Time Series: Homework 4

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Exercise 3.19.

Suppose that $\{X_t\}$ and $\{Y_t\}$ are two zero-mean stationary processes with the same autocovariance function and that $\{Y_t\}$ is an ARMA(p, q) process. Show that $\{X_t\}$ must also be an ARMA(p, q) process. (Hint: If $\phi_1, \dots, \phi_p$ are the AR coefficients for $\{Y_t\}$, show that $\{W_t := X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}\}$ has an autocovariance function which is zero for lags $|h| > q$. Then apply Proposition 3.2.1 to $\{W_t\}$.)

Solution

Let $\gamma(\cdot)$ be the autocovariance function for both $\{X_t\}$ and $\{Y_t\}$.

If $\{Y_t\}$ is an **ARMA** (p, q) process, then there exists $\{Z_t\} \sim \mathbf{WN}\left(0, \sigma^2\right)$ (with $\sigma > 0$) and functions $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$ such that

$$\phi(B) X_t = \theta(B) Z_t.$$

Too avoid ambiguity let $\theta_0 = 1$ and $\theta_q \neq 0$.

Now, define $W_t = \phi(B) X_t$ and $W_t' = \phi(B) Y_t$. Then note that $\mathbf{E}\left(W_t\right) = 0$ and $\mathbf{E}\left(W_t^2\right) < \infty$. In order to prove there exists an autocovariance function for $\{W_t\}$, note that $\{W_t'\}$ is stationary because $W_t' = \theta(B) Z_t$ is a sum of uncorrelated stationary random variables. Therefore, there exists an autocovariance function $\omega(\cdot) = \mathbf{cov}\left(W_t, W_{t+h}\right)$, $\forall t \in \mathbb{Z}$, and from the following calculations we can deduce that the autocovariance function of the process

$\{W_t\}$ is ω too:

$$\begin{aligned}
\mathbf{cov}\left(W_t, W_{t+h}\right) &= \mathbf{E}\left[W_t W_{t+h}\right] \\
&= \sum_{i=0}^{p}\sum_{j=0}^{p} \phi_i \phi_j \mathbf{E}\left[X_{t-i}X_{t+h-j}\right] \\
&= \sum_{i=0}^{p}\sum_{j=0}^{p} \phi_i \phi_j \gamma(h-j+i) \\
&= \sum_{i=0}^{p}\sum_{j=0}^{p} \phi_i \phi_j \mathbf{E}\left[Y_{t-i}Y_{t+h-j}\right] \\
&= \mathbf{E}\left[W'_t W'_{t+h}\right] = \omega(h).
\end{aligned}$$

Then, note that for $|h| > q$, $\mathbf{E}\left[Z_{h-i}Z_{-j}\right] = 0$ for every $i, j \in \{0, \ldots, q\}$ because in order to $\mathbf{E}\left[Z_{h-i}Z_{-j}\right] = \sigma^2 \neq 0$, it must happen that $h - i = -j \iff h = i - j$ but in this case $i - j \leq q$. Therefore,

$$\begin{aligned}
\omega(h) &= \mathbf{E}\left[W'_h W'_0\right] \\
&= \mathbf{E}\left[\theta(B)Z_h \theta(B) Z_0\right] \\
&= \sum_{i=0}^{q}\sum_{j=0}^{q} \theta_i \theta_j \mathbf{E}\left[Z_{h-i}Z_{-j}\right] \\
&= 0.
\end{aligned}$$

Finally, if $i, j \in \{0, \ldots, q\}$, then $\mathbf{E}\left[Z_{q-i}Z_{-j}\right] \neq 0$ only if $i = q, j = 0$. Thus,

$$\omega(q) = \sum_{i=0}^{q}\sum_{j=0}^{q} \theta_i \theta_j \mathbf{E}\left[Z_{q-i}Z_{-j}\right] = \theta_q \theta_0 \sigma^2 \neq 0.$$

Proposition 3.2.1. If $\{W_t\}$ is a zero-mean stationary process with autocovariance function $\omega(\cdot)$ such that $\omega(h) = 0$ for $|h| > q$ and $\omega(q) \neq 0$, then $\{W_t\}$ is an $MA(q)$ process, i.e. there exists a white noise process $\{Z'_t\}$ such that

$$W_t = Z'_t + \theta_1 Z'_{t-1} + \cdots + \theta_q Z'_{t-q}.$$

So from this proposition follows that X_t is and $\mathbf{ARMA}\,(p, q)$ process.

