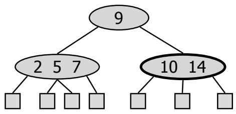
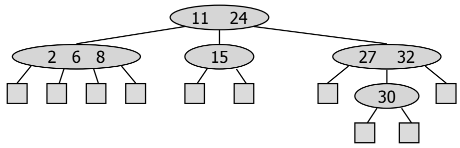
(2,4) Trees



Multi-Way Search Tree

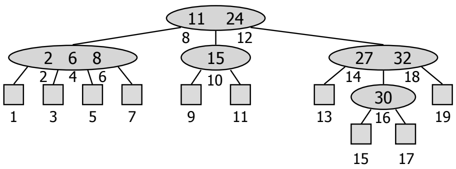


Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\, v_2 \ldots v_d$ storing keys $k_1\, k_2 \ldots k_{d-1}$:
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i ($i = 2, \ldots, d-1$)
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders

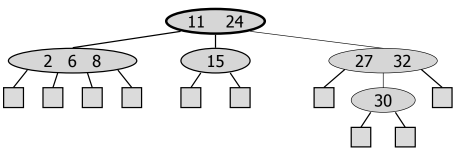
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



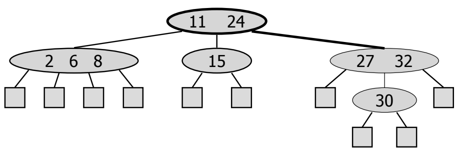
Multi-Way Searching

Searching for 30:



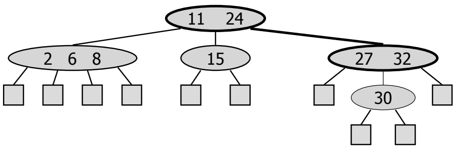
Multi-Way Searching

Searching for 30:



Multi-Way Searching

Searching for 30:



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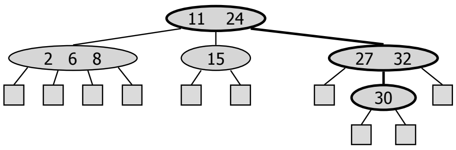
Multi-Way Searching

Searching for 30:



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Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1 v_2 \ldots v_d$ and keys $k_1 k_2 \ldots k_{d-1}$
  - $k = k_i$ ($i = 1, \ldots, d-1$): the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ ($i = 2, \ldots, d-1$): we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully

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(2,4) Trees

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node

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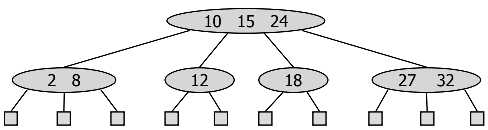
(2,4) Trees



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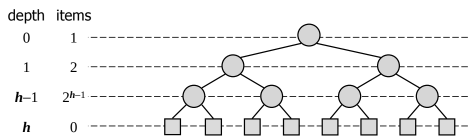
Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - As in proper binary trees, there are at least 2^i items at depth i: $n \geq 2^h - 1$
  - Thus, $h \leq \log(n + 1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



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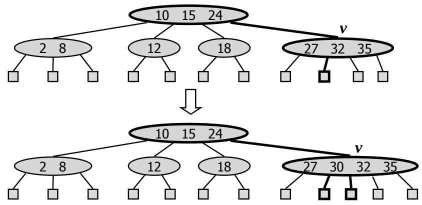
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow



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Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u

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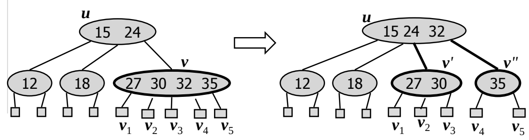
Overflow and Split



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Analysis of Insertion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Finding insertion point takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Inserting the new entry takes $O(1)$ time
  - Dealing with overflow takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits in the worst case.
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

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Exercise

Starting with an empty (2,4) tree, insert items with keys 1,2,3,4,5,6,7

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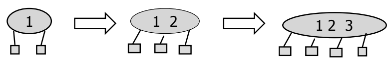
Exercise



