# Introduction to Logic

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# Propositional Logic

Talking Head

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# Propositional Logic Syntax

## Propositional Sentences

Simple Sentences
  express simple facts about the world

Compound sentences
  express logical relationships
  among simpler sentences of which composed

## Simple Sentences

In Propositional Logic, simple sentences take the form of atomic symbols, called *proposition constants*.

By convention (in this course), proposition constants are written as strings of letters, digits, and the special character _.

| Examples: | Non-Examples: |
| --- | --- |
| *raining* | 324567 |
| *r32aining* | *raining-or-snowing* |
| *rAiNiNg* | |
| *raining_or_snowing* | |

## Compound Sentences I

Negations:

$$(\neg p)$$

The argument of a negation is called the *target*.

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Conjunctions:

$$(p \wedge q)$$

The arguments of a conjunction are called *conjuncts*.

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Disjunctions:

$$(p \vee q)$$

The arguments of a disjunction are called *disjuncts*.

## Compound Sentences II

Implications:

$$(p \Rightarrow q)$$

The left argument of an implication is the *antecedent*.
The right argument is the *consequent*.

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Equivalences / Biconditionals:

$$(p \Leftrightarrow q)$$

## Nesting

Note that compound sentences can be nested inside of other compound sentences.

$$((p \wedge q) \wedge r)$$

$$((p \vee q) \vee r)$$

$$(((p \wedge q) \wedge r) \Rightarrow ((p \vee q) \vee r))$$

## Parentheses

Parentheses are messy and sometimes unnecessary.

$$(((p \wedge q) \vee r) \Rightarrow ((p \vee q) \wedge r))$$

Dropping Parentheses makes things simpler.

$(p \wedge q)$ becomes $p \wedge q$

But it can lead to ambiguities.

$((p \wedge q) \vee r)$ becomes $p \wedge q \vee r$
$(p \wedge (q \vee r))$ becomes $p \wedge q \vee r$

## Precedence

Parentheses can be dropped when the structure of an expression can be determined by *precedence*.

$\neg$
$\wedge$
$\vee$
$\Rightarrow \Leftrightarrow$

## Using Precedence

An operand surrounded by two operators associates with the operator of higher precedence. If surrounded by operators of equal precedence, the operand associates with the operator to the right.

$$p \wedge q \vee r \quad \rightarrow \quad ((p \wedge q) \vee r)$$
$$p \vee q \wedge r \quad \rightarrow \quad (p \vee (q \wedge r))$$
$$p \Rightarrow q \Rightarrow r \quad \rightarrow \quad (p \Rightarrow (q \Rightarrow r))$$
$$p \Leftrightarrow q \Leftarrow r \quad \rightarrow \quad (p \Leftrightarrow (q \Rightarrow r))$$
$$\neg p \wedge q \quad \rightarrow \quad ((\neg p) \wedge q)$$

## Propositional Languages

A *propositional vocabulary* is a set/sequence of *proposition constants*.

Given a propositional vocabulary, a *propositional sentence* is either (1) an individual proposition constant or (2) a compound sentence formed from simpler sentences (as previously defined) and that's all.

A *propositional language* is the set of *all* propositional sentences that can be formed from a propositional vocabulary.

# Exercise

Propositional Logic Semantics

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## Truth Assignment

A *propositional truth assignment* is an association between the proposition constants in a propositional language and the truth values *true* or *false*. For simplicity, in what follows we use 1 as a synonym for *true* and 0 as a synonym for *false*.

$$p \xrightarrow{i} 1 \qquad p^i = 1$$
$$q \xrightarrow{i} 0 \qquad q^i = 0$$
$$r \xrightarrow{i} 1 \qquad r^i = 1$$

## Sentential Truth Assignment

A *sentential truth assignment* is an association between arbitrary sentences in a propositional language and the truth values 1 and 0.

$$p^i = 1 \qquad (p \vee q)^i = 1$$
$$q^i = 0 \qquad (q \vee \neg r)^i = 0$$
$$r^i = 1 \qquad ((p \vee q) \wedge \neg(q \vee \neg r))^i = 1$$

Each propositional truth assignment leads to a particular sentential truth assignment by application of operator semantics.

# Negation

Negation:

| $\phi$ | $\neg\phi$ |
|---|---|
| 1 | 0 |
| 0 | 1 |

For example, if the truth value of $p$ is 0, then the truth value of $\neg p$ is 1.