Exercise 3.22.

If $X_t = Z_t - \theta Z_{t-1}$, $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $|\theta| < 1$, show from the prediction equations that the best linear predictor of X_{n+1} in $\overline{\mathrm{sp}}\{X_1, \ldots, X_n\}$ is

$$\hat{X}_{n+1} = \sum_{j=1}^{n} \phi_j X_{n+1-j},$$

where $\phi_1, \ldots, \phi_n$ satisfy the difference equations,

$$-\theta\phi_{j-1} + (1+\theta^2)\phi_j - \theta\phi_{j+1} = 0, \quad 2 \le j \le n-1,$$

with boundary conditions,

$$(1+\theta^2)\phi_n - \theta\phi_{n-1} = 0$$

and

$$(1+\theta^2)\phi_1 - \theta\phi_2 = -\theta.$$

Solution

The prediction equations state that

$$\left\langle X_{n+1} - \sum_{j=1}^{n} \phi_j X_{n+1-j}, X_k \right\rangle = 0, \qquad k = n, n-1, \ldots, 1.$$

Equivalently, we know that if $\{X_t\}$ is a stationary process with autocovariance function $\gamma(\cdot)$, then $\phi = (\phi_1, \ldots, \phi_n)^T$ is the solution for the equation

$$\Gamma_n \phi = \gamma_n,$$

where $\gamma_n = (\gamma(1), \ldots, \gamma(n))^T$ and $\Gamma_n = [\gamma(i-j)]_{i,j=1}^n$. Note that

$$\begin{aligned}
\gamma(0) &= \mathbf{E}\left[X_0 X_0\right] \\
&= \mathbf{E}\left[Z_0 Z_0\right] - \theta\mathbf{E}\left[Z_0 Z_{-1}\right] - \theta\mathbf{E}\left[Z_{-1} Z_0\right] + \theta^2\mathbf{E}\left[Z_{-1} Z_{-1}\right] \\
&= \sigma^2(1+\theta^2).
\end{aligned}$$

$$\begin{aligned}
\gamma(-1) = \gamma(1) &= \mathbf{E}\left[X_1 X_0\right] \\
&= \mathbf{E}\left[Z_1 Z_0\right] - \theta\mathbf{E}\left[Z_1 Z_{-1}\right] - \theta\mathbf{E}\left[Z_0 Z_0\right] + \theta^2\mathbf{E}\left[Z_0 Z_{-1}\right] \cdot \\
&= -\sigma^2\theta.
\end{aligned}$$

Then, for $a = (1+\theta^2)$ and $b = -\theta$, we have the following system:

$$\sigma^2 \begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_n \end{bmatrix} = \sigma^2 \begin{bmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The first line of this system says

$$(1+\theta^2)\phi_1 - \theta\phi_2 = a\phi_1 + b\phi_2 = b = -\theta.$$

From the 2nd to $(n-1)$-th line we have

$$-\theta\phi_{j-1} + (1+\theta^2)\phi_j - \theta\phi_{j+1} = b\phi_{j-1} + a\phi_j - +b\phi_{j+1} = 0.$$

Finally, the n-th line gives us

$$(1+\theta^2)\phi_n - \theta\phi_{n-1} = a\phi_n + b\phi_{n-1} = 0.$$

Exercise 3.23.

Use Definition 3.4.2 and the results of Problem 3.22 to determine the partial autocorrelation function of a moving average of order 1.

