m_t$ has degree $(p-n+1)-1 = p-n$. This can be verified with the following calculation:

$$\begin{aligned}
\nabla^n m_t &= (I-B)(I-B)^{n-1} m_t \\
&= (I-B)\nabla^{n-1} m_t \\
&= \nabla\left(\sum_{k=0}^{p-n+1} d_k t^k\right) \\
&= \sum_{j=0}^{p-n} t^j \cdot \left[d_j - \sum_{k=j}^{p-n+1} \binom{k}{j}(-1)^{k-j} d_k\right].
\end{aligned}$$

Finally, $\nabla^p m_t$ is polynomial of degree 0, and thus, it's a constant function $f_t = K$. Therefore,

$$\begin{aligned}
\nabla^{p+1} m_t &= (I-B)(\nabla^p m_t) \\
&= (I-B)(Kt^0) \\
&= K - BK \\
&= K - K = 0.
\end{aligned}$$

The backwards shift operator evaluated on a constant is the same constant since $f_t = f_{t-1} = K$ for a constant function $f_t$.

## Exercise 1.7.

Let $Z_t, t = 0, \pm 1, \ldots,$ be independent normal random variables each with mean 0 and variance $\sigma^2$ and let $a, b$ and $c$ be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.

(a) $X_t = a + bZ_t + cZ_{t-1}$,

(c) $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$,

(e) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

**Note:** I assumed by mistake that $\sigma^2 = 1$. However, in all of the equations on the following solution, the $\sigma^2$ term can always be factorized without altering the truth value of the propositions.

## Solution part (a)

Using the linearity of the expected value and the variance ($Z_t$'s are independent)

$$\mathbf{E}\, X_t = a + b\mathbf{E}\, Z_t + c\mathbf{E}\, Z_{t-1} = a$$

$$\mathbf{Var}\,(X_t) = b^2\mathbf{Var}\,(Z_t) + c^2\mathbf{Var}\,(Z_{t-1}) = b^2 + c^2$$

$$\implies \mathbf{E}\,|X_t|^2 = \mathbf{Var}\,(X_t) + (\mathbf{E}\, X_t)^2 = a^2 + b^2 + c^2 < \infty$$

Now for the autocovariance function,

$$\begin{aligned}
\gamma_X(r,s) &= \mathbf{E}\,[(X_r - a)(X_s - a)] \\
&= \mathbf{E}\,[(bZ_r + cZ_{r-1})]\mathbf{E}\,[(bZ_s + cZ_{s-1})] \\
&= b^2\mathbf{E}\, Z_r Z_s + bc\mathbf{E}\, Z_r Z_{s-1} + bc\mathbf{E}\, Z_{r-1} Z_s + c^2\mathbf{E}\, Z_{r-1} Z_{s-1}.
\end{aligned}$$

There are two cases where $\gamma_X$ is not zero, and that's because $\mathbf{E}\, Z_r Z_s = 1 \iff r = s$:

$$\begin{aligned}
\gamma_X(t,t) &= b^2\mathbf{E}\, Z_t Z_t + 2bc\mathbf{E}\, Z_t Z_{t-1} + c^2\mathbf{E}\, Z_{t-1} Z_{t-1} \\
&= b^2\mathbf{E}\, Z_t^2 + c^2\mathbf{E}\, Z_{t-1}^2 \\
&= b^2 + c^2,
\end{aligned}$$

and then, by symmetry of $\gamma$,

$$\begin{aligned}
\gamma_X(t,t+1) = \gamma_X(t,t-1) &= b^2\mathbf{E}\, Z_t Z_{t-1} + bc\mathbf{E}\, Z_t Z_{t-2} + bc\mathbf{E}\, Z_{t-1} Z_{t-1} + c^2\mathbf{E}\, Z_{t-1} Z_{t-2} \\
&= bc\mathbf{E}\, Z_{t-1}^2 \\
&= bc.
\end{aligned}$$