For example, if the truth value of $(p \wedge q)$ is 1, then the truth value of $\neg(p \wedge q)$ is 0.

# Conjunction

Conjunction:

| $\phi$ | $\psi$ | $\phi \wedge \psi$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

## Disjunction

Disjunction:

| $\phi$ | $\psi$ | $\phi \vee \psi$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

NB: The type of disjunction here is called *inclusive or*, which says that a disjunction is true if and only if *at least* one of its disjuncts is true. This contrasts with *exclusive or*, which says that a disjunction is true if and only if an odd number of its disjuncts is true.

## Implication

Implication:

| $\phi$ | $\psi$ | $\phi \Rightarrow \psi$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

NB: The semantics of implication here is called *material implication*. An implication is true if the antecedent is false, whether or not there is a connection to the consequent.

*If George Washington is alive, I am a billionaire.*

# Equivalence

Equivalence:

| $\phi$ | $\psi$ | $\phi \Leftrightarrow \psi$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

# Evaluation Procedure

Start with a propositional truth assignment and a sentence.

(1) Replace proposition constants by their truth values.

(2) Use operator semantics to simplify compound sentences with truth values as arguments.

(3) Repeat this inside-out fashion to produce a value for the sentence as a whole.

# Evaluation Example

Interpretation $i$:

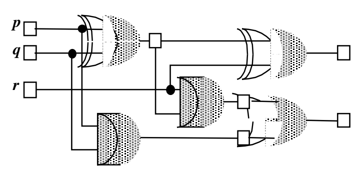
$$p^i = 1$$
$$q^i = 0$$
$$r^i = 1$$

Compound Sentence

$$(p \vee q) \wedge (\neg q \vee r)$$

# More Complex Example

$p^i = 1$
$q^i = 1$
$r^i = 1$




$$(r \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))) \vee (p \wedge q)$$
$$(1 \wedge ((1 \wedge \neg 1) \vee (\neg 1 \wedge 1))) \vee (1 \wedge 1)$$
$$(1 \wedge ((1 \wedge 0) \vee (0 \wedge 1))) \vee (1 \wedge 1)$$
$$(1 \wedge (0 \vee 0)) \vee 1$$
$$(1 \wedge 0) \vee 1$$
$$0 \vee 1$$
$$1$$

## Exercise

## Satisfaction and Falsification

A truth assignment *satisfies* a sentence if and only if it assigns the value 1 to the sentence.

A truth assignment *falsifies* a sentence if and only if it assigns the value 0 to the sentence.

A truth assignment *satisfies* a *set of sentences* if and only if it satisfies *every* element in the set.

A truth assignment *falsifies* a *set of sentences* if and only if it falsifies *at least one* element in the set.

Satisfaction

## Evaluation Versus Satisfaction

Evaluation:

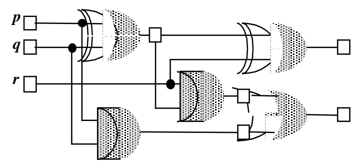
$$\begin{array}{lcl} p^i & = & 1 \\ q^i & = & 0 \end{array} \longrightarrow \begin{array}{lcl} (p \vee q)^i & = & 1 \\ (\neg q)^i & = & 1 \end{array}$$

Satisfaction:

$$\begin{array}{lcl} (p \vee q)^i & = & 1 \\ (\neg q)^i & = & 1 \end{array} \longrightarrow \begin{array}{lcl} p^i & = & 1 \\ q^i & = & 0 \end{array}$$

# Example

$p^i = ?$
$q^i = ?$
$r^i = ?$




$$((r \wedge ((p \wedge \neg q) \vee (\neg p \wedge q))) \vee (p \wedge q))^i = 1$$

# Truth Tables

A *truth table* is a table of all possible truth assignments for the proposition constants in a language.

| $p$ | $q$ | $r$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |
| 0 | 0 | 0 |

One column per constant.

One row per truth assignment.

For a language with $n$ constants, there are $2^n$ truth assignments.

## Satisfaction Procedure

Method to find propositional truth assignments that satisfy a given set of sentences:

(1) Form a truth table for the proposition constants and add columns for each sentence in our set.

(2) Evaluate each sentence for each of the rows of the truth table.

(3) Any row that satisfies all sentences in the set is a solution to the problem.