Solution

Definition 3.4.2. The partial autocorrelation $\alpha(k)$ of $\{X_t\}$ at lag $\boldsymbol{k}$ is

$$\alpha(k) = \phi_{kk}, \qquad k \geq 1,$$

where ϕ_{kk} is uniquely determined by the following system

$$\begin{bmatrix} a & b & 0 & \cdots & 0 \\ b & a & b & \cdots & 0 \\ 0 & b & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} \phi_{n,1} \\ \phi_{n,2} \\ \phi_{n,3} \\ \vdots \\ \phi_{n,n} \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

the standard linear time elimination algorithm for tridiagonal systems gives us the following recursive solution

$$\begin{aligned} \phi_{n,n} &= \alpha_n = \alpha(n), \\ \phi_{n,i} &= \alpha_i - \beta_i \phi_{n,i+1} \qquad i \leq n-1 \end{aligned}$$

with

$$\alpha_i = \begin{cases} \dfrac{b}{a}, & i = 1, \\ \dfrac{-b\alpha_{i-1}}{a - b\beta_{i-1}} & i \geq 2, \end{cases}, \qquad \beta_i = \begin{cases} \dfrac{b}{a}, & i = 1, \\ \dfrac{b}{a - b\beta_{i-1}} & i \geq 2, \end{cases}.$$

We then have,

$$\alpha_2 = \frac{-b\alpha_1}{a - b\beta_1} = \frac{-b^2 a^{-1}}{a - b^2 a^{-1}} = \frac{b^2}{b^2 - a^2}$$

I calculated the first 10 coefficients for this solution

$$\alpha_2 = \frac{b^2}{b^2 - a^2} \qquad \beta_2 = \frac{ab}{a^2 - b^2}$$

$$\alpha_3 = \frac{b^3}{a\left(a^2 - 2b^2\right)} \qquad \beta_3 = \frac{b\left(a^2 - b^2\right)}{a\left(a^2 - 2b^2\right)}$$

$$\alpha_4 = -\frac{b^4}{a^4 - 3a^2b^2 + b^4} \qquad \beta_4 = \frac{ab\left(a^2 - 2b^2\right)}{a^4 - 3a^2b^2 + b^4}$$

$$\alpha_5 = \frac{b^5}{a\left(a^4 - 4a^2b^2 + 3b^4\right)} \qquad \beta_5 = \frac{b\left(a^4 - 3a^2b^2 + b^4\right)}{a\left(a^4 - 4a^2b^2 + 3b^4\right)}$$

$$\alpha_6 = -\frac{b^6}{a^6 - 5a^4b^2 + 6a^2b^4 - b^6} \qquad \beta_6 = \frac{ab\left(a^4 - 4a^2b^2 + 3b^4\right)}{a^6 - 5a^4b^2 + 6a^2b^4 - b^6}$$

$$\alpha_7 = \frac{b^7}{a\left(a^6 - 6a^4b^2 + 10a^2b^4 - 4b^6\right)} \qquad \beta_7 = \frac{b\left(a^6 - 5a^4b^2 + 6a^2b^4 - b^6\right)}{a\left(a^6 - 6a^4b^2 + 10a^2b^4 - 4b^6\right)}$$

$$\alpha_8 = -\frac{b^8}{a^8 - 7a^6b^2 + 15a^4b^4 - 10a^2b^6 + b^8} \qquad \beta_8 = \frac{ab\left(a^6 - 6a^4b^2 + 10a^2b^4 - 4b^6\right)}{a^8 - 7a^6b^2 + 15a^4b^4 - 10a^2b^6 + b^8}$$

$$\alpha_9 = \frac{b^9}{a\left(a^8 - 8a^6b^2 + 21a^4b^4 - 20a^2b^6 + 5b^8\right)} \qquad \beta_9 = \frac{b\left(a^8 - 7a^6b^2 + 15a^4b^4 - 10a^2b^6 + b^8\right)}{a\left(a^8 - 8a^6b^2 + 21a^4b^4 - 20a^2b^6 + 5b^8\right)}$$

$$\alpha_{10} = -\frac{b^{10}}{a^{10} - 9a^8b^2 + 28a^6b^4 - 35a^4b^6 + 15a^2b^8 - b^{10}} \qquad \beta_{10} = \frac{ab\left(a^8 - 8a^6b^2 + 21a^4b^4 - 20a^2b^6 + 5b^8\right)}{a^{10} - 9a^8b^2 + 28a^6b^4 - 35a^4b^6 + 15a^2b^8 - b^{10}}$$

