Time Series: Homework 3

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Exercise 3.1.

Determine which of the following processes are causal and/or invertible

(a) $X_t + .2X_{t-1} - .48X_{t-2} = Z_t$,

(b) $X_t + 1.9X_{t-1} - .88X_{t-2} = Z_t + .2Z_{t-1} + .7Z_{t-2}$,

(c) $X_t + .6X_{t-2} = Z_t + 1.2Z_{t-1}$

Solution: We are going to use Theorems 3.1.1. and 3.1.2. to prove causality and invertibility of the processes.

Solution Item (a)

$\phi(z) = 1 + .2z - .48z^2$ and $\theta(z) = 1$, then

$$\phi(z) = -\frac{12\left(z - \frac{5}{3}\right)\left(z + \frac{5}{4}\right)}{25}$$

which has roots outside of the unit disk so it's causal, and it's also invertible because $\theta(z)$ is never 0.

Solution Item (b)

$\phi(z) = 1 - 1.9z - .88z^2$ and $\theta(z) = 1 + .2z + .7z^2$, then:

The roots of $\phi(z)$ are $\left[-\frac{5\sqrt{713}}{88} - \frac{95}{88} \quad -\frac{95}{88} + \frac{5\sqrt{713}}{88}\right]$ which have module approximately $[2.596, 0.437]$ respectively, so the process cannot be causal.

The roots of $\theta(z)$ are $\left[-\frac{1}{7} - \frac{\sqrt{69}i}{7} \quad -\frac{1}{7} + \frac{\sqrt{69}i}{7}\right]$ which have module approximately $[1.195, 1.195]$ respectively, so the process is invertible.

Solution Item (c)

$\phi(z) = 1 + .6z^2$ and $\theta(z) = 1 + 1.2z$, then:

The roots of $\phi(z)$ are $\left[-\frac{\sqrt{15}i}{3} \quad \frac{\sqrt{15}i}{3}\right]$ which have module approximately $[1.291, 1.291]$ respectively, so the process is causal.

The root of $\theta(z)$ is $\left[-\frac{5}{6}\right]$ which has module approximately $[0.833]$, so the process is not invertible.

Exercise 3.3

Let $\{X_t,\ t = 0, \pm 1, \ldots\}$ be the stationary solution of the non-causal AR(1) equations,

$$X_t = \phi X_{t-1} + Z_t, \qquad \{Z_t\} \sim \mathbf{WN}\,(0, \sigma),\ |\phi| > 1.$$

Show that $\{X_t\}$ also satisfies the causal Ar(1) equations,

$$X_t = \phi^{-1} X_{t-1} + \tilde{Z}_t, \quad \{\tilde{Z}_t\} \sim \mathbf{WN}\,(0, \tilde{\sigma}^2),$$

for a suitably chosen white noise process $\{\tilde{Z}_t\}$. Determine $\tilde{\sigma}^2$.

Solution:

$$X_t = \phi^{-1} X_{t+1} - \phi^{-1} Z_{t+1}$$

In the first place, according to (3.1.14.),

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$$

Then,

$$\begin{aligned}
X_t - \phi^{-1} X_{t-1} &= -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j} + \phi^{-1} \sum_{j=1}^{\infty} \phi^{-j} Z_{t-1+j} \\
&= \phi^{-2} Z_t + \sum_{j=1}^{\infty} (\phi^{-j-2} - \phi^{-j}) Z_{t+j} \\
&= \phi^{-2} Z_t + (\phi^{-2} - 1) \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}.
\end{aligned}$$

Finally, note that if we write $\tilde{Z}_t = \phi^{-2} Z_t + (\phi^{-2} - 1) \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$, then it's clear that since $\phi^{-2} Z_t + (\phi^{-2} - 1) \sum_{j=1}^{k} \phi^{-j} Z_{t+j}$ is a normal distribution that converges in probability to $\tilde{Z}_t$, and

$$\phi^{-2} \mathbf{E}\, Z_t + (\phi^{-2} - 1) \sum_{j=1}^{\infty} \phi^{-j} \mathbf{E}\, Z_{t+j} = 0 < \infty,$$