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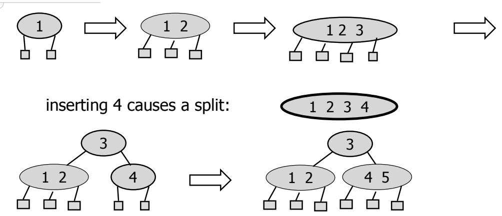
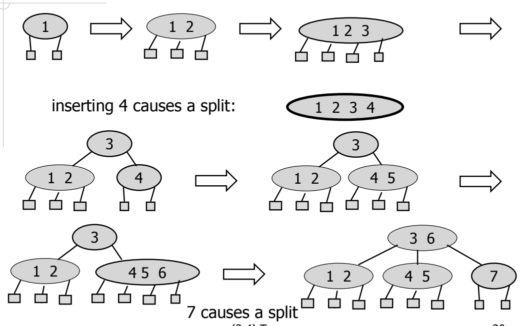
Exercise

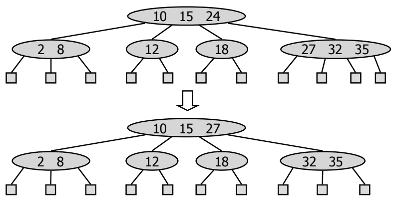


Exercise



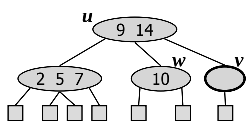
Deletion

- We reduce deletion of an entry to the case where the item is at the node with leaf children
- Otherwise, we replace the entry with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter entry
- Example: to delete key 24, we replace it with 27 (inorder successor)



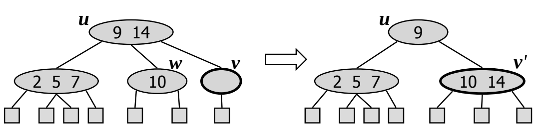
Underflow

- Deleting an entry from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys



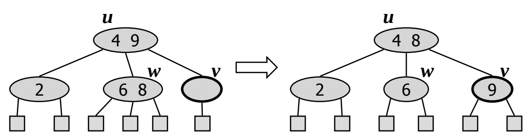
Underflow and Fusion

- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Underflow and Transfer

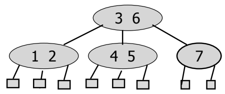
- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs



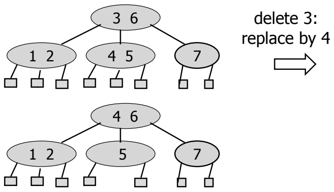
Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the entry
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

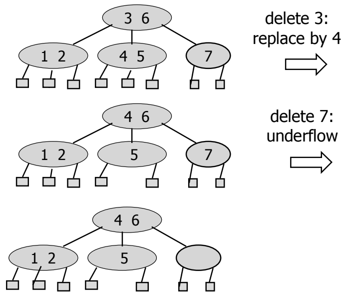
Exercise delete 3,7



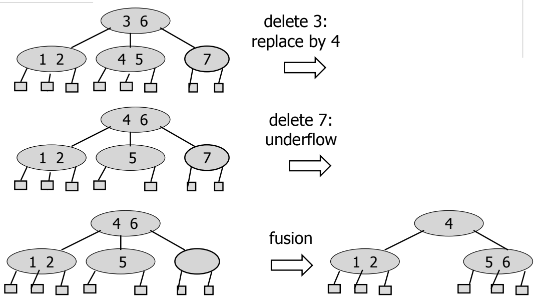
Exercise delete 3,7



Exercise delete 3,7



Exercise delete 3,7



Complexity comparison

- Comparison of hash tables and (2,4) trees

	Search	Insert	Delete	Notes
Hash Table	1 expected	1 expected	1 expected	◈ simple to implement ◈ Requires allocating a lot of memory in advance
(2,4) Tree	$\log n$ worst-case	$\log n$ worst-case	$\log n$ worst-case	◈ complex to implement ◈ dynamic memory use