After substituting back again to $a = 1 + \theta^2$ and $b = -\theta$ we obtain

$$\alpha_2 = -\frac{\theta^2}{\theta^4 + \theta^2 + 1} \qquad \beta_2 = \frac{-\theta^3 - \theta}{\theta^4 + \theta^2 + 1}$$

$$\alpha_3 = -\frac{\theta^3}{\left(\theta^2 + 1\right)\left(\theta^4 + 1\right)} \qquad \beta_3 = -\frac{\theta\left(\theta^4 + \theta^2 + 1\right)}{\left(\theta^2 + 1\right)\left(\theta^4 + 1\right)}$$

$$\alpha_4 = -\frac{\theta^4}{\theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \qquad \beta_4 = \frac{-\theta^7 - \theta^5 - \theta^3 - \theta}{\theta^8 + \theta^6 + \theta^4 + \theta^2 + 1}$$

$$\alpha_5 = -\frac{\theta^5}{\theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \qquad \beta_5 = \frac{-\theta^9 - \theta^7 - \theta^5 - \theta^3 - \theta}{\theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1}$$

$$\alpha_6 = -\frac{\theta^6}{\theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \qquad \beta_6 = \frac{-\theta^{11} - \theta^9 - \theta^7 - \theta^5 - \theta^3 - \theta}{\theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1}$$

$$\alpha_7 = -\frac{\theta^7}{\theta^{14} + \theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1} \qquad \beta_7 = \frac{-\theta^{13} - \theta^{11} - \theta^9 - \theta^7 - \theta^5 - \theta^3 - \theta}{\theta^{14} + \theta^{12} + \theta^{10} + \theta^8 + \theta^6 + \theta^4 + \theta^2 + 1}$$

Finally, I believe that the closed form formula for the autocorrelation function of $\{X_t\}$ is the following

$$\alpha(n) = \frac{-\theta^n}{\sum_{i=0}^{n} \theta^{2i}}$$

Exercise 3.24.

Let $\{X_t\}$ be the stationary solution of $\phi(B)X_t = \theta(B)Z_t$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, $\phi(z) \neq 0$ for all $z \in \mathbb{C}$ such that $|z| = 1$, and $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeroes. If A is any zero-mean random variable in L^2 which is uncorrelated with $\{X_t\}$ and if $|z_0| = 1$, show that the process $\{X_t + Az_0^t\}$ is a complex-valued stationary process (see Definition 4.1.1) and that $\{X_t + Az_0^t\}$ and $\{X_t\}$ both satisfy the equations $(1 - z_0B)\phi(B)X_t = (1 - z_0B)\theta(B)Z_t$.

Solution

Definition 4.1.1. The process $\{X_t\}$ is a complex-valued stationary process if $E|X_t|^2 < \infty$, EX_t is independent of t and $E(X_{t+h}\bar{X}_t)$ is independent of t As already pointed out in Example 2.2.3, Remark 1, the complex-valued random variables x on $(\Omega, \mathcal{F}, P)$ satisfying $E|X|^2 < \infty$ constitute a Hilbert space with the inner product

$$\langle X, Y\rangle = E(X\bar{Y}).$$

- For $\mathbf{E}\left[|X_t + Az_0^t|^2\right]$, since A and X_t are uncorrelated, $\mathbf{E}\left[X_t\overline{A}\right] = \mathbf{E}\left[A\overline{X_t}\right] = 0$. Thus,

$$\begin{aligned} \mathbf{E}\left[|X_t + Az_0^t|^2\right] &= \mathbf{E}\left[(X_t + Az_0^t)(\overline{X_t + Az_0^t})\right] \\ &= \mathbf{E}\left[X_t\overline{X_t}\right] + z_0^t\mathbf{E}\left[X_t\overline{A}\right] + z_0^t\mathbf{E}\left[A\overline{X_t}\right] + |z_0|^{2t}\mathbf{E}\left[A\overline{A}\right] \\ &= \underbrace{\mathbf{E}\left[|X_t|^2\right]}_{<\infty} + 0 + 0 + \underbrace{\mathbf{E}\left[|A|^2\right]}_{<\infty} \\ &< \infty. \end{aligned}$$