$$\begin{aligned} \mathbf{Var}\, \phi^{-2} Z_t + \sum_{j=1}^{\infty} \mathbf{Var}\, [(\phi^{-2} - 1)\phi^{-j} Z_{t+j}] &= \phi^{-4} \mathbf{Var}\, Z_t + (\phi^{-2} - 1)^2 \sum_{j=1}^{\infty} \phi^{-2j} \mathbf{Var}\, Z_{t+j} \\ &= \sigma^2 \phi^{-4} + \sigma^2 (\phi^{-1} - 1)^2 \frac{\phi^{-2}}{1 - \phi^{-2}} \\ &= \sigma^2 (\phi^{-4} - (\phi^{-4} - \phi^{-2})) \\ &= \sigma^2 \phi^{-2} < \infty \end{aligned},$$

it follows that $\tilde{Z}_k$ is the limit in distribution of the previous series, which as we mentioned before, has normal distribution. Therefore,

$$\mathbf{E}\, \tilde{Z}_k = 0, \qquad \mathbf{Var}\, \tilde{Z}_t = \sigma^2 \phi^{-2}.$$

Exercise 3.5.

Let $\{Y_t,\ t = 0, \pm 1, \ldots\}$ be a stationary time series. Show that there exists a stationary solution $\{X_t\}$ of the difference equations,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Y_t + \theta_1 Y_{t-1} + \cdots + \theta_q Y_{t-q},$$

if $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$ for $|z| = 1$. Furthermore if $\phi(z) \neq 0$ for $|z| \leq 1$ show that $\{X_t\}$ is a causal function of $\{Y_t\}$.

Solution:

Note that for the proof of Theorem 3.1.1, the fact that Z_t is white noise is never used. We only use the fact that $|z| \leq 1$ to prove the existence of a power series $\zeta(z)$ such that $\zeta(B)$ exists and satisfies

$$1/\phi(z) = \zeta(z) \implies X_t = \zeta(B)\theta(B) Y_t.$$

However, I don't know exactly why not having roots at the unit circle implies stationarity of the solution.

Exercise 3.8.

The process $X_t = Z_t - Z_{t-1}$, $\{Z_t\} \sim \mathbf{WN}\, \{0, \sigma^2\}$, is not invertible according to Definition 3.1.4. Show however that $Z_t \in \overline{\mathbf{sp}}\{X_j,\ -\infty < j \leq t\}$ by considering the mean square limit of the sequence $\sum_{j=0}^{n} (1 - j/n) X_{t-j}$ as $n \to \infty$.

Solution:

Note that X_t is a **MA** (2) process given by $X_t = \theta(B)Z_t, (1-B)$ and note that $\theta(z) = 1 - z$ has a zero in $z = 1$. Therefore, by theorem 3.1.2., $\{X_t\}$ is not invertible.

$$\begin{aligned} Z_t &= X_t + Z_{t-1} \\ &= \vdots \\ &= \left(\sum_{k=0}^{n} X_{t-k} \right) + Z_{t-n-1} \end{aligned}$$

Now, take $Z_t^{(n)} = \sum_{j=0}^{n} (1 - j/n) X_{t-j}$

$$\begin{aligned} \|Z_t^{(n)} - Z_t\|^2 &= \left\| \left(\sum_{k=0}^{n} X_{t-k} \right) + Z_{t-n-1} - \sum_{j=0}^{n} (1 - j/n) X_{t-j} \right\|^2 \\ &= \left\| Z_{t-n-1} + \sum_{j=0}^{n} \frac{j}{n} X_{t-j} \right\|^2 \\ &= \left\| Z_{t-n-1} + \sum_{j=0}^{n} \frac{j}{n} Z_{t-j} - \sum_{j=0}^{n} \frac{j}{n} Z_{t-j-1} \right\|^2 \\ &= \left\| Z_{t-n-1} + \sum_{j=0}^{n} \frac{j}{n} Z_{t-j} - \sum_{j=1}^{n+1} \frac{j-1}{n} Z_{t-j} \right\|^2 \\ &= \left\| Z_{t-n-1} + \frac{1}{n} Z_t + \sum_{j=1}^{n} \frac{1}{n} Z_{t-j} - Z_{t-n-1} \right\|^2 \\ &= \left\| \sum_{j=0}^{n} \frac{1}{n} Z_{t-j} \right\|^2 \end{aligned}$$