On the other hand, for $|h| > 1$,

$$t \neq t+h, \quad t \neq t+h-1, \quad t-1 \neq t+h, \quad t-1 \neq t+h-1$$

$$\implies \gamma_X(t, t+h) = b^2 \mathbf{E}\, Z_t Z_{t+h} + bc\mathbf{E}\, Z_t Z_{t+h-1} + bc\mathbf{E}\, Z_{t-1} Z_{t+h} + c^2 \mathbf{E}\, Z_{t-1} Z_{t+h-1}$$

$$= 0.$$

Finally, note that $\gamma_X$ is only dependent on the difference $r - s$, and thus, $X_t$ is a stationary process with autocovariance function

$$\gamma(h) = \begin{cases} b^2 + c^2 & h = 0, \\ bc & h = 1, \\ 0 & \text{otherwise.} \end{cases}$$

## Solution part (b)

Again, using the linearity of expectation and variance,

$$\mathbf{E}\, X_t = \cos(ct)\mathbf{E}\, Z_1 + \sin(ct)\mathbf{E}\, Z_2 = 0,$$

$$\begin{aligned} \mathbf{E}\,|X_t|^2 = \mathbf{Var}\,(X_t) &= \cos^2(ct)\mathbf{Var}\, Z_1 + \sin^2(ct)\mathbf{Var}\, Z_2 \\ &= \cos^2(ct) + \sin^2(ct) \\ &= 1. \end{aligned}$$

For the autocovariance function,

$$\begin{aligned} \gamma_X(r, s) &= \mathbf{E}\left[(\cos(cr)Z_1 + \sin(cr)Z_2)(\cos(cs)Z_1 + \sin(cs)Z_2)\right] \\ &= \cos(cr)\cos(cs)\mathbf{E}\, Z_1^2 + \cos(cr)\sin(cs)\mathbf{E}\, Z_1 Z_2 \\ &+ \sin(cr)\cos(cs)\mathbf{E}\, Z_2 Z_1 + \sin(cr)\sin(cs)\mathbf{E}\, Z_2^2 \\ &= \cos(cr)\cos(cs) + \sin(cr)\sin(cs) \\ &= \cos(c(r - s)), \end{aligned}$$

which is only dependent of the value $r - s$, and thus, $\{X_t\}$ is stationary. The autocovariance function can then be defined as

$$\gamma(h) = \cos(c(h))$$

## Solution part (c)

$$\mathbf{E}\, X_t = \cos(ct)\mathbf{E}\, Z_t + \sin(ct)\mathbf{E}\, Z_t = 0,$$

$$
\begin{aligned}
\mathbf{E}\,|X_t|^2 = \mathbf{Var}\,(X_t) &= \cos^2(ct)\mathbf{Var}\,Z_t + \sin^2(ct)\mathbf{Var}\,Z_{t-1} \\
&= \cos^2(ct) + \sin^2(ct) \\
&= 1.
\end{aligned}
$$

Now, we can prove that $\{X_t\}$ is not stationary by taking the case when $r - s = 1$,

$$
\begin{aligned}
\gamma_X(t, t-1) &= \mathbf{E}\left[(\cos(ct)Z_t + \sin(ct)Z_{t-1})(\cos(c(t-1))Z_{t-1} + \sin(c(t-1))Z_{t-2})\right] \\
&= \cos(ct)\cos(c(t-1))\mathbf{E}\,Z_t\mathbf{E}\,Z_{t-1} + \cos(ct)\sin(c(t-1))\mathbf{E}\,Z_tZ_{t-2} \\
&+ \sin(ct)\cos(c(t-1))\mathbf{E}\,Z_{t-1}Z_{t-1} + \sin(ct)\sin(c(t-1))\mathbf{E}\,Z_{t-1}Z_{t-2} \\
&= \sin(ct)\cos(c(t-1))
\end{aligned}
$$

This case depends on the value of $t$ (unless $c$ is a multiple of $\pi$). For example, if $c = \pi/2$, then

$$
\begin{aligned}
\gamma_X(1, 0) &= \sin(\pi/2)\cos(0) = 1, \\
\gamma_X(2, 1) &= \sin(\pi)\cos(\pi/2) = 0.
\end{aligned}
$$

Therefore, $\{X_t\}$ is not stationary.