- For $\mathbf{E}\left[|X_t + Az_0^t|^2\right]$, since $\{X_t\}$ is stationary, $\mathbf{E}\left[X_t\right] = \mu$ for every $t \in \mathbb{Z}$. Therefore,

$$\mathbf{E}\left[X_t + z_0^tA\right] = \mathbf{E}\left[X_t\right] + z_0^t\mathbf{E}\left[A\right] = \mu + 0 = \mu.$$

- For the existence of an autocovariance function,

$$\begin{aligned} \mathbf{E}\left[(X_{t+h} + z_0^tA - \mu)(\overline{X_t + z_0^tA - \mu})\right] = \quad & \mathbf{E}\left[(X_{t+h} - \mu)(\overline{X_t - \mu})\right] + z_0^t\mathbf{E}\left[(X_{t+h} - \mu)\overline{A}\right] \\ & + z_0^t\mathbf{E}\left[A(\overline{X_t} - \mu)\right] + \underbrace{|z_0|^{2t}}_{=1}\mathbf{E}\left[A\overline{A}\right] \\ = \quad & \gamma(h) + 0 + 0 + \mathbf{E}\left[|A|^2\right] \end{aligned}$$

Finally, note that by defining $Y_t = Az_0^t$, then $BY_t = Az_0^{t-1} = \frac{Y_t}{z_0}$. Thus,

$$\begin{aligned}(1 - z_0 B)\phi(B)Y_t &= \phi(B)Y_t - z_0 \sum_{n=0}^{\infty} \phi_i BY_t \\ &= \phi(B)Y_t - \sum_{n=0}^{\infty} \phi_i \frac{z_0 Y_t}{z_0} \\ &= \phi(B)Y_t - \phi(B)Y_t = 0.\end{aligned}$$

Since it's trivially true that $(1 - z_0 B)\phi(B)X_t = (1 - z_0 B)\theta(B)Z_t$ and $(1 - z_0 B)\phi(B)Y_t = 0$, it follows that

$$\begin{aligned}(1 - z_0 B)\phi(B)(X_t + Y_t) &= (1 - z_0 B)\phi(B)X_t + (1 - z_0 B)\phi(B)Y_t \\ &= (1 - z_0 B)\phi(B)X_t + 0 \\ &= (1 - z_0 B)\theta(B)Z_t.\end{aligned}$$

Exercise 5.1.

Let $\{X_\iota\}$ be a stationary process with mean μ . Show that

$$P_{\overline{\text{sp}}|\,1\,X_1,\ldots,X_n|} X_{n+h} = \mu \, + \, P_{\overline{\text{sp}}\{Y_1,\ldots,Y_n\}} Y_{n+h},$$

where $\{Y_t\} = \{X_t - \mu\}$.

Solution

Let $\hat{X}_{n+h} = P_{\overline{\text{sp}}|\,1\,X_1,\ldots,X_n|} X_{n+h}$ and $\hat{Y}_{n+h} = P_{\overline{\text{sp}}\{Y_1,\ldots,Y_n\}} Y_{n+h}$. Then, from the definition of projection, note that for some $\{\psi_i\}_{i=1}^n$,

$$\hat{Y}_{n+h} = \sum_{i=1}^{n} \phi_i Y_{n+1-i}.$$

Therefore,

$$\mathbf{E}\left[X_{n+h} - \mu - \hat{Y}_{n+h}\right] = \mathbf{E}\left[X_{n+h} - \mu\right] - \sum_{i=1} \phi_i \mathbf{E}\left[Y_{n+1-i}\right] = 0$$

Finally, for every $k \in \{1, \ldots, n\}$

$$\begin{aligned}\left\langle X_{n+h} - \mu - \hat{Y}_{n+h},\ X_k \right\rangle &= \mathbf{E}\left[(X_{n+h} - \mu - \hat{Y}_{n+h})(X_k - \mu)\right] \\ &= \mathbf{E}\left[(Y_{n+h} - \hat{Y}_{n+h})(Y_k)\right] \\ &= 0,\end{aligned}$$

and, since $\mathbf{E}[1] = 1$,

$$\left\langle X_{n+h} - \mu - \hat{Y}_{n+h},\ 1 \right\rangle = \mathbf{E}\left[(X_{n+h} - \mu - \hat{Y}_{n+h})(1-1)\right] = 0.$$

From the uniqueness of the projection, it follows that $\hat{X}_{n+h} = \mu + \hat{Y}_{n+h}$.