Finally, by the Law of Large Numbers, $\sum_{j=0}^{n} \frac{1}{n} Z_{t-j} \to \mathbf{E}\, Z_t = 0$ with probability 1. Therefore,

$$\left\| \sum_{j=0}^{n} \frac{1}{n} Z_{t-j} \right\|^2 = \mathbf{E} \left| \sum_{j=0}^{n} \frac{1}{n} Z_{t-j} \right|^2 \xrightarrow{P} 0.$$

It follows that since Z_t is the limit of $Z_t^{(n)} \in \mathbf{sp}\{X_j,\ -\infty < j \le t\}$, $Z_t \in \overline{\mathbf{sp}}\{X_j,\ -\infty < j \le t\}$.

Exercise 3.9.

Suppose $\{X_t\}$ is a two-sided moving average

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \qquad \{Z_t\} \sim \mathbf{WN}\left(0, \sigma^2\right)$$

where $\sum_j |\psi_j| < \infty$. Show that $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$ where $\gamma(\cdot)$ is the autocovariance function of $\{X_t\}$.

Solution:

Note that from the definition,

$$\begin{aligned}
\gamma(h) &= \mathbf{cov}\left(X_t, X_{t+h}\right) \\
&= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_i \psi_j \mathbf{E}\, Z_{t-i} Z_{t+h-j} \\
&= \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}
\end{aligned}$$

Finally,

$$\begin{aligned}
\sum_{h=-\infty}^{\infty} |\gamma(h)| &= \sigma^2 \sum_{h=-\infty}^{\infty} \left| \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \right| \\
&\leq \sigma^2 \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\psi_j| |\psi_{j+h}| \\
&= \sigma^2 \sum_{j=-\infty}^{\infty} |\psi_j| \sum_{h=-\infty}^{\infty} |\psi_{j+h}| \\
&\leq \sigma^2 \left(\sum_{j=-\infty}^{\infty} |\psi_j| \right)^2 < \infty
\end{aligned}$$

Exercise 3.13.

Find the coefficients $\psi_j,\ j = 0, 1, 2, \ldots$, in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

of the **ARMA** $(2,1)$ process,

$$(1-.5B+.04B^2)X_t = (1+.25B)Z_t, \qquad \{Z_t\} \sim \mathbf{WN}\,(0,\sigma^2).$$

Solution:

In the first place note that

$$\phi(z) = 1-.5z+.04z^2 = \frac{(z-10)\left(z-\frac{5}{2}\right)}{25}$$

which doesn't have any zeros in the unit disk. Then, the Taylor series expansion for $1/\phi(z)$ and $|z| \leq 1$, is the following (I used a symbolic calculator again):

$$\begin{aligned}
\phi^{-1}(z) &= \sum_{k=0}^{\infty} z^k \left(\frac{-10^{-k}+4\left(\frac{5}{2}\right)^{-k}}{3}\right) \\
&= \sum_{k=0}^{\infty} z^k \left(\frac{-10^{-k}+4\left(\frac{10}{4}\right)^{-k}}{3}\right). \\
&= \sum_{k=0}^{\infty} z^k \left(\frac{-10^{-k}(4^{1+k}-1)}{3}\right)
\end{aligned}$$

Finally,

$$\begin{aligned}
\psi(z) &= \zeta(z)\theta(z) = \zeta(z) + .25z\zeta(z) \\
&= \sum_{k=0}^{\infty} z^k \left(\frac{-10^{-k}(4^{1+k}-1)}{3}\right) + \frac{1}{4}\sum_{k=0}^{\infty} z^k \left(\frac{-10^{-k}(4^{1+k}-1)}{3}\right) \\
&= \frac{4}{3} + \sum_{k=1}^{\infty} z^k \left(\frac{10^{-k}(4^{1+k}-1)+\frac{1}{4}\cdot 10^{-k+1}(4^{1+k-1}-1)}{3}\right) \\
&= \frac{4}{3} + \frac{1}{3}\sum_{k=1}^{\infty} z^k \frac{(13\cdot 4^k-7)}{2\cdot 10^{-k}}
\end{aligned}$$

Thus,

$$\psi_k = \begin{cases} \dfrac{4}{3}, & k=0 \\ \dfrac{13\cdot 4^k-7}{6\cdot 10^{-k}}, & \text{o.w.} \end{cases}$$