## Satisfaction Problem

Find a truth assignment that satisfies the following set of sentences.

$$\{q \Rightarrow r,\ p \Rightarrow q \wedge r,\ \neg r\}$$

## Satisfaction Example (start)

| $p$ | $q$ | $r$ | $q \Rightarrow r$ | $p \Rightarrow q \wedge r$ | $\neg r$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | | | |
| 1 | 1 | 0 | | | |
| 1 | 0 | 1 | | | |
| 1 | 0 | 0 | | | |
| 0 | 1 | 1 | | | |
| 0 | 1 | 0 | | | |
| 0 | 0 | 1 | | | |
| 0 | 0 | 0 | | | |

## Satisfaction Example (continued)

| $p$ | $q$ | $r$ | $q \Rightarrow r$ | $p \Rightarrow q \wedge r$ | $\neg r$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | | |
| 1 | 1 | 0 | 0 | | |
| 1 | 0 | 1 | 1 | | |
| 1 | 0 | 0 | 1 | | |
| 0 | 1 | 1 | 1 | | |
| 0 | 1 | 0 | 0 | | |
| 0 | 0 | 1 | 1 | | |
| 0 | 0 | 0 | 1 | | |

## Satisfaction Example (continued)

| $p$ | $q$ | $r$ | $q \Rightarrow r$ | $p \Rightarrow q \wedge r$ | $\neg r$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 0 | 0 | 0 | |
| 1 | 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | 0 | |
| 0 | 1 | 1 | 1 | 1 | |
| 0 | 1 | 0 | 0 | 1 | |
| 0 | 0 | 1 | 1 | 1 | |
| 0 | 0 | 0 | 1 | 1 | |

## Satisfaction Example (concluded)

| $p$ | $q$ | $r$ | $q \Rightarrow r$ | $p \Rightarrow q \wedge r$ | $\neg r$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 |

Exercise

Properties of Sentences

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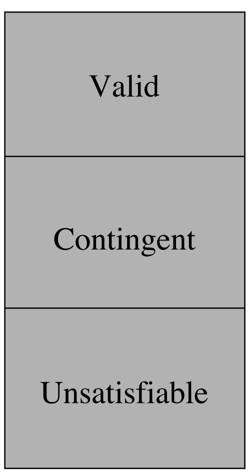
# Properties of Sentences




A sentence is *valid* if and only if *every* interpretation satisfies it.

A sentence is *contingent* if and only if *some* interpretation satisfies it and *some* interpretation falsifies it.

A sentence is *unsatisfiable* if and only if *no* interpretation satisfies it.

# Properties of Sentences

| |
|---|
| Valid |
| Contingent |
| Unsatisfiable |

A sentences is *satisfiable* if and only if it is either valid or contingent.

A sentences is *falsifiable* if and only if it is contingent or unsatisfiable.

# Example of Validity 1

| $p$ | $q$ | $r$ | $(p \Rightarrow q)$ | $(q \Rightarrow r)$ | $(p \Rightarrow q) \vee (q \Rightarrow r)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | | | |
| 1 | 1 | 0 | | | |
| 1 | 0 | 1 | | | |
| 1 | 0 | 0 | | | |
| 0 | 1 | 1 | | | |
| 0 | 1 | 0 | | | |
| 0 | 0 | 1 | | | |
| 0 | 0 | 0 | | | |

## Example of Validity 2

| $p$ | $q$ | $r$ | $(p \Rightarrow q)$ | $(q \Rightarrow r)$ | $(p \Rightarrow q) \vee (q \Rightarrow r)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 0 | 1 | 0 | |
| 1 | 0 | 1 | 0 | 1 | |
| 1 | 0 | 0 | 0 | 1 | |
| 0 | 1 | 1 | 1 | 1 | |
| 0 | 1 | 0 | 1 | 0 | |
| 0 | 0 | 1 | 1 | 1 | |
| 0 | 0 | 0 | 1 | 1 | |

## Example of Validity 3

| $p$ | $q$ | $r$ | $(p \Rightarrow q)$ | $(q \Rightarrow r)$ | $(p \Rightarrow q) \vee (q \Rightarrow r)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 0 | 1 | 0 | |
| 1 | 0 | 1 | 0 | 1 | |
| 1 | 0 | 0 | 0 | 1 | |
| 0 | 1 | 1 | 1 | 1 | |
| 0 | 1 | 0 | 1 | 0 | |
| 0 | 0 | 1 | 1 | 1 | |
| 0 | 0 | 0 | 1 | 1 | |