Exercise 5.5.

Let $\{X_t\}$ be the $\mathbf{MA}(1)$ process of Example 5.2.1. If $|\theta| < 1$, show that as $n \to \infty$,

(a) $\|X_n - \hat{X}_n - Z_n\| \to 0$,

(b) $v_n \to \sigma^2$,

(c) $\theta_{n1} \to \theta$. (Note that $\theta = E(X_{n+1}Z_n)\sigma^{-2}$ and $\theta_{n1} = v_{n-1}^{-1}E(X_{n+1}(X_n - \hat{X}_n))$.)

Solution

In the first place, there exists $K \geq 0$ such that

$$\begin{aligned}
\|X_n - \hat{X}_n - Z_n\| &= \|\theta Z_{t-1} - \theta(X_{n-1} - \hat{X}_{n-1})\| \\
&= |\theta| \|X_{n-1} - \hat{X}_{n-1} - Z_{t-1}\| \\
&= \vdots \\
&= |\theta|^{n-1} \|X_0 - \hat{X}_0 - Z_0\| = |\theta|^{n-1} K
\end{aligned}$$

Therefore, since $|\theta| < 1$, it follows that $\lim_n \|X_n - \hat{X}_n - Z_n\| = 0$.

Now, I'm going to prove using induction that

$$v_n = \sigma^2 \frac{\sum_{i=0}^{n} \theta^{2i}}{\sum_{i=0}^{n-1} \theta^{2i}}.$$

The base case is true because $v_0 = \sigma^2(1+\theta^2)$. For the inductive step,

$$\begin{aligned}
v_{n+1} &= [1+\theta^2 - \theta^2\sigma^2 v_n^{-1}]\sigma^2 \\
&= \left[1+\theta^2 - \theta^2 \frac{\sum_{i=0}^{n-1}\theta^{2i}}{\sum_{i=0}^{n}\theta^{2i}}\right]\sigma^2 \\
&= \sigma^2 \frac{(1+\theta^2)\sum_{i=0}^{n}\theta^{2i} - \theta^2\sum_{i=0}^{n-1}\theta^{2i}}{\sum_{i=0}^{n}\theta^{2i}} \\
&= \sigma^2 \frac{\sum_{i=0}^{n}\theta^{2i} + \sum_{i=1}^{n+1}\theta^{2i} - \sum_{i=1}^{n}\theta^{2i}}{\sum_{i=0}^{n}\theta^{2i}} \\
&= \sigma^2 \frac{\sum_{i=0}^{n+1}\theta^{2i}}{\sum_{i=0}^{n}\theta^{2i}}.
\end{aligned}$$

Therefore, using the partial geometric series expansion ($\theta^2 < 1$), we know that

$$v_n = \sigma^2 \frac{\frac{1-\theta^{2n+2}}{1-r}}{\frac{1-\theta^{2n}}{1-r}} = \sigma^2 \frac{1-\theta^{2n+2}}{1-\theta^{2n}},$$

and we know v_n is decreasing because since $\theta^2 < 1$ that implies

$$\theta^{2n+2} < \theta^{2n+2}$$

$$\implies 1-\theta^{2n} < 1-\theta^{2n}$$

$$\implies \quad \frac{v_{n-1}}{v_n} = \frac{1-\theta^{2n}}{1-\theta^{2n+2}} < 1$$

This also implies that r_n is decreasing. As a matter of fact, this proves that $v_n \to \sigma^2$ and $r_n \to 1$ as $n \to \infty$ because $\theta^{2n} \to 0$.

Finally, since $v_n \to \sigma^2$, it follows that

$$\theta_{n1} = \theta \frac{\sigma^2}{v_{n-1}} \to \theta,$$

as $n \to \infty$.