## Example of Validity 4

| $p$ | $q$ | $r$ | $(p\Rightarrow q)$ | $(q\Rightarrow r)$ | $(p\Rightarrow q)\vee(q\Rightarrow r)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 |

## More Validities

Double Negation:

$$p \Leftrightarrow \neg\neg p$$

deMorgan's Laws:

$$\neg(p\wedge q) \Leftrightarrow (\neg p\vee\neg q)$$
$$\neg(p\vee q) \Leftrightarrow (\neg p\wedge\neg q)$$

Implication Introduction:

$$p \Rightarrow (q \Rightarrow p)$$

Implication Distribution

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

Exercise

Logical Entailment

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## Logical Entailment

A set of premises $\Delta$ *logically entails* a conclusion $\varphi$ (written as $\Delta \models \varphi$) if and only if every interpretation that satisfies the premises also satisfies the conclusion.

$$\{p\} \models (p \vee q)$$

$$\{p\} \not\models (p \wedge q)$$

$$\{p, q\} \models (p \wedge q)$$

## Logical Entailment ≠ Logical Equivalence

$\{p\} \models (p \vee q)$

$\{p \vee q)\} \not\models p$

*Analogy in arithmetic: inequalities rather than equations*

## Truth Table Method

Method for computing whether a set of premises logically entails a conclusion.

(1) Form a truth table for the proposition constants and add a column for the premises and a column for the conclusion.

(2) Evaluate the premises for each row in the table.

(3) Evaluate the conclusion for each row in the table.

(4) If every row that satisfies the premises also satisfies the conclusion, then the premises logically entail the conclusion.

## Example

Does $p$ logically entail $(p \vee q)$?

| $p$ | $q$ | $p$ | $p \vee q$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 |

## Example

Does $p$ logically entail $(p \wedge q)$?

| $p$ | $q$ | $p$ | $p \wedge q$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 |

# Example

Does $\{p,q\}$ logically entail $(p \wedge q)$?

| $p$ | $q$ | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 |

# Example

Problem: $\{(p \Rightarrow q), (m \Rightarrow p \vee q), m\} \models q$?

| $m$ | $p$ | $q$ | $p \Rightarrow q$ | $m \Rightarrow p \vee q$ | $m$ | $q$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 |

## Logical Entailment and Satisfiability

Unsatisfiability Theorem: $\Delta \models \varphi$ if and only if $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Proof: Suppose that $\Delta \models \varphi$. If a truth assignment satisfies $\Delta$, then it must also satisfy $\varphi$. But then it cannot satisfy $\neg\varphi$. Therefore, $\Delta \cup \{\neg\varphi\}$ is unsatisfiable.

Suppose that $\Delta \cup \{\neg\varphi\}$ is unsatisfiable. Then every truth assignment that satisfies $\Delta$ must *fail* to satisfy $\neg\varphi$, i.e. it must satisfy $\varphi$. Therefore, $\Delta \models \varphi$.

Upshot: We can determine logical entailment by determining unsatisfiability.

The Big Game

## The Big Game

*Stanford people always tell the truth, and Berkeley people always lie. Unfortunately, by looking at a person, you cannot tell whether he is from Stanford or Berkeley.*

*You come to a fork in the road and want to get to the football stadium down one fork. However, you do not know which to take. There is a person standing there. What single question can you ask him to help you decide which fork to take?*

## Basic Idea

| *left* | *su* | *Question* | *Response* |
|---|---|---|---|
| 1 | 1 | | |
| 1 | 0 | | |
| 0 | 1 | | |
| 0 | 0 | | |

## Basic Idea

| *left* | *su* | *Question* | *Response* |
|---|---|---|---|
| 1 | 1 | | 1 |
| 1 | 0 | | 1 |
| 0 | 1 | | 0 |
| 0 | 0 | | 0 |

## Basic Idea

| *left* | *su* | *Question* | *Response* |
| --- | --- | --- | --- |
| 1 | 1 | 1 | 1 |
| 1 | 0 | | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | | 0 |

## Basic Idea

| *left* | *su* | *Question* | *Response* |
| --- | --- | --- | --- |
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |

# The Big Game Solved

Question: *Is it the case that the left road the way to the stadium if and only if you are from Stanford?*

$$(\mathit{left} \Leftrightarrow \mathit{su})?$$